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Third Order Upper Bound for the Ground State Energy of the Dilute Bose Gas

Master's Thesis

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Abstract

This thesis mainly consists of two parts. In the first part, we investigate the particle-number structure of a trial state constructed in [13] for the dilute Bose gas in the thermodynamic limit. This trial state arises in the derivation of a third-order upper bound for the ground state energy. We improve earlier estimates for the expected number of particles and its second moment by determining their second-order behavior. The second part is devoted to the energy analysis of the trial state. We review and rewrite the proof of some main theorems in the energy estimate part of [13]. In this way, the thesis gives both a more precise description of the particle-number properties of the trial state and a detailed explanation of how the trial state enters the proof of the third-order energy upper bound.

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1 Introduction

This thesis mainly consists of two parts. In the first part, we investigate the particle-number structure of a trial state constructed in [13] for the dilute Bose gas in the thermodynamic limit. The second part is devoted to the energy analysis of the trial state. We review and rewrite the proof of some main theorems in the energy estimate part of [13].

In the following, we first introduce the setup and the main results from the paper [13] which my work is main based on. We follow similar discussion as in the paper [13] and lecture notes [34, 35].

In quantum mechanics, the systems are described on Hilbert space $\mathcal{H}$. The *states* of a system are associated with vectors $\psi \in \mathcal{H}$ which are normalized so that $\|\psi\| = 1$ and the *observables* are associated with self-adjoint operators on $\mathcal{H}$. If an observable is described by the operator A, then its *expectation* in the state ψ is given by the inner product $\langle \psi, A\psi \rangle$. In every quantum system, one observable has a particularly central role: the *Hamilton operator*, also known as *Hamiltonian*, denoted by H. This is a self-adjoint operator acting on the Hilbert space $\mathcal{H}$, which represents the energy of the system.

We consider a *dilute Bose gas* of N particles confined in a large box $\Lambda_t \in \mathbb{R}^3$, in the *thermodynamic limit regime*, with *Dirichlet boundary condition* and pairwise interaction described by a two-body potential V. Here, a Bose gas is a system of many quantum particles of bosonic type. *Bosons* are particles whose wave function is symmetric with respect to permutations of all the particles. On the other hand, the particles described by the wave function that is antisymmetric under permutations are all called *fermions*. So, in this paper they only concern about Bosons. The systems of N Bosons moving in $\mathbb{R}^3$ can be described on the Hilbert space by

$$L^2_s(\mathbb{R}^{3N}) = L^2(\mathbb{R}^3)^{\otimes_s N} = \{\psi \in L^2_s(\mathbb{R}^{3N}) : \psi(x_{\pi(1)}, \cdots, x_{\pi_N}) = \psi(x_1, \cdots, x_N)\}$$

for all $\pi \in S_N$. In particular, this is a subspace of $L^2(\mathbb{R}^{3N})$ consisting of square-integrable functions such that

$$\int_{\mathbb{R}^{3N}} |\psi(x_1, \cdots, x_N)|^2 dx_1 \cdots dx_N < +\infty$$

that are invariant under the permutations of N particles. A most basic example of a bosonic wave function in $\mathbb{R}^3$ is obtained by taking the product of N copies of the same normalized one-particle wave function $\varphi \in L^2(\mathbb{R}^3)$, namely

$$\psi_N(x_1, \cdots, x_N) = \prod_{j=1}^{N} \varphi(x_j)\,.$$

A Bose gas is called dilute when the particles are, on average, far apart while comparing with the effective range of their interaction. This tells us that although particles do interact, collision are rare, and the gas is sparse on the interaction scale. We can think of Λ_t as a large cube with side length L in $\mathbb{R}^3$, for example

$$\Lambda_t = \left[-\frac{L}{2}, \frac{L}{2}\right]^3,$$

so that its volume is $|\Lambda_t| = L^3$. The thermodynamic limit regime means that both the number of particles N and the volume of the box Λ_t go to infinity, while the density stays fixed, i.e.,

$$N \to \infty, \qquad |\Lambda_t| \to \infty, \qquad \rho = \frac{N}{\Lambda_t} \text{fixed}\,.$$

So, in the way that you enlarge the whole system and add more particles at the same rate, the gas will not become either infinitely crowded or infinitely empty. Hence, this regime is often used to describe a macroscopic homogeneous system. Moreover, the contribution coming from the boundary becomes negligible compared with the contribution from the interior, so the leading-order energy per unit volume is determined by the behavior of the gas in the interior of the system. Then, we impose the Dirichlet boundary conditions to this big thermodynamic box Λ_t, which means that the wave function vanishes at the boundary of the box Λ_t, that is

$$\psi(x_1, \cdots, x_N) = 0$$

if a particle x_j reaches $\partial\Lambda_t$ for all $j = 1, \cdots, N$. Intuitively, the boundary acts like perfectly hard wall: the particles are confined inside the box and the wave function is forced to be zero at the boundary. Mathematically, Dirichlet boundary conditions are important because they determine the domain of the Laplacian $-\Delta$, which is related to the kinetic energy operator which we will introduce in the following.

The Hamiltonian of this system is defined as

$$H_L = \sum_{j=1}^{N} -\Delta_j + \sum_{i<j}^{N} V(x_i - x_j)\,, \tag{1.1}$$

acting on symmetric subspace $L^2_s(\Lambda_t^N)$. The first term is the total kinetic energy of the N particles where $-\Delta_j$ denotes the Laplacian acting on the j-th particle coordinate. The second term corresponds to the potential energy of the system and is given by a sum of pairwise interactions through a two-body potential $V \in L^2(\mathbb{R}^3)$. The potential depends only on the relative distance between particles, which reflects the translation-invariant property of the interaction. We are going to assume that $V \in L^2(\mathbb{R}^3)$ is *radial*, *compactly supported*, *stable* and does not admit *two-body bound states*. Here, potential V is radial if it depends only on the distance from the origin, i.e.,

$$V(x) = V(|x|)\,.$$

So, the interaction is invariant with respect to rotations. Physically, this means that the force between two particles depends only on how far away they are, not on the direction. Compact support means that the interaction vanishes outside the support of potential V, that is,

$$V(x) = 0 \qquad \text{if} \quad x \notin \operatorname{supp} V\,.$$

So, the interaction has finite range. Stability is a fundamental assumption ensuring that the total potential energy cannot become arbitrarily negative as the number of particles grows. In other words, we say that V is stable if there exists a constant $C > 0$ such that

$$\sum_{i,j}^{N} V(x_i - x_j) \geq -CN\,.$$

Stability is a crucial because it excludes a collapse of the system and guarantees a meaningful many-body energy theory. We assume here that there are no two-body bound states means that two particles do not admit negative-energy bound states. In other words, two particles interacting through the potential V cannot form a stable molecule-like bound pair. This assumption is important for the results of the paper [13] because if the potential produced two-body bound states, then particles would tend to cluster into bound pairs and the

energy would be very different from that of a dilute gas. In particular, all non-negative, radial, compactly supported potentials $V \in L^2(\mathbb{R}^3)$ are considered in the paper.

Now, we are going to introduce the scattering length $\mathfrak{a}$, which measure the effective range of the potential $V \in L^2(\mathbb{R}^3)$. In this paper, we only take $\mathfrak{a}$ to be positive. The scattering length $\mathfrak{a}$ is a central quantity in the analysis of dilute Bose gases because it describe the low-energy strength of the interaction. Specifically, in dilute regime, even if the full potential V is complicated, the leading energy behavior depends mostly on $\mathfrak{a}$, not on all details of V. The scattering length $\mathfrak{a}$ is defined through the zero-energy scattering equation

$$\left[-\Delta + \frac{1}{2}V\right] f = 0 \tag{1.2}$$

together with the boundary condition $f(x) \to 1$ as $|x| \to \infty$. Here, f is the corresponding scattering solution. Notice that outside the support of the potential V, the equation (1.2) reduces to

$$-\Delta f = 0,$$

so f is harmonic. In three dimensions, a radial harmonic function that tends to 1 at infinity takes the form

$$f(x) = 1 - \frac{\mathfrak{a}}{|x|}\,.$$

Thus ,the scattering length $\mathfrak{a}$ appears as the constant over $|x|$. Equivalently, the scattering length can also be characterized by the identity

$$8\pi\mathfrak{a} = \int_{\mathbb{R}^3} V(x)f(x)dx\,.$$

Indeed, by the zero energy scattering equation (1.2), we have

$$\int V(x)f(x)dx = 2\int \Delta f(x)dx\,.$$

Since $V \in L^2(\mathbb{R}^3)$ is radial, we can switch to spherical coordinates and define $g(r) = f(x)$ for $|x| = r$. Then we have

$$\Delta f(x) = g''(r) + \frac{2}{r}g'(r) = \frac{1}{r}\partial_r^2(rg(r))$$

so

$$\begin{aligned}\int V(x)f(x)dx = 2\int \Delta f(x)dx &= 2\lim_{R\to\infty}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta \int_0^R dr\, r^2 \left(\frac{1}{r}\partial_r^2(rg(r))\right)\\ &= 8\pi \lim_{R\to\infty}\int_0^R dr\, r\partial_r^2(rg(r))\,.\end{aligned}$$

Integration by parts and using $rg(r) = r - \mathfrak{a}$ for r large enough, we find

$$\int V(x)f(x)dx = 8\pi \lim_{R\to\infty}(R - Rg(R)) = 8\pi\mathfrak{a}\,.$$

This relation is very important because it connects the low-energy parameter $\mathfrak{a}$ directly to the potential V. Moreover, the scattering solution f is bounded, i.e., $f \in L^\infty(\mathbb{R}^3)$, which is very useful in the subsequent analysis.

The ground state energy of the system here is denoted by $E(N, \Lambda_t)$, that is the lowest possible energy of the N-particle system in the box Λ_t. Mathematically, it is the lowest eigenvalue of the Hamiltonian H_L (1.1). Equivalently,

$$E(N, \Lambda_t) = \inf_{\psi \in L^2(\Lambda_t^N) : \|\psi\|=1} \langle \psi, H_L \psi \rangle \ .$$

So, it is obtained by minimizing the energy expectation over all normalized bosonic wave functions. Our main interest is not this total energy $E(N, \Lambda_t)$ itself, but rather the energy per unit volume, i.e., the energy density

$$e(\rho) = \lim_{\substack{N, |\Lambda_t| \to \infty \\ \rho = N/|\Lambda_t|}} \frac{E(N, \Lambda_t)}{|\Lambda_t|}$$

in the thermodynamic limit (1). This is more meaningful than the total energy itself, because total energy varies with the size of the system. In particular, the regime of interest need to assume one more condition, the dilute limit, which is characterized by

$$\rho \mathfrak{a}^3 \to 0 \, .$$

Since ρ has the dimension of an inverse volume (L^{-3}) and $\mathfrak{a}$ has the dimension of a length (L), the quantity $\rho \mathfrak{a}^3$ is dimensionless. It compares the volume scale set by the mean particle density with the two-body interaction scale described by the scattering length $\mathfrak{a}$. In other words, it measures how sparse the gas is on the interaction scale. The condition $\rho \mathfrak{a}^3 \to 0$ means that the average number of particles in a region of size comparable with the scattering length $\mathfrak{a}$ is very small, i.e., particles interact rarely so the gas is dilute. This is precisely the regime in which the celebrated asymptotic expansion of the ground state energy is expected to hold.

The main theorem of this paper [13] is as follows

Theorem 1. *Let $V \in L^2(\mathbb{R}^3)$ be radial, stable and with compact support. Assume that V does not admit two-body bound states and let $\mathfrak{a} > 0$ be its scattering length. Then, there is a constant $C > 0$ such that*

$$e(\rho) \leq 4\pi \mathfrak{a} \rho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho \mathfrak{a}^3} + 8 \left(\frac{4\pi}{3} - \sqrt{3} \rho \mathfrak{a}^3 \log \left(\rho \mathfrak{a}^3 \right) \right) + C \rho \mathfrak{a}^3 \right) \tag{1.3}$$

for all $\rho \mathfrak{a}^3 > 0$ small enough.

History. The estimate in (4.1) is expected to reproduce the ground state energy density of the dilute Bose gas up to errors of order

$$\mathfrak{a} \rho^2 \left(\rho \mathfrak{a}^3 \right)$$

as the dilute limit $\rho \mathfrak{a}^3 \to 0$. Beyond this order, the coefficients are no longer expected to be universal. They should depend on more detailed features of the interaction potential and not only on the scattering length $\mathfrak{a}$.

The expansion appearing on the right-hand side of (4.1) has a long history in the physics literature. The leading term and the next-order correction were predicted by Lee-Huang-Yang in 1957 [26], building on Bogoliubov's earlier work [8]. Shortly afterwards, Wu [36], and independently Hugenholtz-Pines [25] and Sawada [33], identified the logarithmic third-order correction. From a rigorous mathematical perspective, the leading term was first

obtained as an upper bound by Dyson [17], while the corresponding lower bound was proved more than 40 years later by Lieb and Yngvason [30]. The extension to interactions with a negative part was established in [38].

Further progress concerned the Lee-Huang-Yang correction. An upper bound containing this term was proved in [37] for sufficiently regular potentials; an earlier result in [18] reached the Lee-Huang-Yang order but recovered the correct coefficient only in the weak-coupling limit. A more streamlined proof, valid for a broader class of interactions, was later obtained in [2]. The hard-sphere case remains more delicate: the best available upper bound in this setting is due to [4], with an error still of Lee-Huang-Yang size. On the lower-bound side, the Lee-Huang-Yang term was first rigorously derived in [19], and the result was subsequently extended to more general interactions, including hard spheres, in [20]. More recently, a new derivation of the lower bound was given in [22]; this approach also applies to the free energy at low positive temperature, for which a matching upper bound was proved in [23]. The corresponding analysis has further been extended to potentials including hard spheres in [21].

Theorem 1 provides an upper bound for the ground state energy per unit volume of a dilute Bose gas with interaction potential $V \in L^2(\mathbb{R}^3)$, which agrees with the prediction of Wu. To establish Wu's formula completely in the thermodynamic limit, one would still need a matching lower bound. This remains an open and technically demanding problem.

It is worth to emphasize that Wu's correction has already been derived rigorously in the Gross-Pitaevskii regime. In that setting, one considers N particles in the unit torus interacting through a potential whose scattering length is of order $1/N$. The ground state energy and the low-energy excitation spectrum were resolved up to he Lee-Huang-Yang order in [7, 6], using a rigorous implementation of Bogoliubov theory. Earlier results had established the leading-order energy and Bose-Einstein condensation in [29, 28, 27]. More recently, the method of [7, 6] was refined in [15] to capture Wu's logarithmic contribution. Several ideas from [15] will also be used in the work [13], in particular in the section where the kernel of the cubic transformation is introduced. Further related results in the Gross-Pitaevskii regime (and beyond) can be found in [1, 3, 5, 11, 9, 10, 6, 12, 14, 24, 32, 31].

Now we introduce some mathematical tools that will be used throughout the following discussions.

Fock space. In many-body quantum mechanics, it is often inconvenient to work in a Hilbert space with a fixed number of particles, because physical processes may naturally involve states with different particles numbers. So, a variable-particle-number framework is often much more flexible and this motivates the introduction of bosonic Fock space. Instead of describing fixed N-particle states, Fock space collects all n-particle sectors together in a single Hilbert space. For bosons in a region $\Lambda \subset \mathbb{R}^3$, the bosonic Fock space is defined as

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2_s\left(\Lambda^n\right)$$

where $L^2_s(\Lambda^n)$ denotes the space of square-integrable wave functions that are symmetric under permutations of the n particles. A vector in $\mathcal{F}$ is therefore a sequence

$$\Psi = \left\{\psi^{(n)}\right\}_{n \geq 0}$$

with $\psi^{(n)} \in L^2_s(\Lambda^n)$. The inner product on the Fock space is obtained by summing over all the inner products of all sectors: for every $\Psi = \left\{\psi^{(n)}\right\}_{n\geq 0}$ and $\Phi = \left\{\phi^{(n)}\right\}_{n\geq 0}$, we have

$$\langle \Psi,\, \Phi\rangle = \sum_{n\geq 0} \left\langle \psi^{(n)},\, \phi^{(n)} \right\rangle_{L^2(\Lambda^n)} .$$

Since $\langle\cdot,\cdot\rangle$ is indeed an inner product on $\mathcal{F}$, $\mathcal{F}$ is a Hilbert space equipped with norm induced by $\langle\cdot,\cdot\rangle$. Then for a state $\Psi \in \mathcal{F}$, we have

$$\sum_{n\geq 0} \left\|\psi^{(n)}\right\|^2 = \sum_{n\geq 0} \left\langle \psi^{(n)},\, \psi^{(n)} \right\rangle = \langle \Psi,\, \Psi\rangle = \|\Psi\|^2 = 1\,.$$

Hence, $\|\psi^{(n)}\|^2$ can be interpreted as the probability that the system contains exactly n particles. In this way, Fock space describes superpositions of states with different numbers of particles. The operator counting the numbers of particles in the system is the number of particle operator $\mathcal{N}$ such that

$$(\mathcal{N}\Psi)^{(n)} = n\psi^{(n)}$$

for $\Psi = \left\{\psi^{(n)}\right\}_{n\geq 0}$. Notice that $\mathcal{N}$ is an unbounded operator and can be defined as a self-adjoint operator on the dense domain

$$\left\{ \Psi \in \mathcal{F} : \sum_{n\geq 0} n^2 \left\|\psi^{(n)}\right\|^2 < \infty \right\} .$$

Moreover, the states that with a fixed number of particles, i.e., only have one non-zero entry $\psi_N \in \text{Ł}^2_s(\Lambda^N)$, such that

$$\Psi = \{0, 0, \cdots, 0, \psi_N, 0, \cdots\}$$

are the eigenvectors of $\mathcal{N}$. The vacuum

$$\Omega = \{1, 0, 0, \cdots\}$$

is a special eigenvector of $\mathcal{N}$, which represents the state with zero particles.

Creation and annihilation operators. A central tool on Fock space is given by the creation and annihilation operators. For a one-particle wave function $f \in L^2(\Lambda)$, the annihilation operator $a(f)$ removes one particle in the state f, while the creation operator $a^*(f)$ adds one particle in that state. Formally, we defined their action on $\Psi = \left\{\psi^{(n)}\right\}_{n\geq 0}$ sector by sector. On the n-particle component of Ψ, the annihilation operator $a(f)$ is defined by

$$(a(f)\Psi)^n (x_1, \cdots, x_n) = \sqrt{n+1} \int dx \bar{f}(x) \psi^{(n+1)} (x, x_1, \cdots, x_n) ,$$

and the creation operator $a^*(f)$ is given by

$$(a^*(f)\Psi)^n (x_1, \cdots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)} (x_1, \cdots, \not{x}_j, \cdots, x_n) .$$

Then we know that the annihilation operator $a(f)$ maps state with N particles into states with $(N-1)$ particles, while the creation operator $a^*(f)$ maps states with N particles into

states with $(N+1)$ particles. Since changing the number of particles typically increases the size of the vector in an unbounded way, these two operators are unbound. Nevertheless, they can be controlled by the square root of the number of particles operator

$$\|a(f)\Psi\| \leq \|f\|_2 \, \|\mathcal{N}^{1/2}\Psi\|, \qquad \|a^*(f)\Psi\| \leq \|f\|_2 \, \|(\mathcal{N}+1)^{1/2}\Psi\|$$

So, although they are unbounded, they are well behaved on vectors with finite expected number of particles. Moreover, the creation operator $a^*(f)$ is the adjoint of the annihilation operator $a(f)$, i.e.,

$$\langle a^*(f)\Psi,\, \Phi\rangle = \langle \Psi,\, a(f)\Phi\rangle \ .$$

A very important that these two operators satisfy is the *canonical commutation relations*:

$$[a(f), a^*(g)] = \langle f, g\rangle \,, \qquad [a(f), a(g)] = [a^*(f), a^*(g)] = 0 \,. \tag{1.4}$$

The first relation says that annihilating and then creating is not the same as creating and then annihilating; the difference is precisely the one-particle inner product $\langle f, g\rangle$. The other two relations reflect the bosonic symmetry: two creation operators commute, and two annihilation operators also commute.

Second quantization. With creation and annihilation operators. one can represent operators on the Fock space $\mathcal{F}$ very efficiently. This leads to the notion of second quantization. Suppose $A : L^2(\Lambda) \to L^2(\Lambda)$ is a one-particle operator on $L^2(\Lambda)$. Its second quantization, denoted by $d\Gamma(A)$, is the operator acting on the n-particle sector as the sum of A acting on each particle:

$$(d\Gamma(A)\Psi)^{(n)} = \sum_{j=1}^{n} A_j \psi^{(n)}, \tag{1.5}$$

where $A_j = 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1$ is an operator on $L^2(\Lambda^n)$. Thus, the second quantization promotes a one-body operator to the corresponding many-body operator. If $\{f_j\}$ is an orthonormal basis of the one-particle space $L^2(\Lambda_L)$, then $d\Gamma(A)$ can be written in the expression of creation and annihilation operators such that

$$d\Gamma(A) = \sum_{j,\ell \in \mathbb{N}} \langle f_j,\, A f_\ell\rangle \, a^*(f_j) a(f_\ell) \,.$$

In particular, the number operator itself is just the second quantization of the identity:

$$\mathcal{N} = d\Gamma(1) = \sum_{j \in \mathbb{N}} a^*(f_j) a(f_j) \,.$$

This expression has a simple interpretation: for any $j \in \mathbb{N}$, the operator

$$d\Gamma\left(|f_j\rangle \langle f_j|\right) = a^*(f_j) a(f_j)$$

is obtained by second-quantizing the orthogonal projector onto f_j, consequently, it counts the particles occupying the one-particle state f_j, and summing over all basis vectors gives the total number of particle. If one chooses Fourier basis of $L^2(\Lambda_L)$, this becomes

$$\mathcal{N} = \sum_{p \in \Lambda_L^*} a_p^* a_p$$

in the momentum space. Expressing of a_p^* and a_p in position space by using the operator valued distributions such that

$$a_p^* = a(f_p)^* = \int f_p(x) a_x^* \, dx, \qquad a_p = a(f_p) = \int \bar{f}_p(x) a_x \, dx, \tag{1.6}$$

we get

$$\mathcal{N} = \int a_x^* a_x \, dx \, .$$

We can also transform a_x^* and a_x back to the momentum space by

$$a_x^* = \sum_{p \in \Lambda^*} a_p^* e^{-ipx}, \qquad a_x = \sum_{p \in \Lambda^*} a_p e^{ipx} \, . \tag{1.7}$$

By using notations a_p, a_p^* and a_x, a_x^*, we have our canonical commutation relations such that

$$\left[a_p, a_q^*\right] = \delta_{p,q}, \qquad [a_p, a_q] = \left[a_p^*, a_q^*\right] = 0 \tag{1.8}$$

$$\left[a_x, a_y^*\right] = \delta(x-y), \qquad [a_x, a_y] = \left[a_x^*, a_y^*\right] = 0 \tag{1.9}$$

Likewise, if the one-particle operator is the kinetic energy $-\Delta$, then in momentum space its second quantization is given by

$$d\Gamma(-\Delta) = \sum_{p \in \Lambda_L^*} p^2 a_p^* a_p \, .$$

The same idea extends to operators acting on two particles at the same time. If B is a two-body operator on $L^2(\Lambda) \otimes L^2(\Lambda)$, then its second quantization acts on the n-particle sector by summing B over all pairs of particles:

$$(d\Gamma(B)\Psi)^{(n)} = \sum_{i<j} B_{i,j} \psi^{(n)}$$

where $B_{i,j}$ is the operator acting as B on particles i and j at once and as identity on all other particles. In terms of creation and annihilation operators, this becomes a quartic expression:

$$d\Gamma(B) = \frac{1}{2} \sum_{j,k,\ell,m \in \mathbb{N}} \langle f_j \otimes f_m, \, B(f_\ell \otimes f_k) \rangle \, a^*(f_j) a^*(f_m) a(f_\ell) a(f_k) \tag{1.10}$$

For example, if particles interact through a potential $V(x-y)$, then in the momentum space, the associated second-quantized two-body interaction is given by

$$\frac{1}{2} \sum_{p,q,r \in \Lambda_L^*} \hat{V}(r) a_p^* a_q^* a_{p+r} a_{q-r} \, .$$

Notice that this form here is not diagonal while $V(x-y)$ is a multiplication operator in position space. So, the second quantization of $V(x-y)$ can be written more compactly using operator valued distributions (1.6) as

$$\frac{1}{2} \int dx dy \, V(x-y) a_x^* a_y^* a_y a_x \, .$$

This formula expresses a simple physical process: a pair of particles is annihilated and recreated, weighted by the interaction potential between their positions.

With these standard language for many-body quantum theory in mind, we are going to prove the Theorem 1.

Localization. The proof of Theorem 1 relies on standard localization approach. The purpose of this method is to reduce the analysis of the full thermodynamic system to the construction of a suitable trial state in a much smaller but still sufficiently large box which still satisfies the thermodynamic regime (1). More concretely, you begin with a grand canonical trial state ψ_L (By a grand canonical trial state we mean a state in bosonic Fock space, rather than in a fixed particle sector) in a periodic box Λ_L of side length

$$L = \rho^{-\gamma}$$

for some appropriate $\gamma > 0$. The exponent γ specifies the mesoscopic scale at which the system is analyzed. It determines how rapidly the side length of the localization box grows as the density tends to zero. Its role is to ensure that, although the gas is dilute, each of these boxes still contains many particles and therefore captures the relevant many-body behavior. In addition, γ is chosen large enough to make the boundary errors arising from localization sufficiently small. Indeed, at density ρ, the expected number of particles in such a box is

$$N = \rho L^3 .$$

If we choose $\gamma > 1/3$, then (1.11) shows that the number of particles $N \to \infty$ as $\rho \to 0$. Hence, if the trial state ψ_L on such a periodic box is constructed so that it has approximately ρL^3 particles and reproduces, up to small errors, the desired energy per unit volume, one can build a trial state for the full thermodynamic box Λ_t from this periodic box state ψ_L. This is done in two steps: first, we adjust ψ_L into a trial state ψ_L^{dir} which is supported in a slightly larger box with Dirichlet boundary conditions, and then we tile many copies of ψ_L^{dir} to cover the whole volume of the big thermodynamic box Λ_t. In order to prevent the interactions among particles that belong to different boxes, these boxes are separated by thin corridors whose width is larger than the support of the potential V. In this way, the energy of the global trial state is essentially the sum of the energies of the individual boxes, up to controllable boundary and localization errors. Finally, the desired upper bound (4.1) is obtained by passing from the grand canonical setting to the fixed-density thermodynamic energy through the equivalence of ensembles.

The main technical task of the paper [13] is therefore the construction of the grand canonical trial state on the smaller periodic box Λ_L, where again

$$L = \rho^{-\gamma}$$

for a suitable choice of $\gamma > 0$. One wants the box Λ_L to be large enough to contain many particles and approximate the interior energy, but also small enough that the localization procedure remains effective. In particular, to ensure that the error produced when replacing periodic boundary conditions by Dirichlet ones is negligible at an acceptable precision, we eventually need

$$\gamma > \frac{3}{2} .$$

In stead of carrying out the whole analysis on a box whose size depends on ρ, it is more convenient to rescale lengths and work on the fixed unit torus

$$\Lambda = \Lambda_1 = \left[-\frac{1}{2}, \frac{1}{2} \right]^3 .$$

This rescaling is only a change of variables: physically the original box still has side length L, but after rewriting the problem with respect to a unit torus, the dependence on L appears in the coefficients of the Hamiltonian. Indeed, if

$$N = \rho L^3 = \rho^{1-3\gamma}$$

denotes the expected number of particles in Λ_L, then one can rewrite the density and the box size in terms of N as

$$\rho = N^{-1/(3\gamma-1)}, \qquad L = \rho N = N^{\gamma/(3\gamma-1)} = N^{1-\kappa}$$

where

$$\kappa = \frac{2\gamma - 1}{3\gamma - 1}\,.$$

This rescaling does not change the physical regime, it just simply expresses the dilute limit (small ρ) in terms of the expected number of particles (large N), which is the natural variable for Fock-space and Bogoliubov-type analysis. In particular, it makes explicit that the system is dilute in density, while still containing a huge number of particles in each periodic box. For $\kappa \in [0, 2/3)$, the rescaled Hamiltonian takes the form

$$\mathcal{H}_N = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{N^\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) a_{p+r}^* a_{q-r}^* a_q a_p \tag{1.11}$$

acting on the bosonic Fock space

$$\mathcal{F}(\Lambda) = \bigoplus_{n\geq 0} L_s^2\left(\Lambda^n\right)\,.$$

Here, $\Lambda^* = 2\pi\mathbb{Z}^3$ is the dual lattice of the unit torus, so the momenta are discrete because of the periodic boundary conditions. The operators a_p^* and a_p, $p \in \Lambda^*$, are the usual creation and annihilation operators which satisfy canonical commutation relations (1.8). Notice that this operator (1.11) is the same as (1.1) when acting on states with exactly N particles. The factor N^κ/N and $\hat{V}(r/N^{1-\kappa})$ reflect the rescaling from the periodic box Λ_L to the unit torus. Now, let us see how to obtain (1.11) by rescaling the coordinates form (1.1) detailedly.

Originally, the particles live in the box $\Lambda_L = [-L/2, L/2]^3$. We introduce the new coordinates such that

$$x = \frac{1}{L} y\,.$$

Then, $x \in \Lambda = [-1/2, 1/2]^3$ and we have

$$\nabla_y = \frac{1}{L}\nabla_x, \qquad \Delta_y = \frac{1}{L^2}\Delta_x.$$

So, the rescaled Hamiltonian becomes

$$H_L = L^{-2}\left[\sum_{j=1}^N -\Delta_{x_j} + L^2 \sum_{i<j}^N V(x_i - x_j)\right]\,.$$

So, after conjugating by the unitary rescaling map, one gets

$$H_L \sim L^{-2} H_N\,.$$

Now, we switch to the bosonic Fock space $\mathcal{F}(\Lambda)$ by second quantization of the Hamiltonian H_N with $L = N^{1-\kappa}$, we obtain (1.11) and

$$\mathcal{H}_L \sim L^{-2}\mathcal{H}_N\,. \tag{1.12}$$

In this rescaled formulation, the problem is reduced to the construction of a suitable grand canonical trial state on the unit torus, whose expected particle number is of order N and whose energy reproduces the desired asymptotic expansion. The advantage of working on the unit torus Λ is that all relevant scales are now encoded in the N-dependence of the coefficients of $\mathcal{H}_N$. This makes it possible to analyze the condensate and its excitations on the Fock space $\mathcal{F}(\Lambda)$.

The next theorem gives the precise outcome of this construction. It provides a normalized trial $\Psi_N \in \mathcal{F}(\Lambda)$ with the correct particle-number bounds and with an energy expectation matching the expected expansion up to an acceptable error term.

Theorem 2. *Under the same assumptions on V as in Theorem 1, we have that for any $\kappa \in [1/2, 2/3)$ and N large enough, there exists $\Psi_N \in \mathcal{F}(\Lambda)$, $\|\Psi_N\| = 1$ such that*

$$\langle \Psi_N, \mathcal{N}\Psi_N \rangle \geq N, \qquad \left\langle \Psi_N, \mathcal{N}^2\Psi_N \right\rangle \leq CN^2 \tag{1.13}$$

and

$$\begin{aligned} \langle \Psi_N, \mathcal{H}_N\Psi_N \rangle \leq 4\pi\mathfrak{a}N^{1+\kappa} + \frac{512\sqrt{\pi}}{15}\mathfrak{a}^{5/2}N^{5\kappa/2} \\ + 32\pi\left(\frac{4\pi}{3} - \sqrt{3}\right)\mathfrak{a}^4 N^{4\kappa-1}\log\left(N^{3\kappa-2}\right) + CN^{4\kappa-1}. \end{aligned} \tag{1.14}$$

Remark 3. *The assumption $\kappa > 1/2$ ensure that the errors caused by working in a finite box, which are of order $N^{2\kappa}$, are no larger than the remainder term $N^{4\kappa-1}$ in the energy expansion (4.3), and can therefore be absorbed into the final error bound.*

Trial state. The main part of the paper [13] is to prove Theorem 2. A very important task of it is to construct a grand-canonical *trial state* Ψ_N that is accurate enough to reproduce not only the leading-order and Lee-Huang-yang terms, but also the third-order logarithmic correction in the ground state energy. Before estimating the energy, the authors need to make sure that this trial state Ψ_N satisfies the required particle-number. Because the trial state Ψ_N is intended to model an N-particle Bose gas, so its number of particles must be centered and also be sufficiently concentrated at the correct value. Without such information, even a very good energy estimate could be misleading, since it might correspond to the wrong effective density or to a state representing a different physical system. In the paper [13], this is reflected by the following Lemma 4.

Lemma 4. *Consider the trial state Ψ_N such that*

$$\Psi_N = W_{N_0} e^{B^* - B} e^{A^*\Theta - \Theta A}\Omega.$$

Then, we have

$$\langle \Psi_N, \mathcal{N}\Psi_N \rangle \geq N, \quad \left\langle \Psi_N, \mathcal{N}^2\Psi_N \right\rangle \leq CN^2 \tag{1.15}$$

if $\alpha > 0$ and then $n \in \mathbb{N}$ are large enough.

The expectation

$$\langle \Psi_N, \mathcal{N}\Psi_N \rangle \geq N \tag{1.16}$$

gives the average number of particles contained in the trial state and it guarantees that the trial state Ψ has the right number of particles when comparing with the many-body Hamiltonian $\mathcal{H}_N$. The second moment

$$\left\langle \Psi_N, \mathcal{N}^2\Psi_N \right\rangle \leq CN^2 \tag{1.17}$$

measures the fluctuation of the number of particles around its mean, i.e., the variance of the number of particles such that

$$\mathrm{Var}(\mathcal{N}) = \left\langle \Psi_N, \mathcal{N}^2 \Psi_N \right\rangle - (\langle \Psi_N, \mathcal{N} \Psi_N \rangle)^2 \leq C N^2 \,.$$

So, this tells us that the superposition over all the sectors of particles is sharply concentrated near N but not spread out over many sectors.

Although Lemma 4 is enough for the desired upper bound in the paper [13], determining these quantities, $\langle \Psi_N, \mathcal{N} \Psi_N \rangle$, and $\left\langle \Psi_N, \mathcal{N}^2 \Psi_N \right\rangle$, more exactly is significant because it refines our understanding of the trial state Ψ_N itself. For example, precise knowledge of $\langle \Psi_N, \mathcal{N} \Psi_N \rangle$ and $\left\langle \Psi_N, \mathcal{N}^2 \Psi_N \right\rangle$ shows that the trial state Ψ_N is not only accurate for the energy bound, but also structurally well localized in the physical relevant number of particle regime. This is closely related to the role of excitation control in the rigorous proof of Bogoliubov theory, where bounds on number of particle operators are indispensable for turning this transformation into mathematically valid approximations. So, this is sort of a direct way to understand the physical meaning and mathematical sharpness of the trial-state construction underlying the third order upper bound, which may guide some further study for the ground state energy of the dilute Bose gas. Hence, the central task in my work is to calculate the next order term of the number of particles with respect to the trial state Ψ_N, $\langle \Psi_N, \mathcal{N} \Psi_N \rangle$ and the second moment, $\left\langle \Psi_N, \mathcal{N}^2 \Psi_N \right\rangle$, of it. To do this, we should first know exactly what the form of this trial state Ψ_N is.

The construction of Ψ_N is guided by a sequence of physical and mathematical ideas: They first factor out the *Bose-Einstein condensate* by conjugating the Hamiltonian (1.11) with a Weyl operator, then they incorporates pair correlations through a *Bogoliubov transformation*, and finally they adds a carefully designed *cubic transformation*, together with a local cutoff, to capture the next-order interaction effects. Then the final trial state is given by

$$\Psi_N = W_{N_0} e^{B - B^*} e^{A^* \Theta - \Theta A} \Omega \,. \tag{1.18}$$

Now, let me introduce some backgrounds about the concepts that appeared in the construction of the trial state Ψ_N above.

Bose-Einstein condensation. Bose-Einstein condensation refers to the phenomenon that, in a system of many bosons, a macroscopic fraction of the particles occupies the same one-particle quantum state. In other words, for an N-body wave function, if φ denotes the condensate wave function, then the expectation of the number of particles in this state φ is asymptotically equal to N, so that the fraction of particles outside φ becomes negligible as $N \to \infty$. In this sense, Bose-Einstein condensation provides an effective description of the many-body system by a single macroscopic wave function. In the dilute thermodynamic limit that is used in the paper [13], Bose-Einstein condensation describes the low-energy states of the Bose gas. Thus, the role of Bose-Einstein condensation is to capture the leading macroscopic occupation, whereas the physically important corrections to the ground-state energy are generated by correlations beyond the pure condensate picture. In particular, those are encoded by the scattering length and by higher-order excitation effects.

Weyl operators and coherent states. The Weyl operator is a fundamental unitary operator on the bosonic Fock space $\mathcal{F}$. Let $f \in L^2(\Lambda)$ be a one-particle wave function. The associated Weyl operator is defined by

$$W(f) = e^{(a^*(f) - a(f))} \,.$$

Since $a^*(f) - a(f)$ is anti-self-adjoint, the Weyl operator is unitary. In particular, we have

$$W^*(f) = W(-f) = W^{-1}(f)\,.$$

The main role of the Weyl operator is that it generates shifts of the creation and annihilation operators. More precisely, for any $g \in L^2(\Lambda)$, conjugation with the Weyl operator $W(f)$ transforms the annihilation and creation opeartors into shifted versions of themselves such that

$$W^*(f)a(g)W(f) = a(g) + \langle g, f\rangle\,, \qquad W^*(f)a^*(g)W(f) = a^*(g) + \langle f, g\rangle\ .$$

In applications, this is precisely what makes Weyl operators useful for separating Bose-Einstein condensation from its quantum fluctuations. Consequently, the Weyl operators also satisfy the following relation

$$W(f)W(g) = W(g)W(f)e^{2i\mathrm{Im}\langle g,f\rangle}$$

which reflects the underlying canonical commutation structure.

For $f \in L^2(\Lambda)$, the associated coherent state is given by

$$\Psi(f) = W(f)\Omega$$

where Ω denotes the vacuum. So, the coherent states are displaced vacuum states. They are not confined to a single particle-number sector; rather, they are superpositions of sectors containing all possible particle numbers. Using the factorization

$$e^{(a^*(f)-a(f))} = e^{-\|f\|^2/2}e^{a^*(f)}e^{-a(f)}$$

and the fact that $a(f)\Omega = 0$, one obtains that

$$\Psi(f) = e^{-\|f\|^2/2}\left\{1, f, \cdots, \frac{f^{\otimes n}}{\sqrt{n!}}, \cdots\right\}\,.$$

This formula makes clear that coherent states contain contributions from every n-particle sector. The particle-number distribution of a coherent state follows Poisson distribution. More precisely, the probability of finding exactly n particles in the coherent state $\Psi(f)$ is

$$e^{-\|f\|^2}\,\frac{\|f\|^{2n}}{n!}\,.$$

Hence, the expected number of particles is

$$\langle \Psi(f), \mathcal{N}\Psi(f)\rangle = \|f\|^2\,.$$

Another remarkable feature is that coherent states are eigenvectors of all annihilation operators. Indeed, for every $f, g \in L^2(\Lambda)$

$$a(g)\Psi(f) = \langle g, f\rangle\,\Psi(f)\,.$$

This property means that annihilation operators act on coherent states as multiplication by complex numbers. So, coherent states are very similar to classical one and often used in semiclassical analysis which offer a convenient bridge between quantum many-body systems and classical descriptions.

In the paper [13], the Weyl operator identifies the zero-momentum condensate, transforms the Hamiltonian (1.11) into an *excitation Hamiltonian* and thereby prepares the problem for the subsequent Bogoliubov and cubic renormalizations.

More concretely, the Weyl operator in the paper [13] is defined by

$$W_{N_0} = e^{\sqrt{N_0}(a_0^* - a_0)}$$

which satisfies

$$W_{N_0}^* a_p W_{N_0} = a_p + \delta_{p,0}\sqrt{N_0}$$

for all $p \in \Lambda^* = 2\pi\mathbb{Z}^3$.

Bogoliubov transformations. In the analysis of bosonic many-body quantum systems, one is often led to Hamiltonians that are quadradic in creation and annihilation operators. The essential idea is then to perform a linear change of variables that mixes these operators in such a way that the quadratic Hamiltonian acquires a simpler and ideally diagonal form. From a mathematical perspective, there are two basic questions. The first one is that which linear transformations preserve the canonical commutation relations (1.4). Then, under which condition can such a transformation be represented by a unitary operator on bosonic Fock space $\mathcal{F}$. The Bogoliubov Theory answers both questions and thereby provides the structural basis for the applications to Bose gases.

To formulate this theory, let $\mathfrak{h} = L^2(\Lambda)$ for some $\Lambda \subset \mathbb{R}^n$ be the one-particle Hilbert space and let $\mathcal{F}(\mathfrak{h})$ be the bosonic Fock space over $\mathfrak{h}$. For $f, g \in \mathfrak{h}$, we define the antilinear conjugation operator $J : \mathfrak{h} \to \mathfrak{h}$ by $Jf = \bar{f}$. Then we combine creation operator $a^*(f)$ and the annihilation operator $a(f)$ by setting $F = (f, g) \in \mathfrak{h} \oplus \mathfrak{h}$ such that

$$A(F) = a(f) + a^*(Jg) \qquad A^*(F) = a^*(f) + a(Jg)\,.$$

Then the canonical commutation relations can be rewritten as

$$[A(F_1), A^*(F_2)] = \langle F_1, \mathcal{S}F_2\rangle, \qquad \mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $F_1, F_2 \in \mathfrak{h} \oplus \mathfrak{h}$, while $[A(F_1), A(F_2)] \neq 0$ and $[A^*(F_1), A^*(F_2)] \neq 0$.

A Bogoliubov map is a bounded invertible linear operator $\mathcal{V} : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ satisfying

$$\mathcal{V}^*\mathcal{S}\mathcal{V} = \mathcal{S}\,, \qquad \mathcal{V}\mathcal{J} = \mathcal{J}\mathcal{V}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

These two relations express, respectively, the preservation of the symplectic form underlying the canonical commutation relations and the compatibility with conjugation. If we define

$$B(F) = A(\mathcal{V}F)\,, \qquad B^*(F) = A^*(\mathcal{V}F)$$

then the operator $B(F)$ satisfies the same canonical commutation relations as $A(F)$ by $\mathcal{V}\mathcal{J} = \mathcal{J}\mathcal{V}$. Indeed,

$$B^*(F) = A^*(\mathcal{V}F) = A(\mathcal{J}\mathcal{V}F) = A(\mathcal{V}\mathcal{J}F) = B(\mathcal{J}F)\,.$$

By $\mathcal{V}^*\mathcal{S}\mathcal{V} = \mathcal{S}$, we find

$$[B(F_1), B^*(F_2)] = [A(\mathcal{V}F_1), A^*(\mathcal{V}F_2)] = \langle \mathcal{V}F_1, \mathcal{S}\mathcal{V}F_2\rangle = \langle F_1, \mathcal{V}^*\mathcal{S}\mathcal{V}F_2\rangle = \langle F_1, \mathcal{S}F_2\rangle\,.$$

Thus, a Bogoliubov map is exactly a linear change of variables on $\mathfrak{h} \oplus \mathfrak{h}$ that preserves the bosonic algebraic structure.

Every Bogoliubov map $\mathcal{V} : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ can be written in block form as

$$\mathcal{V} = \begin{pmatrix} U & JVJ \\ V & JUJ \end{pmatrix}$$

with bounded operators $U, V : \mathfrak{h} \to \mathfrak{h}$. The conditions defining $\mathcal{V}$ implies

$$U^*U - V^*V = 1\,, \qquad U^*JV = V^*JU$$

and

$$UU^* - JVV^*J = 1\,, \qquad UV^* = JVU^*J\,.$$

The operator U describes the direct part of the transformation, while the off-diagonal term V measures the mixing between annihilation and creation operators. This off-diagonal component is the characteristic feature of Bogoliubov theory, which allows one to absorb pairing terms into a new quasi-particle basis.

At this stage, however, the discussion is only algebraic. In applications, one needs to know whether the transformation can actually be realized by a unitary operator on Fock space $\mathcal{F}(\mathfrak{h})$. A Bogoliubov map $\mathcal{V} : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ is said to be implementable if there exists a unitary operator $\mathbb{U} : \mathcal{F} \to \mathcal{F}$ such that

$$\mathbb{U}^* A(F) \mathbb{U} = A(\mathcal{V}F) \qquad \text{with} \quad \langle \mathbb{U}\Omega, \mathcal{N}\mathbb{U}\Omega \rangle < \infty$$

for all $F \in \mathfrak{h} \oplus \mathfrak{h}$. In this case, $\mathbb{U}$ is called a Bogoliubov transformation. However, preserving the commutation relations at a formal lever is not sufficient. One must also verify that the transformation is well-defined on the Fock space. The criterion for implementability is given by the Shale-Stinespring theorem: if a Bogoliubov map $\mathcal{V} : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ satisfies

$$\mathcal{V}^* \mathcal{S} \mathcal{V} = \mathcal{S}\,, \qquad \mathcal{V}\mathcal{J} = \mathcal{J}\mathcal{V}\,, \qquad \text{with} \quad \mathcal{V} = \begin{pmatrix} U & JVJ \\ V & JUJ \end{pmatrix}$$

for bounded operators $U, V : \mathfrak{h} \to \mathfrak{h}$, then $\mathcal{V}$ is implementable on the bosonic Fock space $\mathcal{F}$ if and only if the V is Hilbert-Schmidt, i.e.,

$$\mathrm{tr}(V^*V) < \infty\,.$$

This condition has direct physical interpretation. The transformed vacuum $\mathbb{U}\Omega$ contains, in general, a nonzero number of particles, and one finds that its expected particle number is exactly

$$\langle \mathbb{U}\Omega, \mathcal{N}\,\mathbb{U}\Omega \rangle = \mathrm{tr}(V^*V)\,.$$

Hence, the Hilbert-Schmidt condition is precisely the requirement that the transformed vacuum still has a finite expected number of particles. In this sense, it is the natural analytic condition that turns a formal symplectic change of variables into a genuine unitary transformation on the Fock space $\mathcal{F}$.

