Counterexamples in Analysis

A catalogue of possible seminar presentations

The topics below explore how familiar conclusions in analysis can fail when an essential hypothesis is missing. They range from unusual continuous functions, derivatives, and convergent sequences to examples in integration, probability, differential equations, operator theory, and partial differential equations. Each talk centres on a concrete counterexample and the theorem that explains its limits.

The level labels in the table indicate the recommended background: B bachelor, B+ advanced bachelor, M master, and M+ advanced master. Difficulty reflects both the prerequisites and the length, abstraction, and technical demands of the main proof. Bachelor topics generally build on core analysis; advanced bachelor topics require greater fluency or additional tools. Master topics usually draw on graduate-level material, while advanced master topics call for a carefully delimited proof goal and closer guidance. The labels are a guide, not a restriction: familiarity with the area matters as much as your stage of study.

How to choose. Start with a problem that interests you, then consult its detailed presentation plan and suggested reading. Check which definitions and theorems you already know, what you would need to learn, and whether you can explain the main construction and proof clearly in 50 minutes. Prefer a topic you can develop in depth over one whose main argument you would have to omit. Discuss your choice with the instructor if you are unsure about the prerequisites or scope.

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No.	Topic	Area	Level
I. Classical real-variable constructions			
1	**The Weierstrass function: continuous everywhere, differentiable nowhere** Opening lecture	Real analysis; Fourier series	B+
2	**The Cantor function: derivative zero almost everywhere, but not constant**	Real analysis; measure theory	B+
3	**Thomae's popcorn function: continuity at every irrational and nowhere else**	Real analysis; Riemann integration	B
4	**A fat Cantor set: nowhere dense, yet of positive measure**	Measure theory; topology	B+
5	**Smooth does not mean analytic: a nonzero flat function**	Real analysis	B
6	**A derivative can be discontinuous and unbounded, but never have a jump**	Differential calculus; real analysis	B+
7	**Rolle's theorem fails for vector-valued functions**	Vector calculus; real analysis	B
8	**Mixed partial derivatives need not commute**	Multivariable calculus; real analysis	B+
9	**Directional derivatives in every direction do not imply differentiability**	Multivariable calculus	B+
10	**Lagrange multipliers can miss a constrained extremum**	Multivariable calculus; optimization	B
11	**Pathological solutions of Cauchy's functional equation**	Real analysis; functional equations	B+
II. Modes of convergence, integration, summation, and probability			
12	**Pointwise convergence does not preserve continuity**	Real analysis	B
13	**Moving spikes and the failure of implications between convergence notions**	Measure theory; L^p spaces	B+
14	**L^p spaces are not ordered on an infinite-measure domain**	Measure theory; function spaces	B

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Counterexamples in Analysis – continued

No.	Topic	Area	Level
15	**Why L^∞ behaves differently: smooth functions are not dense**	Measure theory; functional analysis	B+
16	**Pairwise independence does not imply mutual independence**	Probability; measure theory	B+
17	**What fails in Fubini's theorem without absolute integrability?**	Integration; infinite series	B+
18	**The Cauchy product of two convergent series may diverge**	Infinite series	B+
19	**The Riemann rearrangement theorem**	Infinite series	B
20	**A convergent improper integral whose integrand does not tend to zero**	Improper integrals; oscillation	B
III. Geometry, topology, and elementary dynamics			
21	**Nonuniqueness for an ODE when Lipschitz continuity is removed**	Ordinary differential equations	B
22	**A smooth ODE can develop a finite-time singularity**	Ordinary differential equations	B+
23	**Newton's method can cycle instead of finding a root**	Numerical analysis; nonlinear equations	B
24	**A contraction need not have a fixed point without completeness**	Metric spaces; fixed-point theory	B
25	**A continuous bijection need not have a continuous inverse**	Metric topology	B
26	**A space-filling curve: a continuous surjection from an interval onto a square**	Metric topology; geometric analysis	B+
27	**Connected does not imply path connected: the topologist's sine curve**	Topology in analysis	B+
28	**The Koch snowflake: finite area but infinite perimeter**	Geometric analysis; fractals	B+
29	**The Vitali set: a subset of $[0,1]$ that is not Lebesgue measurable**	Measure theory; set-theoretic methods	B+
IV. Functional and harmonic analysis			
30	**Weak convergence does not imply norm convergence**	Functional analysis	B+
31	**Closed and bounded is not compact in infinite dimensions**	Functional analysis; compactness	B+
32	**Strong operator convergence does not imply norm convergence**	Operator theory	M
33	**The spectrum of an operator need not contain any eigenvalues**	Operator theory; spectral analysis	M
34	**Why completeness is essential in the uniform boundedness principle**	Functional analysis	M
35	**Banach limits: the dual of ℓ^∞ is larger than ℓ^1**	Functional analysis; Hahn–Banach theory	M+
36	**A continuous function whose Fourier series diverges at a point**	Harmonic analysis	M
37	**The Hardy–Littlewood maximal operator is not bounded on L^1**	Real and harmonic analysis	M
V. Sobolev spaces, nonlinear limits, and PDE			
38	**Failure of the critical Sobolev embedding into L^∞**	Sobolev spaces; PDE	M

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Counterexamples in Analysis – continued

No.	Topic	Area	Level
39	**Loss of compactness in Sobolev embeddings: translation and concentration**	Sobolev spaces; compactness	M
40	**Weak limits do not commute with nonlinearities**	PDE; calculus of variations	M
41	**Why distributions cannot be multiplied canonically**	Distribution theory	M+
42	**The Lavrentiev phenomenon: smooth competitors can miss the true infimum**	Calculus of variations	M+
43	**A reentrant corner destroys H^2 elliptic regularity**	Elliptic PDE; Sobolev regularity	M
44	**The backward heat equation is ill-posed**	Parabolic PDE; inverse problems	M
45	**Tychonoff's counterexample to uniqueness for the heat equation**	Partial differential equations	M+
46	**A nonzero harmonic function with boundary trace zero almost everywhere**	Harmonic analysis; elliptic PDE	M

The following pages offer tentative plans for a **50-minute presentation**, followed by approximately 10 minutes for questions and discussion. They are starting points for preparation, not fixed scripts: the proposed structure, internal timings, proof goals, additional remarks, and reading are all recommendations. You may change the order, replace examples, adjust the level of detail, or narrow the scope to suit your background and the audience. **Discuss your proposed plan and any changes with the instructor** before settling on the content.

Using the references. The readings are suggested starting points, not an exclusive list or a requirement to read entire books. Other suitable textbooks, lecture notes, and reliable sources are welcome; discuss your selection with the instructor. Begin with the introductory reading indicated for your topic. Sources marked as advanced or historical are optional background, not prerequisites for the talk. Agree with the instructor on any specialist material you wish to include.

A recommended approach

A useful way to organize a counterexample-based talk is to:

1. Begin with a plausible but false assertion.
2. Introduce the counterexample and explain its construction.
3. Prove the key property that makes it a counterexample.
4. Conclude with a corrected theorem and the hypothesis that repairs the assertion.

This sequence can be adapted; a clear treatment of one central argument is generally more useful than trying to cover every point in the suggested plan.

Agreeing on the scope. For more technical topics, discuss with the instructor which arguments to prove in full and which results to state with a reference. The aim is to agree on a coherent, manageable talk, not to reproduce every item on the following pages.

1 Tentative 50-minute presentation plans

Topic 1 The Weierstrass function: continuity without differentiability

Level: B+ **Area:** Real analysis; lacunary Fourier series
Aim: Construct one function explicitly, prove continuity, and isolate the multiscale oscillation that destroys every derivative.

A continuous graph may seem likely to become smooth when viewed closely enough. The Weierstrass function challenges that intuition: its defining series converges uniformly, yet oscillations persist at every scale and prevent a derivative anywhere.

Weierstrass's theorem. Let $0 < a < 1$, let $b \geq 3$ be an odd integer, and assume $ab > 1 + \frac{3\pi}{2}$. Then

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(\pi b^n x)$$

is continuous on $\mathbb{R}$ and differentiable at no point. The displayed condition is a convenient classical sufficient condition, not the sharp parameter range.

Tentative structure (50 minutes)

Part 1. The false intuition (0–8 min). Review why finite trigonometric sums are smooth and ask whether a uniformly convergent limit of such sums must retain some local smoothness.

Part 2. Construction and continuity (8–18 min). Introduce W, apply the Weierstrass M-test, and prove uniform convergence from $\sum a^n < \infty$.

Part 3. No derivative (18–40 min). Examine difference quotients at the natural scales $h_m = b^{-m}$. Prove the lower-bound lemma separating the mth oscillation from the lower frequencies and controlling the tail.

Part 4. What the example teaches (40–50 min). State sharper parameter results, discuss Hölder regularity, and contrast an explicit example with the Baire-category statement that nowhere differentiable functions are generic in $C[0,1]$.

Key formulas and proof pivots

Uniform convergence is immediate from

$$|a^n \cos(\pi b^n x)| \leq a^n, \qquad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

At spatial scale $h_m \simeq b^{-m}$, the mth mode has amplitude a^m; hence its formal slope has size

$$\frac{a^m}{b^{-m}} = (ab)^m.$$

The rigorous step is to choose signed increments so that this contribution cannot be cancelled by all remaining modes.

Worth mentioning

- Uniform convergence preserves continuity but gives no derivative control.
- The graph has oscillations at every scale and is typically Hölder, but not Lipschitz.
- Do not replace the lower-bound argument by formal termwise differentiation: that series is precisely what is not justified.

Further reading. Begin with the examples in [21] and the uniform-convergence background in [1]. Hardy's paper [24] and [50] are advanced background for the instructor's opening lecture.

Topic 2 The Cantor function and the failure of the naive fundamental theorem

Level: B+ **Area:** Real analysis; measure theory
Aim: Construct a continuous nonconstant function whose derivative vanishes almost everywhere, and identify absolute continuity as the missing hypothesis.

A function whose derivative vanishes everywhere on an interval must be constant, but replacing "everywhere" by "almost everywhere" changes the picture. The Cantor staircase shows how a continuous function can still increase, and why absolute continuity matters in the fundamental theorem of calculus.

Cantor staircase. There exists a continuous nondecreasing function $F : [0,1] \to [0,1]$ such that $F(0) = 0$, $F(1) = 1$, and

$$F'(x) = 0 \qquad \text{for Lebesgue-a.e. } x \in [0,1].$$

Consequently, continuity plus almost-everywhere differentiability does not imply $F(x) - F(0) = \int_0^x F'(t)\,dt$.

Tentative structure (50 minutes)

Part 1. Build the Cantor set (0–12 min). Remove middle thirds, compute the remaining measure, and describe points using ternary digits.

Part 2. Define the staircase (12–25 min). On the Cantor set send ternary digits $0, 2$ to binary digits $0, 1$ and extend constantly across every deleted interval. Prove monotonicity and continuity.

Part 3. Compute the derivative (25–38 min). The complement of the Cantor set is open and has full measure; F is locally constant there, so $F' = 0$ almost everywhere.

Part 4. Repair the theorem (38–50 min). Introduce absolute continuity, state the fundamental theorem for absolutely continuous functions, and interpret F through its singular Stieltjes measure.

Key formulas and theorem

For $x \in C$ with a ternary expansion using only 0 and 2,

$$x = 0.a_1a_2\ldots_3, quada_j \in \{0,2\}, \qquad F(x) = 0.\frac{a_1}{2}\frac{a_2}{2}\ldots_2.$$

The corrected fundamental theorem is

$$F \in AC[0,1] \quad \Longrightarrow \quad F(x) = F(0) + \int_0^x F'(t)\,dt.$$

Since the right-hand side would be constant for the Cantor function, $F \notin AC[0,1]$.

Worth mentioning

- Monotone functions are differentiable almost everywhere, but need not be absolutely continuous.
- The associated measure is singular and supported on a null set.
- Address the two possible ternary expansions at endpoints explicitly.

Further reading. Begin with the Cantor function and absolute continuity in [42]. The treatment of singular measures in [19] is optional advanced background.

Topic 3 Thomae's popcorn function

Level: B **Area:** Real analysis; Riemann integration
Aim: Show that dense discontinuities can coexist with Riemann integrability and that density is not a measure of analytic size.

Dense discontinuities might seem incompatible with Riemann integration. Thomae's function shows otherwise: its values at rational points become small as their denominators grow, leaving enough control for both continuity at irrational points and integrability.

Thomae's example. Define $t : [0,1] \to [0,1]$ by

$$t(x) = \begin{cases} 1/q, & x = p/q \text{ in lowest terms,} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then t is continuous at every irrational, discontinuous at every rational, and Riemann integrable with $\int_0^1 t(x)\,dx = 0$.

Tentative structure (50 minutes)

Part 1. Two dense sets (0–8 min). Compare the Dirichlet function $\mathbf{1}_{\mathbb{Q}}$ with Thomae's denominator-weighted modification.

Part 2. Continuity classification (8–25 min). Prove discontinuity at a rational using irrational approximants. At an irrational, use that only finitely many reduced fractions in $[0,1]$ have denominator at most N.

Part 3. Riemann integrability (25–40 min). Given $\varepsilon > 0$, isolate the finitely many points where $t \geq \varepsilon$, cover them by intervals of small total length, and estimate upper and lower sums.

Part 4. General criterion (40–50 min). State Lebesgue's criterion for Riemann integrability and compare the dense countable discontinuity set with the everywhere discontinuous Dirichlet function.

Key estimate

For each $N \in \mathbb{N}$, the set

$$E_N = \{x \in [0,1] : t(x) \geq 1/N\}$$

is finite. Away from small neighborhoods of E_N, one has $0 \leq t < 1/N$. This single observation proves both continuity at irrationals and the upper-sum estimate.

