Universität Zürich[UZH]

Department of Mathematics

Classical and Higher-Categorical Perspectives on Derived Categories

Master Thesis

submitted by

Hanns-Christian Schwacke

Supervisor:

Prof. Dr. Joseph Ayoub

University of Zurich

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Abstract

Derived categories are obtained by formally inverting quasi-isomorphisms of complexes. They provide a natural framework for derived functors, mapping cones, and triangulated categories, where distinguished triangles encode the derived analogue of exact sequences.

Triangulated categories, however, retain only a 1-categorical shadow of the underlying constructions. Mapping cones are not functorial in a strict sense, and their construction involves choices of homotopies and higher coherences. This motivates the passage to stable ∞-categories, where such coherent data is part of the structure.

This thesis explains the transition from classical derived categories to this higher-categorical viewpoint. We recall the classical theory of complexes, quasi-isomorphisms, cones, and triangulated categories, then introduce simplicial sets, ∞-categories, stable ∞-categories, and spectra. Finally, we indicate how derived categories reappear through stable and simplicial models, including the Dold–Kan correspondence for connective complexes.

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Introduction

Derived categories are a central construction in modern homological algebra. They arise from the idea that, when studying complexes, the relevant notion of equivalence is usually not isomorphism of complexes, but quasi-isomorphism. A quasi-isomorphism is a morphism of complexes which induces isomorphisms on all cohomology groups. Thus, if cohomology is the invariant one wants to preserve, then quasi-isomorphic complexes should be regarded as representing the same object.

The derived category formalizes this idea. Starting with the category of complexes in an abelian category $\mathcal{A}$, one can first pass to the homotopy category $K(\mathcal{A})$, where chain homotopic maps are identified, and then formally inverts the class of quasi-isomorphisms:

$$D(\mathcal{A}) = K(\mathcal{A})[W^{-1}].$$

This construction allows one to treat resolutions as genuine replacements of objects and gives a natural setting for derived functors. For example, an injective resolution $A \to I^\bullet$ becomes an isomorphism in the derived category, so the derived functor $RF(A)$ can be viewed as the image of a suitably chosen replacement rather than as a collection of separate correction terms.

Both the homotopy category $K(\mathcal{A})$ and the derived category $D(\mathcal{A})$ carry triangulated structures. Mapping cones turn morphisms of complexes into distinguished triangles

$$X \longrightarrow Y \longrightarrow \mathrm{Cone}(f) \longrightarrow X[1],$$

and these triangles play the role of exact sequences in the derived setting. In this way, triangulated categories axiomatize the formal behaviour of cone triangles and provide a powerful language for derived exactness.

However, triangulated categories have well-known limitations. One important point is that mapping cones are not functorial in the homotopy category in a strict sense. Constructing maps between cones requires choices of homotopies, and these choices can themselves be related by higher choices. The triangulated category records the resulting 1-categorical pattern of distinguished triangles, but it does not retain the full coherent higher structure behind the construction.

This motivates the higher-categorical viewpoint. In an ∞-category, mapping objects are no longer just sets, but enhanced to spaces. Thus morphisms can have homotopies between them, homotopies between homotopies, and so on. Stable ∞-categories provide the higher-categorical setting in which fiber and cofiber constructions behave like kernels and cokernels in an exact way. A central result is that the homotopy category of a stable ∞-category is naturally triangulated:

$$C \text{ stable} \quad \Longrightarrow \quad hC \text{ triangulated.}$$

From this perspective, triangulated categories appear as 1-categorical shadows of richer stable ∞-categories.

The aim of this thesis is to explain this passage from the classical theory of derived categories to the higher-categorical refinement. The first part recalls the basic constructions of homological algebra: complexes, chain homotopies, quasi-isomorphisms, derived categories, mapping cones, and triangulated categories. The second part introduces the simplicial language needed to define ∞-categories and explains how mapping spaces encode higher coherent information. The third part discusses stable ∞-categories and spectra as a fundamental example of stability. Finally, we indicate how classical derived categories reappear from this higher-categorical viewpoint, for example through the comparison

$$h(\mathrm{Mod}_{HR}) \simeq D(R),$$

where HR is the Eilenberg–Mac Lane spectrum of a ring R. The comparison with HR-module spectra gives a conceptual higher-categorical enhancement of derived categories. However, this point of view also relies on substantial machinery: spectra carry a symmetric monoidal structure given by the smash product, HR has to be regarded as a ring object in spectra, and Mod_{HR} is formed as an ∞-category of module spectra. We therefore also discuss a second, more algebraic perspective, based on simplicial objects. In the case of modules, the Dold–Kan correspondence identifies simplicial R-modules with connective chain complexes. After localization, this gives an algebraic model for the connective derived category:

$$s\,\mathrm{Mod}_R[W^{-1}] \simeq D_{\geq 0}(R).$$

This provides another way to see how derived categories arise from homotopical algebra, without starting directly from the spectral HR-module viewpoint.

1 Classical Homological Algebra and Derived Categories

The aim of this chapter is to recall the classical framework from which stable ∞-categories arise as a refinement. We begin with abelian categories, since they provide the natural setting for chain complexes, homology, exact sequences, resolutions, and derived functors. The material in this chapter follows standard references on homological algebra, especially Weibel's *An Introduction to Homological Algebra* [5]. We recall only the notions needed later: abelian categories, complexes, localization, derived functors, mapping cones, and triangulated categories.

1.1 Abelian Categories

Before defining chain complexes, one has to specify the categorical setting in which kernels, cokernels, images, and exact sequences make sense. This leads to the notion of an abelian category.

Definition 1.1.1. A category $\mathcal{A}$ is called *preadditive* if, for all objects $X, Y \in \mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group, and composition is bilinear. Thus, for morphisms $f, f' : X \to Y$ and $g, g' : Y \to Z$, one has

$$g \circ (f + f') = g \circ f + g \circ f',$$

and

$$(g + g') \circ f = g \circ f + g' \circ f.$$

Definition 1.1.2. A preadditive category $\mathcal{A}$ is called *additive* if it has a zero object and finite biproducts. Thus, for any two objects $X, Y \in \mathcal{A}$, there exists an object

$$X \oplus Y$$

which is both a product and a coproduct of X and Y.

Remark 1.1.3. Additive categories are the minimal setting in which one can add morphisms and speak about zero morphisms. This is already enough to define chain complexes, because the condition

$$d^{n+1} \circ d^n = 0$$

requires the zero morphism. However, to define homology and cohomology one also needs kernels, cokernels, and images. For this one passes to abelian categories.

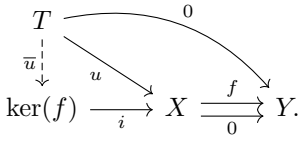
Definition 1.1.4. Let $\mathcal{A}$ be an additive category, and let $f : X \to Y$ be a morphism. A *kernel* of f is a morphism

$$i : \ker(f) \to X$$

which equalizes f and the zero morphism, i.e. $f \circ i = 0$, and is universal with this property. Categorically, i is the equalizer of

$$f : X \to Y \qquad \text{and} \qquad 0 : X \to Y.$$

Thus for every morphism $u : T \to X$ with $f \circ u = 0$, there exists a unique morphism $\overline{u} : T \to \ker(f)$ such that $u = i \circ \overline{u}$. This can be summarized by the diagram



Dually, a *cokernel* of f is a morphism

$$p : Y \to \operatorname{coker}(f)$$

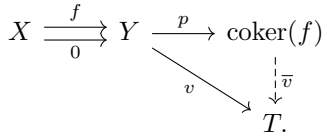
which coequalizes f and the zero morphism, i.e.

$$p \circ f = 0,$$

and is universal with this property. Categorically, p is the coequalizer of

$$f : X \to Y \qquad \text{and} \qquad 0 : X \to Y.$$

Thus for every morphism $v : Y \to T$ with $v \circ f = 0$, there exists a unique morphism $\overline{v} : \operatorname{coker}(f) \to T$ such that $v = \overline{v} \circ p$. This can be summarized by the diagram



Definition 1.1.5. An additive category $\mathcal{A}$ is called an *abelian category* if:

1. every morphism in $\mathcal{A}$ has a kernel and a cokernel;
2. for every morphism f, the canonical morphism

$$\operatorname{coim}(f) \longrightarrow \operatorname{im}(f)$$

is an isomorphism.

Remark 1.1.6. The second condition is what makes abelian categories behave like categories of modules. For modules, the image of a homomorphism may be described either as a quotient of the source or as a subobject of the target:

$$X/\ker(f) \cong \operatorname{im}(f) \subseteq Y.$$

The axiom

$$\operatorname{coim}(f) \cong \operatorname{im}(f)$$

abstracts exactly this familiar fact.

Example 1.1.7. The category of abelian groups **Ab** is an abelian category. More generally, for any ring R, the category of R-modules $\mathbf{Mod_R}$ is an abelian category. Another important example is the category of sheaves of abelian groups on a topological space X denoted $\mathbf{Sh(X)}$.

Remark 1.1.8. These examples explain why abelian categories are useful in homological algebra: they allow one to treat modules and sheaves by the same formal methods. This is one of the main reasons why abelian categories became central in the study.

Definition 1.1.9. A sequence of morphisms in an abelian category

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called *exact at* Y if

$$\operatorname{im}(f) \cong \ker(g).$$

A longer sequence is called *exact* if it is exact at every object where this condition makes sense. In particular, a sequence

$$0 \to X \to Y \to Z \to 0$$

is called *short exact* if it is exact at X, Y, and Z.

1.2 Chain Complexes, Homology, and Chain Homotopy

Let $\mathcal{A}$ be an abelian category. In this subsection we recall the basic homological objects which will later lead to the derived category.

Remark 1.2.1. We use the cochain convention. Thus differentials increase degree: $d^n : C^n \to C^{n+1}$.

Definition 1.2.2. A *cochain complex* in $\mathcal{A}$ is a sequence of objects and morphisms

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \longrightarrow \cdots$$

such that

$$d^{n+1} \circ d^n = 0$$

for all $n \in \mathbb{Z}$. We often denote such a complex by $C^\bullet$.

Remark 1.2.3. The condition $d^{n+1} \circ d^n = 0$ implies $\operatorname{im}(d^{n-1}) \subseteq \ker(d^n)$. Thus a cochain complex is a sequence which is at least partially exact. Its cohomology measures the failure of this inclusion to be an equality.

