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Bose-Einstein condensation of a pseudo-relativistic Bose gas in the Gross-Pitaevskii Limit

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Abstract

We consider a trapped gas of N bosons with dispersion $\sqrt{p^2+m^2}-m$, interacting through a small repulsive potential with scattering length scaling like $1/\sqrt{N}$. We use a quadratic Bogoliubov transformation to find its ground state energy to leading order and prove Bose-Einstein condensation. We define the pseudo-relativistic scattering length for small potentials and prove that the leading order coefficient of the ground state energy depends on the potential only through its scattering length.

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1 Introduction

We consider N pseudo-relativistic bosonic particles of mass $m \geq 0$ constrained in the box $\Lambda = [-1/2; 1/2]^3$ with periodic boundary conditions. The interactions between particles are described by a repulsive potential $V \colon \mathbb{R}^3 \to \mathbb{R}$. The Hamilton operator takes the form

$$H_N = \sum_{j=1}^{N} \left(\sqrt{-\Delta_{x_j} + m^2} - m \right) + \kappa \sum_{i<j}^{N} V_N(x_i - x_j), \tag{1}$$

where V_N is a rescaling of V, and $0 < \kappa < 1$ is a coupling constant. The operator $\sqrt{-\Delta + m^2}$ is defined as the unique non-negative square root of $-\Delta + m^2$ acting on $L^2(\Lambda)$. Equivalently, it is the operator acting by multiplication with $\sqrt{p^2 + m^2}$ in Fourier space. The Hamilton operator (1) acts on $L^2_s(\Lambda^N)$, the space of N-particle wave functions that are symmetric under exchange of variables. We also define the scattering length $\mathfrak{a} > 0$ of the potential κV by the limit

$$\mathfrak{a}^2 = \lim_{|x| \to +\infty} |x|^2 w(x)$$

where $w \in L^2(\mathbb{R}^3)$ is a solution to the massless zero-energy scattering equation

$$\left(\sqrt{-\Delta} + \frac{\kappa}{2} V(x) \right) (1 - w(x)) = 0. \tag{2}$$

Existence and uniqueness of solutions of (2) is proved via Born series in section 4.1. We characterize the Gross-Pitaevskii regime in the pseudorelativistic case by the condition

$$\mathfrak{a}_N = \frac{\mathfrak{a}}{\sqrt{N}},$$

where $\mathfrak{a}_N$ denotes the scattering length of κV_N. This leads to the scaling of the potential given by

$$V_N(x) = \sqrt{N}\, V\left(\sqrt{N} x \right).$$

We believe that this is the correct condition because (as we will prove) the square of the scattering length appears in the leading order coefficient of the ground state energy of (1). This should imitate the approach from [13] in the nonrelativistic case, where the scattering length appears to the first power and thus should scale like $\mathfrak{a}_N = \mathfrak{a}/N$.

The main difference between the above problem and recent works in this field is the pseudorelativistic kinetic energy. So far, extensive work has been done on the spectral analysis of the nonrelativistic Hamilton operator

$$H_N^{(\mathrm{nr})} = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)). \tag{3}$$

With first results by Lieb, Yngvason and Seiringer in [14, 13, 12], the leading order of the ground state energy of $H_N^{(\mathrm{nr})}$ has been established together with Bose-Einstein condensation (BEC) of (approximate) ground states in the zero-momentum state $\psi_0(x) = 1$. More precisely, they proved that the ground state energy $E_N^{(\mathrm{nr})}$ of $H_N^{(\mathrm{nr})}$ satisfies

$$E_N^{(\mathrm{nr})} := \inf_{\Psi \in L^2_s(\Lambda^N), \|\Psi\|_2 = 1} \langle \Psi | H_N | \Psi \rangle = 4\pi \mathfrak{a} N + o(N).$$

Moreover, if Ψ_N denotes the ground state of (3) and $\gamma_N^{(1)}$ denotes its one-particle reduced density matrix such that $\mathrm{Tr}(\gamma_N^{(1)}) = N$, they proved that

$$\lim_{N \to +\infty} 1 - \frac{1}{N} \left\langle \psi_0 \middle| \gamma_N^{(1)} \middle| \psi_0 \right\rangle = 0.$$

In other words, the fraction of particles that are not in the zero-momentum state converges to 0 as $N \to +\infty$. While their approaches used localization techniques, more recent works of Boccato, Brennecke,

Cenatiempo and Schlein [2, 1, 3] used Bogoliubov transformations, i.e. unitary conjugations of $H_N^{(\mathrm{nr})}$, to recover the same results with optimal error estimates. In [4], they also resolved the ground state energy up to order $o(1)$. Later, Hainzl, Schlein and Triay [10] provided a simplified approach to achieve the same resolution of the ground state energy without the smallness assumption on the potential. This also recovered the originally heuristic results obtained by Bogoliubov [5]. On the other hand, Hainzl presented a signifactly more concise proof of BEC in [9] under the assumption of a small potential. This article serves as the main inspiration for the present work.

Physically, we should expect the same results to hold even when taking relativistic corrections into account. One possible approach is to replace the kinetic energy operator $-\Delta$ by the relativistic kinetic energy $\sqrt{-\Delta + m^2} - m$. In this work, we compute the ground state energy of (1) for a fixed mass m. This main result is stated in the following Theorem.

Theorem 1.1. *Suppose $V \in L^\infty(\mathbb{R}^3)$ is non-negative, radially symmetric with compact support inside the box Λ. Assume furthermore that V satisfies*

$$\sum_{p \in 2\pi\mathbb{Z}^3} (1+|p|)|\widehat{V}_N(p)| < +\infty, \quad \int_{\mathbb{R}^3} (1+|p|)|\widehat{V}(p)|dp < +\infty \tag{4}$$

for all $N \in \mathbb{N}$. Then, for sufficiently small coupling constants $\kappa > 0$, the ground state energy E_N (i.e. the lowest eigenvalue) of H_N satisfies

$$E_N = 2\pi^2 \mathfrak{a}^2 N + O(N^{1/2})$$

and the ground state Ψ_N of (1) exhibits complete Bose-Einstein condensation in the zero-momentum state $\psi_0 \equiv 1$ with rate $N^{-1/2}$, that is,

$$0 \leq \ 1 - \frac{1}{N} \left\langle \psi_0 \middle| \gamma_N^{(1)} \middle| \psi_0 \right\rangle \leq \frac{C}{\sqrt{N}}$$

where $\gamma_N^{(1)}$ is the one-particle reduced density matrix of Ψ_N such that $\mathrm{Tr}(\gamma_N^{(1)}) = N$.

Remark. (i) Our procedure also implies that any sequence of approximate ground states exhibits complete BEC in the zero-momentum state; this is precisely stated in Corollary 3.13.

(ii) The assumptions on V are expected to be purely technical. We expect that the same result should hold for unbounded potentials satisfying a milder integrability and summability condition than (4).

(iii) We presume that the smallness of the potential can also be relaxed, though it might require a different approach compared to [10], since a cutoff on the coefficients of the Bogoliubov transformation does not improve the decay of their L^2-norm in the pseudo-relativistic case.

(iv) We expect (but we do not prove) that the next-order corrections to the ground state energy are of the order $O(N^{1/2})$.

The proof of this result is presented in Section 3, with the initial strategy being similar to [9, 10]. We define a unitary conjugation which will extract leading-order contributions from the potential terms in the Hamiltonian. The coefficients of this transformation will be defined according to a "box scattering equation" to model correlations among particles. Mathematically, this allows for the cancellation of off-diagonal terms appearing from the conjugation. Most of the other terms can be bounded using comparably straight-forward Cauchy-Schwartz inequalities. However, while some of the terms prove to be sufficient for an upper bound, they are insufficient for a lower bound. Trying to follow the methods of [9, 10] where the (positive) potential term is used to absorb contributions from several terms, in the pseudo-relativistic regime this leads to the appearance of error terms containing higher powers of the number of excitations operator, which cannot be used to prove condensation. Instead, we will split the conjugation of the potential term in configuration space and expand it to first order. We extract the relevant leading-order contributions to the ground-state energy, while rearranging the other terms into a different positive operator, up to errors that can be efficiently bounded. This is similar to methods of "completing the square", which have previously appeared in the works of Fournais and Solovej [7], [8]. On one hand, this avoids inefficient errors from the conjugation of the potential term, while also

providing simpler and more concise bounds on other terms, absorbing many contributions into this positive operator. The smallness of the potential then allows us to use the positivity for the lower bound. The upper bound on the energy will then only require an upper bound on said positive operator. This procedure allows us to find the leading order of the ground-state energy together with BEC, with (presumably) optimal rate.

In Section 4, we define the scattering length of a potential in the pseudo-relativistic regime. In a first step, we show that the mass m in the Hamiltonian (1) is negligible, in the limit of large N. In other words, the leading order of the ground state energy is independent of m. This effectively allows us to consider a massless scattering equation, containing the pseudo-differential operator $\sqrt{-\Delta}$, which behaves nicely under scaling, similar to regular derivatives. The main difficulty here is posed by the nonlocality of $\sqrt{-\Delta}$, making it rather difficult to define solutions via a variational problem, as done rigorously for example in [15]. Instead, we define a solution via Born series, using the smallness of the potential to guarantee its convergence. The scattering length is then defined as the square root of the coefficient of $1/|x|^2$ in the expansion of the solution, as $|x| \to +\infty$ and connected to the leading-order coefficient of the ground state energy.

2 Formal setup

We follow the definitions from [10], but the objects are standard and can be found in many other references. In order to rewrite the Hamilton operator in second quantization, we define the Fock space

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n),$$

where $L^2_s(\Lambda^n)$ is the subspace of $L^2(\Lambda^n)$ consisting of wave functions that are symmetric with respect to permutations of their variables. For a state $\Psi = \left(\psi^{(n)}\right)_{n\geq 0} \in \mathcal{F}$, we can define the (bosonic) annihilation and creation operators, creating or annihilating a particle with wave function $f \in L^2(\Lambda)$ respectively, by

$$\left(a^\dagger(f)\Psi\right)^{(n)}(x_1,\dots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(x_j)\psi^{(n-1)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n),$$
$$\left(a(f)\Psi\right)^{(n)}(x_1,\dots,x_n) = \sqrt{n+1}\int dx\overline{f(x)}\psi^{(n+1)}(x,x_1,\dots,x_n).$$

A simple computation shows that $a(f)^\dagger = a^\dagger(f)$ as well as the canonical commutation relations

$$\Big[a(f),a(g)\Big] = \Big[a^\dagger(f),a^\dagger(g)\Big] = 0, \qquad \Big[a(f),a^\dagger(g)\Big] = \langle f|g\rangle_{L^2}.$$

Since our system is located in a periodic box, we will use the plane waves $\psi_p(x) = e^{ip\cdot x}$ and denote $a_p = a(\psi_p)$, $a_p^\dagger = a^\dagger(\psi_p)$ for $p \in \Lambda^* = 2\pi\mathbb{Z}^3$. The canonical commutation relations then reduce to

$$\Big[a_p,a_q\Big] = \Big[a_p^\dagger,a_q^\dagger\Big] = 0, \qquad \Big[a_p,a_q^\dagger\Big] = \delta_{pq}.$$

We will also make use of the operator-valued distributions $\check{a}_x, \check{a}_x^\dagger$ satisfying

$$a_p = \int e^{-ip\cdot x}\check{a}_x dx, \qquad a_p^\dagger = \int e^{ip\cdot x}\check{a}_x^\dagger dx, \qquad \int \check{a}_x dx = 0, \qquad \int \check{a}_x^\dagger dx = 0$$

for all $p \in \Lambda^* \setminus \{0\}$. The last two conditions reflect the fact that we define $\check{a}_x, \check{a}_x^\dagger$ so that they act only on the orthogonal complement of the subspace spanned by ψ_0. In particular, we stress that these are different from the traditional operator-valued distributions $a_x, a_x^\dagger$ which are defined as $a_x^\cdot = a^\cdot(\delta_x)$ with the Dirac-Delta distribution δ_x at x. A special operator $\mathcal{N}$ on $\mathcal{F}$, counting the number of particles is defined by

$$\mathcal{N} := \sum_{p\in\Lambda^*} a_p^\dagger a_p.$$

It is also useful to introduce $\mathcal{N}_0$, counting the particles in the zero momentum mode, and $\mathcal{N}_+$ counting the particles with non-zero momentum. In other words, we define

$$\mathcal{N}_0 := a_0^\dagger a_0, \qquad \mathcal{N}_+ := \sum_{p\in\Lambda^*\setminus\{0\}} a_p^\dagger a_p = \int \check{a}_x^\dagger \check{a}_x dx.$$

