Renewal Processes

Master Thesis

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Abstract

This thesis provides a rigorous mathematical treatment of renewal processes and their applications in probability theory and actuarial science. Beginning with the foundational definitions, we establish core limit theorems governing the asymptotic behavior of renewal processes, including the Law of Large Numbers and the Central Limit Theorem for renewal counting processes. The Poisson process is examined as the canonical example, with particular attention to its memoryless structure and the characterization of its independent and stationary increments.

We then develop Blackwell's Renewal Theorem in both the lattice and non-lattice settings. The lattice case is approached via an ergodic argument for a forward recurrence time Markov chain, while the non-lattice case is established through a coupling technique involving a stationary delayed renewal process. Building on these results, we introduce the renewal equation as a unifying analytic framework and prove Smith's Key Renewal Theorem, which characterizes the long-run behavior of solutions to such equations.

In the final part of the thesis, these tools are applied to classical ruin theory in the Cramér–Lundberg model. Under the small risks assumption, we derive Lundberg's inequality and obtain an exact asymptotic formula for the ruin probability via an exponential change of measure. Under the large claims assumption, we analyze subexponential claim distributions and establish that the ruin probability decays polynomially rather than exponentially, with the decay rate governed solely by the tail index of the claim distribution.

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1 Introduction

Renewal theory is a branch of probability theory concerned with a very natural question: Suppose we observe events that occur at random times, with independent and identically distributed waiting times between successive events. How many events will have occurred by time t, and how does this counting process behave in the long run? Although the question is elementary, its systematic study began only in the early twentieth century, motivated by problems in actuarial science and industrial replacement. The Swedish actuary Filip Lundberg was among the first to model insurance claims as a stream of random arrivals, and Harald Cramér later placed these ideas on rigorous probabilistic footing. The work of Feller, Blackwell, and Smith then transformed renewal theory into a mature branch of probability, equipped with powerful limit theorems and a distinctive analytic framework built around the renewal equation. The resulting theory has found applications across a wide range of fields, including:

- Queuing Theory: Customer arrivals at a service center can often be modeled as a renewal process.
- Reliability Theory: The lifetimes of a machine followed by replacements after failure naturally lead to renewal structures.
- Insurance and risk theory: Claims arriving over time provide another important example

In what follows, we first develop the basic definitions and then introduce the limit theorems that allow us to answer the questions above, before turning to their application in ruin theory.

2 Renewal Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Throughout this thesis, we consider a sequence of independent and identically distributed random variables $T_1, T_2, \dots$ taking values in $\mathbb{R}_+$, with the condition that $P(T_1 = 0) < 1$. We denote $\mu := \mathbb{E}[T_1] \in (0, \infty]$ and $F(t) := P[T_1 \leq t]$; that is, μ is the mean of T_1 (possibly infinite) and F its distribution function. For $i \geq 1$, the random variable T_i is called the i-th inter-arrival time, and we define the i-th arrival time (or i-th renewal time) as $S_i = T_1 + \cdots + T_i$. The continuous-time stochastic process $(N_t)_{t\geq 0}$ defined by

$$\forall t \geq 0 \quad N_t = \sum_{k=1}^{\infty} \mathbf{1}_{\{S_k \leq t\}}$$

is called the renewal process with arrival distribution F.

Example 2.1. *Consider a renewal process where each inter-arrival time is deterministic with length 1. Then the distribution of T_1 has the dirac distribution, i.e. $T_1 \sim \delta_1$. Hence, the renewal times occur exactly at time 1, 2, 3,... and the process N_t is given by $N_t = \lfloor t \rfloor$*

Example 2.2. *[5] Consider a renewal process where the inter-arrival times are i.i.d. with the distribution*

$$T_1 \overset{(d)}{=} \begin{cases} \alpha & \text{with probability } \beta, \\ 0 & \text{with probability } 1-\beta, \end{cases}$$

for some fixed $\alpha > 0$ and $0 < \beta \leq 1$. The sequence (T_n) takes values in $\{0, \alpha\}$. Since $\mathbb{P}(T_n = \alpha) = \beta > 0$, the Borel–Cantelli lemma implies that the value α appears infinitely

often, almost surely. We can decompose the sequence into blocks separated by the occurrences of α:

- *Let Y_0 be the number of initial 0's before the first α.*
- *For $i \geq 1$, let Y_i be the number of 0's between the i-th and $(i+1)$-th α.*

By the independence of the T_n, the random variables $Y_0, Y_1, Y_2, \dots$ are i.i.d. The probability that a block has exactly k zeros followed by an α is $(1-\beta)^k\beta$, so $1+Y_i$ follows a geometric distribution with parameter β, supported on $\{1, 2, \dots\}$.

The renewal count N_t is the total number of inter-arrival periods completed by time t. This is equivalent to the number of terms in the sequence (T_n) before we have witnessed $\lfloor t/\alpha \rfloor$ occurrences of α. From the decomposition, this is given by:

$$N_t = Y_0 + \sum_{i=1}^{\lfloor t/\alpha \rfloor} (1 + Y_i).$$

Letting $X_i = 1 + Y_i$ for all $i \geq 0$, where the X_i are i.i.d. Geom(β) *random variables, we obtain the explicit representation:*

$$N_t \overset{(d)}{=} (X_0 - 1) + \sum_{i=1}^{\lfloor t/\alpha \rfloor} X_i.$$

This result shows that for this particular renewal process, the number of renewals up to time t is distributed as a randomly shifted sum of i.i.d. geometric random variables, where the number of terms in the sum is the deterministic integer $\lfloor t/\alpha \rfloor$.

For the remainder of this section, we use our initial understanding of renewal processes to establish some of their basic properties. These properties will later serve as essential tools for introducing and proving various limit theorems.

Proposition 2.3 (Basic properties)**.** *[5] The renewal process N satisfies the following properties. Almost surely,*

(i) $t \mapsto N_t$ *is non-decreasing, right-continuous, with values in* $\mathbb{N}$,

(ii) $\lim_{t\to\infty} N_t = \infty$.

Proof. (i) By assumption $0 < \mathbb{P}(T_1 > 0)$. And we have that

$$\mathbb{P}[T_1 > 0] \leq \mathbb{P}\left[\bigcup_{\alpha\in\mathbb{Q}_+} \{T_1 \geq \alpha\}\right] \leq \sum_{\alpha\in\mathbb{Q}_+} \mathbb{P}[T_1 \geq \alpha]$$

We conclude that there exists $\alpha \in \mathbb{Q}_+$ such that $\mathbb{P}[T_1 \geq \alpha] > 0$. Taking the sum $\sum_{i>0} \mathbb{P}[T_i \geq \alpha] = \infty$. Therefore, by the Borel–Cantelli lemma, $\mathbb{P}[A] = 1$, where $A = \{\omega : T_i(\omega) \geq \alpha \text{ for infinitely many } i\}$. For every $\omega \in A$, $\lim_{n\to\infty} S_n(\omega) = \infty$, and therefore

$$t \mapsto N_t(\omega) = \sum_{k\geq 1} \mathbf{1}_{\{S_k(\omega)\leq t\}}$$

is a non-decreasing function with values in $\mathbb{N}$.

(ii) All the interarrival times $T_1, T_2, \dots$ are finite almost surely. Therefore, all the renewal

times $S_1, S_2, \ldots$ are finite almost surely. When this occurs, we have

$$\lim_{t\to\infty} N_t = \lim_{t\to\infty} \sum_{k\geq 1} \mathbf{1}_{\{S_k \leq t\}} = +\infty.$$

□

The Law of Large Numbers provides a first-order approximation for the renewal process. It confirms that the number of renewals, N_t, grows linearly with time at an average rate of $1/\mu$.

Theorem 2.4 (Law of Large Numbers)**.** *[5] Recall that $\mu = \mathbb{E}[T_1]$. We have*

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \quad a.s.$$

Proof. By the definition of the renewal process, for all $t > 0$, the inequality $S_{N_t} \leq t < S_{N_t+1}$ holds. Dividing by N_t, for large enough t such that $N_t > 0$, we obtain

$$\frac{S_{N_t}}{N_t} \leq \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t}. \tag{1}$$

From Proposition 2.3, we know that $N_t \to \infty$ almost surely as $t \to \infty$. By the Strong Law of Large Numbers for non negative random variables, since the inter-arrival times T_i are i.i.d. with mean μ, we have $S_n/n \to \mu$ a.s. as $n \to \infty$. Since $N_t \to \infty$, this convergence holds along the sequence N_t. We distinguish two cases for μ.

Case 1: $\mu < \infty$. Taking the limit as $t \to \infty$, the lower bound in (1) converges directly:

$$\lim_{t\to\infty} \frac{S_{N_t}}{N_t} = \mu \quad \text{a.s.}$$

For the upper bound in (1), we adjust the indices to apply the SLLN: $\frac{S_{N_t+1}}{N_t} = \frac{S_{N_t+1}}{N_t+1} \cdot \frac{N_t+1}{N_t}$. As $t \to \infty$, $S_{N_t+1}/(N_t+1) \to \mu$ and $(N_t+1)/N_t \to 1$. Thus, the product converges to $\mu \cdot 1 = \mu$. We conclude that $\lim_{t\to\infty} \frac{t}{N_t} = \mu$ a.s., which implies:

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \quad \text{a.s.}$$

Case 2: If $\mu = \infty$ the SLLN implies that $\frac{S_{N_t}}{N_t} \to \infty$ a.s. From the left inequality of (1), we have $\frac{t}{N_t} \geq \frac{S_{N_t}}{N_t} \to \infty$ almost sure. Since $\frac{t}{N_t}$ diverges to infinity, its reciprocal must converge to zero:

$$\lim_{t\to\infty} \frac{N_t}{t} = 0 \quad \text{a.s.}$$

□

A more refined question is to ask about the fluctuations of N_t around its long-run average. We want to understand the "error" term in the approximation $N_t \approx t/\mu$. The Central Limit Theorem provides this second-order approximation. It describes the asymptotic distribution of the deviation of N_t from its mean, showing that when properly scaled, it converges to a standard normal distribution. To establish this, we require a stronger condition than just a finite mean. We must now assume that the inter-arrival times also have a finite variance $\sigma < \infty$.

Theorem 2.5 (Central Limit Theorem)**.** *[5] Assume that* $\mathbb{E}[T_1^2] < \infty$*. Write* $\mu = \mathbb{E}[T_1]$*,* $\sigma^2 = \mathrm{Var}(T_1)$*. Then, assuming* $\sigma > 0$*, we have*

$$\frac{N_t - \frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \xrightarrow[t\to\infty]{law} \mathcal{N}(0,1). \tag{2}$$

Proof. Let $S_n := \sum_{i=1}^n T_i$, then using the central limit theorem we have

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < x\right) = \Phi(x)$$

uniformly. Where Φ denotes the distribution function of the standard normal distribution. For simplicity of notation, we define (Z_t) as the process in the LHS of (2). Now, for given $t > 0$ and fix $x \in \mathbb{R}$, since N_t is integer–valued, we have

$$\mathbb{P}[Z_t < x] = \mathbb{P}\left[N_t < \left\lfloor x\big(\sigma\sqrt{t/\mu^3}\big) + t/\mu \right\rfloor\right].$$

Setting $h(t) := \lfloor x(\sigma\sqrt{t/\mu^3}) + t/\mu \rfloor$, from the basic equivalence $\{N_t < h(t)\} = \{S_{h(t)} > t\}$, we obtain that

$$\mathbb{P}[Z_t < x] = \mathbb{P}[S_{h(t)} > t] = \mathbb{P}\left(\frac{S_{h(t)} - \mu h(t)}{\sigma\sqrt{h(t)}} > \frac{t - \mu h(t)}{\sigma\sqrt{h(t)}}\right).$$

To conclude,it suffices to show $h(t) \to \infty$ and

$$z(t) := \frac{t - \mu h(t)}{\sigma\sqrt{h(t)}} \to -x \quad \text{as } t \to \infty,$$

since in that case the central limit theorem will imply

$$\mathbb{P}\left(\frac{S_{h(t)} - \mu h(t)}{\sigma\sqrt{h(t)}} > z(t)\right) \to 1 - \Phi(-x) = \Phi(x),$$

which means that $\mathbb{P}[Z_t < x] \to \Phi(x)$ and therefore Z_t converges to the standard normal distribution in law as $t \to \infty$. Since $\mu > 0$, $t/\mu \to \infty$ as $t \to \infty$ and for the second term, $x\sigma\sqrt{\frac{t}{\mu^3}}$ is of the order $\sqrt{t}$. Hence we conclude that $h(t) \to \infty$. To show that $\lim_{t\to\infty} z(t) = -x$, we first note that by definition $h(t) = x(\sigma\sqrt{t/\mu^3}) + t/\mu + \epsilon(t)$, where $|\epsilon(t)| < 1$, and hence

$$z(t) = \frac{t - \mu\big[x(\sigma\sqrt{t/\mu^3}) + t/\mu + \epsilon(t)\big]}{\sigma\sqrt{h(t)}} \sim \frac{-\mu x(\sigma\sqrt{t/\mu^3})}{\sigma\sqrt{t/\mu}}.$$

Thus $z(t) \sim -x$ as $t \to \infty$. □

2.1 Poisson Process

The Poisson process is a fundamental stochastic process that serves a dual purpose in probability theory. First, it is a powerful model in its own right for counting random, inde-

pendent events in time, such as radioactive decays, customer arrivals, or insurance claims. Second, and crucially for our purposes, it is the prototypical example of a renewal process. Its structure is so elegant and well-behaved that it provides explicit formulas and exact results where general renewal processes only offer asymptotic limits. Understanding the Poisson process in depth, is therefore essential for developing intuition about the broader class of renewal processes. We start by defining the exponential distribution.

Definition 2.6. *[5] Let $\lambda > 0$, a real random variable T is exponential with parameter λ (we write $T \sim \text{Exp}(\lambda)$) if it has density*

$$f(t) = \lambda e^{-\lambda t} \mathbf{1}_{\{t \geq 0\}}.$$

The exponential distribution exhibits several important analytical properties:

- Mean: $\mathbb{E}[T] = \lambda^{-1}$
- Var(T) = λ^{-2}
- Laplace Transform: For all $t \geq 0$, $\mathcal{L}_T(t) = \mathbb{E}[e^{-tT}] = \frac{\lambda}{\lambda + t}$

The entire structure of the Poisson process is based on the exponential distribution. Its defining characteristic is the memoryless property, which makes it the only continuous distribution suitable for modeling the time between events in a process that "forgets" its history.

Proposition 2.7. *(Memoryless Property) [4] Let $T \sim Exp(\lambda)$. Then for all $x, y \geq 0$, we have*

$$\mathbb{P}(T - y \geq x | T \geq y) = \mathbb{P}(T \geq x)$$

The memoryless property expresses that the conditional distribution of the remaining waiting time, given that no event has occurred by time y, is identical to the original unconditional distribution.

Proof. For $x, y \geq 0$, we compute:

$$\mathbb{P}(T \geq x + y) = \int_{x+y}^{\infty} \lambda e^{-\lambda(x+y)} = e^{-\lambda x} e^{-\lambda y}.$$

Since $\mathbb{P}(T \geq y) = e^{-\lambda y}$, it follows that

$$\mathbb{P}(T - y \geq x | T \geq y) = \frac{\mathbb{P}(T \geq x + y)}{\mathbb{P}(T \geq y)} = \frac{e^{-\lambda x} e^{-\lambda y}}{e^{-\lambda y}} = e^{-\lambda x} = \mathbb{P}(T \geq x).$$

□

The memoryless property is not merely a feature, but actually characterizes the exponential distribution uniquely.

Proposition 2.8. *(Characterization by the Memoryless Property) [4] If T is a non-negative real random variable such that for all $x, y \geq 0$,*

$$\mathbb{P}(T - y \geq x | T \geq y) = \mathbb{P}(T \geq x)$$

then T follows an exponential distribution.

Proof. Let $a = \mathbb{P}(T > 1)$. By iterating the memoryless property, we show that for all $n \in \mathbb{N}$, $\mathbb{P}(T > n) = a^n$. Furthermore, for $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$, the memoryless property

implies

$$\mathbb{P}\left(T > \frac{n}{m}\right) = \mathbb{P}(T > n)^{1/m} = a^{n/m}.$$

By the density of rational numbers in $\mathbb{R}_+$ and the right-continuity of the cumulative distribution function $x \mapsto \mathbb{P}(T \leq x)$, we obtain for all $x \geq 0$, $\mathbb{P}(T > x) = a^x$. Setting $\lambda = -\log(a)$, we recover the survival function of the exponential distribution, $\mathbb{P}(T > x) = e^{-\lambda x}$. Hence, $T \sim Exp(\lambda)$ □

The exponential distribution leads to several other remarkable properties that make exponential random variables particularly tractable. One of the most important consequences, which plays a crucial role in the analysis of Poisson processes, concerns the behavior of minima of independent exponential random variables.

Proposition 2.9. *[4] Let X and Y be two independent and exponential distributed random variables with respective parameters λ and μ, then*

1. *The minimum of X and Y, $Z = \min\{X, Y\}$, has an exponential distribution with rate $\lambda + \mu$;*
$$P(Z > x) = e^{-(\lambda+\mu)x}, \quad x \geq 0.$$
2. $$P(X < Y) = \frac{\lambda}{\lambda + \mu}.$$
3. *The r.v. Z is independent of the two events $\{Z = X\}$ and $\{Z = Y\}$*

Proof. 1. Observing that $Z > x$ if and only if both $X > x$ and $Y > x$, and using independence, we conclude that $\mathbb{P}(Z \geq x) = \mathbb{P}(X \geq x, Y \geq x) = \mathbb{P}(X \geq x)\mathbb{P}(Y \geq x) = e^{-\lambda x}e^{-\mu x} = e^{-(\lambda+\mu)x}$

2. To calculate $\mathbb{P}(X < Y)$, we use the density function of X, given by $f_X(x) = \lambda e^{-\lambda x}$.

$$\begin{aligned}
\mathbb{P}(X < Y) &= \int_0^\infty \mathbb{P}(X < Y \mid X = x) f_X(x) dx \\
&= \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x} dx \\
&= \lambda \int_0^\infty e^{-(\mu+\lambda)x} dx \\
&= \frac{\lambda}{\lambda + \mu}.
\end{aligned}$$

3. To prove independence, it suffices to show that for all $z \geq 0$, $\mathbb{P}(Z \geq z,\ X = Z) = \mathbb{P}(Z \geq z)\mathbb{P}(X = Z)$. Note that $\{X = Z\} = \{X \leq Y\}$ and $\{Z \geq z\} = \{X \geq$

$z\} \cap \{Y \geq z\}$. We compute:

$$\begin{aligned}
\mathbb{P}(Z \geq z,\ X = Z) &= \mathbb{P}(X \geq z,\ Y \geq z,\ X \leq Y) \\
&= \mathbb{P}(X \geq z,\ Y \geq X) \quad \text{(since } X \geq z \text{ and } Y \geq X \text{ imply } Y \geq z) \\
&= \int_z^\infty \mathbb{P}(Y \geq x) f_X(x) dx \quad \text{(by independence)} \\
&= \lambda \int_z^\infty e^{-\mu x} e^{-\lambda x} dx \\
&= \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)z}.
\end{aligned}$$

From standard properties of exponential random variables, we have that $\mathbb{P}(Z \geq z) = e^{-(\lambda+\mu)z}$ and $\mathbb{P}(X = Z) = \mathbb{P}(X \leq Y) = \lambda/(\lambda + \mu)$. Thus, $\mathbb{P}(Z \geq z,\ X = Z) = \mathbb{P}(Z \geq z)\mathbb{P}(X = Z)$ for all $z \geq 0$, proving independence.