The significance of Bogoliubov transformations becomes clear when considering quadratic Hamiltonians. If H is a one-particle self-adjoint operator, then its second its second quantization (1.5) has the form

$$d\Gamma(H) = \sum_j \langle f_j,\, H f_k \rangle\, a^*(f_j) a(f_j) = \sum_j \lambda_j a^*(f_j) a(f_j)$$

provided that $\{f_j\}$ is an orthonormal basis of eigenvectors of H with eigenvalues λ_j, i.e., $Hf_j = \lambda_j f_j$. Such an operator is already diagonal in the number of occupation basis. However, in many-body problems, we are naturally led to quadratic operators of the more general form

$$\begin{aligned} d\Gamma(H) = &\sum_{i,j} \langle f_i, hf_j \rangle \, a^*(f_i)a(f_j) + \sum_{i,j} \langle f_i, kf_j \rangle \, a^*(f_i)a^*(Jf_j) \\ &+ \sum_{i,j} \langle f_i, k^* f_j \rangle \, a(Jf_i)a(f_j) + \sum_{i,j} \langle f_i, JhJf_j \rangle \, a(Jf_i)a^*(Jf_j) \end{aligned}$$

with $h : \mathfrak{h} \to \mathfrak{h}$ self-adjoint and $k : \mathfrak{h} \to \mathfrak{h}$ such that $k^* = JkJ$. Notice that this formula contains pair excitation and pair annihilation terms which do not preserve the number of particles. Bogoliubov theory resolves this by replacing the original operators with new ones obtained through a suitable linear symplectic transformation. Then the transformed Hamiltonian becomes diagonal up to an additive constant, and one can then read off both the ground-state energy correction and the excitation spectrum.

This mechanism appears naturally in the paper [13]. Here, the Bogoliubov transformation is defined by the unitary operator

$$e^{B-B^*} \tag{1.19}$$

where the generator B is a quadratic operator such that

$$B = \frac{1}{2} \sum_{p \geq N^{\kappa/2-\beta/4}} \mu_p a_p a_{-p} \tag{1.20}$$

with $\beta = 2 - 3\kappa$ and coefficients

$$\mu_p = -\frac{1}{4} \log \left(1 + \frac{2N^\kappa \widehat{Vf} \left(p/N^{1-\kappa} \right)}{p^2} \right)$$

. Notice that f in the μ_p is the solution of the zero-energy scattering equation (1.2). Equivalently, μ_p is characterized by

$$\tanh\left(2\mu_p\right) = -\frac{\widehat{Vf}(p/N^{1-\kappa})}{p^2 + \widehat{Vf}(p/N^{1-\kappa})} \, . \tag{1.21}$$

This formula is very important because, as we mentioned before, it shows that the coefficients are chosen so that the Bogoliubov transformation exactly diagonalizes a quadratic Hamiltonian whose potential is the renormalized quantity Vf rather than only the original potential V.

Actually, in the paper [13], the coefficient μ_p are chosen so that a single unitary map (1.19) simultaneously incorporates the short-scale scattering structure and diagonalize the relevant quadratic part of the excitation Hamiltonian. In this sense, the Bogoliubov transformation (1.19) in the paper [13] contains two types of Bogoliubov transformations. The main point is that the same quadratic transformation plays different roles at different momentum scales. For large momentum satisfying $|p| \gg N^{\kappa/2}$, the coefficient μ_p behaves approximately like

$$\mu_p \simeq -\frac{N^\kappa \, \widehat{Vf}(p/N^{1-\kappa})}{2p^2},$$

and therefore has the same leading-order behavior as the Fourier-space solution of the zero-energy scattering equation. In this regime, the main role of the transformation not to diagonalize the low-energy collective excitations, but to incorporate the correct short-range two-body correlations into the trial state. This means that, after the conjugation, the effective coefficients that appear in the transformed Hamiltonian depend on Vf, and hence on the scattering solution f and the scattering length a, rather than only on the Fourier coefficient $\widehat{V}(0)$. In this sense, the short-range correlation information is already encoded in the new variables.

For small momentum $|p| \lesssim N^{\kappa/2}$, the situation is different. Here, the interaction scale and the kinetic scale are comparable, and the correct choice of μ_p is determined by the identity (1.21), which is the relation ensuring that the Bogoliubov transformation diagonalizes a quadratic Hamiltonian with effective potential Vf. Thus, in the low momentum regime, the transformation has the more familiar Bogoliubov meaning: it removes the off-diagonal pairing terms and produces a diagonal quadratic description of the low-energy excitations.

So, the Bogoliubov transformation is doing two jobs at once: at low momentum it diagonalize the quadratic excitation Hamiltonian, while at high momentum it encode the two-body scattering correlations and thus renormalize the effective interaction.

Notice that in (1.20), the Bogoliubov transformation is defined with an infrared cutoff, namely by summing only over momenta such that

$$|p| \geq N^{\kappa/2-\beta/4}, \qquad \text{where} \qquad \beta = 2 - 3\kappa\,.$$

This restriction is justified by the fact that the momentum below this threshold such that $|p| < N^{\kappa/2-\beta/4}$ contribute to energy only at order $N^{4\kappa-1}$ or smaller, and therefore negligible at the level of accuracy considered here.

After conjugation with the Weyl operator and then with the Bogoliubov transformation, the authors obtain a renormalized Hamiltonian given by

$$\widetilde{\mathcal{H}}_N \simeq C_N + \mathcal{K} + \mathcal{C}_N + \mathcal{C}_N^* + \mathcal{V}_N\,. \tag{1.22}$$

Here, the constant C_N already contains the leading term and part of the Lee-Huang-Yang correction of the ground state energy, $\mathcal{K}$ is the kinetic energy operator, $\mathcal{C}_N$ and $\mathcal{C}_N^*$ are cubic in annihilation and creation operations respectively, where

$$\mathcal{C}_N^* = \frac{1}{\sqrt{N}} \sum_{p,r\in\Lambda_+^*} \hat{V}(r/N^{1-\kappa})\sigma_p a_{p+r}^* a_{-r}^* a_p^*$$

with $\sigma_p = \sinh \mu_p$ and $\mathcal{V}_N$ is a quartic potential operator given by

$$\mathcal{V}_N = \frac{1}{2N} \sum_{r,p,q\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) a_{p+r}^* a_q^* a_{q+r} a_p\,.$$

So, after this step, one can see clearly that leftover cubic and quartic terms are not negligible, while in simpler regimes, such as the mean-field regime, they are perturbatively small. This observation precisely motivates the next transformation.

Perturbation theory. Before the cubic transformation, it is useful to introduce the perturbation theory. Perturbation theory is a general method for extracting spectral information about a complicated operator from a simpler one. In quantum mechanics, this idea is particularly natural, because the state space is a Hilbert space $\mathcal{H}$ and the physically

relevant quantities are encoded in the spectral properties of self-adjoint operators. In particular, the Hamiltonian determines both the stationary states and the time evolution of the system, i.e., for a Hamiltonian H, the Schrödinger equation

$$i\partial_t \psi(t) = H\psi(t)$$

defines a unitary evolution on $\mathcal{H}$. In addition, for every normalized eigenvector $\varphi \in \mathcal{H}$ of H, we have $H\varphi = E\varphi$, which give rise to a stationary state $e^{-iEt}\varphi$. Thus, spectral information on H is the basic input for understanding the dynamics and energetics of the problem.

Let H_0 be a reference operator whose spectrum is well understand. Let W be a perturbation and $\lambda \in \mathbb{C}$ be a small parameter. Then we can write the desired operator H_λ as

$$H_\lambda = H_0 + \lambda W\,.$$

So, for small λ, the spectrum of H_λ should be close to that of H_0. One then asks how the eigenvalues and eigenvectors of H_0 are modified when the perturbation is switched on. If E_0 is an eigenvalue of H_0, with normalized eigenvector ψ_0, i.e.,

$$H_0\psi_0 = E_0\psi_0\,.$$

Then we look for

$$H_\lambda\psi(\lambda) = E(\lambda)\psi(\lambda)$$

where

$$E(\lambda) = E_0 + \lambda E^{(1)} + \lambda^2 E^{(2)} + \cdots$$

and

$$\psi(\lambda) = \psi_0 + \lambda\psi^{(1)} + \lambda^2\psi^{(2)} + \cdots\,. \tag{1.23}$$

The idea is to describe the spectrum of the full Hamiltonian H_λ as a perturbative modification of the spectrum of the unperturbed one.

In the non-degenerate case, we have

$$(H_0+\lambda W)(\psi_0+\lambda\psi^{(1)}+\lambda^2\psi^{(2)}+\cdots) = (E_0+\lambda E^{(1)}+\lambda^2 E^{(2)}+\cdots)(\psi_0+\lambda\psi^{(1)}+\lambda^2\psi^{(2)}+\cdots)\,.$$

Now, we compare the powers of λ. For λ^0, we recover the unperturbed equation

$$H_0\psi_0 = E_0\psi_0\,. \tag{1.24}$$

For λ^1, we get

$$H_0\psi^{(1)} + W\psi_0 = E_0\psi^{(1)} + E^{(1)}\psi_0\,.$$

Equivalently,

$$(H_0 - E_0)\psi^{(1)} = -(W - E^{(1)})\psi_0\,.$$

Taking the inner product with ψ_0, by H_0 self-adjoint and (1.24), we find

$$0 = \Big\langle \psi_0,\, (H_0 - E_0)\psi^{(1)} \Big\rangle = \Big\langle \psi_0,\, -(W - E^{(1)})\psi_0 \Big\rangle\,.$$

So,

$$E^{(1)} = \langle \psi_0, W\psi_0\rangle\,.$$

Thus, the first correction to the energy is simply the expectation of the perturbation in the unperturbed state. Physically, this tells us that the first-order perturbation theory keeps the state fixed and only measures how much the new interaction shifts the old energy

level. The first correction to the eigenvector is instead obtained by projecting onto the orthogonal complement of ψ_0. If $\{\psi_n\}_{n\geq 0}$ is an orthonormal basis of eigenvectors of H_0, with eigenvalues E_n, then

$$\psi^{(1)} = \sum_{n\neq 0} \frac{\langle \psi_n, W\psi_0\rangle}{(E_0 - E_n)} \psi_n \,. \tag{1.25}$$

This shows that the perturbation combines the original state ψ_0 with the other eigenvectors of the unperturbed Hamiltonian. The coefficient of this combination is governed both by the matrix element $\psi_n, W\psi_0$ and by the spectral gap $E_n - E_0$. States that are strongly coupled by the perturbation and lie close in energy contribute most significantly.

At second order, follows from the same procedure, we obtain

$$E^{(2)} = \sum_{n\neq 0} \frac{|\langle \psi_n, W\psi_0\rangle|^2}{(E_0 - E_n)} \,. \tag{1.26}$$

For the ground state energy, where $E_n > E_0$ for all $n \neq 0$, this term is non-positive. Hence, the second-order perturbation theory typically lowers the ground state energy. This is consistent with the variational principle: evaluating the perturbed Hamiltonian in the unperturbed ground state ψ_0 yields only a first approximation to the new ground state energy. The actual ground state of the perturbed Hamiltonian is allowed to differ from ψ_0, and this variation leads to an additional energy gain, which is reflected in the second-order correction.

It is useful to rewrite the same formulas abstractly. Let P_0 be the orthogonal projection onto ψ_0 and Q_0 be the complement of P_0 such that

$$P_0 = |\psi_0\rangle \langle \psi_0| \,, \qquad Q_0 = 1 - P_0 \,.$$

On the orthogonal complement of ψ_0, the operator $H_0 - E_0$ is invertible if E_0 is isolated and simple. By (1.25) and (1.26), we get

$$\psi^{(1)} = -\left(Q_0 \left(H_0 - E_0\right) Q_0\right)^{-1} Q_0 W \psi_0 \tag{1.27}$$

and

$$E^{(2)} = -\left\langle \psi_0,\, W \left(Q_0 \left(H_0 - E_0\right) Q_0\right)^{-1} Q_0 W \psi_0 \right\rangle \,. \tag{1.28}$$

Here, we use the assumption that E_0 is simple, which means that E_0 is not degenerate, i.e., the eigenspace for E_0 has dimension one. For the degenerate case, it requires a preliminary step. Suppose E_0 is degenerate. This means that there are several orthonormal vectors $\psi_1, \cdots, \psi_m$ such that

$$H_0 \psi_j = E_0 \psi_j$$

for all $j = 1, \cdots, m$. Then the previous formula (1.26) breaks down, because the denominator $E_0 - E_n$ vanishes inside the degenerate subspace. Then the perturbation must first be diagonalized within the degenerate eigenspace

$$\mathcal{E}_0 = \mathrm{span}\{\psi_1, \cdots, \psi_m\} \,.$$

The first-order energy corrections are then given by the eigenvalues of the matrix $\langle \psi_i, W\psi_j\rangle$ restricted to this subspace $\mathcal{E}_0$. This reflects the fact that the unperturbed Hamiltonian does not distinguish among vectors in the degenerate eigenspace, while the perturbation generally selects preferred linear combinations. Only after this internal diagonalization can one proceed with the usual perturbative expansion.

Although the formal structure of perturbation theory is not very hard, its usefulness crucially depends on the choice of the unperturbed operator. If the reference Hamiltonian H_0 already captures the correct leading physical structure, then perturbation theory may provide accurate asymptotic expansions. If not, the expansion may be misleading even at the leading order. For example, consider free bosons in a unit torus Λ, the non-interacting Hamiltonian

$$H_N^0 = \sum_{j=1}^{N} -\Delta_{x_j}$$

has ground state energy 0, which is attained by the normalized ground state vector $\psi_N^0 = \varphi_0^{\otimes N}$, where $\varphi_0 \in L^2(\Lambda)$ is the zero-momentum orbital with $\varphi_0(x) = 1$ for all $x \in \Lambda$. In this setting, the perturbation theory suggests approximating the interacting ground state energy by the expectation of the full Hamiltonian in the non-interacting ground state ψ_N^0. For example, In the mean-field regime where the Hamiltonian is given by

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^{N} V(x_i - x_j)\,,$$

and, under the assumption that $V \in L^\infty(\mathbb{R}^3)$ and $\widehat{V} \geq 0$, this indeed leads to the correct leading order prediction

$$\left\langle \psi_N^0,\, H_N \psi_N^0 \right\rangle = \frac{N-1}{2}\widehat{V}(0)\,.$$

The situation changes drastically in the Gross-Pitaevskii regime and a dilute gas in the thermodynamic limit. The leading order energy is no longer determined only by the Fourier coefficient $\widehat{V}(0)$ of the original potential V, but by the scattering length a. We take a dilute gas in the thermodynamic limit as an example. We take the expectation of the Hamiltonian (1.1) in the non-interacting ground state ψ_N^0 and we find

$$\left\langle \psi_N^0,\, H_L \psi_N^0 \right\rangle \approx \frac{N(N-1)}{2|\Lambda_t|}\widehat{V}(0)\,.$$

Dividing by the volume and using $\rho = N/|\Lambda_t|$, this gives the energy density

$$e(\rho) = \frac{1}{|\Lambda_t|}\left\langle \psi_N^0,\, H_L \psi_N^0 \right\rangle \approx \frac{1}{2}\widehat{V}(0)\rho^2\,. \tag{1.29}$$

However, Theorem 1 implies that the leading term is given by

$$e(\rho) \sim 4\pi\mathfrak{a}\rho^2\,.$$

Since $8\pi\mathfrak{a} < \widehat{V}(0)$, we know the prediction is too large and the error is again of order ρ^2 which is of the same order as the leading term itself. So, the naive perturbative prediction is wrong already at the leading order. The reason is that the non-interacting ground state $\psi_N^0 = \varphi_0^{\otimes N}$ is completely factorized and therefore misses the short-range two-body interaction encoded by the zero-energy scattering equation. Similar reason for the Gross-Pitaevskii regime.

For this reason, we need first construct a trial state contains the information of interactions. In paper [13], this is done through Weyl and Bogoliubov transformations on the original Hamiltonian $\mathcal{H}_N$, and only afterwards apply perturbative theory to the renormalized excitation Hamiltonian $\widetilde{\mathcal{H}}_N$ (1.22) by viewing $\mathcal{K}$ as the unperturbed Hamiltonian and the remaining term $\mathcal{C}_N + \mathcal{C}_N^* + \mathcal{V}_N$ as the perturbation.

In this renormalized Hamiltonian $\widetilde{\mathcal{H}}_N$ (1.22), the vacuum Ω is the ground state of the unperturbed operator $\mathcal{K}$, and the first-order correction vanishes

$$\langle \Omega,\, \mathcal{C}_N \Omega \rangle = \langle \Omega,\, \mathcal{C}_N^* \Omega \rangle = \langle \Omega,\, \mathcal{V}_N \Omega \rangle = 0$$

because the vacuum expectation of the perturbing terms is zero. For this reason, the first non-trivial contribution comes from second-order perturbation theory.

From this perspective, the cubic correction appearing the trial state is very natural. By (1.27) and $\mathcal{K} = H_0 - E_0$, we obtain

$$\psi^{(1)} = -\left(Q_0 \mathcal{K} Q_0\right)^{-1} Q_0 W \Omega\,.$$

Notice that $Q_0 \mathcal{K} Q_0$ is invertible because Q_0 removes the vacuum direction which is the kernel of $\mathcal{K}$. Then we simply reduce it to

$$\psi^{(1)} = -\mathcal{K}^{-1} W \Omega$$

as $W\Omega \perp \Omega$. Since $\mathcal{C}_N$ and $\mathcal{V}_N$ vanish in the vacuum Ω, by (1.23), we get the trial vector proposed in the paper [13]

$$\begin{aligned} \widetilde{\xi} &= \Omega - \frac{1}{\mathcal{K}} \mathcal{C}_N^* \Omega \\ &= \Omega - \frac{1}{\sqrt{N}\mathcal{K}} \sum_{|p| \geq N^{\kappa/2}, |q| \geq N^{\kappa/2-\beta/4}} \widehat{Vf}(p/N^{1-\kappa}) \sigma_q a_{p+q}^* a_{-p}^* a_{-q}^* \Omega \\ &= \Omega - \frac{1}{\sqrt{N}} \sum_{|p| \geq N^{\kappa/2}, |q| \geq N^{\kappa/2-\beta/4}} \frac{\widehat{Vf}(p/N^{1-\kappa}) \sigma_q}{(p+q)^2 + p^2 + q^2} a_{p+q}^* a_{-p}^* a_{-q}^* \Omega =: \Omega + A^* \Omega\,. \end{aligned} \tag{1.30}$$

Here, the reason for the restriction $|p| \geq N^{\kappa/2}$ and $|q| \geq N^{\kappa/2-\beta/4}$ is that the contribution to the energy of small momenta is negligible as we mentioned above.

Taking expectation of the renormalized Hamiltonian (1.22) with respect to the state $\widetilde{\xi}$, we get

$$\left\langle \widetilde{\xi},\, (\mathcal{K} + \mathcal{C}_N + \mathcal{C}_N^* + \mathcal{V}_N) \widetilde{\xi} \right\rangle = \langle \Omega,\, (A\mathcal{C}_N^* + \mathcal{C}_N A^*) \Omega \rangle + \langle \Omega,\, A(\mathcal{K} + \mathcal{V}_N) A^* \Omega \rangle\,. \tag{1.31}$$

Equation (1.28) with $\mathcal{K} = H_0 - E_0, W = \mathcal{C}_N + \mathcal{C}_N^* + \mathcal{V}_N$ and $A^* = -\mathcal{K}^{-1}\mathcal{C}_N^* \Omega$ implies that

$$E^{(2)} = -\left\langle \Omega,\, \mathcal{C}_N \mathcal{K}^{-1} \mathcal{C}_N^* \Omega \right\rangle = \langle \Omega,\, (A\mathcal{C}_N^* + \mathcal{C}_N A^*) \Omega \rangle + \langle \Omega,\, A\mathcal{K} A^* \Omega \rangle$$

which represents the second-order correction while the left term $\langle \Omega,\, A\mathcal{V}A^*\Omega \rangle$ may formally belong to the third order. Combining (1.31) with the constant term C_N in (1.22), we obtain the correct ground state energy expansion, which also include the third-order contribution in (4.1)

The difficulty is that such a vector $\widetilde{\xi}$ is not normalized

$$\|\widetilde{\xi}\|^2 = 1 + \|A^* \Omega\|^2 \simeq C N^{3\kappa - 1} \tag{1.32}$$

and, in the regime of interest, its norm correction is too large to be neglected. This is why the paper [13] replaces the formal perturbative ansatz by a unitary *cubic transformation*.

Cubic transformations. A cubic transformation is a unitary transformation on bosonic Fock space generated by an operator that is cubic in the creation and annihilation operators. If $\mathcal{F}(\mathfrak{h})$ denotes the bosonic Fock space over a one-particle Hilbert space $\mathfrak{h} = L^2(\Lambda)$ for some $\Lambda \subseteq \mathbb{R}^n$, one may consider a generator X_3 built from suitably controlled linear combination of creation operators a^* and annihilation operators a of total degree three, such as a^*a^*a, a^*aa, aaa or their adjoints. Then we define the corresponding unitary map by

$$U_3 = e^{X_3}$$

with $X_3^* = -X_3$. In contrast with a Bogoliubov transformation, whose quadratic generator keeps operators within the same finite degree (2.14), whereas conjugation with a cubic generator usually produces operators of increasing degree through repeated commutators, so it no longer stays within any fixed finite-degree class. Specifically, let $\mathcal{O}$ be an operator act on bosonic Fock space $\mathcal{F}(\mathfrak{h})$. Then, by Duhamel expansion, we get

$$U_3^* \mathcal{O} U_3 = e^{-X_3} \mathcal{O} e^{X_3} = \mathcal{O} + [\mathcal{O}, X_3] + \frac{1}{2!} [[\mathcal{O}, X_3], X_3] + \cdots . \tag{1.33}$$

For a concrete example, if $\mathcal{O}$ is an annihilation operator $a(f)$ for $f \in \mathfrak{h}$ a one-particle wave function, then after conjugation by U_3, the annihilation operator $a(f)$ no longer stays in the linear combination of annihilation operator $a(f)$ and creation operator $a^*(f)$.

The role of the commutator expansion is to organize the effect of the cubic conjugation and to show that higher-order terms remain perturbative. In the expansion of $U_3 \mathcal{O} U_3$, the successive commutators $[\mathcal{O}, X_3], [[\mathcal{O}, X_3], X_3]$, and so on, measure how the cubic transformation deforms $\mathcal{O}$. In some settings, each new commutator comes with a small parameter carried by the cubic kernel, so that the higher terms in the Duhamel expansion become progressively smaller. This is the mechanism emphasized in the paper [13], where the authors explain that every two additional commutators introduce an extra small factor, preventing the iterated expansion from diverging and making the cubic transformation perturbatively well behaved. Nevertheless, this smallness from the commutator expansion alone is often not enough for a rigorous control, because the resulting operator terms are sensitive to the local distribution of particles in the small spatial regions where the kernels of the transformation are supported. For this reason, cubic transformations are usually combined with a cutoff, inserted either directly in the generator X_3 or in the transformed state, in order to excludes rare configurations with anomalously high local particle density.

A convenient way to formulate such a cutoff is through local number operators. Given a length $\ell > 0$, we introduces an operator

$$\mathcal{N}_\ell(x) = \int_{|x-y| \leq \ell} dy \, a_y^* a_y$$

to count the number of particles in a ball of radius ℓ centered at x, or a smoothed version

$$\mathcal{N}_{\chi_\ell}(x) = \int dy \, \chi_\ell(x-y) a_y^* a_y$$

where $\chi_\ell \in C_c^\infty(\mathbb{R}^3)$ such that $0 \leq \chi_\ell \leq 1$ with $\chi_\ell(x) = 1$ if $|x| \leq \ell$ and $\chi_\ell(x) = 0$ if $|x| \geq 2\ell$. Then we inserts a bounded function Θ of these local observables into the cubic generator. The purpose of Θ is not to change the leading behavior of the transformation and therefore is not expected to contribute significantly to the energy, but to guarantee that the states produced by the cubic unitary satisfy a uniform bound on local particle concentration. Thus, in general, the commutator expansion and the cutoff play complementary roles:

the former controls the growth of the higher-order operator terms produced by the cubic conjugation,while the cutoff controls the local particle density on the short spatial scales where the cubic kernel is concentrated.

To avoid the normalization problem (1.32) of the state $\widetilde{\xi}$ (1.30). The authors replace the linear correction by a unitary transformation generated by the same cubic operator $A^* - A$ defined in (1.30). Formally, the new state ξ is given by

$$\xi = e^{A^* - A}\Omega\,.$$

Notice that $A^* - A$ is anti-sefladjoint, so its exponential is unitary and automatically preserves the norm. Moreover, the first term

$$\langle \Omega,\, [\mathcal{C}_N + \mathcal{C}_N^*, A^* - A]\Omega\rangle = \langle \Omega,\, (\mathcal{C}_N A^* + A\mathcal{C}_N^*)\Omega\rangle$$

and the second term

$$\frac{1}{2!}\,\langle \Omega,\, [[\mathcal{K} + \mathcal{V}_N, A^* - A], A^* - A]\Omega\rangle = \langle \Omega,\, A(\mathcal{K} + \mathcal{V})A^*\Omega\rangle$$

in the Duhamel expansion (1.33) of $\langle \xi,\, (\mathcal{K} + \mathcal{C}_N + \mathcal{C}_N^* + \mathcal{V}_N)\xi\rangle$. Additionally, the higher commutators become progressively smaller. In fact, every time one adds two more commutators with A and A^*, one introduces four additional momentum variables. But in expectation with respect to the vacuum Ω, three new delta constraints also appear, so there is only one new momentum sum remains. This contributes a factor of order $\|\sigma\|_2^2 \simeq N^{3\kappa/2}$. Since each each cubic factor A or A^* carries a prefactor $N^{-1/2}$, every additional pair of commutators yields an extra factor $N^{3\kappa/2-1}$, which is very small because $\kappa < 2/3$. Thus, the unitary cubic transformation generates exactly the terms predicted by the perturbation theory (1.31) with increasingly smaller higher order terms while simultaneously resolving the normalization problem.

As we have introduced above, to turn this formal idea into a rigorous construction, one still need to show the error terms arising in the Duhamel expansion is convergent. A convenient way is to control the local number of particles by a cutoff. Due to the smooth infrared cutoff $|q| \geq N^{\kappa/2-\beta/4}$ in momentum space, the kernel $\check{\sigma}(x)$ decays rapidly once $|x|$ is much larger than the scale

$$\ell_\sigma = N^{-\kappa/2+\beta/4}\,,$$

and the kernel $\check{A}(y,x)$ associated with the cubic operator is similarly negligible when $|x| \gg \ell_\sigma$ or $|y| \gg \ell_\eta$, where

$$\ell_\eta = N^{-\kappa/2}\,.$$

Moreover, these kernels also decay at shorter distances with rate comparable to $|x|^{-5/2}$. As a consequence, their L^2-mass is essentially concentrated on a smaller scale

$$\ell_B = N^{-\kappa/2+\beta/12} \ll \ell_\sigma\,.$$

If one partitions space into boxes B_u of side length ℓ_B, for every $u \in \Lambda_B = \ell_B\mathbb{Z}^3$, and write $\check{\sigma}_u = \check{\sigma}\mathbb{1}_{B_u}$, then one obtains the estimate

$$\sum_{u\in\Lambda_B} \|\check{\sigma}_u\|_2 \leq CN^{3\kappa/4+\epsilon}\,,$$

which, up to the harmless factor N^ϵ, is comparable with the global L^2-norm of $\check{\sigma}$. This shows that the relevant excitations are not spread uniformly over the whole volume, but are concentrated locally in a controlled way.

Motivated by this localization, the final trial state is modified to

$$\xi = e^{A^*\Theta - \Theta A}\Omega$$

with a smooth cutoff $\Theta \in C_c^\infty(\mathbb{R})$ to suppress configurations in which the local number of cubic excitation becomes too large. More precisely, Θ is chosen so that

$$\Theta \mathcal{N}_{\ell_B}(x)\Theta \leq N^\epsilon$$

for every $x \in \Lambda$. This means that, after insertion of the cutoff, the cubic transformation is only allowed to create states whose local number of particles remains controlled on the scale ℓ_B. The cutoff does not change the main energy contribution, because the expected number of particles produced by the cubic transformation in a box of size ℓ_B is very small:

$$\ell_B^3 \left\| A^*\Omega \right\|^2 \simeq \ell_B^3 N^{3\kappa-1} \simeq N^{-\beta/4} \ll 1\,.$$

Hence, the cutoff excludes only very rare local particle configuration of the trial state. This removal has negligible energetic effect, but it is indispensable for estimating commutators involving Θ and for keeping the Duhamel expansion under control. In this way, the cutoff not only maintains the perturbative structure of the trial state, but also, at the same time, supplies the additional local control required for rigorous analysis.

The rest of this thesis is organized as follows. In Chapter 2, we study the expected number of particles in the trial state Ψ_N and derive a refined asymptotic estimate. In Chapter 3, we analyze the second moment of the number operator and prove the corresponding estimate. In Chapter 4, we turn to the energy analysis of the Hamiltonian in the trial state Ψ_N. All the proofs in this section follow the strategy of [13], but rewrite the main estimates in a detailed form.

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2 The Expectation of $\mathcal{N}$

The main purpose of this section is to find more precise estimates on the expectation of number of particles (in this section) and its second moment (in the next section 3) with respect to the trial state Ψ_N in Theorem 2. So, we first introduce this trial state Ψ_N and the coefficients of each operator that we will use in the following estimation.

2.1 The trial state and its coefficients

In this section we introduce the trial state Ψ_N that is used in the proof of the upper bound (4.3) and is also for my calculation of the expectation of number of particles and its second. The state Ψ_N is constructed on the bosonic Fock space $\mathcal{F}(\Lambda)$ given by

$$\Psi_N = W_{N_0} e^{B^* - B} e^{A^*\Theta - \Theta A}\Omega\,. \tag{2.1}$$

where $\Omega \in \mathcal{F}(\Lambda)$ denotes the vacuum vector. The three factors appearing in (2.1) have different roles. The Weyl operator W_{N_0} creates the state in zero-momentum mode. The Bogoliubov transformation e^{B-B^*} incorporates the leading pair correlations and diagonalizes the relevant quadratic part of the excitation Hamiltonian. Finally, the cubic transformation $e^{A^*\Theta-\Theta A}$ introduces higher order correlations, which are needed in order to capture the next correction to the energy.

More precisely, the Weyl operator is defined by

$$W_{N_0} = e^{\sqrt{N_0}(a_0^*-a_0)} \tag{2.2}$$

which satisfies

$$W_{N_0}^* a_p W_{N_0} = a_p + \delta_{p,0}\sqrt{N_0} \tag{2.3}$$

for all $p \in \Lambda^* = 2\pi\mathbb{Z}^3$. So, after conjugation with W_{N_0}, the annihilation operator of the zero momentum a_0 is shifted by the $\sqrt{N_0}$. In this sense, W_{N_0} describes a coherent state containing, on average, N_0 particles with momentum $p = 0$.

The second factor in (2.1) is generated by quadratic operator

$$B = \frac{1}{2}\sum_{p\in\Lambda^*\setminus\{0\}} \mu_p a_p a_{-p}. \tag{2.4}$$

The corresponding unitary transformation e^{B-B^*} is the Bogoliubov transformation. It creates and annihilates pairs of excitations with opposite momenta and is responsible for incorporating the dominant two body correlation structure.

The last factor is generated by the cubic operator

$$A = \sum_{p,q\in\Lambda^*\setminus\{0\}} A_{p,q}\, a_{p+q} a_{-p} a_{-q}\,. \tag{2.5}$$

The unitary transformation $e^{A^*\Theta-\Theta A}$ is a cubic correction to the Bogoliubov ansatz. The operator Θ is a cutoff which restricts configurations with large local particle density. Its presence is essential in the estimates for the Duhamel expansion of the cubic transformation, since it prevents the local number of created excitations from becoming too large as we have introduced above in section 1.

The precise definitions of the coefficients $\mu_p, A_{p,q}$ and of the cutoff Θ will be given in the following subsections.

2.1.1 Coefficients of the Bogoliubov transformation

Definition of coefficients on lattice. We begin with the coefficients entering the Bogoliubov transformation. Let f be the solution of the zero-energy scattering equation (1.2). For $p \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$, we define

$$\eta_{\infty,p} = \frac{-N^\kappa \widehat{Vf}(p/N^{1-\kappa})}{2p^2} \tag{2.6}$$

and

$$\mu_{\infty,p} = -\frac{1}{4}\log\left(1 - 4\eta_{\infty,p}\right)\,. \tag{2.7}$$

For completeness, we set

$$\eta_{\infty,0} = \mu_{\infty,0} = 0\,.$$

The above choice is motivated by the diagonalization of the quadratic Hamiltonian obtained after replaceing the interaction potential V by the renormalized potential Vf. Indeed, a direct computation gives

$$\tanh(2\mu_{\infty,p}) = -\frac{\widehat{Vf}(p/N^{1-\kappa})}{p^2 + \widehat{Vf}(p/N^{1-\kappa})}$$

This is precisely the relation which appears in the diagonalization of a quadratic Hamiltonian with interaction Vf. Thus, the coefficients $\mu_{\infty,p}$ are chosen so that the corresponding Bogoliubov transformation matches the scattering-renormalized quadratic structure.

We shall frequently use the associated hyperbolic functions

$$\sigma_{\infty,p} = \sinh \mu_{\infty,p} \qquad \gamma_{\infty,p} = \cosh \mu_{\infty,p} \,. \tag{2.8}$$

They describe the action of the Bogoliubov transformation on creation and annihilation operators. In terms of $\eta_{\infty,p}$, they are given by

$$\sigma^2_{\infty,p} = \frac{p^2 + N^\kappa \widehat{Vf}(p/N^{1-\kappa}) - \sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}(p/N^{1-\kappa})}}{2\sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}(p/N^{1-\kappa})}} = \frac{1 - 2\eta_{\infty,p} - \sqrt{1 - 4\eta_{\infty,p}}}{2\sqrt{1 - 4\eta_{\infty,p}}} \,,$$

$$\gamma^2_{\infty,p} = \frac{p^2 + N^\kappa \widehat{Vf}(p/N^{1-\kappa}) + \sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}(p/N^{1-\kappa})}}{2\sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}(p/N^{1-\kappa})}} = \frac{1 - 2\eta_{\infty,p} + \sqrt{1 - 4\eta_{\infty,p}}}{2\sqrt{1 - 4\eta_{\infty,p}}} \,,$$

$$\gamma_{\infty,p}\sigma_{\infty,p} = \frac{-N^\kappa \widehat{Vf}(p/N^{1-\kappa})}{2\sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}(p/N^{1-\kappa})}} = \frac{\eta_{\infty,p}}{\sqrt{1 - 4\eta_{\infty,p}}} \,.$$

These identities provide the basic estimates on the Bogoliubov coefficients. They are used to control the number of excitations created by the transformation and to evaluate the contribution of the quadratic part of the Hamiltonian in the paper [13].

Introduction of infrared cutoff. We now modify the coefficients introduced above by removing the very low momentum modes. This modification does not affect the energy at the order relevant for our analysis, but it is useful for obtaining good decay properties in position space.

Let

$$\beta = 2 - 3\kappa > 0 \,. \tag{2.9}$$

Using the momentum-space estimates for $\sigma_{\infty,p}$ from [13, Lemma 3, equation (28)], one obtains

$$\begin{aligned} N^\kappa \sum_{|p|<N^\alpha} \sigma^2_{\infty,p} &\leq C \sum_{|p|<N^\alpha} \left|\widehat{Vf}\left(p/N^{1-\kappa}\right)\right|^2 \left(\frac{N^{\kappa/4}}{|p|^{1/2}}\right)^2 \\ &\leq C N^{3\kappa/2} \int_{|p|<N^\alpha} \frac{1}{|p|}\, dp \leq C N^{3\kappa/2+2\alpha} \leq C N^{4\kappa-1} \,, \end{aligned} \tag{2.10}$$

provided that

$$\alpha \leq \frac{\kappa}{2} - \frac{\beta}{4} \,.$$

This estimate show that the momentum with

$$|p| < N^{\kappa/2-\beta/4}$$

give only a negligible contribution at the precision considered here. Indeed, the l.h.s. of (2.10) is the typical contribution that comes up along with the quadratic term

$$N^{\kappa} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N^{1-\kappa}) a_p^* a_p$$

after the Bogoliubov transformation. The remaining quadratic terms are treated in the same way and give contributions of the same negligible order in this low-momentum region. Hence, for the purpose of resolving the energy up to errors of order $N^{4\kappa-1}$, these very low momentum excitations can be discarded.

We therefore introduce an infrared cutoff in the Bogoliubov coefficients. The cutoff is chosen smooth because smoothness in momentum space gives better decay of the corresponding kernels in position space. Let $\chi_l \in C_c^\infty(\mathbb{R})$ be such that $0 \leq \chi_l \leq 1$ and

$$\chi_l(x) = 0 \quad \text{for } x \leq 1\,; \qquad \chi_l(x) = 1 \quad \text{for } x \geq 2\,. \tag{2.11}$$

We define

$$\chi(p) = \chi_l\left(\frac{|p|}{N^{\kappa/2-\beta/4}}\right). \tag{2.12}$$

The actual coefficients used in the Bogoliubov transformation are then given by

$$\mu_p \coloneqq \mu_{\infty,p}\chi(p) \tag{2.13}$$

for $p \in \Lambda^*$. With these truncated coefficients, the quadratic generator becomes

$$B = \frac{1}{2} \sum_{p \in \Lambda^* \setminus \{0\}} \mu_p a_p a_{-p}. \tag{2.4}$$

This is the operator entering the trial state (2.1).

The action of the corresponding Bogoliubov transformation on the annihilation operator a_p and creation operator a_p^* is given by

$$e^{B-B^*} a_p e^{B^*-B} = \gamma_p a_p + \sigma_p a_{-p}^*\,, \qquad e^{B-B^*} a_p^* e^{B^*-B} = \gamma_p a_p^* + \sigma_p a_{-p} \tag{2.14}$$

where

$$\sigma_p = \sinh \mu_p\,, \qquad \gamma_p = \cosh \mu_p\,. \tag{2.15}$$

Thus the Bogoliubov transformation mixes annihilation operators with creation operators of opposite momentum. This reflects the creation and annihilation of pairs of excitation with momenta p and $-p$.

Definition of coefficients on $\mathbb{R}^3$. For some arguments, it is convenient to work not only with the lattice coefficients but also with their continuous counterparts. For every $p \in \mathbb{R}^3$, we define

$$\eta_\infty(p) = -N^\kappa \frac{\widehat{Vf}(p/N^{1-\kappa})}{2p^2}\,, \qquad \mu_\infty(p) = -\frac{1}{4}\log(1 - 4\eta_\infty(p))\,.$$

We then set

$$\sigma_\infty(p) = \sinh \mu_\infty(p)\,, \qquad \gamma_\infty(p) = \cosh \mu_\infty(p)\,,$$

and, after inserting the same cutoff (2.12), we get

$$\mu(p) = \mu_\infty(p)\chi(p)\,, \qquad \sigma(p) = \sinh\mu(p)\,, \qquad \gamma(p) = \cosh\mu(p)\,.$$

Then, by construction,

$$\mu(p) = \begin{cases} 0\,, & |p| \le N^{\kappa/2-\beta/4}\,, \\ \mu_\infty(p)\,, & |p| \ge 2N^{\kappa/2-\beta/4}\,. \end{cases}$$

Moreover,

$$|\mu(p)| \le |\mu_\infty(p)|$$

for all $p \in \mathbb{R}^3$. Consequently,

$$|\sigma(p)| \le |\sigma_\infty(p)|\,, \qquad 0 \le \gamma(p) - 1 \le \gamma_\infty(p) - 1\,.$$

for all $p \in \mathbb{R}^3$. In particular, in the infrared region $|p| < N^{\kappa/2-\beta/4}$, we have

$$\sigma(p) = 0\,, \qquad \gamma(p) = 1\,.$$

Thus, the Bogoliubov transformation acts trivially on the discarded low-momentum modes, while it agrees with the original scattering-motivated transformation for momenta larger than $2N^{\kappa/2-\beta/4}$.

Localization of kernels in position space. To study the action of cubic transformation

$$e^{A^*\Theta - \Theta A}$$

it is useful to decompose the relevant kernels into localized pieces. This allows us to exploit the cutoff operator Θ, whose role is to prevent too many excitations from being concentrated in a small spatial region and will be defined later in the next subsection 2.1.2. Now, we introduce the spatial scale

$$\ell_B = N^{-\kappa/2+\beta/12}$$

where $\beta = 2 - 3\kappa > 0$ as we defined above (2.9). Let

$$\Lambda_B = \Lambda \cap \ell_B \mathbb{Z}^3$$

be the corresponding lattice of localization centers. For each $u \in \Lambda_B$, we define the cell

$$B_u = \left\{ x \in \Lambda : \inf_{v \in \Lambda_B} |x - v| = |x - u| \right\}.$$

Then the family $\{B_u\}_{u\in\Lambda_B}$ gives a partition of the unit torus Λ, up to boundary sets of measure zero.

We will use this partition to localize the position-space kernels associated with the Bogoliubov coefficients. We first define

$$\check{\sigma}(x) = \sum_{p\in\Lambda^*} \sigma_p e^{ip\cdot x}\,, \qquad \widecheck{\gamma - 1}(x) = \sum_{p\in\Lambda^*} (\gamma_p - 1) e^{ip\cdot x}\,. \tag{2.16}$$

where σ_p and γ_p is defined as above in (2.15). Then, we set

$$\check{\sigma}_u = \check{\sigma}\, \mathbb{1}_{B_u}\,, \qquad (\widecheck{\gamma - 1})_u = \widecheck{\gamma - 1}\, \mathbb{1}_{B_u}$$

By the definition of B_u, we have

$$\check{\sigma} = \sum_{u \in \Lambda_B} \check{\sigma}_u \,, \qquad \widetilde{\gamma - 1} = \sum_{u \in \Lambda_B} (\widetilde{\gamma - 1})_u \,.$$

This localization will be used later to combine the spatial decay of the kernels with the local particle-number control provided by Θ.

The authors also give an upper bound

$$\sum_{u \in \Lambda_B} \|\check{\sigma}_u\|_2 \leq C N^{3\kappa/4+\epsilon} \tag{2.17}$$

for every $\epsilon > 0$ in [13, Lemma 6], which will repeatedly be used in my work.

2.1.2 Coefficients of the cubic transformation

Definition of coefficients $A(p,q)$. We now introduce the coefficients defining the cubic operator. The construction follows the same general idea as in [16]. Let $\chi_l \in C_c^\infty(\mathbb{R})$ be the smooth cutoff defined as above (2.11). Then we set

$$\widetilde{\chi}(p) = \chi_l \left(\frac{|p|}{N^{\kappa/2}} \right) . \tag{2.18}$$

For $p, q \in \mathbb{R}^3 \setminus \{0\}$, we define

$$A(p,q) = N^{-1/2} \widetilde{\chi}(p) \frac{-N^\kappa \widehat{Vf}\big(p/N^{1-\kappa}\big)\sigma(q)}{p^2 + q^2 + (p+q)^2} \,. \tag{2.19}$$

with $\sigma(p)$ given in the above (2.15). If

$$\eta(p) = \frac{-N^\kappa \widehat{Vf}\big(p/N^{1-\kappa}\big)}{2p^2} \widetilde{\chi}(p) \,, \tag{2.20}$$

then

$$A(p,q) = N^{-1/2} \frac{2p^2 \eta(p) \sigma(q)}{p^2 + q^2 + (p+q)^2} \,. \tag{2.21}$$

Notice that the two variables in $A(p,q)$ are treated with different infrared cutoffs. In the variable p, the cutoff $\widetilde{\chi}$ restricts the relevant momenta to the region

$$|p| \gtrsim N^{\kappa/2} \,.$$

By contrast, the factor $\sigma(q)$ already contains the Bogoliubov cutoff, so that in the variable q one keeps momenta down to the lower scale

$$|q| \gtrsim N^{\kappa/2 - \beta/4} \,.$$

For the lattice momenta, we write

$$\eta_p = \eta(p) \,, \qquad A_{p,q} = A(p,q) \,.$$

We also set

$$A_{p,q} = 0$$

whenever $p = 0$ or $q = 0$. The corresponding kernel in position is defined by

$$\check{A}(x,y) = \sum_{p,q \in \Lambda^*} A_{p,q} e^{ipx} e^{iqy} \,. \tag{2.22}$$

Introduction of cutoff Θ. We next define the cutoff that will be inserted into the cubic transformation. Its purpose is to control the number of excitations in small box with side length

$$\ell_B = N^{-\kappa/2+\beta/12}\,.$$

The choice of this scale is adapted to the decay properties of the kernels. Heuristically, the expected number of cubic excitations in a ball of radium ℓ_B is very small. Indeed, since the total number of cubic excitations is of order

$$\|A\|_2^2 \simeq N^{3\kappa-1}\,,$$

the expected number of particles in a region of volume ℓ_B^3 is approximately

$$\ell_B^3 N^{3\kappa-1} \simeq N^{-\beta/4} \ll 1$$

with $\beta = 2-3\kappa$. Thus, the cutoff Θ should not change the leading energy contribution, but it is crucial for controlling the error terms arising in the Duhamel expansion of the cubic transformation.

Now, let $\chi_{\ell_B} \in C_c^\infty(\mathbb{R}^3)$ be radial and satisfy $0 \leq \chi_{\ell_B} \leq 1$ with

$$\chi_{\ell_B}(x) = 1 \text{ if } |x| \leq \ell_B\,, \qquad \chi_{\ell_B}(x) = 0 \text{ if } |x| \geq 2\ell_B\,.$$

For $w \in \Lambda$, we define the smoothed local number of particles operator

$$\mathcal{N}_{\chi_{\ell_B}}(w) \coloneqq \int dv\, \chi_{\ell_B}(w-v) a_v^* a_v\,. \tag{2.23}$$

For comparison, we also introduce the sharp local number operator

$$\mathcal{N}_\ell(w) \coloneqq \int dv\, \mathbb{1}(|w-v| \leq \ell) a_v^* a_v\,. \tag{2.24}$$

The operator $\mathcal{N}_{\chi_{\ell_B}}(w)$ may be regarded as a regularized version of $\mathcal{N}_{\ell_B}(w)$.

Then for $n \in \mathbb{N}$, we define

$$\mathcal{L}_n \coloneqq \int dw \left(\mathcal{N}_{\chi_{\ell_B}}(w)+1\right)^n. \tag{2.25}$$

Let $\Theta \in C_c^\infty(\mathbb{R})$ be a smooth cutoff with $0 \leq \Theta \leq 1$ and

$$\Theta(x) = 1 \quad \text{for } x \leq 1\,, \qquad \Theta(x) = 0 \quad \text{for } x \geq 2\,.$$

For a parameter $\alpha > 0$, we use the cutoff

$$\Theta\left(\frac{\mathcal{L}_n}{N^\alpha}\right)$$

to suppress configurations in which the local number of particles becomes too large.

The commutation relation

$$[\mathcal{N}_{\chi_{\ell_B}}(w), a_x^*] = \chi_{\ell_B}(w-x) a_x^*$$

implies that local number of particle operator $\mathcal{N}_{\chi_{\ell_B}}(w)$ can be moved through finitely many creation a_x^* or annihilation operators a_x at the cost of constants depending only on their number. More precisely, for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$C_k^{-1} \left\| \prod_{i=1}^k a_{x_i} \left(\mathcal{N}_{\chi_{\ell_B}}(w)+1\right)\zeta \right\| \leq \left\| \left(\mathcal{N}_{\chi_{\ell_B}}(w)+1\right) \prod_{i=1}^k a_{x_i}\zeta \right\| \leq \left\| \prod_{i=1}^k a_{x_i} \left(\mathcal{N}_{\chi_{\ell_B}}(w)+1\right)\zeta \right\|$$

for any $\zeta \in \mathcal{F}(\Lambda)$.