Of course, these operators satisfy $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_+$. In this formalism, the so-called second quantization of the Hamilton operator (1) takes the form

$$H_N = \sum_{p\in\Lambda^*} T_p a_p^\dagger a_p + \frac{\kappa}{2}\sum_{p,q,r\in\Lambda^*} \widehat{V}_N(r) a_{p+r}^\dagger a_q^\dagger a_{q+r} a_p, \tag{5}$$

where we use the convention

$$\widehat{V}(p) = \int V(x)e^{-ip\cdot x}dx.$$

The coefficients in the kinetic term are given by

$$T_p = \sqrt{p^2+m^2} - m$$

and the scaling of the potential translates in Fourier space as

$$\widehat{V}_N(p) = \frac{1}{N}\widehat{V}\left(\frac{p}{\sqrt{N}}\right).$$

In (5), one should think of N as a free parameter which has nothing to do with the number of particles. In order to get back the original Hamiltonian (1), we have to restrict (5) to the sector $\{\mathcal{N} = N\}$ consisting of eigenstates of $\mathcal{N}$ having eigenvalue N. As done, for example, in [10], we split the Hamilton operator (5) as

$$H_N = H_0 + H_1 + H_2 + Q_2 + Q_3 + Q_4 \tag{6}$$

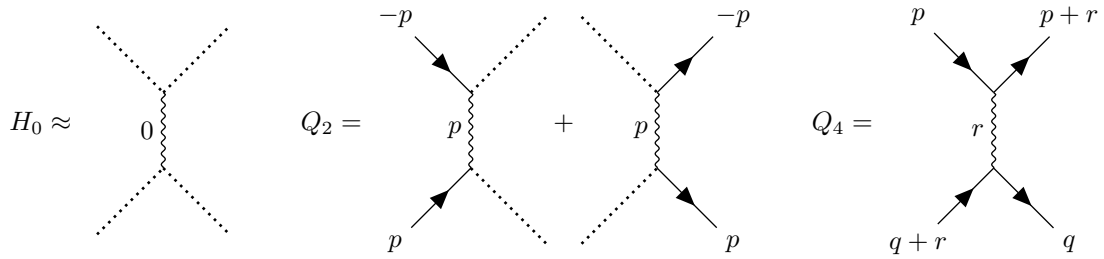
where we set

$$\begin{aligned}
H_0 &= \kappa\frac{N(N-1)}{2}\widehat{V}_N(0),\\
H_1 &= \sum_{p\neq 0} T_p a_p^\dagger a_p,\\
H_2 &= \kappa\sum_{p\neq 0}\widehat{V}_N(p)a_p^\dagger a_p(N-\mathcal{N}_+) - \kappa\frac{\mathcal{N}_+(\mathcal{N}_+-1)}{2}\widehat{V}_N(0),\\
Q_2 &= \frac{\kappa}{2}\sum_{p\neq 0}\widehat{V}_N(p)\left(a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}\right),\\
Q_3 &= \kappa\sum_{q,r,q+r\neq 0}\widehat{V}_N(r)\left(a_{q+r}^\dagger a_{-r}^\dagger a_q a_0 + \text{h.c.}\right),\\
Q_4 &= \frac{\kappa}{2}\sum_{p,q,p+r,q+r\neq 0}\widehat{V}_N(r)a_{p+r}^\dagger a_q^\dagger a_{q+r}a_p,
\end{aligned}$$

and denote by h.c. the hermitian conjugate of the previous term. Since error estimates will be the same for an operator and its hermitian conjugate (in the sense that, for any A, $|\langle\xi|A|\xi\rangle| = |\langle\xi|A^\dagger|\xi\rangle|$), we can almost exclusively restrict our attention to one of them. This splitting is a tedious computation that can be shown using that, on $\{\mathcal{N} = N\}$,

$$a_0^\dagger a_0^\dagger a_0 a_0 = \mathcal{N}_0(\mathcal{N}_0-1) = (N-\mathcal{N}_+)(N-\mathcal{N}_+-1) = N(N-1) + \mathcal{N}_+(\mathcal{N}_+-1) - 2(N-1)\mathcal{N}_+. \tag{7}$$

Note also that in Q_4, $r = 0$ is allowed. The advantage of this separation is that we have extracted the condesate (the zero momentum mode) in all terms. From now on, we will follow the convention that indices appearing in annihilation and creation operators are always non-zero, unless stated otherwise. The following diagram illustrates the terms H_0, Q_2 and Q_4, which, together with H_1, will contribute to the leading order of the ground state energy.



First, let us note that these diagrams serve an illustrative purpose and are not accurate Feynman diagrams. The incoming and outgoing arrows represent incoming and outgoing particles respectively, while the wavy lines represent a transfer of momentum through the potential V. The lines are labeled with the momenta they carry, while the dotted lines carry zero momentum. To compare explicitly with the mathematical expressions, the incoming arrows are exactly the particles in the annihilation operators, while the outgoing arrows are those in creation operators. The diagram for H_0 is also not entirely accurate, due to the identity (7), but it is correct in leading order.

Starting from (6), we will analyze the conjugation by a Bogoliubov transformation, which we define in 3.1, of all terms separately in Section 3.2.

3 Quadratic conjugation

3.1 Definition and analytic properties of the coefficients

We follow closely the ideas of [9, 10]. We consider the conjugation of H_N with an operator $e^{\mathcal{B}}$ where

$$\mathcal{B} = \frac{1}{2}\sum_p \varphi_p a_p^\dagger a_{-p}^\dagger a_0 a_0 - \text{h.c.}. \tag{8}$$

The coefficients φ_p are chosen in such a way that off-diagonal quadratic terms arising from the conjugations of H_1, Q_2 and Q_4 cancel. To this end, denoting by $P = 1 - |\psi_0\rangle\langle\psi_0|$ the orthogonal projection on the orthogonal complement of the state ψ_0, observe that

$$P\left(\sqrt{-\Delta + m^2} - m + \frac{\kappa}{2}V_N\right)P \geq C > 0.$$

Here, $\sqrt{-\Delta + m^2}$ denotes the square root of $-\Delta + m^2$ acting on $L^2(\Lambda)$. Hence we can define

$$\check{\varphi} := -\frac{\kappa}{2}P\left[P\left(\sqrt{-\Delta + m^2} - m + \frac{\kappa}{2}V_N\right)P\right]^{-1}PV_N. \tag{9}$$

Then, the Fourier coefficients φ_p of $\check{\varphi}$ satisfy $\varphi_0 = 0$ as well as the "box scattering equation"

$$T_p\varphi_p + \frac{\kappa}{2}\sum_q \widehat{V}_N(p-q)\varphi_q = -\frac{\kappa}{2}\widehat{V}_N(p) \tag{10}$$

Based on the above, we also define the "box scattering length"

$$\mathfrak{L}_N := \frac{\kappa}{2}N\left[\widehat{V}_N(0) + \sum_p \widehat{V}_N(p)\varphi_p\right] = \frac{\kappa}{2}\widehat{V}(0) + \frac{\kappa}{2}N\sum_p \widehat{V}_N(p)\varphi_p. \tag{11}$$

Actually, this coefficient $\mathfrak{L}_N$ is not the actual scattering length of κV, but closely related to it, as we will discuss in Section 4. The following Lemma yields useful bounds on φ_p and its norms, which will be relevant throughout this work. The proof is similar to the one presented in [10], with some alterations due to the smallness of the potential.

Lemma 3.1. *We have $\varphi_p = \varphi_{-p} \in \mathbb{R}$ and*

$$|\varphi_p| \lesssim \frac{\kappa}{N|p|}$$

for all $p \in 2\pi\mathbb{Z}^3 \setminus \{0\}$. Furthermore,

$$\|\varphi\|_1 \lesssim \kappa, \qquad \|\varphi\|_2 \lesssim \kappa N^{-3/4}, \qquad \|\varphi\|_\infty \lesssim \kappa N^{-1}.$$

Finally, $|\mathfrak{L}_N| \lesssim \kappa$.

Before giving the details, let us outline the proof. The general strategy is to show first that the desired norms are finite and then setting up inequalities which we can solve. First, we will prove the pointwise bound on φ_p by manipulating the scattering equation and using the scaling of the potential, which allows us to calculate norms containing $\widehat{V}_N$. From this, the bound on the ℓ^∞ norm of φ_p automatically follows. To prove the bounds on the other two norms, we bound either norm essentially by itself plus something depending linearly on κ, which allows us to solve inequalities for the desired norms using the smallness of this coupling constant.

Proof. The fact that $\varphi_p = \varphi_{-p} \in \mathbb{R}$ follows from V being non-negative and radial. Because $V \in L^2(\mathbb{R}^3)$, $\check{\varphi}$ defined as in (9) is in L^2, and thus $\|\varphi\|_2 < +\infty$. Multiplying the scattering equation (10) by φ_p and summing over p yields

$$\sum_p T_p\varphi_p^2 = -\frac{\kappa}{2}\sum_{p,q}\widehat{V}_N(p-q)\varphi_p\varphi_q - \frac{\kappa}{2}\sum_p \widehat{V}_N(p)\varphi_p \leq -\frac{\kappa}{2}\sum_p \widehat{V}_N(p)\varphi_p$$

thanks to the positivity of V_N. Thus,

$$\|\sqrt{T}\varphi\|_2^2 \lesssim \kappa \|V_N\|_2 \|\varphi\|_2 < +\infty.$$

In the same spirit, we find also

$$\left\|\sqrt{T}\varphi\right\|_2^2 \leq \frac{\kappa}{2} \sum_p \left| \frac{\widehat{V}_N(p)}{\sqrt{T_p}} \sqrt{T_p}\varphi_p \right| \leq \kappa \left\| \frac{\widehat{V}_N}{\sqrt{T}} \right\|_2 \left\|\sqrt{T}\varphi\right\|_2,$$

or in other words,

$$\left\|\sqrt{T}\varphi\right\|_2 \lesssim \kappa \left\| \frac{\widehat{V}_N}{\sqrt{T}} \right\|_2.$$

For the right hand-side, notice first that we can bound

$$|p| \geq T_p = \sqrt{p^2 + m^2} - m \geq C|p|$$

because for $p \in 2\pi\mathbb{Z}^3 \setminus \{0\}$, $|p| \geq 2\pi > 0$. Hence $1/T_p \lesssim 1/|p|$ and we find

$$\begin{aligned}
\left\| \frac{\widehat{V}_N}{\sqrt{T}} \right\|_2 &\leq \left\| \frac{\widehat{V}_N}{\sqrt{|\cdot|}} \chi_{|\cdot| \leq \sqrt{N}} \right\|_2 + \left\| \frac{\widehat{V}_N}{\sqrt{|\cdot|}} \chi_{|\cdot| > \sqrt{N}} \right\|_2 \\
&\leq \|\widehat{V}_N\|_\infty \left\| \frac{\chi_{|\cdot| \leq \sqrt{N}}}{\sqrt{|\cdot|}} \right\|_2 + \|\widehat{V}_N\|_2 \left\| \frac{\chi_{|\cdot| > \sqrt{N}}}{|\cdot|} \right\|_\infty \\
&\lesssim N^{-1} N^{1/2} + N^{-1/4} N^{-1/2} \\
&\lesssim N^{-1/2},
\end{aligned}$$

which ultimately yields

$$\|\sqrt{T}\varphi\|_2 \lesssim \kappa N^{-1/2}.$$

Reusing the scattering equation (10) we obtain

$$|T_p\varphi_p| \lesssim \kappa |\widehat{V}_N(p)| + \kappa \sum_p |\widehat{V}_N(p-q)|\varphi_q \lesssim \kappa \|\widehat{V}_N\|_\infty + \kappa \left\| \frac{\widehat{V}_N(p-\cdot)}{\sqrt{T}} \right\|_2 \|\sqrt{T}\varphi\|_2 \lesssim \kappa N^{-1},$$

where in the second term we proceeded in the same way as for $\|\widehat{V}_N/\sqrt{T}\|_2$ above. This shows the pointwise bound $|\varphi_p| \lesssim \kappa/(N|p|)$ as well as $\|\varphi\|_\infty \lesssim \kappa N^{-1}$. Next, we use the scattering equation (10) to estimate, for any $r \geq 1$,

$$\begin{aligned}
\|\varphi\|_r &= \left(\sum_p \frac{1}{T_p^r} (T_p\varphi_p)^r \right)^{1/r} \\
&\lesssim \kappa \left(\sum_p \frac{1}{|p|^r} \left((\widehat{V}_N * \varphi)(p) + \widehat{V}_N(p) \right)^r \right)^{1/r} \\
&\lesssim \kappa \left\| \frac{1}{|\cdot|} \widehat{V}_N * \varphi \right\|_r + \kappa \left\| \frac{1}{|\cdot|} \widehat{V}_N \right\|_r. \qquad (12)
\end{aligned}$$

We focus first on $r = 2$. We split

$$\left\| \frac{1}{|\cdot|} \widehat{V}_N \right\|_2^2 = \sum_p \frac{|\widehat{V}_N(p)|^2}{|p|^2} = \sum_{p \leq \sqrt{N}} \frac{|\widehat{V}_N(p)|^2}{|p|^2} + \sum_{p > \sqrt{N}} \frac{|\widehat{V}_N(p)|^2}{|p|^2}.$$

Treating the terms separately, we find

$$\begin{aligned}\left\|\frac{1}{|\cdot|}\widehat{V}_N\right\|_2^2 &\lesssim \|\widehat{V}_N\|_\infty^2 \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|^2}\right\|_1 + \|\widehat{V}_N\|_\infty \|\widehat{V}_N\|_2 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|^2}\right\|_2 \\ &\lesssim N^{-2}N^{1/2} + N^{-1}N^{-1/4}N^{-1/4} \\ &\lesssim N^{-3/2}\end{aligned}$$

as well as

$$\begin{aligned}\left\|\frac{1}{|\cdot|}\widehat{V}_N * \varphi\right\|_2^2 &\lesssim \|\widehat{V}_N * \varphi\|_\infty^2 \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|^2}\right\|_1 + \|\widehat{V}_N * \varphi\|_\infty \|\widehat{V}_N * \varphi\|_2 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|^2}\right\|_2 \\ &\lesssim \left(\|\widehat{V}_N\|_2^2 \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|^2}\right\|_1 + \|\widehat{V}_N\|_2 \|\widehat{V}_N\|_1 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|^2}\right\|_2\right) \|\varphi\|_2^2 \\ &\lesssim \left(N^{-1/2}N^{1/2} + N^{-1/4}N^{1/2}N^{-1/4}\right) \|\varphi\|_2^2 \\ &\lesssim \|\varphi\|_2^2,\end{aligned}$$

which, put together, delivers

$$\|\varphi\|_2 \leq C\kappa(N^{-3/4} + \|\varphi\|_2).$$

Choosing κ sufficiently small (so that $C\kappa \leq 1/2$) we obtain $\|\varphi\|_2 \lesssim \kappa N^{-3/4}$, as desired. In order to bound $\|\varphi\|_1$, we first show that it is finite. Noticing that

$$\left\|\frac{1}{|\cdot|}\widehat{V}_N\right\|_1 \leq \|\widehat{V}_N\|_\infty \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|}\right\|_1 + \|\widehat{V}_N\|_1 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|}\right\|_\infty \lesssim N^{-1}N + N^{1/2}N^{-1/2} \lesssim 1 \tag{13}$$

and combining this with (12) and the fact that $\|\cdot\|_r \lesssim \|\cdot\|_1$ on ℓ^1, we obtain

$$\|\varphi\|_r \lesssim \kappa + \kappa \left\|\frac{1}{|\cdot|}\widehat{V}_N * \varphi\right\|_r. \tag{14}$$