Lebesgue's criterion. A bounded function on $[a,b]$ is Riemann integrable if and only if its set of discontinuities has Lebesgue measure zero.

Worth mentioning

- A dense set can be countable and null; a topological statement need not control measure.
- The lowest-terms condition is essential because it makes the value well defined.
- This is an excellent talk in which to prove every claim completely.

Further reading. Use [1] for continuity and Riemann integration, together with the counterexamples in [21].

Topic 4 A fat Cantor set

Level: B+ **Area:** Measure theory; topology
Aim: Construct a closed nowhere dense subset of $[0,1]$ with positive measure and separate topological from measure-theoretic smallness.

The usual Cantor set has both empty interior and measure zero, so it is tempting to associate these properties. By removing less length at each stage, the fat Cantor construction keeps the empty interior while retaining positive measure.

Smith–Volterra–Cantor set. There exists a compact set $S \subset [0,1]$ with empty interior and

$$|S| = \frac{1}{2}.$$

Thus a nowhere dense set need not have measure zero.

Tentative structure (50 minutes)

Part 1. Recall the ordinary Cantor set (0–8 min). Separate the conclusions "nowhere dense", "uncountable", and "measure zero" rather than treating them as one.

Part 2. Modified deletion scheme (8–23 min). At stage n, remove from each of the 2^{n-1} surviving intervals a centered open interval of length 4^{-n}.

Part 3. Verify all properties (23–39 min). Compute the removed measure, prove compactness, and show that every nonempty interval meets a deleted gap.

Part 4. Consequences (39–50 min). Discuss indicator functions, boundaries of positive measure, and the contrast between Baire category and Lebesgue measure.

Key computation and proof pivots

The total removed length is

$$\sum_{n=1}^{\infty} 2^{n-1}4^{-n} = \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2}, \qquad |S| = 1 - \frac{1}{2} = \frac{1}{2}.$$

The complement is a dense open union of the removed intervals. Hence S is closed and has empty interior, even though half of the original measure remains.

Worth mentioning

- Closed plus empty interior is equivalent here to nowhere dense.
- The construction can be tuned to leave any prescribed measure in $(0,1)$.
- The boundary of S equals S itself, so a boundary may have positive measure.

Further reading. Begin with Cantor-type sets and Lebesgue measure in [42]. For an optional broader comparison of measure and category, consult [39].

Topic 5 A smooth nonanalytic flat function

Level: B **Area:** Real analysis; Taylor series
Aim: Prove that infinite differentiability does not imply local recovery from the Taylor series.

Taylor series record all derivatives at a point, but even that information need not determine a smooth function nearby. The rapid decay of e^{-1/x^2} gives a particularly simple example: every derivative at the origin vanishes, while the function is positive elsewhere.

Flat-function counterexample. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $f \in C^\infty(\mathbb{R})$ and $f^{(k)}(0) = 0$ for every $k \geq 0$, but $f(x) > 0$ for $x \neq 0$. Its Taylor series at the origin is therefore identically zero and does not represent f on any neighborhood of 0.

Tentative structure (50 minutes)

Part 1. Smooth versus analytic (0–8 min). State the definitions and the false inference "all derivatives exist, hence the Taylor series recovers the function."

Part 2. The decisive decay lemma (8–21 min). Prove that $|x|^{-m}e^{-1/x^2} \to 0$ as $x \to 0$ for every $m \geq 0$.

Part 3. All derivatives vanish (21–38 min). Show inductively that away from zero $f^{(k)}(x) = P_k(1/x)e^{-1/x^2}$ for a polynomial P_k, then verify derivatives at zero.

Part 4. Uses and extensions (38–50 min). Construct a compactly supported smooth bump and explain why analytic functions cannot have compact support unless they vanish.

Key formulas and theorem

With $y = 1/x^2$,

$$|x|^{-m}e^{-1/x^2} = y^{m/2}e^{-y} \longrightarrow 0.$$

The contrast is the identity theorem for real-analytic functions: if all derivatives of an analytic function vanish at one point, then the function vanishes on a neighborhood of that point.

Worth mentioning

- Smooth bump functions underpin partitions of unity and localization in PDE.
- Taylor's theorem only gives a useful representation when its remainder is controlled.
- One-sided variants, such as $e^{-1/x}$ for $x > 0$ and 0 for $x \leq 0$, are equally useful.

Further reading. Use [1, 43] for differentiation, Taylor series, and the distinction between smoothness and analyticity. [31] is optional advanced reading on real analytic functions.

Topic 6 A badly discontinuous derivative and Darboux's theorem

Level: B+ **Area:** Differential calculus; real analysis
Aim: Construct an unbounded discontinuous derivative while proving that derivatives still cannot have jump discontinuities.

Differentiability does not make the derivative continuous, or even locally bounded. An oscillatory example shows how badly continuity can fail, while Darboux's theorem reveals a restriction that remains: derivatives cannot skip intermediate values.

Counterexample and repair. Define

$$f(x) = \begin{cases} x^2 \sin(1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is differentiable everywhere, $f'(0) = 0$, and f' is unbounded in every neighborhood of 0. Nevertheless, every derivative has the intermediate value property.

Tentative structure (50 minutes)

Part 1. Differentiate carefully (0–12 min). Compute the derivative away from zero and use the definition of derivative, not the formal formula, at zero.

Part 2. Show the pathology (12–23 min). Choose sequences on which the cosine term equals 1 and -1 to show that f' is unbounded with both signs near zero.

Part 3. Prove Darboux's theorem (23–42 min). For values $f'(a) < \lambda < f'(b)$, apply the extreme-value theorem to $g(x) = f(x) - \lambda x$ and use Fermat's theorem at an interior extremum.

Part 4. Interpretation (42–50 min). Explain why derivatives form a class strictly between arbitrary functions and continuous functions.

Key formulas

For $x \neq 0$,

$$f'(x) = 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2), \qquad f'(0) = \lim_{h \to 0} h \sin(1/h^2) = 0.$$

The second term produces unbounded oscillation, while Darboux's theorem rules out an actual jump.

Worth mentioning

- Distinguish discontinuity, unboundedness, and failure of one-sided limits.
- The proof of Darboux uses continuity of f, not continuity of f'.
- Baire class one functions provide a broader setting for derivatives.

Further reading. Begin with Darboux's theorem in [1, 43]. [8] is a specialist monograph on derivatives, suitable only as optional advanced background.

Topic 7 Rolle's theorem fails for vector-valued functions

Level: B **Area:** Vector calculus; real analysis
Aim: Disprove the naive vector-valued version of Rolle's theorem, identify the scalar order argument that fails, and prove a valid norm estimate.

A particle can travel around a circle and return to its starting point without ever stopping. This simple observation challenges a vector-valued version of Rolle's theorem and raises the question of what survives when scalar values are replaced by vectors.

Failure of vector-valued Rolle. Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the unit-circle parametrization

$$\gamma(t) = (\cos t, \sin t). \tag{1}$$

Then $\gamma(0) = \gamma(2\pi)$, but $\gamma'(t) \neq 0$ for every t.

Tentative structure (50 minutes)

Part 1. Scalar theorem and proof (0–12 min). State Rolle's theorem and the mean-value theorem, then isolate the use of maxima, minima, and the order structure of $\mathbb{R}$.

Part 2. The circle counterexample (12–23 min). Compute the endpoints and derivative of (1), and explain geometrically why a closed regular curve need never stop.

Part 3. Why coordinatewise arguments fail (23–34 min). Apply scalar Rolle separately to the two coordinates and show that the two vanishing derivatives may occur at different times.

Part 4. The correct replacement (34–50 min). Prove the vector-valued mean-value inequality and discuss its Lipschitz consequence.

Decisive computations

$$\gamma'(t) = (-\sin t, \cos t), \tag{2}$$
$$\|\gamma'(t)\| = 1. \tag{3}$$

For every continuously differentiable $F : [a, b] \to \mathbb{R}^m$, the fundamental theorem of calculus gives

$$\|F(b) - F(a)\| \leq \int_a^b \|F'(t)\|\, dt \leq (b - a) \sup_{t \in [a,b]} \|F'(t)\|. \tag{4}$$

Worth mentioning

- No equality with a single intermediate derivative is generally available in dimensions larger than one.
- Applying a linear functional reduces the problem to a valid scalar mean-value statement.
- The counterexample is smooth, periodic, and regular, so the failure is not caused by poor regularity.

Further reading. Use the differentiation chapters of [3, 43] for Rolle's theorem and vector-valued mean-value estimates.

Topic 8 Mixed partial derivatives need not commute

Level: B+ **Area:** Multivariable calculus; real analysis
Aim: Construct a function with both iterated mixed partials at the origin but with different values, and isolate the regularity in the Schwarz–Clairaut theorem.

In routine calculations, the order of partial differentiation is often interchangeable. The example below shows why this requires justification: both mixed partials can exist at the origin and still disagree when the usual continuity hypothesis is absent.

Peano-type counterexample. Define

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

With the convention $f_{xy} = \partial_y(\partial_x f)$, both iterated derivatives exist at the origin, but

$$f_{xy}(0,0) = -1, \qquad f_{yx}(0,0) = 1.$$

Tentative structure (50 minutes)

Part 1. The tempting symmetry (0–8 min). Recall the notation for partial and iterated derivatives and state the false claim that mere existence forces equality.

Part 2. Regularity of the example (8–21 min). Define f, prove continuity, and use a quadratic bound to show differentiability at the origin with $Df(0,0) = 0$.

Part 3. Compute along the axes (21–36 min). Determine $f_x(0,y)$ and $f_y(x,0)$ directly from difference quotients, then differentiate once more at zero.

Part 4. Repair the theorem (36–50 min). Prove Schwarz–Clairaut on a rectangle under C^2 regularity, or prove the local version by applying the one-variable mean value theorem twice.

Decisive estimates and computations

Since $|xy| \le (x^2+y^2)/2$ and $|x^2-y^2| \le x^2+y^2$,

$$|f(x,y)| \le \frac{x^2+y^2}{2} = o\Big(\sqrt{x^2+y^2}\Big).$$

On the coordinate axes one obtains

$$f_x(0,y) = -y, \qquad f_y(x,0) = x,$$

which gives the two opposite mixed derivatives. The failure is therefore not a notational paradox: the relevant second partials are not continuous near the origin.

Worth mentioning

- Fix the order convention for f_{xy} before computing; textbooks differ.
- Existence at one point is far weaker than continuity in a neighborhood.
- The example also warns that evidence from coordinate lines may miss genuinely two-dimensional behavior.

Further reading. Begin with multivariable differential calculus in [3]; [43] provides a complementary treatment of differentiability and mixed derivatives.

Topic 9 Directional derivatives in every direction do not imply differentiability

Level: B+ **Area:** Multivariable calculus; differentiability
Aim: Construct a continuous function with every directional derivative at the origin but no Fréchet derivative, and isolate the missing linear approximation.

Knowing the rate of change along every line through a point may seem enough to describe a derivative there. The difficulty is that these directional rates must fit together into a single linear approximation, something even a continuous function need not allow.

All directions, no derivative. Define

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases} \tag{5}$$

Then f is continuous at the origin and has a directional derivative in every direction, but it is not Fréchet differentiable there.

Tentative structure (50 minutes)

Part 1. Four notions of differentiation (0–11 min). Distinguish partial derivatives, directional derivatives, Gâteaux derivatives, and Fréchet derivatives.

Part 2. Continuity and directional limits (11–25 min). Prove continuity of (5) and compute its derivative along an arbitrary nonzero vector.

Part 3. Failure of linearization (25–38 min). Show that the directional derivatives are homogeneous but not additive, and compare them with the only possible total derivative determined by the partial derivatives.

Part 4. Repairing the claim (38–50 min). Prove or state the theorem that continuous partial derivatives in a neighborhood imply Fréchet differentiability.

Key estimates and obstruction

The estimate

$$|f(x,y)|\le|x| \tag{6}$$

proves continuity at the origin. For $v=(a,b)\neq 0$, direct calculation gives

$$D_vf(0,0)=\lim_{t\to0}\frac{f(ta,tb)}{t}=\frac{a^3}{a^2+b^2}. \tag{7}$$

The partial derivatives would force the candidate linear map $L(a,b)=a$, whereas

$$D_{(1,1)}f(0,0)=\frac{1}{2}\neq L(1,1). \tag{8}$$

Worth mentioning

- Existence of all directional derivatives gives one-dimensional information but no uniform control over directions.
- Homogeneity of the directional derivative map is automatic; additivity is the decisive missing property.
- The example is continuous, so discontinuity is not responsible for the failure.

Further reading. Use the multivariable chapters of [3, 43] for directional and total derivatives. Abstract infinite-dimensional differentiability is not needed for the main example.

Topic 10 Lagrange multipliers can miss a constrained extremum

Level: B **Area:** Multivariable calculus; constrained optimization
Aim: Exhibit a constrained minimum at a singular point where the multiplier equation has no solution, and identify the required constraint qualification.

The Lagrange-multiplier method relies on a constraint describing a regular surface near the extremum. At the tip of a cusp, that geometric picture breaks down: a genuine constrained minimum can exist even though the multiplier equation has no solution.

A missed minimum. Let

$$f(x,y) = x, \qquad g(x,y) = y^2 - x^3. \tag{9}$$

The origin minimizes f on the constraint set $g = 0$, but there is no scalar λ such that $\nabla f(0,0) = \lambda \nabla g(0,0)$.

Tentative structure (50 minutes)

Part 1. The regular theorem (0–12 min). State the one-constraint Lagrange-multiplier theorem and explain the geometric role of a nonzero constraint gradient.

Part 2. The cusp and its minimum (12–24 min). Parametrize the constraint, sketch it, and prove directly that the origin is a constrained minimum.