□

Definition 2.10. *Let $\lambda \in [0, \infty]$. A random variable N is a Poisson random variable with parameter λ if it takes values in $\mathbb{N} \cup \{\infty\}$ and*

$$\forall k \in \mathbb{N}, \quad \mathbb{P}(N = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

We write $N \sim \text{Poi}(\lambda)$.

A renewal process N_t with exponentially distributed inter-arrival times of parameter λ is called a Poisson process. In the following proposition, we establish that the Poisson process indeed has the Poisson distribution, as its name suggests.

Proposition 2.11. *[5] Fix $\lambda > 0$ and assume that $T_1 \sim \text{Exp}(\lambda)$ (i.e. $F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$). In this case, for every fixed $t \geq 0$, we have $N_t \sim \text{Pois}(\lambda t)$.*

Proof. We prove by induction on n, that

$$\forall t \geq 0 \quad \mathbb{P}[N_t = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \tag{3}$$

For $n = 0$, we have $N_t = 0$ if there is no renewal before time t, therefore

$$\mathbb{P}[N_t = 0] = \mathbb{P}[T_1 \geq t] = e^{-\lambda t} = \frac{(\lambda t)^0}{0!} e^{-\lambda t}.$$

Thus the statement for $n = 0$ is true and we use (3) as our Induction Hypothesis.
Let $n \geq 0$ and assume that (3) holds. Fix $t \geq 0$. There are $n + 1$ renewals before time t iff $T_1 < t$ and there are exactly n renewal times between T_1 and t. By conditioning on T_1,

and using independence, we obtain

$$\begin{aligned}\mathbb{P}[N_t = n+1] &= \mathbb{P}[T_1 < t,\, T_1 + \cdots + T_{n+1} \le t,\, T_1 + \cdots + T_{n+2} > t] \\ &= \int_0^\infty \mathbb{P}[s < t,\, s + T_2 + \cdots + T_{n+1} \le t,\, s + T_2 + \cdots + T_{n+2} > t]\, \lambda e^{-\lambda s} ds \\ &= \int_0^t \mathbb{P}[T_2 + \cdots + T_{n+1} \le t - s,\, T_2 + \cdots + T_{n+2} > t - s]\, \lambda e^{-\lambda s} ds \\ &= \int_0^t \mathbb{P}[N_{t-s} = n] \lambda e^{-\lambda s} ds.\end{aligned}$$

By the induction hypothesis, we obtain

$$\mathbb{P}[N_t = n+1] = \int_0^t \mathbb{P}[N_{t-s} = n]\, \lambda e^{-\lambda s}\, ds = \int_0^t \frac{(\lambda(t-s))^n}{n!} e^{-\lambda(t-s)} \lambda e^{-\lambda s}\, ds$$

$$= \left[-\frac{(\lambda(t-s))^{n+1}}{(n+1)!} \right]_0^t e^{-\lambda t} = \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t}.$$

□

The memoryless property of the exponential distribution is the fundamental reason behind the robust structural properties of the Poisson process. Because the inter-arrival times are exponentially distributed, the waiting time until the next event is statistically independent of how much time has elapsed since the last event. This implies that at any time t, the process effectively resets its probabilistic behavior regarding new arrivals. Consequently, the number of events occurring in any future interval $(t, t+s]$ is independent of the history up to time t and depends solely on the duration s. This property is formalized by saying that the Poisson process has independent and stationary increments. Before stating the main theorem, we will first precisely define what these terms mean.

Definition 2.12. *A stochastic process $(X_t)_{t\ge 0}$ with values in $\mathbb{R}$ is said to have independent and stationary increments if*

$$\forall k \ge 1,\ \forall 0 = t_0 < \cdots < t_k \quad X_{t_1} - X_{t_0}, \ldots, X_{t_k} - X_{t_{k-1}} \text{ are independent,}$$

and

$$\forall s < t,\ \forall h \ge 0 \quad X_t - X_s \overset{law}{=} X_{t+h} - X_{s+h}.$$

Theorem 2.13. *[4] The Poisson process $(N_t)_{t\ge 0}$ has independent and stationary increments. Furthermore, for all $t, s \ge 0$ we have $N_{t+s} - N_t \sim \mathrm{Pois}(\lambda s)$.*

The proof relies fundamentally on the following lemma, which provides the conditional joint density of the arrival times $S_1, \ldots, S_k$, given that the Poisson process $N_t = k$. This density allows us to show that the arrival times behave like ordered uniform random variables, which is key to proving the independence of increments.

Lemma 2.14. *[4] Let $(N_t)_{t\ge 0}$ be a Poisson process, and consider its arrival time $S_i = T_1 + \cdots + T_i$ for every i. Then the random vector $(S_1, \ldots, S_k)$ given that $\{N_t = k\}$ has density $f_{S_1,\ldots,S_k}(x_1, \ldots, x_k \mid N_t = k) = \frac{k!}{t^k} \mathbf{1}_{0<x_1<\cdots<x_k<t}$.*

Proof. Define the linear map $h(x_1, \ldots, x_k) = (x_1, x_2 - x_1, \ldots, x_k - x_{k-1})$. Then we have $h^{-1}(T_1, T_2, \ldots, T_k) = (S_1, \ldots, S_k)$. The Jacobian matrix J of the linear transformation h

is triangular with one's on the diagonal. Therefore $|det(J)| = 1$. Hence we have

$$\begin{aligned} f_{S_1,\dots,S_k}(x_1,\dots,x_k) &= f_{T_1,\dots,T_k}(h(x_1,\dots,x_k)) \cdot |det(J)| \\ &= f_{T_1,\dots,T_k}(x_1, x_2 - x_1, \dots, x_k - x_{k-1}) \cdot 1 \\ &= \lambda e^{-\lambda x_1} \mathbf{1}_{x_1>0} \prod_{i=2}^{k} \lambda e^{-\lambda x_i - x_{i-1}} \mathbf{1}_{x_i - x_{i-1} > 0} \\ &= \lambda^k e^{-\lambda(x_1 + x_2 - x_1 + \cdots + x_k - x_{k-1})} \prod_{i=1}^{k} \mathbf{1}_{x_1>0, x_2-x_1>0, \dots, x_k - x_{k-1} > 0} \\ &= \lambda^k e^{-\lambda x_k} \mathbf{1}_{0<x_1<\cdots<x_k}. \end{aligned}$$

Then the conditional density of the random vector $(S_1, \dots, S_k)$ given that $\{N_t = k\}$ is given by

$$\begin{aligned} f_{S_1,\dots,S_n}(x_1,\dots,x_k \mid N_t = k) &= \lim_{\Delta\to 0} \frac{\mathbb{P}(x_1 < S_1 \le x_1 + \Delta, \dots, x_k < S_k \le x_k + \Delta \mid N_t = k)}{\Delta^k} \\ &= \lim_{\Delta\to 0} \frac{\mathbb{P}(x_1 < S_1 \le x_1 + \Delta, \dots, x_k < S_k \le x_k + \Delta, N_t = k)}{\Delta^k \mathbb{P}(N_t = k)} \\ &= \lim_{\Delta\to 0} \frac{\mathbb{P}(x_1 < S_1 \le x_1 + \Delta, \dots, x_k < S_k \le x_k + \Delta, S_{k+1} > t)}{\Delta^k \mathbb{P}(N_t = k)}. \end{aligned}$$

Since the joint density function of the random vector $(S_1, \dots, S_{k+1})$ does not depend on $x_1, \dots . x_k$, we can use it to derive the numerator. For the denominator we apply the result of proposition (2.11). This leads to

$$\frac{\mathbb{P}(S_1 \le x_1, \dots, S_k \le x_k, t < S_{k+1} \le x_{k+1})}{\mathbb{P}(N_t = k)} = \frac{\int_t^\infty \lambda^{k+1} e^{-\lambda x_{k+1}} dx_{k+1}}{\frac{(\lambda t)^k}{k!} e^{-\lambda t}} = \frac{k!}{t^k}$$

We conclude that $f_{S_1,\dots,S_n(x_1,\dots,x_k|N_t=k)} = \frac{k!}{t^k} \mathbf{1}_{0<x_1<\cdots<x_k<t}$. □

Let $U_1, \dots, U_k$ be uniformly distributed independent random variables on the interval $[0, t]$. If we arrange these variables in increasing order, the resulting sequence is denoted by $U_{(1)}, \dots, U_{(k)}$, where $U_{(1)} \le U_{(2)} \le \cdots \le U_{(k)}$. The joint probability density function of the random vector $(U_{(1)}, \dots, U_{(k)})$ is given by $f_{U_{(1)},\dots,U_{(k)}}(x_1, \dots, x_k) = \frac{k!}{t^k} \mathbf{1}_{0<x_1<\cdots<x_k<t}$. Crucially, observe that this density is identical to the conditional density derived in Lemma (2.14).

Proof. (Theorem 2.13) Let $0 = t_0 < t_1 \cdots < t_n$. We define the increments of the process as random variables $X_1, \dots, X_n$ where $X_i = N_{t_i} - N_{t_{i-1}}$ for $i = 1, \dots, n$. Let $x_1, \dots, x_n$ be non-negative integers representing the observed renewals in each interval. We define the total number of events as $k = \sum_{i=1}^n x_i$. To establish the theorem, we must compute the joint probability mass function $P := \mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$. Using the definition of conditional probability, we can condition on the total number of events in the interval $[0, t_n]$. Note that the event $\{X_1 = x_1, \dots, X_n = x_n\}$ implies that the total number of renewals is $N_{t_n} = \sum_{i=1}^n X_i = k$. Thus, we can write $P = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n \mid N_{t_n} = k) \cdot \mathbb{P}(N_{t_n} = k)$.

Given $N_{t_n} = k$, we know from the previous lemma that the k arrival times are distributed as k independent uniform random variables on $[0, t_n]$. Consequently, the problem of counting how many of these k points fall into each disjoint sub-interval $(t_{i-1}, t_i]$ is equivalent to placing k balls into n bins, where the probability of a ball falling into bin i is proportional to the length of the interval. Thus, the conditional distribution is multinomial with parameters k and probabilities $p_1, \ldots, p_n$, where:

$$p_i = \frac{t_i - t_{i-1}}{t_n}.$$

The conditional probability is therefore

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n \mid N_{t_n} = k) = \frac{k!}{x_1! x_2! \cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}.$$

We substitute the multinomial expression and the Poisson probability for $\mathbb{P}(N_{t_n} = k)$ into our original equation for P:

$$P = \left(\frac{k!}{x_1! \cdots x_n!} \prod_{i=1}^{n} \left(\frac{t_i - t_{i-1}}{t_n} \right)^{x_i} \right) \cdot \left(e^{-\lambda t_n} \frac{(\lambda t_n)^k}{k!} \right).$$

Observe that the $k!$ terms cancel out. Since $k = \sum x_i$, we can write $(\lambda t_n)^k = \prod_{i=1}^{n} (\lambda t_n)^{x_i}$. Further observe that $e^{-\lambda t_n} = e^{-\lambda \sum (t_i - t_{i-1})} = \prod_{i=1}^{n} e^{-\lambda(t_i - t_{i-1})}$. Substituting these back into the equation we get

$$P = \frac{1}{x_1! \ldots x_n!} \prod_{i=1}^{n} \left(\frac{t_i - t_{i-1}}{t_n} \right)^{x_i} \prod_{i=1}^{n} (\lambda t_n)^{x_i} \prod_{i=1}^{n} e^{-\lambda(t_i - t_{i-1})}.$$

Notice that the $t_n^{x_i}$ in the denominator of the multinomial probability cancels perfectly with the $t_n^{x_i}$ from the Poisson term. Substituting these back into the equation:

$$P = \frac{1}{x_1! \ldots x_n!} \prod_{i=1}^{n} \left(\frac{t_i - t_{i-1}}{t_n} \right) \prod_{i=1}^{n} (\lambda t_n)^{x_i} \prod_{i=1}^{n} e^{-\lambda(t_i - t_{i-1})}.$$

Notice that the $t_n^{x_i}$ in the denominator of the multinomial probability cancels perfectly with the $t_n^{x_i}$ from the Poisson term. This leaves:

$$P = \prod_{i=1}^{n} \left(\frac{(\lambda(t_i - t_{i-1}))^{x_i}}{x_i!} e^{-\lambda(t_i - t_{i-1})} \right).$$

The joint probability P has factored into a product of terms. Specificaly $P = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i)$, where $X_i \sim Pois(\lambda(t_i - t_{i-1}))$. Since the joint distribution is the product of the marginal distributions, the increments $X_1, \ldots, X_n$ are independent. Furthermore, the marginal distribution of each increment depends only on the length of the interval $t_i - t_{i-1}$, proving stationarity. □

2.2 The Elementary Renewal Theorem

Definition 2.15. *The renewal function is the function $m : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\forall t \geq 0 \quad m(t) = \mathbb{E}[N_t].$$

This function is fundamental because it allows us to use the tools of calculus and analysis to study a probabilistic process. Instead of working with the entire distribution of N_t, we first focus on its mean. To build intuition, let us look at the renewal function for the most canonical renewal process.

Example 2.16. *The renewal function of the Poisson process is given by $m(t) = \mathbb{E}[N_t] = \lambda t$.*

This example is instructive. For a Poisson process, the expected number of arrivals grows linearly with time. A natural question arises: is this linear growth a general property of all renewal processes, or is it specific to the memoryless property of the exponential distribution? The Elementary Renewal Theorem, which we will soon state, will answer this by showing that for a general renewal process, $\frac{m(t)}{t}$ converges to a constant, providing a "long-term average rate" of renewals. Before we can prove such a theorem, we need to establish some basic, foundational properties of the renewal function $m(t)$ itself.

Lemma 2.17. *[5] The renewal function m is non-decreasing, non-negative, and right continuous.*

Proof. Recall that N_t counts the number of renewals (arrivals) up to time t. By construction, $(N_t)_{t\geq 0}$ is a non-decreasing stochastic process, since for $s > 0$ we clearly have $N_{t+s} \geq N_t$ almost surely. Moreover, $N_t \geq 0$ almost surely for every $t \geq 0$. Taking expectations preserves both properties, so the renewal function $m(t) := \mathbb{E}[N_t]$ is also non-decreasing in t and non-negative. For the right-continuity, note that for fixed $t \geq 0$ and $s \downarrow 0$, $N_{t+s} - N_t \downarrow 0$ almost surely and the sequence is non-negative. We may apply the Monotone Convergence Theorem to obtain

$$\lim_{s\downarrow 0} \mathbb{E}[N_{t+s} - N_t] \;=\; \mathbb{E}\left[\lim_{s\downarrow 0}(N_{t+s} - N_t)\right] \;=\; 0.$$

Thus $m(t + s) \to m(t)$ as $s \downarrow 0$, which proves that m is right-continuous. □

Theorem 2.18 (Elementary Renewal Theorem)**.** *[5]*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

The elementary renewal theorem tells us that the expected number of renewals up to time t grows linearly with rate $1/\mu$. Intuitively, this means that in the long run, the renewal process produces on average one renewal every μ time units.

Proof of Elementary Renewal Theorem. By definition of N_t we have $S_{N_t} \leq t < S_{N_t+1}$ where we can write

$$S_{N_t+1} = \sum_{n=1}^{N_t+1} T_n.$$

Then using linearity of the expectation and independence of T_n we obtain

$$\mathbb{E}[S_{N_t+1}] = \mathbb{E}\left[\sum_{n=1}^{N_t+1} T_n\right] = \mathbb{E}\left[\sum_{n\geq 1} T_n \mathbf{1}_{\{S_{n-1}<t\}}\right] = \mathbb{E}\left[\sum_{n\geq 1} \mathbf{1}_{\{S_{n-1}<t\}}\right] \mathbb{E}[T_1]\,.$$

Since $\sum \mathbf{1}_{S_{n-1}<t} = N_t + 1$ we have

$$\mathbb{E}[S_{N_t+1}] = \mathbb{E}[S_{N_t} + 1]\,\mathbb{E}[T_1] = (m(t)+1)\mu.$$

By taking expectation on $S_{N_t} \leq t < S_{N_t+1}$ we have $\mu m(t) \leq t < \mu(m(t)+1)$. Then dividing by t we obtain

$$\frac{\mu m(t)}{t} \leq 1 < \frac{\mu(m(t)+1)}{t}$$

this leads to

$$\frac{1}{\mu} - \frac{1}{t} < \frac{m(t)}{t} \leq \frac{1}{\mu}.$$

As $t \to \infty$, the term $-1/t$ vanishes and we get the desired result. □

3 Blackwell's Renewal Theorem

After establishing the Elementary Renewal Theorem, we have identified the long–run average behavior of a renewal process. A next question to ask is, how many renewals occur in a finite time window once the process has run for a long time? To answer this, we first consider the lattice case of Blackwell's Renewal Theorem and then move on to the non-lattice version.

3.1 Lattice Case

Let us recall the notion of a lattice distribution.

Definition 3.1. *We say that F is lattice if there exists $a > 0$ and such that*

$$\mathbb{P}[T_1 \in a\mathbb{Z}] = 1 \tag{4}$$

In this case the span of F is defined as the largest $a > 0$ such that (4) holds.

The lattice property implies that renewals can only occur at time points that are multiples of the span a. Therefore, unlike the non-lattice case where we can look at any interval $(t, t+h]$, here we are interested in the behavior at these specific renewal points. The relevant question is: What is the long-run probability of a renewal occurring at the point ai for large i?

Theorem 3.2 (Blackwell's Renewal Theorem)**.** *[5] Assume that the law of T_1 is lattice with span a, then the sequence $(m(ai))_{i\in\mathbb{N}}$ satisfies*

$$\lim_{i\to\infty} \big(m(a\cdot i) - m(a\cdot(i-1))\big) = \frac{a}{\mu}.$$

The expression in the theorem, $m(a\cdot i) - m(a\cdot(i-1))$, represents the expected number of renewals in the interval $(a(i-1), ai]$. Since ai is the only point in this interval where a renewal can possibly occur, this difference is exactly the probability of a renewal at ai.

Proof. The proof proceeds in four steps. First, we define a discrete-time Markov chain that captures the state of the renewal process at integer time points. Second, we demonstrate that this chain possesses a unique stationary distribution and derive its form. Third, we relate the stationary distribution to the limiting probability of a renewal event. Finally, we use this relationship to prove the theorem.

We assume that the inter-arrival times T_k are lattice with span 1 and $\mathbb{P}(T_1 = 0) = 0$. Let $\mathcal{R} = \{S_0, S_1, \dots\}$ denote the set of renewal times. We define the process $(X_n)_{n\geq 0}$ as follows:

$$X_n := \inf\{t \in \mathcal{R} : t \geq n\} - n$$

The random variable X_n, called the forward recurrence time, represents the waiting time until the next event, inclusive of the present moment. Note that $X_n = 0$ if and only if a renewal occurs at time n. Let the state space be $S := \mathbb{N}_0$. To show that $(X_n)_{n\geq 0}$ is a Markov chain, we verify that for all $n \in \mathbb{N}$ and for all states $i_0, i_1, ..., i_{n-1}, i, j \in S$, $\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$.

Case 1: If $X_n = i \geq 1$, the next renewal occurs at time $n + i$. At the next time step $n + 1$, the next renewal is still at time $n + i$. Therefore, the remaining waiting time is $(n + i) - (n + 1) = i - 1$. This transition is deterministic and we have $\mathbb{P}(X_{n+1} = i - 1 \mid X_n = i) = 1 = p_{i,i-1}$.