Definition 1.2.4. Let $C^\bullet$ be a cochain complex in $\mathcal{A}$. Its n-th cohomology object is defined by

$$H^n(C^\bullet) := \ker(d^n)/\operatorname{im}(d^{n-1}).$$

Equivalently, in categorical notation,

$$H^n(C^\bullet) = \operatorname{coker}\left(\operatorname{im}(d^{n-1}) \to \ker(d^n)\right).$$

Remark 1.2.5. The complex $C^\bullet$ is exact at C^n precisely when $H^n(C^\bullet) = 0$. Thus cohomology measures the failure of a complex to be exact.

Definition 1.2.6. A morphism of cochain complexes $f : C^\bullet \to D^\bullet$ is a family of morphisms

$$f^n : C^n \to D^n$$

such that, for every n, the square

$$\begin{array}{ccc} C^n & \xrightarrow{d_C^n} & C^{n+1} \\ {\scriptstyle f^n}\downarrow & & \downarrow{\scriptstyle f^{n+1}} \\ D^n & \xrightarrow[d_D^n]{} & D^{n+1} \end{array}$$

commutes. Equivalently,

$$d_D^n \circ f^n = f^{n+1} \circ d_C^n.$$

Remark 1.2.7. Cochain complexes in $\mathcal{A}$, together with morphisms of cochain complexes, form a category denoted $\mathrm{Ch}(\mathcal{A})$.

Definition 1.2.8. Let $f : C^\bullet \to D^\bullet$ be a morphism of cochain complexes. It induces morphisms on cohomology

$$H^n(f) : H^n(C^\bullet) \to H^n(D^\bullet).$$

A morphism f is called a *quasi-isomorphism* if

$$H^n(f) : H^n(C^\bullet) \to H^n(D^\bullet)$$

is an isomorphism for every $n \in \mathbb{Z}$.

Remark 1.2.9. Quasi-isomorphisms are the morphisms which become invisible to cohomology. They play the role of weak equivalences in classical homological algebra. The derived category will be obtained by formally inverting them.

Definition 1.2.10. Let $f, g : C^\bullet \to D^\bullet$ be two morphisms of cochain complexes. A *chain homotopy* from f to g is a family of morphisms

$$h^n : C^n \to D^{n-1}$$

such that, for every n,

$$f^n - g^n = d_D^{n-1} \circ h^n + h^{n+1} \circ d_C^n.$$

If such a family exists, we write

$$f \simeq g.$$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d_C} & C^n & \xrightarrow{d_C} & C^{n+1} & \longrightarrow & \cdots \\ & & {\scriptstyle g}\downarrow\downarrow{\scriptstyle f} & \swarrow{\scriptstyle h^n} & {\scriptstyle g}\downarrow\downarrow{\scriptstyle f} & \swarrow{\scriptstyle h^{n+1}} & {\scriptstyle g}\downarrow\downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow[d_D]{} & D^n & \xrightarrow[d_D]{} & D^{n+1} & \longrightarrow & \cdots \end{array}$$

Remark 1.2.11. The formula

$$f - g = d_D h + h d_C$$

should be understood as saying that the difference between f and g is controlled by the differentials. Thus f and g differ only by a boundary term, and this is why they induce the same maps on cohomology.

Proposition 1.2.12. If two morphisms of cochain complexes $f, g : C^\bullet \to D^\bullet$ are chain homotopic, then they induce the same morphism on cohomology:

$$H^n(f) = H^n(g)$$

for all n.

Proof. By definition of a chain homotopy, we have

$$f^n - g^n = d_D^{n-1} \circ h^n + h^{n+1} \circ d_C^n.$$

Restrict this equality to $\ker(d_C^n) \subseteq C^n$. On $\ker(d_C^n)$, the term

$$h^{n+1} \circ d_C^n$$

vanishes. Hence, on cocycles,

$$f^n - g^n = d_D^{n-1} \circ h^n.$$

Therefore the induced morphism

$$\ker(d_C^n) \to H^n(D^\bullet)$$

given by $f^n - g^n$ is zero, because its image factors through $\mathrm{im}(d_D^{n-1})$. Thus f and g induce the same morphism

$$H^n(C^\bullet) \to H^n(D^\bullet).$$

□

Definition 1.2.13. The *homotopy category of complexes* in $\mathcal{A}$ is the category

$$K(\mathcal{A})$$

whose objects are cochain complexes in $\mathcal{A}$, and whose morphisms are chain-homotopy classes of morphisms of complexes:

$$\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, D^\bullet) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C^\bullet, D^\bullet)/\sim .$$

Remark 1.2.14. The passage $\mathrm{Ch}(\mathcal{A}) \longrightarrow K(\mathcal{A})$ is the first homotopical step. It identifies maps of complexes which induce the same map on cohomology for homotopical reasons. The derived category will be obtained from $K(\mathcal{A})$ by a second step: formally inverting quasi-isomorphisms.

1.3 Localization and the Derived Category

We have passed from the category of complexes $\mathrm{Ch}(\mathcal{A})$ to the homotopy category $K(\mathcal{A})$ by identifying homotopic maps. The next step is to force quasi-isomorphisms to become isomorphisms. This is done by a general categorical construction called localization. For background of this standard catgorical localization construction see Weibel [5].

Definition 1.3.1. Let $\mathcal{C}$ be a category and let $W \subseteq \mathrm{Mor}(\mathcal{C})$ be a class of morphisms. A *localization* of $\mathcal{C}$ at W is a category

$$\mathcal{C}[W^{-1}]$$

together with a functor

$$Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$$

such that:

1. for every $w \in W$, the morphism $Q(w)$ is an isomorphism;
2. if $F : \mathcal{C} \to \mathcal{D}$ is any functor such that $F(w)$ is an isomorphism for every $w \in W$, then F factors uniquely up to unique isomorphism through Q.

In other words, Q is universal among such localization functors.

Remark 1.3.2. Localization is the categorical analogue of a familiar construction from commutative algebra. If R is a ring and $S \subseteq R$ is a multiplicative subset, then the localized ring $S^{-1}R$ is obtained by formally making all elements of S invertible. It is characterized by the universal property that every ring homomorphism $R \to A$ which sends all elements of S to units factors uniquely through $R \to S^{-1}R$.

Theorem 1.3.3. Let $\mathcal{C}$ be a category and let $W \subseteq \mathrm{Mor}(\mathcal{C})$ be a set of morphisms. Then the localization $Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$ exists.

Idea of the construction. One constructs $\mathcal{C}[W^{-1}]$ by formally adjoining an inverse arrow

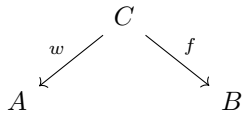
$$s^{-1} : Y \to X$$

for every morphism $s : X \to Y$ in W. One then imposes the relations saying that these new arrows are inverse to the original morphisms:

$$s^{-1} \circ s = \mathrm{id}_X, \qquad s \circ s^{-1} = \mathrm{id}_Y \, .$$

One also imposes the usual relations coming from composition in $\mathcal{C}$. The resulting category has the same objects as $\mathcal{C}$, but its morphisms are generated by ordinary morphisms in $\mathcal{C}$ together with formal inverses of morphisms in W. $\square$

Remark 1.3.4. Morphisms in a localization can often be represented by *roof diagrams*. A roof from A to B is a diagram



where $w \in W$. It should be read as the formal composite

$$f \circ w^{-1} : A \to B.$$

Thus one first replaces A by an object C which is equivalent to A after localization, and then maps C to B.

Definition 1.3.5. Let W be the class of quasi-isomorphisms in $K(\mathcal{A})$. The *derived category* of $\mathcal{A}$ is defined as the localization

$$D(\mathcal{A}) := K(\mathcal{A})[W^{-1}].$$

Thus the derived category is obtained from the homotopy category by formally inverting all quasi-isomorphisms.

Remark 1.3.6. Equivalently, one may first localize the category of complexes $\mathrm{Ch}(\mathcal{A})$ at the quasi-isomorphisms. Since the canonical functor $\mathrm{Ch}(\mathcal{A}) \to K(\mathcal{A})$ is the identity on objects and passes to chain-homotopy classes of morphisms, the resulting derived category is canonically equivalent to the localization of $K(\mathcal{A})$ at the morphisms which induce isomorphisms on cohomology.

Definition 1.3.7. One also defines bounded variants of the derived category. We write

$$D^+(\mathcal{A}), \qquad D^-(\mathcal{A}), \qquad D^b(\mathcal{A})$$

for the derived categories of bounded below, bounded above, and bounded complexes, respectively.

Remark 1.3.8. The passage

$$\mathrm{Ch}(\mathcal{A}) \longrightarrow K(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

has two steps. First, one identifies chain-homotopic maps. Second, one formally inverts quasi-isomorphisms. In the derived category, complexes with the same cohomology are not literally identified, but quasi-isomorphisms between them become isomorphisms. This is the categorical framework in which derived functors become natural.

1.4 Resolutions and Derived Functors

We now return to the classical construction which historically motivates the derived category. In classical homological algebra, derived functors are first defined using resolutions. The derived category later repackages this construction: instead of producing a family of functors

$$R^n F : \mathcal{A} \to \mathcal{B}, \qquad n \geq 0,$$

one constructs a single functor

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B}),$$

and recovers the classical derived functors by taking cohomology:

$$R^n F(A) = H^n(RF(A)).$$

Definition 1.4.1. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. We say that F is *left exact* if for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, the sequence

$$0 \to F(A') \to F(A) \to F(A'')$$

is exact in $\mathcal{B}$. Dually, F is called *right exact* if

$$F(A') \to F(A) \to F(A'') \to 0$$

is exact for every short exact sequence as above. If F is both left exact and right exact, then F is called *exact*.

Definition 1.4.2. Let $\mathcal{A}$ be an abelian category and let $A \in \mathcal{A}$. A *resolution* of A is an exact complex together with a morphism from A into the complex, or from the complex onto A, which identifies A with the only nonzero cohomology object. More concretely, a *right resolution* of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots .$$

Equivalently, it is a quasi-isomorphism

$$A \to I^\bullet,$$

where A is regarded as a complex concentrated in degree 0. Dually, a *left resolution* of A is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0.$$

Equivalently, it is a quasi-isomorphism

$$P_\bullet \to A,$$

where A is again regarded as a complex concentrated in degree 0.

Definition 1.4.3. Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is called *injective* if the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \mathbf{Ab}$$

is exact. Dually, an object $P \in \mathcal{A}$ is called *projective* if the functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$$

is exact.

Definition 1.4.4. Let $A \in \mathcal{A}$. An *injective resolution* of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

such that each I^n is injective. Equivalently, it is a quasi-isomorphism

$$A \to I^\bullet,$$

where A is regarded as a complex concentrated in degree 0, and $I^\bullet$ is a complex of injective objects concentrated in nonnegative degrees.