Part 3. Failure of the multiplier equation (24–35 min). Compute both gradients and show that the necessary condition becomes impossible at the singular point.

Part 4. Constraint qualification (35–50 min). Explain the tangent-line proof in the regular case and compare the cusp with a regular level curve.

The decisive calculation

The cusp admits the parametrization

$$\Gamma(t) = (t^2, t^3), \qquad f(\Gamma(t)) = t^2 \geq 0. \tag{10}$$

At its singular point,

$$\nabla f(0,0) = (1,0), \qquad \nabla g(0,0) = (0,0), \tag{11}$$

and therefore the equation $\nabla f = \lambda \nabla g$ has no solution.

Worth mentioning

- The multiplier theorem supplies a necessary condition only at regular constrained extrema.
- A singular constraint may fail to possess a well-defined tangent line or tangent hyperplane.
- The example distinguishes failure of a theorem's hypotheses from failure of the extremum itself.

Further reading. Use the chapter on implicit functions and extremum problems in [3], especially Lagrange multipliers and the regularity of the constraint.

Topic 11 Pathological solutions of Cauchy's functional equation

Level: B+ **Area:** Real analysis; functional equations
Aim: Construct a discontinuous additive function from a Hamel basis and prove that mild analytic regularity restores ordinary linearity.

The equation $f(x+y) = f(x)+f(y)$ looks as though it should force a straight-line graph. It does under mild regularity assumptions, but a Hamel basis allows additive functions that are discontinuous everywhere, revealing how much those assumptions contribute.

Additive but not real-linear. Choose an irrational number α and a Hamel basis $\mathcal{H}$ of $\mathbb{R}$ over $\mathbb{Q}$ containing 1 and α. Assign values on the basis by

$$A(1) = 0, \qquad A(\alpha) = 1, \qquad A(h) = 0 \quad h \in \mathcal{H} \setminus \{1, \alpha\}, \tag{12}$$

and extend A rationally linearly. Then $A(x+y) = A(x) + A(y)$, but A is not of the form cx.

Tentative structure (50 minutes)

Part 1. What additivity gives for free (0–10 min). Prove $A(0) = 0$, $A(-x) = -A(x)$, and $A(qx) = qA(x)$ for rational q.

Part 2. Hamel-basis construction (10–25 min). Recall finite linear combinations over $\mathbb{Q}$, define the extension, and verify that it is well defined and additive.

Part 3. Why the solution is pathological (25–38 min). Prove that a continuous additive function must equal $xA(1)$ and conclude that the constructed function is discontinuous everywhere.

Part 4. Regularity restores rigidity (38–50 min). State the corresponding results under measurability or boundedness on a set of positive measure, and identify the use of choice in the construction.

Construction and rigidity

Every real number has a unique representation with finitely many nonzero coefficients, and the extension is

$$A\left(\sum_{j=1}^{m} q_j h_j\right) = \sum_{j=1}^{m} q_j A(h_j), \qquad q_j \in \mathbb{Q}, \quad h_j \in \mathcal{H}. \tag{13}$$

If A is continuous at zero and $q_n \in \mathbb{Q}$ converges to x, then

$$A(x) = \lim_{n\to\infty} A(q_n) = \lim_{n\to\infty} q_n A(1) = xA(1). \tag{14}$$

Worth mentioning

- A discontinuous additive function is unbounded on every nonempty interval.
- Lebesgue measurability, Baire measurability, or local boundedness each rules out the pathology.
- The construction depends on a Hamel basis and is therefore nonexplicit in the usual sense.

Further reading. Begin with the additive-function counterexamples in [21]. Consult selected sections of [33] on additive functions and Hamel bases only for optional deeper background.

Topic 12 Pointwise convergence does not preserve continuity

Level: B **Area:** Real analysis; sequences of functions
Aim: Use one elementary sequence to distinguish pointwise and uniform convergence and to state the correct limit theorem.

Each function x^n is continuous on $[0,1]$, yet its pointwise limit has a jump at the right endpoint. Following this elementary sequence makes the distinction between pointwise and uniform convergence concrete, and explains why continuity requires the stronger notion.

Basic counterexample. For $f_n(x) = x^n$ on $[0,1]$,

$$f_n(x) \longrightarrow f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

Every f_n is continuous, but the pointwise limit f is discontinuous.

Tentative structure (50 minutes)

Part 1. Find the limit (0–10 min). Compute it separately for $x < 1$ and $x = 1$; sketch how the transition layer moves toward the endpoint.

Part 2. Locate nonuniformity (10–22 min). Show that $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$ and use the epsilon definition to identify why no single N works near $x = 1$.

Part 3. Prove the uniform limit theorem (22–39 min). Give the three-epsilon proof that a uniform limit of continuous functions is continuous.

Part 4. Other limit operations (39–50 min). Compare integration and differentiation, and state sufficient conditions for each interchange.

Key theorem and estimates

If $f_n \to f$ uniformly and every f_n is continuous at x_0, then

$$|f(x) - f(x_0)| \le |f - f_n|(x) + |f_n(x) - f_n(x_0)| + |f_n - f|(x_0),$$

which proves continuity of f by choosing n first and then x. In the example, $\int_0^1 x^n\,dx = (n+1)^{-1} \to 0$, showing that some other limit operations may still behave well.

Worth mentioning

- Uniform convergence is convergence in the supremum norm.
- Differentiation needs stronger hypotheses, usually convergence of derivatives plus control at one point.
- Dini's theorem is a useful positive result for monotone convergence on compact spaces.

Further reading. Begin with sequences of functions in [1]. [43] gives a complementary treatment of uniform convergence and interchange of limiting operations.

Topic 13 Moving spikes and incompatible convergence notions

Level: B+ **Area:** Measure theory; L^p spaces
Aim: Build two explicit sequences separating pointwise, almost-everywhere, in-measure, and L^1 convergence.

A shrinking spike can disappear at every fixed point without losing any integral, while a moving interval can have vanishing measure and still revisit each point indefinitely. These two examples show why different notions of convergence cannot be used interchangeably.

Two counterexamples. On $(0,1)$, the spike $f_n = n\mathbf{1}_{(0,1/n)}$ converges pointwise to zero but $\|f_n\|_{L^1} = 1$. Conversely, the dyadic typewriter sequence converges to zero in L^1 and in measure but fails to converge pointwise almost everywhere.

Tentative structure (50 minutes)

Part 1. Map the notions (0–10 min). State the definitions and draw only the implications that are actually valid.

Part 2. Concentration spike (10–22 min). Verify pointwise convergence and the constant integral. Explain exactly why no integrable dominating function can exist.

Part 3. Typewriter sequence (22–38 min). Enumerate the dyadic indicators and show that their supports shrink while almost every point is hit once at every scale.

Part 4. Repair the implications (38–50 min). State dominated convergence and the subsequence theorem for convergence in measure; briefly mention uniform integrability.

Key construction and theorems

For $0 \le j < 2^k$, set

$$g_{2^k+j} = \mathbf{1}_{[j2^{-k},(j+1)2^{-k})}, \qquad \|g_{2^k+j}\|_1 = 2^{-k} \longrightarrow 0.$$

Yet for every nondyadic x, the sequence $g_n(x)$ takes the values 0 and 1 infinitely often. If $f_n \to f$ in measure, a subsequence converges to f almost everywhere; the full sequence need not.

Worth mentioning

- Replace the indicator spike by triangular continuous spikes if continuity is desired.
- Almost-everywhere convergence alone does not control integrals.
- The typewriter example explains why subsequences, not full sequences, appear in compactness arguments.

Further reading. Begin with convergence in measure and Lebesgue integration in [42]. [19] is an optional, more abstract account of the convergence theorems.

Topic 14 Non-inclusions between L^p spaces on an infinite-measure domain

Level: B **Area:** Measure theory; function spaces
Aim: Use singularities and tails to prove both failures of inclusion and contrast them with the finite-measure theorem.

On a finite-measure space, increasing the exponent gives a familiar inclusion between L^p spaces. On the real line, local singularities and behavior at infinity pull in opposite directions, so neither inclusion is available in general.

Non-inclusion. Let $1 \le p < q < \infty$. On $\mathbb{R}$ there exist $f \in L^p \setminus L^q$ and $g \in L^q \setminus L^p$. Hence neither $L^p(\mathbb{R})$ nor $L^q(\mathbb{R})$ contains the other.

Tentative structure (50 minutes)

Part 1. Finite measure first (0–12 min). Prove $L^q(\Omega) \subset L^p(\Omega)$ when $|\Omega| < \infty$ by Holder's inequality.

Part 2. A local singularity (12–23 min). Study $x^{-\alpha}$ on $(0,1)$ and choose α so that the pth power is integrable but the qth power is not.

Part 3. A tail at infinity (23–35 min). Study $x^{-\beta}$ on $(1,\infty)$ and reverse the inclusion.

Part 4. General lesson (35–50 min). Separate local integrability from decay at infinity; discuss intersections, sums, and the endpoint L^∞.

Key tests and choices

For $r \ge 1$,

$$\int_0^1 x^{-\alpha r}\,dx < \infty \iff \alpha r < 1, \qquad \int_1^\infty x^{-\beta r}\,dx < \infty \iff \beta r > 1.$$

Choose $1/q < \alpha < 1/p$ and $1/q < \beta < 1/p$. Then

$$x^{-\alpha}\mathbf{1}_{(0,1)} \in L^p \setminus L^q, \qquad x^{-\beta}\mathbf{1}_{(1,\infty)} \in L^q \setminus L^p.$$

Worth mentioning

- The same exponent interval produces opposite conclusions at zero and at infinity.
- On a probability space, $L^q \subset L^p$ for $q > p$.
- Lorentz and weak-L^p spaces refine what happens at borderline exponents.

Further reading. Use the chapter on L^p spaces and Hölder's inequality in [42]. Focus on power-function integrability tests and inclusions; graduate functional analysis is not needed.

Topic 15 Why L^∞ behaves differently: smooth functions are not dense

Level: B+ **Area:** Measure theory; functional analysis
Aim: Contrast the density and separability theory for L^p, $p < \infty$, with two explicit obstructions at the endpoint $p = \infty$.

Approximation by smooth functions works well in L^p for finite p, but the essential-supremum norm is less forgiving. The constant function and a family of characteristic functions expose two endpoint obstructions: failure of compactly supported smooth approximation and failure of separability.

Two endpoint failures. On $\mathbb{R}^d$,

$$\inf_{\varphi \in C_c^\infty(\mathbb{R}^d)} \|\mathbf{1} - \varphi\|_{L^\infty} = 1.$$

Moreover, $L^\infty(0,1)$ is nonseparable: for $0 < s < t < 1$,

$$\left\|\mathbf{1}_{[0,s]} - \mathbf{1}_{[0,t]}\right\|_{L^\infty(0,1)} = 1.$$

Thus the standard density and separability statements for L^p, $p < \infty$, do not extend to $p = \infty$.

Tentative structure (50 minutes)

Part 1. The familiar theorem (0–10 min). Review truncation, approximation by simple functions, and mollification in L^p for $1 \le p < \infty$.

Part 2. Failure of smooth density (10–22 min). Test the constant function $\mathbf{1}$ against compactly supported smooth functions and identify the obstruction outside the support.

Part 3. Failure of separability (22–38 min). Build the uncountable 1-separated family $\{\mathbf{1}_{[0,t]} : 0 < t < 1\}$ and prove that a separable metric space cannot contain such a family.

Part 4. What remains true (38–50 min). Explain weak-* approximation, uniform approximation for uniformly continuous functions, and why convergence in every finite L^p still need not imply convergence in L^∞.

Norm obstruction and weak-* repair

If $\varphi \in C_c^\infty(\mathbb{R}^d)$, then $\varphi = 0$ on a set of positive measure, so $\|\mathbf{1} - \varphi\|_\infty \ge 1$; taking $\varphi = 0$ proves equality. By contrast, smooth cutoffs φ_R with $0 \le \varphi_R \le 1$ and $\varphi_R \to 1$ pointwise satisfy

$$\int_{\mathbb{R}^d} g\,\varphi_R \longrightarrow \int_{\mathbb{R}^d} g \qquad \text{for every } g \in L^1(\mathbb{R}^d),$$

so $\varphi_R \rightharpoonup^* \mathbf{1}$ in L^∞.

Worth mentioning

- Essential supremum ignores null sets, not sets of small positive measure.
- Representatives of an L^∞ class are not pointwise-defined objects.
- The correct endpoint topology is often weak-* rather than norm topology.

Further reading. Begin with L^p spaces and approximation in [42]. [19] provides further background; the weak-* discussion in [7] is an optional advanced extension.

Topic 16 Pairwise independence does not imply mutual independence

Level: B+ **Area:** Probability; measure theory
Aim: Construct three pairwise independent fair bits with a deterministic joint constraint, then distinguish pairwise and mutual factorization.

Checking every pair of random variables can miss a relation involving all of them. Three fair bits linked by a parity constraint provide a small, explicit example in which every pair is independent but the triple is not.

The parity construction. Let U and V be independent fair bits and define

$$X = U, \qquad Y = V, \qquad Z = U \oplus V. \tag{15}$$

Every pair among X, Y, Z is independent, but the triple is not mutually independent.

Tentative structure (50 minutes)

Part 1. Definitions (0–10 min). Formulate independence for two events, finite families of events, and discrete random variables.

Part 2. Construct and verify (10–25 min). Write the four-point probability table and check all three pairwise joint distributions explicitly.

Part 3. Expose the joint constraint (25–36 min). Compare one triple-intersection probability with the product of its marginals and explain the parity relation.

Part 4. Generalization (36–50 min). Use nonzero linear forms on a uniform vector over $\mathbb{F}_2^m$ to build large pairwise independent families, and discuss which probabilistic arguments require mutual independence.