We can now define the cubic transformation. With the kernel $\check{A}$ from (2.22), we set

$$\exp\left(\int dxdydz\, \check{A}(x-y,x-z)a_x^* a_y^* a_z^* \Theta\left(\frac{\mathcal{L}_n}{N^\alpha}\right) - \text{h.c.}\right) =: \exp(A^*\Theta - \text{h.c.})\,. \tag{2.26}$$

Notice that unless specified, the bounds obtained in the following context are uniform in the parameter $\kappa \in [1/2, 2/3)$, with a constant $C > 0$ independent of N. Also, the value of the positive constant C may vary from line to line.

2.2 Main result on $\mathcal{N}$

Theorem 5. *Let*

$$\Psi_N = W_{N_0} e^{B^*-B} e^{A^*\Theta - \Theta A}\Omega.$$

Then, we have

$$\langle \Psi_N, \mathcal{N}\Psi_N\rangle \leq N + CN^{3\kappa-1}, \tag{2.27}$$

if $\alpha > 0$ and $n \in \mathbb{N}$ large enough.

Remark 6. *From the [13, Lemma 12], we know that*

$$\langle \Psi_N, \mathcal{N}\Psi_N\rangle \geq N, \quad \left\langle \Psi_N, \mathcal{N}^2\Psi_N\right\rangle \leq CN^2. \tag{2.28}$$

Here, we want to find a more precise estimate of the expectation of the number of particles. The proof of this statement is very straightforward, just simply follows from the results of [13, Lemma 12].

Proof. With the definition of Weyl operator (2.3), we have

$$\begin{aligned}\langle \Psi_N, \mathcal{N}\Psi_N\rangle &= \left\langle e^{B^*-B}e^{A^*\Theta-\text{h.c.}}\,\Omega, (\mathcal{N}+N_0+\sqrt{N_0}(a_0^*+a_0))e^{B^*-B}e^{A^*\Theta-\text{h.c.}}\,\Omega\right\rangle \\ &= \left\langle e^{B^*-B}e^{A^*\Theta-\text{h.c.}}\,\Omega, (\mathcal{N}+N_0)e^{B^*-B}e^{A^*\Theta-\text{h.c.}}\,\Omega\right\rangle\end{aligned}$$

as the expectation of a_0, a_0^* is zero in the state $e^{B^*-B}e^{A^*\Theta-\text{h.c.}}\,\Omega$. From (2.14), we find

$$e^{B-B^*}\mathcal{N}e^{B^*-B} = \sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p + \sum_{p\in\Lambda_+^*}\gamma_p\sigma_p\left(a_p^*a_{-p}^* + a_pa_{-p}\right) + \sum_{p\in\Lambda_+^*}\sigma_p^2. \tag{2.29}$$

Notice that the expectation of $a_p^*a_{-p}^*$ vanishes in the state $e^{A^*\Theta-\text{h.c.}}\,\Omega$ and $\sum_{p\in\Lambda_+^*}\sigma_p^2$ is a constant, so we only need to calculate the expectation of excitations $\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p$ in the state $\xi = e^{A^*\Theta-\Theta A}\Omega$, i.e.,

$$\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)\left\langle \xi, a_p^*a_p\xi\right\rangle\,.$$

Notice that $\gamma_p^2 = \sigma_p^2 + 1$, so

$$\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)\left\langle \xi,\, a_p^*a_p\xi\right\rangle = \sum_{p\in\Lambda_+^*}\left(2\sigma_p^2+1\right)\left\langle \xi,\, a_p^*a_p\xi\right\rangle = 2\langle\xi,\widetilde{\mathcal{N}}\xi\rangle + \langle\xi,\,\mathcal{N}\xi\rangle \tag{2.30}$$

where $\mathcal{N} = \sum_{p\in\Lambda_+^*}a_p^*a_p$ and $\widetilde{\mathcal{N}} = \sum_{p\in\Lambda_+^*}\sigma_p^2a_p^*a_p$. By [13, Lemma 15 and Lemma 19] and choosing $t = 1$, we simply get

$$\langle\xi,\widetilde{\mathcal{N}}\xi\rangle \leq CN^{3\kappa-1}\,, \qquad \langle\xi,\mathcal{N}\xi\rangle \leq CN^{3\kappa-1}\,.$$

Then, we obtain

$$\sum_{p\in\Lambda_+^*} \left(\gamma_p^2+\sigma_p^2\right)\left\langle \xi,\, a_p^* a_p \xi\right\rangle \leq CN^{3\kappa-1}\,. \tag{2.31}$$

Therefore, we conclude that

$$\langle \Psi_N, \mathcal{N}\Psi_N\rangle \leq N + CN^{3\kappa-1}\,.$$

□

3 The Expectation of $\mathcal{N}^2$

3.1 Main result on $\mathcal{N}^2$

Now, we will calculate the expectation of $\mathcal{N}^2$ in the state Ψ_N.

Theorem 7. *Let*

$$\Psi_N = W_{N_0} e^{B^*-B} e^{A^*\Theta-\Theta A}\Omega.$$

Then, we have

$$\left\langle \Psi_N, \mathcal{N}^2\Psi_N\right\rangle \leq N^2 + CN^{3\kappa}, \tag{3.1}$$

if $\alpha>0$ and $n\in\mathbb{N}$ large enough.

The proof of this statement is much more complicated than the expectation of $\mathcal{N}$. The main idea is just based on the proof of [13, Lemma 14, Lemma 15 and Lemma 19]. To prove theorem 7, we need the following results.

Lemma 8. *Let $\alpha>0$ and choose $n\in\mathbb{N}$ large enough. Then we have*

$$\left\langle e^{t(A^*\Theta-\Theta A)}\Omega,\, \mathcal{N}^2 e^{t(A^*\Theta-\Theta A)}\Omega\right\rangle \leq CN^{6\kappa-2} \tag{3.2}$$

for all $t\in[0,1]$.

Lemma 9. *Let $\alpha>0$ and choose $n\in\mathbb{N}$ large enough. Let*

$$\widetilde{\mathcal{N}} = \sum_{p\in\Lambda_+^*} \sigma_p^2 a_p^* a_p$$

Then we have

$$\left\langle e^{t(A^*\Theta-\Theta A)}\Omega,\, \widetilde{\mathcal{N}}^2 e^{t(A^*\Theta-\Theta A)}\Omega\right\rangle \leq CN^{21\kappa/4-3/2} \tag{3.3}$$

for all $t\in[0,1]$.

With Lemma 8 and Lemma 9, now we can prove Theorem 7.

Proof of Theorem 7. Observe that

$$\begin{aligned} W_{N_0}^*\mathcal{N}^2 W_{N_0} &= \left(\sum_{p\in\Lambda_+^*}\left(W_{N_0}^* a_p^* W_{N_0}\right)\left(W_{N_0}^* a_p W_{N_0}\right)\right)^2 \\ &= \mathcal{N}^2 + 2N_0\mathcal{N} + N_0^2 + N_0(a_0^*+a_0)^2 + \sqrt{N_0}(\mathcal{N}(a_0^*+a_0)+(a_0^*+a_0)\mathcal{N}) + 2N_0^{3/2}(a_0^*+a_0). \end{aligned}$$

Since the expectation of a_0^* and a_0 will vanish in state $e^{B^*-B}e^{A^*\Theta-\Theta A}\Omega$, we have

$$\begin{aligned}\left\langle \Psi_N, \mathcal{N}^2\Psi_N\right\rangle &= \left\langle e^{B^*-B}e^{A^*\Theta-\Theta A}\Omega, W_{N_0}^*\mathcal{N}^2W_{N_0}e^{B^*-B}e^{A^*\Theta-\Theta A}\Omega\right\rangle \\ &= \left\langle e^{B^*-B}e^{A^*\Theta-\Theta A}\Omega, \left(\mathcal{N}^2+2N_0\mathcal{N}+N_0^2\right)e^{B^*-B}e^{A^*\Theta-\Theta A}\Omega\right\rangle.\end{aligned}$$

Now, from (2.29) we obtain

$$\begin{aligned}e^{B-B^*}\mathcal{N}^2e^{B^*-B} &= \left(e^{B-B^*}\mathcal{N}e^{B^*-B}\right)^2 \\ &= \left[\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p+\sum_{p\in\Lambda_+^*}\gamma_p\sigma_p\left(a_p^*a_{-p}^*+a_pa_{-p}\right)+\sum_{p\in\Lambda_+^*}\sigma_p^2\right]^2 \\ &= \left[\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p\right]^2+\left[\sum_{p\in\Lambda_+^*}\gamma_p\sigma_p\left(a_p^*a_{-p}^*+a_pa_{-p}\right)\right]^2+\left[\sum_{p\in\Lambda_+^*}\sigma_p^2\right]^2 \\ &\quad+2\sum_{p,q\in\Lambda_+^*}\sigma_q^2\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p+2\sum_{p,q\in\Lambda_+^*}\sigma_p^3\gamma_p\left(a_p^*a_{-p}^*+a_pa_{-p}\right) \\ &\quad+\sum_{p,q\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)\gamma_p\sigma_p\left[a_p^*a_p\left(a_p^*a_{-p}^*+a_pa_{-p}\right)+\left(a_p^*a_{-p}^*+a_pa_{-p}\right)a_p^*a_p\right]\end{aligned}$$

and

$$e^{B-B^*}2N_0\mathcal{N}e^{B^*-B} = 2N_0\left[\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p+\sum_{p\in\Lambda_+^*}\gamma_p\sigma_p\left(a_p^*a_{-p}^*+a_pa_{-p}\right)+\sum_{p\in\Lambda_+^*}\sigma_p^2\right].$$

Notice that in the state $\xi = e^{A^*\Theta-\Theta A}\Omega$, the particle number of each sector is the multiples of three because the generator $A^*\Theta-\Theta A$ only changes particle numbers by $3k$ from the vacuum Ω for some integer k . So, the operators

$$\begin{gathered}a_p^*a_pa_p^*a_{-p}^*,\quad a_p^*a_pa_pa_{-p},\quad a_p^*a_{-p}^*a_p^*a_p,\quad a_pa_{-p}a_p^*a_p, \\ a_p^*a_{-p}^*,\quad a_pa_{-p},\quad a_p^*a_{-p}^*a_qa_{-q}\quad\text{and}\quad a_pa_{-p}a_q^*a_{-q}\end{gathered}$$

all vanish in the state ξ. Hence, we obtain

$$\begin{aligned}&\left\langle \Psi_N, \mathcal{N}^2\Psi_N\right\rangle \\ &= \left\langle \xi, \left(\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p\right)^2\xi\right\rangle+\left\langle \xi, \sum_{p,q\in\Lambda_+^*}\gamma_p\sigma_p\gamma_q\sigma_q\left(a_p^*a_{-p}^*a_qa_{-q}+a_pa_{-p}a_q^*a_{-q}^*\right)\xi\right\rangle \\ &\quad+2\left(\sum_{q\in\Lambda_+^*}\sigma_q^2+N_0\right)\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)\left\langle \xi, a_p^*a_p\xi\right\rangle+N_0+2N_0\sum_{p\in\Lambda_+^*}\sigma_p^2+\left(\sum_{p\in\Lambda_+^*}\sigma_p^2\right)^2 \\ &= \left\langle \xi, \left(\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)a_p^*a_p\right)^2\xi\right\rangle+\sum_{p,q\in\Lambda_+^*}\gamma_p\sigma_p\gamma_q\sigma_q\left\langle \xi, \left(a_p^*a_{-p}^*a_qa_{-q}+\text{h.c.}\right)\xi\right\rangle \\ &\quad+2N\sum_{p\in\Lambda_+^*}\left(\gamma_p^2+\sigma_p^2\right)\left\langle \xi, a_p^*a_p\xi\right\rangle+N^2.\end{aligned}$$

Since the 3rd term on the r.h.s. is estimated by (2.31) in the proof of Theorem 5 , we have

$$N \sum_{p \in \Lambda_+^*} \left(\gamma_p^2 + \sigma_p^2\right) \left\langle \xi, a_p^* a_p \xi \right\rangle \leq C N^{3\kappa} \,. \tag{3.4}$$

Then, we only need to concern about the first and the second term on the r.h.s. , and we denote them by I and II respectively. For (I), by (2.30), we have

$$\left(\sum_{p \in \Lambda_+^*} \left(\gamma_p^2 + \sigma_p^2\right) a_p^* a_p \right)^2 = \left(\widetilde{\mathcal{N}} + \mathcal{N} \right)^2 = 4\widetilde{\mathcal{N}}^2 + \mathcal{N}^2 + 4\widetilde{\mathcal{N}}\mathcal{N} \leq 6\widetilde{\mathcal{N}}^2 + 3\mathcal{N}^2.$$

Then, by Lemma 8 and Lemma 9, we obtain

$$\begin{aligned} \left\langle \xi, \left(\sum_{p \in \Lambda_+^*} \left(\gamma_p^2 + \sigma_p^2\right) a_p^* a_p \right)^2 \xi \right\rangle &\leq 6 \left\langle \xi, \widetilde{\mathcal{N}}^2 \xi \right\rangle + 3 \left\langle \xi, \mathcal{N}^2 \xi \right\rangle \\ &\leq C N^{21\kappa/4 - 3/2} + C N^{6\kappa - 2} \leq C N^{3\kappa} \,. \end{aligned} \tag{3.5}$$

Now, for (II), observe that

$$\begin{aligned} &\sum_{p,q \in \Lambda_+^*} \gamma_p \sigma_p \gamma_q \sigma_q \left\langle \xi, a_p^* a_{-p}^* a_q a_{-q} \xi \right\rangle \\ &\quad = \sum_{p,q \in \Lambda_+^*} \gamma_p \gamma_q \sigma_p \sigma_q \left\langle a_q^* a_p \xi, a_{-p}^* a_{-q} \xi \right\rangle - \sum_{p \in \Lambda_+^*} \gamma_p^2 \sigma_p^2 \left\langle \xi, a_p^* a_p \xi \right\rangle \\ &\quad \leq \sum_{p,q \in \Lambda_+^*} \gamma_p \gamma_q \sigma_p \sigma_q \left\| a_q^* a_p \xi \right\| \left\| a_{-p}^* a_{-q} \xi \right\| \\ &\quad \leq \left(\sum_{p,q \in \Lambda_+^*} \gamma_p^2 \sigma_q^2 \left\langle \xi, a_p^* a_q a_q^* a_p \xi \right\rangle \right)^{1/2} \left(\sum_{p,q \in \Lambda_+^*} \gamma_q^2 \sigma_p^2 \left\langle \xi, a_{-q}^* a_{-p} a_{-p}^* a_{-q} \xi \right\rangle \right)^{1/2} \\ &\quad \leq \sum_{p,q \in \Lambda_+^*} \left(\sigma_p^2 + 1\right) \sigma_q^2 \left(\left\langle \xi, a_p^* a_q^* a_q a_p \xi \right\rangle + \left\langle \xi, a_p^* a_p \xi \right\rangle \right) \\ &\quad = \sum_{p,q \in \Lambda_+^*} \left(\sigma_p^2 \sigma_q^2 \left\langle \xi, a_p^* a_q^* a_q a_p \xi \right\rangle + \sigma_q^2 \left\langle \xi, a_p^* a_q^* a_q a_p \xi \right\rangle + \sigma_p^2 \sigma_q^2 \left\langle \xi, a_p^* a_p \xi \right\rangle + \sigma_q^2 \left\langle \xi, a_p^* a_p \xi \right\rangle \right) \\ &\quad \leq C \left\langle \xi, \left(\widetilde{\mathcal{N}}^2 + \mathcal{N} \widetilde{\mathcal{N}} \right) \xi \right\rangle + \sum_{q \in \Lambda_+^*} \sigma_q^2 \left\langle \xi, \left(\widetilde{\mathcal{N}} + \mathcal{N} \right) \xi \right\rangle \\ &\quad \leq C N^{3\kappa - \beta/2} \leq C N^{3\kappa} \end{aligned} \tag{3.6}$$

where we apply [13, Lemma 5, Lemma 15 and Lemma 19] together with Cauchy-Schwarz inequality, the fact that $\gamma_p^2 = \sigma_p^2 + 1$ and $\sigma_p = \sigma_{-p}$.

Combining (3.4), (3.5) and (3.6), we conclude that

$$\left\langle \Psi_N, \mathcal{N}^2 \Psi_N \right\rangle \leq N^2 + C N^{3\kappa} \,.$$

□

3.2 Proof of Lemma 8

To prove Lemma 8, we will need the following two key ingredients.

Lemma 10. *Let* $\xi_s = e^{s(A^*\Theta - \Theta A)}\Omega$. *Then*

$$\int_0^t ds\, \langle \xi_s,\, A^*\Theta\xi_s\rangle \leq CN^{3\kappa-1} \tag{3.7}$$

for all $t \in [0,1]$.

Lemma 11. *Let* $\xi_s = e^{s(A^*\Theta - \Theta A)}\Omega$. *Then*

$$\int_0^t ds\, \langle \xi_s,\, A^*\mathcal{N}\Theta\xi_s\rangle \leq CN^{6\kappa-2} \tag{3.8}$$

for all $t \in [0,1]$.

We postpone the proofs of Lemma 10 and Lemma 11 later on. We use these two to prove Lemma 8.

Proof of Lemma 8. We apply Gröwall's Lemma to estimate it. Let

$$f(t) = \langle \xi_t, \mathcal{N}^2\xi_t\rangle$$

where $\xi_t = e^{t(A^*\Theta - \Theta A)}\Omega$. Then, we take derivative of f with respect to the variable t and get

$$\begin{aligned} f'(t) &= \langle \xi_t,\, \left[\mathcal{N}^2, A^*\Theta - \Theta A\right]\xi_t\rangle \\ &= \langle \xi_t,\, \left[\mathcal{N}^2, A^*\Theta\right]\xi_t\rangle - \langle \xi_t,\, \left[\mathcal{N}^2, \Theta A\right]\xi_t\rangle \\ &= 2\mathrm{Re}\,\langle \xi_t,\, \left[\mathcal{N}^2, A^*\Theta\right]\xi_t\rangle \\ &= 12\mathrm{Re}\,\langle \xi_t,\, A^*\mathcal{N}\Theta\xi_t\rangle + 18\mathrm{Re}\,\langle \xi_t,\, A^*\Theta\xi_t\rangle \end{aligned} \tag{3.9}$$

Here, to get the last equality of (3.9) we use the commutators $[\mathcal{N}, \Theta] = 0$ and

$$\begin{aligned} [\mathcal{N}, A^*] &= \sum_{k\in\Lambda^*}\sum_{p,q\in\Lambda_+^*} A_{p,q}\left[a_k^*a_k, a_{p+q}^*a_{-p}^*a_{-q}^*\right] = \sum_{k,p,q\in\Lambda_+^*} A_{p,q}a_k^*\left[a_k, a_{p+q}^*a_{-p}^*a_{-q}^*\right] \\ &= \sum_{k,p,q\in\Lambda_+^*}\delta_{k,p+q}A_{p,q}a_k^*a_{-p}^*a_{-q}^* + \sum_{k,p,q\in\Lambda_+^*}\delta_{k,-p}A_{p,q}a_k^*a_{p+q}^*a_{-q}^* + \sum_{k,p,q\in\Lambda_+^*}\delta_{k,-q}A_{p,q}a_k^*a_{p+q}^*a_{-p}^* \\ &= 3A^* \end{aligned} \tag{3.10}$$

which gives

$$\begin{aligned} \left[\mathcal{N}^2, A^*\Theta\right] &= \left[\mathcal{N}^2, A^*\right]\Theta = \left(\mathcal{N}\left[\mathcal{N}, A^*\right] + \left[\mathcal{N}, A^*\right]\mathcal{N}\right)\Theta \\ &= \left(\mathcal{N}3A^* + 3A^*\mathcal{N}\right)\Theta = \left(3\left(A^*\mathcal{N} + \left[\mathcal{N}, A^*\right]\right) + 3A^*\mathcal{N}\right)\Theta = 6A^*\mathcal{N}\Theta + 9A^*\Theta\,. \end{aligned}$$

We write $f(t)$ in integral form and get

$$f(t) = \int_0^t ds f'(s) + f(0) = 12\mathrm{Re}\int_0^t ds\,\langle \xi_s,\, A^*\mathcal{N}\Theta\xi_s\rangle + 18\mathrm{Re}\int_0^t ds\,\langle \xi_s,\, A^*\Theta\xi_s\rangle\,. \tag{3.11}$$

The results follows directly from Lemma 10 and Lemma 11 and finally applying Gröwall's lemma. □

Proof of Lemma 10. Observe that

$$\begin{aligned}\int_0^t ds\,\langle \xi_s,\, A^*\Theta\xi_s\rangle &= \int_0^t ds\,\langle \xi_s,\, A^* + A^*(\Theta-1)\xi_s\rangle \\ &\leq \langle \Omega, [A, A^*]\Omega\rangle + C\ell_B^3 N^{(3\kappa-1)/2+\epsilon}\left(\sum_{u,v\in\Lambda_B}\left\|\check{A}_{u,v}\right\|_2\right)\int_0^t ds\,\langle \xi_s, \mathcal{N}\xi_s\rangle + CN^{-\nu}\end{aligned} \tag{3.12}$$

for all $\epsilon > 0$ and for $\nu > 0$, where the last inequality is obtained by applying [13, Lemma 11 and Lemma 14] with $K = A$.

Notice that

$$\begin{aligned}|\langle\Omega,\, [A, A^*]\Omega\rangle| \leq C\left\|A^*\Omega\right\|^2 &= C\left\|\sum_{p,q\in\Lambda^*} A_{p,q}\, a^*_{p+q}a^*_{-p}a^*_{-q}\Omega\right\|^2 \\ &\leq C\sum_{p,q,\tilde{p},\tilde{q}\in\Lambda^*}\left|\overline{A_{p,q}}\right|\left|A_{\tilde{p},\tilde{q}}\right|\left\langle\Omega,\, a_{-q}a_{-p}a_{p+q}a^*_{\tilde{p}+\tilde{q}}a^*_{-\tilde{p}}a^*_{-\tilde{q}}\Omega\right\rangle \\ &\leq C\sum_{p,q,\tilde{p},\tilde{q}\in\Lambda^*}\left|\overline{A_{p,q}}\right|\left|A_{\tilde{p},\tilde{q}}\right|\sum_{\pi\in S_3}\delta_{p+q,r_{\pi 1}}\delta_{-p,r_{\pi 2}}\delta_{-q,r_{\pi 3}} \\ &\leq C\sum_{p,q\in\Lambda^*}|A_{p,q}|^2 \leq CN^{3\kappa-1}\end{aligned} \tag{3.13}$$

where $(r_1, r_2, r_3) = (\tilde{p}+\tilde{q}, -\tilde{p}, -\tilde{q})$ and we use [13, Lemma 7] to get the last inequality. Here, for the third line of (3.13), $\left\langle\Omega,\, a_{-q}a_{-p}a_{p+q}a^*_{\tilde{p}+\tilde{q}}a^*_{-\tilde{p}}a^*_{-\tilde{q}}\Omega\right\rangle \neq 0$ only if the annihilation momenta $\{p+q, -p, -q\}$ are exactly the creation momenta $\{\tilde{p}+\tilde{q}, -\tilde{p}, -\tilde{q}\}$ because only then can the created three-particle state be annihilated all the way back to the vacuum.

By [13, Lemma 8 and Lemma 15], we know $\sum_{u,v\in\Lambda_B}\left\|\check{A}_{u,v}\right\|_2 \leq CN^{(3\kappa-1)/2+\epsilon}$ for all $\epsilon > 0$ and $\langle\xi_t, \mathcal{N}\xi_t\rangle \leq CN^{3\kappa-1}$ for all $t \in [0;1]$. Combining them with (3.12) and (3.13), we conclude that

$$\int_0^t ds\,\langle \xi_s,\, A^*\Theta\xi_s\rangle \leq CN^{3\kappa-1} \tag{3.14}$$

for all $t \in [0,1]$. □

Proof of Lemma 11. We prove $t = 1$ and the extension to $t \in [0,1]$ easily follows. Observe that

$$\langle\xi,\, A^*\mathcal{N}\Theta\xi\rangle = \langle\xi,\, A^*\mathcal{N} + A^*\mathcal{N}(\Theta-1)\xi\rangle \leq C\,\langle\xi,\, A^*\mathcal{N}\xi\rangle + CN^{-\nu} \tag{3.15}$$

for $\alpha > 0$ and then $n \in \mathbb{N}$ are chosen large enough in the cutoff $\Theta(\mathcal{L}_n/N^\alpha)$. Here, the last inequality of (3.15) is estimated by using the bound $\left\|A^*(\mathcal{N}+1)^{-3/2}\right\| \leq C\left\|A\right\|_2$ and [13, Lemma 7 and Lemma 11]. In fact,

$$\begin{aligned}\langle\xi,\, A^*\mathcal{N}(\Theta-1)\xi\rangle &= \left\langle A(\mathcal{N}+1)^{-3/2}(\mathcal{N}+1)^{3/2}\xi,\, \mathcal{N}(\Theta-1)\xi\right\rangle \\ &\leq \|A(\mathcal{N}+1)^{-3/2}\|\|(\mathcal{N}+1)^{3/2}\xi\|\,\|\mathcal{N}(\Theta-1)\xi\| \\ &\leq \|A(\mathcal{N}+1)^{-3/2}\|\|(\mathcal{N}+1)^{3/2}\xi\|\|(1-\Theta)\xi\|^{1/2}\|\mathcal{N}^2\xi\|^{1/2}\end{aligned}$$

by Cauchy-Schwarz inequality and the commutator $[\mathcal{N}, \Theta] = 0$. Moreover, [13, Lemma 7 and Lemma 11] tells us that $\|A\|_2^2 \leq CN^{3\kappa-1}$ $\|(1-\Theta)\xi\| \leq CN^{-\alpha/2}$ and $\langle\xi,\, \mathcal{N}^m\xi\rangle \leq CN^{m-m\beta/2}$ for any $m \in \mathbb{N}$ provided $\alpha > 0$ and then $n \in \mathbb{N}$ are chosen large enough. So, we get for any $\nu > 0$

$$\langle\xi,\, A^*\mathcal{N}(\Theta-1)\xi\rangle \leq CN^{-\nu}\,. \tag{3.16}$$

Now, we write $\langle \xi,\, A^*\mathcal{N}\xi\rangle$ in integral form and obtain

$$\begin{aligned}
\langle \xi,\, A^*\mathcal{N}\xi\rangle &= \int_0^1 ds\, \langle \xi_s,\, [A^*\mathcal{N}, A^*\Theta] - [A^*\mathcal{N}, \Theta A]\, \xi_s\rangle \\
&= \int_0^1 ds\, \langle \xi_s,\, \Theta\, [A^*\mathcal{N}, A^*]\, \Theta\xi_s\rangle \qquad (3.17)\\
&\quad + \int_0^1 ds\, \langle \xi_s,\, ((1-\Theta)\, [A^*\mathcal{N}, A^*]\, \Theta + A^*A^*\mathcal{N}(\Theta-1) - A^*(\Theta-1)A^*\mathcal{N})\, \xi_s\rangle \\
&\quad + \int_0^1 ds\, \langle \xi_s,\, \Theta\, [A, A^*\mathcal{N}]\, \Theta\xi_s\rangle \qquad (3.18)\\
&\quad + \int_0^1 ds\, \langle \xi_s,\, (\Theta\, [A, A^*\mathcal{N}]\, (1-\Theta) + (\Theta-1)A^*\mathcal{N}A - A^*\mathcal{N}(\Theta-1)A^*)\, \xi_s\rangle\ .
\end{aligned}$$

Notice that all terms with $(1-\Theta)$ can be estimated similar to (3.16) by $CN^{-\nu}$ for any $\nu > 0$, so we only need to concern about (3.17) and (3.18).

For (3.17), with $[\mathcal{N}, A^*] = 3A^*$ and the definition of A^* in position space (2.26), we follow the proof of [13, Lemma 14] and observe that

$$\begin{aligned}
&\left|\int_0^1 ds\, \langle \xi_s,\, \Theta\, [A^*\mathcal{N}, A^*]\, \Theta\xi_s\rangle\right| = \left|\int_0^1 ds\, \langle \xi_s,\, \Theta A^*\, [\mathcal{N}, A^*]\, \Theta\xi_s\rangle\right| \\
&\leq 3\int_0^1 ds\, |\langle \xi_s,\, \Theta A^*A^*\Theta\xi_s\rangle| \\
&\leq \int_0^1 ds \int dx_1dx_2dx_3dy_1dy_2dy_3\, \left|\check{A}(x_1-x_2, x_1-x_3)\right| \left|\check{A}(y_1-y_2, y_1-y_3)\right| \\
&\qquad\qquad \times \|a_{x_1}a_{x_2}a_{x_3}a_{y_1}a_{y_2}a_{y_3}\Theta\xi_s\| \\
&\leq \sum_{u,v,\tilde{u},\tilde{v}\in\Lambda_B} \int_0^1 ds \Bigg(\int dx_1dx_2dx_3dy_1dy_2dy_3 \mathbb{1}\big(\, |x_1-x_2-u|\,, |x_1-x_3-v|\,, \cdots \\
&\qquad\qquad |y_1-y_2-\tilde{u}|\,, |y_1-y_3-\tilde{v}| \leq \ell_B\big)\, \|a_{x_1}a_{x_2}a_{x_3}a_{y_1}a_{y_2}a_{y_3}\Theta\xi_s\|^2 \Bigg)^{1/2} \\
&\quad \times \left(\int dx_1dx_2dx_3dy_1dy_2dy_3\, \left|\check{A}_{u,v}(x_1-x_2, x_1-x_3)\right|^2 \left|\check{A}_{\tilde{u},\tilde{v}}(y_1-y_2, y_1-y_3)\right|^2 \right)^{1/2}.
\end{aligned} \qquad (3.19)$$

Here, the last inequality of (3.19) is obtained by localizing the kernel $\check{A}$ to the box $B_u \times B_v$ of side length $\ell_B = N^{-\kappa/2+\beta/12}$ and applying Cauchy-Schwarz inequality.

Now, [13, Lemma 11] implies that

$$\int dx_2 \mathbb{1}\left(|x_1-x_2-u| \leq \ell_B\right) \|a_{x_2}\Theta\xi_s\|^2 = \left\|\mathcal{N}_{\ell_B}^{1/2}(x_1-u)\Theta\xi_s\right\|^2 \leq CN^{\epsilon}\, \|\Theta\xi_s\|^2$$

for all $n \in \mathbb{N}$ is large enough. By changing of variables and using [13, Lemma 8], we conclude that

$$\begin{aligned}
\left|\int_0^1 ds\, \langle \xi_s,\, \Theta\, [A^*\mathcal{N}, A^*]\, \Theta\xi_s\rangle\right| &\leq CN^{2\epsilon}\, \langle \xi_s,\, \mathcal{N}^2\xi_s\rangle^{1/2} \left(\sum_{u,v\in\Lambda_B} \left\|\check{A}_{u,v}\right\|_2\right)^2 \\
&\leq CN^{3\kappa-1+\epsilon}\left(\delta\, \langle \xi_s,\, \mathcal{N}^2\xi_s\rangle + \frac{1}{4\delta}\right)
\end{aligned} \qquad (3.20)$$

for all $\epsilon > 0$ and for all $\delta > 0$.

For (3.18), observe that

$$\begin{aligned}[A, A^*\mathcal{N}] &= \int dx_1dx_2dx_3dy_1dy_2dy_3dz\check{A}(y_1-y_2, y_1-y_3)\check{A}(x_1-x_2, x_1-x_3) \\ &\qquad\qquad \times \left[a_{y_1}a_{y_2}a_{y_3}, a^*_{x_1}a^*_{x_2}a^*_{x_3}a^*_z a_z\right] \\ &= \int dx_1dx_2dx_3dy_1dy_2dy_3dz\check{A}(y_1-y_2, y_1-y_3)\check{A}(x_1-x_2, x_1-x_3) \\ &\qquad \times \left(\left[a_{y_1}a_{y_2}a_{y_3}, a^*_{x_1}a^*_{x_2}a^*_{x_3}\right] a^*_z a_z + a^*_{x_1}a^*_{x_2}a^*_{x_3}\left[a_{y_1}a_{y_2}a_{y_3}, a^*_z\right] a_z\right)\end{aligned} \tag{3.21}$$

Here, the second term in the last line of (3.21) can be directly calculated by canonical commutation relations (1.9) and then change of variables ($x_4 = y_1, x_5 = y_2, x_6 = y_3$), i.e.,

$$\begin{aligned}&\int dx_1dx_2dx_3dy_1dy_2dy_3dz\check{A}(y_1-y_2, y_1-y_3)\check{A}(x_1-x_2, x_1-x_3)a^*_{x_1}a^*_{x_2}a^*_{x_3}\left[a_{y_1}a_{y_2}a_{y_3}, a^*_z\right] a_z \\ &= \int dx_1dx_2dx_3dy_1dy_2dy_3dz\check{A}(y_1-y_2, y_1-y_3)\check{A}(x_1-x_2, x_1-x_3) \\ &\qquad\qquad \times a^*_{x_1}a^*_{x_2}a^*_{x_3}\sum_{j=1}^{3}\delta(z-y_j)\Big(\prod_{\ell\neq j}a_{y_\ell}\Big)a_z \\ &= 3\int dx_1\cdots dx_6\check{A}(x_4-x_5, x_4-x_6)\check{A}(x_1-x_2, x_1-x_3)a^*_{x_1}a^*_{x_2}a^*_{x_3}a_{x_4}a_{x_5}a_{x_6} =: 3\,T_{6,2}\,.\end{aligned}$$

For the first term in the last line of (3.21), we first use [13, Equation (120)] and then by the symmetry of kernels defined in [13, Equation (75)] such that

$$\begin{aligned}\check{A}^{(3)}(x,y) &= \check{A}(x,y) + \check{A}(y,x) + \check{A}(-y, x-y) \\ \check{A}^{(6)}(x,y) &= \check{A}^{(3)}(x,y) + \check{A}^{(3)}(-x, -x+y)\end{aligned} \tag{3.22}$$

with changing of variables, we get

$$\begin{aligned}&\int dx_1dx_2dx_3dy_1dy_2dy_3dz\check{A}(y_1-y_2, y_1-y_3)\check{A}(x_1-x_2, x_1-x_3)\left[a_{y_1}a_{y_2}a_{y_3}, a^*_{x_1}a^*_{x_2}a^*_{x_3}\right] a^*_z a_z \\ &= \int dx_1dx_2dx_3dy_1dy_2dy_3dz\check{A}(y_1-y_2, y_1-y_3)\check{A}(x_1-x_2, x_1-x_3) \\ &\qquad \times \Bigg(\sum_{\pi\in S_3}\delta(y_1-x_{\pi(1)})\delta(y_2-x_{\pi(2)})\delta(y_3-x_{\pi(3)})a^*_z a_z \\ &\qquad\quad + \sum_{i_1<i_2}^{3}\sum_{\pi\in S_3}\delta(y_{i_1}-x_{\pi(i_1)})\delta(y_{i_2}-x_{\pi(i_2)})\Big(\prod_{i_3\neq i_1,i_2}a^*_{x_{\pi(i_3)}}a_{y_{i_3}}\Big)a^*_z a_z \\ &\qquad\quad + \sum_{i,j=1}^{3}\delta(y_i-x_j)\Big(\prod_{\ell\neq j}a^*_{x_\ell}\Big)\Big(\prod_{m\neq i}a_{y_m}\Big)a^*_z a_z\Bigg) \\ &= \int dx_1dx_2dx_3dz\check{A}(x_1-x_2, x_1-x_3)^2 a^*_z a_z \\ &\qquad + \int dx_1\cdots dx_4dz\check{A}^{(6)}(x_1-x_2, x_1-x_4)\check{A}^{(3)}(x_1-x_2, x_1-x_3)a^*_{x_3}a_{x_4}a^*_z a_z \\ &\qquad + \int dx_1\cdots dx_5dz\check{A}^{(3)}(x_1-x_4, x_1-x_5)\check{A}^{(3)}(x_1-x_2, x_1-x_3)a^*_{x_2}a^*_{x_3}a_{x_4}a_{x_5}a^*_z a_z \\ &=: T_{2,1} + T_{2,2} + T_{4,1} + T_{4,2} + T_{6,1}\,.\end{aligned}$$

Here, $T_{2,1}, T_{2,2}, T_{4,1}, T_{4,2}$ and $T_{6,1}$ are given by

$$
\begin{aligned}
T_{2,1} &= \int dx_1 \cdots dx_4 \check{A}(x_1 - x_2, x_1 - x_3)^2 a^*_{x_4} a_{x_4} \\
T_{2,2} &= \int dx_1 \cdots dx_4 \check{A}^{(6)}(x_1 - x_2, x_1 - x_4) \check{A}^{(3)}(x_1 - x_2, x_1 - x_3) a^*_{x_3} a_{x_4} \\
T_{4,1} &= \int dx_1 \cdots dx_5 \check{A}^{(6)}(x_1 - x_2, x_1 - x_4) \check{A}^{(3)}(x_1 - x_2, x_1 - x_3) a^*_{x_3} a^*_{x_5} a_{x_4} a_{x_5} \\
T_{4,2} &= \int dx_1 \cdots dx_5 \check{A}^{(3)}(x_1 - x_4, x_1 - x_5) \check{A}^{(3)}(x_1 - x_2, x_1 - x_3) a^*_{x_2} a^*_{x_3} a_{x_4} a_{x_5} \\
T_{6,1} &= \int dx_1 \cdots dx_6 \check{A}^{(3)}(x_1 - x_4, x_1 - x_5) \check{A}^{(3)}(x_1 - x_2, x_1 - x_3) a^*_{x_2} a^*_{x_3} a^*_{x_6} a_{x_4} a_{x_5} a_{x_6} \,,
\end{aligned}
$$

where we apply canonical commutation relations (1.9) with changing of variables to get them. For $T_{2,1}$ and $T_{2,2}$, we switch to momentum space by [13, Equation (64), (78)] and(1.7). We calculate $T_{2,2}$ as an example. Observe that

$$
\begin{aligned}
T_{2,2} &= \int dx_1 \cdots dx_4 \sum_{p,q \in \Lambda^*} A^{(6)}_{p,q} e^{ip\cdot(x_1 - x_2)} e^{iq\cdot(x_1 - x_4)} \sum_{\tilde{p},\tilde{q} \in \Lambda^*} A^{(3)}_{\tilde{p},\tilde{q}} e^{i\tilde{p}\cdot(x_1 - x_2)} e^{i\tilde{q}\cdot(x_1 - x_3)} \\
&\qquad \times \sum_{r \in \Lambda^*} a^*_r e^{-ir\cdot x_3} \sum_{s \in \Lambda^*} a_s e^{is\cdot x_4} \\
&= \sum_{p,q,\tilde{p},\tilde{q},r,s \in \Lambda^*} A^{(6)}_{p,q} A^{(3)}_{\tilde{p},\tilde{q}} a^*_r a_s \int dx_1 \cdots dx_4 \, e^{i(p+q+\tilde{p}+\tilde{q})\cdot x_1} e^{i(-p-\tilde{p})\cdot x_2} e^{i(-\tilde{q}-r)\cdot x_3} e^{i(-q+s)\cdot x_4} \\
&= \sum_{p,q,\tilde{p},\tilde{q},r,s \in \Lambda^*} A^{(6)}_{p,q} A^{(3)}_{\tilde{p},\tilde{q}} a^*_r a_s \delta_{-p-q,\tilde{p}+\tilde{q}} \delta_{-p,\tilde{p}} \delta_{\tilde{q},-r} \delta_{q,s} \\
&= \sum_{p,q \in \Lambda^*} A^{(6)}_{p,q} A^{(3)}_{-p,-q} a^*_q a_q = \sum_{p,q \in \Lambda^*} A^{(6)}_{p,q} A^{(3)}_{p,q} a^*_q a_q \,.
\end{aligned}
\tag{3.23}
$$

Similarly, we have

$$
T_{2,1} = \sum_{p,q \in \Lambda^*} A^2_{p,q} \mathcal{N} \,. \tag{3.24}
$$

By [13, Lemma 8, Lemma 9 and Lemma 15] and Cauchy-Schwarz inequality, we get

$$
\langle \xi_s, \, \Theta T_{2,1} \Theta \xi_s \rangle = \left\langle \xi_s, \sum_{p,q \in \Lambda^*} A^2_{p,q} \mathcal{N} \xi_s \right\rangle \leq C N^{3\kappa - 1} \langle \xi_s, \, \mathcal{N} \xi_s \rangle \leq C N^{6\kappa - 2}, \tag{3.25}
$$

and

$$
\begin{aligned}
\langle \xi_s, \, \Theta T_{2,2} \Theta \xi_s \rangle &= \left\langle \xi_s, \sum_{p,q \in \Lambda^*} A^{(6)}_{p,q} A^{(3)}_{p,q} a^*_q a_q \xi_s \right\rangle \\
&\leq C N^{3\kappa/2 - 1} \left\langle \xi_s, \sum_{q \in \Lambda^*} (1 + \sigma^2_q) a^*_q a_q \xi_s \right\rangle \\
&= C N^{3\kappa/2 - 1} \left\langle \xi_s, \, (\mathcal{N} + \widetilde{\mathcal{N}}) \xi_s \right\rangle \leq C N^{9\kappa/2 - 2} \leq C N^{6\kappa - 2} \,.
\end{aligned}
\tag{3.26}
$$

For $T_{4,1}, T_{4,2}, T_{6,1}$ and $T_{6,2}$, we follow the proof of [13, Lemma 14 and Lemma 19] and we obtain

$$
\begin{aligned}
&|\langle \xi_s, \Theta T_{4,1} \Theta \xi_s \rangle| \\
&\quad \leq \int dx_1 \cdots dx_5 |\check{A}^{(6)}(x_1 - x_2, x_1 - x_4)||\check{A}^{(3)}(x_1 - x_2, x_1 - x_3)| \, \|a_{x_3} a_{x_5} \Theta \xi_s\| \, \|a_{x_4} a_{x_5} \Theta \xi_s\| \\
&\quad \leq \sum_{u,v,\tilde{u},\tilde{v} \in \Lambda_B} \left(\int dx_1 \cdots dx_5 |\check{A}^{(6)}_{u,v}(x_1 - x_2, x_1 - x_4)|^2 \right. \\
&\qquad\qquad \left. \times \mathbb{1}\left(|x_1 - x_2 - \tilde{u}|, |x_1 - x_3 - \tilde{v}| \leq \ell_B\right) \|a_{x_3} a_{x_5} \Theta \xi_s\|^2 \right)^{1/2} \\
&\qquad \times \left(\int dx_1 \cdots dx_5 |\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_1 - x_2, x_1 - x_3)|^2 \right. \\
&\qquad\qquad \left. \times \mathbb{1}\left(|x_1 - x_2 - u|, |x_1 - x_4 - v| \leq \ell_B\right) \|a_{x_4} a_{x_5} \Theta \xi_s\|^2 \right)^{1/2},
\end{aligned}
$$

where we localize the kernels $\check{A}^{(3)}$ and $\check{A}^{(6)}$ to $B_u \times B_v$ and use Cauchy-Schwarz inequality. Then applying [13, Lemma 11] and then by [13, Lemma 8 and Lemma 15], we find, for $j = 3, 6$

$$
|\langle \xi_s, \Theta T_{4,1} \Theta \xi_s \rangle| \leq C N^{\epsilon} \left(\sum_{u,v \in \Lambda_B} \|\check{A}^{(j)}_{u,v}\|_2 \right)^2 \langle \xi_s, \mathcal{N} \xi_s \rangle \leq C N^{6\kappa - 2 + \epsilon} \tag{3.27}
$$

for any $\epsilon > 0$. Similarly, we obtain

$$
\begin{aligned}
&|\langle \xi_s, \Theta T_{4,2} \Theta \xi_s \rangle| \\
&\quad \leq \int dx_1 \cdots dx_5 |\check{A}^{(3)}(x_1 - x_4, x_1 - x_5)||\check{A}^{(3)}(x_1 - x_2, x_1 - x_3)| \, \|a_{x_2} a_{x_3} \Theta \xi_s\| \, \|a_{x_4} a_{x_5} \Theta \xi_s\| \\
&\quad \leq \sum_{u,v,\tilde{u},\tilde{v} \in \Lambda_B} \left(\int dx_1 \cdots dx_5 |\check{A}^{(3)}_{u,v}(x_1 - x_4, x_1 - x_5)|^2 \right. \\
&\qquad\qquad \left. \times \mathbb{1}\left(|x_1 - x_2 - \tilde{u}|, |x_1 - x_3 - \tilde{v}| \leq \ell_B\right) \|a_{x_2} a_{x_3} \Theta \xi_s\|^2 \right)^{1/2} \\
&\qquad \times \left(\int dx_1 \cdots dx_5 |\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_1 - x_2, x_1 - x_3)|^2 \right. \\
&\qquad\qquad \left. \times \mathbb{1}\left(|x_1 - x_4 - u|, |x_1 - x_5 - v| \leq \ell_B\right) \|a_{x_4} a_{x_5} \Theta \xi_s\|^2 \right)^{1/2} \\
&\quad \leq \left(C N^{\epsilon} \sum_{u,v \in \Lambda_B} \|\check{A}^{(j)}_{u,v}\|_2 \right)^2 \leq C N^{3\kappa - 1 + \epsilon} \leq C N^{6\kappa - 2 + \epsilon}
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
&|\langle \xi_s,\, \Theta T_{6,1}\Theta\xi_s\rangle| \\
&\quad\le \int dx_1\cdots dx_6 |\check{A}^{(3)}(x_1-x_4,x_1-x_5)||\check{A}^{(3)}(x_1-x_2,x_1-x_3)|\, \|a_{x_2}a_{x_3}a_{x_6}\Theta\xi_s\|\, \|a_{x_4}a_{x_5}a_{x_6}\Theta\xi_s\| \\
&\quad\le \sum_{u,v,\tilde{u},\tilde{v}\in\Lambda_B} \Bigg(\int dx_1\cdots dx_5 |\check{A}^{(3)}_{u,v}(x_1-x_4,x_1-x_5)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_1-x_2-\tilde{u}|\,,|x_1-x_3-\tilde{v}|\le \ell_B\right) \|a_{x_2}a_{x_3}a_{x_6}\Theta\xi_s\|^2 \Bigg)^{1/2} \\
&\qquad \times \Bigg(\int dx_1\cdots dx_5 |\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_1-x_2,x_1-x_3)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_1-x_4-u|\,,|x_1-x_5-v|\le \ell_B\right) \|a_{x_4}a_{x_5}a_{x_6}\Theta\xi_s\|^2 \Bigg)^{1/2} \\
&\quad\le \left(C N^{\epsilon} \sum_{u,v\in\Lambda_B} \|\check{A}^{(3)}_{u,v}\|_2 \langle \xi,\, \mathcal{N}\xi\rangle^{1/2} \right)^2 \le C N^{6\kappa-2+\epsilon}
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
&|\langle \xi_s,\, \Theta T_{6,2}\Theta\xi_s\rangle| \\
&\quad\le \int dx_1\cdots dx_6 |\check{A}(x_4-x_5,x_4-x_6)||\check{A}(x_1-x_2,x_1-x_3)|\, \|a_{x_1}a_{x_2}a_{x_3}\Theta\xi_s\|\, \|a_{x_4}a_{x_5}a_{x_6}\Theta\xi_s\| \\
&\quad\le \sum_{u,v,\tilde{u},\tilde{v}\in\Lambda_B} \Bigg(\int dx_1\cdots dx_5 |\check{A}_{u,v}(x_4-x_5,x_4-x_6)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_1-x_2-\tilde{u}|\,,|x_1-x_3-\tilde{v}|\le \ell_B\right) \|a_{x_1}a_{x_2}a_{x_3}\Theta\xi_s\|^2 \Bigg)^{1/2} \\
&\qquad \times \Bigg(\int dx_1\cdots dx_5 |\check{A}_{\tilde{u},\tilde{v}}(x_1-x_2,x_1-x_3)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_4-x_5-u|\,,|x_4-x_6-v|\le \ell_B\right) \|a_{x_4}a_{x_5}a_{x_6}\Theta\xi_s\|^2 \Bigg)^{1/2} \\
&\quad\le \left(C N^{\epsilon} \sum_{u,v\in\Lambda_B} \|\check{A}_{u,v}\|_2 \langle \xi,\, \mathcal{N}\xi\rangle^{1/2} \right)^2 \le C N^{6\kappa-2+\epsilon}
\end{aligned}
\tag{3.30}
$$

for any $\epsilon>0$. By (3.25), (3.26) (3.27), (3.28), (3.29) and (3.30), we get

$$
\int_0^1 ds\, \langle \xi_s,\, \Theta\left[A, A^*\mathcal{N}\right]\Theta\xi_s\rangle \le C N^{6\kappa-2+\epsilon} \tag{3.31}
$$

for any $\epsilon>0$.