Using (14) for $r = 1$ together with Young's inequality and bounding $\kappa \leq 1$, we find

$$\|\varphi\|_1 \lesssim 1 + \left\|\frac{1}{|\cdot|}\widehat{V}_N * \varphi\right\|_1 \lesssim 1 + \left\|\frac{1}{|\cdot|}\right\|_4 \|\widehat{V}_N * \varphi\|_{4/3} \lesssim 1 + \|\widehat{V}_N\|_1 \|\varphi\|_{4/3}.$$

The same procedure for $r = 4/3$ yields

$$\|\varphi\|_{4/3} \lesssim 1 + \left\|\frac{1}{|\cdot|}\widehat{V}_N * \varphi\right\|_{4/3} \lesssim 1 + \left\|\frac{1}{|\cdot|}\right\|_4 \|\widehat{V}_N * \varphi\|_2 \lesssim 1 + \|\widehat{V}_N\|_1 \|\varphi\|_2 < +\infty,$$

implying that $\|\varphi\|_1 < +\infty$. To obtain the uniform estimate, we can now bound

$$\begin{aligned}\left\|\frac{1}{|\cdot|}\widehat{V}_N * \varphi\right\|_1 &\lesssim \|\widehat{V}_N * \varphi\|_\infty \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|}\right\|_1 + \|\widehat{V}_N * \varphi\|_1 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|}\right\|_\infty \\ &\lesssim \left(\|\widehat{V}_N\|_\infty \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|}\right\|_1 + \|\widehat{V}_N\|_1 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|}\right\|_\infty\right) \|\varphi\|_1 \\ &\lesssim (N^{-1}N^1 + N^{1/2}N^{-1/2})\|\varphi\|_1 \\ &\lesssim \|\varphi\|_1,\end{aligned} \tag{15}$$

which, applied to (14) with $r = 1$ gives

$$\|\varphi\|_1 \leq C\kappa(1 + \|\varphi\|_1).$$

Also here, choosing κ small enough (so that $C\kappa \leq 1/2$) yields the bound $\|\varphi\|_1 \lesssim \kappa$.
To finish off, we find

$$|\mathfrak{L}_N| \lesssim \kappa|\widehat{V}(0)| + \kappa N \sum_p |\widehat{V}_N(p)||\varphi_p| \lesssim \kappa + \kappa N \|\widehat{V}_N\|_\infty \|\varphi\|_1 \lesssim \kappa,$$

because $\|\varphi\|_1 \lesssim \kappa \leq 1$. □

3.2 Growth of excitations and useful estimates

For completeness, let us mention the Cauchy-Schwarz inequality which will be used throughout this entire section.

Lemma 3.2 (Cauchy-Schwarz)**.** *Let A, B be linear operators acting on a Hilbert space with adjoints $A^\dagger, B^\dagger$. Then, for any normalized vector ξ, we have*

$$2\,|\langle\xi|AB|\xi\rangle| \leq \big\langle\xi\big|AA^\dagger\big|\xi\big\rangle + \big\langle\xi\big|B^\dagger B\big|\xi\big\rangle$$

In short, we write

$$AB + B^\dagger A^\dagger \leq AA^\dagger + B^\dagger B.$$

Proof. By the standard Cauchy-Schwarz inequalities

$$\langle u|v\rangle \leq \|u\|\|v\|, \quad ab \leq \frac{a^2+b^2}{2}$$

we deduce

$$2\,|\langle\xi|AB|\xi\rangle| \leq 2\|A^\dagger\xi\|\|B\xi\| \leq \|A^\dagger\xi\|^2 + \|B\xi\|^2 = \big\langle\xi\big|AA^\dagger\big|\xi\big\rangle + \big\langle\xi\big|B^\dagger B\big|\xi\big\rangle\,,$$

because, for instance, $\|B\xi\|^2 = \langle B\xi|B\xi\rangle = \big\langle\xi\big|B^\dagger B\big|\xi\big\rangle$. □

The following Lemma will be useful later. We state it here primarily because its proof illustrates the general techniques used throughout this work.

Lemma 3.3. *Denote $\Phi = \sum_q \varphi_q a_q a_{-q}$. Then we have*

$$\|\Phi\eta\| \lesssim \kappa N^{-1/2}\|(\mathcal{N}_+ + 1)^{1/2}\eta\|.$$

Proof. We use the canonical commutation relations to write

$$\begin{aligned}\|\Phi\eta\|^2 &= \sum_{p,q}\varphi_p\varphi_q\left\langle\eta\middle|a^\dagger_{-p}a^\dagger_p a_q a_{-q}\middle|\eta\right\rangle \\ &= \sum_{p,q}\varphi_p\varphi_q\left\langle\eta\middle|a^\dagger_{-p}a_q a^\dagger_p a_{-q} + \delta_{p,q}a^\dagger_{-p}a_{-q}\middle|\eta\right\rangle\end{aligned}$$

and apply Cauchy-Schwarz on both terms separately to estimate

$$\|\Phi\eta\|^2 \leq \sum_{p,q}|\varphi_p|^2\|a^\dagger_p a_q\eta\|^2 + \sum_p|\varphi_p|^2\|a_p\eta\|^2.$$

In the second term, we bound $|\varphi_p|^2 \leq \kappa^2 N^{-2}$ by Lemma 3.1, while remaining the sum evaluates to

$$\sum_p\|a_p\eta\|^2 = \sum_p\big\langle\eta\big|a^\dagger_p a_p\big|\eta\big\rangle = \langle\eta|\mathcal{N}_+|\eta\rangle = \|\mathcal{N}_+^{1/2}\eta\|^2. \tag{16}$$

For the first term, we notice that a sum of the form

$$\sum_p\|a^\dagger_p\xi\|^2$$

generally diverges, because

$$\|a^\dagger_p\xi\|^2 = \big\langle\xi\big|a_p a^\dagger_p\big|\xi\big\rangle = \big\langle\xi\big|a^\dagger_p a_p + 1\big|\xi\big\rangle = \|a_p\xi\|^2 + \|\xi\|^2. \tag{17}$$

Instead, we use this with $\xi = a_q\eta$ in order to rewrite

$$\|a^\dagger_p a_q\eta\|^2 = \|a_p a_q\eta\|^2 + \|a_q\eta\|^2$$

and thus obtain

$$\begin{aligned}\sum_{p,q}|\varphi_p|^2\|a_p^\dagger a_q\eta\|^2 &= \sum_{p,q}|\varphi_p|^2\|a_pa_q\eta\|^2+\sum_{p,q}|\varphi_p|^2\|a_q\eta\|^2\\ &\leq \|\varphi\|_\infty^2\|(\mathcal{N}_++1)\eta\|^2+\|\varphi\|_2^2\|\mathcal{N}_+^{1/2}\eta\|^2\\ &\lesssim \kappa^2N^{-2}\|(\mathcal{N}_++1)\eta\|^2+\kappa^2N^{-3/2}\|\mathcal{N}_+^{1/2}\eta\|^2,\end{aligned}$$

having used a similar computation as (16) for the first term. Finally, estimating $\mathcal{N}_+\leq N^{1/2}\mathcal{N}_+^{1/2}$ yields the desired estimate. □

Before we can proceed with the bounds on the separate terms in (6), we would like to control the growth of the number of excitations under $e^{\mathcal{B}}$. While in the non-relativistic case, we would obtain $e^{-\mathcal{B}}(\mathcal{N}_++1)e^{\mathcal{B}}\lesssim\mathcal{N}_++1$, we will have a worse estimate in the pseudo-relativistic case. The fact that $\|N\varphi\|_2^2=N^{1/2}$ provides the heuristic explanation for the appearance of $\mathcal{N}_++N^{1/2}$ in the inequality.

Lemma 3.4. *Let $k\in\mathbb{N}$. On $\{\mathcal{N}=N\}$, we have the operator inequality*

$$e^{-s\mathcal{B}}\left(\mathcal{N}_++N^{1/2}\right)^ke^{s\mathcal{B}}\lesssim\left(\mathcal{N}_++N^{1/2}\right)^k$$

for all $s\in[0;1]$.

Of course, the inequality is not uniform in k. However, we will only need this inequality for $k=1,2$, so this does not pose a problem for our purposes. Let us also remark that in combination with Lemma 3.3, it implies the bound

$$\|\Phi e^{t\mathcal{B}}\xi\|\lesssim\kappa N^{-1/2}\|(\mathcal{N}_++N^{1/2})^{1/2}\xi\|. \tag{18}$$

for any $s\in[0;1]$.

Proof. The proof is a standard Gronwall argument, presented for example in [9]. For simplicity, let us denote $A=\mathcal{N}_++N^{1/2}$. Let $\xi\in\mathcal{F}$ be normalized with $\mathcal{N}\xi=N\xi$ and define

$$f(s):=\left\langle\xi\middle|e^{-s\mathcal{B}}A^ke^{s\mathcal{B}}\middle|\xi\right\rangle.$$

We start by computing $f'(s)$, which is given by

$$f'(s)=\left\langle\xi\middle|e^{-s\mathcal{B}}\Big[A^k,\mathcal{B}\Big]e^{s\mathcal{B}}\middle|\xi\right\rangle$$

To this end, the canonical commutation relations imply that $Aa_p^\dagger=a_p^\dagger(A+1)$, so

$$\Big[A^k,a_p^\dagger a_q^\dagger\Big]=a_p^\dagger\left((A+2)^k-A^k\right)a_q^\dagger=:a_p^\dagger g(\mathcal{N}_+)a_q^\dagger,$$

where we set $g(x)=(x+N^{1/2}+2)^k-(x+N^{1/2})^k$. Hence we can compute

$$\begin{aligned}\Big[A^k,\mathcal{B}\Big]&=\frac{1}{2}\sum_p\varphi_p\Big[A^k,a_p^\dagger a_{-p}^\dagger\Big]a_0a_0+\text{h.c.}\\ &=\frac{1}{2}\sum_p\varphi_pa_p^\dagger g(\mathcal{N}_+)a_{-p}^\dagger a_0a_0+\text{h.c.}\end{aligned}$$

in order to find

$$f'(s)=\frac{1}{2}\sum_p\varphi_p\left\langle\xi\middle|e^{-s\mathcal{B}}a_p^\dagger g(\mathcal{N}_+)a_{-p}^\dagger a_0a_0e^{s\mathcal{B}}\middle|\xi\right\rangle+\text{h.c.}.$$

After rearranging the terms in the inner product, we apply Cauchy-Schwarz to get

$$\begin{aligned}|f'(s)|&\leq\sum_p|\varphi_p|\left|\left\langle g(\mathcal{N}_+)^{1/2}a_pe^{s\mathcal{B}}\xi\middle|g(\mathcal{N}_+)^{1/2}a_{-p}^\dagger a_0a_0e^{s\mathcal{B}}\xi\right\rangle\right|\\ &\lesssim\sum_p|\varphi_p|\left\|g(\mathcal{N}_+)^{1/2}a_pe^{s\mathcal{B}}\xi\right\|\left\|g(\mathcal{N}_+)^{1/2}a_{-p}^\dagger a_0a_0e^{s\mathcal{B}}\xi\right\|.\end{aligned}$$

Let us remove the $a_0 a_0$ operator in the second norm. Since

$$a_0^\dagger a_0^\dagger a_0 a_0 = \mathcal{N}_0(\mathcal{N}_0 - 1) \leq N(N-1) \leq N^2,$$

we find

$$\|a_0 a_0 e^{s\mathcal{B}}\xi\|^2 = \left\langle e^{s\mathcal{B}}\xi \middle| a_0^\dagger a_0^\dagger a_0 a_0 \middle| e^{s\mathcal{B}}\xi \right\rangle \leq N^2 \|e^{s\mathcal{B}}\xi\|^2.$$

Using that $a_0, a_0^\dagger$ commute with $a_{-p}, a_{-p}^\dagger$ and $\mathcal{N}_+$, we thus obtain

$$\left\| g(\mathcal{N}_+)^{1/2} a_{-p}^\dagger a_0 a_0 e^{s\mathcal{B}}\xi \right\| \leq N \left\| g(\mathcal{N}_+)^{1/2} a_{-p}^\dagger e^{s\mathcal{B}}\xi \right\|$$

and hence

$$|f'(s)| \lesssim N \sum_p |\varphi_p| \left\| g(\mathcal{N}_+)^{1/2} a_p e^{s\mathcal{B}}\xi \right\| \left\| g(\mathcal{N}_+)^{1/2} a_{-p}^\dagger e^{s\mathcal{B}}\xi \right\|.$$

Ideally, we would apply $2ab \leq a^2 + b^2$ on the norms, but the last factor is not square-summable in p, so we have to work harder. We may replace $a_{-p}^\dagger$ with a_{-p} with the price of adding another norm, as in (17). More precisely, after commuting $g(\mathcal{N}_+)$ with $a_{-p}^\dagger$, we find

$$\begin{aligned}
\left\| g(\mathcal{N}_+)^{1/2} a_{-p}^\dagger e^{s\mathcal{B}}\xi \right\|^2 &= \left\| a_{-p}^\dagger g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2 \\
&= \left\| a_{-p} g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2 + \left\| g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2 \\
&\leq \left(\left\| a_{-p} g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\| + \left\| g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\| \right)^2
\end{aligned}$$

and so we have

$$\begin{aligned}
|f'(s)| &\lesssim N \sum_p |\varphi_p| \left\| a_p g(\mathcal{N}_+ - 1)^{1/2} e^{s\mathcal{B}}\xi \right\| \left\| a_{-p} g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\| \\
&\quad + N \sum_p |\varphi_p| \left\| a_p g(\mathcal{N}_+ - 1)^{1/2} e^{s\mathcal{B}}\xi \right\| \left\| g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