Pairwise checks and joint failure

For every $i, j \in \{0, 1\}$, one has

$$\mathbb{P}(X = i, Y = j) = \mathbb{P}(X = i, Z = j) = \mathbb{P}(Y = i, Z = j) = \frac{1}{4}. \tag{16}$$

However,

$$\mathbb{P}(X = 0, Y = 0, Z = 0) = \frac{1}{4} \neq \frac{1}{8} = \mathbb{P}(X = 0)\mathbb{P}(Y = 0)\mathbb{P}(Z = 0). \tag{17}$$

Equivalently, the variables satisfy

$$X \oplus Y \oplus Z = 0 \tag{18}$$

with probability one.

Worth mentioning

- Pairwise independence is stronger than being pairwise uncorrelated, but weaker than mutual independence.
- The construction uses only four equally likely outcomes and can be displayed completely in a table.
- Linear algebra over finite fields gives economical families useful in randomized algorithms and derandomization.

Further reading. Begin with Section 4.1 of [23], which distinguishes pairwise and mutual independence through finite examples. [49] is optional measure-theoretic background; martingale theory is not needed.

Topic 17 Failure of Fubini without absolute convergence

Level: B+ **Area:** Integration; double series
Aim: Compute two incompatible iterated sums and identify absolute summability as the hypothesis that makes the order of summation irrelevant.

Changing the order of a finite sum is harmless; for infinite sums, it can change the answer. The double series below makes this failure explicit and shows why absolute convergence is central to interchanging sums and integrals.

Explicit double-series counterexample. For $m, n \geq 1$, define

$$a_{mn} = \begin{cases} 1, & n = m, \\ -1, & n = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_m \sum_n a_{mn} = 0$, whereas $\sum_n \sum_m a_{mn} = 1$.

Tentative structure (50 minutes)

Part 1. The tempting manipulation (0–8 min). Write an infinite rectangular array and ask when rows, columns, and expanding finite rectangles determine the same sum.

Part 2. Compute the counterexample (8–21 min). Every row has sum $1 - 1 = 0$. The first column has sum 1 and every later column has sum 0.

Part 3. Diagnose the failure (21–35 min). Show that $\sum_{m,n} |a_{mn}| = \infty$ and that different exhaustion procedures can see different boundary terms.

Part 4. Correct theorem (35–50 min). State Tonelli for nonnegative functions and Fubini for integrable functions; optionally realize the array as a piecewise constant function on disjoint rectangles.

Key theorem

If

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}| < \infty,$$

then both iterated sums exist and

$$\sum_m \sum_n a_{mn} = \sum_n \sum_m a_{mn} = \lim_{M,N\to\infty} \sum_{m\leq M,\, n\leq N} a_{mn}.$$

The analogous integral hypothesis is $\int_{X\times Y} |f| < \infty$.

Worth mentioning

- Tonelli allows the value $+\infty$ because nonnegative mass cannot cancel.
- Conditional convergence is not merely a technical nuisance: it makes the summation procedure part of the definition.
- Distinguish iterated limits from a two-parameter rectangular limit.

Further reading. Use [3] for series and iterated integration. For the measure-theoretic Fubini–Tonelli theorem, consult selected sections of [42]; abstract product-measure theory can be kept in the background.

Topic 18 The Cauchy product of two convergent series may diverge

Level: B+ **Area:** Infinite series; summability
Aim: Give two convergent series whose Cauchy-product coefficients fail to tend to zero, and prove the role of absolute convergence in Mertens' theorem.

Multiplying two convergent series seems a natural extension of multiplying finite sums. With only conditional convergence, however, the Cauchy-product terms may fail even to tend to zero, making the absolute-convergence assumption in Mertens' theorem essential.

Convergent factors, divergent product. Set

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}. \tag{19}$$

Both series converge by the alternating-series test, but their Cauchy product diverges.

Tentative structure (50 minutes)

Part 1. Finite products and infinite convolution (0–10 min). Derive the Cauchy-product coefficients from finite partial sums and state the tempting but false claim.

Part 2. Choose conditionally convergent factors (10–20 min). Verify convergence of both series in (19) and note the failure of absolute convergence.

Part 3. Analyze the convolution (20–39 min). Rewrite its magnitude as an improper Riemann sum and prove that it converges to π by truncating away from the endpoints.

Part 4. Mertens' repair (39–50 min). State and prove the core estimate showing that convergence of both factors and absolute convergence of one factor force the correct product.

The coefficient obstruction

The nth Cauchy-product coefficient is

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}. \tag{20}$$

Writing $j = k+1$ produces

$$|c_n| = \frac{1}{n+2} \sum_{j=1}^{n+1} \frac{1}{\sqrt{\frac{j}{n+2}\left(1-\frac{j}{n+2}\right)}}, \tag{21}$$

$$\lim_{n\to\infty} |c_n| = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi. \tag{22}$$

Thus c_n does not tend to zero, so the product series cannot converge.

Worth mentioning

- Absolute convergence allows the triangular summation region to be rearranged without changing the limit.
- The endpoint singularities in (21) are integrable but require a short truncation argument.
- Abel's theorem provides another route to identifying the product when the Cauchy product is known to converge.

Further reading. Begin with multiplication of series in Chapter 3 of [43]. [25] is optional advanced background on summability, not required for the counterexample or Mertens' theorem.

Topic 19 The Riemann rearrangement theorem

Level: B **Area:** Infinite series
Aim: Give a constructive permutation of a conditionally convergent series with any prescribed sum.

A conditionally convergent series has a sum, but that sum depends on the order of its terms. By alternating suitable blocks of positive and negative terms, one can steer the partial sums toward any prescribed real value.

Riemann rearrangement theorem. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n| = \infty$, then for every $L \in \mathbb{R}$ there is a permutation π of $\mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\pi(n)} = L.$$

Other permutations can force divergence to $+\infty$, to $-\infty$, or by oscillation.

Tentative structure (50 minutes)

Part 1. Absolute versus conditional (0–10 min). Prove briefly that absolute convergence makes every rearrangement harmless, then introduce the alternating harmonic series.

Part 2. Positive and negative reservoirs (10–21 min). Show that the positive terms sum to $+\infty$, the negative terms to $-\infty$, and both individual terms tend to zero.

Part 3. Greedy construction (21–39 min). Add unused positive terms until the partial sum exceeds L, then unused negative terms until it falls below L, and repeat.

Part 4. Convergence and variants (39–50 min). Prove that overshoots tend to zero; modify the recipe to obtain the three divergent behaviors.

Key construction

For the alternating harmonic series,

$$\sum_{k\geq 1} \frac{1}{2k-1} = +\infty, \qquad \sum_{k\geq 1} -\frac{1}{2k} = -\infty.$$

At the jth crossing of L, the error is bounded by the magnitude of the last term used; these magnitudes tend to zero. Hence the rearranged partial sums converge to L.

Worth mentioning

- The permutation must use every term exactly once; explain why the greedy algorithm does.
- Rearrangement is different from regrouping unless the grouping order is made explicit.
- In finite-dimensional vector series, the geometry of possible rearrangement sums is richer.

Further reading. Use the rearrangement theorem in Chapter 3 of [43]; [1] provides additional undergraduate background on absolute and conditional convergence.

Topic 20 A convergent improper integral whose integrand does not tend to zero

Level: B **Area:** Improper integrals; oscillation
Aim: Prove convergence of the Fresnel integral despite persistent oscillation, and contrast the convergence tests for series and improper integrals.

Convergence of a series forces its terms to tend to zero, but the analogous claim for improper integrals is false. In the Fresnel integral, increasingly rapid oscillation produces enough cancellation for convergence even though the integrand does not decay.

Oscillation defeats the naive term test. The improper integral

$$\int_0^\infty \sin(x^2)\,dx \tag{23}$$

converges conditionally, although $\sin(x^2)$ has no limit as $x \to \infty$.

Tentative structure (50 minutes)

Part 1. Series versus integrals (0–10 min). Recall the Cauchy criteria and prove that convergence of a series forces its terms to vanish.

Part 2. Persistent oscillation (10–20 min). Produce two sequences tending to infinity on which the integrand equals 1 and -1.

Part 3. Convergence by cancellation (20–38 min). Substitute $u = x^2$ and apply Dirichlet's test or integration by parts to the resulting oscillatory integral.

Part 4. Conditional character (38–50 min). Prove failure of absolute convergence and explain how increasing oscillation permits cancellation without pointwise decay.

Convergence and nondecay

The substitution $u = x^2$ gives

$$\int_0^R \sin(x^2)\,dx = \frac{1}{2}\int_0^{R^2} \frac{\sin u}{\sqrt{u}}\,du. \tag{24}$$

The amplitude $u^{-1/2}$ decreases to zero, while the primitives of $\sin u$ are bounded, so Dirichlet's test applies. On the other hand, with

$$x_k = \sqrt{\frac{\pi}{2} + 2\pi k}, \qquad y_k = \sqrt{\frac{3\pi}{2} + 2\pi k}, \tag{25}$$

one has

$$\sin(x_k^2) = 1, \qquad \sin(y_k^2) = -1. \tag{26}$$

Worth mentioning

- Absolute convergence fails because the positive mass on successive half-periods is comparable to a divergent series with terms of order $k^{-1/2}$.
- The value of the Fresnel integral can be evaluated by Gaussian-integral methods, but its value is not needed for the counterexample.
- The example is a first model of cancellation in oscillatory integrals.

Further reading. Use [3] for improper integration and convergence tests. Concentrate on the integration-by-parts estimate; advanced summability theory is unnecessary.

Topic 21 Nonuniqueness for an ODE without a Lipschitz hypothesis

Level: B **Area:** Ordinary differential equations
Aim: Exhibit infinitely many solutions with the same initial data and locate the exact failure of the Picard–Lindelöf argument.

A continuous vector field can admit more than one trajectory through the same initial point. For $y' = \sqrt{|y|}$, a solution may wait at equilibrium before moving away, giving a direct view of what is lost without the Lipschitz hypothesis.

Waiting-time solutions. The initial-value problem

$$y' = \sqrt{|y|}, \qquad y(0) = 0,$$

has infinitely many solutions on $[0, \infty)$. For every $c \geq 0$,

$$y_c(t) = \begin{cases} 0, & 0 \leq t \leq c, \\ (t-c)^2/4, & t \geq c, \end{cases}$$

is a C^1 solution.

Tentative structure (50 minutes)

Part 1. Existence versus uniqueness (0–10 min). State Peano's existence theorem and Picard–Lindelöf's uniqueness theorem, highlighting continuity versus local Lipschitz continuity.

Part 2. Find the nonzero branch (10–21 min). Separate variables away from the equilibrium and obtain $y = (t-c)^2/4$.

Part 3. Glue and verify (21–35 min). Attach the branch after an arbitrary waiting time, check differentiability at $t = c$, and verify the equation pointwise.

Part 4. Mechanism and repair (35–50 min). Compute the failure of a Lipschitz bound at $y = 0$, explain why Gronwall cannot close, and mention Osgood's uniqueness criterion.

Key estimates and theorem

The vector field $F(y) = \sqrt{|y|}$ satisfies

$$\frac{|F(y) - F(0)|}{|y|} = |y|^{-1/2} \longrightarrow \infty.$$

Picard–Lindelöf: if $F(t, y)$ is continuous in t and locally Lipschitz in y, then the initial-value problem has a unique local solution. Here continuity still gives existence, but the missing Lipschitz estimate permits waiting.

Worth mentioning

- Check both the differential equation and C^1 matching at the gluing time.
- The phenomenon occurs because the equilibrium can be left with zero initial velocity.
- Compare with $y' = |y|^\alpha$: nonuniqueness occurs for $0 < \alpha < 1$.

Further reading. Begin with the existence-and-uniqueness chapters of the undergraduate text [10]. Use these to compare the hypotheses of existence and uniqueness theorems with the explicit waiting-time solutions.

Topic 22 A smooth ODE can develop a finite-time singularity

Level: B+ **Area:** Ordinary differential equations; continuation
Aim: Show that smoothness gives local existence and uniqueness but not global existence, and derive a growth condition that prevents finite-time blow-up.

Local existence and uniqueness do not guarantee that a solution survives for all time. The smooth equation $x' = x^2$ provides a simple example: its solution from $x(0) = 1$ grows without bound in finite time, prompting the need for continuation criteria.

Smooth finite-time blow-up. The initial-value problem

$$x'(t) = x(t)^2, \qquad x(0) = 1, \tag{27}$$

has a unique maximal solution on $(-\infty, 1)$, and this solution diverges as $t \uparrow 1$.

Tentative structure (50 minutes)

Part 1. Local theory (0–11 min). Recall the local Picard–Lindelöf theorem and note that the vector field $F(x) = x^2$ is smooth and locally Lipschitz.

Part 2. Solve explicitly (11–23 min). Separate variables, use the initial condition, and verify the resulting formula directly.

Part 3. Maximality and blow-up (23–36 min). Prove that no finite-valued classical extension through $t = 1$ is possible and formulate the continuation criterion.

Part 4. Global hypotheses (36–50 min). Use Gronwall's inequality to show that a global linear-growth bound on the vector field prevents escape to infinity in finite time.

Solution and continuation obstruction

Solving (27) gives

$$x(t) = \frac{1}{1 - t}, \qquad t < 1, \tag{28}$$

and therefore

$$\lim_{t \uparrow 1} x(t) = +\infty. \tag{29}$$

If a general vector field satisfies $|F(t, x)| \le a + b|x|$, Gronwall's inequality yields a finite bound on every bounded time interval and rules out this mechanism.

Worth mentioning

- Local Lipschitz continuity controls uniqueness only while the solution remains in a bounded region.
- A maximal solution of a locally Lipschitz ODE can end at finite time only by leaving every compact subset of the phase space.
- The example complements the waiting-time nonuniqueness example: one violates global existence, the other uniqueness.