Definition 1.4.5. Dually, a *projective resolution* of A is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

such that each P_n is projective.

Remark 1.4.6. A category is said to have *enough injectives*, respectively *enough projectives*, if every object admits a monomorphism into an injective object, respectively an epimorphism from a projective object. In these cases, injective resolutions, respectively projective resolutions, exist.

Definition 1.4.7. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, and assume that $\mathcal{A}$ has enough injectives. For an object $A \in \mathcal{A}$, choose an injective resolution

$$A \to I^\bullet.$$

Applying F gives a cochain complex

$$F(I^0) \to F(I^1) \to F(I^2) \to \cdots .$$

The *right derived functors* of F are defined by

$$R^n F(A) := H^n(F(I^\bullet)).$$

Definition 1.4.8. Dually, let

$$G : \mathcal{A} \to \mathcal{B}$$

be a right exact functor, and assume that $\mathcal{A}$ has enough projectives. For $A \in \mathcal{A}$, choose a projective resolution

$$P_\bullet \to A.$$

The *left derived functors* of G are defined by

$$L_n G(A) := H_n(G(P_\bullet)).$$

Remark 1.4.9. Derived functors measure the failure of a functor to be exact. For instance, if F is left exact, then it preserves the beginning of a short exact sequence, but it may fail to preserve exactness at the final term. The right derived functors $R^n F$ measure this failure.

A nontrivial point is that the objects $R^n F(A)$ do not depend, up to canonical isomorphism, on the chosen injective resolution. Thus the resolution is an auxiliary choice used to compute the derived functors, but the resulting objects are intrinsic.

Example 1.4.10. Let R be a ring. The tensor product functor

$$- \otimes_R N : \mathrm{Mod}_R \to \mathrm{Mod}_R$$

is right exact, but in general it is not exact. Its left derived functors are the Tor groups:

$$\mathrm{Tor}_n^R(M, N) = L_n(- \otimes_R N)(M).$$

Equivalently, one chooses a projective resolution

$$P_\bullet \to M$$

and computes

$$\mathrm{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N).$$

Dually, the functor

$$\mathrm{Hom}_R(M, -) : \mathrm{Mod}_R \to \mathbf{Ab}$$

is left exact, but in general it is not exact. Its right derived functors are the Ext groups:

$$\mathrm{Ext}_R^n(M, N) = R^n \mathrm{Hom}_R(M, -)(N).$$

Equivalently, one chooses an injective resolution

$$N \to I^\bullet$$

and computes

$$\mathrm{Ext}_R^n(M, N) = H^n(\mathrm{Hom}_R(M, I^\bullet)).$$

Thus Tor and Ext arise by deriving the classical functors $\otimes$ and Hom. They measure the failure of these functors to be exact.

Example 1.4.11. Let $R = \mathbb{Z}$, and consider the abelian group $\mathbb{Z}/n\mathbb{Z}$. A projective resolution of $\mathbb{Z}/n\mathbb{Z}$ is given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

To compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N)$, we apply the right exact functor

$$- \otimes_{\mathbb{Z}} N$$

to this projective resolution. By $\mathbb{Z} \otimes_{\mathbb{Z}} N \cong N$ this gives the complex

$$0 \to N \xrightarrow{\cdot n} N \to 0.$$

Therefore

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) \cong \ker\left(N \xrightarrow{\cdot n} N\right),$$

and

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) \cong \mathrm{coker}\left(N \xrightarrow{\cdot n} N\right) \cong N/nN.$$

For $i \geq 2$, one has

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) = 0.$$

In particular, if $N = \mathbb{Z}/m\mathbb{Z}$, then

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \{x \in \mathbb{Z}/m\mathbb{Z} \mid nx = 0\} \cong \mathbb{Z}/\gcd(n, m)\mathbb{Z}.$$

This example shows concretely how Tor measures the failure of tensor product to be exact.

Remark 1.4.12. The universal coefficient theorem and the Künneth formula are classical places where Ext and Tor appear in topology. This illustrates that derived functors are not merely formal objects: they measure the correction terms which occur when ordinary algebraic operations fail to be exact.

Remark 1.4.13. The derived category gives a conceptual reformulation of the construction above. Classically, for a left exact functor $F : \mathcal{A} \to \mathcal{B}$, one chooses an injective resolution $A \to I^\bullet$ and defines $R^n F(A) := H^n(F(I^\bullet))$. The important point is that the resolution is only an auxiliary choice. In the derived category this independence becomes visible. The map $A \to I^\bullet$ is a quasi-isomorphism, where A is viewed as a complex concentrated in degree 0. Hence it becomes an isomorphism in $D^+(\mathcal{A})$:

$$A \simeq I^\bullet \qquad \text{in } D^+(\mathcal{A}).$$

If $A \to J^\bullet$ is another injective resolution, then also

$$A \simeq J^\bullet \qquad \text{in } D^+(\mathcal{A}).$$

Therefore the two resolutions are isomorphic in the derived category:

$$I^\bullet \simeq A \simeq J^\bullet \qquad \text{in } D^+(\mathcal{A}).$$

Thus, from the derived-categorical point of view, one no longer has to remember which resolution was chosen.

Definition 1.4.14. The *total right derived functor*

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

is then defined by applying F after replacing an object by a suitable injective resolution.

Remark 1.4.15. For an object $A \in \mathcal{A}$, one has $RF(A) \simeq F(I^\bullet)$ in $D^+(\mathcal{B})$, and the classical derived functors are recovered by taking cohomology:

$$R^n F(A) = H^n(RF(A)).$$

Thus the family of functors $(R^n F)_{n \geq 0}$ is packaged into one derived functor RF.

Remark 1.4.16. One advantage of the derived-category language is that it suppresses a large amount of classical bookkeeping. For example, suppose one starts with a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

and wants to study the derived functors of a left exact functor F. In the classical approach, one has to choose injective resolutions

$$A' \to I'^{\bullet}, \qquad A \to I^{\bullet}, \qquad A'' \to I''^{\bullet},$$

together with comparison maps between them. Schematically, one tries to build a diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I'^{\bullet} & \longrightarrow & I^{\bullet} & \longrightarrow & I''^{\bullet} & \longrightarrow & 0. \end{array}$$

One then applies F to the chosen resolutions and extracts long exact sequences in cohomology. The resulting maps are independent of the choices, but proving this requires comparison maps, homotopies, and diagram chasing.

In the derived category the same situation is expressed more conceptually. The short exact sequence gives a distinguished triangle

$$A' \to A \to A'' \to A'[1]$$

in $D^+(\mathcal{A})$. Applying the total derived functor gives a triangle

$$RF(A') \to RF(A) \to RF(A'') \to RF(A')[1]$$

in $D^+(\mathcal{B})$, and taking cohomology yields the usual long exact sequence of derived functors. Thus the derived category packages the classical resolution-dependent construction into a functorial formalism.

Example 1.4.17 (The de Rham complex as a resolution)**.** We briefly mention one geometric example of the resolution formalism. This example uses some standard facts from sheaf theory, which we do not develop in detail here. The purpose is only to illustrate how a familiar cohomology theory can arise from the general mechanism of replacing an object by a resolution, applying a functor, and taking cohomology. Let M be a smooth manifold, and let $\underline{\mathbb{R}}$ denote the constant sheaf on M. For every $k \geq 0$, let Ω_M^k denote the sheaf of smooth differential k-forms. There is a complex of sheaves

$$0 \to \underline{\mathbb{R}} \to \Omega_M^0 \xrightarrow{d} \Omega_M^1 \xrightarrow{d} \Omega_M^2 \to \cdots .$$

By the Poincaré lemma, this complex is exact. Thus

$$\underline{\mathbb{R}} \to \Omega_M^{\bullet}$$

is a resolution of the constant sheaf $\underline{\mathbb{R}}$.

Moreover, the sheaves Ω^k_M are fine, and hence acyclic for the global section functor. Therefore this resolution can be used to compute the derived functors of global sections. Applying

$$\Gamma(M, -) : \mathrm{Sh}(M) \to Ab$$

to the resolution gives the global de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots .$$

Hence

$$H^n(M, \underline{\mathbb{R}}) \cong H^n\big(\Gamma(M, \Omega^\bullet_M)\big) = H^n_{\mathrm{dR}}(M).$$

This example illustrates the general mechanism of derived functors. One replaces an object, here the constant sheaf $\underline{\mathbb{R}}$, by a suitable resolution, applies a left exact functor, here global sections, and then takes cohomology. In this way, de Rham cohomology appears naturally as a derived functor computation.

1.5 Mapping Cones

We now introduce the mapping cone construction. It is one of the central constructions in classical homological algebra and provides the basic source of triangles in the homotopy and derived categories. Later, in the ∞-categorical setting, the mapping cone will be replaced by the cofiber.

Definition 1.5.1. Let $C^\bullet$ be a cochain complex. Its *shift* $C^\bullet[1]$ is the cochain complex defined by

$$C[1]^n := C^{n+1},$$

with differential

$$d^n_{C[1]} := -d^{n+1}_C.$$

Remark 1.5.2. The sign in the definition of the shifted differential ensures that

$$d^{n+1}_{C[1]} \circ d^n_{C[1]} = 0.$$

Indeed,

$$d^{n+1}_{C[1]} \circ d^n_{C[1]} = (-d^{n+2}_C) \circ (-d^{n+1}_C) = d^{n+2}_C \circ d^{n+1}_C = 0.$$

Definition 1.5.3. Let $\mathcal{C}$ be an additive category equipped with an autoequivalence

$$[1] : \mathcal{C} \to \mathcal{C}.$$

A *triangle* in $\mathcal{C}$ is a sequence of morphisms

$$X \to Y \to Z \to X[1].$$

Remark 1.5.4. At this point, a triangle is only a sequence of this form. The extra structure of a triangulated category consists of specifying which triangles should be called distinguished and imposing axioms on them. The basic examples of such triangles will come from mapping cones.

Definition 1.5.5. Let $f : X^\bullet \to Y^\bullet$ be a morphism of cochain complexes in an abelian category $\mathcal{A}$. The *mapping cone* of f, denoted $\mathrm{Cone}(f)$, is the cochain complex defined by

$$\mathrm{Cone}(f)^n := Y^n \oplus X^{n+1},$$

with differential

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^{n+1}_X \end{pmatrix}.$$

Explicitly,

$$d^n_{\mathrm{Cone}(f)}(y, x) = \big(d^n_Y(y) + f^{n+1}(x), -d^{n+1}_X(x)\big).$$

Remark 1.5.6. The mapping cone fits into a canonical sequence

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1].$$

Here the map $Y^\bullet \to \mathrm{Cone}(f)$ is given in degree n by $y \mapsto (y, 0)$ and the map $\mathrm{Cone}(f) \to X^\bullet[1]$ is given in degree n by $(y, x) \mapsto x$.