From the first term, we extract the infinity-norm of φ_p and split the rest with Cauchy-Schwarz. Both of the resulting terms are similar (they just differ in $\mathcal{N}_+ - 1$ and $\mathcal{N}_+ + 1$), hence

$$\mathrm{I} \lesssim N \|\varphi\|_\infty \sum_p \left\| a_p g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2 \lesssim \left\| \mathcal{N}_+^{1/2} g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2,$$

where in the last inequality we used Lemma 3.1 and (16). Now, because $g(x+1) \lesssim (x + N^{1/2} + 1)^{k-1}$ and $g(\mathcal{N}_+ + 1)$ commutes with $\mathcal{N}_+$, we conclude that

$$\mathrm{I} \lesssim \left\langle e^{s\mathcal{B}}\xi \middle| \mathcal{N}_+^{1/2} g(\mathcal{N}_+ + 1) \mathcal{N}_+^{1/2} \middle| e^{s\mathcal{B}}\xi \right\rangle \lesssim \left\langle e^{s\mathcal{B}}\xi \middle| \mathcal{N}_+^{1/2} A^{k-1} \mathcal{N}_+^{1/2} \middle| e^{s\mathcal{B}}\xi \right\rangle \leq f(s).$$

On the other hand, for II we must split off φ_p with Cauchy-Schwarz. We find

$$\mathrm{II} \lesssim N^2 \|\varphi\|_2^2 \left\| g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2 + \sum_p \left\| g(\mathcal{N}_+)^{1/2} a_p e^{s\mathcal{B}}\xi \right\|^2.$$

The second term can be treated as above to bound it by $f(s)$. For the first term, we use Lemma 3.1 and again $g(x+1) \lesssim (x + N^{1/2} + 1)^{k-1}$ to obtain

$$N^2 \|\varphi\|_2^2 \left\| g(\mathcal{N}_+ + 1)^{1/2} e^{s\mathcal{B}}\xi \right\|^2 \lesssim N^{1/2} \left\langle e^{s\mathcal{B}}\xi \middle| A^{k-1} \middle| e^{s\mathcal{B}}\xi \right\rangle \lesssim f(s),$$

having used that $1 \leq N^{1/2} \leq A$. In total, we conclude that $|f'(s)| \lesssim f(s)$ and hence, by Gronwall's Lemma, we get

$$f(s) \lesssim f(0) = \left\langle \xi \middle| (\mathcal{N}_+ + N^{1/2})^k \middle| \xi \right\rangle,$$

which concludes the proof. □

3.3 Conjugation of separate terms

In this technical section, we consider the conjugation of all terms appearing in (6). The term H_0 is a constant and thus remains invariant under such conjugation. The conjugation of H_2 yields no leading-order terms and is fully absorbed into the error.

Lemma 3.5. *On $\{\mathcal{N} = N\}$, we have*

$$\pm e^{-\mathcal{B}} H_2 e^{\mathcal{B}} \lesssim \kappa(\mathcal{N}_+ + N^{1/2}).$$

Proof. We recall

$$H_2 = \kappa(N - \mathcal{N}_+) \sum_p \widehat{V}_N(p) a_p^\dagger a_p - \kappa \frac{\mathcal{N}_+(\mathcal{N}_+ - 1)}{2} \widehat{V}_N(0).$$

For the first term, extracting $\widehat{V}_N(p)$ in infinity-norm yields

$$\kappa(N - \mathcal{N}_+) \sum_{p \neq 0} \widehat{V}_N(p) a_p^\dagger a_p \lesssim \kappa \frac{N - \mathcal{N}_+}{N} \sum_p a_p^\dagger a_p \lesssim \kappa \mathcal{N}_+.$$

On the other hand, the second term can be estimated using $\mathcal{N}_+(\mathcal{N}_+ - 1) \leq \mathcal{N}_+^2 \leq N\mathcal{N}_+$, so that

$$\kappa \frac{\mathcal{N}_+(\mathcal{N}_+ - 1)}{2} \widehat{V}_N(0) \lesssim \kappa N^{-1} \mathcal{N}_+^2 \lesssim \kappa \mathcal{N}_+.$$

Put together, this implies that

$$\pm e^{-\mathcal{B}} H_2 e^{\mathcal{B}} \lesssim e^{-\mathcal{B}} \kappa \mathcal{N}_+ e^{\mathcal{B}} \lesssim \kappa(\mathcal{N}_+ + N^{1/2}),$$

having used Lemma 3.4. □

In order to find a lower bound on the ground state energy, we need to follow a fundamentally different approach compared to [9, 10]. In the pseudo-relativistic regime, the weaker decay of $\|\varphi\|_2$ causes the appearance of higher powers of $\mathcal{N}_+$ which cannot be controlled efficiently. More precisely, if the conjugation of Q_4 is expanded as

$$e^{-\mathcal{B}} Q_4 e^{\mathcal{B}} = Q_4 + \int_0^1 e^{-t\mathcal{B}} \Big[Q_4, \mathcal{B} \Big] e^{t\mathcal{B}} dt,$$

the second term produces, apart from leading order terms, also a term that can only be bounded by

$$\frac{1}{N^{1/2}} \mathcal{N}_+^2,$$

which is insufficient to prove condensation. The appearance of such terms is derived later in Lemma 3.10 and used for the upper bound. Instead, we will rewrite the conjugation of Q_4 in an appropriate way, extracting the term which will renormalize the ground state energy while reorganizing everything else into a positive operator which we can use to absorb contributions from the conjugations of H_1, Q_2 and Q_3.

To this end, let us compute $\Big[\check{a}_x^\dagger \check{a}_y^\dagger, \mathcal{B} \Big]$ in position space. Because

$$\mathcal{B} = \frac{1}{2} \int du dv \check{\varphi}(u - v) \check{a}_u^\dagger \check{a}_v^\dagger a_0 a_0 - \text{h.c.},$$

we find

$$\begin{aligned}
\Big[\check{a}_x^\dagger \check{a}_y^\dagger, \mathcal{B} \Big] &= \frac{1}{2} \int du dv \check{\varphi}(u - v) a_0^\dagger a_0^\dagger \Big[\check{a}_u \check{a}_v, \check{a}_x^\dagger \check{a}_y^\dagger \Big] \\
&= \frac{1}{2} \int du dv \check{\varphi}(u - v) a_0^\dagger a_0^\dagger \Big(\check{a}_u \check{a}_x^\dagger \delta(v - y) + \check{a}_u \check{a}_y^\dagger \delta(v - x) + \check{a}_x^\dagger \check{a}_v \delta(u - y) + \check{a}_y^\dagger \check{a}_v \delta(u - x) \Big) \\
&= \check{\varphi}(x - y) a_0^\dagger a_0^\dagger + \Big(\check{a}_x^\dagger \check{a}(\check{\varphi}^y) + \check{a}_y^\dagger \check{a}(\check{\varphi}^x) \Big) a_0^\dagger a_0^\dagger,
\end{aligned}$$

where $f^y(\cdot) = f(\cdot - y)$. This leads to

$$\begin{aligned} e^{-\mathcal{B}}\check{a}_x^\dagger \check{a}_y^\dagger e^{\mathcal{B}} &= \check{a}_x^\dagger \check{a}_y^\dagger + \int_0^1 ds e^{-s\mathcal{B}}\Big[\check{a}_x^\dagger \check{a}_y^\dagger, \mathcal{B}\Big] e^{s\mathcal{B}} \\ &= \check{a}_x^\dagger \check{a}_y^\dagger + \check{\varphi}(x-y)\int_0^1 ds e^{-s\mathcal{B}} a_0^\dagger a_0^\dagger e^{s\mathcal{B}} \\ &\quad + \int_0^1 ds e^{-s\mathcal{B}}\Big(\check{a}_x^\dagger \check{a}(\check{\varphi}^y) + \check{a}_y^\dagger \check{a}(\check{\varphi}^x)\Big) a_0^\dagger a_0^\dagger e^{s\mathcal{B}} \\ &= \check{a}_x^\dagger \check{a}_y^\dagger + \check{\varphi}(x-y) a_0^\dagger a_0^\dagger + \check{\varphi}(x-y) R^\dagger + C^\dagger(x,y), \end{aligned} \tag{19}$$

where the extra terms are defined by

$$R^\dagger := \int_0^1 ds \int_0^s dt e^{-t\mathcal{B}}\Big[a_0^\dagger a_0^\dagger, \mathcal{B}\Big] e^{t\mathcal{B}}, \tag{20}$$

$$C^\dagger(x,y) := \int_0^1 ds e^{-s\mathcal{B}}\Big(\check{a}_x^\dagger \check{a}(\check{\varphi}^y) + \check{a}_y^\dagger \check{a}(\check{\varphi}^x)\Big) a_0^\dagger a_0^\dagger e^{s\mathcal{B}}. \tag{21}$$

Using $a_0^\dagger a_0^\dagger a_0 a_0 = N(N-1) - \mathcal{N}_+(2N - \mathcal{N}_+ - 1)$, this allows us to expand

$$\begin{aligned} e^{-\mathcal{B}} Q_4 e^{\mathcal{B}} &= \frac{\kappa}{2}\int dxdy V_N(x-y) e^{-\mathcal{B}} \check{a}_x^\dagger \check{a}_y^\dagger e^{\mathcal{B}} e^{-\mathcal{B}} \check{a}_y \check{a}_x e^{\mathcal{B}} \\ &= \frac{N(N-1)}{2}\kappa \int dxdy V_N(x-y)|\check{\varphi}(x-y)|^2 \\ &\quad + \frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{\varphi}(x-y) a_0^\dagger a_0^\dagger \check{a}_x \check{a}_y + \text{h.c.}\right) \\ &\quad + S + \mathcal{E}_{Q_4}. \end{aligned} \tag{22}$$

Most notably, the term S on the last line, which is defined by

$$S := \frac{\kappa}{2}\int dxdy V_N(x-y)\Big(\check{a}_x^\dagger \check{a}_y^\dagger + C^\dagger(x,y)\Big)\Big(\check{a}_y \check{a}_x + C(x,y)\Big)$$

is non-negative. The first term, alternatively expressed as

$$\frac{N(N-1)}{2}\kappa \int dxdy V_N(x-y)|\check{\varphi}(x-y)|^2 = \frac{N(N-1)}{2}\kappa \sum_{p,q} \widehat{V}_N(p-q)\varphi_p \varphi_q,$$

will give a contribution to the leading order of the ground state energy, while the second term can be rewritten as

$$\frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{\varphi}(x-y) a_0^\dagger a_0^\dagger \check{a}_x \check{a}_y + \text{h.c.}\right) = \frac{\kappa}{2}\sum_p (\widehat{V}_N * \varphi)(p) a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}$$

and will be combined with similar terms arising from the conjugation of H_1 and Q_2. The error term can be estimated as follows.

Lemma 3.6. *For any $\lambda > 0$, we have*

$$\pm \mathcal{E}_{Q_4} \lesssim \lambda S + \kappa(1 + \lambda^{-1})(\mathcal{N}_+ + N^{1/2}).$$

Proof. An explicit computation shows that the error term consists of five terms and is given by

$$\begin{aligned} \mathcal{E}_{Q_4} &= \frac{\kappa}{2}\int dxdy V_N(x-y) R^\dagger R + \frac{\kappa}{2}\int dxdy V_N(x-y)|\check{\varphi}(x-y)|^2 R^\dagger a_0 a_0 \\ &\quad + \frac{\kappa}{2}\int dxdy V_N(x-y)\check{\varphi}(x-y) R^\dagger \Big[\check{a}_x \check{a}_y + C(x,y)\Big] \\ &\quad + \frac{\kappa}{2}\int dxdy V_N(x-y)\check{\varphi}(x-y) a_0^\dagger a_0^\dagger C(x,y) \\ &\quad - \mathcal{N}_+(2N - \mathcal{N}_+ - 1)\frac{\kappa}{2}\int dxdy V_N(x-y)|\check{\varphi}(x-y)|^2 + \text{h.c.} \\ &=: \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} \end{aligned} \tag{23}$$

The last term V is easy to estimate, since the integral is bounded by $1/N$, cancelling the $(2N - \mathcal{N}_+ - 1)$ and thus leaving only the $\kappa\mathcal{N}_+$. Next, let us estimate R. In fact, since $\left[a_0 a_0, \mathcal{B}\right] = -\Phi(2\mathcal{N}_0 + 1)$, we have

$$R = -\int_0^1 ds \int_0^s dt e^{-t\mathcal{B}} \Phi(2\mathcal{N}_0 + 1) e^{t\mathcal{B}}$$

and hence, using (18) we find

$$\|R\xi\| \lesssim \kappa N^{1/2} \|(\mathcal{N}_+ + N^{1/2})^{1/2}\xi\|. \tag{24}$$

Since $\langle \xi | R^\dagger R | \xi \rangle = \|R\xi\|^2$, this immediately implies that

$$\pm \mathrm{I} \lesssim \kappa^3 \|V_N\|_1 N(\mathcal{N}_+ + N^{1/2}) \lesssim \kappa^3(\mathcal{N}_+ + N^{1/2}). \tag{25}$$

For the second term, we need to work more carefully. We write $\Phi = \sum \varphi_p a_p a_{-p}$ and see

$$\begin{aligned}
\pm \langle \xi | \mathrm{II} | \xi \rangle &\lesssim \pm\kappa \int dxdy V_N(x-y) |\check{\varphi}(x-y)|^2 \left\langle \xi \middle| R^\dagger a_0 a_0 + \text{h.c.} \middle| \xi \right\rangle \\
&\lesssim \pm\kappa \|V_N\|_1 \|\check{\varphi}\|_\infty^2 \left\langle \xi \middle| R^\dagger a_0 a_0 + \text{h.c.} \middle| \xi \right\rangle \\
&\lesssim \kappa^3 N^{-1} \int_0^1 ds \int_0^s dt \sum_p \left| \left\langle \xi \middle| e^{-t\mathcal{B}} \varphi_p a_p^\dagger a_{-p}^\dagger (2\mathcal{N}_0 + 1) e^{t\mathcal{B}} a_0 a_0 \middle| \xi \right\rangle \right| \\
&\lesssim \kappa^3 N^{-1} \int_0^1 ds \int_0^s dt \left(\sum_p \|a_p e^{t\mathcal{B}} \xi\|^2 \right)^{1/2} \left(\sum_p |\varphi_p|^2 \|a_{-p}^\dagger (2\mathcal{N}_0 + 1) e^{t\mathcal{B}}\, a_0 a_0 \xi\|^2 \right)^{1/2}
\end{aligned} \tag{26}$$