Further reading. Begin with [10] for initial-value problems. For continuation and global-existence criteria, use instructor-selected portions of [11, 27]; these are more advanced texts, not preliminary reading.

Topic 23 Newton's method can cycle instead of finding a root

Level: B **Area:** Numerical analysis; nonlinear equations
Aim: Construct a two-cycle for Newton's method and contrast global failure with the local quadratic-convergence theorem near a simple root.

Newton's method is highly effective near a simple root, but an arbitrary starting point need not lead there. For the cubic below, the iterates fall into a two-cycle, illustrating the difference between a local convergence theorem and a global guarantee.

A polynomial two-cycle. For

$$p(x) = x^3 - 2x + 2, \tag{30}$$

Newton's method starting at $x_0 = 0$ alternates forever between 0 and 1, although p has a real zero in $(-2, -1)$.

Tentative structure (50 minutes)

Part 1. Derive Newton's map (0–10 min). Obtain the iteration from tangent-line intersection and state the usual local intuition.

Part 2. Find the two-cycle (10–22 min). Compute two iterations exactly and verify the existence of a real root by the intermediate value theorem.

Part 3. Why the iteration fails (22–35 min). Sketch the polynomial and tangent lines, and explain why root existence alone gives no global convergence guarantee.

Part 4. Local repair (35–50 min). State the local convergence theorem at a simple root and derive the quadratic error estimate from Taylor's theorem.

The cycle and the local estimate

The Newton map is

$$N(x) = x - \frac{x^3 - 2x + 2}{3x^2 - 2}. \tag{31}$$

Direct computation gives

$$N(0) = 1, \qquad N(1) = 0. \tag{32}$$

Near a simple zero r, Taylor's theorem yields an estimate of the form

$$|x_{n+1} - r| \le C|x_n - r|^2 \tag{33}$$

once all iterates remain in a sufficiently small neighborhood of r.

Worth mentioning

- Newton's theorem is local: the initial point and avoidance of critical points matter.
- Other initial conditions can diverge, enter longer cycles, or approach different roots.
- Damping or bracketing methods trade speed for more robust global behavior.

Further reading. Begin with Newton's method and its convergence analysis in [9]. [3] supplies supporting calculus and fixed-point arguments.

Topic 24 A contraction need not have a fixed point without completeness

Level: B **Area:** Metric spaces; fixed-point theory
Aim: Prove that the contraction hypothesis alone does not produce a fixed point, and locate the completeness step in Banach's theorem.

Repeatedly applying a contraction brings successive iterates closer together, but their limit still needs to belong to the space. Halving on $(0, 1]$ makes the missing point visible and explains the role of completeness in Banach's fixed-point theorem.

A fixed point missing from the space. On the metric space $X = (0, 1]$ with the Euclidean distance, define

$$T(x) = \frac{x}{2}. \tag{34}$$

Then $T : X \to X$ is a contraction with constant $1/2$, but T has no fixed point in X.

Tentative structure (50 minutes)

Part 1. Contractions and Picard iteration (0–10 min). Define contractions and fixed points, and compute the standard geometric estimate between successive iterates.

Part 2. The incomplete-space example (10–22 min). Verify that (34) maps X into itself and has contraction constant $1/2$, then solve the fixed-point equation.

Part 3. Follow the escaping limit (22–34 min). Compute the Picard iterates and show that they form a Cauchy sequence whose limit is absent from X.

Part 4. Prove the repaired theorem (34–50 min). Complete the proof of Banach's fixed-point theorem and identify completeness as the step that places the Cauchy limit inside the space.

The missing-limit mechanism

For every $x, y \in X$,

$$|T(x) - T(y)| = \frac{1}{2}|x - y|. \tag{35}$$

Starting from $x_0 \in X$, the iterates satisfy

$$x_n = T^n(x_0) = 2^{-n}x_0 \longrightarrow 0. \tag{36}$$

The sequence is Cauchy, but $0 \notin X$; in the completion $[0, 1]$, the same map has the unique fixed point 0.

Worth mentioning

- The contraction estimate creates a Cauchy sequence before completeness is invoked.
- Closed subsets of complete metric spaces are complete, which is often the practical hypothesis used in applications.
- Banach's theorem gives existence, uniqueness, convergence of the iteration, and an explicit error estimate simultaneously.

Further reading. Use the contraction principle in [43], reviewing complete metric spaces first. No general functional-analysis course is needed for this example.

Topic 25 A continuous bijection need not have a continuous inverse

Level: B **Area:** Metric topology; compactness
Aim: Construct a continuous bijection whose inverse jumps, then prove that compactness of the domain repairs the implication.

A continuous one-to-one parametrization need not remain continuous when read backwards. Wrapping a half-open interval around a circle gives an explicit example: the inverse breaks at the joining point, where compactness of the domain would have prevented the problem.

Wrapping a half-open interval. Define

$$F : [0, 2\pi) \longrightarrow S^1, \qquad F(t) = (\cos t, \sin t). \tag{37}$$

The map F is a continuous bijection, but F^{-1} is not continuous at $(1, 0)$.

Tentative structure (50 minutes)

Part 1. Continuity versus homeomorphism (0–9 min). Recall continuous maps, bijections, inverse maps, and the definition of a homeomorphism.

Part 2. Verify the counterexample (9–23 min). Prove continuity and bijectivity of (37), then test the inverse with a sequence approaching the gluing point.

Part 3. Identify the missing hypothesis (23–34 min). Explain why $[0, 2\pi)$ is not compact and why the same construction cannot work from a compact interval bijectively onto the circle.

Part 4. Compact-to-Hausdorff theorem (34–50 min). Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism by showing that it maps closed sets to closed sets.

Sequential failure of the inverse

Set

$$t_n = 2\pi - \frac{1}{n}. \tag{38}$$

Then

$$F(t_n) \longrightarrow (1, 0) = F(0), \qquad t_n \not\longrightarrow 0 \tag{39}$$

in the topology of $[0, 2\pi)$. Hence the inverse fails the sequential criterion for continuity.

Worth mentioning

- A continuous bijection preserves points but need not preserve the topology.
- Compact subsets of Hausdorff spaces are closed, which is the engine of the repaired theorem.
- The example can also be interpreted as cutting the circle at one point and failing to glue the endpoints topologically.

Further reading. Use the continuity and compactness sections of [38], especially the compact-to-Hausdorff theorem. The algebraic-topology part of the book is not needed.

Topic 26 A space-filling curve

Level: B+ **Area:** Metric topology; geometric analysis
Aim: Construct or rigorously approximate a continuous surjection from an interval onto a square and explain why continuity alone does not preserve dimension.

A curve is often pictured as a thin object, incapable of covering a whole region. Peano's construction shows that continuity alone imposes no such restriction: a continuous map from an interval can fill a square, though it cannot do so injectively.

Peano's theorem. There exists a continuous surjection

$$H : [0,1] \longrightarrow [0,1]^2.$$

It cannot be injective, and it cannot be Lipschitz. A Hilbert–Peano construction may be chosen with a Hölder estimate $|H(s) - H(t)| \le C|s-t|^{1/2}$.

Tentative structure (50 minutes)

Part 1. The dimension intuition (0–8 min). Compare smooth, Lipschitz, and merely continuous images of an interval.

Part 2. Polygonal approximants (8–25 min). Divide the square into 4^n dyadic cells, order them recursively, and join their centers by a path H_n.

Part 3. Pass to the limit (25–39 min). Prove $\|H_{n+1} - H_n\|_\infty \le C2^{-n}$, deduce uniform convergence and continuity, and use nested cells to prove surjectivity.

Part 4. What is impossible (39–50 min). Rule out injectivity by topology and Lipschitz space filling by a covering or area estimate.

Key estimates

The summable displacement estimate

$$\sum_{n=1}^{\infty} \|H_{n+1} - H_n\|_\infty < \infty$$

makes (H_n) uniformly Cauchy. For a Lipschitz map $\gamma : [0,1] \to \mathbb{R}^2$, covering $[0,1]$ by N intervals of length $1/N$ covers the image by N disks of radius $O(N^{-1})$, whose total area is $O(N^{-1}) \to 0$.

Worth mentioning

- A continuous bijection from a compact space to a Hausdorff space is a homeomorphism; $[0,1]$ and $[0,1]^2$ are not homeomorphic.
- The Hölder exponent $1/2$ matches the change from one to two dimensions.
- A careful diagram is more useful here than a long symbolic formula.

Further reading. Begin with the geometric construction of Hilbert's or Peano's curve in [45]. Peano's original French note [40] is optional historical reading.

Topic 27 Connected but not path connected

Level: B+ **Area:** Topology in analysis
Aim: Analyze the closure of an oscillatory graph and separate connectedness from the existence of paths.

A connected set need not allow a continuous path between every pair of its points. The topologist's sine curve makes this distinction visible: taking the closure of an oscillating graph adds a limiting segment without providing paths from the graph to that segment.

Topologist's sine curve. Let

$$G = \{(x, \sin(1/x)) : 0 < x \leq 1\}, \qquad S = \overline{G} = G \cup (\{0\} \times [-1, 1]).$$

Then S is compact and connected but not path connected.

Tentative structure (50 minutes)

Part 1. Draw and define the set (0–10 min). Determine the closure by constructing sequences with $\sin(1/x_n) \to y$ for every $y \in [-1, 1]$.

Part 2. Connectedness (10–21 min). The graph G is the continuous image of the connected interval $(0, 1]$; prove that the closure of a connected set is connected.

Part 3. No joining path (21–40 min). Suppose a path leaves the vertical segment. At its first exit, the positive x-coordinate approaches zero while the forced value $\sin(1/x)$ oscillates; select alternating level sequences to contradict continuity.

Part 4. Local pathology (40–50 min). Identify the two path components and discuss failure of local connectedness along the limiting segment.

Key proposition

If A is connected in a topological space, then $\overline{A}$ is connected. This proves connectedness immediately. For the path obstruction, if $\gamma(t) = (x(t), y(t))$ and $x(t) > 0$, then

$$y(t) = \sin(1/x(t)).$$

Continuity of x near a first exit from $x = 0$ forces it through values on which the right-hand side is alternately 1 and -1, contradicting continuity of y.

Worth mentioning

- Path connectedness always implies connectedness; this example disproves the converse.
- Compactness follows because S is closed and bounded in $\mathbb{R}^2$.
- The example is driven by accumulation of infinite oscillation at one boundary line.

Further reading. Use the connectedness and path-connectedness sections of [38], including the topologist's sine curve. Later algebraic-topology chapters are unnecessary.

Topic 28 The Koch snowflake: finite area but infinite perimeter

Level: B+ **Area:** Geometric analysis; fractals
Aim: Construct a bounded Jordan region whose polygonal approximations have convergent areas but divergent boundary lengths.

The area enclosed by a curve and the length of its boundary can behave very differently. In the Koch snowflake, each stage adds a small amount of area but increases the perimeter, producing a bounded region with an infinitely long boundary.

Finite interior, nonrectifiable boundary. Starting from an equilateral triangle of side length one, replace the middle third of every boundary segment by the other two sides of an outward equilateral triangle. If L_n and A_n denote the perimeter and area after n steps, then

$$L_n \longrightarrow \infty, \qquad A_n \longrightarrow \frac{2\sqrt{3}}{5}. \tag{40}$$

Tentative structure (50 minutes)

Part 1. Iterative construction (0–12 min). Draw the first three stages, count the segments, and record their lengths.

Part 2. Infinite perimeter (12–23 min). Derive the perimeter recurrence and prove divergence geometrically.

Part 3. Finite area (23–38 min). Count the new triangles at every stage, sum their areas, and obtain the finite limit in (40).

Part 4. What length means (38–50 min). Explain rectifiability, self-similarity, and the optional Hausdorff-dimension computation for the boundary.

Length and area calculations

At stage n, the boundary contains $3 \cdot 4^n$ segments of length 3^{-n}, and hence

$$L_n = 3\left(\frac{4}{3}\right)^n. \tag{41}$$

At stage $n \geq 1$, one adds $3 \cdot 4^{n-1}$ triangles of side length 3^{-n}. Therefore

$$A_\infty = \frac{\sqrt{3}}{4}\left(1 + \sum_{n=1}^{\infty} \frac{3 \cdot 4^{n-1}}{9^n}\right), \tag{42}$$

$$A_\infty = \frac{2\sqrt{3}}{5}. \tag{43}$$

The self-similar boundary has dimension

$$d = \frac{\log 4}{\log 3} > 1. \tag{44}$$

Worth mentioning

- Finite planar area imposes no finite-length bound on the boundary.
- The divergent perimeters of the approximations agree with the nonrectifiability of the limiting curve.
- The dimension calculation is optional; the perimeter and area arguments already form a complete counterexample.

Further reading. Use the introductory examples and self-similar constructions in [18]. The length and area calculations are the core; the general theory of fractal dimension is optional.

Topic 29 The Vitali nonmeasurable set

Level: B+ **Area:** Measure theory; set-theoretic methods
Aim: Show that no translation-invariant countably additive extension of length can measure every subset of the line.

Lebesgue measure extends the notion of length far beyond intervals, but it cannot measure every subset of the line. Vitali's choice of representatives modulo rational translations shows how translation invariance and countable additivity force this limitation.

Vitali's theorem. There exists $V \subset [0,1]$ that is not Lebesgue measurable. The construction chooses one representative from each equivalence class of

$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Q}.$$

Tentative structure (50 minutes)

Part 1. Equivalence classes (0–10 min). Verify reflexivity, symmetry, and transitivity; use a choice principle to select one point from each class meeting $[0,1]$.