Remark 1.5.7. The mapping cone should be thought of as a homological replacement for the cokernel. In an abelian category, the cokernel of a morphism $X \to Y$ measures the failure of the morphism to be surjective. For a morphism of complexes, the mapping cone measures the failure of the morphism to be an equivalence from the homological point of view.

Definition 1.5.8. A cochain complex $C^\bullet$ is called *contractible* if $\mathrm{id}_{C^\bullet} \simeq 0$. Equivalently, in reference of Definition 1.2.10 there exists a family of morphisms

$$h^n : C^n \to C^{n-1}$$

such that

$$\mathrm{id}_{C^n} = d^{n-1}_C \circ h^n + h^{n+1} \circ d^n_C$$

for all n.

Definition 1.5.9. A cochain complex $C^\bullet$ is called *acyclic* if $H^n(C^\bullet) = 0$ for all $n \in \mathbb{Z}$.

Remark 1.5.10. Every contractible complex is acyclic, but the converse need not hold in general. Contractibility is a homotopical condition, while acyclicity is a cohomological condition.

Proposition 1.5.11. Let $f : X^\bullet \to Y^\bullet$ be a morphism of cochain complexes. Then f is a homotopy equivalence if and only if $\mathrm{Cone}(f)$ is contractible.

Idea of the proof. The statement expresses that the cone detects whether f is an equivalence in the homotopy category $K(\mathcal{A})$. We give the explicit construction of the contracting homotopy in Appendix A.1.

If f is a homotopy equivalence, then there exists a morphism

$$g : Y^\bullet \to X^\bullet$$

together with homotopies

$$gf \simeq \mathrm{id}_{X^\bullet}, \qquad fg \simeq \mathrm{id}_{Y^\bullet}.$$

Using g and these two homotopies, one constructs explicitly a contracting homotopy of $\mathrm{Cone}(f)$. Hence the cone is contractible.

Conversely, if $\mathrm{Cone}(f)$ is contractible, then the cone triangle

$$X^\bullet \to Y^\bullet \to \mathrm{Cone}(f) \to X^\bullet[1]$$

shows that f becomes an isomorphism in the homotopy category. Equivalently, f admits a homotopy inverse. Thus f is a homotopy equivalence. □

Proposition 1.5.12. Let $f : X^\bullet \to Y^\bullet$ be a morphism of cochain complexes. Then f is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is acyclic.

Proof. There is a short exact sequence of cochain complexes

$$0 \to Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1] \to 0.$$

It induces a long exact sequence in cohomology:

$$\cdots \to H^n(Y^\bullet) \to H^n(\mathrm{Cone}(f)) \to H^n(X^\bullet[1]) \to H^{n+1}(Y^\bullet) \to \cdots.$$

Since

$$H^n(X^\bullet[1]) \cong H^{n+1}(X^\bullet),$$

the connecting morphism in this long exact sequence is, up to the standard sign convention, the morphism induced by f:

$$H^{n+1}(X^\bullet) \to H^{n+1}(Y^\bullet).$$

Therefore the long exact sequence shows that $H^n(\mathrm{Cone}(f)) = 0$ for all n if and only if

$$H^n(f) : H^n(X^\bullet) \to H^n(Y^\bullet)$$

is an isomorphism for all n. Hence $\mathrm{Cone}(f)$ is acyclic if and only if f is a quasi-isomorphism. □

Remark 1.5.13. The two propositions show that the mapping cone detects equivalences at two different levels:

$$f \text{ homotopy equivalence} \quad \Longleftrightarrow \quad \mathrm{Cone}(f) \text{ contractible},$$

and

$$f \text{ quasi-isomorphism} \quad \Longleftrightarrow \quad \mathrm{Cone}(f) \text{ acyclic}.$$

Thus the cone measures the failure of f to be an equivalence. This is the classical precursor of the stable ∞-categorical idea that cofibers measure the failure of a morphism to be an equivalence.

1.6 Triangulated Categories

The mapping cone construction produces sequences of the form

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1].$$

Such sequences are the basic examples of triangles. Triangulated categories axiomatize the formal properties of these cone triangles.

Definition 1.6.1. Let $\mathcal{T}$ be an additive category equipped with an autoequivalence

$$[1] : \mathcal{T} \to \mathcal{T},$$

called the *shift functor*. A *triangle* in $\mathcal{T}$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

A morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & X'[1]. \end{array}$$

Definition 1.6.2. Let $\mathcal{D}$ be an additive category equipped with an autoequivalence

$$T : \mathcal{D} \to \mathcal{D}.$$

A *triangulated category* is the data of such a pair $(\mathcal{D}, T)$ together with a class of triangles, called *distinguished triangles* or *exact triangles*, satisfying the following axioms.

(TR0) Any triangle isomorphic to a distinguished triangle is distinguished.

(TR1) For every object $X \in \mathcal{D}$, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is distinguished.

(TR2) For every morphism $f : X \to Y$ in $\mathcal{D}$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to T(X).$$

(TR3) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is distinguished if and only if the rotated triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$$

is distinguished.

(TR4) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ be distinguished triangles. Suppose we are given morphisms $u : X \to X'$ and $v : Y \to Y'$ such that $v \circ f = f' \circ u$. Then there exists a morphism

$$w : Z \to Z'$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \vdots{\scriptstyle w} & & \downarrow{\scriptstyle T(u)} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & T(X'). \end{array}$$

In other words, u and v can be completed to a morphism of distinguished triangles.

(TR5) The *octahedral axiom*. Suppose we are given composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ together with distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \xrightarrow{h'} T(X),$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \xrightarrow{k'} T(Y),$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Y' \xrightarrow{l'} T(X).$$

Then there exists a distinguished triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} T(Z')$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \xrightarrow{h'} & T(X) \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \| \\ X & \xrightarrow[g\circ f]{} & Z & \xrightarrow[l]{} & Y' & \xrightarrow[l']{} & T(X) \\ {\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow[g]{} & Z & \xrightarrow[k]{} & X' & \xrightarrow[k']{} & T(Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle l} & & \| & & \downarrow{\scriptstyle T(h)} \\ Z' & \xrightarrow[u]{} & Y' & \xrightarrow[v]{} & X' & \xrightarrow[w]{} & T(Z'). \end{array}$$

Remark 1.6.3. The numbering of the triangulated category axioms is not completely uniform in the literature. We use the labels (TR0)–(TR5) only as internal references within this thesis. They correspond to the standard axioms of triangulated categories, up to harmless changes of numbering and convention.

Remark 1.6.4. The octahedral axiom expresses the compatibility of distinguished triangles with composition. Given composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, one can complete f, g, and $g \circ f$ to distinguished triangles by (TR2). The axiom says that the three new objects appearing in these triangles are not independent: they themselves fit into a fourth distinguished triangle.

Proposition 1.6.5. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then

$$g \circ f = 0.$$

By rotating the triangle, one also obtains $h \circ g = 0$ and $T(f) \circ h = 0$.

Proof. By the identity triangle axiom, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is distinguished. By (TR1) and (TR4) we get

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\ {\scriptstyle \mathrm{id}_X}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{T(X)}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & T(X). \end{array}$$

The commutativity of the middle square gives $g \circ f = 0$, because the map $X \to 0 \to Z$ is the zero morphism. The other identities follow by rotations of the triangle. □

Definition 1.6.6. Let $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ from a triangulated category $\mathcal{D}$ to $\mathcal{A}$ is called *cohomological* if, for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$, the sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

is exact.

Remark 1.6.7. Because distinguished triangles can be rotated, a cohomological functor actually turns a distinguished triangle into a long exact sequence

$$\cdots \to H(T^{-1}Z) \to H(X) \to H(Y) \to H(Z) \to H(TX) \to \cdots .$$

Thus distinguished triangles play the role of exact sequences inside a triangulated category.

Proposition 1.6.8. Let $\mathcal{D}$ be a triangulated category. For every object $W \in \mathcal{D}$, the functor

$$\mathrm{Hom}_{\mathcal{D}}(W, -) : \mathcal{D} \to \mathbf{Ab}$$

is cohomological. Similarly, the contravariant functor

$$\mathrm{Hom}_{\mathcal{D}}(-, W) : \mathcal{D}^{op} \to \mathbf{Ab}$$

is cohomological.

Proof. We prove the statement for $\mathrm{Hom}_{\mathcal{D}}(W,-)$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a distinguished triangle. We have to show that

$$\mathrm{Hom}_{\mathcal{D}}(W,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{D}}(W,Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{D}}(W,Z)$$

is exact.

First, by the previous proposition, $g \circ f = 0$. Hence $g_* \circ f_* = 0$, so

$$\mathrm{im}(f_*) \subseteq \ker(g_*).$$

Conversely, let $u \in \ker(g_*)$. Thus $u : W \to Y$ is a morphism such that $g \circ u = 0$. Using the axioms of a triangulated category, one obtains a morphism of distinguished triangles

$$\begin{array}{ccccccc} W & \xrightarrow{\mathrm{id}_W} & W & \longrightarrow & 0 & \longrightarrow & T(W) \\ {\scriptstyle v}\vdots & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle T(v)} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & T(X). \end{array}$$

The commutativity of the first square gives $f \circ v = u$. Therefore

$$u = f_*(v),$$

so $u \in \mathrm{im}(f_*)$. Hence

$$\ker(g_*) \subseteq \mathrm{im}(f_*),$$

and the sequence is exact.

The proof for $\mathrm{Hom}_{\mathcal{D}}(-,W)$ is dual. □

Remark 1.6.9. Although distinguished triangles are not short exact sequences, they behave analogously to long exact sequences after applying Hom-functors as in classical homological algebra.

Lemma 1.6.10. Let $f : X^\bullet \to Y^\bullet$ be a morphism of cochain complexes. Then the rotated triangle

$$Y^\bullet \xrightarrow{\alpha(f)} \mathrm{Cone}(f) \xrightarrow{\beta(f)} X^\bullet[1] \xrightarrow{-f[1]} Y^\bullet[1]$$

is isomorphic in $K(\mathcal{A})$ to the cone triangle of $\alpha(f) : Y^\bullet \to \mathrm{Cone}(f)$. Explicitly:

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{\alpha(f)} & \mathrm{Cone}(f) & \xrightarrow{\beta(f)} & X^\bullet[1] & \xrightarrow{-f[1]} & Y^\bullet[1] \\ \| & & \| & & \downarrow{\scriptstyle u} & & \| \\ Y^\bullet & \xrightarrow[\alpha(f)]{} & \mathrm{Cone}(f) & \xrightarrow[\alpha(\alpha(f))]{} & \mathrm{Cone}(\alpha(f)) & \xrightarrow[\beta(\alpha(f))]{} & Y^\bullet[1]. \end{array}$$

In particular, the rotation of a cone triangle is again distinguished in $K(\mathcal{A})$.