Case 2: If $X_n = 0$, then a renewal occurs at time n (i.e. $n = S_k$ for some k). The next renewal will occur at $S_{k+1} = n + T_{k+1}$, where T_{k+1} is the next inter-arrival time. The forward recurrence time at $n + 1$ is therefore $X_{n+1} = S_{k+1} - (n + 1) = (n + T_{k+1}) - (n + 1) = T_{k+1} - 1$. Since the inter-arrival times T_k are i.i.d and independent of the history $X_0, \dots, X_{n-1}$, the distribution of X_{n+1} depends only on the distribution of T. Specifically: $\mathbb{P}(X_{n+1} = j \mid X_n = 0) = \mathbb{P}(T_{k+1} - 1 = j) = \mathbb{P}(T_1 = j + 1)$. Combining these cases, we formally state the transition probabilities $p_{i,j} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$:

$$p_{i,j} = \begin{cases} 1 & \text{if } i \geq 1 \text{ and } j = i - 1 \\ \mathbb{P}(T_1 = j + 1) & \text{if } i = 0 \text{ and } j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The chain $(X_n)_{n\geq 0}$ is irreducible since for any k such that $\mathbb{P}(T_1 = k + 1) > 0$, we can transition from $0 \to k$, and then deterministically from $k \to k - 1 \to \cdots \to 0$. Since the span of the distribution of T_1 is 1, the greatest common divisor of the return times to state 0 is 1, implying the chain is aperiodic. Recall that a stationary distribution π must satisfy $\pi = \pi P$. Using the transition probabilities we get $\pi_j = \pi_{j+1} p_{j+1,j} + \pi_0 p_{0,j} = \pi_{j+1} + \pi_0 \mathbb{P}(T_1 = j + 1)$. Rearranging this yields a recurrence relation for the difference between consecutive terms:

$$\pi_j - \pi_{j+1} = \pi_0 \mathbb{P}(T = j + 1)$$

To find a explicit form for π_j we sum up the differences from 0 to $j - 1$ on both sides:

$$\sum_{k=0}^{j-1} (\pi_k - \pi_{k+1}) = \sum_{k=0}^{j-1} \pi_0 \mathbb{P}(T = k + 1).$$

On the LHS the sum collapses to $\pi_0 - \pi_j$, because the intermediate terms cancel out. For the RHS we factor out π_0 and observe the sum

$$\pi_0 \sum_{k=0}^{j-1} \mathbb{P}(T = k+1) = \pi_0 \mathbb{P}(T \leq j).$$

This leads to $\pi_0 - \pi_j = \pi_0 \mathbb{P}(T \leq j)$. By first rearrange and then using the complement rule we get

$$\pi_j = \pi_0 - \pi_0 \mathbb{P}(T \leq j) = \pi_0(1 - \mathbb{P}(T \leq j)) = \pi_0 \mathbb{P}(T > j), \quad \text{for} \quad j \geq 1 \tag{5}$$

By using the tail-sum formula for expectation, $\mathbb{E}[T_1] = \sum_{j=0}^{\infty} \mathbb{P}(T_1 > j)$ we obtain

$$1 = \pi_0 + \sum_{j=1}^{\infty} \pi_j = \pi_0 \left(1 + \sum_{j=1}^{\infty} \mathbb{P}(T_1 > j) \right) = \pi_0 \sum_{j=0}^{\infty} \mathbb{P}(T_1 > j) = \pi_0 \mathbb{E}[T_1]$$

hence

$$\pi_0 = \frac{1}{\mathbb{E}[T_1]},$$

Using equation (5) the unique stationary distribution π is given by

$$\forall j \in S \quad \pi_j = \frac{\mathbb{P}(T_1 > j)}{\mathbb{E}[T_1]}.$$

Since the chain $(X_n)_{n \geq 0}$ is irreducible, aperiodic and admits a stationary distribution π, for every distribution μ on S

$$\lim_{n \to \infty} \mathbb{P}_\mu [X_n = 0] = \pi_0 = \frac{1}{\mu}.$$

By construction of the Markov chain, $\mathbb{P}(X_n = 0) = \mathbb{P}(\textit{Renewal at time } n)$ and hence we can now evaluate the limit of the expected number of renewals in an interval.

$$\begin{aligned} m(n+h) - m(n) = \mathbb{E}[N_{n+h} - N_n] &= \mathbb{E}\left[\sum_{j=1}^{h} \mathbf{1}_{\text{Renewal at } n+j} \right] \\ &= \sum_{j=1}^{h} \mathbb{E}[\mathbf{1}_{\text{Renewal at } n+j}] = \sum_{j=1}^{h} \mathbb{P}(\text{Renewal at } n+j) \\ &\xrightarrow[n \to \infty]{} \sum_{j=1}^{h} \frac{1}{\mu} = \frac{h}{\mu}. \end{aligned}$$

where we used that a renewal process with lattice distribution has integer valued jump times and the interval $(t, t+h]$ contains exactly h integers. □

3.2 Non-Lattice Case

Theorem 3.3 (Blackwell's Renewal Theorem)**.** *[5] Assume that the law of T_1 is non-lattice and $\mu < \infty$, then for all $h \geq 0$*

$$\lim_{t\to\infty} \big(m(t+h) - m(t)\big) = \frac{h}{\mu}$$

Our goal is to prove Blackwell's Renewal Theorem using a coupling argument. This approach involves first constructing a "perfect," easily analyzable stationary renewal process. We will then demonstrate that our ordinary process, $N(t)$, eventually "forgets" its initial state and behaves like this perfect process. The proof presented here closely follows the exposition in Armand Riera's lecture notes, Statistique des Processus.

A crucial component of this proof relies on the properties of non-lattice distributions. Intuitively, if the time between events is not restricted to a discrete grid (i.e., the distribution is non-lattice), looking far enough into the future ensures a positive probability that a renewal event will occur within any given time window of length δ, no matter how small δ is. The following lemma formalizes this concept and plays a significant role in the proof.

Lemma 3.4. *[3] Suppose F is non-lattice. Then for each δ there exists T such that if $t \geq T$*

$$m(t+\delta) - m(t) > 0.$$

Equivalently, for a pure renewal process $N = (N_t)_{t\in\mathbb{R}_+}$ it holds that

$$\mathbb{P}[N_{t+\delta} - N_t > 0] > 0$$

for all $t \geq T$.

Proof. See Appendix C □

To build the "perfect" stationary renewal process mentioned in our coupling argument, we must first introduce the concept of a delayed renewal process.

Definition 3.5. *(Delayed Renewal Process) Let $T_1^*, T_2, T_3, \ldots$ be a sequence of mutually independent, non-negative random variables. Suppose that the first inter-arrival time T_1^* has a cumulative distribution function G, and the subsequent inter-arrival times T_n for $n \geq 2$ are identically distributed with a cumulative distribution function F. Let the arrival time S_k^* be defined as:*

$$S_k^* := T_1^* + T_2 + \cdots + T_k \quad \forall k \geq 1.$$

The counting process $(N_t^*)_{t\geq 0}$, representing the number of renewals up to time t, is defined by:

$$N_t^* := \sum_{k\geq 1} \mathbf{1}_{S_k^* \leq t} \quad \forall t \geq 0.$$

This process is called a delayed renewal process. Furthermore, its associated delayed renewal function is defined as:

$$m^*(t) := \mathbb{E}[N_t^*].$$

To understand the practical necessity of this generalization, it is helpful to see how delayed renewal processes arise naturally from standard ones. Suppose we observe a standard renewal process, but rather than starting our observation at its inception at time $t = 0$, we begin observing at some later fixed time $t > 0$.

Let $(N_u)_{u\geq 0}$ be a standard renewal process with arrival times $(S_n)_{n\geq 1}$ and inter-arrival times $(T_n)_{n\geq 1}$. Consider the shifted counting process $(\bar{N}_s)_{s\geq 0}$, which counts the number of renewals in the interval $(t, t+s]$; $\bar{N}_s := N_{t+s} - N_t$, $s \geq 0$. We can characterize this new process by its own sequence of inter-arrival times, $(\bar{T}_i)_{i\geq 0}$. The time until the first observed arrival in this shifted frame is the forward recurrence time (also known as the "excess life" at time t), which is the residual time from t until the next immediate arrival S_{N_t+1}. We define this as $\bar{T}_1 := S_{N_t+1} - t$. The subsequent inter-arrival times correspond simply to the original inter-arrival times, shifted by the random number of arrivals N_t that occurred prior to time t: $\bar{T}_i := T_{N_t+i}$, $\forall i \geq 2$. The new arrival epochs are given by $\bar{S}_k := \bar{T}_1 + \cdots + \bar{T}_k$, allowing us to express the shifted process as:

$$\bar{N}_s = \sum_{k=1}^{\infty} \mathbf{1}_{\bar{S}_k \leq s}.$$

Therefore, $(\bar{N}_s)_{s\geq 0}$ forms a delayed renewal process where the initial delay $\bar{T}_1$ has the distribution of the excess life at time t, while all subsequent inter-arrival times $(\bar{T}_i)_{i\geq 2}$ retain their original distribution F. Having seen how a delayed renewal process naturally arises from shifting the time origin, we can ask whether there exists a specific initial delay that makes the resulting delayed renewal process stationary, in the sense that $m^*(t) = t/\mu$ holds exactly for every $t \geq 0$ rather than only in the limit. The following lemma identifies the unique distribution of the first arrival time achieving this.

Lemma 3.6. *[4] Let $(N_t^*)_{t\geq 0}$ be a delayed renewal process where the inter-arrival times T_n for $n \geq 2$ have a cumulative distribution function F with a finite mean $\mu \in (0, \infty)$. Suppose the first inter-arrival time T_1^* has the cumulative distribution function G given by:*

$$G(t) := \frac{1}{\mu} \int_0^t (1 - F(s))\, ds, \quad \text{for } t \geq 0. \tag{6}$$

Then the renewal function $m^*(t) = \frac{t}{\mu}$ for all $t \geq 0$. Moreover, if we fix $t \geq 0$ and set $\bar{T}_1 = S^*_{N_t^*+1} - t$ and $\bar{T}_i = S^*_{N_t^*+i} - S^*_{N_t^*+i-1} = T_{N_t^*+i}$ for $i \geq 2$, then

$$(\bar{N}_s)_{s\geq 0} \overset{(d)}{=} (N_s^*)_{s\geq 0}.$$

Where $\bar{N}_s$ denotes the renewal process constructed from the inter-arrival times $(\bar{T}_i)_{i\geq 1}$ and S_n^* denotes the n-the arrival time of $(N_t^*)_{t\geq 0}$.

Proof. We begin by applying the fundamental definition of the renewal function $m^*(t) = E[N_t^*]$. By linearity of expectation, we can express this as the sum of the probabilities $P(S_k^* \leq t)$ for all $k \geq 1$. Finally, we identify each term as the distribution function of the k-th arrival time S_k^*, which is given by the convolution of the initial delay G and $(k-1)$ subsequent arrivals from F. More precisely:

$$m^*(t) = \mathbb{E}(N_t^*) = \mathbb{E}\left[\sum_{k\geq 1} \mathbf{1}_{S_k^* \leq t}\right] = \sum_{k\geq 1} \mathbb{P}(S_k^* \leq t) = \sum_{k\geq 1} \mathbb{P}(T_1^* + S_{k-1} \leq t) \tag{7}$$

$$= \sum_{k\geq 1} \left(G * F^{*(k-1)}\right)(t) \tag{8}$$

In the next step, we apply the Laplace-Stieltjes Transform, abbreviated LST, to equation (7). We denote the Laplace transform of the Lebesgue-Stieltjes measure induced by the CDF F as $\tilde{F}(s) = \int_0^\infty e^{-st} dF(t)$, which is well-defined for $Re(s) \geq 0$. The LST is a powerful tool because it maps the convolution of distributions to the pointwise product of their transform. Applying the LST to our renewal equation $m^*(t) = \sum_{k=1}^{\infty} \left(G * F^{*(k-1)}\right)(t)$, we get:

$$\tilde{m}^*(s) = \sum_{k=1}^{\infty} \tilde{G}(s) \cdot \tilde{F}(s)^{k-1} = \tilde{G}(s) \cdot \left[\sum_{k=1}^{\infty} \tilde{F}(s)^{k-1}\right]. \tag{9}$$

Our goal is now to find simple expressions for the two parts of this product.

The second part of our product is an infinite sum. By re-indexing with $j = k - 1$, we can see that this is a standard geometric series. $\sum_{k=1}^{\infty} \tilde{F}(s)^{k-1} = \sum_{j=0}^{\infty} \tilde{F}(s)^j = 1 + \tilde{F}(s) + \tilde{F}(s)^2 + \cdots$, and thus converges to

$$\frac{1}{1 - \tilde{F}(s)}.$$

For the term $\tilde{G}$ we must find the LST of our specific delay distribution $G(t) = \frac{1}{\mu} \int_0^t (1 - F(u))du$. First we find its probability density function $g(t)$ by differentiating $G(t)$:

$$g(t) = G'(t) = \frac{1}{\mu}(1 - F(t)).$$

Now we find its LST,

$$\tilde{G}(s) = \int_0^\infty e^{-st} g(t)dt = \int_0^\infty e^{-st} \frac{1 - F(t)}{\mu} dt.$$

Using the standard identity $\int_0^\infty e^{-st}(1 - F(t))dt = \frac{1-\tilde{F}}{s}$ we get

$$\tilde{G}(s) = \frac{1}{\mu}\left[\frac{1 - F(s)}{s}\right].$$

We now substitute these two results into equation (9):

$$\tilde{m}^*(s) = \left[\frac{1 - \tilde{F}(s)}{\mu s}\right] \cdot \left[\frac{1}{1 - \tilde{F}(s)}\right] = \frac{1}{\mu s}.$$

To recover the function $m^*(t)$ we observe that

$$\int_0^\infty e^{-st} d\left(\frac{t}{\mu}\right) = \frac{1}{\mu} \int_0^\infty e^{-st} dt = \frac{1}{\mu s},$$

hence we conclude that $m^*(t) = \frac{t}{\mu}$. It remains to show that $(\bar{N}_s)_{s\geq 0}$ has the same distribution as $(N_s^*)_{s\geq 0}$. Since the renewal process is a deterministic measurable functional of its inter-arrival sequence, it suffices to show that

$$(\bar{T}_1, \bar{T}_2, \bar{T}_3, ...) \stackrel{d}{=} (T_1^*, T_2^*, T_3^*, ...). \tag{10}$$

Where on the right hand side $T_1^* \sim G$, $(T_i^*)_{i\geq 2}$ are i.i.d. with distribution F, and T_1^* is

independent of $(T_i^*)_{i\geq 2}$. We proceed in two steps. We first establish the marginal $\bar{T}_1 \stackrel{d}{=} T_1^*$, and then derive the full joint distribution by a direct conditioning argument on the value of N_t^*. Fix $x \geq 0$, by decomposing the event $\{\bar{T}_1 \geq x\}$ according to the value of N_t^*, we obtain

$$\mathbb{P}(\bar{T}_1 \geq x) = \mathbb{P}(T_1^* \geq x+t) + \sum_{k\geq 1} \mathbb{P}(S_k^* \leq t, S_k^* + T_{k+1} \geq x+t).$$

Indeed, on the event $\{N_t^* = 0\}$ no arrival has occurred by time t, so the next arrival is T_1^* itself, and since $\bar{T}_1 = T_1^* - t$, the condition $\bar{T}_1 \geq x$ thus reads $T_1^* \geq x+t$. On the event $\{N_t^* = k\}$ for $k \geq 1$, the k-th arrival has occurred at time $S_k^* \leq t$ and the $(k+1)$-th has not, so $\bar{T}_1 = S_k^* + T_{k+1} - t$, and $\bar{T}_1 \geq x$ reads $S_k^* + T_{k+1} \geq x+t$. The first term is computed directly from the definition of G:

$$\mathbb{P}(T_1^* \geq x+t) = 1 - G(x+t) = \frac{1}{\mu}\int_{x+t}^{\infty}(1-F(s))ds.$$

The second term is handled by conditioning on S_k^*. For each $k \geq 1$, the variable T_{k+1} is independent of S_k^* and has distribution F, so

$$\mathbb{P}(S_k^* \leq t, S_k^* + T_{k+1} \geq x+t) = \mathbb{E}\left[\mathbf{1}_{S_k^* \leq t}(1 - F(x+t-S_k^*))\right].$$

Recalling that the law of S_k^* is $G * F^{*(k-1)}$, we may rewrite this expectation as an integral against this measure:

$$\mathbb{E}\left[\mathbf{1}_{S_k^* \leq t}(1 - F(x+t-S_k^*))\right] = \int_0^t (1-F(x+t-s))d(G*F^{*(k-1)})(s).$$

Summing over $k \geq 1$ and since all integrands are non-negative, using Tonelli's theorem to interchange sum and integral, we obtain

$$\sum_{k\geq 1}\mathbb{P}(S_k^* \leq t, S_k^* + T_{k+1} \geq x+t) = \int_0^t (1-F(x+t-s))dm^*(s)$$

since $m^*(s) = \sum_{k\geq 1}(G * F^{*(k-1)})(s)$ by (7). Since $m^*(s) = s/\mu$, so $dm^*(s) = \mu^{-1}ds$, and therefore

$$\int_0^t (1-F(x+t-s))dm^*(s) = \mu^{-1}\int_0^t (1-F(x+t-s))ds = \mu^{-1}\int_x^{x+t}(1-F(u))du.$$

Where the second equality follows from the substitution $u = x+t-s$, $du = -ds$. Combining the two terms, we conclude

$$\mathbb{P}(\bar{T}_1 \geq x) = \mu^{-1}\int_{x+t}^{\infty}(1-F(s))ds + \mu^{-1}\int_x^{x+t}(1-F(s))ds = \mu^{-1}\int_x^{\infty}(1-F(s))ds.$$

Recognizing the right-hand side as $1 - G(x) = \mathbb{P}(T_1^* \geq x)$, we have shown that $\bar{T}_1 \stackrel{d}{=} T_1^*$. It remains to show that $\bar{T}_1$ is independent of $(\bar{T}_i)_{i\geq 2}$ and that $(\bar{T}_i)_{i\geq 2}$ is i.i.d. with distribution

F. Fix $x \geq 0$ and a Borel set $A \subset \mathbb{R}_+^{\mathbb{N}}$. We decompose according to the value of N_t^*:

$$\mathbb{P}(\bar{T}_1 > x, (\bar{T}_i)_{i\geq 2} \in A) = \sum_{k\geq 0} \mathbb{P}(N_t^* = k, \bar{T}_1 > x, (\bar{T}_i)_{i\geq 2} \in A).$$

For each $k \geq 0$, on the event $\{N_t^* = k\}$ we have, by definition, that $\bar{T}_1 = T_1^* + T_2 + \cdots + T_{k+1} - t$ and $\bar{T}_i = T_{k+i}$ for $i \geq 2$. Moreover, $\{N_t^* = k\} = \{S_k^* \leq t < S_{k+1}^*\}$ is determined by $(T_1^*, T_2, \ldots, T_{k+1})$. Consequently, both the event $\{N_t^* = k\}$ and the variable $\bar{T}_1$ are measurable with respect to $\sigma(T_1^*, T_2, \ldots, T_{k+1})$, whereas $(\bar{T}_i)_{i\geq 2} = (T_{k+i})_{i\geq 2}$ is measurable with respect to $\sigma(T_{k+2}, T_{k+3}, \ldots)$. Since $T_1^*, T_2, T_3, \ldots$ are mutually independent, the σ-algebras $\sigma(T_1^*, T_2, \ldots, T_{k+1})$ and $\sigma(T_{k+2}, T_{k+3}, \ldots)$ are independent. Moreover, since $(T_i)_{i\geq 2}$ are i.i.d. with distribution F, the sequence $(T_{k+i})_{i\geq 2}$ has the same distribution as $(T_i^*)_{i\geq 2}$. Therefore,

$$\mathbb{P}(N_t^* = k, \bar{T}_1 > x, (\bar{T}_i)_{i\geq 2} \in A) = \mathbb{P}(N_t^* = k, \bar{T}_1 > x) \cdot \mathbb{P}((T_i^*)_{i\geq 2} \in A).$$

Summing over $k \geq 0$,

$$\mathbb{P}(\bar{T}_1 > x, (\bar{T}_i)_{i\geq 2} \in A) = \mathbb{P}(\bar{T}_1 > x) \cdot \mathbb{P}((T_i^*)_{i\geq 2} \in A). \tag{11}$$

The identity (11) shows simultaneously that $\bar{T}_1$ is independent of $(\bar{T}_i)_{i\geq 2}$ and that $(\bar{T}_i)_{i\geq 2} \stackrel{d}{=} (T_i^*)_{i\geq 2}$. □

Corollary 3.7. *Under the hypothesis of Lemma 3.6, the forward recurrence time process*

$$Z^*(s) := S^*_{N_s^*+1} - s, \quad s \geq 0,$$

is strictly stationary: for $t \geq 0$, $(Z^*(t+s))_{s\geq 0} \stackrel{d}{=} (Z^*(s))_{s\geq 0}$.