Combining (3.20) and (3.32), we obtain

$$
\int_0^t ds\, \langle \xi_s,\, A^*\mathcal{N}\Theta\xi_s\rangle \le C N^{6\kappa-2+\epsilon} + C N^{3\kappa-1+\epsilon}\left(\delta \langle \xi_s,\, \mathcal{N}^2\xi_s\rangle + \frac{1}{4\delta}\right) \tag{3.32}
$$

for any $\epsilon>0$ and $\delta>0$. Taking an appropriate δ, say $\delta = N^{-3\kappa+1-\epsilon}$, choosing $\epsilon>0$ sufficiently small, and apply Gröwall's lemma, we conclude the proof of Lemma 11. □

3.3 Proof of Lemma 9

This proof follows from the proof of [13, Lemma 19]. Let $g := \check{\sigma} * \check{\sigma}$. By the definition of $\mathcal{N}_k = \int dzdz'\, k(z-z')a_z^* a_{z'}$ in [13] and then switch to momentum space using (1.7), we have

$$\begin{aligned}\mathcal{N}_g &= \int dzdz'\, g(z-z')a_z^* a_{z'} \\ &= \int dzdz' \sum_{r\in\Lambda^*} \hat{g}_r e^{ir\cdot(z-z')} \left(\sum_{p\in\Lambda^*} a_p^* e^{-ip\cdot z}\right)\left(\sum_{q\in\Lambda^*} a_q e^{iq\cdot z'}\right) \\ &= \sum_{p,q,r\in\Lambda^*} \hat{g}_r a_p^* a_q \int dzdz'\, e^{i(r-p)\cdot z} e^{i(-r+q)\cdot z'} \\ &= \sum_{p,q,r\in\Lambda^*} \delta_{r,p}\delta_{r,q}\hat{g}_r a_p^* a_p = \sum_{p\in\Lambda^*} \sigma_p^2 a_p^* a_p = \widetilde{\mathcal{N}}\,.\end{aligned} \tag{3.33}$$

Moreover, by [13, Lemma 6] and Cauchy-Schwarz inequality, we have the following bound

$$\|\check{\sigma}\|_1 = \int |\check{\sigma}(x)|\, dx \leq \sum_{u\in\Lambda_B} \int |\check{\sigma}_u(x)|\, dx \leq C\ell_B^{3/2} \sum_{u\in\Lambda_B} \|\check{\sigma}_u\|_2 \leq CN^{\beta/8}\,.$$

Then we also obtain

$$\|\hat{g}\|_\infty \leq \|g\|_1 = \int |\check{\sigma} * \check{\sigma}(x)|\, dx \leq \|\check{\sigma}\|_1^2 \leq CN^{\beta/4} \tag{3.34}$$

and

$$\mathcal{N}_g = \int dzdz'\, g(z-z')a_z^* a_{z'} \leq \|\hat{g}\|_\infty \sum_{p\in\Lambda^*} a_p^* a_p \leq CN^{\beta/4}\mathcal{N} \tag{3.35}$$

by proceeding in a similar way as (3.33) and then using (3.34).

Let $\xi_t = e^{t(A^*\Theta - \Theta A)}\Omega$ for $t\in[0,1]$ and we write $\left\langle \xi_t, \widetilde{\mathcal{N}}^2\xi_t\right\rangle$ in integral form

$$\begin{aligned}\left\langle \xi_t, \widetilde{\mathcal{N}}^2\xi_t\right\rangle &= \left\langle \xi_t, \mathcal{N}_g^2\xi_t\right\rangle \\ &= 2\mathrm{Re}\int_0^t ds\, \left\langle \xi_s, \left(\left[\mathcal{N}_g^2, A^*\right] + \left[\mathcal{N}_g^2, A^*\right](\Theta-1) + A^*\left[\mathcal{N}_g^2, (\Theta-1)\right]\right)\xi_s\right\rangle\,.\end{aligned} \tag{3.36}$$

Notice that, with [13, Lemma 11] and the inequalities $\|A(\mathcal{N}+1)^{-3/2}\| \leq CN^{(3\kappa-1)/2}$ and (3.35), all terms including $(\Theta-1)$ can be bounded by $N^{-\nu}$ for any $\nu>0$ if we choose $\alpha>0$ and then $n\in\mathbb{N}$ sufficiently large . This is also true for the following proof, so we just absorb all of these terms that will arise later on into the error term $N^{-\nu}$. From [13, Equation (127)], we know

$$[\mathcal{N}_g, A^*] = \int dzdz'dx_1dx_2\, g(z-z')\check{A}^{(3)}(x_1-x_2, x_1-z')a_z^* a_{x_1}^* a_{x_2}^*. \tag{3.37}$$

$$[A, \mathcal{N}_g] = \int dx_1dx_2dzdz'\, \check{A}^{(3)}(x_1-x_2, x_1-z)g(z-z')a_{x_1}^* a_{x_2}^* a_{z'}^*. \tag{3.38}$$

With (3.36), we expand $\langle \xi_s,\, [\mathcal{N}_g^2, A^*]\, \xi_s \rangle$ once more and obtain

$$\begin{aligned}\langle \xi_t, \mathcal{N}_g^2 \xi_t \rangle &= \int_0^t ds_1 \int_0^{s_1} ds_2 \, \langle \xi_{s_2},\, [[\mathcal{N}_g^2, A^*\Theta - \Theta A]\,, A^*\Theta - \Theta A]\, \xi_{s_2} \rangle \\ &\leq 2\mathrm{Re} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \langle \xi_{s_2},\, [[\mathcal{N}_g^2, A^*]\,, -A]\, \xi_{s_2} \rangle \\ &\quad + 2\mathrm{Re} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \langle \xi_{s_2},\, [A, [A, \mathcal{N}_g^2]]\, \xi_{s_2} \rangle + N^{-\nu}\end{aligned} \tag{3.39}$$

for any $\nu > 0$. Now, observe that

$$\begin{aligned}[[\mathcal{N}_g^2, A^*]\,, -A] &= [A, [\mathcal{N}_g^2, A^*]] = [A, \mathcal{N}_g\, [\mathcal{N}_g, A^*]] + [A, [\mathcal{N}_g, A^*]\, \mathcal{N}_g] \\ &= \mathcal{N}_g\, [A, [\mathcal{N}_g, A^*]] + [A, [\mathcal{N}_g, A^*]]\, \mathcal{N}_g + [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g] + [A, \mathcal{N}_g]\, [\mathcal{N}_g, A^*]\end{aligned}$$

and

$$[A, [A, \mathcal{N}_g^2]] = [A, [A, \mathcal{N}_g]\, \mathcal{N}_g] + [A, \mathcal{N}_g\, [A, \mathcal{N}_g]] = 4\, [A, \mathcal{N}_g]\, [A, \mathcal{N}_g]\ .$$

By Cauchy-Schwarz inequality, we have

$$\langle \xi_{s_2},\, [A, [\mathcal{N}_g, A^*]]\, \mathcal{N}_g \xi_{s_2} \rangle \leq C\, \langle \xi_{s_2},\, \mathcal{N}_g\, [A, [\mathcal{N}_g, A^*]]\, \xi_{s_2} \rangle\ ,$$

$$\langle \xi_{s_2},\, [A, \mathcal{N}_g]\, [\mathcal{N}_g, A^*]\, \xi_{s_2} \rangle \leq C\, \langle \xi_{s_2},\, [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g]\, \xi_{s_2} \rangle = C\, \| [A, \mathcal{N}_g]\, \xi_{s_2} \|^2$$

and

$$\langle \xi_{s_2}, [A, \mathcal{N}_g]\, [A, \mathcal{N}_g]\, \xi_{s_2} \rangle \leq C\, \| [\mathcal{N}_g, A^*]\, \xi_{s_2} \|^2 + C\, \| [A, \mathcal{N}_g]\, \xi_{s_2} \|^2 \leq C\, \| [A, \mathcal{N}_g]\, \xi_{s_2} \|^2\ .$$

Then from (3.39), we have

$$\begin{aligned}\langle \xi_t, \mathcal{N}_g^2 \xi_t \rangle &\leq C\mathrm{Re} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \langle \xi_{s_2},\, \mathcal{N}_g\, [A, [\mathcal{N}_g, A^*]]\, \xi_{s_2} \rangle \\ &\quad + C\mathrm{Re} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \| [A, \mathcal{N}_g]\, \xi_{s_2} \|^2 + N^{-\nu}\end{aligned} \tag{3.40}$$

for any $\nu > 0$.

For the second term on the r.h.s. of (3.40), we apply [13, Lemma 11] to insert the cutoff Θ and choose $\alpha > 0$ and then $n \in \mathbb{N}$ large enough and obtain

$$\begin{aligned}&\| [A, \mathcal{N}_g]\, \xi_{s_2} \|^2 = \langle \xi_{s_2},\, [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g]\, \xi_{s_2} \rangle \\ &\quad = \langle \xi_{s_2},\, \Theta\, [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g]\, \Theta \xi_{s_2} \rangle + \langle \xi_{s_2},\, (1-\Theta)\, [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g]\, \Theta \xi_{s_2} \rangle \\ &\qquad + \langle \xi_{s_2},\, \Theta\, [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g]\, (1-\Theta) \xi_{s_2} \rangle + \langle \xi_{s_2},\, (1-\Theta)\, [\mathcal{N}_g, A^*]\, [A, \mathcal{N}_g]\, (1-\Theta) \xi_{s_2} \rangle \\ &\quad \leq \| [A, \mathcal{N}_g]\, \Theta \xi_{s_2} \|^2 + N^{-\nu}\end{aligned}$$

for any $\nu > 0$. Then it suffices to estimate

$$\begin{aligned}&\| [A, \mathcal{N}_g]\, \Theta \xi_{s_2} \|^2 \\ &= \left\| \int dy dx_1 dx_2 dz dz'\, \check{A}^{(3)}(x_1 - x_2, x_1 - z)\, \check{\sigma}(y - z)\, \check{\sigma}(y - z') a_{x_1} a_{x_2} a_{z'} \Theta \xi_{s_2} \right\|^2 \\ &\leq \int dy d\tilde{y} dx_1 dx_2 d\tilde{x}_1 d\tilde{x}_2 dz d\tilde{z}\, \check{A}^{(3)}(x_1 - x_2, x_1 - z)\, \check{A}^{(3)}(\tilde{x}_1 - \tilde{x}_2, \tilde{x}_1 - \tilde{z})\, \check{\sigma}(y - z)\, \check{\sigma}(\tilde{y} - \tilde{z}) \\ &\qquad \times \left\| \int dz'\, \check{\sigma}(y - z') a_{x_1} a_{x_2} a_{z'} \Theta \xi_{s_2} \right\| \left\| \int d\tilde{z}'\, \check{\sigma}(\tilde{y} - \tilde{z}') a_{\tilde{x}_1} a_{\tilde{x}_2} a_{\tilde{z}'} \Theta \xi_{s_2} \right\|\end{aligned}$$

by Cauchy-Schwarz inequality. Now, we localize $\check{\sigma}$ and the kernel $\check{A}^{(3)}$ and then use Cauchy-Schwarz inequality again to obtain

$$
\begin{aligned}
&\|[A,\mathcal{N}_g]\,\Theta\xi_{s_2}\|^2 \\
&\quad\leq \sum_{u,v,w,\tilde{u},\tilde{v},\tilde{w}\in\Lambda_B}\Bigg(\int dyd\tilde{y}dx_1dx_2d\tilde{x}_1d\tilde{x}_2dzd\tilde{z}|\check{A}^{(3)}_{\tilde{u},\tilde{v}}(\tilde{x}_1-\tilde{x}_2,\tilde{x}_1-\tilde{z})|^2|\check{\sigma}_{\tilde{w}}(\tilde{y}-\tilde{z})|^2 \\
&\qquad\times \mathbb{1}\left(|x_1-x_2-u|\,,|x_1-z-v|\,,|y-z-w|\leq \ell_B\right)\left\|\int dz'\,\check{\sigma}(y-z')a_{x_1}a_{x_2}a_{z'}\Theta\xi_{s_2}\right\|^2\Bigg)^{1/2} \\
&\qquad\quad\times\Bigg(\int dyd\tilde{y}dx_1dx_2d\tilde{x}_1d\tilde{x}_2dzd\tilde{z}|\check{A}^{(3)}_{u,v}(x_1-x_2,x_1-z)|^2|\check{\sigma}_{w}(y-z)|^2 \\
&\qquad\times \mathbb{1}\left(|\tilde{x}_1-\tilde{x}_2-\tilde{u}|\,,|\tilde{x}_1-\tilde{z}-\tilde{v}|\,,|\tilde{y}-\tilde{z}-\tilde{w}|\leq \ell_B\right)\left\|\int d\tilde{z}'\,\check{\sigma}(\tilde{y}-\tilde{z}')a_{\tilde{x}_1}a_{\tilde{x}_2}a_{\tilde{z}'}\Theta\xi_{s_2}\right\|^2\Bigg)^{1/2}.
\end{aligned}
$$

By [13, Lemma 11] and then [13, Lemma 6, Lemma 8 and Lemma 15], we get

$$
\begin{aligned}
\|[A,\mathcal{N}_g]\,\Theta\xi_{s_2}\|^2 &\leq \left(CN^{\epsilon}\ell_B^{3/2}\left(\sum_{u,v\in\Lambda_B}\|\check{A}^{(3)}_{u,v}\|_2\right)\left(\sum_{w\in\Lambda_B}\|\check{\sigma}_w\|_2\right)\langle\xi_{s_2},\,\mathcal{N}\xi_{s_2}\rangle^{1/2}\right)^2 \\
&\leq CN^{21\kappa/4-3/2+\epsilon}
\end{aligned}
\tag{3.41}
$$

for any $\epsilon>0$.

Now, we estimate the first term on the r.h.s. of (3.40). By [13, Equation (128)], we find

$$
\begin{aligned}
&\mathcal{N}_g\,[A,[\mathcal{N}_g,A^*]] \\
&\quad=\int d\alpha dz_4dy_1dy_2dy_3dzdz'dx_1dx_2\,g(\alpha-z_4)\,g(z-z')\,\check{A}(y_1-y_2,y_1-y_3) \\
&\qquad\qquad\times\check{A}^{(3)}(x_1-x_2,x_1-z')a^*_{z_3}a_{z_4}\left[a_{y_1}a_{y_2}a_{y_3},a^*_za^*_{x_1}a^*_{x_2}\right] \\
&\quad=\mathcal{N}_g\left(T_0+T_2+T_{4,1}+T_{4,2}\right)
\end{aligned}
\tag{3.42}
$$

with

$$
\mathcal{N}_gT_0=\int dz_3dz_4dzdz'dx_1dx_2\,g(z_3-z_4)\,g(z-z')\check{A}^{(3)}(x_1-x_2,x_1-z')\check{A}^{(6)}(z-x_1,z-x_2)a^*_{z_3}a_{z_4}\,,
$$

$$
\begin{aligned}
\mathcal{N}_gT_2=&-\int dz_3dz_4dzdz'dx_1dx_2dx_3\,g(z_3-z_4)\,g(z-z') \\
&\qquad\times\check{A}^{(6)}(x_1-x_2,x_1-z')\check{A}^{(6)}(x_3-x_2,x_3-z)a^*_{z_3}a_{z_4}a^*_{x_1}a_{x_3} \\
&+\int dz_3dz_4dzdz'dx_1dx_2dx_3\,g(z_3-z_4)\,g(z-z') \\
&\qquad\times\check{A}^{(3)}(x_1-x_2,x_1-z')\check{A}^{(6)}(x_3-x_2,x_3-x_1)a^*_{z_3}a_{z_4}a^*_za_{x_3}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{N}_gT_{4,1}=&\int dz_3dz_4dzdz'dx_1dx_2dx_3dx_4\,g(z_3-z_4)\,g(z-z') \\
&\qquad\times\check{A}^{(6)}(x_1-x_2,x_1-z')\check{A}^{(3)}(x_3-x_4,x_3-x_1)a^*_{z_3}a_{z_4}a^*_za^*_{x_2}a_{x_3}a_{x_4} \\
\mathcal{N}_gT_{4,2}=&\int dz_3dz_4dzdz'dx_1dx_2dx_3dx_4\,g(z_3-z_4)\,g(z-z') \\
&\qquad\times\check{A}^{(3)}(x_1-x_2,x_1-z')\check{A}^{(3)}(x_3-x_4,x_3-z)a^*_{z_3}a_{z_4}a^*_{x_1}a^*_{x_2}a_{x_3}a_{x_4}\,.
\end{aligned}
$$

For $\mathcal{N}_g T_{4,1}$ and $\mathcal{N}_g T_{4,2}$, by the canonical commutation relations (1.9), we obtain quartic terms and sextic terms given by

$$\begin{aligned}
\widetilde{T}_{4,1} &= \int dz_3 dz dz' dx_1 dx_2 dx_3 dx_4 \, g(z_3 - z)\, g(z - z') \\
&\qquad \times \check{A}^{(6)}(x_1 - x_2, x_1 - z')\check{A}^{(3)}(x_3 - x_4, x_3 - x_1) a^*_{z_3} a^*_{x_2} a_{x_3} a_{x_4} \\
\widetilde{T}_{4,2} &= \int dz_3 dz dz' dx_1 dx_2 dx_3 dx_4 \, g(z_3 - x_2)\, g(z - z') \\
&\qquad \times \check{A}^{(6)}(x_1 - x_2, x_1 - z')\check{A}^{(3)}(x_3 - x_4, x_3 - x_1) a^*_{z_3} a^*_{z} a_{x_3} a_{x_4} \\
\widetilde{T}_{4,3} &= \int dz_3 dz dz' dx_1 dx_2 dx_3 dx_4 \, g(z_3 - x_1)\, g(z - z') \\
&\qquad \times \check{A}^{(3)}(x_1 - x_2, x_1 - z')\check{A}^{(3)}(x_3 - x_4, x_3 - z) a^*_{z_3} a^*_{x_2} a_{x_3} a_{x_4} \\
\widetilde{T}_{4,4} &= \int dz_3 dz dz' dx_1 dx_2 dx_3 dx_4 \, g(z_3 - x_2)\, g(z - z') \\
&\qquad \times \check{A}^{(3)}(x_1 - x_2, x_1 - z')\check{A}^{(3)}(x_3 - x_4, x_3 - z) a^*_{z_3} a^*_{x_1} a_{x_3} a_{x_4}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{T}_{6,1} &= \int dz_3 dz_4 dz dz' dx_1 dx_2 dx_3 dx_4 \, g(z_3 - z_4)\, g(z - z') \\
&\qquad \times \check{A}^{(6)}(x_1 - x_2, x_1 - z')\check{A}^{(3)}(x_3 - x_4, x_3 - x_1) a^*_{z_3} a^*_{z} a^*_{x_2} a_{z_4} a_{x_3} a_{x_4} \\
\widetilde{T}_{6,2} &= \int dz_3 dz_4 dz dz' dx_1 dx_2 dx_3 dx_4 \, g(z_3 - z_4)\, g(z - z') \\
&\qquad \times \check{A}^{(3)}(x_1 - x_2, x_1 - z')\check{A}^{(3)}(x_3 - x_4, x_3 - z) a^*_{z_3} a^*_{x_1} a^*_{x_2} a_{z_4} a_{x_3} a_{x_4} \, .
\end{aligned}$$

For $\mathcal{N}_g T_0$ and $\mathcal{N}_g T_2$, we directly pass them to momentum space by (2.22) and (1.7). Then we obtain

$$\begin{aligned}
\mathcal{N}_g T_0 &= \int dz_3 dz_4 dz dz' dx_1 dx_2 \sum_{r \in \Lambda^*} \hat{g}_r e^{ir\cdot(z_3 - z_4)} \sum_{\tilde{r} \in \Lambda^*} \hat{g}_{\tilde{r}} e^{i\tilde{r}\cdot(z - z')} \sum_{p,q \in \Lambda^*} A^{(3)}_{p,q} e^{ip\cdot(x_1 - x_2)} e^{iq\cdot(x_1 - z')} \\
&\qquad \times \sum_{\tilde{p},\tilde{q} \in \Lambda^*} A^{(6)}_{\tilde{p},\tilde{q}} e^{i\tilde{p}\cdot(z - x_1)} e^{i\tilde{q}\cdot(z - x_2)} \sum_{k \in \Lambda^*} a^*_k e^{-ik\cdot z_3} \sum_{\tilde{k} \in \Lambda^*} a_{\tilde{k}} e^{i\tilde{k}\cdot z_4} \\
&= \sum_{p,q,r \in \Lambda^*} \hat{g}_r \hat{g}_{-q} A^{(3)}_{p,q} A^{(6)}_{p,q} a^*_r a_r + \sum_{q,r \in \Lambda^*} \hat{g}_r \hat{g}_{-q} \left(A^{(6)}_{r-q,q} \right)^2 a^*_r a_r + \sum_{p,r \in \Lambda^*} \hat{g}^2_r A^{(3)}_{p,-r} A^{(6)}_{p,-r} a^*_r a_r \, .
\end{aligned}$$

Since $g = \check{\sigma} * \check{\sigma}$, $\hat{g} = \sigma^2$. By $\sigma_p = \sigma_{-p}$, Cauchy Schwarz inequality and [13, Lemma 9, Equation (80)], we find

$$\begin{aligned}
&|\langle \xi_{s_2}, \sum_{p,q,r \in \Lambda^*} \hat{g}_r \hat{g}_{-q} A^{(3)}_{p,q} A^{(6)}_{p,q} a^*_r a_r \xi_{s_2} \rangle| \leq |\langle \xi_{s_2}, \sum_{r \in \Lambda^*} \sigma^2_r a^*_r a_r \xi_{s_2} \rangle| \sum_{p,q \in \Lambda^*} \sigma^2_q |A^{(3)}_{p,q}||A^{(6)}_{p,q}| \\
&\quad \leq |\langle \xi_{s_2}, \widetilde{\mathcal{N}} \xi_{s_2} \rangle| \left(\sum_{p,q \in \Lambda^*} \sigma^2_q |A^{(3)}_{p,q}|^2 \right)^{1/2} \left(\sum_{p,q \in \Lambda^*} \sigma^2_q |A^{(6)}_{p,q}|^2 \right)^{1/2} \leq C N^{3\kappa - 1} |\langle \xi_{s_2}, \widetilde{\mathcal{N}} \xi_{s_2} \rangle| \, , \\
&|\langle \xi_{s_2}, \sum_{q,r \in \Lambda^*} \hat{g}_r \hat{g}_{-q} \left(A^{(6)}_{r-q,q} \right)^2 a^*_r a_r \xi_{s_2} \rangle| \leq |\langle \xi_{s_2}, \sum_{r \in \Lambda^*} \sigma^2_r a^*_r a_r \xi_{s_2} \rangle| \sum_{q \in \Lambda^*} \sigma^2_q |A^{(6)}_{r-q,q}|^2 \\
&\quad \leq C N^{3\kappa - 1} |\langle \xi_{s_2}, \widetilde{\mathcal{N}} \xi_{s_2} \rangle|
\end{aligned}$$

and

$$|\langle\xi_{s_2},\sum_{p,r\in\Lambda^*}\hat{g}_r^2A^{(3)}_{p,-r}A^{(6)}_{p,-r}a_r^*a_r\xi_{s_2}\rangle|\leq|\langle\xi_{s_2},\sigma_r^2a_r^*a_r\xi_{s_2}\rangle|\sum_{p,r\in\Lambda^*}\sigma_r^2|A^{(3)}_{p,-r}||A^{(6)}_{p,-r}|$$
$$\leq|\langle\xi_{s_2},\widetilde{\mathcal{N}}\xi_{s_2}\rangle|\left(\sum_{p,r\in\Lambda^*}\sigma_r^2|A^{(3)}_{p,-r}|^2\right)^{1/2}\left(\sum_{p,r\in\Lambda^*}\sigma_r^2|A^{(6)}_{p,-r}|^2\right)^{1/2}\leq CN^{3\kappa-1}|\langle\xi_{s_2},\widetilde{\mathcal{N}}\xi_{s_2}\rangle|\,.$$

Applying [13, Lemma 19], we get

$$|\langle\xi_{s_2},\mathcal{N}_gT_0\xi_{s_2}\rangle|\leq CN^{6\kappa-2}\,. \tag{3.43}$$

Similarly, we find

$$\mathcal{N}_gT_2=-\sum_{p,q,r\in\Lambda^*}\sigma_r^2\hat{g}_qA^{(6)}_{p,q}A^{(6)}_{-p,-q}a_r^*a_ra_{p+q}^*a_{p+q}+\sum_{p,q,r\in\Lambda^*}\sigma_r^2\hat{g}_qA^{(3)}_{p,q}A^{(6)}_{p,q}a_r^*a_ra_{-q}^*a_{-q}\,.$$

By [13, Lemma 9, Equation (79)] and then applying Lemma 8, we obtain

$$\begin{aligned}|\langle\xi_{s_2},\mathcal{N}_gT_2\xi_{s_2}\rangle|&\leq\|\hat{g}\|_\infty\sum_{p,q\in\Lambda^*}|A^{(6)}_{p,q}|^2|\langle\xi_{s_2},\sum_{r\in\Lambda^*}a_r^*a_ra_{p+q}^*a_{p+q}\xi_{s_2}\rangle|\\&\quad+\|\hat{g}\|_\infty\sum_{p,q\in\Lambda^*}|A^{(6)}_{p,q}|^2|\langle\xi_{s_2},\sum_{r\in\Lambda^*}a_r^*a_ra_{p+q}^*a_{p+q}\xi_{s_2}\rangle|\\&\leq CN^{\beta/4}N^{3\kappa/2-1}\left(1+\sigma_q^2\right)\langle\xi_{s_2},\tilde{\mathcal{N}}a_q^*a_q\xi_{s_2}\rangle\\&\leq CN^{-\beta/4}\left\langle\xi_{s_2},\left(\tilde{\mathcal{N}}^2+\mathcal{N}^2\right)\xi_{s_2}\right\rangle\\&\leq CN^{-\beta/4}\left\langle\xi_{s_2},\tilde{\mathcal{N}}^2\xi_{s_2}\right\rangle+CN^{6\kappa-2-\beta/4}\,.\end{aligned} \tag{3.44}$$

For $\widetilde{T}_{4,1}$, we follow the estimation for $T_{4,1}$ in the proof of [13, Lemma 19]. So, we first inset the cutoff Θ by [13, Lemma 11] and then localize $\check{\sigma}$ and kernels $\check{A}^{(3)},\check{A}^{(6)}$ and apply Cauchy-Schwarz to obtain

$$\begin{aligned}&|\langle\xi_{s_2},\Theta\widetilde{T}_{4,1}\Theta\xi_{s_2}\rangle|\\&\leq\int dydzdz'dx_1\cdots dx_4\,|\check{\sigma}(y-z)|\,|g(z-z')|\,|\check{A}^{(6)}(x_1-x_2,x_1-z')|\\&\qquad\times|\check{A}^{(3)}(x_3-x_4,x_3-x_1)|\,|\langle\int dz_3\check{\sigma}(y-z_3)a_{z_3}a_{x_2}\Theta\xi_{s_2},\,a_{x_3}a_{x_4}\Theta\xi_{s_2}\rangle|\\&\leq\sum_{u,v,\tilde{u},\tilde{v},w\in\Lambda_B}\left(\int dydzdz'dx_1\cdots dx_4|\check{\sigma}_w(y-z)|^2|g(z-z')|\,|\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_3-x_4,x_3-x_1)|^2\right.\\&\qquad\left.\times\mathbb{1}\left(|x_1-x_2-u|,|x_1-z'-v|\leq\ell_B\right)\left\|\int dz_3\check{\sigma}(y-z_3)a_{z_3}a_{x_2}\Theta\xi_{s_2}\right\|^2\right)^{1/2}\\&\quad\times\left(\int dydzdz'dx_1\cdots dx_4|g(z-z')|\,|\check{A}^{(6)}_{u,v}(x_1-x_2,x_1-z')|^2\right.\\&\qquad\left.\times\mathbb{1}\left(|y-z-w|,|x_3-x_4-\tilde{u}|,|x_3-x_1-\tilde{v}|\leq\ell_B\right)\|a_{x_3}a_{x_4}\Theta\xi_{s_2}\|^2\right)^{1/2}.\end{aligned}$$

By [13, Lemma 11] and then use (3.34) and [13, Lemma 8, Lemma 6 and Lemma 19], we find for any $\epsilon > 0$, $j = 3, 6$,

$$\begin{aligned} |\langle \xi_{s_2},\, \Theta \widetilde{T}_{4,1} \Theta \xi_{s_2} \rangle| &\le C\ell_B^3 \, \|g\|_1 \left(\sum_{u,v \in \Lambda_B} \|\check{A}^{(j)}_{u,v}\|_2 \right)^2 \left(\sum_{w \in \Lambda_B} \|\check{\sigma}_w\|_2 \right) \left\langle \xi_{s_2},\, \widetilde{\mathcal{N}} \xi_{s_2} \right\rangle^{1/2} \\ &\le C N^{9\kappa/4 - 1/2 + \epsilon} \,. \end{aligned} \tag{3.45}$$

For $\widetilde{T}_{4,2}$, we estimate similarly as $\widetilde{T}_{4,1}$ and obtain, for $j = 3, 6$,

$$\begin{aligned} &|\langle \xi_{s_2},\, \Theta \widetilde{T}_{4,2} \Theta \xi_{s_2} \rangle| \\ &\le \int dy dy' dz' dx_1 \cdots dx_4 \, |\check{\sigma}(y - x_2)| \, |\check{\sigma}(y' - z')| \, |\check{A}^{(6)}(x_1 - x_2, x_1 - z')| \\ &\qquad \times |\check{A}^{(3)}(x_3 - x_4, x_3 - x_1)| \left| \langle \int dz_3 dz \check{\sigma}(y - z_3) |\check{\sigma}(y' - z) | a_{z_3} a_z \Theta \xi_{s_2},\, a_{x_3} a_{x_4} \Theta \xi_{s_2} \rangle \right| \\ &\le \sum_{u,v,w,\tilde{u},\tilde{v},\tilde{w} \in \Lambda_B} \Bigg(\int dy dy' dz dz' dx_1 \cdots dx_4 |\check{\sigma}_w(y - x_2)|^2 |\check{\sigma}_{\tilde{w}}(y' - z')|^2 |\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_3 - x_4, x_3 - x_1)|^2 \\ &\qquad \times \mathbb{1}\left(|x_1 - x_2 - u|, |x_1 - z' - v| \le \ell_B \right) \left\| \int dz_3 dz \check{\sigma}(y - z_3) |\check{\sigma}(y' - z)| a_{z_3} a_z \Theta \xi_{s_2} \right\|^2 \Bigg)^{1/2} \\ &\quad \times \Bigg(\int dy dy' dz dz' dx_1 \cdots dx_4 |g(z - z')| \, |\check{A}^{(6)}_{u,v}(x_1 - x_2, x_1 - z')|^2 \\ &\qquad \times \mathbb{1}\left(|y - x_2 - w|, |y' - z' - \tilde{w}|, |x_3 - x_4 - \tilde{u}|, |x_3 - x_1 - \tilde{v}| \le \ell_B \right) \|a_{x_3} a_{x_4} \Theta \xi_{s_2}\|^2 \Bigg)^{1/2} \\ &\le C N^{\epsilon} \ell_B^{9/2} \left(\sum_{u,v \in \Lambda_B} \|\check{A}^{(j)}_{u,v}\|_2 \right)^2 \left(\sum_{w \in \Lambda_B} \|\check{\sigma}_w\|_2 \right)^2 \left\langle \xi_{s_2},\, \widetilde{\mathcal{N}}^2 \xi_{s_2} \right\rangle^{1/2} \\ &\le C N^{9\kappa/8 - 1/4 + \epsilon} \left(\delta \left\langle \xi_{s_2},\, \widetilde{\mathcal{N}}^2 \xi_{s_2} \right\rangle + \frac{1}{4\delta} \right) \end{aligned}$$

for any $\epsilon > 0$ and for any $\delta > 0$. If we take $\delta = N^{-(3\kappa - 1)}$, then we obtain

$$|\langle \xi_{s_2},\, \Theta \widetilde{T}_{4,2} \Theta \xi_{s_2} \rangle| \le C N^{-15\kappa/8 + 3/4 + \epsilon} \left\langle \xi_{s_2},\, \widetilde{\mathcal{N}}^2 \xi_{s_2} \right\rangle + C N^{33\kappa/8 - 5/4 + \epsilon} \,. \tag{3.46}$$

Since $\widetilde{T}_{4,3}$ and $\widetilde{T}_{4,4}$ are similar, we only calculate $\widetilde{T}_{4,3}$ as an example. Then

$$
\begin{aligned}
&|\langle \Theta\xi_{s_2}, \widetilde{T}_{4,3}\Theta\xi_{s_2}\rangle| \\
&\leq \int dydz_1dz_2dx_1\cdots dx_4\,|\check{\sigma}(y-x_1)|\,|g(z_1-z_2)|\,|\check{A}^{(3)}(x_1-x_2,x_1-z_2)| \\
&\qquad\qquad \times |\check{A}^{(3)}(x_3-x_4,x_3-z_1)|\left|\langle\int dz_3\,\check{\sigma}(y-z_3)a_{z_3}a_{x_2}\Theta\xi_{s_2},\, a_{x_3}a_{x_4}\Theta\xi_{s_2}\rangle\right| \\
&\leq \sum_{u,v,\tilde{u},\tilde{v},w\in\Lambda_B}\Bigg(\int dydz_1dz_2dx_1\cdots dx_4\,|\check{\sigma}_w(y-x_1)|^2|g(z_1-z_2)|\,|\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_3-x_4,x_3-z_1)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_1-x_2-u|,|x_1-z_2-v|\leq \ell_B\right)\left\|\int dz_3\,\check{\sigma}(y-z_3)a_{z_3}a_{x_2}\Theta\xi_{s_2}\right\|^2\Bigg)^{1/2} \\
&\qquad \times\Bigg(\int dydz_1dz_2dx_1\cdots dx_4\,|g(z_1-z_2)|\,|\check{A}^{(3)}_{u,v}(x_1-x_2,x_1-z_2)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_3-x_4-\tilde{u}|,|x_3-z_1-\tilde{v}|,|y-x_1-w|\leq \ell_B\right)\|a_{x_3}a_{x_4}\Theta\xi_{s_2}\|^2\Bigg)^{1/2} \\
&\leq CN^{3\epsilon/2}\ell_B^3\,\|g\|_1\left(\sum_{u,v\in\Lambda_B}\|\check{A}^{(3)}_{u,v}\|_2\right)^2\left(\sum_{w\in\Lambda_B}\|\check{\sigma}_w\|_2\right)\left\langle\xi_{s_2},\widetilde{\mathcal{N}}^2\xi_{s_2}\right\rangle^{1/2} \\
&\leq CN^{9\kappa/4-1/2+\epsilon}
\end{aligned}
\tag{3.47}
$$

for any $\epsilon>0$.

Now, we estimate $\widetilde{T}_{6,1}$ still using the same approach as above. Then, for $j=3,6$, we find

$$
\begin{aligned}
&|\langle \Theta\xi_{s_2}, \widetilde{T}_{6,1}\Theta\xi_{s_2}\rangle| \\
&\leq \int dy_1dy_2dz_2dx_1\cdots dx_4|\check{\sigma}(y_2-z_2)|\,|\check{A}^{(6)}(x_1-x_2,x_1-z_2)|\,|\check{A}^{(3)}(x_3-x_4,x_3-x_1)| \\
&\qquad\qquad \times\left|\langle\int dz_3dz_1\,\check{\sigma}(y_1-z_3)a_{z_3}a_{z_1}a_{x_2}\Theta\xi_{s_2},\,\int dz_4\,\check{\sigma}(y_1-z_4)a_{z_4}a_{x_3}a_{x_4}\Theta\xi_{s_2}\rangle\right| \\
&\leq \sum_{u,v,\tilde{u},\tilde{v},w\in\Lambda_B}\Bigg(\int dy_1dy_2dz_2dx_1\cdots dx_4|\check{\sigma}_w(y_2-z_2)|^2|\check{A}^{(6)}_{u,v}(x_1-x_2,x_1-z_2)|^2 \\
&\qquad\qquad \times \mathbb{1}\left(|x_3-x_4-\tilde{u}|,|x_3-x_1-\tilde{v}|\leq \ell_B\right)\left\|\int dz_4\,\check{\sigma}(y_1-z_4)a_{z_4}a_{x_3}a_{x_4}\Theta\xi_{s_2}\right\|^2\Bigg)^{1/2} \\
&\qquad\times\Bigg(\int dy_1dy_2dz_2dx_1\cdots dx_4|\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_3-x_4,x_3-x_1)|^2\,\mathbb{1}(|x_1-x_2-u|, \\
&\qquad\qquad \cdots|x_1-z_2-v|,|y_2-z_2-w|\leq \ell_B)\left\|\int dz_3dz_1\,\check{\sigma}(y_1-z_3)a_{z_3}a_{z_1}a_{x_2}\Theta\xi_{s_2}\right\|^2\Bigg)^{1/2} \\
&\leq CN^{3\epsilon/2}\ell_B^3\,\|g\|_1\left(\sum_{u,v\in\Lambda_B}\|\check{A}^{(j)}_{u,v}\|_2\right)^2\left(\sum_{w\in\Lambda_B}\|\check{\sigma}_w\|_2\right)\left\langle\xi_{s_2},\widetilde{\mathcal{N}}\xi_{s_2}\right\rangle^{1/2}\left(\delta\left\langle\xi_{s_2},\widetilde{\mathcal{N}}^2\xi_{s_2}\right\rangle+\frac{1}{4\delta}\right) \\
&\leq C\delta N^{3\kappa-1+\epsilon}\left\langle\xi_{s_2},\widetilde{\mathcal{N}}^2\xi_{s_2}\right\rangle+C\delta^{-1}N^{3\kappa-1+\epsilon}
\end{aligned}
$$

for any $\epsilon > 0$ and for any $\delta > 0$. If we take $\delta = N^{-(3\kappa-1)}$, then we get

$$|\langle \Theta\xi_{s_2}, \widetilde{T}_{6,1}\Theta\xi_{s_2}\rangle| \leq C\langle \xi_{s_2}, \widetilde{\mathcal{N}}^2\xi_{s_2}\rangle + CN^{6\kappa-2+\epsilon}\,. \tag{3.48}$$

For $\widetilde{T}_{6,2}$, we also bound it as before. Then we find

$$\begin{aligned}
&|\langle \Theta\xi_{s_2}, \widetilde{T}_{6,2}\Theta\xi_{s_2}\rangle| \\
&\leq \int dydz_1dz_2dx_1\cdots dx_4 |g(z_1-z_2)|\,|\check{A}^{(3)}(x_1-x_2,x_1-z_2)|\,|\check{A}^{(3)}(x_3-x_4,x_3-z_1)| \\
&\qquad\qquad \times \left|\langle \int dz_3\,\check{\sigma}(y-z_3)a_{z_3}a_{z_1}a_{x_2}\Theta\xi_{s_2},\, \int dz_4\,\check{\sigma}(y-z_4)a_{z_4}a_{x_3}a_{x_4}\Theta\xi_{s_2}\rangle\right| \\
&\leq \sum_{u,v,\tilde{u},\tilde{v},w\in\Lambda_B} \left(\int dy_1dy_2dz_2dx_1\cdots dx_4 |g(z_1-z_2)|\,|\check{A}^{(3)}_{u,v}(x_1-x_2,x_1-z_2)|^2 \right. \\
&\qquad\qquad \left. \times \mathbb{1}\left(|x_3-x_4-\tilde{u}|,|x_3-z_1-\tilde{v}|\leq \ell_B\right) \left\| \int dz_4\,\check{\sigma}(y-z_4)a_{z_4}a_{x_3}a_{x_4}\Theta\xi_{s_2}\right\|^2 \right)^{1/2} \\
&\qquad \times \left(\int dy_1dy_2dz_2dx_1\cdots dx_4 |g(z_1-z_2)|\,|\check{A}^{(3)}_{\tilde{u},\tilde{v}}(x_3-x_4,x_3-z_1)|^2 \right. \\
&\qquad\qquad \left. \times \mathbb{1}(|x_1-x_2-u|,|x_1-z_2-v|,\leq \ell_B) \left\| \int dz_3\,\check{\sigma}(y_1-z_3)a_{z_3}a_{z_1}a_{x_2}\Theta\xi_{s_2}\right\|^2 \right)^{1/2} \\
&\leq CN^{2\epsilon}\,\|g\|_1 \left(\sum_{u,v\in\Lambda_B} \|\check{A}^{(j)}_{u,v}\|_2 \right)^2 \left\langle \xi_{s_2}, \widetilde{\mathcal{N}}\xi_{s_2}\right\rangle \\
&\leq CN^{6\kappa-2+\beta/4+\epsilon}
\end{aligned} \tag{3.49}$$

for any $\epsilon > 0$.

Putting (3.43), (3.44) (3.45), (3.46), (3.47), (3.48) and (3.49)together, we get

$$\langle \xi_{s_2}, \mathcal{N}_g\,[A,[\mathcal{N}_g,A^*]]\,\xi_{s_2}\rangle \leq C\langle \xi_{s_2}, \widetilde{\mathcal{N}}^2\xi_{s_2}\rangle + CN^{21\kappa/4-3/2+\epsilon} \tag{3.50}$$

for any $\epsilon > 0$. Combining (3.41) and (3.50) and applying Gröwall's lemma, we conclude that

$$\left\langle \xi_t, \widetilde{\mathcal{N}}^2\xi_t\right\rangle \leq CN^{21\kappa/4-3/2}\,.$$

□

Remark 12. *Notice that this result* $\left\langle \xi_t, \widetilde{\mathcal{N}}^2\xi_t\right\rangle \leq CN^{21\kappa/4-3/2}$ *is not optimal, but it is enough to show Theorem 7. One may get a more precise estimate by expanding terms such as* $\widetilde{T}_{6,2}$ *more times.*

4 Energy estimates

In this part, we revisit the energy estimate part in [13]. The main theorem of the [13] is stated as follows.

Theorem 1. *Let $V \in L^2(\mathbb{R}^3)$ be radial, stable and with compact support. Assume that V does not admit two-body bound states and let $\mathfrak{a} > 0$ be its scattering length. Then, there is a constant $C > 0$ such that*

$$e(\rho) \leq 4\pi\mathfrak{a}\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}^3} + 8\left(\frac{4\pi}{3} - \sqrt{3}\rho\mathfrak{a}^3 \log\left(\rho\mathfrak{a}^3\right)\right) + C\rho\mathfrak{a}^3\right) \tag{4.1}$$

for all $\rho\mathfrak{a}^3 > 0$ small enough.

To prove Theorem 1, it suffices to prove Theorem 2 which tells us that under the same assumptions as in Theorem 1, if there exists a good trial state on the rescaled unit torus Λ, then we get an upper bound on the energy density on this torus Λ. This is the content of the following theorem.

Theorem 2. *Under the same assumptions on V as in Theorem 1, we have that for any $\kappa \in [1/2, 2/3)$ and N large enough, there exists $\Psi_N \in \mathcal{F}(\Lambda)$, $\|\Psi_N\| = 1$ such that*

$$\langle \Psi_N, \mathcal{N}\Psi_N \rangle \geq N, \qquad \left\langle \Psi_N, \mathcal{N}^2\Psi_N \right\rangle \leq CN^2 \tag{4.2}$$

and

$$\begin{aligned} \langle \Psi_N, \mathcal{H}_N\Psi_N \rangle \leq 4\pi\mathfrak{a}N^{1+\kappa} + \frac{512\sqrt{\pi}}{15}\mathfrak{a}^{5/2}N^{5\kappa/2} \\ + 32\pi\left(\frac{4\pi}{3} - \sqrt{3}\right)\mathfrak{a}^4 N^{4\kappa-1}\log\left(N^{3\kappa-2}\right) + CN^{4\kappa-1}\,. \end{aligned} \tag{4.3}$$

Now, we explain how the upper bound obtained in Theorem 2 for the rescaled Hamiltonian $\mathcal{H}_N$ on the unit torus Λ implies the desired upper bound on the thermodynamic ground state energy density $e(\rho)$.