As usual, we split $\|a_p^\dagger \eta\|^2 = \|a_p \eta\|^2 + \|\eta\|^2$ and take φ in the infinity norm for the first term, while we take it in the 2-norm for the second term. More precisely,

$$\begin{aligned}
\sum_p |\varphi_p|^2 \|a_{-p}^\dagger (2\mathcal{N}_0 + 1) e^{t\mathcal{B}}\, a_0 a_0 \xi\|^2 &\lesssim \sum_p |\varphi_p|^2 \|a_{-p} (2\mathcal{N}_0 + 1) e^{t\mathcal{B}}\, a_0 a_0 \xi\|^2 \\
&\quad + \sum_p |\varphi_p|^2 \|(2\mathcal{N}_0 + 1) e^{t\mathcal{B}}\, a_0 a_0 \xi\|^2 \\
&\lesssim \|\varphi\|_\infty^2 N^4 \|(\mathcal{N}_+ + N^{1/2})^{1/2} \xi\|^2 + \|\varphi\|_2^2 N^4 \|\xi\|^2 \\
&\lesssim \kappa^2 N^2 \left\langle \xi \middle| \mathcal{N}_+ + N^{1/2} \middle| \xi \right\rangle + \kappa^2 N^{5/2},
\end{aligned}$$

having used Lemma 3.1, Lemma 3.4 and the estimates $\mathcal{N}_0 \leq N$, $a_0^\dagger a_0^\dagger a_0 a_0 \leq N^2$. On the other hand, the first factor in (26) is simply estimated by

$$\sum_p \|a_p e^{t\mathcal{B}} \xi\|^2 \lesssim \left\langle \xi \middle| \mathcal{N}_+ + N^{1/2} \middle| \xi \right\rangle$$

by virtue of Lemma 3.4. Ultimately, this yields

$$\pm \mathrm{II} \lesssim \kappa^3(\mathcal{N}_+ + N^{1/2}).$$

For III, we use Cauchy-Schwarz to obtain

$$\begin{aligned}
\mathrm{III} &= \frac{\kappa}{2} \int dxdy V_N(x-y) \check{\varphi}(x-y) R^\dagger (a_x a_y + C(x,y)) \\
&\lesssim \delta\kappa \int dxdy V_N(x-y) |\check{\varphi}(x-y)|^2 R^\dagger R \\
&\qquad + \delta^{-1} \frac{\kappa}{2} \int dxdy V_N(x-y) (a_x^\dagger a_y^\dagger + C^\dagger(x,y))(a_x a_y + C(x,y)) \\
&\lesssim \delta\kappa^3 (\mathcal{N}_+ + N^{1/2}) + \delta^{-1} S,
\end{aligned}$$

where the bound for the first term comes from (25) together with $\|\check{\varphi}\|_\infty^2 \lesssim 1$ and we choose $\delta = \lambda^{-1}$. For the last term on the second line of (23), we use the definition of $C(x,y)$ from (21) to find

$$\mathrm{IV} = \frac{\kappa}{2}\int_0^1 ds \int dxdy V_N(x-y)\check{\varphi}(x-y)a_0^\dagger a_0^\dagger e^{-s\mathcal{B}}a_0 a_0 a^\dagger(\check{\varphi}^y)\check{a}_x e^{s\mathcal{B}} + \text{ symm.},$$

where "symm." denotes terms with exchanged x's and y's. We use Cauchy-Schwarz to split between $a^\dagger(\check{\varphi}^y)$ and $\check{a}_x$. The second term produced from this can be estimated through

$$\delta^{-1}\kappa\int_0^1 ds\int dxdy V_N(x-y)|\check{\varphi}(x-y)|e^{-s\mathcal{B}}\check{a}_x^\dagger\check{a}_x e^{s\mathcal{B}} \lesssim \delta^{-1}\kappa^2 N^{-1}(\mathcal{N}_+ + N^{1/2}),$$

where the N^{-1} appears from the integral of V_N. On the other hand, using that

$$\int dy a^\dagger(\check{\varphi}^y)a(\check{\varphi}^y) = \sum_p |\varphi_p|^2 a_p^\dagger a_p \leq \|\varphi\|_\infty^2 \mathcal{N}_+,$$

the first term will be estimated by

$$\begin{aligned}\delta\kappa\int_0^1 ds\int dxdy V_N(x-y)|\check{\varphi}(x-y)|a_0^\dagger a_0^\dagger e^{-s\mathcal{B}}a_0 a_0 a^\dagger(\check{\varphi}^y)a(\check{\varphi}^y)a_0^\dagger a_0^\dagger e^{s\mathcal{B}}a_0 a_0 \\ \lesssim \delta\kappa^2 N^{-1}N^4\|\varphi\|_\infty^2(\mathcal{N}_+ + N^{1/2}) \\ \lesssim \delta\kappa^4 N^1(\mathcal{N}_+ + N^{1/2}).\end{aligned}$$

Choosing $\delta = \kappa^{-1}N^{-1}$ concludes the proof. □

The advantage of this procedure is that we can now bound the conjugation of Q_3 more efficiently.

Lemma 3.7. *We have, for any $\lambda > 0$,*

$$\pm e^{-\mathcal{B}}Q_3 e^{\mathcal{B}} \lesssim \lambda S + \kappa(1+\lambda^{-1})(\mathcal{N}_+ + N^{1/2}).$$

Proof. We pass to position space, where

$$Q_3 = \kappa\int dxdy V_N(x-y)\check{a}_x^\dagger\check{a}_y^\dagger\check{a}_x a_0 + \text{h.c.}.$$

Using (19) we find

$$\begin{aligned}e^{-\mathcal{B}}Q_3 e^{\mathcal{B}} &= \kappa\int dxdy V_N(x-y)\left(\check{a}_x^\dagger\check{a}_y^\dagger + C^\dagger(x,y) + \check{\varphi}(x-y)(a_0^\dagger a_0^\dagger + R^\dagger)\right)e^{-\mathcal{B}}\check{a}_x a_0 e^{\mathcal{B}} \\ &= \kappa\int dxdy V_N(x-y)\left(\check{a}_x^\dagger\check{a}_y^\dagger + C^\dagger(x,y)\right)e^{-\mathcal{B}}\check{a}_x a_0 e^{\mathcal{B}} \\ &\quad + \kappa\int dxdy V_N(x-y)\check{\varphi}(x-y)(a_0^\dagger a_0^\dagger + R^\dagger)e^{-\mathcal{B}}\check{a}_x a_0 e^{\mathcal{B}}.\end{aligned}$$

In the second term, the y-integral is independent of x, and the remaining x-integral is $\int dx\check{a}_x = 0$, because the distribution-valued operators act only on the orthogonal complement of the zero-momentum state. For the first term, we use Cauchy-Schwarz to find

$$\begin{aligned}\kappa\int dxdy V_N(x-y)\left(\check{a}_x^\dagger\check{a}_y^\dagger + C^\dagger(x,y)\right)e^{-\mathcal{B}}\check{a}_x a_0 e^{\mathcal{B}} &\lesssim \lambda S + \lambda^{-1}\kappa\int dxdy V_N(x-y)e^{-\mathcal{B}}a_0^\dagger\check{a}_x^\dagger\check{a}_x a_0 e^{\mathcal{B}} \\ &\lesssim \lambda S + \lambda^{-1}\kappa N\|V_N\|_1 e^{-\mathcal{B}}\,\mathcal{N}_+ e^{\mathcal{B}} \\ &\lesssim \lambda S + \lambda^{-1}\kappa(\mathcal{N}_+ + N^{-1/2}),\end{aligned}$$

having used that $\mathcal{N}_+$ commutes with $\mathcal{N}_0$, as well as $\mathcal{N}_0 \leq N$. □

Next, we extract the contribution from Q_2.

Lemma 3.8. *We have*

$$e^{-\mathcal{B}}\ Q_2 e^{\mathcal{B}} = Q_2 + N(N-1)\kappa \sum_p \widehat{V}_N(p)\varphi_p + \mathcal{E}_{Q_2},$$

where, for any $\lambda > 0$, the error can be estimated with

$$\pm\mathcal{E}_{Q_2} \lesssim \lambda S + \kappa(1+\lambda^{-1})(\mathcal{N}_+ + N^{1/2}).$$

Proof. We pass to position space and use (19) so that we find

$$\begin{aligned} e^{-\mathcal{B}} Q_2 e^{\mathcal{B}} &= \frac{\kappa}{2}\int dxdy V_N(x-y) e^{-\mathcal{B}} \check{a}_x^\dagger \check{a}_y^\dagger e^{\mathcal{B}} e^{-\mathcal{B}} a_0 a_0 e^{\mathcal{B}} + \text{h.c.} \\ &= \frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{a}_x^\dagger \check{a}_y^\dagger + C^\dagger(x,y)\right)(a_0 a_0 + \tilde{R}) \\ &\quad + \frac{\kappa}{2}\int dxdy V_N(x-y)\check{\varphi}(x-y)(a_0^\dagger a_0^\dagger + R^\dagger)(a_0 a_0 + \tilde{R}) + \text{h.c.} \end{aligned} \tag{27}$$

where we defined

$$\tilde{R} := \int_0^1 ds e^{-s\mathcal{B}}\Big[a_0 a_0, \mathcal{B}\Big] e^{s\mathcal{B}}$$

so that $e^{-\mathcal{B}} a_0 a_0 e^{\mathcal{B}} = a_0 a_0 + \tilde{R}$. We consider the two contributions separately. The term on the first line can be rearranged to

$$\begin{aligned} &\frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{a}_x^\dagger \check{a}_y^\dagger + C^\dagger(x,y)\right)(a_0 a_0 + \tilde{R}) + \text{h.c.} \\ &\quad = Q_2 + \frac{\kappa}{2}\int dxdy V_N(x-y) C^\dagger(x,y) a_0 a_0 + \frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{a}_x^\dagger \check{a}_y^\dagger + C^\dagger(x,y)\right)\tilde{R} + \text{h.c.} \end{aligned}$$

The first error can be bounded in the same way as the term IV in the proof of Lemma 3.6, while the second term can be estimated with Cauchy-Schwarz via

$$\pm\frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{a}_x^\dagger \check{a}_y^\dagger + C^\dagger(x,y)\right)\tilde{R} + \text{h.c.} \lesssim \lambda S + \kappa\lambda^{-1}\int dxdy V_N(x-y)\tilde{R}^\dagger \tilde{R}$$

and the second appearing term can be treated similarly to I from Lemma 3.6, since $\tilde{R}$ also satisfies (24). Using that $a_0^\dagger a_0^\dagger a_0 a_0 = N(N-1) - \mathcal{N}_+(2N - \mathcal{N}_+ - 1)$, the term appearing on the second line of (27) can be rewritten as

$$\begin{aligned} &\frac{\kappa}{2}\int dxdy V_N(x-y)\check{\varphi}(x-y)(a_0^\dagger a_0^\dagger + R^\dagger)(a_0 a_0 + \tilde{R}) \\ &\quad = \frac{N(N-1)}{2}\kappa\int dxdy V_N(x-y)\check{\varphi}(x-y) + \text{h.c.} \\ &\qquad + \frac{\kappa}{2}\int dxdy V_N(x-y)\check{\varphi}(x-y)\left(R^\dagger a_0 a_0 + a_0^\dagger a_0^\dagger \tilde{R}\right) + \frac{\kappa}{2}\int dxdy V_N(x-y)\check{\varphi}(x-y) R^\dagger \tilde{R} + \text{h.c.} \\ &\qquad + \mathcal{N}_+(2N - \mathcal{N}_+ - 1)\frac{\kappa}{2}\int dxdy V_N(x-y)(\check{\varphi}(x-y) + \text{h.c.}). \end{aligned}$$

The first term can be treated just like II from Lemma 3.6, while the second term is similar to I. The integral in the term on the last line is bounded by $1/N$, cancelling the $(2N - \mathcal{N}_+ - 1)$. Rewriting the term proportional to $N(N-1)$ in momentum space yields the result. □

It is left to extract the important terms from the conjugation of the kinetic energy H_1. Here, we will need to work in momentum space and go up to second order in the expansion.