Proof. Fix $t \geq 0$ and let $(\bar{N}_s)_{s\geq 0}$ be the renewal process constructed from $\bar{T}_1 = S^*_{N_t^*+1} - t$, $T_{N_t^*+i}$ for $i \geq 2$. By construction, the forward recurrence time of (N^*) at time $t+s$ equals the forward recurrence time of $(\bar{N})$ at time s:

$$Z^*(t+s) = \bar{Z}(s) \quad \textit{for all} \quad s \geq 0.$$

The map sending a renewal process to its forward recurrence time process is measurable, so Lemma 3.6 gives

$$(Z^*(t+s))_{s\geq 0} = (\bar{Z}(s))_{s\geq 0} \stackrel{d}{=} (Z^*(s))_{s\geq 0}.$$

□

Proof. (Blackwell's Renewal Theorem) Let $T_1, T_2, \ldots,$ be i.i.d. non-negative random variables with $0 < \mathbb{E}[T_i] = \mu < \infty$ and distribution function F of non-lattice type. Let $S_n = \sum_{i=1}^n T_i$ with the corresponding renewal process $N_t = \sum_{k\geq 1} \mathbf{1}_{S_k \leq t}$. Let the random variables $T_0', T_1', \ldots,$ be independent, also independent of $(T_i)_{i\geq 1}$. Here, $T_1', T_2', \ldots,$ all have distribution F and T_0' has density $f_0(y) = (1 - F(y))/\mu \ \forall y \geq 0$. Then $S_n' = \sum_{i=0}^n T_i'$ with the corresponding renewal process $N_t' = \sum_{k\geq 0} \mathbf{1}_{S_k \leq t}$ is a delayed renewal process and by Lemma 3.6 this process is stationary. Our goal is to show that, with probability one, a renewal from $(S_n)_{n\geq 1}$ comes arbitrarily close to one from $(S_n')_{n\geq 0}$ and can then be coupled

to coincide. For each $i \geq 0$ let

$$Z(S_i) := \inf\{S'_j - S_i : S'_j - S_i \geq 0, j \geq 0\} = S'_{N'_{S_i}+1} - S_i,$$

i.e. the waiting time from S_i until the next S'-renewal. For any fixed $\epsilon > 0$, let $A_i = \{Z(S_j) < \epsilon \quad \textit{for some} \quad j \geq i\}$. We have $A_0 \supset A_1 \supset \cdots \supset \bigcap_{i=0}^{\infty} A_i = A_\infty = \{Z(S_j) < \epsilon \quad \textit{i.o.}\}$. First we claim that $\mathbb{P}(A_i)$ is independent of i. For this we check that the law of the process $(Z(S_j)_{j\geq i})$ does not depend on i, i.e.,

$$(Z(S_j))_{j\geq i} \overset{d}{=} (Z(S_j))_{j\geq 0}. \tag{12}$$

Fix $i,\ k \geq 0$ and Borel sets $\Lambda_0, \cdots, \Lambda_k \subset \mathcal{B}(\mathbb{R}_+)$. To establish (12), we condition on $(S_{i+j})_{0\leq j\leq k}$. Since the process $(S'_l)_{l\geq 0}$ is independent of $(T_n)_{n\geq 1}$, and hence of $(S_n)_{n\geq 1}$, the random vector $(Z(s_{i+j}))_{0\leq j\leq k}$ is independent of $(S_n)_{n\geq 1}$. Disintegrating against the joint law of $(S_{i+j})_{0\leq j\leq k}$ gives

$$\begin{aligned}&\mathbb{P}(Z(S_i) \in \Lambda_0, \cdots, Z(S_{i+k}) \in \Lambda_k)\\ &= \int \mathbb{P}(Z(s_{i+j}) \in \Lambda_j, 0 \leq j \leq k)\mathbb{P}(S_{i+j} \in ds_{i+j}, 0 \leq j \leq k).\end{aligned}$$

Because the forward-recurrence-time process $t \mapsto Z(t)$ of the stationary renewal process is itself strictly stationary, the integrand is invariant under the shift $s_{i+j} \mapsto s_{i+j} - s_i$:

$$\mathbb{P}(Z(s_{i+j}) \in \Lambda_j, 0 \leq j \leq k) = \mathbb{P}(Z(s_{i+j} - s_i) \in \Lambda_j, 0 \leq j \leq k).$$

Substituting and using that the increments $(S_{i+j} - S_i)_{0\leq j\leq k} = (0, T_{i+1}, T_{i+1} + T_{i+2}, ...)$ have the same joint distribution as $(S_j)_{0\leq j\leq k} = (0, T_1, T_1 + T_2, ...)$ because the T_n are i.i.d. we obtain

$$\begin{aligned}&\int \mathbb{P}(Z(s_{i+j} - s_i) \in \Lambda_j, 0 \leq j \leq k)\mathbb{P}(S_{i+j} \in ds_{i+j}, 0 \leq j \leq k)\\ &= \mathbb{P}(Z(S_0) \in \Lambda_0, Z(S_1) \in \Lambda_1, ..., Z(S_k) \in \Lambda_k).\end{aligned}$$

This proves (12) and thus $\mathbb{P}(A_i) = \mathbb{P}(A_0)$, as asserted. We therefore conclude

$$\lim_{i\to\infty} \mathbb{P}(A_i) = \mathbb{P}(A_\infty) = \mathbb{P}(A_0).$$

The next step is to show that $\mathbb{P}(A_0|T'_0 = t)$[1] > 0. Fix a value of t and write

$$\mathbb{P}(A_0|T'_0 = t) = \mathbb{P}\left[\bigcup_{j=0}^{\infty}[Z(S_j) < \epsilon|T'_0 = t]\right] = \mathbb{P}\left[\bigcup_{j=0}^{\infty}[t + \sum_{i=1}^{n} T'_i \in (S_j, S_j + \epsilon) \textit{ for some } n \geq 1]\right].$$

[1]Since T'_0 is a continuous random variable, the symbol $\mathbb{P}(\cdot|T'_0 = t)$ is to be understood as a regular conditional probability. By the independence of T'_0 from $(T'_i)_{i\geq 1}$ and $(T_n)_{n\geq 1}$, a version of this conditional probability is given explicitly by $\varphi(t) := \mathbb{P}(t + \sum_{i=1}^{n} T'_i \in (S_j, S_j + \epsilon) \textit{for some } j \geq 0, n \geq 1)$, in which $t \geq 0$ is a deterministic constant. Indeed, Fubini's theorem applied to the product law of T'_0 and $(T'_i)_{i\geq 1}$, $(T_n)_{n\geq 0}$ yields $\mathbb{P}(A_0|T'_0 \in B) = \int_B \varphi(t)F_0(dt)$ for every Borel set $B \subset \mathbb{R}_+$ which is the defining property of a version of $\mathbb{P}(A_0|T'_0 = t)$. In what follows, every expression of the form $\mathbb{P}(\cdot|T'_0 = t)$ is to be read as the corresponding version, and all subsequent identities hold for F_0-almost every $t \geq 0$, which suffices since the variable t is ultimately integrated against $F_0(dt)$.

Bounding the union below by its j-term, for any fixed j, this is

$$\mathbb{P}(A_0|T_0' = t) \geq \mathbb{P}[t + \sum_{i=1}^{n} T_i' \in (S_j, S_j + \epsilon) \text{ for some } n \geq 1]$$

Conditioning on S_j and using independence of $(T_i')_{i\geq 1}$ from $(T_n)_{n\geq 1}$

$$\mathbb{P}(A_0|T_0' = t) \geq \int_0^\infty \mathbb{P}\left[\sum_{i=1}^{n} T_i' \in (s-t, s-t+\epsilon) \text{ for some } n \geq 1\right] \mathbb{P}(S_j \in ds).$$

Restricting the integration to $s > T + t$ where T is as in Lemma 3.4, the integrand, which equals $\mathbb{P}(N'_{s-t+\epsilon} - N'_{s-t} > 0)$, is strictly positive. Since $S_j \to \infty$ almost surely as $j \to \infty$, we can choose j large enough that $\mathbb{P}(S_j > T + t) > \frac{1}{2}$. For this choice, the integrand is strictly positive,

$$\mathbb{P}[A_0|T_0' = t] = \int_{T+t}^{\infty} \mathbb{P}(N'_{s-t+\epsilon} - N'_{s-t} > 0)\mathbb{P}(S_j \in ds) > 0.$$

We next assert $\mathbb{P}(A_\infty|T_0' = t) = 0$ *or* 1, which is a direct consequence of Hewitt-Savage (see Appendix A.6) applied to the i.i.d sequence of random vectors $\{(T_1, T_1'), (T_2, T_2'), ..., \}$ since

$$\mathbb{P}(A_\infty|T_0' = t) = \mathbb{P}\left[\limsup_{j\to\infty} \bigcup_{n=1}^{\infty} [t + \sum_{i=1}^{n} T_i' \in (S_j, S_j + \epsilon)]\right].$$

Now let $\Lambda := \{t : \mathbb{P}(A_\infty|T_0' = t) = 1\}$, and we show $F_0(\Lambda) = 1$. Suppose for contradiction that $F_0(\Lambda) < 1$, equivalently $F_0(\Lambda^c) > 0$. Then

$$\mathbb{P}(A_0) = \int \mathbb{P}(A_0|T_0' = t)F_0(dt)$$

and

$$\mathbb{P}(A_0) = \mathbb{P}(A_\infty) = \int \mathbb{P}(A_\infty|T_0' = t)F_0(dt) = \int_\Lambda 1F(dt) + \int_{\Lambda^c} 0F_0(dt) = F_0(\Lambda)$$

since we have that $\mathbb{P}(A_\infty|T_0' = t) = 0$ for $t \in \Lambda^c$. Note that $A_0 \supset A_\infty$ so on Λ we have $\mathbb{P}(A_0|T_0' = t) = 1$, hence

$$F_0(\Lambda) = \mathbb{P}(A_0) = \int \mathbb{P}(A_0|T_0' = t)F_0(dt) = \int_\Lambda 1F_0(dt) + \int_{\Lambda^c} \mathbb{P}(A_0|T_0' = t)F_0(dt).$$

Recall $\mathbb{P}(A_0|T_0' = t) > 0$ for all t. If $F_0(\Lambda) < 1$ so that $F_0(\Lambda^c) > 0$, then the above gives a contradiction. Since $F_0(\Lambda) = 1$, we have

$$\mathbb{P}(A_\infty) = \int \mathbb{P}(A_\infty|T_0' = t)F_0(dt) = \int_\Lambda \mathbb{P}(A_\infty|T_0' = t)F_0(dt) = \int_\Lambda 1F_0(dt) = F_0(\Lambda) = 1.$$

And therefore we can conclude that $\mathbb{P}(A_0) = \mathbb{P}(A_\infty) = 1$.
Let $\tau = min\{i : Z(S_i) < \epsilon\}$ and $\tau' = min\{j : S_j' \geq S_\tau\}$. We construct a new process $(\bar{N}_t)_{t\geq 0}$, which coincides with the original process $(N_t)_{t\geq 0}$ up to time S_τ and with the

delayed process $(N'_t)_{t\geq 0}$ afterwards:

$$\bar{N}_t = \begin{cases} N_t, & \text{if } t \leq S_\tau, \\ N_{S_\tau} + N'_{t+Z_\tau} - N'_{S'_{\tau'}}, & \text{if } t > S_\tau. \end{cases}$$

By construction $\bar{N}_t$ has the same law as N_t, but from time S_τ onward, its renewals follow those of N'_t, shifted by a small amount less than ϵ.

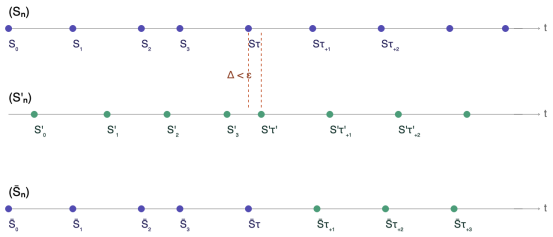

Figure 1: Coupling process

$$\begin{aligned} m(t+\delta) - m(t) &= \mathbb{E}\big[\bar{N}_{t+\delta} - \bar{N}_t\big] \\ &= \mathbb{E}\big[N_{(t+\delta)\wedge S_\tau} - N_{t\wedge S_\tau}\big] + \mathbb{E}\big[N'_{t+Z_\tau+\delta} - N'_{t+Z_\tau}\big] - \mathbb{E}\Big[N'_{(t+Z_\tau+\delta)\wedge S'_\tau} - N'_{(t+Z_\tau)\wedge S'_\tau}\Big] \end{aligned}$$

To bound the first term, fix $a > 0$. Then

$$\begin{aligned} \mathbb{E}\big[N_{(t+\delta)\wedge S_\tau} - N_{t\wedge S_\tau}\big] &\leq \mathbb{E}\big[(N_{t+\delta} - N_t)\mathbf{1}_{\{t\leq S_\tau\}}\big] \leq a\cdot\mathbb{P}(t\leq S_\tau) + \mathbb{E}\big[(N_{t+\delta} - N_t)\mathbf{1}_{\{N_{t+\delta}-N_t>a\}}\big] \\ &\leq a\cdot\mathbb{P}(t\leq S_\tau) + \mathbb{E}\left[N_\delta \mathbf{1}_{N_\delta \geq a}\right]. \end{aligned}$$

Then taking $t\to\infty$ we observe

$$\limsup_{t\to\infty} \mathbb{E}\left[N_{(t+\delta)\wedge S_\tau} - N_{t\wedge S_T}\right] \leq a\cdot\limsup_{t\to\infty}\mathbb{P}(t\leq S_\tau) + \mathbb{E}\left[N_\delta \mathbf{1}_{N_\delta\geq a}\right] = \mathbb{E}\left[N_\delta \mathbf{1}_{N_\delta\geq a}\right]$$

Where we used that $S_\tau < \infty$ a.s.. For $a\to\infty$ we observe

$$\limsup_{t\to\infty} \mathbb{E}\left[N_{(t+\delta)\wedge S_\tau} - N_{t\wedge S_\tau}\right] = 0$$

With the same argument applied to the third term we get

$$\lim_{t\to\infty} \mathbb{E}\left[N'_{(t+Z_\tau+\delta)\wedge S'_\tau} - N'_{(t+Z_\tau)\wedge S'_\tau}\right] = 0$$

With the help of Lemma 3.6 and using that $Z_\tau \in (0,\epsilon)$ we find that

$$m'(t+\delta) - m'(t+\epsilon) \leq \mathbb{E}\left[N'_{t+Z_\tau+\delta} - N'_{t+Z_\tau}\right] \leq m'(t+\epsilon) - m'(t)$$

$$\frac{\delta-\epsilon}{\mu} \leq \mathbb{E}\left[N'_{t+Z_\tau+\delta} - N'_{t+Z_\tau}\right] \leq \frac{\delta+\epsilon}{\mu}$$

$$\frac{\delta-\epsilon}{\mu} \leq \liminf_{t\to\infty} \mathbb{E}\left[N'_{t+Z_\tau+\delta} - N'_{t+Z_\tau}\right] \leq \limsup_{t\to\infty} \mathbb{E}\left[N'_{t+Z_\tau+\delta} - N'_{t+Z_\tau}\right] \leq \frac{\delta+\epsilon}{\mu}$$

From here we can conclude that

$$\frac{\delta-\epsilon}{\mu} \leq \liminf_{t\to\infty} m(t+\delta)-m(t) \leq \limsup_{t\to\infty} m(t+\delta)-m(t) \leq \frac{\delta+\epsilon}{\mu}$$

for $\epsilon \to 0$ the proof is complete □

4 Renewal Equation

One of the main reasons, why renewal processes can be analyzed so successfully is the close connection to analytic methods. Many probabilistic quantities of interest, for instance the excess distribution, (i.e. the remaining waiting time until the next renewal) can be expressed through the renewal equation. This analytic bridge allows us to translate probabilistic questions into functional equations which can then be studied and solved using tools from analysis.

Before turning to the renewal equation itself, we first introduce some analytic tools that will be essential in the subsequent discussion. In particular, we recall the notion of the Lebesgue–Stieltjes measure, as well as the convolution operator. These concepts provide the natural framework in which renewal equations can be formulated and solved. And they will serve as the foundation for the analytic treatment of renewal processes.

Lemma 4.1. *[5] Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a right continuous non-decreasing function. There exists a unique measure ν_g on $\mathbb{R}_+$ such that*

$$\forall t \geq 0 \quad \nu_g([0,t]) = g(t)$$

Proof. [1] The uniqueness of ν_g is a consequence of Corollary A.4, since the class $C = \{(0,x]; x \in \mathbb{R}_+\}$ is closed under finite intersections and generates $B(\mathbb{R}_+)$. To show existence, we set, for every $A \subset \mathbb{R}_+$,

$$\nu_g^*(A) = \inf\left\{\sum_{i\in\mathbb{N}} (g(b_i)-g(a_i)) : A \subset \bigcup_{i\in\mathbb{N}} (a_i,b_i]\right\}$$

Then ν_g^* is an outer measure and the intervals $(0,\alpha]$ are in $\mathcal{M}(\nu_g^*)$. It follows that the σ-field $\mathcal{M}(\nu_g^*)$ contains $\mathcal{B}(\mathbb{R})$, and the restriction of ν_g^* to $\mathcal{B}(\mathbb{R}_+)$ is a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, which we denote by ν_g. Clearly, ν_g is a finite measure. To complete the proof, we have to verify that $\nu_g((0,x]) = g(x)$ for every $x \in \mathbb{R}_+$. To this end, it suffices to check that $\nu_g((a,b]) = g(b) - g(a)$ for every $a < b$. The bound

$$\nu_g((a,b]) \leq g(b) - g(a)$$

is immediate from the construction of ν_g^*.

To get the reverse inequality, let $((x_i,y_i])_{i\in\mathbb{N}}$ be a countable cover of $(a,b]$, and let $\epsilon \in (0, b-a)$. For every $i \in \mathbb{N}$, we can find $y_i' > y_i$ such that $g(y_i') \leq g(y_i) + \epsilon 2^{-i}$. By compactness, we can find N_ϵ large enough so that $[a+\epsilon, b]$ is covered by the finite collection $((x_i,y_i'))_{i\in\{1,\dots,N_\epsilon\}}$. An elementary argument shows that

$$g(b)-g(a+\epsilon) \leq \sum_{i=1}^{N_\epsilon} (g(y_i')-g(x_i)) \leq \sum_{i=1}^{\infty} (g(y_i')-g(x_i)) \leq \sum_{i=1}^{\infty} (g(y_i)-g(x_i)) + \epsilon$$

Since ϵ was arbitrary and $g(a+\epsilon) \to g(a)$ as $\epsilon \to 0$, we get

$$g(b) - g(a) \leq \sum_{i=1}^{\infty} (g(y_i) - g(x_i))$$

which gives the lower bound $\nu_g((a,b]) \geq g(b) - g(a)$ and completes the proof. □

The renewal function $m(t) = \mathbb{E}[N_t]$ is by definition a right-continuous and non-decreasing function. It therefore satisfies the conditions for g in the above lemma. This lemma allows us to associate a unique measure ν_m with $m(t)$, called the renewal measure, such that $\nu_m((0,t]) = m(t)$ for all $t \geq 0$. The following proposition provides a more general and intuitive probabilistic representation for this renewal measure, defining it for any measurable set B.