Proof of Theorem 1. We use the localization result from [2, Proposition 1.2]. Let $R > 0$ be the radius of the support of V, and choose $\ell, L > 0$ such that $R < \ell < L$. Let $\Psi_L \in \mathcal{F}(\Lambda_L)$ be a normalized grand-canonical state satisfying

$$\langle \Psi_L, \mathcal{N}\Psi_L \rangle \geq \rho(1 + c\rho)(L + 2\ell + R)^3, \qquad \langle \Psi_L, \mathcal{N}^2\Psi_L \rangle \leq C\rho^2(L + 2\ell + R)^6. \tag{4.4}$$

Then the localization estimate gives

$$e(\rho) \leq \frac{\langle \Psi_L, \mathcal{H}\Psi_L \rangle}{L^3} + \frac{C}{L^4\ell}\langle \Psi_L, \mathcal{N}\Psi_L \rangle,$$

where $\mathcal{H}$ denotes the Hamiltonian on $\mathcal{F}(\Lambda_L)$ with $\Lambda_L^* = 2\pi\mathbb{Z}^3/L$, namely,

$$\mathcal{H} = \sum_{p \in \Lambda_L^*} p^2 a_p^* a_p + \frac{1}{2}\sum_{p,q,r \in \Lambda_L^*} \widehat{V}(r)\, a_{p+r}^* a_{q-r}^* a_q a_p, \qquad .$$

The condition (4.4) means that the periodic trial state Ψ_L has to contain slightly more particles than ρL^3. Indeed, in the localization procedure, one first enlarges the box in order to impose Dirichlet boundary conditions, and then one separates different boxes by corridors so that particles in different boxes do not interact. The additional length $2\ell + R$ accounts for these two effects. Hence the trial state in the periodic box has to represent density ρ in the larger effective box of side length $L + 2\ell + R$. This is the reason why we introduce an auxiliary density $\widetilde{\rho} > \rho$.

Applying Theorem 2 with density $\widetilde{\rho}$ and rescaling lengths to the unit torus, we obtain, for any $\gamma > 1$, a normalized state $\Psi_L \in \mathcal{F}(\Lambda_L)$ with

$$L = \widetilde{\rho}^{-\gamma}, \qquad N = \widetilde{\rho} L^3 = \widetilde{\rho}^{1-3\gamma}.$$

such that

$$\langle \Psi_L, \mathcal{N} \Psi_L \rangle \geq \widetilde{\rho} L^3, \qquad \langle \Psi_L, \mathcal{N}^2 \Psi_L \rangle \leq C \widetilde{\rho}^2 L^6. \tag{4.5}$$

Since the rescaled Hamiltonian $\mathcal{H}_N$ is unitarily equivalent to $L^2 \mathcal{H}$, by Theorem 2, we have

$$\begin{aligned} \frac{\langle \Psi_L, \mathcal{H} \Psi_L \rangle}{L^3} &= \frac{\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle}{L^5} \\ &\leq 4\pi \mathfrak{a} \widetilde{\rho}^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\widetilde{\rho}\, \mathfrak{a}^3} + 8 \left(\frac{4\pi}{3} - \sqrt{3} \right) \widetilde{\rho}\, \mathfrak{a}^3 \log(\widetilde{\rho}\, \mathfrak{a}^3) + C \widetilde{\rho}\, \mathfrak{a}^3 \right). \end{aligned} \tag{4.6}$$

We may replace $\log(\widetilde{\rho})$ by $\log(\widetilde{\rho}\, \mathfrak{a}^3)$, because

$$\left| \widetilde{\rho}\, \mathfrak{a}^3 \big(\log(\widetilde{\rho}\, \mathfrak{a}^3) - \log(\widetilde{\rho}) \big) \right| = \left| \widetilde{\rho}\, \mathfrak{a}^3 \log(\mathfrak{a}^3) \right| \leq C \widetilde{\rho}\, \mathfrak{a}^3,$$

We now choose the buffer length $\ell = L^\alpha$, for $0 < \alpha < 1$. Since $\ell = L^\alpha = \widetilde{\rho}^{-\alpha\gamma}$, for $\widetilde{\rho} > 0$ sufficiently small, we have $\ell > R$. Moreover, because $0 < \alpha < 1$, we also have $\ell < L$. By Cauchy-Schwarz and (4.5), we obtain

$$\langle \Psi_L, \mathcal{N} \Psi_L \rangle \leq \langle \Psi_L, \mathcal{N}^2 \Psi_L \rangle^{1/2} \| \Psi_L \| \leq C \widetilde{\rho} L^3.$$

Therefore, we get

$$\frac{C}{L^4 \ell} \langle \Psi_L, \mathcal{N} \Psi_L \rangle \leq \frac{C}{L^4 L^\alpha} \cdot \widetilde{\rho} L^3 = C \frac{\widetilde{\rho}}{L^{1+\alpha}} = C \widetilde{\rho}^{1+\gamma(1+\alpha)}. \tag{4.7}$$

Combining (4.6), and (4.7), we obtain

$$e(\rho) \leq 4\pi \mathfrak{a} \widetilde{\rho}^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\widetilde{\rho}\, \mathfrak{a}^3} + 8 \left(\frac{4\pi}{3} - \sqrt{3} \right) \widetilde{\rho}\, \mathfrak{a}^3 \log(\widetilde{\rho}\, \mathfrak{a}^3) + C \widetilde{\rho}\, \mathfrak{a}^3 \right) + C \widetilde{\rho}^{1+\gamma(1+\alpha)}. \tag{4.8}$$

We choose $\widetilde{\rho}$ so that the first condition in (4.4) matches (4.5). Namely, we impose

$$\widetilde{\rho} L^3 = \rho (1 + c\rho)(L + 2\ell + R)^3.$$

Dividing by L^3, we obtain

$$\begin{aligned} \widetilde{\rho} &= \rho(1 + c\rho) \left(1 + \frac{2\ell}{L} + \frac{R}{L} \right)^3 = \rho(1 + c\rho) \left(1 + \frac{2L^\alpha}{L} + \frac{R}{L} \right)^3 \\ &= \rho(1 + c\rho) \left(1 + 2\widetilde{\rho}^{\gamma(1-\alpha)} + R\widetilde{\rho}^\gamma \right)^3. \end{aligned} \tag{4.9}$$

To justify that such a choice is possible, we want to use the implicit function theorem. Define

$$F(\rho, \widetilde{\rho}) = \widetilde{\rho} - \rho(1 + c\rho) \left(1 + 2\widetilde{\rho}^{\gamma(1-\alpha)} + R\widetilde{\rho}^\gamma \right)^3.$$

Then

$$F(0, 0) = 0, \qquad \partial_{\widetilde{\rho}} F(0, 0) = 1.$$

Hence, by the implicit function theorem, for $\rho > 0$ sufficiently small there exists a unique solution $\widetilde{\rho} = \widetilde{\rho}(\rho)$ near zero such that $F(\rho, \widetilde{\rho}) = 0$. Therefore, squaring (4.9), we find

$$\widetilde{\rho}^2 \leq \rho^2 \left(1 + C\rho + C\rho^{\gamma(1-\alpha)}\right). \tag{4.10}$$

Substituting (4.10) into (4.8), we obtain

$$\begin{aligned} e(\rho) \leq 4\pi \mathfrak{a} \rho^2 \Bigg(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho\, \mathfrak{a}^3} + 8 \left(\frac{4\pi}{3} - \sqrt{3} \right) \rho\, \mathfrak{a}^3 \log(\rho\, \mathfrak{a}^3) \Bigg) \\ + C \left(\rho^3 + \rho^{2+\gamma(1-\alpha)} + \rho^{1+\gamma(1+\alpha)} \right). \end{aligned} \tag{4.11}$$

In fact, the optimal choice of α satisfies

$$2 + \gamma(1-\alpha) = 1 + \gamma(1+\alpha).$$

Solving this equation gives $\alpha = 1/(2\gamma)$. Then

$$\rho^{2+\gamma(1-\alpha)} = \rho^{\gamma+3/2} = \rho^{\gamma(1+\alpha)+1}.$$

To ensure the error is negligible, we need $\gamma \geq 3/2$. This concludes the proof.

□

So, the following contents are devoted to build up the proof of Theorem 2. The proof is divided into two main parts. The first part is to construct a proper trial state $\Psi_N = W_{N_0} e^{B-B^*} e^{A^*\Theta - \Theta A} \Omega$ on the rescaled unit torus Λ with the correct number of particles 4.2. This has been done in previous sections. Then we compute the energy of the Hamiltonian

$$\mathcal{H}_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{N^\kappa}{2N} \sum_{p,q,r \in \Lambda^*} \hat{V}(r/N^{1-\kappa}) a_{p+r}^* a_{q-r}^* a_q a_p \tag{4.12}$$

with respect to this trial state Ψ_N, i.e, $\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle$. To reach this goal, we first conjugate the Hamiltonian $\mathcal{H}_N$ by the Weyl operator W_{N_0} and then apply the Bogoliubov transformation e^{B^*-B} defined in 2.1.1. Then we obtain the renormalized excitation Hamiltonian $\widetilde{\mathcal{H}}_N$ containing constant term C_N, kinetic energy operator $\mathcal{K}$, cubic operator $\mathcal{C}_N, \mathcal{C}_N^*$, quartic operator $\mathcal{V}_N$ with some negligible terms $\mathcal{Q}, \widetilde{\mathcal{C}}_N^*, \widetilde{\mathcal{C}}_N, \widetilde{\mathcal{V}}_N$ and some error term of order $N^{4\kappa-1} + N^{2\kappa}$. This is summarized in [13, Lemma 13], and we restate it as the following lemma.

Lemma 13. *We have*

$$\begin{aligned} &\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle = \langle \xi,\, e^{-B^*+B} W_{N_0}^* \mathcal{H}_N W_{N_0} e^{B^*-B} \xi \rangle \\ &\quad = 4\pi \mathfrak{a} N^{1+\kappa} \\ &\qquad + \frac{1}{2} \sum_{p \in \Lambda_+^*} \left(\sqrt{|p|^4 + 2p^2 N^\kappa \widehat{V f}(\frac{p}{N^{1-\kappa}})} - p^2 - N^\kappa \widehat{V f}(\frac{p}{N^{1-\kappa}}) + \frac{N^{2\kappa} \widehat{V f}(\frac{p}{N^{1-\kappa}})^2}{2\,|p|^2} \right) \\ &\qquad - \frac{N^\kappa \|\sigma\|_2^2}{N} \sum_{p \in \Lambda^*} \hat{V}(p/N^{1-\kappa}) \eta_{\infty,p} - N^{\kappa-1} \sum_{p,r \in \Lambda^*} \hat{V}(r/N^{1-\kappa}) \eta_{\infty,p+r} \sigma_p^2 \\ &\qquad + \langle \xi, (\mathcal{K} + \mathcal{C}_N^* + \mathcal{C}_N + \mathcal{V}_N) \xi \rangle + \langle \xi, (\mathcal{Q} + \widetilde{\mathcal{C}}_N^* + \widetilde{\mathcal{C}}_N + \widetilde{\mathcal{V}}_N) \xi \rangle + O(N^{4\kappa-1} + N^{2\kappa}) \end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
\mathcal{K} &= \sum_{p\in\Lambda^*} p^2 a_p^* a_p \,, \\
\mathcal{C}_N^* &= \frac{N^\kappa N_0^{1/2}}{N} \sum_{p,r\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) \sigma_p a_{p+r}^* a_{-r}^* a_{-p}^* \,, \\
\mathcal{V}_N &= \frac{N^\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) a_{p+r}^* a_{q-r}^* a_q a_p \,, \\
\widetilde{\mathcal{C}}_N^* &= \frac{N^\kappa N_0^{1/2}}{N} \sum_{p,r\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) ((\gamma_{p+r}\gamma_r - 1)\sigma_p + \sigma_{p+r}\sigma_r\gamma_p) a_{p+r}^* a_{-r}^* a_{-p}^* \,, \\
\widetilde{\mathcal{V}}_N &= \frac{N^\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) (\gamma_{p+r}\gamma_{q-r}\gamma_q\gamma_p - 1 + \sigma_{p+r}\sigma_{q-r}\sigma_q\sigma_p + 2\gamma_{p+r}\sigma_{q-r}\sigma_q\gamma_p) \\
&\qquad\qquad \times a_{p+r}^* a_{q-r}^* a_q a_p \\
&\quad + \frac{N^\kappa}{N} \sum_{p,q,r\in\Lambda^*} \hat{V}(r/N^{1-\kappa}) \gamma_{p+r}\sigma_{q-r}\gamma_q\sigma_p a_{p+r}^* a_{-p}^* a_{-q+r} a_q
\end{aligned} \tag{4.14}$$

and $\mathcal{Q}$ is a quadratic operator satisfying

$$\pm\mathcal{Q} \leq CN^\kappa(\mathcal{N} + \tilde{\mathcal{N}}) \tag{4.15}$$

with $\tilde{\mathcal{N}} = \sum_{p\in\Lambda^*} \sigma_p^2 a_p^* a_p$.

After a careful calculation, the expectations of negligible terms with respect to the state $\xi = e^{A^*\Theta - \Theta A}\Omega$ can be absorbed into the error of order $N^{4\kappa-1}$ by using [13, Lemma 15, Lemma 18, Lemma 19 and Lemma 20]. We combine all these results into the following lemma.

Lemma 14. *The following bounds hold provided in [13, Lemma 11], we choose α and then n large enough. We have*

$$\left\langle e^{t(A^*\Theta-\Theta A)}\Omega,\, \mathcal{N} e^{t(A^*\Theta-\Theta A)}\Omega \right\rangle \leq CN^{3\kappa-1} \tag{4.16}$$

for all $t \in [0;1]$.

Let $h \in L^2(\Lambda)$, real-valued. We assume that $h(x) = \sum_{u\in\Lambda_B} h_u(x)$, where $h_u \in L^2(\Lambda)$ for all $u \in \Lambda_B$, with $h_u(x) = 0$, if $|x-u| > 10\ell_B$. Then, for every $\epsilon > 0$, we have

$$\begin{aligned}
&\left\langle e^{t(A^*\Theta-\Theta A)}\Omega,\, \mathcal{N}_{h\star h} e^{t(A^*\Theta-\Theta A)}\Omega \right\rangle \\
&\qquad\qquad \leq C\left\langle \Omega,\, [A,[\mathcal{N}_{h\star h}, A^*]]\,\Omega\right\rangle + CN^{-5\beta/8+\epsilon}\left(\sum_{u\in\Lambda_B} \|h_u\|_2\right)^2
\end{aligned} \tag{4.17}$$

for all $t \in [0;1]$. Here, we used the notation

$$h \star h(x) = \int h(y-x)h(y)dy$$

($h\star h$ is the usual convolution, if $h(-y) = h(y)$). In particular, recalling that $\widetilde{\mathcal{N}} = \mathcal{N}_{\check{\sigma}\star\check{\sigma}}$, we find for all $t \in [0;1]$,

$$\left\langle e^{t(A^*\Theta-\Theta A)}\Omega, \widetilde{\mathcal{N}} e^{t(A^*\Theta-\Theta A)}\Omega \right\rangle \leq CN^{3\kappa-1} \,.$$

Furthermore, we have

$$\left|\left\langle e^{A^*\Theta-\Theta A}\Omega,\, \widetilde{\mathcal{C}}_N e^{A^*\Theta-\Theta A}\Omega\right\rangle\right| \leq CN^{4\kappa-1}$$

and

$$\left\langle e^{A^*\Theta-\Theta A}\Omega,\, \widetilde{\mathcal{V}}_N e^{A^*\Theta-\Theta A}\Omega\right\rangle \leq CN^{4\kappa-1}\,.$$

In this thesis, We skip the proofs of Lemma 13 and Lemma 14. We will give a more detailed version of estimating the expectation of the cubic operator $\mathcal{C}_N, \mathcal{C}_N^*$, the quartic operator $\mathcal{V}_N$ and the kinetic energy operator $\mathcal{K}$ in the state $\xi = e^{A^*\Theta-\Theta A}\Omega$ based on the original proofs in [13, Lemma 24, Lemma 25 and Lemma 26]. Then we put these three results together in [13, Lemma 27], whose proof is also skipped since it is technical. Finally, we include a detailed version of proof of Theorem 2 built on [13, Lemma 13 and Lemma 27].

4.1 Bound for the cubic operator $\mathcal{C}_N$ and $\mathcal{C}_N^*$

Now, we will show the estimate of the expectation the cubic operator $\mathcal{C}_N$ and $\mathcal{C}_N^*$ in the state ξ. Let us first recall the following Lemma which is from [13, Lemma 24].

Lemma 15. *In position space, we write the operator $\mathcal{C}_N^*$ from (4.13) as*

$$\mathcal{C}_N^* = N_0^{1/2} \int dxdydz N^{2-2\kappa} V(N^{1-\kappa}(x-y))\check{\sigma}(x-z) a_x^* a_y^* a_z^*. \tag{4.18}$$

Recalling the definition [13, equation (75)] of the kernel $\check{A}^{(6)}$, we find

$$\begin{aligned}\left\langle e^{A^*\Theta-\Theta A}\Omega,\, (\mathcal{C}_N + \mathcal{C}_N^*) e^{A^*\Theta-\Theta A}\Omega\right\rangle \\ = 2N_0^{1/2} \int dxdy N^{2-2\kappa} V(N^{1-\kappa}x)\check{\sigma}(y)\check{A}^{(6)}(x,y) + O(N^{4\kappa-1}).\end{aligned} \tag{4.19}$$

Proof of Lemma 15. We first recall that $V_N(x) = N^{2-2\kappa}V(N^{1-\kappa}x)$ and that from [13, Equation (134)]

$$[a_x^* a_y^*, -A] = \mathcal{T}_1(x,y) + \mathcal{T}_3(x,y),$$

where

$$\begin{aligned}\mathcal{T}_1(x,y) &= \int dz\, \check{A}^{(6)}(z-y, z-x)\, a_z, \\ \mathcal{T}_3(x,y) &= \int dx_1 dx_2 \left[\check{A}^{(3)}(x_1-x_2, x_1-x)\, a_y^* a_{x_1} a_{x_2} + \check{A}^{(3)}(x_1-x_2, x_1-y)\, a_x^* a_{x_1} a_{x_2}\right].\end{aligned}$$

We denote $G = A^*\Theta - \Theta A$ and $\xi_s = e^{sG}\Omega$ for each $s \in [0,1]$. Then

$$\langle \xi, \mathcal{C}_N \xi\rangle = N_0^{1/2} \int dxdydz\, V_N(x-y)\check{\sigma}(x-z)\, \langle \Omega, e^{-G} a_x^* a_y^* a_z^* e^G \Omega\rangle. \tag{4.20}$$

We first define, for fixed x, y, z,

$$F(s) = \langle \Omega, e^{-sG} a_x^* a_y^* e^{sG} e^{-G} a_z^* \xi\rangle.$$

Then

$$\begin{aligned}F(1) = \langle \Omega, e^{-G} a_x^* a_y^* a_z^* \xi\rangle = \langle \Omega, e^{-G} a_x^* a_y^* a_z^* e^G \Omega\rangle, \\ F(0) = \langle \Omega, a_x^* a_y^* e^{-G} a_z^* \xi\rangle = 0,\end{aligned}$$

which gives

$$F(1) = \int_0^1 ds\, F'(s).$$

Observe that

$$\frac{d}{ds}\left(e^{-sG} a_x^* a_y^* e^{sG}\right) = e^{-sG}[a_x^* a_y^*, G] e^{sG},$$

and we obtain

$$F'(s) = \left\langle \Omega, e^{-sG}[a_x^* a_y^*, G] e^{sG} e^{-G} a_z^* \xi \right\rangle = \left\langle \xi_s, [a_x^* a_y^*, G] e^{(s-1)G} a_z^* \xi \right\rangle. \tag{4.21}$$

Next we expand the commutator involved:

$$[a_x^* a_y^*, G] = [a_x^* a_y^*, A^*\Theta - \Theta A] = [a_x^* a_y^*, A^*\Theta] + [\Theta A, a_x^* a_y^*]$$

where

$$\begin{aligned}
[a_x^* a_y^*, A^*\Theta] &= [a_x^* a_y^*, A^*]\Theta + A^*[a_x^* a_y^*, \Theta] = A^*[a_x^* a_y^*, \Theta]\,, \\
[\Theta A, a_x^* a_y^*] &= \Theta[A, a_x^* a_y^*] + [\Theta, a_x^* a_y^*]A = [A, a_x^* a_y^*] + (\Theta - 1)[A, a_x^* a_y^*] + [\Theta, a_x^* a_y^*]A
\end{aligned}$$

because $[a_x^* a_y^*, A^*] = 0$. Therefore

$$[a_x^* a_y^*, G] = [A, a_x^* a_y^*] + (\Theta - 1)[A, a_x^* a_y^*] + A^*[a_x^* a_y^*, \Theta] + [\Theta, a_x^* a_y^*]A. \tag{4.22}$$

Notice that the first term on the r.h.s. of (4.22) is the main contribution. The remaining terms are cutoff-error terms: the second one contains the factor $\Theta - 1$, and the last two contain commutators with Θ. As in discussed in [13], these terms can be collected into error $\mathcal{E}_\Theta$. Moreover, for every $\nu > 0$, after choosing $\alpha > 0$ and then $n \in \mathbb{N}$ sufficiently large, we have $\pm\mathcal{E}_\Theta \leq C_\nu N^{-\nu}$. Thus, inserting (4.22) into (4.21), we obtain

$$\langle \xi, \mathcal{C}_N \xi \rangle = N_0^{1/2} \int_0^1 ds \int dx dy dz\, V_N(x-y) \check{\sigma}(x-z) \left\langle \xi_s, [A, a_x^* a_y^*] e^{(s-1)G} a_z^* \xi \right\rangle + \mathcal{E}_\Theta.$$

Using $[A, a_x^* a_y^*] = \mathcal{T}_1(x,y) + \mathcal{T}_3(x,y)$, we write

$$\langle \xi, \mathcal{C}_N \xi \rangle = S_1 + S_3 + \mathcal{E}_\Theta,$$

where

$$\begin{aligned}
S_1 &\coloneqq N_0^{1/2} \int_0^1 ds \int dx dy dz\, V_N(x-y) \check{\sigma}(x-z) \left\langle \xi_s, \mathcal{T}_1(x,y) e^{(s-1)G} a_z^* \xi \right\rangle, \\
S_3 &\coloneqq N_0^{1/2} \int_0^1 ds \int dx dy dz\, V_N(x-y) \check{\sigma}(x-z) \left\langle \Theta\xi_s, \mathcal{T}_3(x,y) e^{(s-1)G} a_z^* \xi \right\rangle.
\end{aligned}$$

Since $\mathcal{C}_N^*$ is the adjoint of $\mathcal{C}_N$, we obtain

$$\left\langle \xi, (\mathcal{C}_N + \mathcal{C}_N^*)\xi \right\rangle = 2\,\mathrm{Re}(S_1 + S_3) + \mathcal{E}_\Theta.$$

Now we estimate the term S_3. Using $N_0^{1/2} \leq CN^{1/2}$, the unitarity of e^{tG}, and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
|S_3| &\leq CN_0^{1/2} \int_0^1 ds \int dx dy\, |V_N(x-y)| \left\| e^{(1-s)G} \mathcal{T}_3^*(x,y) \Theta \xi_s \right\| \left\| \int dz\, \check{\sigma}(x-z)\, a_z^* \xi \right\| \\
&\leq CN^{1/2} \int_0^1 ds \left(\int dx dy\, |V_N(x-y)| \left\| \mathcal{T}_3^*(x,y) \Theta \xi_s \right\|^2 \right)^{1/2} \\
&\qquad \times \left(\int dx dy\, |V_N(x-y)| \left\| \int dz\, \check{\sigma}(x-z)\, a_z^* \xi \right\|^2 \right)^{1/2}.
\end{aligned} \tag{4.23}$$

For the second factor, we first observe that

$$\int dy\,|V_N(y)| = \int dy\,N^{2-2\kappa}|V(N^{1-\kappa}y)| = N^{2-2\kappa}N^{-3+3\kappa}\int d\widetilde{y}\,|V(\widetilde{y})| \leq CN^{-1+\kappa}\,,$$

Also, by using canonical commutation relation, we get

$$\begin{aligned}\int dx\left\|\int dz\,\check{\sigma}(x-z)\,a_z^*\xi\right\|^2 &= \int dx\int dzdz'\,\check{\sigma}(x-z)\check{\sigma}(x-z')\left\langle\xi,a_{z'}a_z^*\xi\right\rangle\\ &= \int dx\int dzdz'\,\check{\sigma}(x-z)\check{\sigma}(x-z')\left\langle\xi,a_z^*a_{z'}\xi\right\rangle + \int dxdz\,|\check{\sigma}(x-z)|^2\\ &= \langle\xi,\widetilde{N}\xi\rangle + \int dxdz|\check{\sigma}(x-z)|^2 \leq CN^{3\kappa/2}\end{aligned}$$

because

$$\int dxdz|\check{\sigma}(x-z)|^2 = \|\check{\sigma}\|_2^2 = \sum_{p\in\Lambda^*}|\sigma_p|^2 \leq CN^{3\kappa/2}\,.$$

Therefore, by using the bound

$$\int dxdy\,|V_N(x-y)|\left\|\mathcal{T}_3^*(x,y)\Theta\xi_s\right\|^2 \leq CN^{4\kappa-1-\beta+\varepsilon},$$

from [13, Equation (141)], we obtain, for every $\varepsilon>0$,

$$|S_3| \leq CN^{1/2}\int_0^1 ds\left(CN^{4\kappa-1-\beta+\varepsilon}\right)^{1/2}\left(CN^{3\kappa/2}N^{-1+\kappa}\right)^{1/2} \leq CN^{4\kappa-1}N^{-\beta/4+\varepsilon}.$$

We now turn to S_1. Notice that

$$\begin{aligned}S_1 &= N_0^{1/2}\int_0^1 ds\int dxdydzdz'\,V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)\\ &\qquad\qquad\times\left\langle\xi_s,a_{z'}e^{-(1-s)G}a_z^*\xi\right\rangle\\ &= N_0^{1/2}\int_0^1 ds\int dxdydzdz'\,V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)\\ &\qquad\qquad\times\left\langle\xi,e^{(1-s)G}a_{z'}e^{-(1-s)G}a_z^*\xi\right\rangle.\end{aligned}\tag{4.24}$$

Now define

$$g_{z'}(t) := e^{tG}a_{z'}e^{-tG},\qquad \forall\, t\in[0,1].$$

Then $g_{z'}(0)=a_{z'}$ and

$$\begin{aligned}g'_{z'}(t) &= e^{tG}[G,a_{z'}]e^{-tG} = e^{tG}\Big([A^*\Theta,a_{z'}]-[\Theta A,a_{z'}]\Big)e^{-tG}\\ &= e^{tG}\Big([A^*,a_{z'}]+[A^*,a_{z'}](\Theta-1)+A^*[\Theta,a_{z'}]-[\Theta,a_{z'}]A\Big)e^{-tG}\end{aligned}\tag{4.25}$$

The first term on the r.h.s. of (4.25) gives the main contribution. The remaining terms contain either $\Theta-1$ or a commutator with Θ, and hence are absorbed into $\mathcal{E}_\Theta$. Thus

$$\begin{aligned}g_{z'}(1-s) &= g_{z'}(0)+\int_0^{1-s}dt\,f'_{z'}(t)\\ &= a_{z'}+\int_0^{1-s}dt\,e^{tG}[A^*,a_{z'}]e^{-tG}+\mathcal{E}_\Theta.\end{aligned}\tag{4.26}$$

Substituting (4.26) into S_1, we obtain

$$S_1 = N_0^{1/2} \int dxdydzdz' \, V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)\big\langle \xi, a_{z'}a_z^*\xi\big\rangle$$
$$+ N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int dxdydzdz' \, V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)$$
$$\times \big\langle \xi, e^{tG}[A^*, a_{z'}]e^{-tG}a_z^*\xi\big\rangle + \mathcal{E}_\Theta. \tag{4.27}$$

For the second term on the r.h.s. of (4.27), we notice

$$\big\langle \xi, e^{tG}[A^*, a_{z'}]e^{-tG}a_z^*\xi\big\rangle = \big\langle \xi_{1-t}, [A^*, a_{z'}]e^{-tG}a_z^*e^{tG}\xi_{1-t}\big\rangle. \tag{4.28}$$

Then we expand $e^{-tG}a_z^*e^{tG}$. So we define, for $0 \le \tau \le t$

$$g_z(\tau) \coloneqq e^{-\tau G}a_z^*e^{\tau G}\,.$$

Then $g_z(0) = a_z^*$ and

$$g_z'(\tau) = e^{-\tau G}[a_z^*, G]e^{\tau G} = e^{-\tau G}[a_z^*, A^*\Theta - \Theta A]e^{\tau G}.$$

Similarly as before, we obtain

$$e^{-tG}a_z^*e^{tG} = a_z^* + \int_0^t d\tau \, e^{-\tau G}[A, a_z^*]e^{\tau G} + \mathcal{E}_\Theta. \tag{4.29}$$

Now substituting (4.29) into (4.28), and then back into (4.27), gives

$$S_1 = N_0^{1/2} \int dxdydz \, V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z-y,z-x)$$
$$+ N_0^{1/2} \int dxdydzdz' \, V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)\big\langle \xi, a_z^*a_{z'}\xi\big\rangle$$
$$+ N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int dxdydzdz' \, V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)$$
$$\times \big\langle \xi_{1-t}, [A^*, a_{z'}]a_z^*\xi_{1-t}\big\rangle$$
$$+ N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int_0^t d\tau \int dxdydzdz' \, V_N(x-y)\check{\sigma}(x-z)\check{A}^{(6)}(z'-y,z'-x)$$
$$\times \big\langle \xi_{1-t}, [A^*, a_{z'}]e^{-\tau G}[A, a_z^*]e^{\tau G}\xi_{1-t}\big\rangle + \mathcal{E}_\Theta\,,$$

where we use canonical commutation relation $a_{z'}a_z^* = a_z^*a_{z'} + \delta(z-z')$ for the first term on the r.h.s. of (4.27). By using the symmetry of the kernel of $\check{A}^{(6)}$ and change of variables, we can write

$$S_1 = S_{1,1} + S_{1,2} + S_{1,3} + S_{1,4} + \mathcal{E}_\Theta,$$

where

$$S_{1,1} \coloneqq N_0^{1/2} \int dxdy \, V_N(y)\check{\sigma}(x)\check{A}^{(6)}(y,x),$$
$$S_{1,2} \coloneqq N_0^{1/2} \int dxdydzdz' \, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z')\big\langle \xi, a_z^*a_{z'}\xi\big\rangle,$$
$$S_{1,3} \coloneqq N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int dxdydzdz' \, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z')$$
$$\times \big\langle \xi_{1-t}, [A^*, a_{z'}]a_z^*\xi_{1-t}\big\rangle,$$
$$S_{1,4} \coloneqq N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int_0^t d\tau \int dxdydzdz' \, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z')$$
$$\times \big\langle \xi_{1-t}, [A^*, a_{z'}]e^{-\tau G}[A, a_z^*]\xi_{1-t+\tau}\big\rangle.$$

Now we estimate the error terms $S_{1,2}$, $S_{1,3}$, and $S_{1,4}$. Recall that

$$\mathcal{N}_g \coloneqq \int dz\, dz'\, g(z-z') a_z^* a_{z'},$$
$$\ddot{\mathcal{N}}_g \coloneqq \int dz\, dz'\, g(z-z')[A^*, a_z][a_{z'}^*, A].$$

With this notation, we claim that

$$\int dx \left\| \int dz'\, \check{A}^{(6)}(y, x-z') a_{z'} \xi_s \right\|^2 = \left\| \mathcal{N}_{A_y \star A_y}^{1/2} \xi_s \right\|^2, \tag{4.30}$$
$$\int dx \left\| \int dz\, \check{A}^{(6)}(y, x-z)[a_z^*, A] \xi_s \right\|^2 = \left\| \ddot{\mathcal{N}}_{A_y \star A_y}^{1/2} \xi_s \right\|^2. \tag{4.31}$$

Indeed, for the first identity (4.30), we only have to change dummy variables. We compute

$$\begin{aligned}
&\int dx \left\| \int dz'\, A_y(x-z') a_{z'} \xi_s \right\|^2 \\
&= \int dx \left\langle \int dz\, A_y(x-z) a_z \xi_s,\ \int dz'\, A_y(x-z') a_{z'} \xi_s \right\rangle \\
&= \int dz\, dz' \left(\int dx\, A_y(x-z) A_y(x-z') \right) \langle \xi_s, a_z^* a_{z'} \xi_s \rangle \\
&= \int dz\, dz'\, (A_y \star A_y)(z-z') \langle \xi_s, a_z^* a_{z'} \xi_s \rangle \\
&= \langle \xi_s, \mathcal{N}_{A_y \star A_y} \xi_s \rangle = \left\| \mathcal{N}_{A_y \star A_y}^{1/2} \xi_s \right\|^2
\end{aligned}$$

For the second identity (4.31), we proceed similarly

$$\begin{aligned}
&\int dx \left\| \int dz\, A_y(x-z)[a_z^*, A] \xi_s \right\|^2 \\
&= \int dx\, dz\, dz'\, A_y(x-z) A_y(x-z') \langle \xi_s, [A^*, a_z][a_{z'}^*, A] \xi_s \rangle \\
&= \int dz\, dz'\, (A_y \star A_y)(z-z') \langle \xi_s, [A^*, a_z][a_{z'}^*, A] \xi_s \rangle \\
&= \langle \xi_s, \ddot{\mathcal{N}}_{A_y \star A_y} \xi_s \rangle = \left\| \ddot{\mathcal{N}}_{A_y \star A_y}^{1/2} \xi_s \right\|^2.
\end{aligned}$$

Now, it remains to estimate $\left\| \mathcal{N}_{A_y \star A_y}^{1/2} \xi_s \right\|^2$ and $\left\| \ddot{\mathcal{N}}_{A_y \star A_y}^{1/2} \xi_s \right\|^2$. We use a Duhamel expansion, expanding twice as in the estimate for the local number operator. We first compute the constant term. Let

$$g_y \coloneqq A_y \star A_y.$$

Then

$$[\mathcal{N}_{g_y}, A^*] = \int dz\, d\widetilde{z}\, g_y(z-\widetilde{z})[a_z^* a_{\widetilde{z}}, A^*] = \int dz\, d\widetilde{z}\, g_y(z-\widetilde{z}) a_z^* [a_{\widetilde{z}}, A^*],$$

because a_z^* commutes with A^*. Therefore

$$\begin{aligned}
[A, [\mathcal{N}_{g_y}, A^*]] &= \int dz\, d\widetilde{z}\, g_y(z-\widetilde{z}) [A, a_z^* [a_{\widetilde{z}}, A^*]] \\
&= \int dz\, d\widetilde{z}\, g_y(z-\widetilde{z}) \big([A, a_z^*][a_{\widetilde{z}}, A^*] + a_z^* [A, [a_{\widetilde{z}}, A^*]] \big).
\end{aligned} \tag{4.32}$$

When taking the vacuum expectation, the second term on the r.h.s. of (4.32) vanishes, because of the remaining creation operator a_z^*. Thus

$$\big\langle \Omega, [A, [\mathcal{N}_{g_y}, A^*]]\Omega\big\rangle = \int dz\, d\widetilde{z}\, g_y(z-\widetilde{z})\big\langle \Omega, [A, a_z^*][a_{\widetilde{z}}, A^*]\Omega\big\rangle. \tag{4.33}$$

We now compute the two commutators explicitly. We have

$$\begin{aligned}
[A, a_z^*] &= \int dx_1 dx_2 dx_3\, \check{A}(x_1-x_2, x_1-x_3)[a_{x_1}a_{x_2}a_{x_3}, a_z^*] \\
&= \int dx_1 dx_2\, \check{A}^{(3)}(x_1-x_2, x_1-z)a_{x_1}a_{x_2}, \\
[a_{\widetilde{z}}, A^*] &= \int dx_1 dx_2 dx_3\, \check{A}(x_1-x_2, x_1-x_3)[a_{\widetilde{z}}, a_{x_1}^* a_{x_2}^* a_{x_3}^*] \\
&= \int dx_1 dx_2\, \check{A}^{(3)}(x_1-x_2, x_1-\widetilde{z})a_{x_1}^* a_{x_2}^*.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&[A, a_z^*][a_{\widetilde{z}}, A^*] \\
&= \int dx_1 dx_2\, d\widetilde{x}_1 d\widetilde{x}_2\, \check{A}^{(3)}(x_1-x_2, x_1-z)\check{A}^{(3)}(\widetilde{x}_1-\widetilde{x}_2, \widetilde{x}_1-\widetilde{z})a_{x_1}a_{x_2}a_{\widetilde{x}_1}^* a_{\widetilde{x}_2}^*.
\end{aligned}$$

Taking the vacuum expectation and using the canonical commutation relations,

$$\big\langle \Omega, a_{x_1}a_{x_2}a_{\widetilde{x}_1}^* a_{\widetilde{x}_2}^*\Omega\big\rangle = \delta(x_1-\widetilde{x}_1)\delta(x_2-\widetilde{x}_2) + \delta(x_1-\widetilde{x}_2)\delta(x_2-\widetilde{x}_1),$$

we obtain

$$\begin{aligned}
&\big\langle \Omega, [A, a_z^*][a_{\widetilde{z}}, A^*]\Omega\big\rangle \\
&= \int dx_1 dx_2\, \check{A}^{(3)}(x_1-x_2, x_1-z)\Big[\check{A}^{(3)}(x_1-x_2, x_1-\widetilde{z}) + \check{A}^{(3)}(x_2-x_1, x_2-\widetilde{z})\Big] \\
&= \int dx_1 dx_2\, \check{A}^{(3)}(x_1-x_2, x_1-z)\check{A}^{(6)}(x_1-x_2, x_1-\widetilde{z}) \\
&= \int dx_1 dx_2\, \check{A}^{(3)}(x_1-x_2, x_1-\widetilde{z})\check{A}^{(6)}(z-x_1, z-x_2)
\end{aligned}$$

We first rewrite the kernel $g_y = A_y \star A_y$ in a more useful form. Recall that

$$A_y(x) = \check{A}^{(6)}(y, x), \qquad g_y(z-\widetilde{z}) = (A_y \star A_y)(z-\widetilde{z}).$$

Then, a simple change of variable gives

$$\begin{aligned}
g_y(z-\widetilde{z}) &= \int d\widetilde{y}\, A_y\big(\widetilde{y}-(z-\widetilde{z})\big)A_y(\widetilde{y}) = \int d\widetilde{y}\, \check{A}^{(6)}\big(y, \widetilde{y}-(z-\widetilde{z})\big)\check{A}^{(6)}(y, \widetilde{y}) \\
&= \int dz'\, \check{A}^{(6)}(y, z'-\widetilde{z})\check{A}^{(6)}(y, z'-z).
\end{aligned}$$

Therefore, the constant term in the double Duhamel expansion of $\big\|\mathcal{N}_{A_y\star A_y}^{1/2}\xi_s\big\|^2$ is

$$\begin{aligned}
\big\langle \Omega, [A, [\mathcal{N}_{A_y\star A_y}, A^*]]\Omega\big\rangle = \int dx_1 dx_2 dx_3 dz dz'\, &\check{A}^{(6)}(y, z'-z)\check{A}^{(6)}(y, z'-x_3) \\
&\times \check{A}^{(3)}(x_1-x_2, x_1-x_3)\check{A}^{(6)}(z-x_1, z-x_2).
\end{aligned} \tag{4.34}$$

We now switch to momentum space using

$$\check{A}^{(j)}(x,y) = \sum_{p,q\in\Lambda^*} A^{(j)}_{p,q} e^{ip\cdot x} e^{iq\cdot y}, \qquad j \in \{3,6\}.$$

Inserting this representation into (4.34), we obtain

$$\begin{aligned}
&\big\langle \Omega, [A, [\mathcal{N}_{A_y \star A_y}, A^*]]\Omega \big\rangle \\
&= \int dx_1 dx_2 dx_3 dz dz' \sum_{\substack{p_i,q_i\in\Lambda^* \\ i=1,\dots,4}} A^{(6)}_{p_1,q_1} A^{(6)}_{p_2,q_2} A^{(3)}_{p_3,q_3} A^{(6)}_{p_4,q_4} \\
&\qquad \times e^{ip_1\cdot y} e^{iq_1\cdot(z'-z)} e^{ip_2\cdot y} e^{iq_2\cdot(z'-x_3)} e^{ip_3\cdot(x_1-x_2)} e^{iq_3\cdot(x_1-x_3)} e^{ip_4\cdot(z-x_1)} e^{iq_4\cdot(z-x_2)} \\
&= \sum_{\substack{p_i,q_i\in\Lambda^* \\ i=1,\dots,4}} A^{(6)}_{p_1,q_1} A^{(6)}_{p_2,q_2} A^{(3)}_{p_3,q_3} A^{(6)}_{p_4,q_4} e^{i(p_1+p_2)\cdot y} \\
&\qquad \times \delta(p_3+q_3-p_4)\delta(-p_3-q_4)\delta(-q_2-q_3)\delta(-q_1+p_4+q_4)\delta(q_1+q_2)\,. \qquad (4.35)
\end{aligned}$$

Plug in the constraints $q_2 = -q_1, q_3 = q_1, q_4 = -p_3,$ and $p_4 = p_3 + q_3$, and we find

$$\begin{aligned}
\big\langle \Omega, [A, [\mathcal{N}_{A_y \star A_y}, A^*]]\Omega \big\rangle &= \sum_{p,q,r,s\in\Lambda^*} A^{(6)}_{r,q} A^{(6)}_{s,-q} A^{(3)}_{p,q} A^{(6)}_{p+q,-p} e^{i(r+s)\cdot y} \\
&= \sum_{p,q,r,s\in\Lambda^*} A^{(6)}_{r,q} A^{(6)}_{s,-q} A^{(3)}_{p,q} A^{(6)}_{p,q} e^{i(r+s)\cdot y}
\end{aligned}$$

where we used the symmetry of the kernel $A^{(6)}$ to rewrite $A^{(6)}_{p+q,-p} = A^{(6)}_{p,q}$. Now,

$$\begin{aligned}
\Big|\big\langle \Omega, [A, [\mathcal{N}_{A_y \star A_y}, A^*]]\Omega \big\rangle\Big| &\le \sum_{p,q,r,s\in\Lambda^*} |A^{(6)}_{r,q}||A^{(6)}_{s,-q}||A^{(3)}_{p,q}||A^{(6)}_{p,q}| \\
&\le CN \sum_{p,q\in\Lambda^*} (1+|\sigma_q|)^2 |A^{(3)}_{p,q}||A^{(6)}_{p,q}| \\
&\le CN \Big(\sum_{p,q\in\Lambda^*} (1+|\sigma_q|)^2 |A^{(3)}_{p,q}|^2 \Big)^{1/2} \Big(\sum_{p,q\in\Lambda^*} (1+|\sigma_q|)^2 |A^{(6)}_{p,q}|^2 \Big)^{1/2} \\
&\le CN \sum_{j=3,6} \sum_{p,q\in\Lambda^*} (1+2|\sigma_q|+|\sigma_q|^2) |A^{(j)}_{p,q}|^2 \le CN^{3\kappa}\,, \qquad (4.36)
\end{aligned}$$

by Cauchy–Schwarz inequality and results from [13, Lemma 9].

Now, we define

$$D^{(6)}(x) := \sup_{y\in\Lambda} |\check{A}^{(6)}(y,x)| = \sup_{y\in\Lambda} |A_y(x)|.$$

For $u \in \Lambda_B$, set

$$D^{(6)}_u(x) := D^{(6)}(x) \mathbb{1}_{B_u}(x).$$

Then, we have

$$\begin{aligned}\sup_{y\in\Lambda}\sum_{u\in\Lambda_B}\|(A_y)_u\|_2 &= \sup_{y\in\Lambda}\sum_{u\in\Lambda_B}\left(\int|(A_y)_u(x)|^2\,dx\right)^{1/2}\\ &\le \sum_{u\in\Lambda_B}\left(\int_{B_u}\sup_{y\in\Lambda}|\check{A}^{(6)}(y,x)|^2\,dx\right)^{1/2}\\ &= \sum_{u\in\Lambda_B}\left(\int_{B_u}|D^{(6)}(x)|^2\,dx\right)^{1/2}\\ &= \sum_{u\in\Lambda_B}\|D_u^{(6)}\|_2 \le CN^{3\kappa/4+1/2+\varepsilon}\,,\end{aligned}\tag{4.37}$$

where we used [13, the equation after (81)].

By [13, Lemma 19], together with (4.36) and (4.37), we obtain, for every $\varepsilon>0$,

$$\sup_{y\in\Lambda}\left\|\mathcal{N}_{A_y\star A_y}^{1/2}\xi_s\right\|^2 \le CN^{3\kappa}+CN^{-5\beta/8+\varepsilon}\left(N^{3\kappa/4+1/2+\varepsilon}\right)^2 \le CN^{3\kappa}\big(1+N^{-\beta/8+\varepsilon}\big).$$

Moreover, by [13, the remark after Lemma 19], we have, for every $\varepsilon>0$,

$$\sup_{y\in\Lambda}\left\|\ddot{\mathcal{N}}_{A_y\star A_y}^{1/2}\xi_s\right\|^2 \le CN^{3\kappa-\beta/8+\varepsilon}.\tag{4.38}$$

Choosing $\varepsilon>0$ sufficiently small, we conclude that

$$\begin{aligned}&\sup_{y\in\Lambda}\left\|\mathcal{N}_{A_y\star A_y}^{1/2}\xi_s\right\| \le CN^{3\kappa/2},\\ &\sup_{y\in\Lambda}\left\|\ddot{\mathcal{N}}_{A_y\star A_y}^{1/2}\xi_s\right\| \le CN^{3\kappa/2}, \qquad \forall\, s\in[0,1].\end{aligned}\tag{4.39}$$

Observe that

$$\int dx\left\|\int dz\,\check{\sigma}(x-z)a_z\xi_s\right\|^2 = \left\|\mathcal{N}_{\check{\sigma}\star\check{\sigma}}^{1/2}\xi_s\right\|^2 = \left\|\widetilde{N}^{1/2}\xi_s\right\|^2 \le CN^{3\kappa-1},\tag{4.40}$$

$$\int dx\left\|\int dz\,\check{\sigma}(x-z)[a_z^*,A]\xi_s\right\|^2 = \left\|\ddot{\mathcal{N}}_{\check{\sigma}\star\check{\sigma}}^{1/2}\xi_s\right\|^2 \le CN^{-5\beta/8+\varepsilon}\left(\sum_{u\in\Lambda_B}\|\check{\sigma}_u\|_2\right)^2 \le CN^{3\kappa-1}.\tag{4.41}$$

where we used [13, Lemma 19 and Lemma 6] for the first and the second line respectively.

We now combine the estimates above with $\|V_N\|_1=\int dy\,|V_N(y)|\le CN^{-1+\kappa}$ to get

$$\begin{aligned}|S_{1,2}| &\le N_0^{1/2}\int dxdy\,|V_N(y)|\left|\left\langle\int dz\,\check{\sigma}(x-z)a_z\xi,\ \int dz'\,\check{A}^{(6)}(y,x-z')a_{z'}\xi\right\rangle\right|\\ &\le CN^{1/2}\int dy\,|V_N(y)|\int dx\left\|\int dz\,\check{\sigma}(x-z)a_z\xi\right\|\left\|\int dz'\,\check{A}^{(6)}(y,x-z')a_{z'}\xi\right\|\\ &\le CN^{1/2}\|V_N\|_1\left(\int dx\left\|\int dz\,\check{\sigma}(x-z)a_z\xi\right\|^2\right)^{1/2}\\ &\qquad\times\sup_{y\in\Lambda}\left(\int dx\left\|\int dz'\,\check{A}^{(6)}(y,x-z')a_{z'}\xi\right\|^2\right)^{1/2}\\ &\le CN^{1/2}N^{-1+\kappa}N^{(3\kappa-1)/2}N^{3\kappa/2} \le CN^{4\kappa-1}\end{aligned}$$

The term $S_{1,4}$ is estimated in the same spirit. Using Cauchy-Schwarz, we find

$$\begin{aligned}|S_{1,4}| &\leq CN^{1/2}\|V_N\|_1 \int_0^1 ds \int_0^{1-s} dt \int_0^t d\tau \sup_{y\in\Lambda} \big\|\ddot{\mathcal{N}}_{A_y\star A_y}^{1/2}\xi_{1-t}\big\| \, \big\|\ddot{\mathcal{N}}_{\check{\sigma}\star\check{\sigma}}^{1/2}\xi_{1-t+\tau}\big\|. \\ &\leq CN^{1/2}N^{-1+\kappa}N^{3\kappa/2}N^{(3\kappa-1)/2} \leq CN^{4\kappa-1}\,.\end{aligned}$$

by (4.39) and (4.41).