Proof. With $\mathrm{Cone}(f)^n = Y^n \oplus X^{n+1}$, we have

$$\mathrm{Cone}(\alpha(f))^n = \mathrm{Cone}(f)^n \oplus Y^{n+1} = Y^n \oplus X^{n+1} \oplus Y^{n+1}.$$

The differential is given by

$$d^n_{\mathrm{Cone}(\alpha(f))}(y, x, y') = \big(d^n_Y y + f^{n+1}x + y', -d^{n+1}_X x, -d^{n+1}_Y y'\big).$$

Define

$$u : X^\bullet[1] \to \mathrm{Cone}(\alpha(f))$$

in degree n by

$$u^n(x) = \big(0, x, -f^{n+1}(x)\big).$$

We first check that u is a morphism of complexes. Since the differential on $X^\bullet[1]$ is $-d_X$, we compute

$$d^n_{\mathrm{Cone}(\alpha(f))} u^n(x) = d^n_{\mathrm{Cone}(\alpha(f))}(0, x, -f^{n+1}x)$$

$$= \big(f^{n+1}x - f^{n+1}x, -d^{n+1}_X x, d^{n+1}_Y f^{n+1}x\big) = \big(0, -d^{n+1}_X x, f^{n+2}d^{n+1}_X x\big).$$

Using that f is a morphism of complexes, $d^{n+1}_Y f^{n+1} = f^{n+2} d^{n+1}_X$. On the other hand,

$$u^{n+1}\big(-d^{n+1}_X x\big) = \big(0, -d^{n+1}_X x, -f^{n+2}(-d^{n+1}_X x)\big) = \big(0, -d^{n+1}_X x, f^{n+2}d^{n+1}_X x\big).$$

Hence

$$d_{\mathrm{Cone}(\alpha(f))} \circ u = u \circ d_{X[1]},$$

so u is a morphism of complexes.

Now consider the diagram

$$\begin{array}{ccccccc}
Y^\bullet & \xrightarrow{\alpha(f)} & \mathrm{Cone}(f) & \xrightarrow{\beta(f)} & X^\bullet[1] & \xrightarrow{-f[1]} & Y^\bullet[1] \\
\| & & \| & & \downarrow{\scriptstyle u} & & \| \\
Y^\bullet & \xrightarrow[\alpha(f)]{} & \mathrm{Cone}(f) & \xrightarrow[\alpha(\alpha(f))]{} & \mathrm{Cone}(\alpha(f)) & \xrightarrow[\beta(\alpha(f))]{} & Y^\bullet[1].
\end{array}$$

The first square commutes trivially. The last square also commutes strictly, since

$$\beta(\alpha(f))^n u^n(x) = \beta(\alpha(f))^n(0, x, -f^{n+1}x) = -f^{n+1}x = (-f[1])^n(x).$$

For the middle square, the two morphisms

$$\alpha(\alpha(f)), \quad u \circ \beta(f) : \mathrm{Cone}(f) \to \mathrm{Cone}(\alpha(f))$$

are not equal as morphisms of complexes, but they are chain homotopic. Indeed, for $(y, x) \in \mathrm{Cone}(f)^n$, one has

$$\alpha(\alpha(f))^n(y, x) = (y, x, 0)$$

and

$$u^n\beta(f)^n(y,x) = u^n(x) = (0, x, -f^{n+1}x).$$

Thus

$$\alpha(\alpha(f))^n(y,x) - u^n\beta(f)^n(y,x) = (y, 0, f^{n+1}x).$$

Define

$$s^n : \mathrm{Cone}(f)^n \to \mathrm{Cone}(\alpha(f))^{n-1}$$

by

$$s^n(y,x) = (0,0,y).$$

Then

$$d^{n-1}_{\mathrm{Cone}(\alpha(f))}s^n(y,x) = d^{n-1}_{\mathrm{Cone}(\alpha(f))}(0,0,y) = (y, 0, -d^n_Y y),$$

while

$$s^{n+1}d^n_{\mathrm{Cone}(f)}(y,x) = s^{n+1}(d^n_Y y + f^{n+1}x, -d^{n+1}_X x) = (0,0,d^n_Y y + f^{n+1}x).$$

Adding gives

$$d^{n-1}_{\mathrm{Cone}(\alpha(f))}s^n(y,x) + s^{n+1}d^n_{\mathrm{Cone}(f)}(y,x) = (y, 0, f^{n+1}x).$$

Hence

$$\alpha(\alpha(f)) - u \circ \beta(f) = d_{\mathrm{Cone}(\alpha(f))}s + s\, d_{\mathrm{Cone}(f)}.$$

So the middle square commutes in the homotopy category $K(\mathcal{A})$.

Finally, the projection

$$p : \mathrm{Cone}(\alpha(f)) \to X^\bullet[1], \qquad p^n(y,x,y') = x,$$

satisfies

$$p \circ u = \mathrm{id}_{X^\bullet[1]}.$$

Moreover, $u \circ p$ is homotopic to $\mathrm{id}_{\mathrm{Cone}(\alpha(f))}$. Indeed, with

$$S^n(y,x,y') = (0,0,y),$$

one checks in the same way that

$$\mathrm{id}_{\mathrm{Cone}(\alpha(f))} - u \circ p = d_{\mathrm{Cone}(\alpha(f))}S + S\, d_{\mathrm{Cone}(\alpha(f))}.$$

Thus u is an isomorphism in $K(\mathcal{A})$

□

Theorem 1.6.11. Let $\mathcal{A}$ be an abelian category. The homotopy category $K(\mathcal{A})$ is triangulated. The shift functor is given by the shift of complexes $C^\bullet \mapsto C^\bullet[1]$, and the distinguished triangles are the triangles isomorphic to mapping cone triangles

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1].$$

Sketch of proof. This is the standard triangulated structure on the homotopy category of complexes; see Weibel, Chapter 10.2 [5]. We indicate how the axioms of a triangulated category are verified. The point is that the axioms abstract the formal behaviour of mapping cone triangles.

(TR0) By definition, the class of distinguished triangles consists of all triangles isomorphic to mapping cone triangles. Hence any triangle isomorphic to a distinguished triangle is again distinguished.

(TR1) For every complex $X^\bullet$, the mapping cone of

$$\mathrm{id}_{X^\bullet} : X^\bullet \to X^\bullet$$

is contractible. Therefore it is isomorphic to the zero object in $K(\mathcal{A})$. The cone triangle of $\mathrm{id}_{X^\bullet}$ therefore gives a distinguished triangle isomorphic to

$$X^\bullet \xrightarrow{\mathrm{id}} X^\bullet \to 0 \to X^\bullet[1].$$

(TR2) Every morphism $f : X^\bullet \to Y^\bullet$ has a mapping cone. Hence every morphism can be completed to the distinguished triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \to \mathrm{Cone}(f) \to X^\bullet[1].$$

(TR3) This follows from Lemma 1.6.10.

(TR4) The axiom on morphisms of triangles is verified by comparing mapping cone triangles. More precisely, suppose we are given a commutative square in $K(\mathcal{A})$

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f} & Y^\bullet \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X'^\bullet & \xrightarrow[f']{} & Y'^\bullet. \end{array}$$

After choosing representatives and, if necessary, a homotopy witnessing the commutativity of the square, one constructs a morphism

$$\mathrm{Cone}(f) \to \mathrm{Cone}(f')$$

which completes the diagram to a morphism of cone triangles. This is a standard mapping-cone verification, and we will not spell out the sign computations here.

(TR5) The octahedral axiom is the most technical part. For composable morphisms

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet,$$

one considers the three cone triangles associated to f, g, and $g \circ f$. The required fourth distinguished triangle is obtained by an explicit comparison of these mapping cones. This gives the compatibility condition required by the octahedral axiom.

The direct cone-level verification of (TR4) and (TR5) is lengthy. Conceptually, these axioms are explained by the general fact that the homotopy category of a stable ∞-category carries a canonical triangulated structure; see Lurie's *Higher Algebra*, Theorem 1.1.2.14 [1]. In the case of complexes, this recovers the usual triangulated structure on $K(\mathcal{A})$, with distinguished triangles given by mapping cone triangles. □

Example 1.6.12. The derived category $D(\mathcal{A})$ is also triangulated. Its distinguished triangles are induced from the distinguished triangles in $K(\mathcal{A})$ after localizing at quasi-isomorphisms.

Remark 1.6.13. Distinguished triangles generalize the role of short exact sequences in homological algebra. For example, a short exact sequence of complexes

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

gives rise to a distinguished triangle

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

in the derived category. Moreover, cohomological functors send distinguished triangles to exact sequences. Thus triangulated categories retain a large part of the exactness formalism of classical homological algebra.

Remark 1.6.14 (The limitation of the triangulated formalism)**.** However, the triangulated formalism has an important limitation. Given a morphism $X \to Y$ the axiom (TR2) guarantees the existence of a distinguished triangle

$$X \to Y \to Z \to X[1],$$

but the third object Z is not characterized by a universal property. In particular, it is not defined as a cokernel, pushout, or colimit inside the triangulated category. It is only determined up to non-unique isomorphism.

This is the categorical reason behind the non-functoriality of cones. Suppose, for example, that one has a morphism between two maps. At the level of complexes, this involves a commutative square. After passing to the homotopy category, such a square may commute only up to a chosen homotopy

$$H : v \circ f \simeq f' \circ u.$$

The axioms of a triangulated category only ensure that, after such data is given, there exists a morphism between the corresponding third objects. They do not specify a canonical choice, and different choices of homotopy H may lead to different morphisms. Thus triangulated categories remember the formal properties of mapping cones, but not the coherent choices from which these cones arise.

The following chapters introduce a higher-categorical framework in which this lost coherence is retained. In that setting, the analogue of the cone will be defined by a universal property rather than only by the existence of a distinguished triangle.

2 Simplicial Sets and ∞-Categories

There are several models for higher category theory. In this thesis we use the model of simplicial sets, more precisely the model of ∞-categories as simplicial sets satisfying the inner horn filling condition. The material in this chapter follows Lurie's *Higher Topos Theory*, especially Chapter 1 [4], and Gallauer's notes on ∞-categories [6]. The aim is not to develop the full theory of simplicial sets, but only to introduce the constructions needed later: horns, Kan complexes, ∞-categories, nerves, homotopy categories, and mapping spaces.