Proposition 4.2. *[5] Let m be the renewal function associated with a renewal process, and let ν_m denote the measure on $\mathbb{R}_+$ corresponding to m. Then ν_m admits the following probabilistic representation: for every measurable $B \subset \mathbb{R}_+$,*

$$\nu_m(B) = \mathbb{E}\big[\textit{Number of renewals in } B\big] = \mathbb{E}\left(\sum_{k\geq 1} \mathbf{1}_{\{S_k \in B\}}\right).$$

In particular, for every measurable bounded function $h : \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int_{\mathbb{R}_+} h \, d\nu_m = \mathbb{E}\left[\sum_{k\geq 1} h(S_k)\right].$$

Proof. Define a measure ν' on $\mathbb{R}_+$ by

$$\nu'(B) := \mathbb{E}\left(\sum_{k\geq 1} \mathbf{1}_{\{S_k \in B\}}\right), \qquad B \subset \mathbb{R}_+ \text{ measurable.}$$

For $B = [0,t]$, we have $\nu'([0,t]) = m(t)$ for all $t \geq 0$. Since the measure ν_m satisfying $\nu_m([0,t]) = m(t)$ is unique, it follows that $\nu_m = \nu'$. The integral identity then follows from the definition of ν' and linearity of expectation. □

Definition 4.3 (Convolution operator)**.** *Let G be a right-continuous non-decreasing function on $\mathbb{R}_+$. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be measurable such that for all $t \geq 0$*

$$\int_0^t |h(t-s)| \, dG(s) < \infty$$

or h is measurable and non-negative. Then for every $t \geq 0$, define

$$(h * G)(t) = \int_0^t h(t-s) \, dG(s).$$

The following remark shows how the convolution operator is used to analyze sums of random variables.

Remark 4.4. *If X, Y are two independent random variables on $\mathbb{R}_+$ with distribution func-*

tions F_X, F_Y respectively, then

$$F_{X+Y} = F_X * F_Y$$

Proof.

$$\begin{aligned} F_{X+Y}(t) &= \mathbb{P}(X+Y \le t) \\ &= \mathbb{P}(X \le t-Y) \\ &= \int_0^\infty \mathbb{P}(X \le t-y)\, \mathbf{1}_{\{y \le t\}}\, dF_Y(y) \\ &= \int_0^t F_X(t-y)\, dF_Y(y) \\ &= (F_X * F_Y)(t). \end{aligned}$$

□

Having established the convolution operator, we now have the principal tool required to define the central equation of renewal theorey.

This integral equation, known as the renewal equation, provides a general framework for modeling many quantities of interest. It uses the convolution to relate an unknown function $g(t)$ to a known function $h(t)$ and the underlying inter-arrival distribution F.

As we will see in Proposition 4.6, the renewal function $m(t)$ is itself a solution to this type of equation. The Key Renewal Theorem, which is a major objective of our study, is dedicated to finding the asymptotic behavior of the solution $g(t)$

Definition 4.5. *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be measurable and locally bounded (i.e. $\forall t \ge 0$, $h|_{[0,t]}$ is bounded). Let $g : \mathbb{R}_+ \to \mathbb{R}$ be measurable such that for all $t \ge 0$*

$$\int_0^t |g(t-s)|\, dF(s) < \infty.$$

We say that g is a solution of the (h, F) renewal equation if

$$\forall t \ge 0 \quad g(t) = h(t) + \int_0^t g(t-s)\, dF(s),$$

i.e. $g = h + g * F$.

Proposition 4.6. *[5] m is a solution of the (F, F) renewal equation, i.e. $m = F + m * F$.*

Proof. For $t \ge 0$, we have

$$m(t) = \mathbb{E}\left(\sum_{k\ge1} \mathbf{1}_{\{T_1+\cdots+T_k \le t\}}\right) = \sum_{k\ge1} \mathbb{P}(T_1+\cdots+T_k \le t) = \mathbb{P}[T_1 \le t] + \sum_{k\ge2} \underbrace{\mathbb{P}[T_1+\cdots+T_k \le t]}_{(\star)}.$$

Now, for the term $(\star)$ using that T_1 is independent of $(T_2 + T_3 + ...)$ and condition on $T_1 = s$:

$$(\star) = \int_0^t \mathbb{P}\left[s + T_2 + \cdots + T_k \le t\right] dF(s) = \int_0^t \mathbb{P}\left[T_2 + \cdots + T_k \le t - s\right] dF(s).$$

Thus,

$$m(t) = F(t) + \int_0^t \sum_{k\geq 2} \mathbb{P}\left[T_2 + \cdots + T_k \leq t - s\right] dF(s) = F(t) + \int_0^t m(t-s) dF(s) = F(t) + (m * F)(t)$$

which is precisely the renewal equation. □

Example 4.7 (Age Process)**.** *Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution F. Denote by $(A_t)_{t\geq 0}$ the age process of $(N_t)_{t\geq 0}$ defined by $A_t = t - S_{N_t}$. Set $a_x(t) = \mathbb{P}(A_t \leq x)$ for $t \geq 0$. First we can separate the term $\mathbb{P}(A_t \leq x) = \mathbb{P}(T_1 > t, A_t \leq x) + \mathbb{P}(T_1 \leq t, A_t \leq x)$ then we can analyze each term separately:*
For the first term we immediately have $\mathbb{P}(T_1 \geq t, A_t \leq x) = \mathbf{1}_{t\leq x}(1 - F(t))$
For the second term we use an auxiliary function and the renewal structure of the process. $\psi_t(t_1, t_2, ...) = \sum_{n\geq 0} \mathbf{1}_{t_1+t_2+\cdots+t_n \leq t < t_1+t_2+\cdots+t_{n+1}}(t - (t_1 + \cdots + t_n))$
By definition we have that $A_t = \psi_t(t_1, t_2, ...)$. With the help of this auxiliary function we get:

$$\begin{aligned}
\mathbb{P}[T_1 \leq t, A_t \leq x] &= \mathbb{P}[T_1 \leq t, \psi_t(T_1, T_2, \ldots) \leq x] \\
&= \int_0^t \mathbb{P}[\psi_t(s, T_2, \ldots) \leq x] \, dF(s) \\
&= \int_0^t \mathbb{P}[\psi_{t-s}(T_2, \ldots) \leq x] \, dF(s) \\
&= \int_0^t a_x(t-s) \, dF(s) \;=\; (a_x * F)(t).
\end{aligned}$$

Then we can conclude that $a_x(t) = \mathbf{1}_{\{t\leq x\}}\big(1 - F(t)\big) + (a_x * F)(t)$.

With the definition and the above example in mind, the next question to understand is how to find the form of the solution of the renewal equation. If we assume g to be the solution, then we have $g = h + g * F$. Now we replace g on the right hand side with the expression of g itself and obtain $g = h + (h + g * F) * F$. Replacing g inductively by $h + g \star F$ and assuming convergence of the resulting expression, we would get:

$$g = h + h * F + h * F^{*2} + h * F^{*3} + \cdots = h + h * m$$

This heuristic can be turned into the following rigorous result.

Proposition 4.8. *[5] Let $h : \mathbb{R}_+ \to \mathbb{R}$ be measurable and locally bounded. Then there exists a unique $g : \mathbb{R}_+ \to \mathbb{R}$, measurable and locally bounded, solution of*

$$g = h + g * F,$$

given by $g = h + h * m$.

Proof. The proof consists of two parts. First, we show that $g = h + h * m$ is indeed a solution (existence), and second, we show that this solution is the only possible one (uniqueness). We propose the solution $g = h + h * m$ and verify that it satisfies the renewal equation $g = h + g * F$. We substitute our proposed g into the right-hand side of the equation. Then using the distributive property of convolution and factoring out

h we observe that $h + g * F = h + (h + h * m) * F = h + h * (F + m * F)$. We know from Proposition (4.6) that the renewal function $m(t)$ itself satisfies the renewal equation $m = F + m * F$. Substituting this into our expression gives exactly our proposed solution $h + h * (F + m * F) = h + h * m = g$. The solution g is measurable and locally bounded because h and m are, and the convolution of such functions preserves these properties.

To prove uniqueness, let g_1 and g_2 be two solutions to the (h, F) renewal equation. Then $g_1 - g_2 = (g_1 - g_2) * F$ and therefore, by induction, $g_1 - g_2 = (g_1 - g_2) * F^{*n}$ for every $n \geq 1$. Fix $t \geq 0$. For every $n \geq 1$, we have

$$|g_1(t) - g_2(t)| = \left| \int_0^t (g_1 - g_2)(t - s) dF^{*n}(s) \right| \leq \sup_{[0,t]} |g_1 - g_2| \int_0^t dF^{*n}(s),$$

where the integral term is equal to $\mathbb{P}[T_1 + \cdots + T_n \leq t]$ which converges to 0 as n tends to infinity. Hence $g_1 = g_2$. □

5 Smith Key Renewal Theorem

To derive the general theorem, we first analyze the behavior of $g(t)$ for a simple choice of h. Let $h(t) = \mathbf{1}_{[a,b]}(t)$ for $0 \leq a \leq b$. For this specific h and for every $t > b$, the solution $g(t)$ becomes:

$$g(t) = \int_0^t \mathbf{1}_{[a,b]}(t - s)\, dm(s).$$

The integrand is non-zero only when $a \leq t - s \leq b$, which is equivalent to $t - b \leq s \leq t - a$. For $t > b$, the integral simplifies to:

$$g(t) = \int_{t-b}^{t-a} dm(s) = m(t - a) - m(t - b).$$

To find the limit of $g(t)$ as $t \to \infty$, we invoke Blackwell's Renewal Theorem, which states that for a non-lattice distribution F with mean μ, and any fixed $k > 0$:

$$\lim_{t\to\infty} [\, m(t + k) - m(t) \,] = \frac{k}{\mu}$$

Applying this theorem to our expression for $g(t)$ by setting the "time" to $t - b$ and the "interval length" to $b - a$, we obtain:

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} [\, m((t - b) + (b - a)) - m(t - b) \,] = \frac{b - a}{\mu}$$

We observe that the numerator, $b - a$, is simply the integral of the function $h(t)$:

$$\int_0^\infty h(s)\, ds = \int_a^b 1\, ds = b - a$$

Therefore, for $h(t) = \mathbf{1}_{[a,b]}(t)$, we have established the result:

$$\lim_{t\to\infty} g(t) = \frac{1}{\mu} \int_0^\infty h(s)\, ds$$

Theorem 5.1 (Smith Key Renewal Theorem)**.** *[5] Assume that F is non-lattice. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing integrable function. Then*

$$\lim_{t\to\infty} (h * m)(t) \;=\; \frac{1}{\mu}\int_0^\infty h(u)\,du.$$

Proof. We present the proof in two steps. First, we show the theorem for the simple function

$$h(t) = \sum_{k\geq 0} c_k \mathbf{1}_{\left[\frac{k}{n},\frac{k+1}{n}\right)}(t), \qquad c_k \geq 0, \quad \sum_{k\geq 0} c_k < \infty,$$

and then we extend the result to general non-increasing integrable functions by approximation. Let $n \geq 1$ such that $F(1/n) < 1$. By Fubini Theorem, for every $t \geq 0$, we have

$$\begin{aligned}
\int_0^t h(t-s)\,dm(s) &= \int_0^t \sum_{k\geq 0} c_k \mathbf{1}_{\left[\frac{k}{n},\frac{k+1}{n}\right)}(t-s)\,dm(s) \\
&= \sum_{k\geq 0} c_k \int_0^t \mathbf{1}_{\left[\frac{k}{n},\frac{k+1}{n}\right)}(t-s)\,dm(s) \\
&= \sum_{k\geq 0} c_k \int_{t-\frac{k+1}{n}}^{t-\frac{k}{n}} dm(s) \\
&= \sum_{k\geq 0} c_k \left[m\big(t-\tfrac{k}{n}\big) - m\big(t-\tfrac{k+1}{n}\big)\right]. \qquad (13)
\end{aligned}$$

Let $h_k(t) := c_k\left[m\left(t-\frac{k}{n}\right) - m\left(t-\frac{k+1}{n}\right)\right]$. By Proposition 4.6 we have $m = F + m * F$. Hence, for all $t \geq 1$, we have

$$\begin{aligned}
1 \geq F(t) &= m(t) - \int_0^t F(t-s)dm(s) \\
&= \int_0^t dm(s) - \int_0^t F(t-s)dm(s) \\
&= \int_0^t (1-F(t-s))dm(s) \\
&\geq \int_{t-\frac{1}{n}}^t (1-F(t-s))dm(s) \\
&\geq \int_{t-\frac{1}{n}}^t 1 - F(\frac{1}{n})dm(s) \\
&= \left(1-F(\frac{1}{n})\right)\left(m(t) - m(t-\frac{1}{n})\right).
\end{aligned}$$

We used that $F(t-s) \leq F(\frac{1}{n})$ since for $s \in \left[t-\frac{1}{n}, t\right]$, we have $t-s \in \left[0,\frac{1}{n}\right]$ and F is non-decreasing. Thus for every $t \geq 1$ and $k \geq 0$,

$$h_k(t) \leq \frac{c_k}{1-F(\frac{1}{n})}.$$

Using dominated convergence and Blackwell

$$\lim_{t\to\infty}\sum_{k\geq 0}h_k(t)=\sum_{k\geq 0}\lim_{t\to\infty}h_k(t)=\sum_{k\geq 0}c_k\frac{1}{n\mu}$$

we can conclude that

$$\lim_{t\to\infty}\int_0^t h(t-s)dm(s)=\sum_{k\geq 0}\frac{c_k}{n\mu}=\frac{1}{\mu}\int_0^\infty h(u)du.$$

For the second part of the proof let $h\geq 0$ be a non-increasing, integrable function. We approximate h from above and below by two simple functions $\underline{h}_n$ and $\overline{h}_n$ defined as:

$$\underline{h}_n(t)=h\left(\frac{\lfloor nt\rfloor+1}{n}\right)\quad\text{and}\quad\bar{h}_n(t)=h\left(\frac{\lfloor nt\rfloor}{n}\right).$$

Both $\underline{h}_n$ and $\overline{h}_n$ are simple functions of the form considered in the first part of the proof, so the renewal limit theorem already applies to them. Therefore,

$$\lim_{t\to\infty}(\underline{h}_n*m)(t)=\int_0^\infty\underline{h}_n(s)ds,\quad\lim_{t\to\infty}(\bar{h}_n*m)(t)=\int_0^\infty\bar{h}_n(s)ds.$$

For $n\geq 1$, the convolution is monotone with respect to h, so we have

$$(\underline{h}_n*m)(t)\leq(h*m)(t)\leq(\bar{h}_n*m)(t),\quad\forall t\geq 0.$$

Taking limits as $t\to\infty$, we obtain

$$\int_0^\infty\underline{h}_n(s)ds\leq\liminf_{t\to\infty}(h*m)(t)\leq\limsup_{t\to\infty}(h*m)(t)\leq\int_0^\infty\bar{h}_n(s)ds.$$

Because h is integrable and non-increasing, the step functions $\underline{h_n}$ and $\overline{h_n}$ converge to h pointwise and are uniformly dominated by $h(0)$. Moreover, the bounds $\overline{h}_n(t)\leq h(t-\frac{1}{n})\leq h(t-1)$ for large t ensure domination by an integrable function.
Hence by the dominated convergence theorem, both integral converge:

$$\lim_{n\to\infty}\int_0^\infty\underline{h}_n(s)ds=\int_0^\infty h(s)ds,\quad\lim_{n\to\infty}\int_0^\infty\overline{h}_n(s)ds=\int_0^\infty h(s)ds$$

By taking the limit as $n\to\infty$ in the inequality chain, the Squeeze Theorem implies:

$$\frac{1}{\mu}\int_0^\infty h(s)ds\leq\liminf_{t\to\infty}(h*m)(t)\leq\limsup_{t\to\infty}(h*m)(t)\leq\frac{1}{\mu}\int_0^\infty h(s)ds$$

Since the lower and upper limits are identical, the limit must exist and be equal to this value. By using the first part of the proof, we finally have

$$\lim_{t\to\infty}(h*m)(t)=\frac{1}{\mu}\int_0^\infty h(s)ds.$$

□

Example 5.2 (Repair of a machine)**.** *Let $(U_i,V_i)_{i\geq 1}$ be i.i.d. with $U_i,V_i\geq 0$, and set*

$T_i := U_i + V_i$. Assume $\mathbb{E}[U_1] + \mathbb{E}[V_1] < \infty$ and that the distribution F of T_1 is non-arithmetic. Start the system in operation at $t = 0$, and define

$$Y_t := \mathbf{1}\{\textit{machine operational at } t\}, \qquad g(t) := \mathbb{P}(Y_t = 1), \qquad h(t) := \mathbb{P}(U_1 > t).$$

Assume that for $t \geq 0$,

$$g(t) = h(t) + \int_0^t g(t-s)\, dF(s),$$

then we show

$$\lim_{t\to\infty} g(t) = \frac{\mathbb{E}[U_1]}{\mathbb{E}[U_1] + \mathbb{E}[V_1]}.$$

Proof. The function h is non-increasing and $h \geq 0$ with

$$\int_0^\infty h(t)\, dt = \int_0^\infty \mathbb{P}(U_1 > t)\, dt = \mathbb{E}[U_1] < \infty,$$

hence h is directly Riemann integrable. Since F is non-lattice and g is a solution of the equation $g = h + g * F$, we know by Smith's key renewal theorem that

$$\lim_{t\to\infty} g(t) = \frac{1}{\mathbb{E}[T_1]} \int_0^\infty h(s)\, ds = \frac{1}{\mathbb{E}[U_1] + \mathbb{E}[V_1]} \int_0^\infty h(s)\, ds = \frac{\mathbb{E}[U_1]}{\mathbb{E}[U_1] + \mathbb{E}[V_1]}.$$

□

6 Ruin Theory

The study of ruin probability is a central topic in classical risk theory. It seeks to quantify the likelihood that an insurer's reserve becomes negative due to random claim occurrences. This problem was first formalized in the early twentieth century by the Swedish actuary Filip Lundberg. He proposed a stochastic model to describe the evolution of an insurance company's surplus over time. Later, Harald Cramér extended Lundberg's ideas, giving rise to what is now known as the Cramér–Lundberg model.

In this section, we will briefly discuss Lundberg's model as a classical application of renewal processes in risk theory. The model describes the evolution of an insurer's surplus as premiums accumulate over time while claims arrive randomly according to a renewal process. Within this framework, quantities such as the probability of ruin can be analyzed using renewal-theoretic methods, illustrating the deep connection between stochastic process theory and actuarial risk analysis.