It remains to treat $S_{1,3}$. We expand this term once more. For fixed z, z', define

$$g_{z,z'}(\tau) \coloneqq e^{-\tau G}[A^*, a_{z'}]a_z^* e^{\tau G}, \qquad 0 \leq \tau \leq 1-t.$$

Then $g_{z,z'}(0) = [A^*, a_{z'}]a_z^*$ and

$$g'_{z,z'}(\tau) = e^{-\tau G}\big[[A^*, a_{z'}]a_z^*, G\big]e^{\tau G}.$$

Since $G = A^*\Theta - \Theta A$, the main term in this commutator comes from the part $-\Theta A$ and the terms containing $\Theta - 1$ or commutators with Θ are again absorbed into the cutoff error $\mathcal{E}_\Theta$. Consequently, up to $\mathcal{E}_\Theta$

$$\begin{aligned}g_{z,z'}(1-t) &= g_{z,z'}(0) + \int_0^{1-t} d\tau\, g'_{z,z'}(\tau) \\ &= [A^*, a_{z'}]a_z^* + \int_0^{1-t} d\tau\, e^{-\tau G}\big[A, [A^*, a_{z'}]a_z^*\big]e^{\tau G}. \end{aligned} \tag{4.42}$$

Taking vacuum expectation, we have

$$\big\langle \xi_{1-t}, [A^*, a_{z'}]a_z^*\xi_{1-t}\big\rangle = \big\langle \Omega, g_{z,z'}(1-t)\Omega\big\rangle.$$

Substituting (4.42), the constant term vanishes:

$$\big\langle \Omega, [A^*, a_{z'}]a_z^*\Omega\big\rangle = 0,$$

because $a_z^*\Omega$ is a one-particle vector, while $[A^*, a_{z'}]$ contains two creation operators. Hence

$$\begin{aligned}S_{1,3} = N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int_0^{1-t} d\tau \int dxdydzdz'\, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y, x-z') \\ \times \big\langle \xi_\tau, \big[A, [A^*, a_{z'}]a_z^*\big]\xi_\tau\big\rangle + \mathcal{E}_\Theta.\end{aligned}$$

Notice that

$$\begin{aligned}[A^*, a_{z'}]a_z^* &= \int dx_1dx_2dx_3\, \check{A}^{(3)}(x_1-x_2, x_1-x_3)[a_{x_1}^*a_{x_2}^*a_{x_3}^*, a_{z'}]a_z^* \\ &= \int dx_1dx_2\, \check{A}^{(3)}(x_1-x_2, x_1-z')a_z^*a_{x_1}^*a_{x_2}^*.\end{aligned}$$

We get

$$\begin{aligned}S_{1,3} = N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int_0^{1-t} d\tau \int dxdydzdz'\, dx_1dx_2\, V_N(y)\check{\sigma}(x-z) \\ \times \check{A}^{(6)}(y, x-z')\check{A}^{(3)}(x_1-x_2, x_1-z')\big\langle \xi_\tau, [A, a_z^*a_{x_1}^*a_{x_2}^*]\xi_\tau\big\rangle.\end{aligned}$$

Since this is of similar form as [13, Equation (127)], if we choose

$$g(z) \coloneqq N_0^{1/2} \int dxdy\, V_N(y)\check{\sigma}(x)\check{A}^{(6)}(y, x-z).$$

Therefore, we can split

$$S_{1,3} = T_0 + T_2 + T_{4,1} + T_{4,2} + \mathcal{E}_\Theta$$

where

$$\begin{aligned}
T_0 &\coloneqq \frac{N_0^{1/2}}{3} \int dxdydzdz'\, dx_1dx_2\, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z') \\
&\qquad \times \check{A}^{(6)}(x_1-x_2,x_1-z')\check{A}^{(3)}(x_1-x_2,x_1-z). \\
T_2 &\coloneqq -N_0^{1/2}\int_0^1 ds\int_0^{1-s} dt\int_0^{1-t} d\tau \int dxdydzdz'\, dx_1dx_2dx_3\, V_N(y)\check{\sigma}(x-z) \\
&\qquad \times \check{A}^{(6)}(y,x-z')\check{A}^{(6)}(x_1-x_2,x_1-z')\check{A}^{(6)}(x_3-x_2,x_3-z)\big\langle \xi_\tau, a^*_{x_1}a_{x_3}\xi_\tau\big\rangle \\
&\quad + N_0^{1/2}\int_0^1 ds\int_0^{1-s} dt\int_0^{1-t} d\tau \int dxdydzdz'\, dx_1dx_2dx_3\, V_N(y)\check{\sigma}(x-z) \\
&\qquad \times \check{A}^{(6)}(y,x-z')\check{A}^{(3)}(x_1-x_2,x_1-z')\check{A}^{(6)}(x_3-x_1,x_3-x_2)\big\langle \xi_\tau, a^*_{z}a_{x_3}\xi_\tau\big\rangle \\
T_{4,1} &\coloneqq N_0^{1/2}\int_0^1 ds\int_0^{1-s} dt\int_0^{t} d\tau \int dxdydzdz'\, dx_1\cdots dx_4\, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z') \\
&\qquad \times \check{A}^{(6)}(x_1-x_2,x_1-z')\check{A}^{(3)}(x_3-x_4,x_3-x_1)\big\langle \xi_\tau, a^*_{z}a^*_{x_2}a_{x_3}a_{x_4}\xi_\tau\big\rangle \qquad (4.43) \\
T_{4,2} &\coloneqq N_0^{1/2}\int_0^1 ds\int_0^{1-s} dt\int_0^{1-t} d\tau \int dxdydzdz'\, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z') \\
&\qquad \times \big\langle \xi_\tau, a^*_{x_1}a^*_{x_2}a_{x_3}a_{x_4}\xi_\tau\big\rangle.
\end{aligned}$$

We now switch T_0 and T_2 to momentum space. We get

$$\begin{aligned}
T_0 &= \frac{N_0^{1/2}}{3}\int dxdydzdz'\, dx_1dx_2 \sum_{r,k\in\Lambda^*} \widehat{V}_N(r)\sigma_k e^{ir\cdot y}e^{ik\cdot(x-z)} \sum_{\substack{p_i,q_i\in\Lambda^*\\ i=1,2,3}} A^{(6)}_{p_1,q_1}A^{(6)}_{p_2,q_2}A^{(3)}_{p_3,q_3} \\
&\quad \times e^{ip_1\cdot y}e^{iq_1\cdot(x-z')}e^{ip_2\cdot(x_1-x_2)}e^{iq_2\cdot(x_1-z')}e^{ip_3\cdot(x_1-x_2)}e^{iq_3\cdot(x_1-z)}. \\
&= \frac{N_0^{1/2}}{3}\sum_{r,k,p\in\Lambda^*}\widehat{V}_N(r)\sigma_k A^{(6)}_{r,k}A^{(3)}_{p,k}A^{(6)}_{p,k},
\end{aligned}$$

where we properly change the variables. Notice that

$$\|\widehat{V}_N\|_\infty = \sup_{r\in\Lambda^*}\left|\int dx\, V_N(x)e^{-ir\cdot x}\right| \le \int dx\,|V_N(x)| = \|V_N\|_1,$$

and $\|V_N\|_1 \le CN^{-1+\kappa}$. Then

$$\begin{aligned}
|T_0| &\le CN^{1/2}\|\widehat{V}_N\|_\infty \sum_{r,k,p\in\Lambda^*}|\sigma_k||A^{(6)}_{r,k}||A^{(3)}_{p,k}||A^{(6)}_{p,k}| \\
&\le CN^{1/2}\|V_N\|_1 \sum_{p,k\in\Lambda^*}|\sigma_k|\Big(\sum_{r\in\Lambda^*}|A^{(6)}_{r,k}|\Big)|A^{(3)}_{p,k}||A^{(6)}_{p,k}| \\
&\le CN^{1/2}N^{-1+\kappa}\sum_{p,k\in\Lambda^*}|\sigma_k|N^{1/2}(1+|\sigma_k|)|A^{(3)}_{p,k}||A^{(6)}_{p,k}| \\
&\le CN^{\kappa}\sum_{p,k\in\Lambda^*}\big(1+3|\sigma_k|^2\big)|A^{(3)}_{p,k}||A^{(6)}_{p,k}|.
\end{aligned}$$

Using Cauchy-Schwarz and the bounds of [13, Lemma 9],

$$\begin{aligned}|T_0| &\leq CN^{\kappa}\Big[\Big(\sum_{p,k\in\Lambda^*}|A^{(3)}_{p,k}|^2\Big)^{1/2}\Big(\sum_{p,k\in\Lambda^*}|A^{(6)}_{p,k}|^2\Big)^{1/2}\\ &\qquad + 3\Big(\sum_{p,k\in\Lambda^*}\sigma_k^2|A^{(3)}_{p,k}|^2\Big)^{1/2}\Big(\sum_{p,k\in\Lambda^*}\sigma_k^2|A^{(6)}_{p,k}|^2\Big)^{1/2}\Big]\\ &\leq CN^{\kappa}\big(CN^{3\kappa-1}+3CN^{3\kappa-1}\big)\leq CN^{4\kappa-1}. \end{aligned} \tag{4.44}$$

We now turn to T_2. First we simplify the time integral. The region of integration is

$$0\leq s\leq 1,\qquad 0\leq t\leq 1-s,\qquad 0\leq\tau\leq 1-t.$$

Since the integrand in T_2 depends only on τ, we fix τ. Then

$$0\leq\tau\leq 1,\qquad 0\leq t\leq 1-\tau,\qquad 0\leq s\leq 1-t.$$

Therefore, for any function F,

$$\begin{aligned}\int_0^1 ds\int_0^{1-s}dt\int_0^{1-t}d\tau\,F(\tau) &= \int_0^1 d\tau\,F(\tau)\int_0^{1-\tau}dt\int_0^{1-t}ds\\ &= \int_0^1 d\tau\,F(\tau)\int_0^{1-\tau}dt\,(1-t)\\ &= \frac{1}{2}\int_0^1 d\tau\,(1-\tau^2)F(\tau)\,.\end{aligned}$$

Using the Fourier representations, we obtain

$$T_2=\frac{N_0^{1/2}}{2}\int_0^1 d\tau\,(1-\tau^2)\sum_{r,k,p\in\Lambda^*}\widehat{V}_N(r)\sigma_k A^{(6)}_{r,k}A^{(3)}_{p,k}A^{(6)}_{p,k}\big\langle\xi_\tau,\big(a^*_{p+k}a_{p+k}+a^*_pa_p+a^*_ka_k\big)\xi_\tau\big\rangle.$$

Here we used the symmetry relations

$$A^{(6)}_{p,k}=A^{(6)}_{k,p}=A^{(6)}_{-p-k,k}=A^{(6)}_{p,-p-k},$$

together with

$$A^{(j)}_{-p,-q}=A^{(j)}_{p,q},\qquad j=3,6.$$

By $\|\widehat{V}_N\|_\infty\leq\|V_N\|_1$ and [13, Lemma 9], we have

$$\begin{aligned}|T_2| &\leq CN^{1/2}\int_0^1 d\tau\,(1-\tau^2)\|\widehat{V}_N\|_\infty\sum_{r,k,p\in\Lambda^*}|\sigma_k||A^{(6)}_{r,k}||A^{(3)}_{p,k}||A^{(6)}_{p,k}|\\ &\qquad\times\big\langle\xi_\tau,\big(a^*_{p+k}a_{p+k}+a^*_pa_p+a^*_ka_k\big)\xi_\tau\big\rangle\\ &\leq CN^{\kappa}\int_0^1 d\tau\,(1-\tau^2)\sum_{p,k\in\Lambda^*}|\sigma_k|(1+|\sigma_k|)|A^{(3)}_{p,k}||A^{(6)}_{p,k}|\\ &\qquad\times\big\langle\xi_\tau,\big(a^*_{p+k}a_{p+k}+a^*_pa_p+a^*_ka_k\big)\xi_\tau\big\rangle.\end{aligned}$$

Since $|\sigma_k|(1+|\sigma_k|) \le C\|\sigma\|_\infty(1+|\sigma_k|)$, we can bound, for example,

$$\begin{aligned}
&\sum_{p,k\in\Lambda^*} |\sigma_k|(1+|\sigma_k|)|A^{(3)}_{p,k}||A^{(6)}_{p,k}|\langle \xi_\tau, a^*_{p+k}a_{p+k}\xi_\tau\rangle \\
&\le C\|\sigma\|_\infty \Big(\sum_{p,k\in\Lambda^*} (1+|\sigma_k|)^2 |A^{(3)}_{p,k}|^2 \langle \xi_\tau, a^*_{p+k}a_{p+k}\xi_\tau\rangle\Big)^{1/2} \\
&\qquad \times \Big(\sum_{p,k\in\Lambda^*} |A^{(6)}_{p,k}|^2 \langle \xi_\tau, a^*_{p+k}a_{p+k}\xi_\tau\rangle\Big)^{1/2}. \qquad (4.45)
\end{aligned}$$

The same argument applied to the $a^*_p a_p$- and $a^*_k a_k$-terms. Using the momentum-space bounds from [13, Lemma 9], we get

$$\begin{aligned}
\sum_{p,k\in\Lambda^*} |A^{(6)}_{p,k}|^2 \langle \xi_\tau, a^*_{p+k}a_{p+k}\xi_\tau\rangle &\le CN^{3\kappa/2-1} \sum_{\widetilde{p}\in\Lambda^*} (1+\sigma^2_{\widetilde{p}}) \langle \xi_\tau, a^*_{\widetilde{p}}a_{\widetilde{p}}\xi_\tau\rangle \\
&\le CN^{3\kappa/2-1}\langle \xi_\tau, (\mathcal{N}+\widetilde{\mathcal{N}})\xi_\tau\rangle \le CN^{3\kappa/2-1}N^{3\kappa-1} \\
&= CN^{9\kappa/2-2}.
\end{aligned}$$

Similarly,

$$\sum_{p,k\in\Lambda^*} |A^{(3)}_{p,k}|^2 \langle \xi_\tau, a^*_k a_k\xi_\tau\rangle \le CN^{3\kappa/2-1} \sum_{k\in\Lambda^*} (1+\sigma^2_k)\langle \xi_\tau, a^*_k a_k\xi_\tau\rangle \le CN^{9\kappa/2-2}.$$

The same estimate holds for the term with $a^*_p a_p$. Therefore, we obtain

$$|T_2| \le CN^\kappa \int_0^1 d\tau\, (1-\tau^2)\, N^{\beta/4} CN^{9\kappa/2-2} \le CN^{4\kappa-1-\beta/4}. \qquad (4.46)$$

Here we used $\beta = 2-3\kappa$ in the last step.

To estimate $T_{4,1}$. We set

$$D^{(6)}(x) := \sup_{y\in\Lambda} |\check{A}^{(6)}(y,x)|.$$

Starting from the position-space expression (4.43), inserting the cutoff Θ, and applying Cauchy–Schwarz, we obtain

$$\begin{aligned}
|T_{4,1}| \le CN^{1/2} \int_0^1 d\tau\,(1-\tau^2) \int dxdydzdz'\, dx_1\cdots dx_4\, |V_N(y)|\, D^{(6)}(x-z') \\
\times |\check{A}^{(6)}(x_1-x_2, x_1-z')|\, |\check{A}^{(3)}(x_3-x_4, x_3-x_1)| \\
\times \Big\| \int dz\, \check{\sigma}(x-z)\, a_z a_{x_2}\Theta\xi_\tau \Big\| \, \big\| a_{x_3}a_{x_4}\Theta\xi_\tau\big\|. \qquad (4.47)
\end{aligned}$$

We now localize the kernels $D^{(6)}$, $\check{A}^{(6)}$, and $\check{A}^{(3)}$ in boxes of side length ℓ_B. For $w,u,v\in\Lambda_B$, define

$$D^{(6)}_w(x) := D^{(6)}(x)\mathbb{1}_{B_w}(x),$$

and

$$\check{A}^{(j)}_{u,v}(x,y) := \check{A}^{(j)}(x,y)\mathbb{1}_{B_u}(x)\mathbb{1}_{B_v}(y), \qquad j=3,6.$$

Then (4.47) can be estimated by

$$
\begin{aligned}
|T_{4,1}| \leq CN^{1/2}\|V_N\|_1 \int_0^1 d\tau\,(1-\tau^2) \sum_{\substack{w,u_1,v_1\\ u_2,v_2\in\Lambda_B}} \\
\times \Bigg(\int dx\,dz'\,dx_1\cdots dx_4\,|D_w^{(6)}(x-z')|^2 |\check{A}^{(3)}_{u_1,v_1}(x_3-x_4,x_3-x_1)|^2 \\
\times \mathbb{1}(|x_1-x_2-u_2|,|x_1-z'-v_2|\leq k\ell_B) \Big\| \int dz\,\check{\sigma}(x-z) a_z a_{x_2} \Theta\xi_\tau \Big\|^2 \Bigg)^{1/2} \\
\times \Bigg(\int dx\,dz'\,dx_1\cdots dx_4\,|\check{A}^{(6)}_{u_2,v_2}(x_1-x_2,x_1-z')|^2 \\
\times \mathbb{1}(|x_3-x_4-u_1|,|x_3-x_1-v_1|,|x-z'-w|\leq k\ell_B) \big\| a_{x_3} a_{x_4} \Theta\xi_\tau \big\|^2 \Bigg)^{1/2}.
\end{aligned}
$$

Using the cutoff estimate $\Theta\mathcal{N}_{\ell_B}(x)\Theta \leq CN^{\varepsilon}$ and applying [13, Lemma 8 and Lemma 9], we get

$$
\begin{aligned}
|T_{4,1}| &\leq CN^{\varepsilon}N^{1/2}\|V_N\|_1 \ell_B^3 \sum_{w\in\Lambda_B} \|D_w^{(6)}\|_2 \sum_{u,v\in\Lambda_B} \|\check{A}^{(3)}_{u,v}\|_2 \sum_{u,v\in\Lambda_B} \|\check{A}^{(6)}_{u,v}\|_2 \big\langle \xi_\tau, \widetilde{\mathcal{N}}\xi_\tau \big\rangle^{1/2} \\
&\leq CN^{\varepsilon}N^{1/2}N^{-1+\kappa}\ell_B^3 N^{3\kappa/4+1/2+\varepsilon} N^{3\kappa/2-1+\varepsilon} N^{3\kappa/2-1+\varepsilon} N^{(3\kappa-1)/2} \\
&\leq CN^{4\kappa-1-\beta/8+\varepsilon}. \qquad (4.48)
\end{aligned}
$$

In particular, $|T_{4,1}| \leq CN^{4\kappa-1}$ after choosing $\varepsilon > 0$ sufficiently small.

For $T_{4,2}$, we notice that

$$
\begin{aligned}
T_{4,2} = N_0^{1/2} \int_0^1 ds \int_0^{1-s} dt \int_0^{1-t} d\tau \int dx dy dz dz'\, V_N(y)\check{\sigma}(x-z)\check{A}^{(6)}(y,x-z') \\
\times \big\langle \xi_\tau, [A^*, a_{z'}][A, a_z^*]\xi_\tau \big\rangle,
\end{aligned}
$$

Then, by Cauchy-Schwarz,

$$
\begin{aligned}
|T_{4,2}| \leq CN^{1/2} \int dy\,|V_N(y)| \int_0^1 ds \int_0^{1-s} dt \int_0^{1-t} d\tau \int dx \\
\times \Big\| \int dz'\,\check{A}^{(6)}(y,x-z')[a_{z'}^*, A]\xi_\tau \Big\| \Big\| \int dz\,\check{\sigma}(x-z)[A, a_z^*]\xi_\tau \Big\|.
\end{aligned}
$$

Hence, using the bounds already obtained for $\ddot{\mathcal{N}}_{A_y\star A_y}$ and $\ddot{\mathcal{N}}_{\check{\sigma}\star\check{\sigma}}$,

$$
\begin{aligned}
|T_{4,2}| &\leq CN^{1/2}\|V_N\|_1 \sup_{y\in\Lambda} \big\|\ddot{\mathcal{N}}^{1/2}_{A_y\star A_y}\xi_\tau\big\| \, \big\|\ddot{\mathcal{N}}^{1/2}_{\check{\sigma}\star\check{\sigma}}\xi_\tau\big\| \\
&\leq CN^{1/2}N^{-1+\kappa}N^{3\kappa/2}N^{(3\kappa-1)/2} \leq CN^{4\kappa-1}. \qquad (4.49)
\end{aligned}
$$

Combining (4.44), (4.46), (4.48), and (4.49), we obtain

$$
|S_{1,3}| \leq CN^{4\kappa-1}.
$$

Together with the estimates of $S_{1,2}$ and $S_{1,4}$, this gives

$$
|S_1| \leq CN^{4\kappa-1},
$$

which concludes the proof. □

4.2 Bound for the quartic operator $\mathcal{V}_N$

The following lemma from [13, Lemma 25] gives an upper bound of the potential energy operator $\mathcal{V}_N$.

Lemma 16. *From (4.14), recall the operator*

$$\mathcal{V}_N = \frac{1}{2}\int dxdyN^{2-2\kappa}V(N^{1-\kappa}(x-y))a_x^*a_y^*a_xa_y\,. \tag{4.50}$$

We have

$$\begin{aligned}\left\langle e^{A^*\Theta-\Theta A}\Omega,\, \mathcal{V}_N e^{A^*\Theta-\Theta A}\Omega\right\rangle \\ = \frac{1}{2}\int dxdyN^{2-2\kappa}V(N^{1-\kappa}x)\check{A}^{(6)}(x,y)^2 + O(N^{4\kappa-1})\,.\end{aligned} \tag{4.51}$$

Proof of Lemma 16. Let $G = A^*\Theta - \Theta A$ and $\xi_s = e^{sG}\Omega$ for $s \in [0,t]$ and $t \in [0,1]$. Define

$$f_1(s) = e^{-sG}\mathcal{V}_N e^{sG}, \qquad \forall s \in [0,t], t \in [0,1].$$

We first write the expectation of the potential energy in Duhamel form.

$$\begin{aligned}f_1'(s) &= e^{-sG}[\mathcal{V}_N, G]e^{sG} = e^{-sG}[\mathcal{V}_N, A^*\Theta - \Theta A]e^{sG}\\ &= e^{-sG}[\mathcal{V}_N, A^*\Theta]e^{sG} - e^{-sG}[\mathcal{V}_N, \Theta A]e^{sG}\\ &= e^{-sG}\big([\mathcal{V}_N, A^*]\Theta + A^*[\mathcal{V}_N, \Theta]\big)e^{sG} - e^{-sG}\big([\mathcal{V}_N, \Theta]A + \Theta[\mathcal{V}_N, A]\big)e^{sG}\end{aligned} \tag{4.52}$$

Since $\mathcal{V}_N$ and Θ are both multiplication operators in position space on each fixed particle-number sector, we have $[\mathcal{V}_N, \Theta] = 0$. Hence (4.52) becomes

$$\begin{aligned}f_1'(s) &= e^{-sG}\Big([\mathcal{V}_N, A^*]\Theta - \Theta[\mathcal{V}_N, A]\Big)e^{sG}\\ &= e^{-sG}\Big([\mathcal{V}_N, A^*] + [A, \mathcal{V}_N]\Big)e^{sG} + \mathcal{E}_\Theta(s)\\ &= e^{-sG}\Big([\mathcal{V}_N, A^*] + \text{h.c.}\Big)e^{sG} + \mathcal{E}_\Theta(s),\end{aligned} \tag{4.53}$$

where the cutoff error is given by

$$\mathcal{E}_\Theta(s) = e^{-sG}\Big([\mathcal{V}_N, A^*](\Theta - 1) + (\Theta - 1)[A, \mathcal{V}_N]\Big)e^{sG}.$$

Therefore, using the fundamental theorem of calculus and the fact that $\mathcal{V}_N\Omega = 0$, we obtain

$$\begin{aligned}\langle \xi_t, \mathcal{V}_N\xi_t\rangle &= \langle \Omega, f_1(t)\Omega\rangle\\ &= \left\langle \Omega, \left(f_1(0) + \int_0^t f_1'(s)\,ds\right)\Omega\right\rangle\\ &= \int_0^t ds\,\left\langle \xi_s, \Big([\mathcal{V}_N, A^*] + \text{h.c.}\Big)\xi_s\right\rangle + \mathcal{E}_\Theta.\end{aligned} \tag{4.54}$$

Here $\mathcal{E}_\Theta$ denotes all contributions containing a factor $1-\Theta$. By [13, Lemma 11] and by the same argument used for, the cutoff errors are negligible. More precisely, if $\alpha > 0$ is fixed and then $n \in \mathbb{N}$ is chosen large enough in the definition of Θ, we get $\pm\mathcal{E}_\Theta \leq CN^{4\kappa-1}$. We now compute the commutator appearing in (4.54). Recall that

$$V_N(x) = N^{2-2\kappa}V(N^{1-\kappa}x),$$

so that

$$\mathcal{V}_N = \frac{1}{2}\int dx\, dy\, V_N(x-y) a_x^* a_y^* a_x a_y.$$

Using the position-space representation of A^*, we have

$$[\mathcal{V}_N, A^*] = \frac{1}{2}\int dx\, dy\, dx_1\, dx_2\, dx_3\, V_N(x-y)\check{A}(x_1-x_2, x_1-x_3)[a_x^* a_y^* a_x a_y, a_{x_1}^* a_{x_2}^* a_{x_3}^*].$$

A simple application of the canonical commutation relation gives

$$\begin{aligned}[a_x^* a_y^* a_x a_y, a_{x_1}^* a_{x_2}^* a_{x_3}^*] &= a_x^* a_y^* [a_x a_y, a_{x_1}^* a_{x_2}^* a_{x_3}^*] \\ &= a_x^* a_y^* \left(a_x [a_y, a_{x_1}^* a_{x_2}^* a_{x_3}^*] + [a_x, a_{x_1}^* a_{x_2}^* a_{x_3}^*] a_y \right).\end{aligned}$$

By the canonical commutation relation again and by the symmetry of the kernels of $\check{A}^{(6)}$ and $\check{A}^{(3)}$, we obtain

$$\begin{aligned}[\mathcal{V}_N, A^*] + \text{h.c.} &= \frac{1}{2}\int dx\, dy\, dz\, V_N(x-y)\check{A}^{(6)}(x-y, x-z) a_x^* a_y^* a_z^* + \text{h.c.} \\ &\quad + \int dx\, dy\, dz_1\, dz_2\, V_N(x-y)\check{A}^{(3)}(z_1-z_2, z_1-x) a_x^* a_y^* a_{z_1}^* a_{z_2}^* a_y + \text{h.c.} \\ &=: T_3 + T_5.\end{aligned}$$

We first estimate the term T_5. Introduce

$$\mathcal{W}_N := \frac{1}{2}\int dx\, dy\, |V_N(x-y)| a_x^* a_y^* a_x a_y.$$

When estimating $\langle \xi_s, T_5 \xi_s \rangle$, we insert the cutoff Θ with a correction which is again absorbed by $\mathcal{E}_\Theta$. Then, we can write

$$\begin{aligned}|\langle \xi_s, T_5 \xi_s \rangle| &\leq C \int dx\, dy\, dz_1\, dz_2\, |V_N(x-y)| |\check{A}^{(3)}(z_1-z_2, z_1-x)| \| a_{z_1} a_{z_2} a_x a_y \Theta \xi_s \| \, \| a_y \Theta \xi_s \| + \mathcal{E}_\Theta. \\ &\leq C \sum_{u,v \in \Lambda_B} \left(\int dx\, dy\, dz_1\, dz_2\, |V_N(x-y)| |\check{A}^{(3)}_{u,v}(z_1-z_2, z_1-x)|^2 \| a_y \Theta \xi_s \|^2 \right)^{1/2} \\ &\quad \times \left(\int dx\, dy\, dz_1\, dz_2\, |V_N(x-y)| \mathbb{1}\big(|z_1 - z_2 - u| \leq \ell_B,\, |z_1 - x + v| \leq \ell_B \big) \right. \\ &\qquad \left. \times \| a_{z_1} a_{z_2} a_x a_y \Theta \xi_s \|^2 \right)^{1/2} + \mathcal{E}_\Theta,\end{aligned}$$

where we localize the kernel $\check{A}^{(3)}$ on boxes of side length ℓ_B, $\check{A}^{(3)} = \sum_{u,v \in \Lambda_B} \check{A}^{(3)}_{u,v}$, and use Cauchy–Schwarz inequality. Using [13, Lemma 8, Lemma 11, Lemma 15], we obtain

$$\begin{aligned}|\langle \xi_s, T_5 \xi_s \rangle| &\leq C N^\varepsilon \|V_N\|_1 \left(\sum_{u,v \in \Lambda_B} \|\check{A}^{(3)}_{u,v}\|_2 \right) \| \mathcal{N}^{1/2} \xi_s \| \, \| \mathcal{W}_N^{1/2} \xi_s \| + \mathcal{E}_\Theta \\ &\leq C N^{4\kappa - 1} + C N^{-\beta + \varepsilon} \langle \xi_s, \mathcal{W}_N \xi_s \rangle\end{aligned}$$

for all $s \in [0, t]$.

We now turn to the term T_3. Notice that T_3 has exactly the same structure as the cubic operator $\mathcal{C}_N$ appearing in [13, Equation (150)] with the replacement of $N_0^{1/2} \check{\sigma}(x-z)$

by $\check{A}^{(6)}(x-y,x-z)/2$. Therefore, we follow the same argument in [13, Equation (150)] up to the change of kernel described above:

$$\langle \xi_s, T_3 \xi_s \rangle = \mathrm{Re}\big(\widetilde{S}_1 + \widetilde{S}_3\big) + \mathcal{E}_\Theta,$$

where

$$\widetilde{S}_1 = \int_0^s d\tau \int dx\, dy\, dz\, V_N(x-y)\check{A}^{(6)}(x-y,x-z) \left\langle \xi_\tau, \mathcal{T}_1(x,y) e^{-(1-\tau)G} a_z^* \xi_s \right\rangle,$$
$$\widetilde{S}_3 = \int_0^s d\tau \int dx\, dy\, dz\, V_N(x-y)\check{A}^{(6)}(x-y,x-z) \left\langle \xi_\tau, \mathcal{T}_3(x,y) e^{-(1-\tau)G} a_z^* \xi_s \right\rangle.$$

Here $\mathcal{T}_1(x,y)$ and $\mathcal{T}_3(x,y)$ denote the corresponding operators appearing in the expansion used for the cubic term $\mathcal{C}_N$.

For $\widetilde{S}_3$, we first insert the cutoff Θ where the error produced is absorbed in $\mathcal{E}_\Theta$. Hence

$$\begin{aligned}
|\widetilde{S}_3| &\leq \int_0^s d\tau \int dx\, dy\, dz\, |V_N(x-y)| |\check{A}^{(6)}(x-y,x-z)| \|\mathcal{T}_3(x,y)\Theta\xi_\tau\| \|e^{(1-\tau)G} a_z^* \xi_s\| + \mathcal{E}_\Theta \\
&\leq \int_0^s d\tau \bigg(\int dx\, dy\, |V_N(x-y)| \|\mathcal{T}_3(x,y)\Theta\xi_\tau\|^2 \bigg)^{1/2} \\
&\qquad \times \bigg(\int dx\, dy\, |V_N(x-y)| \bigg\| \int dz\, \check{A}^{(6)}(x-y,x-z) a_z^* \xi_s \bigg\|^2 \bigg)^{1/2} + \mathcal{E}_\Theta, \qquad (4.55)
\end{aligned}$$

where we used the fact that $e^{-(1-\tau)G}$ is unitary and Cauchy-Schwarz inequality. By [13, Lemma 22], the first factor in (4.55) is bounded by

$$\bigg(\int dx\, dy\, |V_N(x-y)| \|\mathcal{T}_3(x,y)\Theta\xi_\tau\|^2 \bigg)^{1/2} \leq C N^{(4\kappa-1-\beta+\varepsilon)/2}. \qquad (4.56)$$

For the second factor in (4.55), we set $\widetilde{y} = x-y$, and $A_{\widetilde{y}}(x-z) = \check{A}^{(6)}(\widetilde{y}, x-z)$ to get

$$\begin{aligned}
&\int dx\, dy\, |V_N(x-y)| \bigg\| \int dz\, \check{A}^{(6)}(x-y,x-z) a_z^* \xi_s \bigg\|^2 \\
&\quad = \int d\widetilde{y}\, dx\, |V_N(\widetilde{y})| \bigg\| \int dz\, A_{\widetilde{y}}(x-z) a_z^* \xi_s \bigg\|^2 \\
&\quad \leq \|V_N\|_1 \sup_y \int dx \bigg\| \int dz\, A_y(x-z) a_z^* \xi_s \bigg\|^2. \qquad (4.57)
\end{aligned}$$

To estimate the last supremum, we observe that

$$\begin{aligned}
\int dx \bigg\| \int dz\, A_y(x-z) a_z^* \xi_s \bigg\|^2 &= \int dz\, dz' \int dx\, A_y(x-z)\overline{A_y(x-z')} \langle \xi_s, a_z a_{z'}^* \xi_s \rangle \\
&= \int dz\, dz' \int dx\, A_y(x-z)\overline{A_y(x-z')} \langle \xi_s, (a_{z'}^* a_z + \delta(z-z'))\xi_s \rangle \\
&= \langle \xi_s, \mathcal{N}_{A_y * A_y} \xi_s \rangle + \|A_y\|_2^2.
\end{aligned}$$

where

$$\mathcal{N}_{A_y * A_y} = \int dz\, dz'\, (A_y * A_y)(z-z') a_{z'}^* a_z.$$

Consequently,

$$\sup_y \int dx \left\| \int dz\, A_y(x-z) a_z^* \xi_s \right\|^2 \leq \sup_y \|\mathcal{N}_{A_y * A_y}^{1/2} \xi_s\|^2 + \sup_y \|A_y\|_2^2 \leq CN^{3\kappa} + CN^{3\kappa/2+1} \leq CN^{3\kappa/2+1}. \tag{4.58}$$

In the last step of (4.58) we used [13, Equation (156) and the last bound in Lemma 9]. Combining (4.55), (4.56), (4.57), and (4.58), we find

$$\begin{aligned} |\widetilde{S}_3| &\leq CN^{(4\kappa-1-\beta+\varepsilon)/2} \|V_N\|_1^{1/2} N^{3\kappa/4+1/2} + \mathcal{E}_\Theta \\ &\leq CN^{4\kappa-1-\beta/4+\varepsilon} + \mathcal{E}_\Theta \leq CN^{4\kappa-1-\beta/4+\varepsilon}. \end{aligned} \tag{4.59}$$

We now estimate $\widetilde{S}_1$. Proceeding in the same way of deriving [13, Equation (155)]:

$$\begin{aligned} \widetilde{S}_1 = \int_0^s d\tau \int dx\, dy\, dz\, dz'\, V_N(x-y) \check{A}^{(6)}(x-y, x-z) \check{A}^{(6)}(z'-y, z'-x) \\ \times \left\langle \xi_s, e^{(1-\tau)G} a_{z'} e^{-(1-\tau)G} a_z^* \xi_s \right\rangle. \end{aligned}$$

By changing variables and using the symmetry of the kernel of $\check{A}^{(6)}$, we write

$$\begin{aligned} \widetilde{S}_1 &= s \int dx\, dy\, V_N(y) |\check{A}^{(6)}(y, x)|^2 \\ &\quad + s \int dx\, dy\, dz\, dz'\, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \langle \xi, a_z^* a_{z'} \xi \rangle \\ &\quad + \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int dx\, dy\, dz\, dz'\, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \langle \xi_{1-\tau_2}, [A^*, a_{z'}] a_z^* \xi_{1-\tau_2} \rangle \\ &\quad + \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int dx\, dy\, dz\, dz'\, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \\ &\qquad \times \langle \xi_{1-\tau_2}, [A^*, a_{z'}] e^{-\tau G} [A, a_z^*] \xi_{1-\tau_2+\tau_3} \rangle \\ &:= \widetilde{S}_{1,1} + \widetilde{S}_{1,2} + \widetilde{S}_{1,3} + \widetilde{S}_{1,4}. \end{aligned}$$

The first term $\widetilde{S}_{1,1}$ is the main contribution. It has the same form as the term appearing in [13, Equation (165)], up to the replacement of the corresponding kernel. Thus we focus on the remaining terms. For $\widetilde{S}_{1,2}$ and $\widetilde{S}_{1,4}$, we proceed as in the estimates [13, Equation (157) and (158)]. The only difference is that the terms

$$N^{1/2} \left\| \int dz\, \check{\sigma}(x-z) a_z \xi \right\| \quad \text{and} \quad N^{1/2} \|\mathcal{N}_{\check{\sigma} * \check{\sigma}}^{1/2} \xi_{1-t-\tau}\|$$

are replaced by

$$\sup_y \|\mathcal{N}_{A_y * A_y}^{1/2} \xi_s\|, \quad \text{and} \quad \sup_y \|\ddot{\mathcal{N}}_{A_y * A_y}^{1/2} \xi_s\|,$$

respectively. Therefore, [13, Equation (156)] gives

$$\begin{aligned} |\widetilde{S}_{1,2}| &\leq C \|V_N\|_1 \left(\sup_y \|\mathcal{N}_{A_y * A_y}^{1/2} \xi_s\| \right)^2 \leq CN^{4\kappa-1}; \\ |\widetilde{S}_{1,4}| &\leq C \|V_N\|_1 \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \left(\sup_y \|\ddot{\mathcal{N}}_{A_y * A_y}^{1/2} \xi_s\| \right)^2 \leq CN^{4\kappa-1}. \end{aligned}$$

It remains to control $\widetilde{S}_{1,3}$. We expand the commutator once more as in [13, Equation (160)] and write

$$\widetilde{S}_{1,3} = \widetilde{T}_0 + \widetilde{T}_2 + \widetilde{T}_{4,1} + \widetilde{T}_{4,2} + \mathcal{E}_\Theta \,, \tag{4.60}$$

where

$$\begin{aligned}
\widetilde{T}_0 &= \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{1-\tau_2} d\tau_3 \int dx dy dz dz' dx_1 dx_2 \, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \\
&\qquad \times \check{A}^{(3)}(x_1 - x_2, x_1 - z') \check{A}^{(6)}(x_1 - x_2, x_1 - z) \\
\widetilde{T}_2 &= \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{1-\tau_2} d\tau_3 \int dx dy dz dz' dx_1 dx_2 dx_3 \, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \\
&\qquad \times \check{A}^{(6)}(x_1 - x_2, x_1 - z') \check{A}^{(6)}(x_3 - x_2, x_3 - z) \langle \xi_{\tau_3}, a^*_{x_1} a_{x_3} \xi_{\tau_3} \rangle \\
&\quad + \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{1-\tau_2} d\tau_3 \int dx dy dz dz' dx_1 dx_2 dx_3 \, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \\
&\qquad \times \check{A}^{(6)}(x_1 - x_2, x_1 - z') \check{A}^{(6)}(x_3 - x_1, x_3 - x_2) \langle \xi_{\tau_3}, a^*_z a_{x_3} \xi_{\tau_3} \rangle
\end{aligned}$$

$$\begin{aligned}
\widetilde{T}_{4,1} &= \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{1-\tau_2} d\tau_3 \int dx \, dy \, dz \, dz' \, dx_1 \cdots dx_4 \, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \\
&\qquad \times \check{A}^{(6)}(x_1 - x_2, x_1 - z') \check{A}^{(3)}(x_3 - x_4, x_3 - x_1) \left\langle \Theta \xi_{\tau_3}, a^*_z a^*_{x_2} a_{x_3} a_{x_4} \Theta \xi_{\tau_3} \right\rangle \\
\widetilde{T}_{4,2} &= \int_0^s d\tau_1 \int_0^{1-\tau_1} d\tau_2 \int_0^{1-\tau_2} d\tau_3 \int dx \, dy \, dz \, dz' \, V_N(y) \check{A}^{(6)}(y, x-z) \check{A}^{(6)}(y, x-z') \\
&\qquad \times \langle \xi_{\tau_3}, [A^*, a_{z'}][A, a^*_z] \xi_{\tau_3} \rangle \,.
\end{aligned}$$

For $\widetilde{T}_0$ and $\widetilde{T}_2$, we switch to momentum space as in the derivation of [13, Equation (161) and (162)]. We get

$$\begin{aligned}
\widetilde{T}_0 &= \frac{1}{2} \left(s - \frac{s^3}{3} \right) \int dx \, dy \, dz \, dz' \, dx_1 \, dx_2 \sum_{r \in \Lambda^*} \widehat{V}_N(r) e^{ir \cdot y} \sum_{\substack{p_i, q_i \in \Lambda^* \\ i = 1, \dots, 4}} A^{(6)}_{p_1, q_1} e^{ip_1 \cdot y} e^{iq_1 \cdot (x - z)} \\
&\qquad \times A^{(6)}_{p_2, q_2} e^{ip_2 \cdot y} e^{iq_2 \cdot (x - z')} A^{(3)}_{p_3, q_3} e^{ip_3 \cdot (x_1 - x_2)} e^{iq_3 \cdot (x_1 - z')} A^{(6)}_{p_4, q_4} e^{ip_4 \cdot (x_1 - x_2)} e^{iq_4 \cdot (x_1 - z)} \\
&= \frac{1}{2} \left(s - \frac{s^3}{3} \right) \sum_{\substack{r \in \Lambda^* \\ p_i, q_i \in \Lambda^*}} \widehat{V}_N(r) A^{(6)}_{p_1, q_1} A^{(6)}_{p_2, q_2} A^{(3)}_{p_3, q_3} A^{(6)}_{p_4, q_4} \int dx \, dy \, dz \, dz' \, dx_1 \, dx_2 \, e^{i(q_1 + q_2) \cdot x} \\
&\qquad \times e^{i(r + p_1 + p_2) \cdot y} e^{-i(q_1 + q_4) \cdot z} e^{-i(q_2 + q_3) \cdot z'} e^{i(p_3 + q_3 + p_4 + q_4) \cdot x_1} e^{-i(p_3 + p_4) \cdot x_2} \\
&= \frac{s}{2} \left(1 - \frac{s^2}{3} \right) \sum_{r, q, k, p \in \Lambda^*} \widehat{V}_N(r) A^{(6)}_{q, k} A^{(6)}_{-q - r, -k} A^{(3)}_{p, -k} A^{(6)}_{-p, k} \\
&= \frac{s}{2} \left(1 - \frac{s^2}{3} \right) \sum_{r, q, k, p \in \Lambda^*} \widehat{V}_N(r) A^{(6)}_{q, k} A^{(6)}_{q + r, k} A^{(3)}_{p, k} A^{(6)}_{p, k} \,,
\end{aligned}$$

where we use change of variable in the last step.