2.1 Simplicial sets

Definition 2.1.1. Let Δ denote the category of finite ordinals

$$[n] := \{0 < 1 < \cdots < n\}, \qquad n \geq 0,$$

and weakly order-preserving maps. Thus a morphism $\alpha : [n] \to [m]$ in Δ is a map satisfying

$$i \leq j \quad \Longrightarrow \quad \alpha(i) \leq \alpha(j).$$

The category Δ is called the *simplex category*.

Definition 2.1.2. A *simplicial set* is a presheaf on Δ, i.e. a contravariant functor

$$C : \Delta^{op} \to \mathbf{Set}.$$

We denote the category of simplicial sets by $\mathbf{sSet} := \mathrm{Fun}(\Delta^{op}, \mathbf{Set})$. If C is a simplicial set, we write

$$C_n := C([n]).$$

The elements of C_n are called the *n-simplices* of C. Thus a simplicial set consists of sets of simplices in all dimensions, together with restriction maps

$$C(\alpha) : C_m \to C_n$$

for every order-preserving map $\alpha : [n] \to [m]$ in Δ.

Definition 2.1.3. For each $0 \leq i \leq n$, let

$$\delta^i : [n-1] \to [n]$$

be the unique injective order-preserving map whose image misses i. Since a simplicial set is contravariant, this induces a map

$$d_i := C(\delta^i) : C_n \to C_{n-1}.$$

The maps d_i are called the *face maps*. Geometrically, d_i removes the i-th vertex of an n-simplex. Similarly, for each $0 \leq i \leq n$, let

$$\sigma^i : [n+1] \to [n]$$

be the unique surjective order-preserving map which hits i twice. This induces a map

$$s_i := C(\sigma^i) : C_n \to C_{n+1}.$$

The maps s_i are called the *degeneracy maps*. Geometrically, s_i repeats the i-th vertex. Degenerate simplices may therefore be interpreted as simplices containing redundant or identity-like directions.

Remark 2.1.4. Every morphism in Δ can be written as a composite of the coface maps δ^i and the codegeneracy maps σ^i. Consequently, the structure of a simplicial set may equivalently be described by the face and degeneracy maps, subject to the simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad \text{for } i < j,$$

$$s_i s_j = s_{j+1} s_i \qquad \text{for } i \leq j,$$

and

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ \mathrm{id}, & i = j \text{ or } i = j+1, \\ s_j d_{i-1}, & i > j+1. \end{cases}$$

These identities express the compatibility between deleting and repeating vertices.

Definition 2.1.5. For $n \geq 0$, the *standard n-simplex* is the simplicial set

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n])$$

represented by $[n]$.

Definition 2.1.6. For $0 \leq i \leq n$, the *i-th face* of Δ^n denoted $\partial_i \Delta^n \subseteq \Delta^n$ is the simplicial subset generated by the simplex

$$\delta^i : [n-1] \to [n],$$

Definition 2.1.7. The *boundary* of Δ^n, denoted $\partial \Delta^n \subseteq \Delta^n$ is the simplicial subset generated by all faces $\partial_i \Delta^n$.

Definition 2.1.8. The *i-th horn* of Δ^n, denoted $\Lambda^n_i \subseteq \Delta^n$ is the simplicial subset generated by all faces of Δ^n except the i-th one. Thus

$$\Lambda^n_i = \bigcup_{\substack{0 \leq j \leq n \\ j \neq i}} \partial_j \Delta^n.$$

The horn Λ^n_i is called *inner* if $0 < i < n$ and *outer* otherwise.

Remark 2.1.9. The notion of a horn will be essential for the definition of an ∞-category. Indeed, ∞-categories are precisely those simplicial sets in which every inner horn admits a filler.

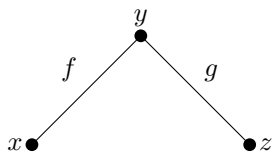
Example 2.1.10. Consider the inner horn $\Lambda^2_1 \subseteq \Delta^2$. Geometrically, one may think of Λ^2_1 as a triangle with its lower edge removed. A map of simplicial sets

$$\Lambda^2_1 \to C$$

into a simplicial set C amounts to the data of two composable 1-simplices

$$x \xrightarrow{f} y \xrightarrow{g} z$$

in C, i.e. two edges $f, g \in C_1$ satisfying $d_0(f) = d_1(g) = y$. The horn Λ^2_1 and such a map may be visualized as follows:



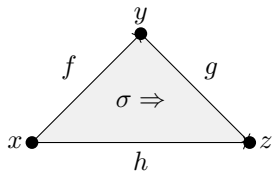
Definition 2.1.11. An *extension* of the map $\Lambda^2_1 \to C$ is then a 2-simplex in C.

$$\Delta^2 \to C$$

Remark 2.1.12. It fills the horn in 2.1.10 by providing a missing edge

$$h : x \to z$$

together with a 2-simplex σ witnessing the compatibility of h with the composable pair (f, g). This may be drawn as:



Remark 2.1.13. The wording "a missing edge" and "a chosen two-simplex" is deliberate. In an ordinary category, a composable pair of morphisms has a uniquely determined composite. By contrast, a horn in a simplicial set does not in general determine a unique extension. Rather, an extension consists of the choice of an additional edge together with a chosen 2-simplex filling the triangle. Thus this additional edge behaves like a composite, but only relative to the chosen filler. This idea is the first indication that horn fillers encode coherent composition data in a simplicial set.

2.2 The Singular Complex and Geometric Realization

An important example is provided by the classical singular complex for Algebraic Topology.

Definition 2.2.1. The functor

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$$

associates to every topological space X a simplicial set $\mathrm{Sing}(X)$ defined by

$$\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X).$$

the n-simplex a continous map $|\Delta^n| \to X$.

Remark 2.2.2. Here

$$|\Delta^n| = \left\{(x_0, \ldots, x_n) \in [0,1]^{n+1} \;\middle|\; x_0 + \cdots + x_n = 1\right\}$$

is the standard topological n-simplex.

Definition 2.2.3. The *geometric realization* of a simplicial set K is the topological space

$$|K| = \left(\coprod_{n \geq 0} K_n \times |\Delta^n| \right) / \sim,$$

where K_n is given the discrete topology and the equivalence relation is generated by the face and degeneracy maps of K.

Remark 2.2.4. The geometric realization construction defines a functor

$$|-| : \mathbf{sSet} \to \mathbf{Top}.$$

It is left adjoint to the singular complex functor

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet},$$

so that

$$|-| \dashv \mathrm{Sing}\,.$$

Remark 2.2.5. Heuristically, the geometric realization of K is obtained by taking one topological n-simplex for every n-simplex of K and gluing these simplices together according to the combinatorial instructions encoded by the face and degeneracy maps.

Proposition 2.2.6. For every topological space X, the counit map

$$\varepsilon_X : |\,\mathrm{Sing}(X)| \to X$$

is a weak homotopy equivalence, i.e. $\pi_n(|\,\mathrm{Sing}(X)|) \cong \pi_n(X)$ for all n.

Remark 2.2.7. This shows that, up to weak homotopy equivalence, a topological space may be replaced by its singular simplicial set. In this sense, simplicial sets provide a combinatorial model for spaces and homotopy types.

2.3 Kan Complexes and ∞-Categories

The boundary and horn inclusions introduced above allow us to formulate lifting conditions for simplicial sets. These conditions lead to two central notions: Kan complexes and ∞-categories. We now introduce both definitions and explain their basic interpretation.

Definition 2.3.1. A simplicial set X is called a *Kan complex* if for every $0 \leq i \leq n$, every map

$$\Lambda^n_i \xrightarrow{f} X$$

admits an extension to a map

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f} & X \\ \downarrow & \nearrow_{\tilde{f}} & \\ \Delta^n & & \end{array}$$

Equivalently, every horn in X admits a filler.

Remark 2.3.2. From the viewpoint of higher category theory, they are the simplicial sets in which all morphisms are invertible up to higher homotopy, and hence they may also be thought of as models for ∞-groupoids. (∞-groupoids are the ∞-categories such that all morphisms have an inverse) This is in accordance with Grothendieck's homotopy hypothesis, which informally asserts that spaces and ∞-groupoids represent the same homotopy-theoretic objects.

Proposition 2.3.3. For every topological space X, the singular complex $\mathrm{Sing}(X)$ is a Kan complex.

Proof. Let $\Lambda^n_i \to \mathrm{Sing}(X)$ be a horn in $\mathrm{Sing}(X)$. By the adjunction $|-| \dashv \mathrm{Sing}$, this corresponds to a continuous map

$$|\Lambda^n_i| \xrightarrow{f} X.$$

It therefore suffices to show that this map extends to a continuous map

$$\begin{array}{ccc} |\Lambda^n_i| & \xrightarrow{f} & X \\ \downarrow & \nearrow & \\ |\Delta^n| & & \end{array}$$

Now the horn $|\Lambda^n_i|$ is a retract of the simplex $|\Delta^n|$ in the category of topological spaces, i.e. a continous map r

$$r : |\Delta^n| \to |\Lambda^n_i|$$

such that the composite

$$|\Lambda^n_i| \hookrightarrow |\Delta^n| \xrightarrow{r} |\Lambda^n_i|$$

is the identity on $|\Lambda_i^n|$, meaning it extends via $f \circ r$. Passing back across the adjunction yields an extension

$$\Delta^n \to \mathrm{Sing}(X).$$

Hence every horn in $\mathrm{Sing}(X)$ admits a filler, and $\mathrm{Sing}(X)$ is a Kan complex. □

Definition 2.3.4. A simplicial set X is called an ∞-*category* if for every $n \geq 2$ and every inner horn inclusion

$$\Lambda_i^n \hookrightarrow \Delta^n, \qquad 0 < i < n,$$

every map $\Lambda_i^n \to X$ admits an extension to a map $\Delta^n \to X$.

Remark 2.3.5. Equivalently, every inner horn in C admits a filler. This is Lurie's model of ∞-categories by simplicial sets, also called quasi-categories; see [4, Section 1.1.2].

Remark 2.3.6. Kan complexes are special kind of ∞-categories.

$$\{\text{Kan complexes}\} \subset \{\infty\text{-categories}\} \subset \{\text{Simplicial sets}\}$$

Heuristically, inner horn fillers encode composition, while the additional fillers in a Kan complex correspond to invertibility up to higher homotopy.