Definition 6.1. *Let $(W_i)_{i\geq 1}$ be i.i.d random variables with values in $\mathbb{R}_+$ representing the claims occuring at random inter-arrival times T_i. Let u be the initial capital and c the constant premium over time. Then we define the following surplus process:*

$$U(t) = u + ct - \sum_{i=1}^{N_t} W_i$$

The main quantity of interest in this section is the ruin probability. That is the probability that the reserve becomes negative at some time:

Proposition 6.2. *[4] The ruin probability is given by:*

$$\Psi(u) = \mathbb{P}(\exists t \geq 0 : U(t) < 0 | U(0) = u) = \mathbb{P}\left(\sup_{n\geq 0} \sum_{i=1}^{n} (W_i - cT_i) > u\right)$$

Proof. Between claims, $U(t)$ increases linearly at rate c. Since no claims are paid. The only time $U(t)$ decreases are the claims arrival time T_i. Hence, ruin can only occur at an arrival time. Thus,

$$\{\exists t \geq 0 : U(t) < 0\} \iff \{\exists n \geq 0 : U(S_n) < 0\}$$

$$\begin{aligned} U(S_n) &= u + cS_n - \sum_{i=1}^{N_{S_n}} W_i = u + c\sum_{i=1}^{n} T_i - \sum_{i=1}^{N_{S_n}} W_i \\ &= u + c\sum_{i=1}^{n} T_i - \sum_{i=1}^{n} W_i = u - \sum_{i=1}^{n} (W_i - cT_i). \end{aligned}$$

$$u - \sum_{i=1}^{n} (W_i - cT_i) < 0 \iff u < \sum_{i=1}^{n} (W_i - cT_i)$$

We can conclude that:

$$\Psi(u) = \mathbb{P}\left(\exists n \geq 0 : \sum_{i=1}^{n} (W_i - cT_i) > u\right) = \mathbb{P}\left(\sup_{n\geq 0} \sum_{i=1}^{n} (W_i - cT_i) > u\right).$$

□

To determine the practical conditions under which a company can actually survive, in the sense that the ruin probability is strictly less than one, we must first analyze the asymptotic drift of the surplus process over time.

Proposition 6.3. *[4] Let $\{U(t)\}_{t\geq 0}$ be the surplus process, and assume $\mathbb{E}[W_1] < \infty$ and $\mathbb{E}[T_1] = \mu < \infty$. Then the asymptotic drift satisfies*

$$\frac{U(t)}{t} \xrightarrow[t\to\infty]{} c - \frac{\mathbb{E}[W_1]}{\mu} \quad \textit{almost surely.}$$

Consequently, if the net profit condition is violated (i.e. $\rho = \frac{c\mu}{\mathbb{E}[W_1]} - 1 \leq 0$), then ruin occurs with probability one for any initial capital u.

Proof. We write

$$\frac{U(t)}{t} = \frac{u + ct - \sum_{i=1}^{N_t} W_i}{t} = \frac{u}{t} + c - \frac{\sum_{i=1}^{N_t}}{t} = \frac{u}{t} + c - \frac{N_t}{t} \cdot \frac{1}{N_t} \sum_{i=1}^{N_t} W_i$$

Since $\{N_t\}_{t\geq 0}$ is a renewal process with inter-arrival mean μ, we have that $N_t \to \infty$ a.s. as $t \to \infty$. In particular, we may assume $N_t \geq 1$ for all sufficiently large t, almost surely, so the expression $\frac{1}{N_t} \sum_{i=1}^{N_t} W_i$ is well-defined a.s. for large t. By the strong law of large

numbers applied along the a.s. diverging random index N_t,

$$\frac{1}{N_t}\sum_{i=1}^{N_t} W_i \xrightarrow{t\to\infty} \mathbb{E}[W_1] \quad a.s$$

By the elementary renewal theorem $\lim_{t\to\infty}\frac{N_t}{t} = \frac{1}{\mu}$ a.s. as $t\to\infty$, and trivially $\frac{u}{t}\to 0$. Combining these three limits gives

$$\lim_{t\to\infty}\frac{U(t)}{t} = c - \frac{1}{\mu}\mathbb{E}[W_1] \quad a.s$$

For the second part of the proposition we distinguish two cases:

Case 1: $\rho < 0$. This is equivalent to $\frac{c\mu}{\mathbb{E}[W_1]} < 1$, i.e $c < \frac{\mathbb{E}[W_1]}{\mu}$. By the first part of the proof, $U(t)/t \to c - \mathbb{E}[W_1]/\mu < 0$ a.s., so $U(t)\to -\infty$ a.s. In particular, for any $u \geq 0$, there exists a.s. a finite time at which $U(t) < 0$, hence $\Psi(u) = 1$.

Case 2: $\rho = 0$. Define the i.i.d. increments $Z_i := W_i - cT_i$. Then $\mathbb{E}[Z_i] = \mathbb{E}[W_1] - c\mu = 0$, where the last equality holds precisely when $\rho = 0$. Moreover, the assumption $\mathbb{E}[W_1] < \infty$ and $\mu < \infty$ immediately gives $\mathbb{E}[|Z_i|] \leq \mathbb{E}[W_i] + c\mathbb{E}[T_i] = \mathbb{E}[W_1] + c\mu < \infty$. Since $W_i > 0$ and $T_i > 0$, the increment Z_i is not a.s. constant. For any i.i.d. sequence with finite mean equal to zero and non-constant increments, $\limsup_{n\to\infty} X_n = +\infty$ a.s., where $X_n = \sum_{i=1}^n Z_i$. In particular $\sup_{n\geq 0} X_n = +\infty > u$ a.s. for every $u \in \mathbb{R}$, and by Proposition 6.2 we conclude $\Psi(u) = 1$. □

6.1 Small Risks

We say that the risk process operates in the small risks regime if there exists a constant $\gamma_0 > 0$ such that the exponential moment of the claim size W_1 is finite, namely $\mathbb{E}[\exp(\gamma_0 W_1)] < \infty$. This integrability condition immediately implies that $\mathbb{E}[\exp(\gamma W_1)] < \infty$ for all $\gamma \leq \gamma_0$, and in particular rules out heavy-tailed claim distributions such as the Pareto distribution studied in Section 6.3.

To analyze the asymptotic behavior of the ruin probability, we introduce the logarithmic moment generating function of the net loss per period. Define the function $\kappa{:}[0,\gamma_0] \to \mathbb{R}\cup\{\infty\}$ by

$$\kappa(\gamma) = \log \mathbb{E}[\exp\left(\gamma(W_1 - cT_1)\right)].$$

To understand why κ is the natural object to study, observe that the ruin event is driven by the partial sums $S_n = \sum_{i=1}^n (W_i - cT_i)$ growing large. By independence of the inter-arrival periods, $\mathbb{E}[\exp(\gamma S_n)] = e^{n\kappa(\gamma)}$, so the sign of $\kappa(\gamma)$ controls whether the exponentially-tilted process $e^{\gamma S_n}$ grows or decays with n. When $\kappa(\gamma) < 0$, the quantity $e^{\gamma S_n}$ decays to zero exponentially fast in n, which suppresses the probability of ruin. We verify that κ is convex: for $\gamma_1, \gamma_2 \in [0,\gamma_0]$ and $\lambda \in [0,1]$, Hölder's inequality gives

$$\mathbb{E}[\exp((\lambda\gamma_1 + (1-\lambda)\gamma_2)(W_1 - cT_1))] \leq \mathbb{E}[\exp(\gamma_1(W_1 - cT_1))]^{\lambda} \cdot \mathbb{E}[\exp(\gamma_2(W_1 - cT_1))]^{1-\lambda},$$

and taking logarithms yields $\kappa(\lambda\gamma_1 + (1-\lambda)\gamma_2) \leq \lambda\kappa(\gamma_1) + (1-\lambda)\kappa(\gamma_2)$. Furthermore, $\kappa(0) = \log \mathbb{E}[1] = 0$. We define the adjustment coefficient, or Lundberg exponent γ^* as the unique positive solution to the equation $\kappa(\gamma) = 0$, assuming such a solution exists.

The adjustment coefficient dictates the exponential decay rate of the ruin probability. The cornerstone of ruin theory under small risks is Lundberg's inequality, which provides an upper bound on the ruin probability $\Psi(u)$ for a given initial capital u.

Proposition 6.4. *(Lundberg's Inequality) [4] Let $\gamma > 0$ be a constant such that $\kappa(\gamma) \leq 0$. Then, the probability of ultimate ruin is bounded by:*

$$\Psi(u) \leq \exp(-\gamma u), \quad \forall u \geq 0$$

Proof. Fix a $\gamma > 0$ such that $\kappa(\gamma) \leq 0$. Define the natural filtration $\mathcal{F}_n = \sigma((W_k, T_k), k = 1, ..., n)$ for $n \geq 1$, and set $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Consider the discrete-time stochastic process $(Z_n)_{n\geq 0}$ defined by $Z_0 = 1$ and, for $n \geq 1$,

$$Z_n = \exp\left(\gamma \sum_{k=1}^{n} (W_k - cT_k)\right).$$

Each Z_n is $\mathcal{F}_n$-measurable by construction. Since $c > 0$ and $T_k \geq 0$, we have $\exp(-\gamma c T_k) \leq 1$, and therefore

$$\mathbb{E}[Z_n] = \prod_{k=1}^{n} \mathbb{E}[\exp(\gamma(W_k - cT_k))] \leq \mathbb{E}[\exp(\gamma W_1)]^n < \infty,$$

where we used independence, identical distribution of (W_k, T_k) and the small risks assumption. We now show that $(Z_n)_{n\geq 0}$ is an $\mathcal{F}_n$-super-martingale. Since the pairs (W_k, T_k) are i.i.d., the increment $W_{n+1} - cT_{n+1}$ is independent of $\mathcal{F}_n$. Therefore,

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = Z_n \mathbb{E}[\exp(\gamma(W_1 - cT_1))] = Z_n \exp(\kappa(\gamma)).$$

Since $\kappa(\gamma) \leq 0$, we have $\exp(\kappa(\gamma)) \leq 1$, and so $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \leq Z_n$. This establishes that (Z_n) is an $\mathcal{F}_n$-super-martingale. Defne the stopping time $\tau_u = \inf\{n \geq 0 : Z_n > \exp(\gamma u)\}$, representing the index of the first claim that causes ruin. The probability of ultimate ruin is $\Psi(u) = \mathbb{P}(\tau_u < \infty)$. For each fixed $n \geq 0$, the stopping time $n \wedge \tau_u$ is bounded and hence the super-martingale property applied at each step gives $\mathbb{E}[Z_{n\wedge\tau_u}] \leq \mathbb{E}[Z_0] = 1$. We decompose this expectation over the events $\{\tau_u > n\}$ and $\{\tau_u \leq n\}$:

$$\mathbb{E}[Z_{n\wedge\tau_u}] = \mathbb{E}[Z_n \mathbf{1}_{\tau_u > n}] + \mathbb{E}[Z_{\tau_u} \mathbf{1}_{\tau_u \leq n}].$$

Since $Z_n > 0$ almost surely, the first term is non-negative. On the event $\{\tau_u \leq n\}$, the definition of τ_u gives $Z_{\tau_u} > \exp(\gamma u)$, so

$$\mathbb{E}[Z_{n\wedge\tau_n}] \geq \exp(\gamma u)\mathbb{P}(\tau_u \leq n).$$

Combining with the super-martingale bound yields $\mathbb{P}(\tau_u \leq n) \leq \exp(-\gamma u)$ for all $n \geq 0$. Finally, since $\{\tau_u \leq n\} \to \{\tau_u < \infty\}$ as $n \to \infty$, the monotone convergence theorem for probabilities gives

$$\Psi(u) = \mathbb{P}(\tau_u < \infty) = \lim_{n\to\infty} \mathbb{P}(\tau_u \leq n) \leq \exp(-\gamma u).$$

□

The super-martingale bound derived above is universally applicable under the small risks condition. However, if we assume the existence of the adjustment coefficient γ^*, we can refine our analysis to obtain both a two-sided bound (for bounded claims) and an exact formula.

Corollary 6.5. *[4] If the claim sizes W_i are deterministically bounded by a constant $b > 0$, then:*

$$\exp\left(-\gamma^*(u+b)\right) \leq \Psi(u) \leq \exp\left(-\gamma^* u\right).$$

Proof. Let us substitute $\gamma = \gamma^*$ into our process (Z_n). Because $\kappa(\gamma^*) = 0$, the process $Z_n = \exp(\gamma^* \sum_{i=1}^n (W_i - cT_i))$ is now an exact $\mathcal{F}_n$-martingale. Applying the optional stopping theorem to the bounded stopping time $n \wedge \tau_u$ gives exactly $1 = \mathbb{E}[Z_{n\wedge\tau_u}] = \mathbb{E}[Z_n \mathbf{1}_{\tau_u > n}] + \mathbb{E}[Z_{\tau_u} \mathbf{1}_{\tau_u \leq n}]$. Because the claims are bounded by b, the "overshoot" of the process past the ruin threshold u is constrained. Consequently, $Z_{\tau_u} < \exp(\gamma^*(u+b))$ when $\tau_u < \infty$. Expanding the expectation:

$$1 \leq \mathbb{P}(\tau_u < n) \exp(\gamma^*(u+b)) + \mathbb{E}[\mathbf{1}_{\tau_u > n} Z_n]$$

Rearranging this algebraic relation provides $1 - \mathbb{E}[\mathbf{1}_{\tau_u > n} Z_n] \leq \Psi(u) \exp(\gamma^*(u+b))$ for all $n \geq 1$. To conclude the proof, we must show that $\lim_{n\to\infty} \mathbb{E}[\mathbf{1}_{\tau_u > n} Z_n] = 0$. Assuming a positive safety loading, $\rho > 0$, implies $\mathbb{E}[W_1 - cT_1] < 0$. By the Strong Law of Large Numbers, the sum $\sum_{i=1}^n (W_i - cT_i)$ diverges to $-\infty$ almost surely. By the continuity of the exponential function $\lim_{n\to\infty} Z_n = 0$ almost surely. Furthermore, on the set $\{\tau_u > n\}$, ruin has not yet occurred, meaning $Z_n \leq \exp(\gamma^* u) \leq \exp(\gamma^*(u+b))$. This deterministic constant serves as an integrable dominating function. By Lebesgue's Dominated Convergence Theorem, we may pass the limit inside the integral

$$\lim_{n\to\infty} \mathbb{E}[Z_n \mathbf{1}_{\tau_u > n}] = \mathbb{E}\left[\lim_{n\to\infty} Z_n \mathbf{1}_{\tau_u > n}\right] = \mathbb{E}[0] = 0.$$

□

Even without the assumption of bounded jumps, the martingale technique reveals an elegant exact closed-form expression for the ruin probability.

Corollary 6.6. *[4] The exact probability of ruin is given by:*

$$\Psi(u) = \frac{\exp(-\gamma^* u)}{\mathbb{E}\left[\exp\left(\gamma^* \sum_{i=1}^{\tau_u} (W_i - cT_i) - \gamma^* u\right) \mid \tau_u < \infty\right]}$$

Proof. Retaining $\gamma = \gamma^*$ and the martingale decomposition $1 = \mathbb{E}[Z_{\tau_u} \mathbf{1}_{\tau_u \leq n}] + \mathbb{E}[Z_n \mathbf{1}_{\tau_u > n}]$, we apply the limits as $n \to \infty$. By the Dominated convergence Theorem as applied in the proof of corollary (6.5), $\lim_{n\to\infty} \mathbb{E}[Z_n \mathbf{1}_{\tau_u > n}] = 0$. For the first term, let us define the sequence of random variables $Y_n = Z_{\tau_u} \mathbf{1}_{\tau_u \leq n}$. Since Z_{τ_u} is strictly positive, and the indicator functions satisfy $\mathbf{1}_{\tau_u \leq n} \leq \mathbf{1}_{\tau_u \leq n+1}$ pointwise, the sequence $(Y_n)_{n\geq 1}$ is non-negative and monotonically increasing. Furthermore, $\lim_{n\to\infty} \mathbf{1}_{\tau_u \leq n} = \mathbf{1}_{\tau_u < \infty}$. We apply Lebesgue's Monotone Convergence Theorem to interchange the limit and the expectation:

$$\lim_{n\to\infty} \mathbb{E}[Z_{\tau_u} \mathbf{1}_{\tau_u \leq n}] = \mathbb{E}\left[\lim_{n\to\infty} Z_{\tau_u} \mathbf{1}_{\tau_u \leq n}\right] = \mathbb{E}\left[Z_{\tau_u} \mathbf{1}_{\tau_u < \infty}\right]$$

Taking the limit of the entire martingale decomposition equation, we find $1 = \mathbb{E}[Z_{\tau_u} \mathbf{1}_{\tau_u < \infty}]$. we rewrite this absolute expectation as a conditional expectation multiplied by the prob-

ability of the conditioning event:

$$\mathbb{E}[Z_{\tau_u}\mathbf{1}_{\tau_u<\infty}] = \mathbb{E}[Z_{\tau_u} \mid \tau_u < \infty]\Psi(u).$$

Since this overall expectation equals 1, we isolate $\Psi(u)$ to find:

$$\Psi(u) = \frac{1}{\mathbb{E}[Z_{\tau_u} \mid \tau_u < \infty]}$$

Expanding the definition of Z_{τ_u} directly yields the stated result. □

6.2 The Poisson Case

In this section, we restrict our attention to the classical Cramér-Lundberg model, where the claim arrival process $(N_t)_{t\geq 0}$ is strictly a homogeneous Poisson process with constant intensity $\lambda > 0$.

Crucially, in the first part of this section, we do not require the "small risks" assumption. This means that the claims do not need to possess finite exponential moments. Because we do not need the small risks assumption for the first part, we can use these results later in the large claims section. We only require that the claims $(W_i)_{i\geq 1}$ are independent, identically distributed, strictly positive random variables with a finite first moment $\mathbb{E}[W_1] < \infty$ and a cumulative distribution function $H(x) = \mathbb{P}(W_1 \leq x)$. We denote the ruin probability by $\Psi(u) = \mathbb{P}\left(\sup_{n\geq 0}\left(\sum_{i=1}^{n} W_i - cT_i\right) > u\right)$, and the survival probability by $\bar{\Psi}(u) = 1-\Psi(u) = \mathbb{P}\left(\sup_{n\geq 0}\left(\sum_{i=1}^{n} W_i - cT_i\right) \leq u\right)$. We assume the net profit condition holds, meaning the safety loading $\rho = \frac{c}{\lambda\mathbb{E}[W_1]} - 1$ is strictly positive i.e. $\rho > 0$.

Theorem 6.7. *[4] The survival probability $\bar{\Psi}(u)$ satisfies the integral equation:*

$$\bar{\Psi}(u) = \bar{\Psi}(0) + \frac{\lambda}{c}\int_0^u (1 - H(y))\bar{\Psi}(u-y)dy$$

Proof. Taking into account that $S_n := \sum_{i=1}^{n} W_i - cT_i$ and $(T_i)_{i\geq 1}$ are i.i.d random variables, one has

$$\begin{aligned}
\bar{\Psi}(u) &= \mathbb{P}\left(\sup_{n\geq 0}\sum_{i=1}^{n} W_i - cT_i \leq u\right) = \mathbb{P}\left(W_1 - cT_1 \leq u, \sup_{n\geq 2}\sum_{i=1}^{n} W_i - cT_i \leq u\right) \\
&= \mathbb{P}\left(W_1 - cT_1 \leq u, \sup_{n\geq 2}\sum_{i=2}^{n} W_i - cT_i \leq u - (W_1 - cT_1)\right) \\
&= \mathbb{E}[\mathbf{1}_{W_1 - cT_1 \leq u}\bar{\Psi}(u - W_1 + cT_1)].
\end{aligned}$$

Since $T_1 \sim Exp(\lambda)$ is independent of $W_1 \sim f_W$, writing out the expectation with the product density $\lambda e^{-\lambda t} f_W(y)$ gives

$$\bar{\Psi}(u) = \int_0^\infty \lambda\exp(-\lambda t)\left(\int_0^{u+ct} \bar{\Psi}(u + ct - y)dH(y)\right)dt$$

To solve this, we apply the change of variables $z = u + ct$, which implies $dt = \frac{dz}{c}$. The

domain of integration for $t \in [0, \infty)$ maps to $z \in [u, \infty)$. Substituting this yields:

$$\bar{\Psi}(u) = \frac{\lambda}{c} \exp\left(\frac{\lambda}{c} u\right) \int_u^\infty \exp\left(-\frac{\lambda}{c} z\right) \left(\int_0^z \bar{\Psi}(z-y) dH(y)\right) dz$$

Because $\bar{\Psi}$ is continuous and bounded, we can apply the Fundamental Theorem of Calculus by differentiating both sides with respect to u. Using the product rule and Leibniz's rule:

$$\bar{\Psi}'(u) = \frac{\lambda}{c} \bar{\Psi}(u) - \frac{\lambda}{c} \int_0^u \bar{\Psi}(u-y) dH(y).$$

We integrate this derivative over the interval $[0, u]$:

$$\bar{\Psi}(u) - \bar{\Psi}(0) = \frac{\lambda}{c} \int_0^u \bar{\Psi}(t) dt - \frac{\lambda}{c} \int_0^u \left(\int_0^t \bar{\Psi}(t-y) dH(y)\right) dt.$$

By Fubini's Theorem, we exchange the order of integration in the double integral. Integrating dt first, and recognizing that $\int_0^u \bar{\Psi}(u-z) H(z) dz$ emerges from the integration by parts of the cumulative distribution, we arrive at the simplified expression:

$$\bar{\Psi}(u) - \bar{\Psi}(0) = \frac{\lambda}{c} \int_0^u (1 - H(y)) \bar{\Psi}(u-y) dy$$

which concludes the proof. □

The integral equation obtained in Theorem 6.7 is already close to a renewal equation, but it is not yet in the standard form required to apply the tools developed in Chapter 4. To bridge this gap, we reframe the equation in the language of renewal theory. The key observation is that the kernel $\frac{\lambda}{c}(1 - H(y))$ appearing in the convolution integral is, up to a normalizing constant, the density of a probability distribution. This motivates introducing the integrated tail distribution of the claim sizes:

$$I(t) = \frac{1}{\mathbb{E}[W_1]} \int_0^t \mathbb{P}(W_1 > y) dy = \frac{1}{\mathbb{E}[W_1]} \int_0^t (1 - H(y)) dy$$

That I is indeed a valid CDF follows from the identity $\int_0^\infty (1 - H(y)) dy = \mathbb{E}[W_1] < \infty$.