Part 2. Rational translates (10–23 min). For $q \in \mathbb{Q} \cap [-1,1]$, prove that the sets $V+q$ are pairwise disjoint and that their union covers $[0,1]$.

Part 3. Measure contradiction (23–38 min). The union lies in $[-1,2]$. If $m(V)=0$, countable additivity gives a null union; if $m(V)>0$, the union has infinite measure.

Part 4. Interpretation (38–50 min). Locate the uses of choice, translation invariance, and countable additivity; explain why Lebesgue measure is complete but not universal.

Decisive inclusions

With $Q_0 = \mathbb{Q} \cap [-1,1]$,

$$[0,1] \subseteq \bigcup_{q \in Q_0} (V+q) \subseteq [-1,2].$$

If all translates were measurable, translation invariance would give each the same measure. Countable additivity then yields either 0 or $+\infty$, contradicting the displayed bounds.

Worth mentioning

- Pairwise disjointness depends on choosing exactly one representative per class.
- Outer measure exists for every set, but nonmeasurable sets fail Carathéodory's criterion.
- The conclusion is not that "length is inconsistent", but that its desired properties cannot hold simultaneously on the full power set.

Further reading. Begin with nonmeasurable sets in [42]. [19] gives a more abstract account; Vitali's original Italian note [48] is optional historical reading.

Topic 30 Weak convergence does not imply norm convergence

Level: B+ **Area:** Functional analysis
Aim: Use the standard basis of a Hilbert space to explain what weak convergence detects and what it loses.

Weak convergence tests a sequence against each fixed continuous linear functional, rather than measuring its distance from the limit. The standard basis of ℓ^2 illustrates the distinction: every fixed test tends to zero, while the vectors retain unit norm.

Orthonormal counterexample. In ℓ^2, the standard basis satisfies

$$e_n \rightharpoonup 0, \qquad \|e_n\|_{\ell^2} = 1 \quad \text{for every } n.$$

Thus weak convergence does not imply strong convergence, even for a bounded sequence in a Hilbert space.

Tentative structure (50 minutes)

Part 1. Two topologies (0–12 min). Define norm and weak convergence in a Banach space; specialize to inner products in a Hilbert space.

Part 2. Verify the example (12–25 min). For $y \in \ell^2$, show $\langle e_n, y\rangle = y_n \to 0$, while the norms remain one.

Part 3. Positive structure (25–39 min). Prove weak lower semicontinuity of the norm in Hilbert space and the criterion "weak convergence plus convergence of norms implies strong convergence."

Part 4. Why analysts use weak limits (39–50 min). Mention weak compactness of bounded sequences in reflexive spaces and the need for compactness to recover strong convergence.

Key identity and theorem

If $x_n \rightharpoonup x$ in a Hilbert space and $\|x_n\| \to \|x\|$, then

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\,\mathrm{Re}\langle x_n, x\rangle \longrightarrow 0.$$

In the counterexample the weak limit has norm zero, while the sequence norms stay equal to one.

Worth mentioning

- Every coordinate moving to zero is not, by itself, the definition of weak convergence; test against an arbitrary element of the dual.
- Weak limits are unique because the dual separates points.
- Compact operators send weakly convergent sequences to norm-convergent sequences.

Further reading. Begin with weak and strong convergence in [32], reviewing Hilbert spaces and bounded linear functionals first. [7, 13] provide optional graduate-level background.

Topic 31 Closed and bounded is not compact in infinite dimensions

Level: B+ **Area:** Functional analysis; compactness
Aim: Disprove the infinite-dimensional Heine–Borel property and derive the general obstruction from Riesz's lemma.

In finite dimensions, closed and bounded sets are compact; an infinite-dimensional space has room for a bounded sequence whose terms stay apart. The unit vectors in ℓ^2 exhibit this directly, and Riesz's lemma extends the obstruction beyond Hilbert spaces.

Failure of Heine–Borel. The closed unit ball of an infinite-dimensional normed space is never compact in the norm topology. In ℓ^2, the sequence (e_n) lies in the unit ball and satisfies

$$\|e_n - e_m\|_{\ell^2} = \sqrt{2} \qquad (n \neq m),$$

so it has no convergent subsequence.

Tentative structure (50 minutes)

Part 1. Finite-dimensional benchmark (0–10 min). Recall equivalent forms of compactness in metric spaces and the Heine–Borel theorem in $\mathbb{R}^d$.

Part 2. Concrete sequence (10–21 min). Use pairwise separation of the unit vectors to rule out Cauchy, hence convergent, subsequences.

Part 3. General proof (21–39 min). State and prove Riesz's lemma; iteratively choose unit vectors separated from the span of their predecessors.

Part 4. Consequences (39–50 min). Explain why compact embeddings and compact operators must create extra regularity, tightness, or decay.

Riesz's lemma

If Y is a proper closed subspace of a normed space X and $0 < \alpha < 1$, there exists $x \in X$ such that

$$\|x\| = 1, \qquad \operatorname{dist}(x, Y) > \alpha.$$

Applying the lemma to $Y_n = \operatorname{span}\{x_1, \ldots, x_n\}$ produces a bounded sequence with no Cauchy subsequence.

Worth mentioning

- Sequential compactness and compactness agree here because normed spaces are metric.
- The unit ball may be weakly compact even when it is not norm compact.
- A normed space has compact closed unit ball if and only if it is finite dimensional.

Further reading. Begin with normed spaces, compactness, and finite dimension in [32]. [7, 13] are optional graduate-level alternatives; focus on Riesz's lemma and the elementary ℓ^2 example.

Topic 32 Strong operator convergence does not imply norm convergence

Level: M **Area:** Operator theory
Aim: Separate pointwise convergence on vectors from uniform convergence on the unit ball.

An operator sequence may converge on every fixed vector without approaching its limit uniformly on the unit ball. Coordinate projections on ℓ^2 show the difference: each vector is approximated increasingly well, but there are always unit vectors beyond the retained coordinates.

Projection counterexample. For $x = (x_j)_{j\geq 1} \in \ell^2$, set

$$P_n x = (x_1, \ldots, x_n, 0, 0, \ldots).$$

Then $P_n \to I$ strongly, but $\|P_n - I\| = 1$ for every n.

Tentative structure (50 minutes)

Part 1. Operator topologies (0–12 min). Define weak, strong, and norm operator convergence and write their quantifiers in the correct order.

Part 2. Strong convergence (12–23 min). For each fixed $x \in \ell^2$, identify the tail norm and use convergence of the defining series.

Part 3. Norm obstruction (23–35 min). Test $I - P_n$ on e_{n+1} and prove the matching upper bound from orthogonal projection.

Part 4. When convergence improves (35–50 min). Prove uniform convergence on compact subsets for a uniformly bounded strongly convergent family; contrast finite and infinite dimensions.

Quantifiers and computation

$$\forall x \in \ell^2 : \quad \|(I - P_n)x\|_2^2 = \sum_{j>n} |x_j|^2 \longrightarrow 0,$$

whereas

$$\|I - P_n\| = \sup_{\|x\|_2 \leq 1} \|(I - P_n)x\|_2 = 1.$$

The maximizing test vector depends on n; this dependence is the entire obstruction to interchanging $\forall x$ with a uniform supremum.

Worth mentioning

- In finite dimensions, strong convergence of operators implies norm convergence.
- Strong convergence plus collective compactness can yield stronger conclusions.
- This example is also the prototype for spectral and numerical truncation limits.

Further reading. Use [13] for operator topologies. [41] is a complementary graduate reference; restrict attention to bounded operators and the projection example.

Topic 33 The spectrum of an operator need not contain any eigenvalues

Level: M **Area:** Operator theory; spectral analysis
Aim: Compute the full spectrum of a concrete self-adjoint operator, prove that its point spectrum is empty, and construct approximate eigenvectors.

For matrices, spectral values are eigenvalues, but infinite-dimensional operators need not follow this pattern. Multiplication by x on $L^2(0,1)$ has no eigenvectors, while functions concentrated near a point of $[0,1]$ provide approximate ones.

Purely continuous spectrum. Let $M : L^2(0,1) \to L^2(0,1)$ be

$$(Mf)(x) = xf(x).$$

Then M is bounded and self-adjoint,

$$\sigma(M) = [0,1], \qquad \sigma_{\mathrm{p}}(M) = \varnothing.$$

Thus even a self-adjoint operator can have a nonempty spectrum without a single eigenvalue.

Tentative structure (50 minutes)

Part 1. Finite-dimensional intuition (0–8 min). Recall eigenvalues, invertibility, resolvents, and why matrices have spectrum consisting only of eigenvalues.

Part 2. No eigenfunctions (8–19 min). Show that $Mf = \lambda f$ forces f to be supported on the null set $\{x = \lambda\}$.

Part 3. Compute the spectrum (19–38 min). Construct the bounded inverse for $\lambda \notin [0,1]$ and normalized functions concentrating near each $\lambda \in [0,1]$ to prove failure of bounded invertibility there.

Part 4. Interpretation (38–50 min). Introduce approximate point and continuous spectrum, state Weyl's criterion, and contrast this example with compact self-adjoint operators.

Resolvent and Weyl sequence

For $\lambda \notin [0,1]$,

$$(M - \lambda I)^{-1} g(x) = \frac{g(x)}{x - \lambda}, \qquad \|(M - \lambda I)^{-1}\| \le \frac{1}{\operatorname{dist}(\lambda, [0,1])}.$$

For $\lambda \in [0,1]$, let $I_n = (\lambda - n^{-1}, \lambda + n^{-1}) \cap (0,1)$ and $u_n = |I_n|^{-1/2} \mathbf{1}_{I_n}$. Then

$$\|u_n\|_2 = 1, \qquad \|(M - \lambda I)u_n\|_2 \le n^{-1}.$$

Worth mentioning

- Approximate eigenvectors may concentrate while no genuine eigenvector exists.
- The spectrum of a bounded operator is always nonempty and compact over $\mathbb{C}$.
- Continuous spectrum is fundamental for differential operators on unbounded domains.

Further reading. Begin with bounded operators and spectral theory in [13]. Consult [41] for multiplication operators; the general theory of unbounded operators is not required.

Topic 34 Completeness in the uniform boundedness principle

Level: M **Area:** Functional analysis; Baire category
Aim: Give an explicit failure on an incomplete space and then expose the exact point where completeness enters the theorem.

The uniform boundedness principle turns separate bounds for each vector into a common bound on operator norms. On the incomplete space c_{00}, a simple family of coordinate functionals defeats that conclusion and makes the completeness assumption tangible.

Counterexample on c_{00}. Let $X = c_{00}$ carry the supremum norm and define $T_n : X \to \mathbb{R}$ by $T_n x = n x_n$. Then

$$\forall x \in X : \quad \sup_n |T_n x| < \infty, \qquad \text{but} \qquad \|T_n\| = n \longrightarrow \infty.$$

Thus pointwise boundedness need not imply uniform boundedness when the domain is incomplete.

Tentative structure (50 minutes)

Part 1. State Banach–Steinhaus (0–10 min). Write the pointwise hypothesis and operator-norm conclusion with their quantifiers.

Part 2. Verify the counterexample (10–22 min). Every $x \in c_{00}$ has finite support, while testing on e_n gives $\|T_n\| = n$.

Part 3. Find the missing hypothesis (22–39 min). Note that c_{00} is dense but not closed in c_0. Prove the theorem using the closed sets $E_k = \{x : \sup_n \|T_n x\| \le k\}$ and Baire category.

Part 4. Interpretation (39–50 min). Explain why pointwise information becomes uniform only when the ambient space cannot be written as a countable union of nowhere dense sets.

Uniform boundedness principle

If X is Banach, Y is normed, and $T_n \in \mathcal{L}(X, Y)$ satisfy

$$\forall x \in X : \quad \sup_n \|T_n x\|_Y < \infty,$$

then $\sup_n \|T_n\|_{\mathcal{L}(X,Y)} < \infty$. Once some E_k contains a ball, linearity converts boundedness on that ball into a global operator-norm bound.

Worth mentioning

- The completion c_0 contains sequences for which $n x_n$ is unbounded; pointwise boundedness does not extend to the completion.
- Banach–Steinhaus also proves existence of pathological objects, as in Topic 36.
- This talk should include the Baire step, not only state it.

Further reading. Use [13] for uniform boundedness and Baire category. [44] is a more advanced alternative, rather than additional required reading.

Topic 35 Banach limits: the dual of ℓ^∞ is larger than ℓ^1

Level: M+ **Area:** Functional analysis; Hahn–Banach theory
Aim: Construct a positive shift-invariant extension of the usual limit and prove that it cannot arise from an ℓ^1 coefficient sequence.

Can the usual limit be extended to bounded sequences that do not converge while preserving positivity and shift invariance? Banach limits do exactly this, but they cannot be expressed using ℓ^1 coefficients, revealing functionals beyond that familiar description of the dual.

Existence of a Banach limit. There is a bounded linear functional $L : \ell^\infty \to \mathbb{R}$ such that

$$L(\mathbf{1}) = 1, \qquad x_n \geq 0 \Rightarrow L(x) \geq 0, \qquad L(Sx) = L(x),$$

where $(Sx)_n = x_{n+1}$, and $L(x) = \lim_n x_n$ whenever the usual limit exists. No such functional has the form

$$L(x) = \sum_{n=1}^{\infty} a_n x_n \qquad \text{with } a \in \ell^1.$$

Tentative structure (50 minutes)

Part 1. The false duality guess (0–8 min). Recall $(\ell^1)^* = \ell^\infty$ and ask whether dualizing again gives $(\ell^\infty)^* = \ell^1$.

Part 2. Build a dominating functional (8–22 min). For bounded real sequences, define the upper block-average functional p below and prove its sublinearity using Fekete's lemma.