2.4 The Nerve of an Ordinary Category

The nerve of an ordinary category is the first fundamental example of an ∞-category. It provides a canonical way of regarding an ordinary category as a simplicial set. In this way, ordinary categories embed into simplicial sets as a particularly rigid class of ∞-categories, rigid in the sense that composition is strictly defined, associativity holds on the nose, and no nontrivial higher homotopical data is present. Thus ordinary category theory is recovered as the strict shadow of a much richer higher-categorical framework.

Definition 2.4.1. Let $\mathcal{C}$ be an ordinary category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are chains of n composable morphisms in $\mathcal{C}$,

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n.$$

Equivalently,

$$N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}),$$

since a functor $[n] \to \mathcal{C}$ is uniquely determined by the corresponding sequence of composable arrows.

Remark 2.4.2. The face maps of $N(\mathcal{C})$ are obtained by deleting one object in the chain. At the two ends, this simply removes the first or last morphism,

while for $0 < i < n$ the face map d_i removes the object c_i and replaces the two adjacent arrows by their composite. Thus

$$d_i(c_0 \to \cdots \to c_n) = (c_0 \to \cdots \to c_{i-1} \to c_{i+1} \to \cdots \to c_n),$$

where in the interior case the new arrow from c_{i-1} to c_{i+1} is given by the composite of the two original arrows, i.e. $f_{i+1} \circ f_i$.

Similarly, the degeneracy maps are obtained by repeating an object and inserting an identity morphism. Thus s_i replaces c_i by two consecutive copies of c_i connected by id_{c_i}.

Remark 2.4.3. The preceding discussion shows that the nerve encodes the basic categorical structure: objects, morphisms, identities, and composition. The following results make precise how ordinary categories sit inside simplicial sets.

Remark 2.4.4. The following two results are standard; see [4, Proposition 1.1.2.2] and [6, Proposition 1.20].

Proposition 2.4.5. The nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. A functor $F : \mathcal{C} \to \mathcal{D}$ clearly induces a simplicial map

$$N(F) : N(\mathcal{C}) \to N(\mathcal{D}).$$

Conversely, a simplicial map

$$\varphi : N(\mathcal{C}) \to N(\mathcal{D})$$

is determined by its values on 0-simplices and 1-simplices. These give, respectively, a map on objects and a map on morphisms. Since φ commutes with degeneracy maps, it preserves identity morphisms. Since 2-simplices in the nerve encode composition, φ also preserves composition. Thus φ comes from a unique functor $\mathcal{C} \to \mathcal{D}$. Hence the natural map

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathbf{sSet}}(N(\mathcal{C}), N(\mathcal{D}))$$

is a bijection. Therefore N is fully faithful. □

Lemma 2.4.6. Let $\mathcal{C}$ be a small category. Then every inner horn

$$\Lambda^n_i \to N(\mathcal{C}), \qquad 0 < i < n,$$

admits a unique filler.

Proof. Let $\mathrm{Sp}^n \subseteq \Delta^n$ denote the spine, i.e. the simplicial subset generated by the adjacent edges

$$0 \to 1, \quad 1 \to 2, \quad \ldots, \quad n-1 \to n.$$

Since the horn is inner, one has

$$\mathrm{Sp}^n \subseteq \Lambda^n_i.$$

Indeed, for $n = 2$ this is equality, and for $n \geq 3$ an inner horn contains all edges. The restriction of the horn to the spine therefore determines a string of composable morphisms in $\mathcal{C}$:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n.$$

This string determines a unique n-simplex

$$\overline{f} : \Delta^n \to N(\mathcal{C}).$$

Hence, if $\overline{f}$ restricts to the given horn, then it is necessarily the unique filler. It remains to check that

$$\overline{f}|_{\Lambda^n_i} = f.$$

For $n = 2$ there is nothing to check. Assume $n \geq 3$. It suffices to check the equality on every face

$$\partial_j \Delta^n \subseteq \Lambda^n_i, \qquad j \neq i.$$

On such a face, it is enough to check equality on the spine

$$\mathrm{Sp}^{n-1} \subseteq \partial_j \Delta^n.$$

Most adjacent edges of this spine are already adjacent edges in Sp^n, where the two maps agree by construction. The only possible issue occurs for the edge passing over the omitted vertex j, namely the edge

$$j - 1 \to j + 1.$$

For $\overline{f}$, this edge is the composite $f_{j+1} \circ f_j$. But the 2-simplex on the vertices

$$j - 1 < j < j + 1$$

lies inside the inner horn Λ^n_i. Therefore the original horn also forces the edge $j - 1 \to j + 1$ to be the composite

$$f_{j+1} \circ f_j.$$

Thus $\overline{f}$ agrees with the given horn on each face of Λ^n_i. Hence $\overline{f}$ is the unique filler. Therefore every inner horn in $N(\mathcal{C})$ admits a unique filler. □

Theorem 2.4.7 (Characterization of nerves)**.** A simplicial set K is isomorphic to the nerve of a small category if and only if every inner horn in K admits a unique filler.

Proof. The forward direction is the preceding lemma. The converse is the essential image statement for the nerve functor: if every inner horn in K admits a unique filler, then one constructs a category $\mathcal{C}_K$ with objects K_0, morphisms K_1, identities given by degenerate edges, and composition defined by unique fillers of inner 2-horns. The uniqueness of fillers for inner 3-horns gives associativity, and higher unique fillers imply that every simplex of K is uniquely determined by its spine. Hence

$$K \cong N(\mathcal{C}_K).$$

This characterization of nerves is standard; see Lurie's *Higher Topos Theory*, Proposition 1.1.2.2 [4]. □

Remark 2.4.8. The essential point is that we now require *unique* fillers. Thus nerves of ordinary categories are precisely those simplicial sets in which every inner horn admits a unique filler. In contrast, general ∞-categories require the existence of fillers, not uniqueness.

Definition 2.4.9. An ordinary category $\mathcal{C}$ is called a *groupoid* if every morphism in $\mathcal{C}$ is an isomorphism. An ∞-category C is called an *∞-groupoid* if every morphism in C is an equivalence. Equivalently, as a simplicial set, an ∞-groupoid is a Kan complex.

Proposition 2.4.10. Let $\mathcal{C}$ be an ordinary category. Then its nerve $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

Proof. Recall that $N(\mathcal{C})$ is always an ∞-category, since every inner horn has a unique filler. Thus the only extra condition for $N(\mathcal{C})$ to be a Kan complex is that the outer horns also admit fillers.

Assume first that $N(\mathcal{C})$ is a Kan complex. Let $f : x \to y$ be a morphism in $\mathcal{C}$. We show that f is invertible.

Consider the outer horn $\Lambda^2_0 \to N(\mathcal{C})$ with vertices x, y, x, whose edge $x \to y$ is f and whose edge $x \to x$ is id_x. Since $N(\mathcal{C})$ is Kan, this horn has a filler. The missing edge gives a morphism

$$g : y \to x$$

such that

$$g \circ f = \mathrm{id}_x \,.$$

Similarly, filling the outer horn $\Lambda^2_2 \to N(\mathcal{C})$ with edge $x \to y$ equal to f and edge $y \to y$ equal to id_y gives a morphism

$$h : y \to x$$

such that

$$f \circ h = \mathrm{id}_y \,.$$

Then

$$g = g \circ \mathrm{id}_y = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_x \circ h = h.$$

Hence $g = h$, and f is an isomorphism. Therefore $\mathcal{C}$ is a groupoid.

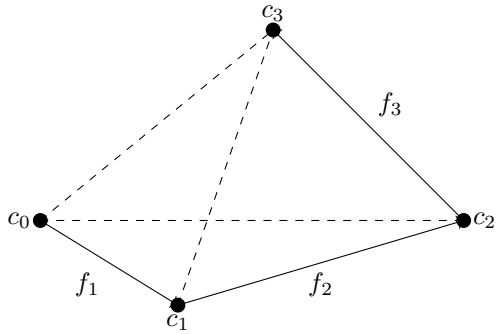
Conversely, assume that $\mathcal{C}$ is a groupoid. We already know that inner horns in $N(\mathcal{C})$ have unique fillers. It remains to consider outer horns. In an outer horn, the missing edge can be recovered using the inverse of a known morphism, since all morphisms in $\mathcal{C}$ are invertible. Once the missing edge is determined, the remaining simplex is uniquely determined by composition in $\mathcal{C}$. Hence every horn admits a filler, and $N(\mathcal{C})$ is a Kan complex. □

Remark 2.4.11. This proposition explains the slogan that Kan complexes are the ∞-categorical analogue of groupoids. Ordinary categories embed into simplicial sets by the nerve, and among them, the groupoids are exactly those whose nerves are Kan complexes.

Example 2.4.12. We illustrate the unique inner horn filling property of the nerve in the case $n = 3$. Consider a chain of composable morphisms in an ordinary category

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3.$$

Illustrated:



$$d_3: \quad c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2, \qquad \text{long edge } = f_2 \circ f_1,$$

$$d_0: \quad c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3, \qquad \text{long edge } = f_3 \circ f_2,$$

$$d_2: \quad c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_3 \circ f_2} c_3, \quad \text{long edge } = (f_3 \circ f_2) \circ f_1,$$

$$d_1: \quad c_0 \xrightarrow{f_2 \circ f_1} c_2 \xrightarrow{f_3} c_3, \quad \text{long edge } = f_3 \circ (f_2 \circ f_1).$$

Now consider the inner horn $\Lambda^3_1 \to N(\mathcal{C})$ obtained by forgetting the face d_1. The remaining faces d_0, d_2, and d_3 already determine the missing face uniquely. Indeed, d_3 determines the composite $f_2 \circ f_1$, while d_0 determines the composite $f_3 \circ f_2$, and d_2 forces the long edge to be $(f_3 \circ f_2) \circ f_1$. By associativity in the ordinary category $\mathcal{C}$, this is equal to

$$f_3 \circ (f_2 \circ f_1),$$

which is exactly the long edge required for the missing face d_1. Hence the horn admits a unique filler. This shows, in a concrete case, how inner horn fillers in the nerve encode strict composition together with strict associativity.