Corollary 6.8. *[4] The ruin probability $\Psi(u)$ satisfies the defective renewal equation:*

$$\Psi(u) = \frac{1 - I(u)}{1 + \rho} + \frac{1}{1 + \rho} \Psi * I(u),$$

and the probability of ruin with zero inital capital is $\Psi(0) = \frac{\lambda \mathbb{E}[W_1]}{c}$.

Proof. We evaluate the limit of the integral equation from Theorem (6.7) as $u \to \infty$. Since eventual survival is guaranteed for an infinitely large initial capital under the positive safety loading assumption, $\lim_{u \to \infty} \bar{\Psi}(u) = 1$. We must evaluate $\lim_{u \to \infty} \int_0^u (1 - H(y)) \bar{\Psi}(u-y) dy$. Let us extend the integrand to $[0, \infty)$ by writing

$$\int_0^u (1 - H(y)) \bar{\Psi}(u-y) dy = \int_0^\infty (1 - H(y)) \bar{\Psi}(u-y) \mathbf{1}_{y \leq u} dy.$$

Define $f_u(y) := (1 - H(y)) \bar{\Psi}(u-y) \mathbf{1}_{y \leq u}$. This sequence is bounded by the dominating

function $g(y) = 1 - H(y)$, which satisfies

$$\int_0^\infty g(y)dy = \int_0^\infty \mathbb{P}(W_1 > y)dy = \mathbb{E}[W_1] < \infty,$$

and therefore Lebesgue integrable. For each fixed $y \geq 0$, as $u \to \infty$ we have $\mathbf{1}_{y \leq u} \to 1$ and $\bar{\Psi}(u-y) \to 1$, so $f_u(y) \to 1 - H(y)$ pointwise. By Lebesgue's Dominated Convergence Theorem we may pass the limit inside the integral:

$$\lim_{u\to\infty} \int_0^\infty f_u(y)dy = \int_0^\infty (1 - H(y))dy = \mathbb{E}[W_1].$$

Taking $u \to \infty$ in Theorem 6.7 now gives $1 = \bar{\Psi}(0) + \frac{\lambda}{c}\mathbb{E}[W_1]$, and rearranging yields $\Psi(0) = 1 - \bar{\Psi}(0) = \frac{\lambda \mathbb{E}[W_1]}{c}$. We now derive the renewal equation for $\Psi(u)$. Substituting $\bar{\Psi}(u) = 1 - \Psi(u)$ into Theorem 6.7:

$$1 - \Psi(u) = 1 - \Psi(0) + \frac{\lambda}{c}\int_0^u (1 - H(y))(1 - \Psi(u-y))dy.$$

Rearranging and substituting $\Psi(0) = \frac{\lambda \mathbb{E}[W_1]}{c}$:

$$-\Psi(u) = -\frac{\lambda \mathbb{E}[W_1]}{c} + \frac{\lambda}{c}\int_0^u (1 - H(y))dy - \frac{\lambda}{c}\int_0^u (1 - H(y))\Psi(u-y)dy.$$

Multiplying by -1 and recalling the integrated tail distribution $I(u) = \frac{1}{\mathbb{E}[W_1]}\int_0^u (1 - H(y))dy$, so that $\frac{\lambda}{c}\int_0^u (1 - H(y))dy = \frac{\lambda \mathbb{E}[W_1]}{c} I(u)$:

$$\Psi(u) = \frac{\lambda \mathbb{E}[W_1]}{c}(1 - I(u)) + \frac{\lambda \mathbb{E}[W_1]}{c}\int_0^u \Psi(u-y)dI(y).$$

Since $\rho = \frac{c}{\lambda \mathbb{E}[W_1]} - 1$ implies $\frac{\lambda \mathbb{E}[W_1]}{c} = \frac{1}{1+\rho}$, we obtain the defective renewal equation. □

Corollary 6.8 expresses $\Psi(u)$ as a solution to a defective renewal equation, meaning the measure $dF = dI/(1+\rho)$ has total mass $1/(1+\rho) < 1$. This defectiveness is precisely what allows $\Psi(u) \to 0$ as $u \to \infty$, but it also prevents a direct application of Smith's Key Renewal Theorem, which requires a proper probability measure. To recover a proper renewal equation, we exploit the existence of the adjustment coefficient γ^*, which was estabished in Section 6.1 as the unique root of $\kappa(\gamma) = 0$. The idea is to perform a "exponential change of measure": defining $g(u) = \exp(\gamma^* u)\Psi(u)$ effectively tilts the defective measure $dI/(1+\rho)$ by the expoential weight $\exp(\gamma^* y)$, producing a new measure

$$dJ(y) = \frac{\exp(\gamma^* y)dI(y)}{1+\rho}$$

The condition $\kappa(\gamma^*) = 0$ is precisely what guarantees that J integrates to 1. Hence turning the defective equation into a proper one.

Theorem 6.9. *[4] If the adjustment coefficient γ^* exists, then:*

$$\lim_{u\to\infty} \exp(\gamma^* u)\Psi(u) = \frac{\rho \mathbb{E}[W_1]}{\gamma^* \int_0^\infty x \exp(\gamma^* x)\mathbb{P}(W_1 > x)dx}.$$

Proof. Define $g(u) = \exp(\gamma^* u)\Psi(u)$ and $h(u) = \exp(\gamma^* u)\frac{1-I(u)}{1+\rho}$. Multiplying the defective renewal equation by $\exp(\gamma^* u)$ transforms it into

$$g(u) = h(u) + \int_0^u g(u-y)dJ(y)$$

where the new measure is defined as

$$dJ(y) = \frac{\exp(\gamma^* y)dI(y)}{1+\rho} = \frac{\exp(\gamma^* y)\mathbb{P}(W_1 > y)}{(1+\rho)\mathbb{E}[W_1]}dy.$$

For this to be a proper renewal equation, J must be a valid probability measure i.e. $\int_0^\infty dJ(y) = 1$. We evaluate this via integration by parts:

$$\int_0^\infty dJ(y) = \frac{1}{(1+\rho)\mathbb{E}[W_1]}\left(\frac{\exp(\gamma^* y)}{\gamma^*}\mathbb{P}(W_1 > y)|_0^\infty + \int_0^\infty \frac{\exp(\gamma^* y)}{\gamma^*} f_{W_1}(y)dy\right)$$

Taking into account that for $0 < \gamma^* < \delta$, we obtain $\mathbb{P}(W_1 > y) \leq \mathbb{E}[\exp(\delta W_1)]\exp(-\delta y) \to 0$ as $y \to \infty$. Thus, the upper boundary term vanishes, leaving

$$\int_0^\infty dJ(y) = \frac{1}{(1+\rho)\mathbb{E}[W_1]}\left(-\frac{1}{\gamma^*} + \frac{\mathbb{E}[\exp(\gamma^* W_1)]}{\gamma^*}\right) \tag{14}$$

Recall the adjustment coefficient satisfies $\lambda(\mathbb{E}[\exp(\gamma^* W_1)] - 1) - c\gamma^* = 0$. Substituting the safety loading $\rho = \frac{c}{\lambda\mathbb{E}[W_1]} - 1$ gives the identity $\mathbb{E}[\exp(\gamma^* W_1)] - 1 = \gamma^*(1+\rho)\mathbb{E}[W_1]$. Plugging this into the equation (14) yields exactly 1. Thus, (h, J) is a proper renewal equation. Because $h(u)$ is bounded by a decaying exponential function, it is directly Riemann integrable. Furthermore, since J is a proper, non-lattice probability distribution with a finite mean, the conditions for Smith's Key Renewal Theorem are satisfied. Applying the theorem yields:

$$\lim_{u\to\infty} g(u) = \frac{\int_0^\infty h(t)dt}{\int_0^\infty x dJ(x)}.$$

Expanding the denominator gives $\int_0^\infty x dJ(x) = \frac{1}{(1+\rho)\mathbb{E}[W_1]}\int_0^\infty x\exp(\gamma^* x)\mathbb{P}(W_1 > x)dx$. To evaluate the numerator $\int_0^\infty h(u)du = \frac{1}{1+\rho}\int_0^\infty \exp(\gamma^* u)(1-I(u))du$, we apply Fubini's Theorem to exchange the integrals:

$$\int_0^\infty \exp(\gamma^* u)(1-I(u))du = \frac{1}{\mathbb{E}[W_1]}\int_0^\infty \mathbb{P}(W_1 > t)\left(\int_0^t \exp(\gamma^* u)du\right)dt.$$

Evaluating the inner integral yields $\frac{\exp(\gamma^* t)-1}{\gamma^*}$. Expanding this expected value and utilizing the fact that $\kappa(\gamma^*) = 0$ simplifies the expression entirely to $\frac{\rho\mathbb{E}[W_1]}{\gamma^*}$. Substituting the evaluated numerator and denominator into Smith's theorem result yields the final limit

$$\lim_{u\to\infty}\exp(\gamma^* u)\Psi(u) = \frac{\rho\mathbb{E}[W_1]}{\gamma^*\int_0^\infty x\exp(\gamma^* x)\mathbb{P}(W_1 > x)dx}.$$

□

6.3 Large Claims

Up to this point, our analysis of ruin probabilities has relied fundamentally on the standard or light-tailed claim size distributions. We now relax the moment assumptions on the claim sizes while retaining the Poisson arrival structure of Section 6.2, in order to investigate the impact of large claims on the surplus process. This chapter explores models where claim sizes exhibit heavy tails, meaning the probability of extreme losses decays slower than exponentially. We will examine the mathematical properties of subexponential distributions and establish how the asymptotic behavior of the ruin probability $\Psi(u)$ fundamentally changes when a portfolio is exposed to these catastrophic risks. In particular, the defective renewal equation derived in Corollary 6.8 under the Poisson assumption remains the key analytic tool, and the results of this section should be understood within that setting.

Definition 6.10. *[2] We say that a random variable W is subexponential if i.i.d random variables $(W_i)_{i\geq 1}$ having the same distribution as W for all $n \geq 1$ satisfy the following condition*

$$\lim_{z\to\infty} \frac{\mathbb{P}\left(\sum_{j=1}^n W_j > z\right)}{\mathbb{P}(W_1 > z)} = n \tag{15}$$

The condition (15) says that the probability of the sum exceeding a large threshold z is asymptotically n times the probability that a single variable exceeds z. Intuitively, the event $\{\sum_{j=1}^n W_j > z\}$ is driven by exactly one of the n terms being anomalously large while the others remain of typical size.

Example 6.11. *[2] Let W be a positive random variable such that for any $z \geq 0$*

$$\mathbb{P}(W > z) = \frac{1}{(1+z)^\alpha} \quad \text{and } \alpha > 0.$$

This is the Pareto distribution, whose tail decays polynomially rather than exponentially. In particular $\mathbb{E}[\exp(\gamma W)] = \infty$ for all $\gamma > 0$. Let us show by induction that W satisfies the condition (15). The base case $n = 1$ is trivial. Assuming that the property (15) holds for $n-1$, we will check this condition for n. To this end we set $\bar{F}_n(z) = \mathbb{P}(S_n > z)$, where $S_n = \sum_{j=1}^n W_j$. We have

$$\begin{aligned}
\bar{F}_n(z) &= \mathbb{P}\left(\sum_{j=1}^{n-1} W_j > z - W_n\right) \\
&= \mathbb{E}\left[\bar{F}_{n-1}(z - W_n)\mathbf{1}_{W_n \leq z}\right] + \mathbb{P}(W_n > z) \\
&= \int_0^z \bar{F}_{n-1}(z-t)dF(t) + \bar{F}(z),
\end{aligned}$$

where F is the distribution function of W and $\bar{F}(z) = \bar{F}_1(z)$. So, we obtain that

$$\frac{\bar{F}_n(z)}{\bar{F}(z)} = 1 + \int_0^z \frac{\bar{F}_{n-1}(z-t)}{\bar{F}(z)} dF(t). \tag{16}$$

Fix $0 < r < 1$ and split the integral into

$$I_{1,r}(z) = \int_0^{rz} \frac{\bar{F}_{n-1}(z-t)}{\bar{F}(z)} dF(t), \qquad I_{2,r}(z) = \int_{rz}^z \frac{\bar{F}_{n-1}(z-t)}{\bar{F}(z)} dF(t).$$

For $I_{1,r}$: since $\mathbb{P}(W > z) = (1+z)^{-\alpha}$, one computes for any fixed $t \geq 0$,

$$\frac{\bar{F}(z-t)}{\bar{F}(z)} = \left(\frac{1+z}{1+z-t}\right)^{\alpha} \to 1 \quad as \quad z \to \infty,$$

and therefore by the induction hypothesis $\bar{F}_{n-1}(z-t)/\bar{F}(z) \to n-1$ for each fixed t. For $t \leq rz$ the uniform bound

$$\frac{\bar{F}_{n-1}(z-t)}{\bar{F}(z)} \leq \frac{\bar{F}_{n-1}((1-r)z)}{\bar{F}(z)} \leq \frac{n-1}{(1-r)^{\alpha}}$$

holds for all large z by the induction hypothesis, providing the dominating function needed to apply the dominated convergence theorem. We conclude $\lim_{z\to\infty} I_{1,r}(z) = n-1$ for each fixed $r \in (0,1)$.

For $I_{2,r}(z)$: for any $z > 0$,

$$I_{2,r}(z) \leq \frac{F(z) - F(rz)}{\bar{F}(z)} = \left(\frac{1}{(1+rz)^{\alpha}} - \frac{1}{(1+z)^{\alpha}}\right)(1+z)^{\alpha}.$$

This expression converges to $r^{-\alpha}-1$ as $z \to \infty$. Taking $r \to 1$ gives $\limsup_{r\to 1} \limsup_{z\to\infty} I_{2,r}(z) = 0$. Combining both estimates and taking $r \to 1$ in 16 yields $\bar{F}_n(z)/\bar{F}(z) \to n$, completing the induction.

The inductive proof technique used here, splitting the integral at rz and handling the two parts separately, is the same strategy used in the proof of the next lemma. That result is not just a generalization of the example. It is a technical lemma that will be essential to justify the interchange of limit and series in the proof of the main theorem.

Lemma 6.12. *[2] Let W be a subexponential random variable. Then for any $\epsilon > 0$ it exists $K = K(\epsilon) > 0$ such that for any $n \geq 1$,*

$$\sup_{z\geq 0} \frac{\bar{F}_n(z)}{\bar{F}(z)} \leq K(1+\epsilon)^n, \tag{17}$$

where $\bar{F}_n(z) = \mathbb{P}(S_n > z)$, $\bar{F}(z) = \bar{F}_1(z)$, $S_n = \sum_{j=1}^n W_j$ and $(W_j)_{j\geq 1}$ are i.i.d. random variables of the same distribution as W.

Proof. Define $\alpha_n = \sup_{z\geq 0} \frac{\bar{F}_n(z)}{\bar{F}(z)}$. We establish a recursive bound on α_n. Writing $S_{n+1} = S_n + W_{n+1}$ and conditioning on W_{n+1},

$$\begin{aligned}
\frac{\bar{F}_{n+1}(z)}{\bar{F}(z)} &= 1 + \frac{\int_0^z \mathbb{P}(S_n > z-t)dF(t)}{\bar{F}(z)} \\
&\leq 1 + \alpha_n \frac{\mathbb{P}(W_1 + W_2 > z, W_2 \leq z)}{\bar{F}(z)} \\
&= 1 + \alpha_n \left(\frac{\mathbb{P}(S_2 > z)}{\bar{F}(z)} - 1\right).
\end{aligned}$$

Since W is subexponential, $\mathbb{P}(S_2 > z)/\bar{F}(z) \to 2$ as $z \to \infty$. Fix $\epsilon > 0$. There exists $T = T(\epsilon) > 0$ such that

$$\sup_{z\geq T} \left(\frac{\mathbb{P}(S_2 > z)}{\bar{F}(z)} - 1\right) \leq (1+\epsilon).$$

Therefore,

$$\alpha_{n+1} \leq \sup_{0\leq z\leq T} \frac{\bar{F}_n(z)}{\bar{F}(z)} + \sup_{z\geq T} \frac{\bar{F}_n(z)}{\bar{F}(z)} \leq \alpha_n(1+\epsilon) + K_0,$$

where $K_0 = 1+1/\mathbb{P}(W > T)$. This inequality means that for $n \geq 1$, $\alpha_{n+1} = (1+\epsilon)\alpha_n+\beta_{n+1}$ with $\alpha_1 = 1$ and $\beta_{n+1} = \alpha_{n+1} - (1+\epsilon)\alpha_n \leq K_0$. We can resolve this equation and find that

$$\alpha_n = (1+\epsilon)^{n-1}\alpha_1 + \sum_{j=2}^{n}(1+\epsilon)^j\beta_{n-j} \leq (1+\epsilon)^{n-1} + K_0\sum_{j=2}^{n}(1+\epsilon)^j \leq K(1+\epsilon)^n,$$

where $K = 1 + K_0/\epsilon$. From here we obtain (17). □

Lemma (6.12) gives us uniform geometric control over the tails of S_n. The next step is to connect the ruin probability to the distribution I introduced in Corollary 6.8. Recall that the integrated tail distribution of the claim size is

$$I(t) = \frac{1}{\mathbb{E}[W_1]}\int_0^t \mathbb{P}(W_1 > y)dy,$$

and that Corollary 6.8 established that $\Psi(u)$ satisfies the defective renewal equation

$$\Psi(u) = \frac{1-I(u)}{1+\rho} + \frac{1}{1+\rho}(\Psi * I)(u).$$

The following proposition shows that the survival probability can be expressed as an explicit series in convolutions of I, which will be the key to deriving its asymptotic behavior.