Lemma 3.9. *We have*

$$e^{-\mathcal{B}} H_1 e^{\mathcal{B}} = H_1 + N(N-1)\sum_p T_p \varphi_p^2 + \sum_p T_p \varphi_p \left(a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}\right) + \mathcal{E}_{H_1}$$

where the error satisfies

$$\pm\mathcal{E}_{H_1} \lesssim \lambda S + \kappa(1+\lambda^{-1})(\mathcal{N}_+ + N^{1/2}).$$

Proof. We write

$$e^{-\mathcal{B}}H_1e^{\mathcal{B}} = H_1 + \Big[H_1,\mathcal{B}\Big] + \int_0^1 dt\int_0^t ds e^{-s\mathcal{B}}\Big[[H_1,\mathcal{B}],\mathcal{B}\Big]e^{s\mathcal{B}}.$$

It is easy to compute the first commutator, evaluating to

$$\Big[H_1,\mathcal{B}\Big] = \sum_p T_p\varphi_p a_p^\dagger a_{-p}^\dagger a_0a_0 + \text{h.c.}.$$

The second commutators involves more terms and is split into parts. As presented, for instance, in [10, Lemma 8], we make use of the relations

$$\begin{aligned}\Big[a_{-q}a_q, a_p^\dagger a_{-p}^\dagger\Big] &= (\delta_{p,q}+\delta_{p,-q})(1+a_p^\dagger a_p + a_{-p}^\dagger a_{-p}),\\ \Big[a_0a_0, a_0^\dagger a_0^\dagger\Big] &= 2(2\mathcal{N}_0+1),\\ a_0^\dagger a_0^\dagger a_0a_0 &= \mathcal{N}_0(\mathcal{N}_0-1) = N(N-1) - 2N\mathcal{N}_+ + \mathcal{N}_+(\mathcal{N}_++1),\end{aligned}$$

to deduce

$$\begin{aligned}\Big[[H_1,\mathcal{B}],\mathcal{B}\Big] &= -\frac{\kappa}{2}\sum_{p,q}T_p\varphi_p\varphi_q\Big[a_p^\dagger a_{-p}^\dagger a_0a_0, a_0^\dagger a_0^\dagger a_{-q}a_q\Big] + \text{h.c.}\\ &= \frac{\kappa}{2}\sum_{p,q}T_p\varphi_p\varphi_q\left(\Big[a_{-q}a_q,a_p^\dagger a_{-p}^\dagger\Big]a_0^\dagger a_0^\dagger a_0a_0 - a_p^\dagger a_{-p}^\dagger a_{-q}a_q\Big[a_0a_0,a_0^\dagger a_0^\dagger\Big]\right) + \text{h.c.}\\ &= 2N(N-1)\kappa\sum_p T_p\varphi_p^2\\ &\quad + 2\kappa\sum_p T_p\varphi_p^2\Big(\mathcal{N}_+(\mathcal{N}_++1) - 2N\mathcal{N}_+\Big)\\ &\quad + 4\kappa\sum_p T_p\varphi_p^2\mathcal{N}_0(\mathcal{N}_0-1)a_p^\dagger a_p\\ &\quad - \kappa\sum_{p,q}T_p\varphi_p\varphi_q a_p^\dagger a_{-p}^\dagger a_{-q}a_q(2\mathcal{N}_0+1) + \text{h.c.}.\end{aligned}\tag{28}$$

The term on the first line, together with the first commutator and H_1 give all appearing terms (note that for the term on the first line, the integral produces a factor $1/2$). Thus, it is left to estimate the remaining terms. Using that

$$\sum_p T_p\varphi_p^2 \lesssim \kappa^2 N^{-1},$$

the terms on the second line are easily bounded by $\kappa^3(\mathcal{N}_++1)$. For the term on the third line, we use that $|T_p\varphi_p^2| \lesssim \kappa^2N^{-2}$, which also bounds this term by $\kappa^3(\mathcal{N}_++1)$. It is left to treat the final term on the last line of (28). With the notation

$$R_s = -e^{-s\mathcal{B}}\Phi(2\mathcal{N}_0+1)e^{s\mathcal{B}},$$

where $\Phi = \sum_q \varphi_q a_q a_{-q}$, we denote by

$$E := -\kappa\int_0^1 dt\int_0^t ds\sum_p T_p\varphi_p e^{-s\mathcal{B}}a_p^\dagger a_{-p}^\dagger e^{s\mathcal{B}}R_s + \text{h.c.}$$

the remaining term we have to bound. Observe that the term R_s satisfies the same estimate (24) that R, defined in (20), does. Denoting $f = 1+\check{\varphi}$ in position space, the scattering equation (10) implies that

$$T_p\varphi_p = -\frac{\kappa}{2}\left(\widehat{V}_N * \widehat{f}\right)(p).$$

Inserting this into E and passing to position space, we find

$$E = \frac{\kappa^2}{2}\int_0^1 dt\int_0^t ds\int dxdyV_N(x-y)f(x-y)e^{-s\mathcal{B}}\check{a}_x^\dagger\check{a}_y^\dagger e^{s\mathcal{B}}R_s + \text{h.c.}.$$

We expand the conjugation of $\check{a}_x^\dagger \check{a}_y^\dagger$ so as to recover some terms from (19), namely

$$e^{-s\mathcal{B}}\check{a}_x^\dagger \check{a}_y^\dagger e^{s\mathcal{B}} = e^{-\mathcal{B}}\check{a}_x^\dagger \check{a}_y^\dagger e^{\mathcal{B}} - \int_s^1 dt e^{-t\mathcal{B}}\Big[\check{a}_x^\dagger \check{a}_y^\dagger, \mathcal{B}\Big] e^{t\mathcal{B}} =: \mathrm{A} - \mathrm{B}.$$

Rearranging the terms coming from A appropriately, we find

$$\begin{aligned} E_{\mathrm{A}} = \frac{\kappa^2}{2}\int_0^1 dt \int_0^t ds \int dxdy V_N(x-y) f(x-y)\left(\check{a}_x^\dagger \check{a}_y^\dagger + C^\dagger(x,y)\right) R_s \\ + \frac{\kappa^2}{2}\int_0^1 dt \int_0^t ds \int dxdy V_N(x-y) f(x-y)\check{\varphi}(x-y)\left(a_0^\dagger a_0^\dagger + R^\dagger\right) R_s + \text{h.c.}. \end{aligned}$$

Both of these terms are similar to those treated in the proof of Lemma 3.6 because $\|f\|_\infty \lesssim 1$ and they can be treated just as illustrated there. On the other hand, the estimate for the terms arising from B is slightly more technical, since we cannot recover the positive operator S. Defining

$$\begin{aligned} \tilde{R}_s^\dagger &:= \int_s^1 d\tau \int_0^\tau dt e^{-t\mathcal{B}}\Big[a_0^\dagger a_0^\dagger, \mathcal{B}\Big] e^{t\mathcal{B}}, \\ \tilde{C}_s^\dagger(x,y) &:= \int_s^1 d\tau e^{-\tau\mathcal{B}}\Big(\check{a}_x^\dagger \check{a}(\check{\varphi}^y) + \check{a}_y^\dagger \check{a}(\check{\varphi}^x)\Big) a_0^\dagger a_0^\dagger e^{\tau\mathcal{B}}, \end{aligned}$$

we still recover the form

$$\mathrm{B} = \check{\varphi}(x-y)(a_0^\dagger a_0^\dagger + \tilde{R}_s^\dagger) + \tilde{C}_s^\dagger(x,y).$$

This leads to

$$\begin{aligned} E_{\mathrm{B}} = \frac{\kappa^2}{2}\int_0^1 dt \int_0^t ds \int dxdy V_N(x-y) f(x-y)\check{\varphi}(x-y)(a_0^\dagger a_0^\dagger + \tilde{R}_s^\dagger) R_s \\ + \frac{\kappa^2}{2}\int_0^1 dt \int_0^t ds \int dxdy V_N(x-y) f(x-y)\tilde{C}_s^\dagger(x,y) R_s + \text{h.c.}. \end{aligned}$$

The term on the first line is treated like the terms I, II from Lemma 3.6, while the term on the second line can be bounded with a similar procedure to IV from the same Lemma. We omit the details. □

Finally, we also provide an upper bound for S.

Lemma 3.10. *We have*

$$0 \leq S \lesssim Q_4 + \kappa^3 N^{1/2} + \kappa^3 N^{-1/2}\mathcal{N}_+^2.$$

Proof. We expand

$$S = Q_4 + \frac{\kappa}{2}\int dxdy V_N(x-y) C^\dagger(x,y) C(x,y) + \frac{\kappa}{2}\int dxdy V_N(x-y)\left(\check{a}_x^\dagger \check{a}_y^\dagger C(x,y) + \text{h.c.}\right).$$

The first term is part of the upper bound and the last term can be estimated by the first two using Cauchy-Schwarz. For the second term, we observe

$$\begin{aligned} &\int dxdy V_N(x-y) C^\dagger(x,y) C(x,y) \\ &\quad = \int_0^1 ds \int_0^1 dt \int dxdy V_N(x-y) e^{-s\mathcal{B}}\check{a}_x^\dagger a(\check{\varphi}^y) a_0^\dagger a_0^\dagger e^{s\mathcal{B}} e^{-t\mathcal{B}} a_0 a_0 a^\dagger(\check{\varphi}^y)\check{a}_x e^{t\mathcal{B}} \\ &\qquad + \text{ similar terms} \\ &\quad \lesssim \int_0^1 ds \int dxdy V_N(x-y) e^{-s\mathcal{B}} a_0^\dagger a_0^\dagger \check{a}_x^\dagger a(\check{\varphi}^y) a^\dagger(\check{\varphi}^y)\check{a}_x a_0 a_0 e^{s\mathcal{B}}, \end{aligned}$$

having used Cauchy-Schwarz to separate between $e^{s\mathcal{B}}$ and $e^{-t\mathcal{B}}$. The similar terms mentioned on the first line consist of switched x's and y's or different exponents in the conjugations, neither of which change the estimate. Using

$$a(\check{\varphi}^y) a^\dagger(\check{\varphi}^y) = a^\dagger(\check{\varphi}^y) a(\check{\varphi}^y) + \|\varphi\|_2^2 \leq \|\varphi\|_2^2(\mathcal{N}_+ + 1),$$

we hence deduce that

$$\begin{aligned}
&\int dxdy V_N(x-y)C^\dagger(x,y)C(x,y)\\
&\quad\lesssim \int_0^1 ds \int dxdy V_N(x-y) e^{-s\mathcal{B}} a_0^\dagger a_0^\dagger \check{a}_x^\dagger \|\varphi\|_2^2 (\mathcal{N}_+ + 1)\check{a}_x a_0 a_0 e^{s\mathcal{B}}\\
&\quad\lesssim N^2 \|V_N\|_1 \|\varphi\|_2^2 (\mathcal{N}_+ + N^{1/2})^2\\
&\quad\lesssim \kappa^2 N^{1/2} + \kappa^2 N^{-1/2}\mathcal{N}_+^2,
\end{aligned}$$

which concludes the proof. □

3.4 The ground state energy and Bose-Einstein condensation

In this section, we combine the results from the previous section so as to prove most of the statements in our main Theorem 1.1. From here on, the procedure is "standard" and can be found in many references, including [9, 10, 2, 4, 1, 3]. The following Proposition will serve as the main ingredient.

Proposition 3.11. *For sufficiently small $\kappa > 0$, we have*

$$e^{-\mathcal{B}} H_N e^{\mathcal{B}} \geq \mathfrak{L}_N N + \frac{1}{2} H_1 - C N^{1/2}$$

for some constant $C > 0$.

Proof. Combining Lemmas 3.5, 3.6, 3.7, 3.8, 3.9 and (22), we conclude that

$$\begin{aligned}
e^{-\mathcal{B}} H_N e^{\mathcal{B}} = H_0 &+ N(N-1)\sum_p \left(T_p \varphi_p^2 + \kappa \widehat{V}_N(p)\varphi_p + \frac{\kappa}{2}(\widehat{V}_N * \varphi)(p)\varphi_p \right)\\
&+ \sum_p \left(T_p \varphi_p + \frac{\kappa}{2}\widehat{V}_N(p) + \frac{\kappa}{2}(\widehat{V}_N * \varphi)(p) \right) a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}\\
&+ H_1 + S + \mathcal{E}
\end{aligned} \tag{29}$$

with $\mathcal{E}$ satisfying

$$\pm\mathcal{E} \leq C\lambda S + C\kappa(1+\lambda^{-1})(\mathcal{N}_+ + N^{1/2})$$

for any $\lambda > 0$. Choosing first λ sufficiently small so that $C\lambda \leq 1/2$ and then κ sufficiently small so that $C\kappa(1+\lambda^{-1}) \leq 1/2$, we conclude that

$$\pm\,\mathcal{E} \leq \frac{1}{2}(S + \mathcal{N}_+ + N^{1/2}). \tag{30}$$

Using the scattering equation (10), the off-diagonal quadratic term completely cancels, while the constant term reduces to

$$\begin{aligned}
&H_0 + N(N-1)\sum_p \left(T_p \varphi_p^2 + \kappa \widehat{V}_N(p)\varphi_p + \frac{\kappa}{2}(\widehat{V}_N * \varphi)(p)\varphi_p \right)\\
&\qquad = \kappa \frac{N(N-1)}{2}\left(\widehat{V}_N(0) + \sum_p \widehat{V}_N(p)\varphi_p \right) = \mathfrak{L}_N(N-1).
\end{aligned}$$

Using that $\mathcal{N}_+ \leq H_1$, the error term is completely absorbed into $H_1 + S$, and the remaining part of S is bounded by 0 from below. □

The next Theorem proves an expression for the leading order of the ground state energy, which we will use later in Section 4.

Theorem 3.12. *For sufficiently small $\kappa > 0$, the ground state energy E_N of H_N is given by*

$$E_N = \mathfrak{L}_N N + O(N^{1/2}).$$

Proof. The lower bound from Proposition 3.11 immediately implies that

$$E_N \geq \mathfrak{L}_N N - CN^{1/2}.$$

On the other hand, starting from (29) and (30), we argue as in the previous proof to find

$$e^{-\mathcal{B}} H_N e^{\mathcal{B}} = \mathfrak{L}_N N + H_1 + S + \mathcal{E}$$

with $\mathcal{E} \lesssim S + \mathcal{N}_+ + N^{1/2}$. Together with Lemma 3.10 we deduce

$$e^{-\mathcal{B}} H_N e^{\mathcal{B}} \leq \mathfrak{L}_N N + C(H_1 + N^{1/2} + Q_4 + N^{-1/2}\mathcal{N}_+^2).$$

Taking the expectation of this inequality in the state ψ_0 we conclude

$$E_N \leq \left\langle \psi_0 \middle| e^{-\mathcal{B}} H_N e^{\mathcal{B}} \middle| \psi_0 \right\rangle \leq \mathfrak{L}_N N + CN^{1/2}$$

as desired. □

Finally, the next Corollary proves BEC in the zero-momentum state for any sequence of approximate ground states. Its proof is based on the lower bound in Proposition 3.11.