We next compute the time coefficient appearing in $\widetilde{T}_2$. In the triple integral we have

$$0 \le \tau_1 \le s, \qquad 0 \le \tau_2 \le 1 - \tau_1, \qquad 0 \le \tau_3 \le 1 - \tau_2.$$

For fixed τ_3, we have

$$0 \le \tau_1 \le s, \qquad 0 \le \tau_2 \le \min\{1 - \tau_1, 1 - \tau_3\}.$$

We denote

$$I_s(\tau_3) = \int_0^s d\tau_1 \int_0^{\min\{1-\tau_1, 1-\tau_3\}} d\tau_2.$$

and distinguish the following two cases. If $s \le \tau_3$, then for all $0 \le \tau_1 \le s$, $1-\tau_1 \ge 1-\tau_3$. Thus

$$I_s(\tau_3) = \int_0^s d\tau_1 \, (1-\tau_3) = s(1-\tau_3).$$

If $0 \le \tau_3 \le s$, then we split the integral at $\tau_1 = \tau_3$. For $0 \le \tau_1 \le \tau_3$, the minimum is $1-\tau_3$, while for $\tau_3 \le \tau_1 \le s$, the minimum is $1-\tau_1$. Hence

$$I_s(\tau_3) = \int_0^{\tau_3} d\tau_1 \, (1-\tau_3) + \int_{\tau_3}^s d\tau_1 \, (1-\tau_1) = \tau_3(1-\tau_3) + \left[\tau_1 - \frac{\tau_1^2}{2}\right]_{\tau_3}^s = s(1-\tau_3) - \frac{(s-\tau_3)^2}{2}.$$

Combining the two cases, we obtain

$$I_s(\tau_3) = s(1-\tau_3) - \frac{(s-\tau_3)^2}{2} \mathbb{1}_{\{\tau_3 \le s\}}.$$

We now spell out the momentum-space contributions in $\widetilde{T}_2$. Observe that

$$\begin{aligned}
&\int dx\, dy\, dz\, dz'\, dx_1\, dx_2\, dx_3 \sum_{r\in\Lambda^*} \widehat{V}_N(r) e^{ir\cdot y} \sum_{\substack{p_i,q_i\in\Lambda^* \\ i=1,\dots,4}} A^{(6)}_{p_1,q_1} e^{ip_1\cdot y} e^{iq_1\cdot(x-z)} A^{(6)}_{p_2,q_2} e^{ip_2\cdot y} e^{iq_2\cdot(x-z')} \\
&\qquad \times A^{(3)}_{p_3,q_3} e^{ip_3\cdot(x_1-x_2)} e^{iq_3\cdot(x_1-z')} A^{(6)}_{p_4,q_4} e^{ip_4\cdot(x_3-x_2)} e^{iq_4\cdot(x_3-z)} \sum_{s\in\Lambda^*} a_s^* e^{-is\cdot x_1} \sum_{t\in\Lambda^*} a_t e^{it\cdot x_3} \\
&= \sum_{\substack{r,s,t\in\Lambda^* \\ p_i,q_i\in\Lambda^*}} \widehat{V}_N(r) A^{(6)}_{p_1,q_1} A^{(6)}_{p_2,q_2} A^{(3)}_{p_3,q_3} A^{(6)}_{p_4,q_4} a_s^* a_t \int dx\, dy\, dz\, dz'\, dx_1\, dx_2\, dx_3\, e^{i(q_1+q_2)\cdot x} e^{i(r+p_1+p_2)\cdot y} \\
&\qquad \times e^{-i(q_1+q_4)\cdot z} e^{-i(q_2+q_3)\cdot z'} e^{i(p_3+q_3-s)\cdot x_1} e^{-i(p_3+p_4)\cdot x_2} e^{i(p_4+q_4+t)\cdot x_3} \\
&= \sum_{r,q,k,p\in\Lambda^*} \widehat{V}_N(r) A^{(6)}_{q,k} A^{(6)}_{q+r,k} A^{(3)}_{p,k} A^{(6)}_{p,k} a^*_{p+k} a_{p+k}. \qquad (4.61)
\end{aligned}$$

Here we used change of variables and the symmetry of the $A^{(6)}$-kernel to write, for instance,

$$A^{(6)}_{-r-q,-k} = A^{(6)}_{q+r,k}, \qquad A^{(6)}_{p,-k} = A^{(6)}_{p,k}.$$

Similarly, we get the following observation

$$\begin{aligned}
&\int dx\, dy\, dz\, dz'\, dx_1\, dx_2\, dx_3 \sum_{r,s,t,p_i,q_i\in\Lambda^*} \widehat{V}_N(r) A^{(6)}_{p_1,q_1} A^{(6)}_{p_2,q_2} A^{(3)}_{p_3,q_3} A^{(6)}_{p_4,q_4} a_s^* a_t e^{ir\cdot y} e^{ip_1\cdot y} e^{iq_1\cdot(x-z)} \\
&\qquad \times e^{ip_2\cdot y} e^{iq_2\cdot(x-z')} e^{ip_3\cdot(x_1-x_2)} e^{iq_3\cdot(x_1-z')} e^{ip_4\cdot(x_3-x_1)} e^{iq_4\cdot(x_3-x_2)} e^{-is\cdot z} e^{it\cdot x_3} \\
&= \sum_{r,q,k,p\in\Lambda^*} \widehat{V}_N(r) A^{(6)}_{q,k} A^{(6)}_{q+r,k} A^{(3)}_{p,k} A^{(6)}_{p,k} a_k^* a_k. \qquad (4.62)
\end{aligned}$$

Recall that, from [13, Equation (77)]

$$A^{(6)}_{p,k} = A^{(3)}_{p,k} + A^{(3)}_{-p-k,k}.$$

Using the change of variables $\widetilde{p} = -p-k$ and [13, Equation (78)], we have

$$A^{(3)}_{-p-k,k} A^{(6)}_{p,k} a^*_{p+k} a_{p+k} = A^{(3)}_{\widetilde{p},k} A^{(6)}_{-\widetilde{p}-k,k} a^*_{-\widetilde{p}} a_{-\widetilde{p}} = A^{(3)}_{p,k} A^{(6)}_{p,k} a^*_p a_p. \qquad (4.63)$$

Combining (4.61), (4.62), and (4.63), we obtain

$$\widetilde{T}_2 = \int_0^1 d\tau \left[s(1-\tau) - \frac{(s-\tau)^2}{2} \mathbb{1}_{\{\tau \leq s\}} \right] \sum_{r,q,k,p \in \Lambda^*} \widehat{V}_N(r) A^{(6)}_{q,k} A^{(6)}_{q+r,k} A^{(3)}_{p,k} A^{(6)}_{p,k} \\ \times \left\langle \xi_\tau, \left(a_p^* a_p + a_k^* a_k + a_{p+k}^* a_{p+k} \right) \xi_\tau \right\rangle . \tag{4.64}$$

By [13, Lemma 9, Lemma 15, Lemma 19 and Equation (79)], we get

$$\begin{aligned} |\widetilde{T}_0| &\leq C \|\widehat{V}_N\|_\infty \sum_{r,q,k,p \in \Lambda^*} |A^{(6)}_{q,k}| |A^{(6)}_{q+r,k}| |A^{(3)}_{p,k}| |A^{(6)}_{p,k}| \\ &\leq C N^\kappa \sum_{k,p \in \Lambda^*} (1 + |\sigma_k|^2) |A^{(3)}_{p,k}| |A^{(6)}_{p,k}| \\ &\leq C N^\kappa \sum_{k,p \in \Lambda^*} (1 + |\sigma_k|^2) \left(|A^{(3)}_{p,k}|^2 + |A^{(6)}_{p,k}|^2 \right) \leq C N^{4\kappa - 1}. \end{aligned} \tag{4.65}$$

and

$$\begin{aligned} |\widetilde{T}_2| &\leq C \|\widehat{V}_N\|_\infty N \sum_{k,p \in \Lambda^*} (1 + |\sigma_k|)^2 |A^{(3)}_{p,k}| |A^{(6)}_{p,k}| \left\langle \xi_\tau, \left(a_p^* a_p + a_k^* a_k + a_{p+k}^* a_{p+k} \right) \xi_\tau \right\rangle \\ &\leq C N^\kappa \|\sigma\|_\infty^2 \Bigg[\sum_{p,k \in \Lambda^*} \sum_{j=3,6} |A^{(j)}_{p,k}|^2 \left\langle \xi_\tau, a_p^* a_p \xi_\tau \right\rangle + \sum_{p,k \in \Lambda^*} \sum_{j=3,6} |A^{(j)}_{p,k}|^2 \left\langle \xi_\tau, a_k^* a_k \xi_\tau \right\rangle \\ &\qquad + \sum_{k,p \in \Lambda^*} |A^{(3)}_{p+k,-p}| |A^{(6)}_{p+k,-k}| \| a_{p+k} \xi_\tau \| \, \| a_{p+k} \xi_\tau \| \Bigg] \\ &\leq C N^{\kappa + \beta/4} \Bigg[\sum_{j=3,6} \sum_{p \in \Lambda^*} N^{3\kappa/2 - 1} (1 + |\sigma_p|^2) \langle \xi_\tau, a_p^* a_p \xi_\tau \rangle + \sum_{j=3,6} \sum_{k \in \Lambda^*} N^{3\kappa/2-1} (1 + |\sigma_k|^2) \langle \xi_\tau, a_k^* a_k \xi_\tau \rangle \\ &\quad + \left(\sum_{k,p \in \Lambda^*} |A^{(3)}_{p+k,-p}|^2 \langle \xi_\tau, a_{p+k}^* a_{p+k} \xi_\tau \rangle \right)^{1/2} \left(\sum_{k,p \in \Lambda^*} |A^{(6)}_{p+k,-k}|^2 \langle \xi_\tau, a_{p+k}^* a_{p+k} \xi_\tau \rangle \right)^{1/2} \Bigg] \\ &\leq C N^{\kappa + \beta/4} N^{3\kappa/2 - 1} \langle \xi_\tau, (\mathcal{N} + \widetilde{\mathcal{N}}) \xi_\tau \rangle \leq C N^{\kappa + \beta/4} N^{3\kappa/2 - 1} N^{3\kappa - 1} \\ &\leq C N^{4\kappa - 1 - \beta/4}, \end{aligned} \tag{4.66}$$

where we used $\|\widehat{V}_N\|_\infty N \leq C N^\kappa$, $(1 + |\sigma_k|)^2 \leq C \|\sigma\|_\infty^2$, [13, Equation (43) and (79)] Here the factor $N^{\beta/4}$ comes from the bound $\|\sigma\|_\infty^2 \leq C N^{\beta/4}$.

We now estimate the quartic terms. Set

$$D^{(6)}(x) := \sup_y |\check{A}^{(6)}(y, x)|.$$

Then, from the expression for $\widetilde{T}_{4,1}$, we have

$$
\begin{aligned}
|\widetilde{T}_{4,1}| &\leq C \int_0^s d\tau \int dxdydz'dx_1 \cdots dx_4 \, |V_N(y)| \, D^{(6)}(x-z') \|a_{x_3} a_{x_4} \Theta \xi_\tau\| \\
&\quad \times |\check{A}^{(6)}(x_1 - x_2, x_1 - z')| |\check{A}^{(3)}(x_3 - x_4, x_3 - x_1)| \sup_{\widetilde{y}} \left\| \int dz \, \check{A}^{(6)}(\widetilde{y}, x - z) a_z a_{x_2} \Theta \xi_\tau \right\| \\
&\leq C \int_0^s d\tau \sum_{w,u,v,\widetilde{u},\widetilde{v} \in \Lambda_B} \Bigg(\int dxdydz'dx_1 \cdots dx_4 \, |V_N(y)| \, |D_w^{(6)}(x-z')|^2 |\check{A}^{(3)}_{u,v}(x_3 - x_4, x_3 - x_1)|^2 \\
&\qquad \times \mathbb{1}\big(|x_1 - x_2 - \widetilde{u}|, |x_1 - z' - \widetilde{v}| \leq \ell_B\big) \sup_{\widetilde{y}} \left\| \int dz \, \check{A}^{(6)}(\widetilde{y}, x - z) a_z a_{x_2} \Theta \xi_\tau \right\|^2 \Bigg)^{1/2} \\
&\quad \times \Bigg(\int dxdydz'dx_1 \cdots dx_4 \, |V_N(y)| |\check{A}^{(6)}_{\widetilde{u},\widetilde{v}}(x_1 - x_2, x_1 - z')|^2 \\
&\qquad \times \mathbb{1}\big(|x_3 - x_4 - u|, |x_3 - x_1 - v|, |x - z' - w| \leq \ell_B\big) \|a_{x_3} a_{x_4} \Theta \xi_\tau\|^2 \Bigg)^{1/2}.
\end{aligned}
$$

The local cutoff gives a factor N^ε, while the localized boxes produce the factor $\ell_B^{-9/2}$. Therefore,

$$
\begin{aligned}
|\widetilde{T}_{4,1}| &\leq C \|V_N\|_1 \left(\sum_{w \in \Lambda_B} \|D_w^{(6)}\|_2 \right) \left(\sum_{u,v \in \Lambda_B} \|\check{A}^{(3)}_{u,v}\|_2 \right) \left(\sum_{\widetilde{u},\widetilde{v} \in \Lambda_B} \|\check{A}^{(6)}_{\widetilde{u},\widetilde{v}}\|_2 \right) \\
&\quad \times N^\varepsilon \ell_B^{-9/2} \sup_y \|\mathcal{N}^{1/2}_{A_y * A_y} \xi_\tau\| \, \|\mathcal{N}^{1/2} \xi_\tau\|.
\end{aligned}
$$

Using the bounds on the localized kernels, the estimate $\|V_N\|_1 \leq C N^{-1+\kappa}$, the identity $\ell_B = N^{-\kappa/2 + \beta/12}$, and the bounds

$$
\sup_y \|\mathcal{N}^{1/2}_{A_y * A_y} \xi_\tau\| \leq C N^{3\kappa/4 + 1/2}, \qquad \|\mathcal{N}^{1/2} \xi_\tau\| \leq C N^{(3\kappa - 1)/2},
$$

we find

$$
|\widetilde{T}_{4,1}| \leq C N^{4\kappa - 1 - \beta/8 + \varepsilon}. \tag{4.67}
$$

The second quartic term is simpler. Using again the bound on $\ddot{\mathcal{N}}_{A_y * A_y}$, we get

$$
|\widetilde{T}_{4,2}| \leq C \|V_N\|_1 \int_0^s d\tau \left(\sup_y \|\ddot{\mathcal{N}}^{1/2}_{A_y * A_y} \xi_\tau\| \right)^2 \leq C N^{-1+\kappa} N^{3\kappa} \leq C N^{4\kappa - 1}. \tag{4.68}
$$

Combining (4.65), (4.66), (4.67), and (4.68), and choosing $\varepsilon > 0$ sufficiently small, we obtain $|\widetilde{S}_{1,3}| \leq C N^{4\kappa - 1}$. Together with the bounds for $\widetilde{S}_{1,2}$, $\widetilde{S}_{1,4}$, and $\widetilde{S}_3$, this yields

$$
\langle \xi_s, T_3 \xi_s \rangle \leq s \int dx \, dy \, V_N(y) |\check{A}^{(6)}(y, x)|^2 + C N^{4\kappa - 1}, \qquad \forall \, s \in [0, 1].
$$

Returning to the Duhamel representation for $\langle \xi_t, \mathcal{V}_N \xi_t \rangle$, and using [13, Equation (166), (167) and (168)], we obtain

$$
\langle \xi_t, \mathcal{V}_N \xi_t \rangle \leq \frac{t^2}{2} \int dxdy \, V_N(y) |\check{A}^{(6)}(y, x)|^2 + C N^{-\beta + \varepsilon} \int_0^t ds \, \langle \xi_s, \mathcal{W}_N \xi_s \rangle + C N^{4\kappa - 1}. \tag{4.69}
$$

Notice that the only difference between $\mathcal{V}_N$ and $\mathcal{W}_N$ is that $\mathcal{W}_N$ contains $|V_N|$ instead of V_N, namely

$$
\mathcal{W}_N = \frac{1}{2} \int dx \, dy \, |V_N(x - y)| a_x^* a_y^* a_x a_y.
$$

Therefore, applying exactly the same argument with V_N replaced by $|V_N|$, we get

$$\langle \xi_t, \mathcal{W}_N \xi_t \rangle \leq \frac{t^2}{2} \int dxdy \, |V_N(y)| \, |\check{A}^{(6)}(y,x)|^2 + CN^{-\beta+\varepsilon} \int_0^t ds \, \langle \xi_s, \mathcal{W}_N \xi_s \rangle + CN^{4\kappa-1}. \tag{4.70}$$

Since

$$\| \widehat{|V|} \|_\infty = \sup_k \left| \int dx \, |V(x)| e^{-ik\cdot x} \right| \leq \|V\|_1 \leq C,$$

we now estimate the first term on the right-hand side of (4.70)

$$\begin{aligned} &\int dxdy \, |V_N(y)| \, |\check{A}^{(6)}(y,x)|^2 \\ &= \int dxdy \sum_{r\in\Lambda^*} \widehat{|V_N|}(r) e^{ir\cdot y} \sum_{p,q,p',q'\in\Lambda^*} A^{(6)}_{p,q} \overline{A^{(6)}_{p',q'}} e^{i(p-p')\cdot y} e^{i(q-q')\cdot x} \\ &= \sum_{p,q,r\in\Lambda^*} \widehat{|V_N|}(r) A^{(6)}_{p,q} \overline{A^{(6)}_{p+r,q}} \, . \end{aligned}$$

Moreover, since $|V_N(y)| = N^{2-2\kappa} |V(N^{1-\kappa} y)|$, we get, by the change of variables $\widetilde{y} = N^{1-\kappa} y$,

$$\begin{aligned} \widehat{|V_N|}(r) &= \int dy \, |V_N(y)| e^{-ir\cdot y} = N^{2-2\kappa} \int dy \, |V(N^{1-\kappa} y)| e^{-ir\cdot y} \\ &= N^{-1+\kappa} \int d\widetilde{y} \, |V(\widetilde{y})| e^{-i \frac{r}{N^{1-\kappa}} \cdot \widetilde{y}} = N^{-1+\kappa} \widehat{|V|} \left(\frac{r}{N^{1-\kappa}} \right) . \end{aligned}$$

Therefore,

$$\begin{aligned} \int dx \, dy \, |V_N(y)| \, |\check{A}^{(6)}(y,x)|^2 &\leq N^{-1+\kappa} \| \widehat{|V|} \|_\infty \sum_{p,q,r\in\Lambda^*} |A^{(6)}_{p,q}| |A^{(6)}_{p+r,q}| \\ &= CN^{-1+\kappa} \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(6)}_{p,q}| \right)^2 . \end{aligned} \tag{4.71}$$

It remains to estimate the last sum. From [13, Equation (62)], $|A^{(3)}_{p,q}| \leq CN^{-1/2} |\eta_p| \, |\sigma_q|$, we first get

$$\begin{aligned} \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{p,q}| \right)^2 &\leq CN^{-1} \sum_{q\in\Lambda^*} |\sigma_q|^2 \left(\sum_{p\in\Lambda^*} |\eta_p| \right)^2 \\ &\leq CN^{-1} \|\sigma\|_2^2 \|\eta\|_1^2 \leq CN^{-1} N^{3\kappa/2} N^2 = CN^{3\kappa/2+1} \, , \end{aligned} \tag{4.72}$$

where we used the bounds $\|\eta\|_1 \leq CN$, and $\|\sigma\|_2^2 \leq CN^{3\kappa/2}$. Moreover,

$$\sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(6)}_{p,q}| \right)^2 \tag{4.73}$$

$$\begin{aligned} &\leq C \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{p,q}| \right)^2 + C \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{q,p}| \right)^2 + C \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{-p-q,p}| \right)^2 \\ &+ C \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{-p-q,q}| \right)^2 + C \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{q,-p-q}| \right)^2 + C \sum_{q\in\Lambda^*} \left(\sum_{p\in\Lambda^*} |A^{(3)}_{p,-p-q}| \right)^2 . \end{aligned} \tag{4.74}$$

The first term on the r.h.s. of (4.74) is bounded by (4.72). For the second one on the r.h.s. of (4.74), we use $|A^{(3)}_{p,q}| \leq CN^{-1/2}|\eta_p|\,|\sigma_q|$ again:

$$\begin{aligned}\sum_{q\in\Lambda^*}\left(\sum_{p\in\Lambda^*}|A^{(3)}_{q,p}|\right)^2 &\leq CN^{-1}\sum_{q\in\Lambda^*}|\eta_q|^2\left(\sum_{p\in\Lambda^*}|\sigma_p|\right)^2\\ &\leq CN^{-1}\|\eta\|_2^2\|\sigma\|_1^2 \leq CN^{-1}N^{3\kappa/2}N^2 = CN^{3\kappa/2+1}.\end{aligned}$$

The remaining terms on the r.h.s. of (4.74) can be treated similarly. Therefore,

$$\sum_{q\in\Lambda^*}\left(\sum_{p\in\Lambda^*}|A^{(6)}_{p,q}|\right)^2 \leq CN^{3\kappa/2+1}. \tag{4.75}$$

Combining (4.71) with (4.75), we obtain

$$\int dx\,dy\,|V_N(y)|\,|\check{A}^{(6)}(y,x)|^2 \leq CN^{-1+\kappa}N^{3\kappa/2+1} = CN^{5\kappa/2}. \tag{4.76}$$

Inserting (4.76) into (4.70), and choosing $\varepsilon > 0$ sufficiently small, we get

$$\langle\xi_t, \mathcal{W}_N\xi_t\rangle \leq CN^{5\kappa/2} + C\int_0^t ds\,\langle\xi_s, \mathcal{W}_N\xi_s\rangle\,.$$

By Grönwall's lemma, using $t\in[0,1]$ and $N^{-\beta+\varepsilon}\ll 1$, we conclude that

$$\langle\xi_t, \mathcal{W}_N\xi_t\rangle \leq CN^{5\kappa/2}, \qquad t\in[0,1]. \tag{4.77}$$

Finally, inserting (4.77) into (4.69), we find

$$\langle\xi_t, \mathcal{V}_N\xi_t\rangle \leq \frac{t^2}{2}\int dx\,dy\,V_N(y)|\check{A}^{(6)}(y,x)|^2 + CN^{-\beta+\varepsilon}N^{5\kappa/2} + CN^{4\kappa-1}.$$

Since $\beta = 2-3\kappa$ and $\kappa < 2/3$, choosing $\varepsilon > 0$ sufficiently small gives

$$N^{-\beta+\varepsilon}N^{5\kappa/2} = N^{11\kappa/2-2+\varepsilon} \leq CN^{4\kappa-1}.$$

Therefore, for any $t\in[0,1]$

$$\langle\xi_t, \mathcal{V}_N\xi_t\rangle \leq \frac{t^2}{2}\int dx\,dy\,V_N(y)|\check{A}^{(6)}(y,x)|^2 + CN^{4\kappa-1}$$

as required.

□

4.3 Bound for the kinetic operator $\mathcal{K}$

An estimate of the kinetic energy operator is proved in the next Lemma from [13, Lemma 26].

Lemma 17. *We have*

$$\begin{aligned}&\left\langle e^{A^*\Theta-\Theta A}\Omega,\, \mathcal{K}e^{A^*\Theta-\Theta A}\Omega\right\rangle\\ &\qquad = -N^{1/2}\int dxdy\,N^{2-2\kappa}(Vf)(N^{1-\kappa}x)\check{\sigma}(y)\check{A}^{(6)}(x,y) + O(N^{4\kappa-1})\,.\end{aligned}$$

Proof of Lemma 17. Let $\xi_t = e^{tG}\Omega$, with $G = A^*\Theta - \Theta A$, and define

$$g(t) := \langle \xi_t, \mathcal{K}\xi_t \rangle.$$

Then

$$\begin{aligned} g'(t) &= \langle \xi_t, [\mathcal{K}, G]\xi_t \rangle = \langle \xi_t, [\mathcal{K}, A^*\Theta]\xi_t \rangle - \langle \xi_t, [\mathcal{K}, \Theta A]\xi_t \rangle \\ &= \langle \xi_t, \big([\mathcal{K}, A^*]\Theta + A^*[\mathcal{K}, \Theta]\big)\xi_t \rangle - \langle \xi_t, \big([\mathcal{K}, \Theta]A + \Theta[\mathcal{K}, A]\big)\xi_t \rangle \\ &= 2\mathrm{Re}\, \langle \xi_t, [\mathcal{K}, A^*]\Theta\xi_t \rangle + 2\mathrm{Re}\, \langle \xi_t, A^*[\mathcal{K}, \Theta]\xi_t \rangle. \end{aligned}$$

Since $\mathcal{K}\Omega = 0$, we have $g(0) = 0$. Hence

$$\begin{aligned} g(t) &= g(0) + \int_0^t ds\, g'(s) = 2\mathrm{Re} \int_0^t ds\, \langle \xi_s, [\mathcal{K}, A^*]\Theta\xi_s \rangle + 2\mathrm{Re} \int_0^t ds\, \langle \xi_s, A^*[\mathcal{K}, \Theta]\xi_s \rangle \\ &= 2\mathrm{Re} \int_0^t ds\, \langle \xi_s, [\mathcal{K}, A^*]\xi_s \rangle + 2\mathrm{Re} \int_0^t ds\, \langle \xi_s, A^*[\mathcal{K}, \Theta]\xi_s \rangle + 2\mathrm{Re} \int_0^t ds\, \langle \xi_s, [\mathcal{K}, A^*](\Theta - 1)\xi_s \rangle. \end{aligned} \tag{4.78}$$

By [13, Lemma 11], the last term on the r.h.s. of (4.78) can be absorbed into $\mathcal{E}_\Theta$ because

$$\left| 2\mathrm{Re} \int_0^t ds\, \langle \xi_s, [\mathcal{K}, A^*](\Theta - 1)\xi_s \rangle \right| \leq CN^{4\kappa - 1},$$

provided the parameter $\alpha > 0$ is sufficiently large. Then we compute the commutator $[\mathcal{K}, A^*]$:

$$\begin{aligned} [\mathcal{K}, A^*] &= \sum_{k,p,q \in \Lambda^*} k^2 A_{p,q} \big[a_k^* a_k, a_{p+q}^* a_{-p}^* a_{-q}^*\big]. \\ &= -N^{\kappa - \frac{1}{2}} \sum_{p,q \in \Lambda^*} \widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big)\, \widetilde{\chi}(p) \sigma_q a_{p+q}^* a_{-p}^* a_{-q}^* \\ &= 2N^{-1/2} \sum_{p,q \in \Lambda^*} p^2 \eta_p \sigma_q a_{p+q}^* a_{-p}^* a_{-q}^* =: \sum_{p,q \in \Lambda^*} A_{\mathcal{K},p,q} a_{p+q}^* a_{-p}^* a_{-q}^*, \end{aligned} \tag{4.79}$$

Observe that, for any vector $\xi \in \mathcal{F}$,

$$\begin{aligned} \Big\| (\mathcal{N}+1)^{-3/2} [\mathcal{K}, A^*] \xi \Big\| &= \left\| \sum_{p,q \in \Lambda^*} A_{\mathcal{K},p,q} (\mathcal{N}+1)^{-3/2} a_{p+q}^* a_{-p}^* a_{-q}^* \xi \right\| \\ &= \left\| \sum_{p,q \in \Lambda^*} A_{\mathcal{K},p,q} a_{p+q}^* a_{-p}^* a_{-q}^* (\mathcal{N}+4)^{-3/2} \xi \right\| \leq C \|A_{\mathcal{K},p,q}\|_{\ell^2}\,. \end{aligned}$$

Then we estimate the ℓ^2-norm of the kernel $A_{\mathcal{K},p,q}$. We compute

$$\begin{aligned} \|A_{\mathcal{K}}\|_{\ell^2}^2 &= \sum_{p,q \in \Lambda^*} |A_{\mathcal{K},p,q}|^2 = N^{2\kappa - 1} \sum_{p,q \in \Lambda^*} \Big| \widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big) \Big|^2 \widetilde{\chi}(p)^2 |\sigma_q|^2 \\ &= N^{2\kappa - 1} \left(\sum_{p \in \Lambda^*} \Big| \widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big) \Big|^2 \widetilde{\chi}(p)^2 \right) \left(\sum_{q \in \Lambda^*} |\sigma_q|^2 \right). \end{aligned}$$

By [13, Lemma 4],

$$\sum_{p \in \Lambda^*} \Big| \widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big) \Big|^2 \widetilde{\chi}(p)^2 \leq CN^{3-3\kappa},$$

and by [13, Equation (45)], $\sum_{q\in\Lambda^*}|\sigma_q|^2 \leq CN^{3\kappa/2}$, we get

$$\|A_{\mathcal{K}}\|_{\ell^2}^2 \leq CN^{2\kappa-1}N^{3-3\kappa}N^{3\kappa/2} = CN^{2+\kappa/2}.$$

Hence

$$\left\|(\mathcal{N}+1)^{-3/2}[\mathcal{K},A^*]\right\| \leq CN^{1+\kappa/4}. \tag{4.80}$$

We first estimate

$$\begin{aligned}|\langle\xi_s,[\mathcal{K},A^*]\xi_s\rangle| &\leq \left\|(\mathcal{N}+1)^{3/2}\xi_s\right\|\left\|(\mathcal{N}+1)^{-3/2}[\mathcal{K},A^*]\xi_s\right\| \\ &\leq CN^{\frac{3}{2}\cdot\frac{3\kappa}{2}}N^{1+\frac{\kappa}{4}} = CN^{1+\frac{5\kappa}{2}}.\end{aligned} \tag{4.81}$$

where we used [13, Equation (96)] together with (4.80). Hence, for $0\leq t\leq 1$,

$$\left|\int_0^t ds\ \langle\xi_s,[\mathcal{K},A^*]\xi_s\rangle\right| \leq CN^{1+\frac{5\kappa}{2}}. \tag{4.82}$$

Then we estimate the term $\int_0^t ds\ \langle\xi_s,A^*[\mathcal{K},\Theta]\xi_s\rangle$. Recall that the cutoff Θ acts on the Fock space $\mathcal{F}=\bigoplus_{m\geq 0}L^2_s(\Lambda^m)$. and it preserves the number of particles. Therefore, if

$$\Psi=\{\psi^{(m)}\}_{m\geq 0}\in\mathcal{F},$$

then, on the m-particle sector, Θ acts as a multiplication operator:

$$\left(\Theta\left(\frac{\mathcal{L}_n}{N^\alpha}\right)\Psi\right)^{(m)}(x_1,\ldots,x_m) = F_m(x_1,\ldots,x_m)\,\psi^{(m)}(x_1,\ldots,x_m), \tag{4.83}$$

where

$$F_m(x_1,\ldots,x_m) = \Theta\left(\frac{1}{N^\alpha}\int_\Lambda dw\ \left(\sum_{j=1}^m \chi_{\ell_B}(w-x_j)+1\right)^n\right). \tag{4.84}$$

Here $\mathcal{L}_n$ and χ_{ℓ_B} are defined by

$$\mathcal{L}_n = \int_\Lambda dw\ \left(\mathcal{N}_{\chi_{\ell_B}}(w)+1\right)^n$$

and

$$\mathcal{N}_{\chi_{\ell_B}}(w) = \int_\Lambda dv\,\chi_{\ell_B}(w-v)a_v^*a_v.$$

Let us verify explicitly why Θ acts as the multiplication operator (4.84). On the m-particle sector, we have

$$\begin{aligned}\left(\mathcal{N}_{\chi_{\ell_B}}(w)\psi^{(m)}\right)(x_1,\ldots,x_m) &= \left(\int_\Lambda dv\,\chi_{\ell_B}(w-v)a_v^*a_v\right)\psi^{(m)}(x_1,\ldots,x_m) \\ &= \int_\Lambda dv\sum_{j=1}^m \chi_{\ell_B}(w-v)\delta(v-x_j)\psi^{(m)}(x_1,\ldots,x_m) \\ &= \sum_{j=1}^m \chi_{\ell_B}(w-x_j)\psi^{(m)}(x_1,\ldots,x_m).\end{aligned}$$

Therefore

$$\left(\mathcal{L}_n\psi^{(m)}\right)(x_1,\dots,x_m) = \left[\int_\Lambda dw\,\left(\sum_{j=1}^m \chi_{\ell_B}(w-x_j)+1\right)^n\right]\psi^{(m)}(x_1,\dots,x_m).$$

Applying the smooth function $\Theta(\cdot/N^\alpha)$ gives exactly (4.83) and (4.84).

Moreover, the kinetic energy operator $\mathcal{K}$ also preserves the number of particles. On the m-particle sector,

$$(\mathcal{K}\Psi)^{(m)} = \sum_{j=1}^m(-\Delta_{x_j})\psi^{(m)}.$$

Hence the commutator $[\mathcal{K},\Theta]$ on the m-particle sector is given by

$$\begin{aligned}([\mathcal{K},\Theta]\Psi)^{(m)} &= \sum_{j=1}^m(-\Delta_{x_j})\left(F_m\psi^{(m)}\right) - F_m\sum_{j=1}^m(-\Delta_{x_j})\psi^{(m)}\\ &= -\sum_{j=1}^m(\Delta_{x_j}F_m)\psi^{(m)} - 2\sum_{j=1}^m\nabla_{x_j}F_m\cdot\nabla_{x_j}\psi^{(m)}. \end{aligned}\tag{4.85}$$

using integration by parts. Equivalently, as an operator on $L^2_s(\Lambda^m)$,

$$[\mathcal{K},\Theta] = -2\sum_{j=1}^m(\nabla_{x_j}F_m)\cdot\nabla_{x_j} - \sum_{j=1}^m\Delta_{x_j}F_m. \tag{4.86}$$

We now compute the derivative of F_m, by the chain rule,

$$\begin{aligned}\nabla_{x_j}F_m = \frac{n}{N^\alpha}\Theta'\left(\frac{1}{N^\alpha}\int_\Lambda dw\,\left(\sum_{i=1}^m\chi_{\ell_B}(x_i-w)+1\right)^n\right)\\ \times\int_\Lambda dw\,\left(\sum_{i=1}^m\chi_{\ell_B}(x_i-w)+1\right)^{n-1}\nabla\chi_{\ell_B}(x_j-w).\end{aligned}\tag{4.87}$$

By the definition of Θ, the derivative Θ' is supported in the interval $[1,2]$. Therefore, we can write

$$[\mathcal{K},\Theta] = \mathbb{1}_{[1,\infty)}\left(\frac{\mathcal{L}_n}{N^\alpha}\right)[\mathcal{K},\Theta],$$

and

$$\Theta'\left(\frac{1}{N^\alpha}\int_\Lambda dw\,\left(\sum_{i=1}^m\chi_{\ell_B}(x_i-w)+1\right)^n\right)\neq 0$$

if and only if

$$1\le\frac{1}{N^\alpha}\int_\Lambda dw\,\left(\sum_{i=1}^m\chi_{\ell_B}(x_i-w)+1\right)^n\le 2\,.$$

This implies that

$$\int_\Lambda dw\,\left(\sum_{i=1}^m\chi_{\ell_B}(x_i-w)+1\right)^n\le 2N^\alpha. \tag{4.88}$$

Notice that

$$\sum_{i=1}^m\chi_{\ell_B}(x_i-w)+1\ge 1.$$

Hence, for every $w \in \Lambda$,

$$\left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n-1} \leq \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n}.$$

We shall also use the elementary bounds

$$\|\nabla \chi_{\ell_B}\|_\infty = \sup_x |\nabla \chi_{\ell_B}(x)| = \ell_B^{-1} \sup_x |\nabla \chi(x)| \leq C\ell_B^{-1}, \tag{4.89}$$

and $\|\Theta'\|_\infty \leq C$, since $\Theta \in C_c^\infty(\mathbb{R})$. We now estimate $\nabla_{x_j} F_m$. From (4.87),

$$\begin{aligned}
\left|\nabla_{x_j} F_m(x_1, \ldots, x_m)\right|^2 &\leq \frac{n^2}{N^{2\alpha}} \left|\Theta'\left(\frac{1}{N^\alpha}\int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^n\right)\right|^2 \\
&\qquad \times \left|\int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n-1} \nabla \chi_{\ell_B}(x_j - w)\right|^2 \\
&\leq \frac{n^2}{N^{2\alpha}} \|\Theta'\|_\infty^2 \left|\int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n-1}\right|^2 \|\nabla \chi_{\ell_B}\|_\infty^2 \\
&\leq \frac{C_n}{N^{2\alpha}} \|\nabla \chi_{\ell_B}\|_\infty^2 \left[\int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n}\right]^2 \\
&\leq C_n \ell_B^{-2}.
\end{aligned} \tag{4.90}$$

where $C_n > 0$ is dependent on $n \in \mathbb{N}$. Next we estimate $\Delta_{x_j} F_m$.

$$\begin{aligned}
\Delta_{x_j} F_m = \left(\frac{n}{N^\alpha}\right)^2 \Theta''&\left(\frac{1}{N^\alpha}\int_\Lambda dw \sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1^n\right) \\
&\times \left[\int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n-1}\right]^2 \left(\nabla \chi_{\ell_B}(x_j - w)\right)^2 \\
+ \frac{n(n-1)}{N^\alpha} \Theta'&\left(\frac{1}{N^\alpha}\int_\Lambda dw \sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1^n\right) \\
&\times \int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n-2} \left(\nabla \chi_{\ell_B}(x_j - w)\right)^2 \\
+ \frac{n}{N^\alpha} \Theta'&\left(\frac{1}{N^\alpha}\int_\Lambda dw \sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1^n\right) \\
&\times \int_\Lambda dw \left(\sum_{i=1}^{m} \chi_{\ell_B}(x_i - w) + 1\right)^{n-1} \Delta \chi_{\ell_B}(x_j - w).
\end{aligned}$$

Using (4.88), (4.89), and $\|\Delta \chi_{\ell_B}\|_\infty \leq C\ell_B^{-2}$, we obtain

$$\left|\Delta_{x_j} F_m(x_1, \ldots, x_m)\right| \leq C_n \ell_B^{-2}. \tag{4.91}$$

Now,

$$\left\| \frac{1}{m+1} \sum_{j=1}^{m} [-\Delta_{x_j}, F_m] \xi_s^{(m)} \right\| \tag{4.92}$$

$$\leq \left\| \frac{2}{m+1} \sum_{j=1}^{m} (\nabla_{x_j} F_m) \cdot \nabla_{x_j} \xi_s^{(m)} \right\| + \left\| \frac{1}{m+1} \sum_{j=1}^{m} (\Delta_{x_j} F_m) \xi_s^{(m)} \right\|$$

$$\leq \frac{2}{m+1} \sum_{j=1}^{m} \left\| (\nabla_{x_j} F_m) \cdot \nabla_{x_j} \xi_s^{(m)} \right\| + \frac{1}{m+1} \sum_{j=1}^{m} \left\| (\Delta_{x_j} F_m) \xi_s^{(m)} \right\|. \tag{4.93}$$

We estimate the two terms separately. First, for each $j = 1, \ldots, m$,

$$\begin{aligned} \left\| (\nabla_{x_j} F_m) \cdot \nabla_{x_j} \xi_s^{(m)} \right\|^2 &= \int_{\Lambda^m} \left| (\nabla_{x_j} F_m)(x_1, \ldots, x_m) \cdot (\nabla_{x_j} \xi_s^{(m)})(x_1, \ldots, x_m) \right|^2 dx_1 \cdots dx_m \\ &\leq \int_{\Lambda^m} \left| \nabla_{x_j} F_m(x_1, \ldots, x_m) \right|^2 \left| \nabla_{x_j} \xi_s^{(m)}(x_1, \ldots, x_m) \right|^2 dx_1 \cdots dx_m \\ &\leq C \ell_B^{-2} \left\| \nabla_{x_j} \xi_s^{(m)} \right\|_2^2, \end{aligned} \tag{4.94}$$

where we used (4.90) in the last step. Similarly, using (4.90), we obtain

$$\begin{aligned} \left\| (\Delta_{x_j} F_m) \xi_s^{(m)} \right\|^2 &= \int_{\Lambda^m} \left| (\Delta_{x_j} F_m)(x_1, \ldots, x_m) \xi_s^{(m)}(x_1, \ldots, x_m) \right|^2 dx_1 \cdots dx_m \\ &= \int_{\Lambda^m} \left| \Delta_{x_j} F_m(x_1, \ldots, x_m) \right|^2 \left| \xi_s^{(m)}(x_1, \ldots, x_m) \right|^2 dx_1 \cdots dx_m \\ &\leq C \ell_B^{-4} \left\| \xi_s^{(m)} \right\|_2^2. \end{aligned} \tag{4.95}$$

Combining (4.93), (4.94), (4.95), and using Cauchy–Schwarz, we find

$$\begin{aligned} \left\| \frac{1}{\mathcal{N}+1} [\mathcal{K}, \Theta] \xi_s \right\|^2 &= \sum_{m \geq 0} \left\| \frac{1}{m+1} \sum_{j=1}^{m} [-\Delta_{x_j}, F_m] \xi_s^{(m)} \right\|^2 \\ &\leq C \ell_B^{-2} \sum_{m \geq 0} \frac{1}{m+1} \sum_{j=1}^{m} \left\| \nabla_{x_j} \xi_s^{(m)} \right\|_2^2 + C \ell_B^{-4} \sum_{m \geq 0} \left\| \xi_s^{(m)} \right\|_2^2 \\ &\leq C \ell_B^{-4} \left\| (\mathcal{K}+1)^{1/2} \xi_s \right\|^2 \end{aligned} \tag{4.96}$$

because $\|\xi_s\| = 1$ and $\ell_B^{-2} \leq \ell_B^{-4}$. Therefore,

$$\begin{aligned} \left| \int_0^t ds \, \langle \xi_s, A^* [\mathcal{K}, \Theta] \xi_s \rangle \right| &\leq \int_0^t ds \, \left\| (\mathcal{N}+1) \mathbb{1}_{[1,\infty)} \left(\frac{\mathcal{L}_n}{N^\alpha} \right) A \xi_s \right\| \left\| \frac{1}{\mathcal{N}+1} [\mathcal{K}, \Theta] \xi_s \right\| \\ &\leq C \ell_B^{-2} \int_0^t ds \, \left\| (\mathcal{N}+1) \mathbb{1}_{[1,\infty)} \left(\frac{\mathcal{L}_n}{N^\alpha} \right) A \xi_s \right\| \left\| (\mathcal{K}+1)^{1/2} \xi_s \right\|. \end{aligned} \tag{4.97}$$

By [13, the remark after Lemma 11], for every $\nu > 0$, if $\alpha > 0$ and $n \in \mathbb{N}$ are chosen sufficiently large, we have

$$\left\| (\mathcal{N}+1) \mathbb{1}_{[1,\infty)} \left(\frac{\mathcal{L}_n}{N^\alpha} \right) A \xi_s \right\| = \left\langle \xi_s, A^* (\mathcal{N}+1)^2 \mathbb{1}_{[1,\infty)} \left(\frac{\mathcal{L}_n}{N^\alpha} \right) A \xi_s \right\rangle^{1/2} \leq C N^{-\nu}. \tag{4.98}$$

Inserting (4.98) into (4.97), we obtain

$$\left| \int_0^t ds \, \langle \xi_s, A^*[\mathcal{K}, \Theta] \xi_s \rangle \right| \leq C N^{-\nu} \int_0^t ds \, \left\| (\mathcal{K}+1)^{1/2} \xi_s \right\|. \tag{4.99}$$

Combining (4.78), (4.82), and (4.99), we find that, for all $t \in [0,1]$,

$$\langle \xi_t, \mathcal{K} \xi_t \rangle \leq C N^{1+\frac{5\kappa}{2}} + C N^{-\nu} \int_0^t ds \, \left\| (\mathcal{K}+1)^{1/2} \xi_s \right\|.$$

By Grönwall's lemma, we obtain

$$\langle \xi_t, \mathcal{K} \xi_t \rangle \leq C N^{1+\frac{5\kappa}{2}} \qquad \text{for all } t \in [0,1]. \tag{4.100}$$

Plugging (4.100) back into (4.99), and choosing $\alpha > 0$ and $n \in \mathbb{N}$ sufficiently large, we obtain

$$\left| \int_0^t ds \, \langle \xi_s, A^*[\mathcal{K}, \Theta] \xi_s \rangle \right| \leq C N^{4\kappa - 1}.$$

Therefore,

$$\langle \xi_t, \mathcal{K} \xi_t \rangle = 2\mathrm{Re} \int_0^t ds \, \langle \xi_s, [\mathcal{K}, A^*] \xi_s \rangle + O(N^{4\kappa-1}). \tag{4.101}$$

We now switch the commutator $[\mathcal{K}, A^*]$ to position space. From (4.79), we have

$$\begin{aligned}
[\mathcal{K}, A^*] &= -N^{\kappa - \frac{1}{2}} \sum_{p,q \in \Lambda^*} \widetilde{\chi}(p) \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) \sigma_q \int_{\Lambda^3} dx dy dz \, e^{-i(p+q)\cdot x} e^{ip\cdot y} e^{iq\cdot z} a_x^* a_y^* a_z^* \\
&= -N^{1/2} \int_{\Lambda^3} dx dy dz \, N^{2-2\kappa} (Vf)\big(N^{1-\kappa}(x-y)\big) \check{\sigma}(x-z) a_x^* a_y^* a_z^* \\
&\qquad + \int_{\Lambda^3} dx dy dz \, \check{A}_{\mathcal{K}}^{\mathrm{low}}(x-y, x-z) a_x^* a_y^* a_z^*. \\
&=: \mathrm{I} + \mathrm{II}
\end{aligned}$$

where $A_{\mathcal{K}}^{\mathrm{low}}$ is defined in [13, Equation (83)]. For II, we apply [13, Lemma 14]:

$$\begin{aligned}
\left| \int_0^t ds \, \langle \xi_s, \mathrm{II} \, \xi_s \rangle \right| &\leq \left| \left\langle \Omega, \int_{\Lambda^3} dx dy dz \, \check{A}_{\mathcal{K}}^{\mathrm{low}}(x-y, x-z) [A, a_x^* a_y^* a_z^*] \Omega \right\rangle \right| \\
&\quad + C \ell_B^3 N^{\frac{3\kappa-1}{2} + \varepsilon} \left(\sum_{u,v \in \Lambda_B} \left\| \check{A}_{\mathcal{K},u,v}^{\mathrm{low}} \right\|_2 \right) \int_0^t ds \, \langle \xi_s, \mathcal{N} \xi_s \rangle + C N^{-\nu}.
\end{aligned} \tag{4.102}$$

The first term on the right-hand side is bounded by $C \|A\|_2 \|A_{\mathcal{K}}^{\mathrm{low}}\|_2$. Using $\|A\|_2 \leq C N^{\frac{3\kappa-1}{2}}$, and $\|A_{\mathcal{K}}^{\mathrm{low}}\|_2 \leq C N^{\frac{5\kappa-1}{2}+\varepsilon}$, we obtain, for all $\varepsilon > 0$,

$$C \|A\|_2 \|A_{\mathcal{K}}^{\mathrm{low}}\|_2 \leq C N^{4\kappa - 1 + \varepsilon}. \tag{4.103}$$

For the second term in (4.102), we use

$$\ell_B = N^{-\kappa/2 + \beta/12}, \qquad \sum_{u,v \in \Lambda_B} \left\| \check{A}_{\mathcal{K},u,v}^{\mathrm{low}} \right\|_2 \leq C N^{\frac{5\kappa-1}{2}+\varepsilon},$$

and the number estimate

$$\int_0^t ds \, \langle \xi_s, \mathcal{N} \xi_s \rangle \leq C N^{3\kappa - 1}.$$

Therefore

$$\begin{aligned} & C\ell_B^3 N^{\frac{3\kappa-1}{2}+\varepsilon} \left(\sum_{u,v\in\Lambda_B} \left\| \check{A}^{\mathrm{low}}_{\mathcal{K},u,v} \right\|_2 \right) \int_0^t ds \, \langle \xi_s, \mathcal{N}\xi_s \rangle \\ & \leq C N^{-\frac{3\kappa}{2}+\frac{\beta}{4}} N^{\frac{3\kappa-1}{2}+\varepsilon} N^{\frac{5\kappa-1}{2}+\varepsilon} N^{3\kappa-1} \\ & = C N^{4\kappa-1-\frac{\beta}{4}+2\varepsilon}. \end{aligned} \tag{4.104}$$

Combining (4.102), (4.103), and (4.104), we obtain that

$$\left| \int_0^t ds \, \langle \xi_s, \mathrm{II}\, \xi_s \rangle \right| \leq C N^{4\kappa-1}. \tag{4.105}$$

Hence the term II can be absorbed into the error.

It remains to analyze I. This term has the same structure as the cubic operator $\mathcal{C}_N^*$. The only differences are that the prefactor and the potential. Therefore, proceeding exactly as in the proof of [13, Lemma 24], we obtain, for every $s \in [0,1]$,

$$2\mathrm{Re}\, \langle \xi_s, [\mathcal{K}, A^*]\xi_s \rangle = -2sN^{1/2} \int dxdy \, N^{2-2\kappa} (Vf)(N^{1-\kappa}x) \check{\sigma}(y) \check{A}^{(6)}(x,y) + O(N^{4\kappa-1}).$$

Therefore,

$$\langle \xi_t, \mathcal{K}\xi_t \rangle = -t^2 N^{1/2} \int_{\Lambda^2} dxdy \, N^{2-2\kappa} (Vf)(N^{1-\kappa}x) \check{\sigma}(y) \check{A}^{(6)}(x,y) + O(N^{4\kappa-1}) \,,$$

which concludes the proof.

□

4.4 Proof of Theorem 2

We now combine all the estimates for the cubic operator $\mathcal{C}_N$, potential energy operator $\mathcal{V}_N$ and the kinetic energy operator $\mathcal{K}$ into the following lemma from [13, Lemma 27].