Proposition 6.13. *[2] We work in the Cramér-Lundberg model with Poisson claim arrivals of intensity $\lambda > 0$, as in Section 6.2. We assume that the distribution function F_W for the random amounts $(W_i)_{i\geq 1}$ has a density f_W and $\mathbb{E}[W_1] < \infty$,and we assume that the net profit condition $\rho = \frac{c}{\lambda\mathbb{E}[W_1]} - 1 > 0$ is satisfied. Then the survival probability $\bar{\Psi}(u)$ can be expressed as the absolutely convergent series*

$$\bar{\Psi}(u) = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}\frac{I^{*n}(u)}{(1+\rho)^n}, \tag{18}$$

where I^{*n} denotes the n-fold convolution of the distribution I.

Before proving this, we verify that the series is absolutely convergent. Since $I^{*n}(u) \leq 1$ for all n and u,

$$\sum_{n=0}^{\infty}\frac{I^{*n}(u)}{(1+\rho)^n} \leq \sum_{n=0}^{\infty}\frac{1}{(1+\rho)^n} = \frac{1+\rho}{\rho} < \infty,$$

where the geometric series converges since $\rho > 0$ implies $1/(1+\rho) < 1$. Hence (18) is well-defined and bounded, and in particular $\Psi(u) \leq 1$ for all u.

Proof. Let us denote the right part in the equality (18) by $\tilde{\Psi}(u)$, i.e.,

$$\tilde{\Psi}(u) = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}\frac{I^{*n}}{(1+\rho)^n}.$$

It is clear that this function is bounded, i.e.

$$\tilde{\Psi}(u) \leq \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} \frac{1}{(1+\rho)^n} = \frac{\frac{\rho}{1+\rho}}{1-\frac{1}{1+\rho}} = 1.$$

Moreover, one can see that this function satisfies the defective renewal equation. Indeed, unpacking the series and using the recursive integral definition of convolutions $I^{*n}(u) = \int_0^u I^{*(n-1)}(u-t)dI(t)$:

$$\begin{aligned}\tilde{\Psi}(u) &= \frac{\rho}{1+\rho} + \frac{\rho}{(1+\rho)^2} I(u) + \frac{\rho}{1+\rho} \sum_{n=2}^{\infty} \frac{1}{(1+\rho)^n} \int_0^u I^{*(n-1)}(u-t)dI(t) \\ &= \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \int_0^u \tilde{\Psi}(u-t)dI(t).\end{aligned}$$

We now show that $\tilde{\Psi}(u) = \bar{\Psi}(u)$. To this end, we set $\delta(u) = \tilde{\Psi}(u) - \bar{\Psi}(u)$. We have already seen that $|\delta(u)| \leq 2$ for all $u \geq 0$. Let u be a fixed positive number. Denote by $M_u = \sup_{0 \leq t \leq u} |\delta(t)|$. Then there exists $0 \leq t_0 \leq u$ such that $M_u = |\delta(t_0)|$ because the function $\delta(\cdot)$ is continous on the interval $[0, u]$. Therefore,

$$\begin{aligned}M_u = |\delta(t_0)| &= \frac{1}{1+\rho} \left| \int_0^{t_0} \delta(t_0 - t)dI(t) \right| \\ &\leq \frac{1}{1+\rho} \int_0^{t_0} |\delta(t_0 - t)|dI(t) \\ &\leq \frac{1}{1+\rho} M_u I(t_0) \leq \frac{1}{1+\rho} M_u.\end{aligned}$$

Taking into account that $1/(1+\rho) < 1$, we get that $M_u = 0$ for all $u > 0$. Therefore, $\bar{\Psi}(u) = \tilde{\Psi}(u)$ for any $u > 0$. □

We now have all the ingredients in place for the main theorem of this section. The series representation in (18) expresses $\Psi(u)$ as

$$\Psi(u) = 1 - \bar{\Psi}(u) = \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} \frac{1 - I^{*n}(u)}{(1+\rho)^n}.$$

The asymptotic behavior of $\Psi(u)$ as $u \to \infty$ therefore reduces to understanding the ratio $(1 - I^{*n}(u))/(1 - I(u))$ as $u \to \infty$. By the subexponential condition applied to I, this ratio converges to n for each fixed n. Lemma 6.12 provides the uniform geometric bound needed to interchange the limit with the series. The conclusion is the following theorem, which is the central result of this section.

Theorem 6.14. *We assume that all the conditions in Proposition 6.13 hold and the distribution function $I(\cdot)$ is subexponential. Then*

$$\lim_{u \to \infty} \frac{\Psi(u)}{1 - I(u)} = \rho^{-1}.$$

Proof. From Proposition 6.13, we have directly that

$$\Psi(u) = 1 - \bar{\Psi}(u) = \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} \frac{1}{(1+\rho)^n} (1 - I^{*n}(u)).$$

Thus, taking into account that

$$\frac{\rho}{1+\rho} \sum_{n=1}^{\infty} n \left(\frac{1}{1+\rho} \right)^n = \rho^{-1},$$

we have

$$\Delta(u) = \frac{\Psi(u)}{1 - I(u)} - \rho^{-1} = \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} \frac{1}{(1+\rho)^n} \sigma_n(u),$$

where

$$\sigma_n(u) = \frac{1 - I^{*n}(u)}{1 - I(u)} - n.$$

Now we fix $\varepsilon > 0$ such that $\theta = \frac{1}{1+\rho}(1+\varepsilon) < 1$. Then, by Lemma 6.12, we obtain that there is a positive constant K such that for any $n \geq 1$ and for all $u > 0$

$$\left(\frac{1}{1+\rho} \right)^n |\sigma_n(u)| \leq K\theta^n + \left(\frac{1}{1+\rho} \right)^n n.$$

Moreover, for any $n \geq 1$

$$\lim_{u \to \infty} \sigma_n(u) = 0.$$

Then the dominated convergence theorem directly implies that

$$\lim_{u \to \infty} \Delta(u) = \frac{\rho}{1+\rho} \sum_{n=1}^{\infty} \left(\frac{1}{1+\rho} \right)^n \lim_{u \to \infty} \sigma_n(u) = 0.$$

This proofs the theorem. □

To appreciate what this theorem says, it is worth contrasting it with the light-tailed result of Theorem 6.9. In the small risks regime, the ruin probability decays exponentially: $\Psi(u) \sim C \exp(-\gamma^* u)$, with the rate γ^* determined by the full moment generating function of the claims. In the subexponential regime, the decay is instead governed by the tail of I, which is polynomial for Pareto-type claims. There is no adjustment coefficient, no exponential decay, and the entire martingale approach is unavailable. The following example makes the asymptotic explicit for Pareto claims.

Example 6.15. *Consider the Cramér-Lundberg model with Pareto-distributed claims,*

$$F_W(z) = \mathbb{P}(W \leq z) = 1 - \frac{1}{(1+z)^{1+\alpha}}, \quad z \geq 0,\ \alpha > 0.$$

Then $\mathbb{E}[W] = 1/\alpha$*, and the integrated tail distribution is*

$$I(z) = 1 - \frac{1}{(1+z)^{\alpha}}.$$

We verified in Example 6.11 that $1 - I(z) = (1+z)^{-\alpha}$ *is subexponential. Theorem 6.14*

therefore gives

$$\Psi(u) \sim \rho^{-1}(1 - I(u)) = \frac{1}{\rho(1+u)^{\alpha}} \qquad as \quad u \to \infty,$$

or equivalently $(1+u)^{\alpha}\Psi(u) \to \rho^{-1}$. *The ruin probability decays polynomially with exponent* α, *which is determined entirely by the tail index of the claim distribution. As* α *decreases, meaning heavier tails, the decay becomes slower and the ruin probability for large initial capital becomes larger. In particular, even an arbitrarily large initial capital* u *cannot make the ruin probability negligible when* α *is small, in contrast to the exponential decay of the light-tailed case.*

7 Conclusion

This thesis has developed the theory of renewal processes from its foundational principles through to sophisticated applications in actuarial risk theory. The journey began with establishing the basic structural properties of renewal processes and their fundamental limit theorems. The Law of Large Numbers confirmed the intuitive result that renewals occur at a long-run average rate of $1/\mu$, while the Central Limit Theorem provided a second-order description of the fluctuations around this mean, revealing that properly scaled deviations converge to a standard normal distribution.

The Poisson process emerged as the central example throughout, serving both as an analytically tractable special case and as an important model in its own right. Its memoryless structure, embodied by the exponential inter-arrival distribution, was shown to be the key property behind its elegant characterization through independent and stationary increments.

Blackwell's Renewal Theorem deepened our understanding of the local behavior of renewal processes, showing that the expected number of renewals in a window of fixed length h converges to h/μ as the window is shifted far into the future. The two approaches taken, a Markov chain ergodic argument in the lattice case and a probabilistic coupling construction in the non-lattice case, illustrated the richness of techniques available in renewal theory. The renewal equation then provided the analytic bridge connecting probabilistic quantities of interest to functional equations amenable to classical analysis. Smith's Key Renewal Theorem unified these threads by characterizing the asymptotic behavior of solutions to such equations, with the machine repair example demonstrating how naturally these results arise in practice.

The final chapter applied this machinery to the Cramér–Lundberg model of insurance risk. Two fundamentally different regimes were identified and analyzed. In the small risks regime, the exponential decay of the ruin probability was established through martingale methods, with Lundberg's inequality providing a sharp upper bound and an exact asymptotic formula derived via an exponential change of measure. In the large claims regime, the assumption of subexponential claim distributions led to a qualitatively different picture: the ruin probability decays only polynomially, at a rate governed entirely by the tail index of the claim distribution. The contrast between these two regimes underscores a central message of modern risk theory, namely that the tail behavior of the claim distribution is the decisive factor in determining the long-run solvency of an insurer.

Taken together, the results of this thesis illustrate the power and versatility of renewal theory as a mathematical framework. Its blend of probabilistic intuition and analytic rigor makes it an indispensable tool not only in risk theory but across a wide range of applied

probability, from queuing systems to reliability engineering. Future directions could include the study of Markov renewal processes, where inter-arrival times depend on an underlying Markov chain, or the extension of the ruin theory results to more general surplus models with dependent claim arrivals.

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Declaration on the use of generative AI. Sonnet 4.6, (Claude, Anthropic, 2026) was employed as an auxiliary tool in the formulation of certain proofs and in improving the clarity of the text. The author retains full responsibility for all mathematical content.

A Probability Theory Fundamentals

Lemma A.1. *(Borel-Cantelli). [1] Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events.*

(i) If $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) < \infty$, then

$$\mathbb{P}\big(\limsup A_n\big) = 0$$

or equivalently,

$$\{n \in \mathbb{N} : \omega \in A_n\} \text{ is a.s. finite.}$$

(ii) If $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$ and if the events A_n, $n \in \mathbb{N}$, are independent, then

$$\mathbb{P}\big(\limsup A_n\big) = 1$$

or equivalently

$$\{n \in \mathbb{N} : \omega \in A_n\} \text{ is a.s. infinite.}$$

Definition A.2 (Laplace-Stieltjes Transform)**.** *Let X be a non-negative random variable with cumulative distribution function (CDF) $F(t) = P(X \le t)$.*

*The **Laplace-Stieltjes transform** (LST) of the distribution function $F(t)$ is the function $\tilde{F}(s)$ defined for $s \ge 0$ as the expected value of e^{-sX}:*

$$\tilde{F}(s) = \mathbb{E}\big[e^{-sX}\big] = \int_0^\infty e^{-st}\, dF(t).$$

Theorem A.3 (Monotone Class Theorem)**.** *[1] Let $\mathcal{C} \subset \mathcal{P}(E)$ be closed under finite intersections. Then $\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C})$. Consequently, if $\mathcal{M}$ is any monotone class such that $\mathcal{C} \subset \mathcal{M}$, we have also $\sigma(\mathcal{C}) \subset \mathcal{M}$.*

Corollary A.4. *[1] Let μ and ν be two measures on $(E, \mathcal{A})$. Suppose that there exists a class $\mathcal{C} \subset \mathcal{A}$, which is closed under finite intersections, such that $\sigma(\mathcal{C}) = \mathcal{A}$ and $\mu(A) = \nu(A)$ for every $A \in \mathcal{C}$.*

1. *If $\mu(E) = \nu(E) < \infty$, then we have $\mu = \nu$.*
2. *If there exists an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{C}$ such that $E = \bigcup_{n\in\mathbb{N}} E_n$ and $\mu(E_n) = \nu(E_n) < \infty$ for every $n \in \mathbb{N}$, then $\mu = \nu$.*

Definition A.5. *[3] A set $A \subset \mathbb{R}^\infty$ is called invariant if*

$$(x_1, x_2, \dots, x_i, \dots, x_j, \dots) \in A \iff (x_1, x_2, \dots, x_j, \dots, x_i, \dots) \in A$$

for any i, j. This is equivalent to supposing A is invariant under permutations involving finitely many coordinates:

$$(x_1, x_2, \dots, x_k, \dots) \in A \iff (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}, \dots) \in A$$

for any k and any permutation π of $\{1, 2, \dots, k\}$.

Theorem A.6. *(Hewitt-Savage) [3] If $\zeta = \{\zeta_n, n \ge 1\}$ is an i.i.d. sequence, then for any invariant set A*

$$\mathbb{P}[\zeta \in A] = 0 \text{ or } 1$$

B Markov Chain Theory and Properties

This appendix summarizes the fundamental definitions and convergence results for discrete-time Markov chains used in the proof of Theorem (3.2).

Definition B.1. *[5] We call distribution on S a probability measure μ on S. It is identified with a collection $\mu = (\mu_x)_{x\in S}$ of numbers satisfying*

(i) $\forall x \in S\ \mu_x \geq 0$, *and*

(ii) $\sum_{x\in S} \mu_x = 1$.

Definition B.2. *[5] A transition probability is a collection $P = (p_{x,y})_{x,y\in S}$ such that:*

(i) $\forall x, y \in S \quad p_{x,y} \geq 0$, *and*

(ii) $\forall x \in S \quad \sum_{y\in S} p_{x,y} = 1$.

Definition B.3. *[5] Let P be a transition probability, μ a distribution on S. A sequence $X = (X_n)_{n\geq 0}$ of random variables with values in S is a Markov Chain with initial distribution μ and transition probability P if for every $n \geq 0$ and $x_0, \ldots, x_n \in S$*

$$\boxed{\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n) = \mu_{x_0} p_{x_0,x_1} \cdots p_{x_{n-1},x_n}.}$$

In this case, we write $X \sim \mathrm{MC}(\mu, P)$.

Definition B.4. *[5] Let $x \in S$. The period of x is defined by*

$$d_x = \gcd\{n > 0 : p_{xx}^{(n)} > 0\}.$$

By convention $\gcd(\emptyset) = \infty$.

Definition B.5. *[5] We say that P is aperiodic if for every $x \in S$*

$$d_x = 1$$

Definition B.6 (Stationary Distribution)**.** *[5] Let π be a distribution on S, we say that π is stationary (for P) if*

$$\pi = \pi P.$$

Definition B.7. *[5] The equivalence classes of $\leftrightarrow$ are called communication classes of P. If P has a single unique communication class, we say that P is irreducible.*

Theorem B.8. *[5] Assume that P is irreducible, aperiodic, and admits a stationary distribution π. Then for every distribution μ on S and $x \in S$*

$$\lim_{n\to\infty} \mathbb{P}[X_n = x] = \pi_x$$

C Proof of Lemma 3.4

[3] For a measure G on $\mathbb{R}$, say x is a point of increase of G if $G(x-\epsilon, x+\epsilon) > 0$ for all $\epsilon > 0$. The assumption that F is not arithmetic means the points of increase of F are not a subset of

$$L(h) := \{0, h, 2h, \ldots\}$$

for any $h > 0$. We write $pi(G)$ for the set consisting of points of increase of G. Let $\Sigma = \bigcup_{n=0}^{\infty} pi(F^{n*})$ so that $\Sigma \subset pi(U)$.

Proof. [3] First observe that if F_1, F_2 are distributions on $\mathbb{R}$ then $a \in pi(F_1), b \in pi(F_2)$

implies $a+b \in pi(F_1 * F_2)$. If ζ_i is a rv with distribution F_i, $i=1,2$, then

$$\begin{aligned}\mathbb{P}[\zeta_1+\zeta_2 \in (a+b-\epsilon, a+b+\epsilon] &\geq \mathbb{P}[\zeta_1 \in (a-\frac{\epsilon}{2}, a+\frac{\epsilon}{2}), \zeta_2 \in (b-\frac{\epsilon}{2}, b+\frac{\epsilon}{2})] \\ &= F_1(a-\frac{\epsilon}{2}, a+\frac{\epsilon}{2}) \cdot F_2(b-\frac{\epsilon}{2}, b+\frac{\epsilon}{2}) > 0.\end{aligned}$$

From this follows that if

$$a, b \in \Sigma, a < b, h = b-a > 0 \; then \; na+mh \in \Sigma \tag{19}$$

for $0 \leq m \leq n$. To check this, observe that if $a, b \in \Sigma$ then for some integers r, s

$$a \in pi(F^{r*}), \quad b \in pi(F^{s*})$$

and therefore

$$na+mh = (n-m)a + mb \in pi(F^{((n-m)r+ms)*}) \subset \Sigma.$$

Now suppose $t \in [na, na+nh]$. From (19) it follows t must be within distance h of some point in Σ. But the intervals $[na, na+nh]$ overlap for n large; i.e., $(n+1)a < na+nh$ if $ah^{-1} < n$ so any $t \geq T = a^2 h$ is within distance h of a point in Σ. So if $\delta > h$

$$m(t+\delta) - m(t) > 0$$

for $t \geq T$. It now remains to show h can be chosen arbitrarily small. We proceed by contradiction. We wish to show

$$inf\{|b-a|, a, b \in \Sigma\} = 0.$$

Suppose otherwise and that $\inf\{|b-a|, a, b \in \Sigma\} = h_0 > 0$. There exist, then, $a, b \in \Sigma$, $a < b$ and $h_0 \leq b-a = h \leq 2h_0$. It follows that

$$[na, na+nh] \cap \Sigma = \{na+mh, m = 0, 1, \ldots, n\}, \tag{20}$$

since if there is a point p in $[na, na+nh] \cap \Sigma$ not of the form given by the right side of (20), it would be at distance less than $h/2 \leq h_0$ from some point $na + k_0 h \in \Sigma$, which contradicts the definition of h_0.

Suppose $n \geq a/h$. Then on the one hand $(n+1)a \in \Sigma$ and on the other $(n+1)a \in [na, na+nh]$. Thus $(n+1)a$ belongs to the left side of (20) and hence to the right side.

Therefore, $(n+1)a = na+mh$ for some $m \in \{0, 1, \ldots, n\}$, so $a = mh$ and $a \in L(h)$. We conclude from (20) that

$$[na, na+nh] \cap \Sigma \subset L(h). \tag{21}$$

To finish, take $c \in pi(F)$. Since the intervals $[na, na+nh]$ overlap for large n, there exist k and n such that

$$c + ka \in [na, na+nh].$$

Since $c \in pi(F)$ and $a \in \Sigma$, we have $c+ka \in \Sigma$, and, further, from (21), we have $c+ka \in L(h)$. Since $a \in L(h)$, we conclude that $c \in L(h)$. Hence $pi(F) \subset L(h)$ which contradicts the fact that F is non-arithmetic.

To prove $m(t+\delta) - m(t) > 0$ for $t \geq T$ iff $\mathbb{P}[N_{t+\delta} - N_t > 0] > 0$ for $t \geq T$ we observe

$m(t+\delta) - m(t) > 0$ iff there is some r such that $F^{r*}(t, t+\delta) > 0$. Since

$$\begin{aligned}\mathbb{P}[N_{t+\delta} - N_t > 0] &= \mathbb{P}\{\cup_0^\infty [S_n \in (t, t+\delta)]\} \\ &\geq \mathbb{P}[S_n \in (t, t+\delta)] = F^{r*}(t, t+\delta),\end{aligned}$$

the result follows. □