Corollary 3.13. *Let $\kappa > 0$ be sufficiently small and let $\Psi_N \in L^2_s(\Lambda^N)$ be a normalized sequence of approximate ground states, in the sense that there exists a constant $C > 0$ so that*

$$\langle \Psi_N | H_N | \Psi_N \rangle \leq \mathfrak{L}_N N + CN^{1/2}.$$

Then:

(i) There exist constants $K_1, K_2 > 0$ (depending on C but not on N or Ψ_N) such that

$$\langle \Psi_N | \mathcal{N}_+ | \Psi_N \rangle \leq K_1 \langle \Psi_N | H_1 | \Psi_N \rangle \leq K_2 N^{1/2}.$$

(ii) Denoting by $\gamma_N^{(1)}$ the one-particle reduced density matrix of Ψ_N, we have

$$0 \leq 1 - \frac{1}{N} \left\langle \psi_0 \middle| \gamma_N^{(1)} \middle| \psi_0 \right\rangle \leq \frac{K_2}{N^{1/2}}.$$

In other words, the sequence Ψ_N exhibits Bose-Einstein condensation in the state ψ_0 with rate $N^{-1/2}$.

(iii) The ground state exhibits Bose-Einstein condensation in the state ψ_0 with rate $N^{-1/2}$.

Proof. From Proposition 3.11 and the assumption on Ψ_N, we find

$$\mathfrak{L}_N N + \frac{1}{2} \langle \Psi_N | H_1 | \Psi_N \rangle - CN^{1/2} \leq \langle \Psi_N | H_N | \Psi_N \rangle \leq \mathfrak{L}_N N + CN^{1/2}$$

which implies that

$$\langle \Psi_N | H_1 | \Psi_N \rangle \leq CN^{1/2}$$

and thereby the first part, because $\mathcal{N}_+ \lesssim H_1$. The second part follows from combining

$$\left\langle \psi_0 \middle| \gamma_N^{(1)} \middle| \psi_0 \right\rangle = \langle \Psi_N | \mathcal{N}_0 | \Psi_N \rangle = N - \langle \Psi_N | \mathcal{N}_+ | \Psi_N \rangle$$

with the first part. The third part follows from the second by the previous Theorem. □

Together, Theorem 3.12 and Corollary 3.13 imply most of the statements of Theorem 1.1. In the last section, we connect the coefficient $\mathfrak{L}_N$ with the scattering length $\mathfrak{a}$ of κV.

4 The scattering length

In this section, we study the definition and the properties of the scattering length of a potential in the pseudo-relativistic regime. Essentially, we would like to find some value $\mathfrak{L}$ so that

$$\mathfrak{L}_N \to \mathfrak{L} \tag{31}$$

with a sufficiently fast rate. Recall that, with the notation $f = 1 + \check{\varphi}$ in position space,

$$\mathfrak{L}_N = \frac{\kappa}{2} \int_\Lambda N V_N(x) f(x) dx.$$

We extend this definition to the whole space. For reasons that we make precise in Section 4.2, we consider a *massless* scattering equation. Namely, given a potential κV, we consider the *massless zero-energy scattering equation*

$$\sqrt{-\Delta} u(x) + \frac{\kappa}{2} V(x) u(x) = 0 \tag{32}$$

on $\mathbb{R}^3$ under the condition that $1 - u \in L^2(\mathbb{R}^3)$. To make sure the action of $\sqrt{-\Delta}$ is well defined, we use the convention that

$$\sqrt{-\Delta}(1) = 0.$$

If $\overline{f}$ is a solution to equation (32), we define

$$\mathfrak{L} = \frac{\kappa}{2} \int_{\mathbb{R}^3} V(x) \overline{f}(x) dx. \tag{33}$$

Notice also that $\overline{f}(\sqrt{N}x)$ solves the massless zero-energy scattering equation for κV_N, because

$$\sqrt{-\Delta}\Big(\overline{f}(\sqrt{N}\cdot)\Big)(x) = \sqrt{N}\left(\sqrt{-\Delta}\,\overline{f}\right)(\sqrt{N}x),$$

see, for example, [6, Lemma 1.2.5].
In the non-relativistic case, where there is no square root on the Laplacian, this is easier to handle by making use of the locality of $-\Delta$. A rigorous treatment can be found in [15]. In essence, one restricts the scattering equation onto a bounded domain, solves it there and then sends the boundary to infinity. This is not as easy in the pseudo-relativistic case because of the aforementioned non-locality of $\sqrt{-\Delta}$. Instead, we will prove existence of solutions by defining them via Born series in Section 4.1. Once we have found a solution, the scattering length is found in an analogous way to the non-relativistic situation. Finally, we detail the convergence of (31) in Section 4.2.

4.1 Existence and uniqueness

We recall that we use the convention

$$\widehat{f}(p) = \int_{\mathbb{R}^3} f(x) e^{-ip\cdot x} dx$$

for $f : \mathbb{R}^3 \to \mathbb{C}$. We start by showing uniqueness.

Proposition 4.1. *The scattering equation* (32) *has at most one solution under the given conditions.*

Proof. Let f, g be two solutions of (32) for which $1 - f, 1 - g \in L^2(\mathbb{R}^3)$ and set $u = f - g$. Then, $u \in L^2(\mathbb{R}^3)$ and it satisfies

$$\sqrt{-\Delta} u(x) + v(x) u(x) = 0$$

where $v = \kappa V/2$. We have to show that $u \equiv 0$. In fact, let us observe that the positivity of v implies

$$0 = \left\langle u \middle| \sqrt{-\Delta} + v \middle| u \right\rangle \geq \left\langle u \middle| \sqrt{-\Delta} \middle| u \right\rangle \geq 0. \tag{34}$$

In Fourier space, the last term is

$$\left\langle u \middle| \sqrt{-\Delta} \middle| u \right\rangle = C \int_{\mathbb{R}^3} |p| |\widehat{u}(p)|^2 dp$$

which is zero, due to (34). The only way for this to happen is that $\widehat{u} \equiv 0$, and thus $u \equiv 0$. □

To prove existence, we define the solution of the scattering equation by its Born series. Let $w = 1 - \overline{f}$, which formally satisfies the equation

$$w(x) = -\frac{1}{\sqrt{-\Delta}} v(x) + \frac{1}{\sqrt{-\Delta}} \Big(v(x) w(x) \Big)$$

where $v = \kappa V/2$. We can iterate this equation to a series $w(x) = -\sum_{n=0}^{+\infty} g_n(x)$, where formally

$$\begin{aligned} g_0(x) &= \frac{1}{\sqrt{-\Delta}} v(x), \\ g_{n+1}(x) &= \frac{1}{\sqrt{-\Delta}} \Big(v(x) g_n(x) \Big). \end{aligned}$$

Our goal is to show that this series converges. To this end, we describe the action of $1/\sqrt{-\Delta}$ as a convolution operator. While we follow the method via Fourier transforms, the same result can be found in [6, Section 1.4] by different analytic methods.

Lemma 4.2. *For $\eta \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have*

$$\frac{1}{\sqrt{-\Delta}} \eta = -\frac{1}{2\pi^2} \frac{1}{|\cdot|^2} * \eta. \tag{35}$$

Furthermore, we have $\frac{1}{\sqrt{-\Delta}} \eta \in L^{3/2+\varepsilon}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for any $\varepsilon > 0$.

Proof. Let $\eta \in L^1 \cap L^\infty$. Denoting by $\mathcal{F}$ the Fourier transform, we have

$$\mathcal{F}\left(\frac{1}{\sqrt{-\Delta}} \eta \right)(p) = \frac{1}{|p|} \widehat{\eta}(p) = \mathcal{F}(\omega * \eta)(p)$$

where ω is a distribution with Fourier transform $1/|p|$. It follows that (see, e.g. [17, Theorem 2.3])

$$\omega(x) = -\frac{1}{2\pi^2} \frac{1}{|x|^2}.$$

This implies the desired expression. Moreover, applying Young's equality to the RHS of (35) in infinity- and $(3/2 + \varepsilon)$-norm while also splitting

$$\frac{1}{|x|^2} = \frac{\chi_{|x|<1}}{|x|^2} + \frac{\chi_{|x|>1}}{|x|^2}$$

implies that $\frac{1}{\sqrt{-\Delta}} \eta \in L^{3/2+\varepsilon}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. □

Notice that $v \in L^1 \cap L^\infty$ by our assumptions, and hence also $v g_n \in L^1 \cap L^\infty$. Thus the sequence g_n is well-defined. The following Theorem shows that there exists a solution to the massless zero-energy scattering equation.

Theorem 4.3. *For sufficiently small $\kappa > 0$, the series*

$$w(x) = -\sum_{n=0}^{+\infty} g_n(x) \tag{36}$$

converges in L^2, and $1 - w$ solves the massless zero-energy scattering equation.

Proof. With the previous Lemma we find

$$g_{n+1}(x) = C \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} v(y) g_n(y) dy.$$

The Hardy-Littlewood-Sobolev inequality (see, e.g. [11, Section 4.3]) now implies that

$$\|g_{n+1}\|_q \lesssim \|v g_n\|_p$$

whenever $1 < p < q < +\infty$ and

$$\frac{1}{p} - \frac{1}{q} = 1 - \frac{2}{3}.$$

The choice $p = 6/5$, $q = 2$ satisfies these conditions, and furthermore by Hölder we find

$$\|g_{n+1}\|_2 \leq C\|v\|_3\|g_n\|_2.$$

Remembering that $v = \kappa V/2$, we can choose κ small enough so that

$$\|g_{n+1}\|_2 \leq \frac{1}{2}\|g_n\|_2 \tag{37}$$

for all $n \in \mathbb{N}$. This implies that the series converges absolutely in L^2. Finally, by the same inequality, the operator

$$g \mapsto \frac{1}{\sqrt{-\Delta}}(vg)$$

is continuous in L^2. Thus we have

$$w(x) = -g_0(x) - \sum_{n=1}^{+\infty} g_n(x) = -g_0(x) - \frac{1}{\sqrt{-\Delta}}\left(v(x)\sum_{n=1}^{+\infty} g_{n-1}(x)\right) = -g_0(x) + \frac{1}{\sqrt{-\Delta}}\Big(v(x)w(x)\Big),$$

or in other words, $1 - w$ solves the massless zero-energy scattering equation. □

The following Lemma describes the main contribution to w for large $|x|$.

Lemma 4.4. *We have*

$$\lim_{|x|\to+\infty} |x|^2 w(x) = \frac{1}{2\pi^2}\mathfrak{L}.$$

Proof. Let $R > 2$ and $|x| > R$. It is simple to check that for any $y \in \mathbb{R}^3$ with $|y| \leq 1$, we have the inequality

$$\left|\frac{|x|^2}{|x-y|^2} - 1\right| \lesssim \frac{1}{R}.$$

Thus we have

$$-|x|^2 g_0(x) = \frac{1}{2\pi^2}\int \frac{|x|^2}{|x-y|^2}v(y)dy = \frac{1}{2\pi^2}\int v(y)dy + O(1/R) \to \frac{1}{2\pi^2}\int v(y)dy$$

and similarly

$$-|x|^2 g_{n+1}(x) = \frac{1}{2\pi^2}\int \frac{|x|^2}{|x-y|^2}v(y)g_n(y)dy = \frac{1}{2\pi^2}\int v(y)g_n(y)dy + O(1/R) \to \frac{1}{2\pi^2}\int v(y)g_n(y)dy,$$

where we used the inequality (37) to make sure that the $1/R$-term is summable in n. Hence

$$\lim_{|x|\to+\infty} |x|^2 w(x) = \frac{1}{2\pi^2}\int v(y)dy + \sum_{n=0}^{+\infty}\frac{1}{2\pi^2}\int v(y)g_n(y)dy = \frac{1}{2\pi^2}\int v(y)(1-w(y))dy = \frac{\mathfrak{L}}{2\pi^2},$$

where the last equality follows from (33) and uniqueness. □

Hence we define the scattering length $\mathfrak{a} > 0$ of κV by the equation

$$\mathfrak{a}^2 = \frac{\mathfrak{L}}{2\pi^2} = \frac{\kappa}{4\pi^2}\int V(x)\overline{f}(x)dx > 0, \tag{38}$$

if $\overline{f}$ solves the massless zero-energy scattering equation. Equivalently, we define $\mathfrak{a}$ so that

$$\overline{f}(x) = 1 - \frac{\mathfrak{a}^2}{|x|^2} + o(1/|x|^2)$$

for large $x \in \mathbb{R}^3$.

4.2 Convergence

Finally, we want to show the convergence (31). To this end, we need to compare the coefficients φ_p, which satisfy (10), with coefficients $\tilde{\varphi}_p$ satisfying $\tilde{\varphi}_0 = 0$ and the "massless" box scattering equation

$$|p|\tilde{\varphi}_p + \frac{\kappa}{2}\sum_q \widehat{V}_N(p-q)\tilde{\varphi}_q = -\frac{\kappa}{2}\widehat{V}_N(p) \tag{39}$$

for $p \neq 0$. The existence of such coefficients is proven by the same argument as for φ_p in (9). Let us denote by $\tilde{\mathfrak{L}}_N$ the box scattering length corresponding to $\tilde{\varphi}_p$, i.e.

$$\tilde{\mathfrak{L}}_N := \frac{\kappa}{2}N\left[\widehat{V}_N(0) + \sum_p \widehat{V}_N(p)\tilde{\varphi}_p\right] = \frac{\kappa}{2}\widehat{V}(0) + \frac{\kappa}{2}N\sum_p \widehat{V}_N(p)\tilde{\varphi}_p.$$

The next Lemma shows that $\mathfrak{L}_N$ and $\tilde{\mathfrak{L}}_N$ are sufficiently close to each other.