Part 3. Apply Hahn–Banach (22–39 min). Extend the ordinary value on constant sequences under the domination $L \leq p$; derive positivity, norm one, shift invariance, and agreement with ordinary limits.

Part 4. Prove nonrepresentation (39–50 min). Evaluate on the standard unit vectors and on $\mathbf{1}$, then discuss nonuniqueness and finitely additive measures.

Construction and contradiction

Set

$$p(x) = \lim_{N\to\infty} \sup_{m\geq 1} \frac{1}{N} \sum_{k=0}^{N-1} x_{m+k}.$$

The limit exists because the suprema of the unnormalized block sums form a subadditive sequence. Moreover $p(x - Sx) = p(Sx - x) = 0$, so any Hahn–Banach extension dominated by p is shift invariant. If $L(x) = \sum a_n x_n$, then

$$a_n = L(e_n) = 0 \quad \text{for every } n, \qquad \text{but} \qquad L(\mathbf{1}) = 1,$$

a contradiction.

Worth mentioning

- A Banach limit assigns value 0 to $((-1)^n)$ by shift invariance.
- Banach limits are not unique; ordinary convergence determines only a small subspace.
- The construction is nonconstructive and uses Hahn–Banach, hence a form of choice.

Further reading. Begin with Hahn–Banach and Banach-space duality in [13]. Lorentz's paper [35] is optional research-level reading on Banach limits and almost convergence.

Topic 36 A continuous function with a divergent Fourier series

Level: M **Area:** Harmonic analysis; functional analysis
Aim: Use the Dirichlet kernel and uniform boundedness to prove existence of a continuous function whose Fourier partial sums diverge at a prescribed point.

A continuous periodic function need not be recovered pointwise by its Fourier partial sums. The growing L^1 norm of the Dirichlet kernel, combined with uniform boundedness, turns this failure into an existence proof and highlights the value of Fejér summation.

du Bois-Reymond phenomenon. There exists $f \in C(\mathbb{T})$ such that the sequence of Fourier partial sums $(S_N f(0))_N$ is unbounded, hence divergent. Continuity alone does not guarantee pointwise convergence of the Fourier series.

Tentative structure (50 minutes)

Part 1. Fourier partial sums (0–10 min). Derive the convolution formula with the Dirichlet kernel and regard evaluation at zero as a linear functional on $C(\mathbb{T})$.

Part 2. Kernel growth (10–25 min). Prove or carefully estimate $\|D_N\|_{L^1} \simeq \log N$ by splitting near the zeros and using $|\sin(t/2)| \simeq |t|$ near zero.

Part 3. Existence by Banach–Steinhaus (25–38 min). The functionals $L_N f = S_N f(0)$ have unbounded norms, so some continuous f has $\sup_N |L_N f| = \infty$.

Part 4. Fejér's repair (38–50 min). Replace partial sums by Cesàro means and prove uniform convergence using positivity and concentration of the Fejér kernels.

Key formulas

$$S_N f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-t) D_N(t)\, dt, \qquad D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(t/2)}.$$

Moreover,

$$\|L_N\|_{C(\mathbb{T})^*} = \frac{1}{2\pi}\|D_N\|_1 \simeq \log N.$$

The equality can be obtained by continuously approximating the sign of D_N.

Worth mentioning

- The argument proves existence but does not display the pathological f explicitly.
- The bad behavior comes from cancellation and the growing L^1 mass of D_N.
- Carleson's theorem is much deeper and concerns almost-everywhere convergence for L^2 data.

Further reading. Begin with classical Fourier series and Dirichlet kernels in [29]. [50] is an advanced reference for further divergence phenomena.

Topic 37 The Hardy–Littlewood maximal operator is not bounded on L^1

Level: M **Area:** Real and harmonic analysis
Aim: Give the elementary endpoint counterexample, prove the weak $(1,1)$ replacement, and explain why maximal estimates govern differentiation of integrals.

Taking local averages might seem unlikely to destroy integrability. The maximal operator nevertheless gives the indicator of $[0,1]$ a nonintegrable tail, showing why weak L^1 control, rather than an L^1 norm bound, is the appropriate endpoint estimate.

Strong endpoint failure. For the uncentered maximal operator

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy$$

on $\mathbb{R}$, take $f = \mathbf{1}_{[0,1]}$. Then

$$Mf(x) \geq \frac{1}{x} \qquad (x > 1),$$

so $f \in L^1(\mathbb{R})$ but $Mf \notin L^1(\mathbb{R})$. Nevertheless, $M : L^1 \to L^{1,\infty}$ is bounded.

Tentative structure (50 minutes)

Part 1. Why maximal averages arise (0–8 min). Introduce local averages and their role in controlling pointwise limits of approximations to the identity.

Part 2. Compute the counterexample (8–19 min). For $x > 1$, choose $I = [0, x]$ and integrate the resulting $1/x$ tail.

Part 3. Prove the endpoint substitute (19–38 min). For $E_\lambda = \{Mf > \lambda\}$, choose intervals with average above λ and use a one-dimensional covering lemma to prove the weak estimate.

Part 4. Consequences (38–50 min). State the strong L^p theorem for $p > 1$, explain interpolation, and connect the maximal estimate to the Lebesgue differentiation theorem.

Correct endpoint statement

There is an absolute constant C such that

$$\big|\{x : Mf(x) > \lambda\}\big| \leq \frac{C}{\lambda} \|f\|_{L^1}, \qquad \lambda > 0.$$

Equivalently, $\|Mf\|_{L^{1,\infty}} \leq C\|f\|_1$. For $1 < p \leq \infty$,

$$\|Mf\|_{L^p} \leq C_p \|f\|_{L^p},$$

so the distinction between strong and weak type occurs exactly at the endpoint.

Worth mentioning

- The obstruction is a nonintegrable tail, not a singularity of f.
- Centered and uncentered maximal operators differ only by harmless constants here.
- Weak L^1 is large enough to contain $x^{-1}\mathbf{1}_{(1,\infty)}$.

Further reading. Use [22] for maximal functions and the weak-$(1,1)$ estimate. Hardy and Littlewood's original paper [26] is optional historical reading.

Topic 38 Failure of the critical Sobolev embedding into L^∞

Level: M **Area:** Sobolev spaces; PDE
Aim: Construct a normalized logarithmic spike and identify the correct borderline substitute for an L^∞ estimate.

At the critical exponent, Sobolev control no longer supplies a uniform bound on function values. Logarithmic spikes make this failure explicit: their heights grow without bound while their $W^{1,n}$ norms remain bounded, pointing toward weaker borderline estimates.

Critical failure. Let $n \geq 2$ and let $B_1 \subset \mathbb{R}^n$. There is no constant C such that

$$\|u\|_{L^\infty(B_1)} \leq C\|u\|_{W^{1,n}(B_1)} \qquad \text{for all } u \in W_0^{1,n}(B_1).$$

Tentative structure (50 minutes)

Part 1. Scaling prediction (0–10 min). Recall the subcritical Sobolev exponent and show why $p = n$ is invariant under spatial scaling.

Part 2. Define the Moser sequence (10–22 min). Use a radial logarithmic profile with a constant core of radius ε.

Part 3. Compute the norms (22–38 min). Evaluate the radial gradient integral and show that the $W^{1,n}$ norm stays bounded while the supremum diverges.

Part 4. Borderline repair (38–50 min). State the Trudinger–Moser exponential integrability theorem and mention the embedding into BMO.

Explicit sequence

Writing $L_\varepsilon = \log(1/\varepsilon)$, set

$$u_\varepsilon(x) = \begin{cases} L_\varepsilon^{(n-1)/n}, & |x| \leq \varepsilon, \\ \dfrac{\log(1/|x|)}{L_\varepsilon^{1/n}}, & \varepsilon < |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Then $\|u_\varepsilon\|_\infty = L_\varepsilon^{(n-1)/n} \to \infty$, whereas

$$\|\nabla u_\varepsilon\|_n^n = \frac{\omega_{n-1}}{L_\varepsilon} \int_\varepsilon^1 \frac{dr}{r} = \omega_{n-1}.$$

Poincaré's inequality controls the remaining L^n term, so the full $W_0^{1,n}(B_1)$ norm is bounded.

Worth mentioning

- In one dimension, $W^{1,1}$ does control the supremum; the restriction $n \geq 2$ matters.
- The Trudinger–Moser replacement controls $\int e^{\alpha|u|^{n/(n-1)}}$ below a sharp threshold.
- The sequence illustrates concentration at one point, not oscillation or translation.

Further reading. Begin with the embedding results in [2]. Moser's paper [37] concerns the sharper exponential-integrability theorem and is optional research-level background, not needed for the unbounded example.

Topic 39 Loss of compactness in Sobolev embeddings

Level: M **Area:** Sobolev spaces; compactness
Aim: Display the two fundamental defects—translation and concentration—and contrast them with the Rellich theorem.

A bounded Sobolev sequence can fail to have a strongly convergent subsequence for quite different reasons. Translations move its mass away, while critical rescalings concentrate it near one point; these examples explain the limits of compact embedding theorems.

Two noncompactness mechanisms. Translations destroy compactness on $\mathbb{R}^d$, while critical dilations destroy compactness even on bounded domains. For $d \geq 3$ and $2^* = 2d/(d-2)$, the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous but not compact.

Tentative structure (50 minutes)

Part 1. Compact embedding (0–10 min). Define compactness of the inclusion map and state Rellich–Kondrachov for bounded domains and subcritical exponents.

Part 2. Escape by translation (10–22 min). For a fixed compactly supported bump, set $u_k(x) = \phi(x - ke_1)$; norms are unchanged and mass leaves every compact set.

Part 3. Critical concentration (22–39 min). Place a rescaled bump near $x_0 \in \Omega$ and compute the scale-invariant gradient and critical norms.

Part 4. Structural conclusion (39–50 min). Introduce the concentration–compactness trichotomy: compactness, vanishing, and concentration/dichotomy.

Critical scaling

For $\phi \in C_c^\infty(B_1)$ and large λ, define

$$u_\lambda(x) = \lambda^{(d-2)/2}\phi(\lambda(x - x_0)).$$

Then

$$\|\nabla u_\lambda\|_2 = \|\nabla \phi\|_2, \qquad \|u_\lambda\|_{2^*} = \|\phi\|_{2^*}, \qquad u_\lambda \rightharpoonup 0.$$

Strong convergence in L^{2^*} is impossible because the norm does not tend to zero.

Worth mentioning

- On a bounded domain, translation is blocked but small-scale concentration remains.
- For $q < 2^*$, the corresponding embedding is compact under standard domain hypotheses.
- Symmetry, confinement, or normalization can sometimes remove a defect of compactness.

Further reading. Use [2] for Sobolev and compact embeddings. Lions's paper [34] is an optional research-level extension on concentration–compactness, not a proof requirement for the scaling examples.

Topic 40 Weak limits do not commute with nonlinearities

Level: M **Area:** PDE; calculus of variations
Aim: Compute a quadratic defect generated by oscillation and motivate refined descriptions of weakly convergent sequences.

Weak convergence captures linear tests but can miss information carried by nonlinear expressions. The oscillations of $\sin(nx)$ disappear weakly, yet their squares retain a nonzero average, offering a concrete starting point for understanding nonlinear limits.

Oscillatory defect. On $\mathbb{T} = (0, 2\pi)$, let $u_n(x) = \sin(nx)$. Then

$$u_n \rightharpoonup 0 \quad \text{in } L^2(\mathbb{T}), \qquad u_n^2 \rightharpoonup \frac{1}{2} \quad \text{in } L^2(\mathbb{T}).$$

Hence squaring is not weakly sequentially continuous.

Tentative structure (50 minutes)

Part 1. Verify weak convergence (0–12 min). First test against trigonometric polynomials, then use their density in L^2 and uniform boundedness of u_n.

Part 2. Compute the nonlinear limit (12–23 min). Use the double-angle identity and the same oscillatory argument for $\cos(2nx)$.

Part 3. Compare with positive results (23–38 min). Show that strong L^2 convergence would imply L^1 convergence of the squares; recall weak lower semicontinuity.

Part 4. PDE relevance (38–50 min). Discuss Reynolds stresses, defect measures, Young measures, and why compensated compactness requires special differential structure.

Decisive identity

$$u_n^2(x) = \sin^2(nx) = \frac{1}{2} - \frac{1}{2}\cos(2nx).$$

Thus the weak limit remembers the average energy of the oscillations even though the oscillations themselves disappear. In contrast, if $u_n \to u$ strongly in L^2, then

$$\|u_n^2 - u^2\|_1 \le \|u_n - u\|_2\big(\|u_n\|_2 + \|u\|_2\big) \longrightarrow 0.$$

Worth mentioning

- Weak convergence retains all linear observables, not nonlinear ones.
- The defect is positive and matches weak lower semicontinuity of the squared norm.
- Do not claim that products of arbitrary weakly convergent sequences converge.

Further reading. Begin with weak convergence in [7]. Ball's chapter [5] is optional advanced reading on Young measures; the oscillatory example can be proved without that theory.

Topic 41 Why distributions cannot be multiplied canonically

Level: M+ **Area:** Distribution theory
Aim: Use mollifiers to show that the square of a Dirac mass has no distributional limit and explain the structural obstruction to a universal product.

Smoothing distributions makes ordinary multiplication possible, but removing the smoothing need not give a well-defined product. The Dirac mass and the Heaviside function expose the difficulty through divergent or regularization-dependent products, motivating the restrictions of classical distribution theory.

Regularization obstruction. Let $\rho \in C_c^\infty(\mathbb{R})$, $\int \rho = 1$, and $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(x/\varepsilon)$. Although $\rho_\varepsilon \to \delta_0$ in $\mathcal{D}'$, the sequence ρ_ε^2 has no limit in $\mathcal{D}'$; its action on a nonnegative test function with $\varphi(0) > 0$ diverges like ε^{-1}.