Lemma 18. *Let $\mathcal{C}_N$ be the cubic operator defined in (4.14), $\mathcal{V}_N$ the quartic potential energy operator and $\mathcal{K}$ the quadratic kinetic energy operator. Then, we have*

$$\begin{aligned} & \langle \xi, (\mathcal{K} + \mathcal{C}_N + \mathcal{C}_N^* + \mathcal{V}_N)\xi \rangle \\ & \quad = N^{\kappa-1} \sum_{p,q\in\Lambda^*} \hat{V}(p/N^{1-\kappa})(\eta_{\infty,p} + \eta_{\infty,p+q})\sigma_q^2 \\ & \qquad - N^{4\kappa-1} \sum_{|p|,|q|\geq N^{\kappa/2}} (\widehat{Vf}(p/N^{1-\kappa}) + \widehat{Vf}((p+q)/N^{1-\kappa})) \\ & \qquad\qquad \times \frac{\widehat{Vf}(p/N^{1-\kappa})\, \widehat{Vf}(q/N^{1-\kappa})\, \widehat{Vf}((p+q)/N^{1-\kappa})}{4p^2q^2(p^2+q^2+(p+q)^2)} \\ & \qquad + N^{4\kappa-1} \sum_{|p|,|q|\geq N^{\kappa/2}} (\widehat{Vf}(p/N^{1-\kappa}) + \widehat{Vf}((p+q)/N^{1-\kappa})) \\ & \qquad\qquad \times \frac{(p\cdot q)\, \widehat{Vf}(p/N^{1-\kappa})\, \widehat{Vf}(q/N^{1-\kappa})^2}{4q^4p^2(p^2+q^2+(p+q)^2)} + O(N^{4\kappa-1}) \,. \end{aligned}$$

By using Lemma 13, Lemma 18 and Lemma 14, we are now able to prove Theorem 2.

Proof of Theorem 2. By Lemma 13 with Lemma 14 and Lemma 18, we obtain, for any $\kappa \geq 1/2$,

$$
\begin{aligned}
&\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle \\
&\quad = 4\pi \mathfrak{a} N^{1+\kappa} \\
&\qquad + \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[\sqrt{|p|^4 + 2p^2 N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)} - p^2 - N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) + \frac{N^{2\kappa} \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)^2}{2|p|^2} \right] \\
&\qquad - N^{4\kappa-1} \sum_{\substack{p,q \in \Lambda^* \\ |p|,|q| \geq N^{\kappa/2}}} \left[\widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) + \widehat{Vf}\left(\frac{p+q}{N^{1-\kappa}}\right) \right] \frac{\widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) \widehat{Vf}\left(\frac{q}{N^{1-\kappa}}\right) \widehat{Vf}\left(\frac{p+q}{N^{1-\kappa}}\right)}{4p^2 q^2 \left(p^2 + q^2 + (p+q)^2\right)} \\
&\qquad + N^{4\kappa-1} \sum_{\substack{p,q \in \Lambda^* \\ |p|,|q| \geq N^{\kappa/2}}} \left[\widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) + \widehat{Vf}\left(\frac{p+q}{N^{1-\kappa}}\right) \right] \frac{(p \cdot q) \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) \widehat{Vf}\left(\frac{q}{N^{1-\kappa}}\right)^2}{4q^4 p^2 \left(p^2 + q^2 + (p+q)^2\right)} \\
&\qquad + O(N^{4\kappa-1}) \\
&\quad =: 4\pi \mathfrak{a} N^{1+\kappa} + S_1 + S_2 + S_3 + O(N^{4\kappa-1}).
\end{aligned}
$$

We first consider the term S_1. Observe that,

$$
\begin{aligned}
&\sqrt{|p|^4 + 2p^2 N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)} - p^2 - N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) + \frac{N^{2\kappa} \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)^2}{2|p|^2} \\
&= p^2 \left[\sqrt{1 + \frac{2N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2}} - 1 - \frac{N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2} + \frac{N^{2\kappa} \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)^2}{2p^4} \right].
\end{aligned}
\tag{4.106}
$$

We now distinguish two momentum regions.

Case 1: $|p| > N^{1-\kappa}$**.** Since $\widehat{Vf}$ is bounded, we have

$$
\left| \frac{N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2} \right| \leq C \frac{N^\kappa}{N^{2-2\kappa}} = C N^{3\kappa-2}. \tag{4.107}
$$

Since $\kappa \in [1/2, 2/3)$, we have $3\kappa - 2 < 0$. Hence the quantity in (4.107) is an inverse power of N and is therefore small for N large. In particular, we may use the Taylor expansion of $\sqrt{1+2x}$ around $x = 0$:

$$
\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + O(x^3).
$$

Taking

$$
x = \frac{N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2},
$$

and inserting this into (4.106), we obtain

$$
\begin{aligned}
&\left| p^2 \left[\sqrt{1 + \frac{2N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2}} - 1 - \frac{N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2} + \frac{N^{2\kappa} \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)^2}{2p^4} \right] \right| \\
&\qquad \leq C p^2 \left| \frac{N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)}{p^2} \right|^3 \leq C \frac{N^{3\kappa}}{|p|^4}.
\end{aligned}
\tag{4.108}
$$

We now sum the estimate (4.108) over the high momentum region $|p| > N^{1-\kappa}$. We obtain

$$\sum_{\substack{p\in\Lambda_+^* \\ |p|>N^{1-\kappa}}} \left| \sqrt{|p|^4 + 2p^2 N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)} - p^2 - N^\kappa \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) + \frac{N^{2\kappa}\widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right)^2}{2p^2} \right| \leq C \sum_{\substack{p\in\Lambda_+^* \\ |p|>N^{1-\kappa}}} \frac{N^{3\kappa}}{|p|^4}. \tag{4.109}$$

Using the standard lattice-sum estimate

$$\sum_{\substack{p\in\Lambda_+^* \\ |p|>R}} \frac{1}{|p|^4} \leq C \int_R^\infty dr\, \frac{r^2}{r^4} = \frac{C}{R},$$

with $R = N^{1-\kappa}$, we find

$$\sum_{\substack{p\in\Lambda_+^* \\ |p|>N^{1-\kappa}}} \frac{N^{3\kappa}}{|p|^4} \leq C N^{3\kappa} N^{-1+\kappa} = C N^{4\kappa-1}.$$

Hence the contribution of momenta $|p| > N^{1-\kappa}$ is negligible.

Case 2: $|p| \leq N^{1-\kappa}$**.** In this case, we introduce, for $t \in [0,1]$, the function

$$g_p(t) := \sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)} - p^2 - N^\kappa \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right) + \frac{N^{2\kappa}}{2p^2}\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)^2.$$

We compute the first derivative of g_p. By the chain rule,

$$\begin{aligned} g_p'(t) &= \frac{1}{2}\left(p^4 + 2p^2 N^\kappa \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\right)^{-1/2} 2p^2 N^\kappa \nabla\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot\frac{p}{N^{1-\kappa}} \\ &\quad - N^\kappa \nabla\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot\frac{p}{N^{1-\kappa}} + \frac{N^{2\kappa}}{2p^2}\cdot 2\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\nabla\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot\frac{p}{N^{1-\kappa}} \\ &= N^\kappa \frac{p}{N^{1-\kappa}}\cdot\nabla\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\left[\frac{p^2}{\sqrt{p^4 + 2p^2 N^\kappa \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)}} - 1 + \frac{N^\kappa}{p^2}\widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\right]. \end{aligned} \tag{4.110}$$

Let us now compute the derivative of g_p at $t = 0$. We write

$$\widetilde{p}_t := \frac{tp}{N^{1-\kappa}}.$$

Therefore

$$\nabla\widehat{Vf}(\widetilde{p}_t) = -i\int_{\mathbb{R}^3} x\, V(x) f(x) e^{-i\widetilde{p}_t\cdot x}\, dx.$$

In particular,

$$\nabla\widehat{Vf}(0) = -i\int_{\mathbb{R}^3} x\, V(x) f(x)\, dx = 0.$$

because V and f are radial, the function $xV(x)f(x)$ is odd, therefore vanishes. Hence, from (4.110),

$$g_p'(0) = 0. \tag{4.111}$$

By the Duhamel expansion, we can write

$$S_1 = \frac{1}{2} \sum_{\substack{p\in\Lambda_+^* \\ |p|\leq N^{1-\kappa}}} \int_0^1 ds \int_0^s dt\, g_p''(t) + \frac{1}{2} \sum_{\substack{p\in\Lambda_+^* \\ |p|\leq N^{1-\kappa}}} g_p(0) + O(N^{4\kappa-1}).$$

Then, we control $g_p''(t)$ more explicitly. We differentiate (4.110) once more to get

$$\begin{aligned} g_p''(t) = N^{2\kappa-1} p\cdot\nabla\left[\nabla\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot\frac{p}{N^{1-\kappa}}\right]\left[\frac{|p|^2}{\sqrt{|p|^4+2|p|^2N^\kappa\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)}}-1+\frac{N^\kappa}{|p|^2}\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\right] \\ + N^{2\kappa}\left(\nabla\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot\frac{p}{N^{1-\kappa}}\right)^2\left[-\frac{|p|^4}{|p|^4+2|p|^2N^\kappa\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)^{3/2}}+\frac{1}{|p|^2}\right]. \end{aligned} \tag{4.112}$$

Denote by $D^2\widehat{V f}$ the Hessian matrix of $\widehat{V f}$, then

$$\begin{aligned} p\cdot\nabla\left[\nabla\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot p\right] &= p^T\left[D^2\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\right]p \\ &= \sum_{i,j=1}^3 \partial_i\partial_j\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)p_ip_j. \end{aligned}$$

Consequently,

$$p\cdot\nabla\left[\nabla\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot\frac{p}{N^{1-\kappa}}\right] = \frac{1}{N^{1-\kappa}}p^T\left[D^2\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\right]p. \tag{4.113}$$

We now pass to position space: for $k\in\mathbb{R}^3$,

$$\widehat{V f}(k) = \int V(x)f(x)e^{-ik\cdot x}\,dx.$$

Hence

$$\partial_i\widehat{V f}(k) = -i\int x_iV(x)f(x)e^{-ik\cdot x}\,dx,$$

and

$$\partial_i\partial_j\widehat{V f}(k) = -\int x_ix_jV(x)f(x)e^{-ik\cdot x}\,dx.$$

Therefore, we have

$$\begin{aligned} \nabla\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot p &= -i\int (p\cdot x)V(x)f(x)e^{-i\frac{tp}{N^{1-\kappa}}\cdot x}\,dx, \\ p^T\left[D^2\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\right]p &= -\int (p\cdot x)^2V(x)f(x)e^{-i\frac{tp}{N^{1-\kappa}}\cdot x}\,dx. \end{aligned}$$

Since V has compact support and f is bounded, we have

$$\begin{aligned} \left|\nabla\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\cdot p\right| &\leq C|p|\int |x|\,|V(x)|\,|f(x)|\,dx \leq C|p|, \\ \left|p^T\left[D^2\widehat{V f}\left(\frac{tp}{N^{1-\kappa}}\right)\right]p\right| &\leq C|p|^2\int |x|^2|V(x)|\,|f(x)|\,dx \leq C|p|^2. \end{aligned} \tag{4.114}$$

In particular, by (4.113),

$$\left| p \cdot \nabla \left[\nabla \widehat{Vf} \left(\frac{tp}{N^{1-\kappa}} \right) \cdot \frac{p}{N^{1-\kappa}} \right] \right| \leq C \frac{|p|^2}{N^{1-\kappa}}. \tag{4.115}$$

Now, if $|p| \leq KN^{\kappa/2}$, where $K > 0$ is fixed to be large, we have

$$\begin{aligned} &\left| \frac{|p|^2}{\sqrt{|p|^4 + 2|p|^2 N^\kappa \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)}} - 1 + \frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right) \right| \\ &= \left| \frac{1}{\sqrt{1 + 2\frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)}} - 1 + \frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right) \right| \leq C \frac{N^\kappa}{|p|^2}. \end{aligned} \tag{4.116}$$

Moreover,

$$\left| -\frac{|p|^4}{|p|^4 + 2|p|^2 N^\kappa \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)^{3/2}} + \frac{1}{|p|^2} \right| = \frac{1}{|p|^2} \left| -\frac{1}{\left(1 + 2\frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\right)^{3/2}} + 1 \right| \leq \frac{C}{|p|^2}. \tag{4.117}$$

Combining (4.112), (4.114), (4.115), (4.116), and (4.117), we get, for $|p| \leq KN^{\kappa/2}$,

$$|g_p''(t)| \leq CN^{3\kappa-2}|p|^2 \frac{N^\kappa}{|p|^2} + CN^{4\kappa-2}|p|^2 \frac{1}{|p|^2} \leq CN^{4\kappa-2}.$$

If $|p| > KN^{\kappa/2}$, where $K > 0$ is chosen sufficiently large, we have $\frac{N^\kappa}{|p|^2} \leq \frac{1}{K^2}$, and hence the quantity

$$\frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)$$

is small, provided K is large enough. Therefore we can use the Taylor expansion

$$(1 + 2x)^{-1/2} = 1 - x + O(x^2).$$

With

$$x = \frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right),$$

we get

$$\left| \frac{1}{\sqrt{1 + 2\frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)}} - 1 + \frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right) \right| \leq C \frac{N^{2\kappa}}{|p|^4} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)^2 \leq C \frac{N^{2\kappa}}{|p|^4}. \tag{4.118}$$

Similarly, using

$$(1 + 2x)^{-3/2} = 1 + O(x),$$

we obtain

$$\left| -\frac{1}{|p|^2 \left(1 + 2\frac{N^\kappa}{|p|^2} \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right)\right)^{3/2}} + \frac{1}{|p|^2} \right| \leq \frac{C}{|p|^2} \frac{N^\kappa}{|p|^2} \left| \widehat{Vf}\left(\frac{tp}{N^{1-\kappa}}\right) \right| \leq C \frac{N^\kappa}{|p|^4}. \tag{4.119}$$

Inserting (4.118) and (4.119) into the expression for $g_p''(t)$, we find

$$|g_p''(t)| \leq CN^{3\kappa-2}|p|^2\frac{N^{2\kappa}}{|p|^4} + CN^{4\kappa-2}|p|^2\frac{N^{\kappa}}{|p|^4} \leq CN^{4\kappa-2}\frac{N^{\kappa}}{|p|^2}.$$

Therefore, we obtain

$$S_1 = \frac{1}{2}\sum_{\substack{p\in\Lambda_+^* \\ |p|\leq N^{1-\kappa}}} \left[\sqrt{|p|^4 + 16\pi\mathfrak{a}N^{\kappa}p^2} - p^2 - 8\pi\mathfrak{a}N^{\kappa} + \frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{2|p|^2}\right] + O(N^{4\kappa-1}).$$

Let

$$h(p) = \sqrt{|p|^4 + 16\pi\mathfrak{a}N^{\kappa}p^2} - p^2 - 8\pi\mathfrak{a}N^{\kappa} + \frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{2|p|^2}.$$

Since $\Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$, every lattice point $p \in \Lambda_+^*$ has an associated cube $p + [-\pi,\pi]^3$. These cubes tile $\mathbb{R}^3$, and each cube has volume $(2\pi)^3$. Hence, trivially,

$$h(p) = \frac{1}{(2\pi)^3}\int_{[-\pi,\pi]^3} d\zeta\, h(p)\,.$$

Then

$$\begin{aligned}\sum_{p\in\Lambda_+^*} h(p) &= \frac{1}{(2\pi)^3}\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\, h(p) \\ &= \frac{1}{(2\pi)^3}\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\, h(p+\zeta) + \frac{1}{(2\pi)^3}\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\, \big(h(p) - h(p+\zeta)\big) \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} dp\, h(p) + \frac{1}{(2\pi)^3}\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\, \big(h(p) - h(p+\zeta)\big).\end{aligned} \tag{4.120}$$

Changing to spherical coordinates gives

$$\begin{aligned}\left|\int_{[-\pi,\pi]^3} dp\, h(p)\right| &\leq C\int_{-\pi}^{\pi} dr\, \left(\sqrt{r^4 + 16\pi\mathfrak{a}N^{\kappa}r^2} - r^2 - 8\pi\mathfrak{a}N^{\kappa} + \frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{2r^2}\right) r^2 \\ &\leq CN^{2\kappa} \leq CN^{4\kappa-1}.\end{aligned} \tag{4.121}$$

We now rewrite h as a radial function. Set $r = |p|$. Then $h(p) = H(r)$, where

$$H(r) = \sqrt{r^4 + 16\pi\mathfrak{a}N^{\kappa}r^2} - r^2 - 8\pi\mathfrak{a}N^{\kappa} + \frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{2r^2}. \tag{4.122}$$

Define

$$k(s) \coloneqq p + s\zeta, \qquad \rho(s) \coloneqq |k(s)| = |p + s\zeta|, \qquad s \in [0,1].$$

Then

$$\frac{d^2}{ds^2}h(p+s\zeta) = \frac{d^2}{ds^2}H(\rho(s)) = H''(\rho(s))\big(\rho'(s)\big)^2 + H'(\rho(s))\rho''(s). \tag{4.123}$$

We compute $\rho'(s)$ and $\rho''(s)$ explicitly. Since $\rho(s) = |k(s)| = \sqrt{k(s)\cdot k(s)}$, we have

$$\rho'(s) = \frac{k(s)\cdot\zeta}{\rho(s)}\,.$$

Differentiating again gives

$$\rho''(s) = \frac{d}{ds}\left(\frac{k(s)\cdot\zeta}{\rho(s)}\right) = \frac{\zeta\cdot\zeta}{\rho(s)} - \frac{(k(s)\cdot\zeta)\rho'(s)}{\rho(s)^2} = \frac{|\zeta|^2}{\rho(s)} - \frac{(k(s)\cdot\zeta)^2}{\rho(s)^3}\,.$$

Thus,

$$\frac{d^2}{ds^2}h(p+s\zeta) = H''(\rho(s))\frac{(k(s)\cdot\zeta)^2}{\rho(s)^2} + H'(\rho(s))\left(\frac{|\zeta|^2}{\rho(s)} - \frac{(k(s)\cdot\zeta)^2}{\rho(s)^3}\right). \tag{4.124}$$

From (4.122), we compute

$$\begin{aligned} H'(\rho) &= \frac{1}{2}\big(\rho^4 + 16\pi\mathfrak{a}N^\kappa\rho^2\big)^{-1/2}\big(4\rho^3 + 32\pi\mathfrak{a}N^\kappa\rho\big) - 2\rho - \frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{\rho^3} \\ &= \frac{2\big(\rho^2 + 8\pi\mathfrak{a}N^\kappa\big)}{\sqrt{\rho^2 + 16\pi\mathfrak{a}N^\kappa}} - 2\rho - \frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{\rho^3}. \end{aligned} \tag{4.125}$$

Define $\widetilde{\rho} \coloneqq \frac{8\pi\mathfrak{a}N^\kappa}{\rho^2}$. Since $\rho \le CN^{1-\kappa}$, we have

$$\widetilde{\rho} \ge \frac{8\pi\mathfrak{a}N^\kappa}{N^{2-2\kappa}} \ge 8\pi\mathfrak{a}N^{3\kappa-2}\,.$$

For small $\widetilde{\rho}$, Taylor expansion

$$\frac{1+x}{\sqrt{1+2x}} = 1 + \frac{x^2}{2} + O(x^3).$$

gives

$$\frac{1}{\rho}H'(\rho) = \left(\frac{1+\widetilde{\rho}}{\sqrt{1+2\widetilde{\rho}}} - 1 - \frac{\widetilde{\rho}^2}{2}\right) = O(\widetilde{\rho}^3).$$

In particular, we have the bound

$$\left|\frac{1}{\rho}H'(\rho)\right| \le C\widetilde{\rho}^3 \le C\widetilde{\rho}^2 = C\frac{N^{2\kappa}}{\rho^4}\,, \tag{4.126}$$

for small $\widetilde{\rho}$. If $\widetilde{\rho}$ is large, the same weaker estimate follows directly from (4.125).

We next compute $H''(\rho)$. Differentiating (4.125), we obtain

$$\begin{aligned} H''(\rho) &= \frac{4\rho\sqrt{\rho^2 + 16\pi\mathfrak{a}N^\kappa} - 2(\rho^2 + 8\pi\mathfrak{a}N^\kappa)(\rho^2 + 16\pi\mathfrak{a}N^\kappa)^{-1/2}\,\rho}{\rho^2 + 16\pi\mathfrak{a}N^\kappa} - 2 + \frac{3N^{2\kappa}(8\pi\mathfrak{a})^2}{\rho^4} \\ &= \frac{2\rho(\rho^2 + 24\pi\mathfrak{a}N^\kappa)}{(\rho^2 + 16\pi\mathfrak{a}N^\kappa)^{3/2}} - 2 + \frac{3N^{2\kappa}(8\pi\mathfrak{a})^2}{\rho^4} \\ &= 2\left(\frac{1+3\widetilde{\rho}}{(1+2\widetilde{\rho})^{3/2}} - 1 + \frac{3}{2}\widetilde{\rho}^2\right) \end{aligned}$$

For small $\widetilde{\rho}$, Taylor expansion

$$\frac{1+3x}{(1+2x)^{3/2}} = 1 - \frac{3}{2}x^2 + O(x^3),$$

gives

$$H''(\rho) = O(\widetilde{\rho}^3).$$

Again, we get the weaker bound

$$|H''(\rho)| \leq C\widetilde{\rho}^3 \leq C\widetilde{\rho}^2 = C\frac{N^{2\kappa}}{\rho^4} \tag{4.127}$$

for small $\widetilde{\rho}$. If $\widetilde{\rho}$ is not small, the same estimate follows directly from (4.127).

From (4.124), (4.126) and (4.127), we have

$$\begin{aligned}\left|\frac{d^2}{ds^2}h(p+s\zeta)\right| &\leq |H''(\rho(s))|\frac{(k(s)\cdot\zeta)^2}{\rho(s)^2} + \left|\frac{H'(\rho(s))}{\rho(s)}\right|\left(|\zeta|^2 + \frac{(k(s)\cdot\zeta)^2}{\rho(s)^2}\right) \\ &\leq C\frac{N^{2\kappa}}{|p+s\zeta|^4}.\end{aligned} \tag{4.128}$$

Now, by Duhamel expansion

$$\begin{aligned}&\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\,\big(h(p)-h(p+\zeta)\big) = -\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\int_0^1 ds\,\frac{d}{ds}h(p+s\zeta) \\ &= -\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\int_0^1 ds\int_0^s dt\,\frac{d^2}{dt^2}h(p+t\zeta) - \sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\int_0^1 ds\;\frac{d}{ds}h(p+s\zeta)\Big|_{s=0}.\end{aligned}$$

The second term vanishes. Indeed,

$$\frac{d}{ds}h(p+s\zeta)\Big|_{s=0} = \nabla h(p+s\zeta)\cdot\zeta|_{s=0} = \nabla h(p)\cdot\zeta\,.$$

Since h is radial, i.e. $h(p) = H(|p|)$, we have

$$\nabla h(p) = H'(|p|)\nabla|p| = H'(|p|)\frac{p}{|p|}.$$

Thus

$$\nabla h(-p) = -H'(|p|)\frac{p}{|p|} = -\nabla h(p),$$

which means that $\nabla h(p)$ is odd with respect to p. Since the lattice $\Lambda_+^* = 2\pi\mathbb{Z}^3\setminus\{0\}$ is symmetric under $p\mapsto -p$, we obtain $\sum_{p\in\Lambda_+^*}\nabla h(p)\cdot\zeta = 0$. Consequently,

$$\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\int_0^1 ds\;\frac{d}{ds}h(p+s\zeta)\Big|_{s=0} = 0.$$

Therefore, using (4.128), we arrive at

$$\begin{aligned}\left|\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\,\big(h(p)-h(p+\zeta)\big)\right| &\leq C\sum_{p\in\Lambda_+^*}\int_{[-\pi,\pi]^3} d\zeta\int_0^1 ds\int_0^s dt\,\frac{N^{2\kappa}}{|p+t\zeta|^4} \\ &\leq CN^{2\kappa}\sum_{p\in\Lambda_+^*}\frac{1}{|p|^4} \leq CN^{2\kappa}\int_1^\infty dr\,\frac{r^2}{r^4} \leq CN^{2\kappa}\,.\end{aligned} \tag{4.129}$$

Combining (4.120), (4.121), and (4.129), we arrive at

$$\begin{aligned}
S_1 &= \frac{1}{2(2\pi)^3}\int_{|p|\le R} dp\left[\sqrt{|p|^4+16\pi\mathfrak{a}N^\kappa p^2}-p^2-8\pi\mathfrak{a}N^\kappa+\frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{2|p|^2}\right]+O(N^{4\kappa-1})\\
&= \frac{1}{16\pi^3}\int_{|p|\le R} dp\left[\sqrt{|p|^4+16\pi\mathfrak{a}N^\kappa p^2}-p^2-8\pi\mathfrak{a}N^\kappa+\frac{N^{2\kappa}(8\pi\mathfrak{a})^2}{2|p|^2}\right]+O(N^{4\kappa-1})\\
&= \frac{(8\pi\mathfrak{a}N^\kappa)^{5/2}}{16\pi^3}\int_{|p|\le R} d\widetilde{p}\left[\sqrt{|\widetilde{p}|^4+2|\widetilde{p}|^2}-\widetilde{p}^2-1+\frac{1}{2|\widetilde{p}|^2}\right]+O(N^{4\kappa-1})\\
&= \frac{(8\pi\mathfrak{a}N^\kappa)^{5/2}}{16\pi^3}4\pi\int_0^R dr\, r^2\left[\sqrt{r^4+2r^2}-r^2-1+\frac{1}{2r^2}\right]+O(N^{4\kappa-1})\\
&= \frac{(8\pi\mathfrak{a}N^\kappa)^{5/2}}{4\pi^2}\int_0^R dr\left[r^3\sqrt{r^2+2}-r^4-r^2+\frac{1}{2}\right]+O(N^{4\kappa-1})
\end{aligned}$$

where we denote $R := N^{1-3\kappa/2}/\sqrt{8\pi\mathfrak{a}}$. Notice that

$$\int_0^R r^3\sqrt{r^2+2}\,dr = \left[\frac{1}{5}(r^2+2)^{5/2}-\frac{2}{3}(r^2+2)^{3/2}\right]_0^R. \tag{4.130}$$

Then, we obtain

$$\begin{aligned}
S_1 &= \frac{(8\pi\mathfrak{a}N^\kappa)^{5/2}}{4\pi^2}\left[\frac{1}{5}(r^2+2)^{5/2}-\frac{2}{3}(r^2+2)^{3/2}-\frac{r^5}{5}-\frac{r^3}{3}+\frac{r}{2}\right]_0^R+O(N^{4\kappa-1})\\
&= \frac{(8\pi\mathfrak{a}N^\kappa)^{5/2}}{4\pi^2}\left[\frac{1}{5}(R^2+2)^{5/2}-\frac{2}{3}(R^2+2)^{3/2}-\frac{R^5}{5}-\frac{R^3}{3}+\frac{R}{2}+\frac{8\sqrt{2}}{15}\right]+O(N^{4\kappa-1}).
\end{aligned}$$

By the binomial expansion

$$(1+x)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n}x^n, \qquad |x|<1,$$

we have

$$\begin{aligned}
\frac{1}{5}(R^2+2)^{5/2} &= \frac{R^5}{5}\left(1+\frac{2}{R^2}\right)^{5/2} = \frac{R^5}{5}\left(1+\frac{5}{R^2}+\frac{15}{2R^4}+O(R^{-6})\right)\\
&= \frac{R^5}{5}+R^3+\frac{3}{2}R+O(R^{-1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{2}{3}(R^2+2)^{3/2} &= \frac{2R^3}{3}\left(1+\frac{2}{R^2}\right)^{3/2} = \frac{2R^3}{3}\left(1+\frac{3}{R^2}+O(R^{-4})\right)\\
&= \frac{2R^3}{3}+2R+O(R^{-1}).
\end{aligned}$$

Therefore, all polynomial terms in R cancel:

$$\frac{1}{5}(R^2+2)^{5/2}-\frac{2}{3}(R^2+2)^{3/2}-\frac{R^5}{5}-\frac{R^3}{3}+\frac{R}{2} = O(R^{-1}).$$

Hence, after letting $R \to \infty$, we arrive at

$$\begin{aligned} S_1 &= \frac{(8\pi\mathfrak{a}N^\kappa)^{5/2}}{4\pi^2}\frac{8\sqrt{2}}{15} + O(N^{4\kappa-1}) \\ &= \frac{512\sqrt{\pi}}{15}\,\mathfrak{a}^{5/2}N^{5\kappa/2} + O(N^{4\kappa-1}). \end{aligned} \tag{4.131}$$

We now turn to S_2. Set

$$\widetilde{p} = \frac{p}{N^{1-\kappa}}, \qquad \widetilde{q} = \frac{q}{N^{1-\kappa}}.$$

Firstly,

$$\begin{aligned} \sum_{\substack{|p|>N^{1-\kappa} \\ N^{\kappa/2}<|q|\le N^{1-\kappa}}} \frac{1}{|p|^4|q|^2} &\le C\int_{|p|>N^{1-\kappa}} dp\,\frac{1}{|p|^4}\int_{N^{\kappa/2}<|q|\le N^{1-\kappa}} dq\,\frac{1}{|q|^2} \\ &= C\int_{|\widetilde{p}|>1} d\widetilde{p}\,\frac{1}{|\widetilde{p}|^4}\int_{N^{3\kappa/2-1}<|\widetilde{q}|\le 1} d\widetilde{q}\,\frac{1}{|\widetilde{q}|^2} \le C. \end{aligned} \tag{4.132}$$

Secondly, we have

$$\begin{aligned} &\sum_{|p|,|q|>N^{1-\kappa}} \Big|\widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big)\Big|\,\frac{1}{|p|^2|q|^4} \\ &\le C\int_{|p|,|q|>N^{1-\kappa}} dp\,dq\,\Big|\widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big)\Big|\,\frac{1}{|p|^2|q|^4} \\ &= C\int_{|\widetilde{p}|,|\widetilde{q}|>1} d\widetilde{p}\,d\widetilde{q}\,|\widehat{Vf}(\widetilde{p})|\frac{1}{|\widetilde{p}|^2|\widetilde{q}|^4} \\ &\le C\int_{|\widetilde{q}|>1} d\widetilde{q}\,\frac{1}{|\widetilde{q}|^4}\left(\int_{|\widetilde{p}|>1} d\widetilde{p}\,|\widehat{Vf}(\widetilde{p})|^2\right)^{1/2}\left(\int_{|\widetilde{p}|>1} d\widetilde{p}\,\frac{1}{|\widetilde{p}|^4}\right)^{1/2} \le C. \end{aligned} \tag{4.133}$$

Hence, for the leading logarithmic contribution to S_2, it is enough to consider

$$N^{\kappa/2} \le |p|, |q| \le N^{1-\kappa}.$$

Observe that

$$\begin{aligned} \widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big) - 8\pi\mathfrak{a} &= \int V(x)f(x)e^{-i\frac{p}{N^{1-\kappa}}\cdot x}\,dx - \int_{\mathbb{R}^3} V(x)f(x)\,dx \\ &= \int V(x)f(x)\left(e^{-i\frac{p}{N^{1-\kappa}}\cdot x} - 1\right)dx \end{aligned} \tag{4.134}$$

By Taylor expansion,

$$e^{-i\frac{p}{N^{1-\kappa}}\cdot x} = 1 - i\frac{p}{N^{1-\kappa}}\cdot x + O\left(\Big|\frac{p}{N^{1-\kappa}}\cdot x\Big|^2\right).$$

Then,

$$\Big|\widehat{Vf}\Big(\frac{p}{N^{1-\kappa}}\Big) - 8\pi\mathfrak{a}\Big| \le \left|-i\int V(x)f(x)\frac{p}{N^{1-\kappa}}\cdot x\,dx\right| + C\frac{|p|^2}{N^{2-2\kappa}}\int |x|^2|V(x)|\,|f(x)|\,dx.$$

Since V and f are radial, the function $V(x)f(x)$ is even, while $x \mapsto p\cdot x$ is odd. Hence

$$\int_{\mathbb{R}^3} V(x)f(x)\,p\cdot x\,dx = 0.$$

Therefore, using that V has compact support and f is bounded, we obtain

$$\left|\widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right) - 8\pi\mathfrak{a}\right| \leq C\frac{|p|^2}{N^{2-2\kappa}}\int_{\mathbb{R}^3}|x|^2|V(x)|\,|f(x)|\,dx \leq C\frac{|p|^2}{N^{2-2\kappa}}. \tag{4.135}$$

In the region $|p| \leq N^{1-\kappa}$ this bound is at most of order one, and more importantly it gives a small error whenever it is inserted in the summable expressions appearing in S_2.

Thus, using (4.135) together with the bounds in (4.132) and (4.133), we can replace all factors $\widehat{Vf}\left(\frac{\cdot}{N^{1-\kappa}}\right)$ appearing in S_2 by $\widehat{Vf}(0) = 8\pi\mathfrak{a}$, at the cost of an error that can be absorbed into $O(N^{4\kappa-1})$. Therefore

$$\begin{aligned} S_2 = &-N^{4\kappa-1} \sum_{\substack{p,q\in\Lambda_+^* \\ N^{\kappa/2}\leq|p|,|q|\leq N^{1-\kappa}}} \frac{2(8\pi\mathfrak{a})^4}{4|p|^2|q|^2\big(|p|^2+|q|^2+|p+q|^2\big)} + O(N^{4\kappa-1}) \\ = &-2048\,\pi^4\mathfrak{a}^4 N^{4\kappa-1} \sum_{\substack{p,q\in\Lambda_+^* \\ N^{\kappa/2}\leq|p|,|q|\leq N^{1-\kappa}}} \frac{1}{|p|^2|q|^2\big(|p|^2+|q|^2+|p+q|^2\big)} + O(N^{4\kappa-1}). \end{aligned} \tag{4.136}$$

We now consider S_3. The argument is similar to the one used for S_2. The only point that requires some care is the region

$$|p| > N^{1-\kappa}, \qquad N^{\kappa/2} \leq |q| \leq N^{1-\kappa}.$$

This region cannot be estimated directly by absolute values, because the factor $p \cdot q$ is not sufficiently integrable in p. To handle it, we use a symmetrization argument in q. For simplicity, set

$$F_p := \widehat{Vf}\left(\frac{p}{N^{1-\kappa}}\right), \qquad D_+(p,q) := |p|^2+|q|^2+|p+q|^2, \qquad D_-(p,q) := |p|^2+|q|^2+|p-q|^2.$$

We also define

$$\Phi(p,q) := \frac{(F_p + F_{p+q})(p\cdot q)F_pF_q^2}{4|p|^2|q|^4 D_+(p,q)}.$$

Then, replacing q by $-q$, and using $F_{-q} = F_q$ because Vf is radial, we have

$$\Phi(p,-q) = -\frac{(F_p + F_{p-q})(p\cdot q)F_pF_q^2}{4|p|^2|q|^4 D_-(p,q)}.$$

Since Vf is radial, its Fourier transform is even, i.e., $F_{-q} = F_q$. Therefore, for fixed p, we can symmetrize the sum in q:

$$\sum_{q\in\Lambda^*}\Phi(p,q) = \frac{1}{2}\sum_{q\in\Lambda^*}\big(\Phi(p,q)+\Phi(p,-q)\big).$$

Observe that

$$\begin{aligned} \Phi(p,q)+\Phi(p,-q) &= \frac{F_pF_q^2(p\cdot q)}{4|p|^2|q|^4}\left(\frac{F_p+F_{p+q}}{D_+(p,q)} - \frac{F_p+F_{p-q}}{D_-(p,q)}\right) \\ &= \frac{F_pF_q^2(p\cdot q)}{4|p|^2|q|^4}\left[(F_p+F_{p+q})\left(\frac{1}{D_+(p,q)}-\frac{1}{D_-(p,q)}\right) + \frac{F_{p+q}-F_{p-q}}{D_-(p,q)}\right] \\ &=: \mathrm{I} + \mathrm{II} \end{aligned} \tag{4.137}$$

For I, observe that

$$\left|\frac{1}{D_+(p,q)}-\frac{1}{D_-(p,q)}\right|=\frac{|D_-(p,q)-D_+(p,q)|}{D_+(p,q)D_-(p,q)}=\frac{4|p\cdot q|}{D_+(p,q)D_-(p,q)}\leq C\frac{|p||q|}{|p|^4}=C\frac{|q|}{|p|^3}.$$

Since the factors F_p, F_q, F_{p+q} are bounded, we obtain

$$|\mathrm{I}|\leq C\frac{|p||q|}{|p|^2|q|^4}\frac{|q|}{|p|^3}=C\frac{1}{|p|^4|q|^2}.$$

Consequently,

$$\sum_{\substack{|p|>N^{1-\kappa}\\ N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}}|\mathrm{I}|\leq C\int_{|p|>N^{1-\kappa}}dp\,\frac{1}{|p|^4}\int_{N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}dq\,\frac{1}{|q|^2}\leq C. \tag{4.138}$$

For II, we use the mean value theorem,

$$|F_{p+q}-F_{p-q}|\leq\frac{|(p+q)-(p-q)|}{N^{1-\kappa}}\sup_{\theta\in[-1,1]}\left|\nabla\widehat{V f}\left(\frac{p+\theta q}{N^{1-\kappa}}\right)\right|.$$

Then, applying Cauchy-Schwarz, we obtain

$$\begin{aligned}\sum_{\substack{|p|>N^{1-\kappa}\\ N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}}|\mathrm{II}|&\leq CN^{\kappa-1}\sum_{N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}\frac{1}{|q|^2}\left(\sum_{|p|>N^{1-\kappa}}\frac{1}{|p|^6}\right)^{1/2}\\&\quad\times\left(\sum_{|p|>N^{1-\kappa}}\sup_{\theta\in[-1,1]}\left|\nabla\widehat{V f}\left(\frac{p+\theta q}{N^{1-\kappa}}\right)\right|^2\right)^{1/2}\\&\leq CN^{\kappa-1}\sum_{N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}\frac{1}{|q|^2}\left(\sum_{|p|>N^{1-\kappa}}\frac{1}{|p|^6}\right)^{1/2}N^{(3-3\kappa)/2}\leq C\,,\end{aligned} \tag{4.139}$$

where we use [13, Lemma 4] and

$$\begin{aligned}\sum_{|p|>N^{1-\kappa}}\frac{1}{|p|^6}&\leq C\int_{N^{1-\kappa}}^{\infty}dr\,\frac{r^2}{r^6}=CN^{-3+3\kappa},\\\sum_{N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}\frac{1}{|q|^2}&\leq C\int_{N^{\kappa/2}}^{N^{1-\kappa}}dr\leq CN^{1-\kappa}.\end{aligned}$$

Combining (4.138) and (4.139), we get

$$N^{4\kappa-1}\sum_{\substack{|p|>N^{1-\kappa}\\ N^{\kappa/2}\leq|q|\leq N^{1-\kappa}}}|\Phi(p,q)|\leq CN^{4\kappa-1}.$$

Hence this part of S_3 is negligible. Therefore, up to an error $O(N^{4\kappa-1})$, it remains only to estimate S_3 in the region

$$N^{\kappa/2}\leq|p|,|q|\leq N^{1-\kappa}.$$

In this region, by the replacement estimate (4.135), we may replace every factor

$$\widehat{Vf}\left(\frac{\cdot}{N^{1-\kappa}}\right)$$

by $8\pi\mathfrak{a}$, with an error that is absorbed into $O(N^{4\kappa-1})$. We obtain

$$\begin{aligned} S_3 &= N^{4\kappa-1} \sum_{\substack{p,q\in\Lambda_+^* \\ N^{\kappa/2}\le|p|,|q|\le N^{1-\kappa}}} \frac{2(8\pi\mathfrak{a})^4(p\cdot q)}{4|p|^2|q|^4\big(|p|^2+|q|^2+|p+q|^2\big)} + O(N^{4\kappa-1}) \\ &= 2048\pi^4\mathfrak{a}^4 N^{4\kappa-1} \sum_{\substack{p,q\in\Lambda_+^* \\ N^{\kappa/2}\le|p|,|q|\le N^{1-\kappa}}} \frac{p\cdot q}{|p|^2|q|^4\big(|p|^2+|q|^2+|p+q|^2\big)} + O(N^{4\kappa-1}). \end{aligned} \tag{4.140}$$

Combining (4.136) and (4.140), we get

$$S_2+S_3 = 2048\pi^4\mathfrak{a}^4 N^{4\kappa-1} \sum_{\substack{p,q\in\Lambda_+^* \\ N^{\kappa/2}\le|p|,|q|\le N^{1-\kappa}}} \frac{p\cdot q-|q|^2}{|p|^2|q|^4\big(|p|^2+|q|^2+|p+q|^2\big)} + O(N^{4\kappa-1}).$$

We now replace the sum by an integral. Proceeding as in [15], the error given by such a replacement is bounded by $O(N^{2\kappa-\beta/2})$. Therefore

$$\begin{aligned} S_2+S_3 &= \frac{2048\pi^4\mathfrak{a}^4}{2(2\pi)^6} N^{4\kappa-1} \int_{N^{\kappa/2}\le|p|,|q|\le N^{1-\kappa}} dp\,dq\, \frac{p\cdot q-|q|^2}{|p|^2|q|^4\big(|p|^2+|q|^2+p\cdot q\big)} \\ &\qquad + O(N^{2\kappa-\beta/2}) + O(N^{4\kappa-1}) \\ &= \frac{16\mathfrak{a}^4}{\pi^2} N^{4\kappa-1} \int_{1\le|\widetilde{p}|,|\widetilde{q}|\le N^{\beta/2}} d\widetilde{p}\,d\widetilde{q}\, \frac{\widetilde{p}\cdot\widetilde{q}-|\widetilde{q}|^2}{|\widetilde{p}|^2|\widetilde{q}|^4\big(|\widetilde{p}|^2+|\widetilde{q}|^2+\widetilde{p}\cdot\widetilde{q}\big)} + O(N^{4\kappa-1})\,. \end{aligned} \tag{4.141}$$

For simplicity, we drop the tildes in the following. We can remove the restriction on p since the error caused in the region $|p|<1, |p|>N^{\beta/2}$ is of order $N^{4\kappa-1}$, so negligible.

Let $\omega := |q|$. By rotational symmetry, we may assume $q=\omega e_3$, with $e_3=(0,0,1)$. Then we set $p=\omega x$. It follows that

$$dp=\omega^3 dx, \qquad p\cdot q=\omega^2 x_3, \qquad |p|^2=\omega^2|x|^2, \qquad |q|^2=\omega^2.$$

Hence

$$\begin{aligned} &\int_{1\le|q|\le N^{\beta/2}} dq \int_{\mathbb{R}^3} dp\, \frac{p\cdot q-|q|^2}{|p|^2|q|^4\big(|p|^2+|q|^2+p\cdot q\big)} \\ &\quad = 4\pi \int_1^{N^{\beta/2}} d\omega\,\omega^2 \int_{\mathbb{R}^3} dx\,\omega^3 \frac{\omega^2(x_3-1)}{\omega^2|x|^2\omega^4\omega^2(|x|^2+1+x_3)} \\ &\quad = 4\pi \int_1^{N^{\beta/2}} \frac{d\omega}{\omega} \int_{\mathbb{R}^3} dx\, \frac{x_3-1}{|x|^2(|x|^2+1+x_3)} \\ &\quad = 2\pi\log(N^\beta)\,\mathrm{I}, \end{aligned} \tag{4.142}$$

where

$$\mathrm{I} := \int_{\mathbb{R}^3} dx\, \frac{x_3-1}{|x|^2(|x|^2+1+x_3)}.$$

We now compute I using the Feynman parameter identity: for $a, b > 0$,

$$\frac{1}{ab} = \int_0^1 \frac{ds}{\big(sa + (1-s)b\big)^2},$$

Taking $a = |x|^2,$ and $b = |x|^2 + x_3 + 1$, we get

$$I = \int_0^1 ds \int dx \, \frac{x_3 - 1}{\big(|x|^2 + (1-s)x_3 + (1-s)\big)^2}.$$

Completing the square gives

$$|x|^2 + (1-s)x_3 + (1-s) = \left|x + \frac{1-s}{2}e_3\right|^2 + \frac{(1-s)(3+s)}{4}.$$

Setting $y = x + \frac{1-s}{2}e_3$. gives

$$x_3 - 1 = y_3 - \frac{1-s}{2} - 1 = y_3 - \frac{3-s}{2}.$$

Therefore

$$\mathrm{I} = \int_0^1 ds \int dy \, \frac{y_3 - \frac{3-s}{2}}{\left(|y|^2 + \frac{(1-s)(3+s)}{4}\right)^2} = -\frac{1}{2}\int_0^1 ds\,(3-s)\int_{\mathbb{R}^3} \frac{dy}{\left(|y|^2 + \frac{(1-s)(3+s)}{4}\right)^2}$$

because the term containing y_3 is odd and therefore integrates to zero. We now use the standard integral identity

$$\int_{\mathbb{R}^3} \frac{dy}{(|y|^2 + m^2)^2} = \frac{\pi^2}{m}\,.$$

Therefore,

$$\mathrm{I} = -\frac{1}{2}\int_0^1 ds\,(3-s)\,\frac{\pi^2}{\frac{1}{2}\sqrt{(1-s)(3+s)}} = -\pi^2 \int_0^1 ds\,\frac{3-s}{\sqrt{(1-s)(3+s)}}.$$

To calculate the integral, we set $u \coloneqq 1-s$. Then $3-s = 2+u, 3+s = 4-u,$ and $ds = -du$. Then

$$\mathrm{I} = -\pi^2 \int_0^1 du\,\frac{2+u}{\sqrt{u(4-u)}}.$$

Next, set $u = 4\sin^2\theta$. Since $u \in [0,1]$, we have

$$\theta \in \left[0, \frac{\pi}{6}\right], \qquad du = 8\sin\theta\cos\theta\,d\theta,$$

and

$$\sqrt{u(4-u)} = \sqrt{4\sin^2\theta \cdot 4\cos^2\theta} = 4\sin\theta\cos\theta.$$

Therefore,

$$\begin{aligned}
\mathrm{I} &= -\pi^2 \int_0^{\pi/6} \frac{2 + 4\sin^2\theta}{4\sin\theta\cos\theta}\, 8\sin\theta\cos\theta\,d\theta \\
&= -4\pi^2 \int_0^{\pi/6} \big(1 + 2\sin^2\theta\big)\,d\theta \\
&= -4\pi^2 \left[2\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/6} = -\pi^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)
\end{aligned}$$

Substituting this into (4.142), we find

$$\int_{1\leq|q|\leq N^{\beta/2}} dq \int_{\mathbb{R}^3} dp \, \frac{p\cdot q - |q|^2}{|p|^2|q|^4\big(|p|^2+|q|^2+p\cdot q\big)} = -2\pi^3\left(\frac{4\pi}{3}-\sqrt{3}\right)\log(N^\beta).$$

Consequently, from (4.141), we conclude that

$$\begin{aligned} S_2+S_3 &= \frac{16\mathfrak{a}^4}{\pi^2}N^{4\kappa-1}\left[-2\pi^3\left(\frac{4\pi}{3}-\sqrt{3}\right)\log(N^\beta)\right]+O(N^{4\kappa-1}) \\ &= -32\pi\left(\frac{4\pi}{3}-\sqrt{3}\right)\mathfrak{a}^4N^{4\kappa-1}\log(N^\beta)+O(N^{4\kappa-1}). \end{aligned}$$

This completes the proof. □

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