Lemma 4.5. *We have*

$$|\mathfrak{L}_N - \tilde{\mathfrak{L}}_N| \lesssim \kappa N^{-1/2}.$$

Proof. Inserting the definitions, the first terms cancel and we find

$$\mathfrak{L}_N - \tilde{\mathfrak{L}}_N = \kappa N \sum_p \widehat{V}_N(p)(\varphi_p - \tilde{\varphi}_p).$$

We extract $\widehat{V}_N(p)$ in infinity-norm, which leaves us with the estimate

$$|\mathfrak{L}_N - \tilde{\mathfrak{L}}_N| \lesssim \kappa\|\varphi - \tilde{\varphi}\|_1.$$

We would like to show that the right-hand side behaves like $\kappa N^{-1/2}$. From Lemma 3.1, which also applies to $\tilde{\varphi}_p$, we already know that $\|\varphi - \tilde{\varphi}\|_1 \lesssim \kappa$. By combining equations (10) and (39), we deduce

$$|p|(\varphi_p - \tilde{\varphi}_p) = \varepsilon_p\varphi_p - \frac{\kappa}{2}\left[\widehat{V}_N * (\varphi - \tilde{\varphi})\right](p)$$

where $\varepsilon_p = |p| - \sqrt{p^2 + m^2} + m$. It is easily checked that $|\varepsilon_p| \lesssim 1$, which then implies that

$$\sum_p |\varphi_p - \tilde{\varphi}_p| \lesssim \sum_p \frac{1}{|p|}|\varphi_p| + \kappa\sum_p \frac{1}{|p|}|(\widehat{V}_N * (\varphi - \tilde{\varphi}))(p)|.$$

The second term can be controlled following the same procedure as in (15), which yields

$$\kappa\sum_p \frac{1}{|p|}|(\widehat{V}_N * (\varphi - \tilde{\varphi}))(p)| \lesssim \kappa\|\varphi - \tilde{\varphi}\|_1.$$

On the other hand, using the scattering equation (10) in a similar fashion as in (12), we find

$$\sum_p \frac{1}{|p|}|\varphi_p| \lesssim \kappa\sum_p \frac{|\widehat{V}_N(p)|}{|p|^2} + \kappa\sum_p \frac{|\widehat{V}_N * \varphi|(p)}{|p|^2}.$$

The first term can be treated as in (13), namely

$$\sum_p \frac{1}{|p|^2}|\widehat{V}_N(p)| \lesssim \|\widehat{V}_N\|_\infty \left\|\frac{\chi_{|\cdot|\leq\sqrt{N}}}{|\cdot|^2}\right\|_1 + \|\widehat{V}_N\|_1 \left\|\frac{\chi_{|\cdot|>\sqrt{N}}}{|\cdot|^2}\right\|_\infty \lesssim N^{-1/2}.$$

Finally, because $\|\widehat{V}_N * \varphi\|_r \lesssim \|\varphi\|_1\|\widehat{V}_N\|_r \lesssim \|\widehat{V}_N\|_r$, also the other term will satisfy the same estimate. This shows that

$$\|\varphi - \tilde{\varphi}\|_1 \leq C\kappa(N^{-1/2} + \|\varphi - \tilde{\varphi}\|_1).$$

Choosing $\kappa > 0$ small enough so that $C\kappa \leq 1/2$ concludes the proof. □

We have shown that the box scattering length of a potential is, asymptotically, independent of the mass. Next, we want to show that the box scattering length actually approximates the real scattering length of κV. To do this, we follow the ideas of [16, p.93], which involve the approximation of integrals by their Riemann sums. We first make this precise.

Lemma 4.6. *Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable functions so that*

$$(1 + |\cdot|)f(\cdot) \in L^1(\mathbb{R}^3), \quad \partial^\alpha f, \partial^\alpha g \in L^\infty(\mathbb{R}^3), \sum_{p\in\mathbb{Z}^3} |p||f(\lambda p)| < +\infty$$

for all $|\alpha| \leq 2$ and all $\lambda > 0$ sufficiently small. Then, we have that

$$\left| \sum_{p\in\mathbb{Z}^3} \lambda^3 \frac{f(\lambda p)}{|\lambda p|} - \int \frac{f(x)}{|x|} dx \right| \leq C\lambda \tag{40}$$

as $\lambda \to 0$, where the constant C depends only on the L^1-norm of $(1 + |\cdot|)f$ and the L^∞-norms of $\partial^\alpha f$. More generally, we have

$$\left| \sum_{p\in\mathbb{Z}^3} \lambda^3 \frac{f(\lambda p)}{|\lambda p|} g(\lambda p - a) - \int \frac{f(x)}{|x|} g(x - a) dx \right| \leq C\lambda \tag{41}$$

as $\lambda \to 0$, for any $a \in \mathbb{R}^3$ (which is allowed to depend on λ). Also here, the constant C is independent of a and only depends on the norms listed above, as well as the L^∞-norms of $\partial^\alpha g$.

Proof. We start by demonstrating (40). The first step is to write (recall $\Lambda = [-1/2; 1/2]^3$)

$$\begin{aligned}
\sum_{p\in\mathbb{Z}^3} \lambda^3 \frac{f(\lambda p)}{|\lambda p|} - \int \frac{f(x)}{|x|} dx &= \sum_{p\in\mathbb{Z}^3} \int_{\lambda p + \lambda\Lambda} \frac{f(\lambda p)}{|\lambda p|} - \frac{f(x)}{|x|} dx \\
&= \sum_{p\in\mathbb{Z}^3} \lambda^3 \int_\Lambda \frac{f(\lambda p)}{|\lambda p|} - \frac{f(\lambda p + \lambda x)}{|\lambda p + \lambda x|} dx \\
&= E_\ell + E_m + E_h
\end{aligned}$$

where E_- are the same expression, except the sum runs over the index sets

$$M_\ell = \{|\lambda p| < k\}, \quad M_m = \{k < |\lambda p| < K\}, \quad M_h = \{|\lambda p| > K\}$$

respectively and we choose $k, K > 0$ later. The contributions coming from M_ℓ and M_h can be bounded by simply using triangle inequalities, giving the bounds

$$|E_\ell| \lesssim k^2, \quad |E_h| \lesssim K^{-2}$$

having also used that $|\cdot| f \in L^1$. For E_m, we Taylor-expand the second part of the integrand around λp to second order. The first order contribution will vanish by symmetry of the integral domain and the fact that $x \mapsto x$ is odd. We find

$$|E_m| = \left| \sum_{k<|\lambda p|<K} \lambda^3 \int_\Lambda \lambda^2 x \cdot D^2 \left(\frac{f(\xi)}{|\xi|} \right) x dx \right| \leq \sum_{k<|\lambda p|<K} \lambda^5 \int_\Lambda \left\| D^2 \left(\frac{f(\xi)}{|\xi|} \right) \right\| dx$$

where D^2 denotes the second derivative (i.e. Hessian matrix) and $\xi = \xi(x, p, \lambda) \in \lambda p + \lambda\Lambda$ arises from Taylor's Theorem. Computing the second derivative and bounding all occurences of $\partial^\alpha f$ with their L^∞-norms, we find that

$$\left\| D^2 \left(\frac{f(\xi)}{|\xi|} \right) \right\| \lesssim \frac{1}{|\xi|} + \frac{1}{|\xi|^2} + \frac{1}{|\xi|^3} \lesssim \frac{1}{\lambda(|p| - 1/2)} + \frac{1}{\lambda^2(|p| - 1/2)^2} + \frac{1}{\lambda^3(|p| - 1/2)^3}.$$

This bound is independent of x, so the integral disappears. The sums over p can now be estimated by

$$\sum_{k<|\lambda p|<K} \frac{1}{\lambda(|p|-1/2)} \lesssim \lambda^{-1} \sum_{|p|<K/\lambda} \frac{1}{|p|} \lesssim K^2\lambda^{-3},$$
$$\sum_{k<|\lambda p|<K} \frac{1}{\lambda^2(|p|-1/2)^2} \lesssim \lambda^{-2} \sum_{|p|<K/\lambda} \frac{1}{|p|^2} \lesssim K\lambda^{-3},$$
$$\sum_{k<|\lambda p|<K} \frac{1}{\lambda^3(|p|-1/2)^3} \lesssim \lambda^{-3} \sum_{k/\lambda<|p|<K/\lambda} \frac{1}{|p|^3} \lesssim \log(K/k)\lambda^{-3},$$

which we can put together to find that

$$|E_m| \lesssim \left(K^2 + K + \log(K/k)\right)\lambda^2.$$

The choice $K = \lambda^{-1/2}$ and $k = \lambda^{1/2}$ guarantees that all terms are of order at most λ.
To prove (41), we use exactly the same technique, except that we bound occurences of g with their L^∞-norm whenever possible. This makes sure that the bound stays independent of a (because the L^∞-norm of g does not depend on a translation) while keeping mostly the same error bounds due to the product rule when applying Taylor's Theorem. □

The statement of equation (41) can be interpreted in the sense that multiplying a "nice" function (here f) with something bounded (here g) does not change the convergence of the Riemann sums. This is exactly what we need for the following Theorem, which yields the final result.

Theorem 4.7. *We have*

$$|\mathfrak{L}_N - \mathfrak{L}| \lesssim N^{-1/2}.$$

Proof. By the previous Lemma, it suffices to show the estimate for $\mathfrak{L}_N$ replaced by $\tilde{\mathfrak{L}}_N$. Writing the Born series (36) in Fourier space, as well as iterating the box scattering equation (39) and inserting these into the expressions for $\tilde{\mathfrak{L}}_N$ and $\mathfrak{L}$, we arrive at the expansions

$$\tilde{\mathfrak{L}}_N = \frac{\kappa}{2}\widehat{V}(0) + \left(\frac{\kappa}{2}\right)^2 N^{-3/2} \sum_{q\neq 0} \frac{\widehat{V}(q/\sqrt{N})^2}{|q/\sqrt{N}|} - \sum_{k\geq 1}(-1)^k \left(\frac{\kappa}{2}\right)^{k+2} I_k^{\mathrm{box}}(0)$$
$$\mathfrak{L} = \frac{\kappa}{2}\widehat{V}(0) + \left(\frac{\kappa}{2}\right)^2 \frac{1}{(2\pi)^3} \int dq \frac{\widehat{V}(q)^2}{|q|} - \sum_{k\geq 1}(-1)^k \left(\frac{\kappa}{2}\right)^{k+2} I_k^{\mathrm{space}}(0)$$

where, for $a \in \mathbb{R}^3$, we defined

$$I_k^{\mathrm{box}}(a) = \sum_{q_0,\dots,q_k\neq 0} N^{-3(k+1)/2} \frac{\widehat{V}(q_0/\sqrt{N})}{|q_0/\sqrt{N}|} \left[\prod_{j=1}^k \frac{\widehat{V}((q_{j-1}-q_j)/\sqrt{N})}{|q_j/\sqrt{N}|}\right] \widehat{V}(q_k/\sqrt{N} - a)$$
$$I_k^{\mathrm{space}}(a) = \frac{1}{(2\pi)^{3(k+1)}} \int dq_0 \cdots dq_k \frac{\widehat{V}(q_0)}{|q_0|} \left[\prod_{j=1}^k \frac{\widehat{V}(q_{j-1}-q_j)}{|q_j|}\right] \widehat{V}(q_k - a). \tag{42}$$

One observes that the sums appearing in I_k^{box} are exactly Riemann sums for the integrals in I_k^{space}. By the same procedure as in the proof of Theorem 4.3, one can show that there is a constant $C > 0$ (which depends on the potential) so that

$$|I_k^{\mathrm{box}}(a)|, |I_k^{\mathrm{space}}(a)| \leq C^k$$

uniformly in a. Using the smallness of κ, this assures the convergence of all appearing sums and integrals. The $(2\pi)^{-3}$ factors come from the scaling of the mesh size, as $q \in 2\pi\mathbb{Z}^3$. To show the desired convergence, we have to show that

$$|I_k^{\mathrm{box}}(a) - I_k^{\mathrm{space}}(a)| \leq \frac{C^k}{\sqrt{N}}$$

for all k, uniformly in a (the convergence of the single sum to the single integral will follow by the same argument). Let us also remark that to obtain our result, we of course only need to focus on $a = 0$. However, the a allows us to use the recursive definition of the I_k's in the form

$$I_{k+1}^{\mathrm{box}}(a) = N^{-3/2} \sum_q \frac{I_k^{\mathrm{box}}(q/\sqrt{N})}{|q/\sqrt{N}|} \widehat{V}(q/\sqrt{N} - a),$$
$$I_{k+1}^{\mathrm{space}}(a) = \frac{1}{(2\pi)^3} \int dq \frac{I_k^{\mathrm{space}}(q)}{|q|} \widehat{V}(q - a).$$

This motivates the induction on k. We can start from $k = 0$ by defining I_0 as in (42), treating an empty product as equal to 1. Since $\widehat{V}$ satisfies all desired assumptions, we get that

$$|I_0^{\mathrm{box}}(a) - I_0^{\mathrm{space}}(a)| \leq \frac{C}{\sqrt{N}}$$

by Lemma 4.6. For the induction step, we observe that $I_k^{\mathrm{space}}(q)$ also satisfies the desired regularity assumptions, so we can apply Lemma 4.6 after using the induction hypothesis as follows,

$$\begin{aligned} I_{k+1}^{\mathrm{box}}(a) &= N^{-3/2} \sum_q \frac{I_k^{\mathrm{space}}(q/\sqrt{N})}{|q/\sqrt{N}|} \widehat{V}(q/\sqrt{N} - a) + O(N^{-1/2}) \\ &= \frac{1}{(2\pi)^3} \int dq \frac{I_k^{\mathrm{space}}(q)}{|q|} \widehat{V}(q - a) + O(N^{-1/2}) \\ &= I_{k+1}^{\mathrm{space}}(a) + O(N^{-1/2}), \end{aligned}$$

with the error being independent of a and only picking up norms of $\widehat{V}$. □

Combining Theorem 4.7 with (38) and applying it to Theorem 3.12, we deduce that the ground state energy E_N of H_N is given by

$$E_N = 2\pi^2 \mathfrak{a}^2 N + O(N^{1/2}),$$

which concludes the proof of Theorem 1.1.

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The diagrams in Section 2 have been created using the `tikz-feynhand` package, see J. Ellis' original guide at https://doi.org/10.1016/j.cpc.2016.08.019 and M. Dohse's reference guide at https://doi.org/10.48550/arXiv.1802.00689.

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