Tentative structure (50 minutes)

Part 1. What is defined (0–10 min). Recall distributional convergence, derivatives, and the valid product of a distribution with a smooth function.

Part 2. Square a mollifier (10–23 min). Compute the pairing of ρ_ε^2 with φ and prove divergence.

Part 3. Heaviside paradox (23–38 min). A consistent regularization gives $H_\varepsilon H_\varepsilon' \to \frac{1}{2}\delta_0$, while different relative regularizations can change the result; formal algebraic rules lead to contradictions.

Part 4. Restricted repairs (38–50 min). Mention products under wave-front-set conditions and Colombeau algebras, making clear which classical property is weakened.

Key computation

For $\varphi \in C_c^\infty$,

$$\langle \rho_\varepsilon^2, \varphi \rangle = \frac{1}{\varepsilon}\int_{\mathbb{R}} \rho(y)^2 \varphi(\varepsilon y)\, dy = \frac{\varphi(0)}{\varepsilon}\int \rho^2 + O(1).$$

Also, if $H_\varepsilon' = \rho_\varepsilon$, then

$$H_\varepsilon H_\varepsilon' = \frac{1}{2}(H_\varepsilon^2)' \longrightarrow \frac{1}{2}\delta_0.$$

Worth mentioning

- The obstruction concerns a universal product; many particular distributional products are well defined.
- Renormalization subtracts divergent pieces but introduces additional choices.
- The Schwartz impossibility theorem formalizes the conflict among multiplication, differentiation, and agreement with ordinary functions.

Further reading. Use selected distribution-theory sections of [28] for background. Schwartz's note [46] gives the original obstruction; [12] is an optional research-level extension. Agree on the scope with the instructor.

Topic 42 The Lavrentiev phenomenon

Level: M+ **Area:** Calculus of variations
Aim: Show that norm density of smooth maps need not imply density of their energies and explain why conforming approximation can converge to the wrong variational value.

Approximating a function in norm need not approximate its energy. In Manià's example, restricting a variational problem to smoother competitors changes the infimum, showing why density alone does not justify an approximation scheme for minimization.

Manià's example. For

$$J(u) = \int_0^1 |u'(x)|^6 \big(u(x)^3 - x\big)^2 \, dx, \qquad u(0) = 0, \quad u(1) = 1,$$

one has

$$\min_{u \in W^{1,1}} J(u) = J(x^{1/3}) = 0, \qquad \inf_{u \in W^{1,\infty}} J(u) > 0.$$

This strict inequality is a Lavrentiev gap.

Tentative structure (50 minutes)

Part 1. Direct method intuition (0–10 min). Explain why one normally approximates Sobolev competitors by smooth or Lipschitz maps and expects the energy to follow.

Part 2. Singular minimizer (10–21 min). Verify that $u_*(x) = x^{1/3}$ is absolutely continuous, satisfies the boundary data, and makes the integrand identically zero.

Part 3. The gap estimate (21–40 min). Present the core lemma: a Lipschitz competitor cannot track $x^{1/3}$ sufficiently closely near zero without paying a fixed amount through the factor $|u'|^6$. Prove the lemma in a carefully selected interval.

Part 4. Consequences (40–50 min). Discuss failure of energy convergence under smooth approximation and the implications for conforming finite-element methods.

Proof pivots

The minimizer satisfies

$$u_*^3 - x = 0, \qquad u_*'(x) = \frac{1}{3} x^{-2/3} \in L^1(0,1) \setminus L^\infty(0,1).$$

Although Lipschitz maps are dense in $W^{1,1}$, the integrand has coupled superlinear growth in u' and $u^3 - x$; convergence in $W^{1,1}$ does not control J.

Worth mentioning

- State the admissible classes and boundary conditions before speaking of a gap.
- The phenomenon concerns infima over nested classes, not failure of existence alone.
- For a full proof of the positive Lipschitz lower bound, assign one precise gap lemma rather than hiding it behind numerical plots.

Further reading. Begin with the discussion of the Lavrentiev phenomenon in [14]. Manià's original note [36] and [6] are specialist sources to consult selectively with the instructor.

Topic 43 A reentrant corner destroys H^2 elliptic regularity

Level: M **Area:** Elliptic PDE; Sobolev regularity
Aim: Construct an H_0^1 solution with smooth L^2 forcing whose second derivatives are not square integrable, and identify boundary geometry as the missing hypothesis.

Smooth forcing does not by itself guarantee full regularity up to the boundary. A reentrant corner supports a singular solution whose second derivatives are not square integrable, making the geometry of the domain an essential part of elliptic estimates.

Corner singularity. Let

$$\Omega = \{(r,\theta) : 0 < r < 1,\ 0 < \theta < 3\pi/2\}.$$

Choose a smooth radial cutoff η equal to 1 near $r = 0$ and vanishing near $r = 1$, and set

$$u(r,\theta) = \eta(r) r^{2/3} \sin(2\theta/3).$$

Then $u \in H_0^1(\Omega)$, the forcing $f = -\Delta u$ is smooth up to the polygonal boundary and belongs to $L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Tentative structure (50 minutes)

Part 1. The expected estimate (0–10 min). Recall the weak Dirichlet problem and the global estimate $\|u\|_{H^2} \lesssim \|f\|_2$ on sufficiently smooth domains.

Part 2. Find the singular harmonic mode (10–22 min). Separate variables in a sector of opening ω to obtain $r^{\pi/\omega} \sin(\pi\theta/\omega)$.

Part 3. Cut off and compute (22–39 min). Verify the zero trace and H^1 integrability, show that the cutoff error f is smooth away from the corner, and prove divergence of the H^2 integral.

Part 4. Correct regularity theorem (39–50 min). State the estimate for $C^{1,1}$ domains and for convex polygons, then discuss corner expansions and finite element mesh refinement.

The decisive exponent

For a sector of opening ω, the first Dirichlet singular exponent is $\alpha = \pi/\omega$. Here $\alpha = 2/3$, and near the corner

$$|\nabla u| \simeq r^{\alpha-1}, \qquad |D^2 u| \simeq r^{\alpha-2}.$$

Consequently,

$$\int_0^\varepsilon r^{2\alpha-1}\, dr < \infty, \qquad \int_0^\varepsilon r^{2\alpha-3}\, dr = \infty,$$

which is exactly $u \in H^1$ but $u \notin H^2$.

Worth mentioning

- The forcing is regular; the singularity is created entirely by the boundary geometry.
- A cutoff converts the local harmonic mode into a global weak Dirichlet solution.
- Reentrant corners are also responsible for reduced finite-element convergence rates.

Further reading. Use [17] for the regularity theorem on smooth domains. [15] is the specialist reference for corner singularities; only selected material is needed for the explicit sector example.

Topic 44 The backward heat equation is ill-posed

Level: M **Area:** Parabolic PDE; inverse problems
Aim: Use the Fourier spectrum of the Dirichlet Laplacian to prove that backward heat evolution fails continuous dependence and to motivate regularization.

Forward heat flow damps high-frequency modes, but reversing it requires amplifying those same modes. Small errors in terminal data can therefore become large errors in the reconstructed past, giving an explicit failure of continuous dependence.

Unbounded backward evolution. Let $S(T)$ be the time-T solution map for

$$u_t = u_{xx} \quad \text{on } (0,\pi), \qquad u(0,t) = u(\pi,t) = 0.$$

For $g_n(x) = e^{-n^2 T}\sin(nx)$,

$$\|g_n\|_{L^2} \longrightarrow 0, \qquad S(T)^{-1} g_n = \sin(nx),$$

whose L^2 norm is constant. Hence $S(T)^{-1}$ is unbounded on its range.

Tentative structure (50 minutes)

Part 1. Hadamard well-posedness (0–8 min). Separate existence, uniqueness, and continuous dependence; predict which property backward diffusion should lose.

Part 2. Forward spectral solution (8–22 min). Expand in the sine basis and derive the damping factor $e^{-n^2 t}$ for the nth mode.

Part 3. Instability and range (22–38 min). Use g_n to disprove continuity and characterize terminal data that actually lie in the range of $S(T)$.

Part 4. Regularization (38–50 min). Explain spectral cutoff and Tikhonov regularization, and show how an a priori smoothness bound yields conditional stability.

Fourier amplification

For $u_0 = \sum_{n\ge 1} a_n \sin(nx)$,

$$S(T)u_0 = \sum_{n\ge 1} e^{-n^2 T} a_n \sin(nx).$$

Thus terminal coefficients b_n admit an L^2 predecessor precisely when

$$\sum_{n\ge 1} e^{2n^2 T} |b_n|^2 < \infty.$$

High-frequency measurement errors are amplified exponentially, so no uniform stability estimate can hold on all of L^2.

Worth mentioning

- On the range of $S(T)$ the predecessor is unique; the failure is stability and, for generic terminal data, existence.
- This is distinct from Tychonoff nonuniqueness in Topic 45.
- Regularization deliberately suppresses modes that the inverse would amplify most.

Further reading. Begin with heat evolution in [17]. [30] supplies optional background on ill-posedness and regularization; a general inverse-problems course is not required.

Topic 45 Tychonoff's nonuniqueness for the heat equation

Level: M+ **Area:** Partial differential equations
Aim: Construct a nonzero smooth solution with zero initial data and show why growth at spatial infinity is part of uniqueness.

Zero initial data may seem to rule out any later heat evolution. On the whole line, however, that conclusion also needs control at spatial infinity: Tychonoff's construction produces a nonzero smooth solution when such a restriction is absent.

Tychonoff counterexample. There exists $u \in C^\infty(\mathbb{R} \times [0,\infty))$, not identically zero, such that

$$u_t = u_{xx} \quad \text{on } \mathbb{R} \times (0,\infty), \qquad u(x,0) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Such a solution necessarily violates the standard bounded or Gaussian-growth uniqueness classes.

Tentative structure (50 minutes)

Part 1. Where the usual proof hides growth (0–10 min). Review the maximum-principle or heat-kernel uniqueness argument and identify the uncontrolled boundary at infinity.

Part 2. Choose flat time data (10–20 min). Let $g(t) = e^{-1/t^2}$ for $t > 0$ and $g(t) = 0$ for $t \le 0$; all derivatives vanish at zero.

Part 3. Build the solution (20–39 min). Define the power series below, prove local uniform convergence of it and all required derivatives using estimates on $g^{(k)}$, and differentiate term by term.

Part 4. Interpretation (39–50 min). Verify zero initial data and nontriviality; explain why the construction grows too rapidly as $|x| \to \infty$ for uniqueness theorems to apply.

Core series and index shift

Set

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Once termwise differentiation is justified,

$$u_{xx} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2} = \sum_{j=0}^{\infty} \frac{g^{(j+1)}(t)}{(2j)!} x^{2j} = u_t.$$

Moreover $u(x,0) = 0$, while $u(0,t) = g(t) > 0$ for $t > 0$.

Worth mentioning

- The convergence estimates are the technical heart and should not be omitted.
- There is no contradiction with uniqueness in bounded, tempered, or suitably Gaussian-growth classes.
- The example shows that behavior at infinity can determine well-posedness even for a local PDE.

Further reading. Use [17] for heat-equation background, not as the sole source for this construction. Consult Tychonoff's French paper [47] and selected material from [20] with the instructor; both require advanced preparation.

Topic 46 Zero boundary trace almost everywhere does not determine a harmonic function

Level: M **Area:** Harmonic analysis; elliptic PDE
Aim: Use the Poisson kernel to separate almost-everywhere radial boundary values from admissible boundary data in a uniqueness class.

Almost-everywhere boundary values may appear to determine a harmonic function just as continuous boundary data do. A Poisson kernel concentrated at one boundary point shows the missing restriction: it has zero radial trace elsewhere but remains positive throughout the disk.

Poisson-kernel counterexample. In the unit disk, define

$$u(re^{i\theta}) = P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Then u is positive and harmonic, but

$$\lim_{r\uparrow 1} u(re^{i\theta}) = 0 \quad \text{for every } \theta \neq 0.$$

Thus the radial boundary trace is zero almost everywhere although $u \not\equiv 0$.

Tentative structure (50 minutes)

Part 1. Naive uniqueness claim (0–8 min). Recall uniqueness for continuous Dirichlet data and ask whether almost-everywhere boundary agreement is enough.

Part 2. Poisson kernel (8–22 min). Derive or verify harmonicity, positivity, and the normalization

$$(2\pi)^{-1}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1.$$

Part 3. Boundary behavior (22–35 min). Compute the limit for $\theta \neq 0$ and the blow-up $P_r(0) = (1+r)/(1-r)$ at the exceptional point.

Part 4. Correct uniqueness class (35–50 min). State Poisson representation for bounded harmonic functions and for harmonic h^p, $p > 1$; interpret the example as the Poisson integral of a singular boundary measure.

Key norm distinction

The kernel has constant L^1 mass but concentrates at one point:

$$\|P_r\|_{L^1(\mathbb{T})} = 2\pi, \qquad \|P_r\|_{L^p(\mathbb{T})} \longrightarrow \infty \quad (p > 1).$$

This is why an h^1-type bound still permits singular boundary mass, whereas h^p control for $p > 1$ forces an L^p boundary function and restores uniqueness from an a.e. trace.

Worth mentioning

- The exceptional boundary point has measure zero but carries all of the limiting mass.
- The maximum principle requires control up to the boundary; it cannot be applied to this unbounded example as if the trace were continuous.
- This is the harmonic analogue of concentration phenomena elsewhere in the seminar.

Further reading. Begin with Poisson integrals and harmonic Hardy spaces in [4]. [16] concerns analytic Hardy spaces and is optional related background, not the main source for the harmonic counterexample.

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