Example 2.4.13 (Outer horns need not have fillers)**.** We now consider an outer horn in dimension 3. Let

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3$$

be composable morphisms in an ordinary category, and suppose in addition that we are given a morphism

$$g : c_1 \to c_3$$

such that

$$g \circ f_1 = (f_3 \circ f_2) \circ f_1.$$

Now looking at the case for outer horns ($i = 0$) we get the following:

$$\Lambda^3_0 \to N(\mathcal{C})$$

whose faces are:

$$d_3 : \quad c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2,$$

$$d_2 : \quad c_0 \xrightarrow{f_1} c_1 \xrightarrow{g} c_3,$$

and

$$d_1 : \quad c_0 \xrightarrow{f_2 \circ f_1} c_2 \xrightarrow{f_3} c_3.$$

These faces are compatible precisely because

$$g \circ f_1 = f_3 \circ (f_2 \circ f_1).$$

However, to extend this horn to a 3-simplex, the missing face

$$d_0 : \quad c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3$$

would have to agree with the face d_2, and this would force

$$g = f_3 \circ f_2.$$

But this need not be true in a general category: from

$$g \circ f_1 = (f_3 \circ f_2) \circ f_1$$

one cannot in general cancel f_1. Thus the horn need not admit any filler. This shows that outer horns in the nerve of an ordinary category are not, in general, forced to have fillers. Hence the nerve of an ordinary category need not be a Kan complex.

Remark 2.4.14. In the higher-categorical setting, an ordinary category is often referred to as a 1-*category*, in order to distinguish it from higher categories. The full faithfulness of the nerve functor also means that the functor is a complete invariant of an ordinary category. For this reason, one often does not distinguish notationally between an ordinary category and its nerve when working inside simplicial sets.

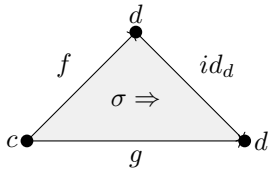
2.5 The homotopy category hC

The nerve functor embeds ordinary categories into simplicial sets. Conversely, every ∞-category has an associated ordinary category, called its homotopy category. It is obtained by keeping the same objects but passing from morphisms to homotopy classes of morphisms. Thus the homotopy category may be viewed as the 1-categorical shadow of an ∞-category.

Definition 2.5.1. Let C be an ∞-category, and let $f, g : c \to d$ be 1-simplices in C. We say that f and g are *homotopic* if there exists a 2-simplex

$$\sigma : \Delta^2 \to C$$

whose boundary is given by the following diagram:



In other words, f and g are homotopic if they are connected by a 2-simplex whose third edge is the degenerate edge at d.

Remark 2.5.2. Dually, one may define left homotopy by requiring the degenerate edge to lie at the source. In an ∞-category, the left and right notions agree, so one simply speaks of homotopic morphisms.

Proposition 2.5.3. Let C be an ∞-category. Homotopy defines an equivalence relation on the set of 1-simplices $f : x \to y$ with fixed source x and target y.

Proof. Reflexivity is given by the degenerate 2-simplex obtained from $f : x \to y$. This gives a 2-simplex whose edges are

$$x \xrightarrow{f} y \xrightarrow{\mathrm{id}_y} y$$

and whose long edge is again f. Hence $f \sim f$.

We prove symmetry by using one inner horn filler. Suppose $f \sim g$. Thus there is a 2-simplex

$$\sigma : \Delta^2 \to C$$

with boundary

$$x \xrightarrow{f} y \xrightarrow{\mathrm{id}_y} y, \qquad x \xrightarrow{g} y.$$

Consider a 3-simplex with vertices

$$x, y, y, y.$$

We specify three of its faces as follows:

- the face on vertices $(0,1,2)$ is the given 2-simplex σ;
- the face on vertices $(0,1,3)$ is the degenerate 2-simplex witnessing $f \sim f$;
- the face on vertices $(1,2,3)$ is the degenerate 2-simplex on id_y.

These three faces define an inner horn

$$\Lambda^3_1 \to C.$$

Since C is an ∞-category, this horn admits a filler

$$\Delta^3 \to C.$$

The missing face, namely the face on vertices $(0,2,3)$, has boundary

$$x \xrightarrow{g} y \xrightarrow{\mathrm{id}_y} y, \qquad x \xrightarrow{f} y.$$

Thus it is a homotopy $g \sim f$. Hence the relation is symmetric.

Transitivity is similar. Suppose $f \sim g$ and $g \sim h$. Choose 2-simplices witnessing these two homotopies. We form an inner horn

$$\Lambda^3_2 \to C$$

with vertices x, y, y, y whose faces are:

- the face $(0,1,2)$, witnessing $f \sim g$;
- the face $(0,2,3)$, witnessing $g \sim h$;
- the face $(1,2,3)$, the degenerate 2-simplex on id_y.

Filling this inner horn gives a 3-simplex. Its missing face is the face $(0,1,3)$, whose boundary is

$$x \xrightarrow{f} y \xrightarrow{\mathrm{id}_y} y, \qquad x \xrightarrow{h} y.$$

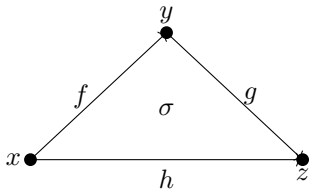
Hence this missing face witnesses $f \sim h$. Therefore the relation is transitive. $\square$

Definition 2.5.4. Let C be an ∞-category. The homotopy category hC is the ordinary category defined as follows:

1. The objects of hC are the objects of C.
2. For objects $x, y \in C$, the morphisms $\mathrm{Hom}_{hC}(x,y)$ are the homotopy classes of 1-simplices $f : x \to y$ in C, written $[f]$.
3. If $x \xrightarrow{f} y \xrightarrow{g} z$ are composable 1-simplices, then choose an inner horn filler

$$\sigma : \Delta^2 \to C$$

whose edges are f and g.



The third edge of σ determines a composite, and composition in hC is given by

$$[g] \circ [f] = [g \circ f].$$

This is independent of the choice of filler.

4. Identities are given by the degenerate edges: $\mathrm{id}_x = [s_0(x)]$.

   See Lurie's HTT [4, Section 1.2.3].

Remark 2.5.5. Thus hC is the 1-categorical shadow of the ∞-category C: it gives back the objects and morphisms, but forgets the higher homotopical data. Equivalently,

$$\mathrm{Hom}_{hC}(x, y) = \pi_0 \,\mathrm{Map}_C(x, y).$$

Although we have not defined $Map_C(x, y)$ yet. See next section. Nevertheless it should hint the reader the homotopic nature of the Hom-space of a given ∞-category C.

Definition 2.5.6. Let C be an ∞-category. A morphism

$$u : x \to y$$

in C is called an *equivalence* if its image

$$[u] : x \to y$$

is an isomorphism in the homotopy category hC, i.e., u is an equivalence in ∞-category if it has an inverse up to homotop

Proposition 2.5.7. The homotopy category construction defines a functor

$$h : \mathbf{sSet} \to \mathbf{Cat}$$

which is left adjoint to the nerve functor

$$N : \mathbf{Cat} \to \mathbf{sSet}.$$

Remark 2.5.8. In particular, if $\mathcal{C}$ is an ordinary category, then

$$h(N(\mathcal{C})) \cong \mathcal{C}.$$

Thus the homotopy category construction recovers an ordinary category from its nerve.

2.6 Inner Fibrations and Mapping Spaces

We next introduce the notion of mapping spaces, which replaces the classical Hom-set by a space, thereby enriching the notion of morphisms to the higher-categorical setting. Before doing so, it is necessary to show some lifting-theoretic notion for maps of simplicial sets. For the construction and basic properties of mapping spaces in an ∞-category, see Lurie's *Higher Topos Theory*, Section 1.2.2 [4].

Definition 2.6.1. A map of simplicial sets

$$p : X \to S$$

is called an *inner fibration* if it has the right lifting property (RLP) with respect to all inner horn inclusions

$$\Lambda^n_i \hookrightarrow \Delta^n, \qquad 0 < i < n.$$

Equivalently, for every such commutative diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S \end{array}$$

with $0 < i < n$, there exists a dotted arrow making the diagram commute.

Remark 2.6.2. A simplicial set C is an ∞-category if and only if its structure map $C \to \Delta^0$ is an inner fibration.

Example 2.6.3. We begin with the ordinary categorical prototype to ease in the higher-categorical definiton. Let $\mathcal{C}$ be an ordinary category and let $c, d \in \mathcal{C}$ be two objects. Consider the source-target functor

$$(s, t) : \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C} \times \mathcal{C},$$

which sends a morphism in $\mathcal{C}$ to its source and target. Then the set of morphisms from c to d may be recovered as the pullback

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c, d) & \longrightarrow & \mathrm{Fun}([1], \mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ [0] & \xrightarrow[(c,d)]{} & \mathcal{C} \times \mathcal{C} \end{array}$$

Thus $\mathrm{Hom}_{\mathcal{C}}(c, d)$ may be viewed as the fiber of the source-target map over the pair (c, d). Note technically $\mathrm{Hom}_{\mathcal{C}}(c, d)$ is viewed as the discrete category.

Remark 2.6.4. For an ∞-category, one defines the simplicial set of morphisms in the same manner. Since the category of simplicial sets is cartesian closed,

meaning that for every simplicial set K the functor $-\times K$ admits a right adjoint, for any simplicial sets K and C there exists an internal Hom simplicial set

$$\mathrm{Fun}(K, C).$$

It is characterized by the natural bijection

$$\mathrm{Hom}_{\mathbf{sSet}}\big(A, \mathrm{Fun}(K, C)\big) \cong \mathrm{Hom}_{\mathbf{sSet}}(A \times K, C)$$

for every simplicial set A. Equivalently, the n-simplices of $\mathrm{Fun}(K, C)$ are maps

$$\Delta^n \times K \to C.$$

In particular, $\mathrm{Fun}(\Delta^1, C)$ may be regarded as the simplicial set of arrows in C. See next section for concrete construction.

Definition 2.6.5. Let C be an ∞-category and let $c, d \in C_0$ be objects. The *mapping space* from c to d, denoted

$$\mathrm{Map}_C(c, d),$$

is defined by the pullback square

$$\begin{array}{ccc} \mathrm{Map}_C(c,d) & \longrightarrow & \mathrm{Fun}(\Delta^1, C) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow[(c,d)]{} & \mathrm{Fun}(\partial\Delta^1, C). \end{array}$$

Here the right vertical map is induced by restriction along the inclusion

$$\partial\Delta^1 \hookrightarrow \Delta^1,$$

and one identifies

$$\mathrm{Fun}(\partial\Delta^1, C) \cong C \times C.$$

The bottom horizontal map picks out the pair of objects (c, d).

Remark 2.6.6. The objects of the simplicial set $\mathrm{Map}_C(c, d)$ are precisely all the morphisms $f : c \to d$ in C. More generally, a 1-simplex of $\mathrm{Map}_C(c, d)$ is a homotopy between two such morphisms, and higher simplices encode higher homotopies. Heuristically, you have homotopies of homotopies of homotopies of...

Proposition 2.6.7. Let C be an ∞-category and let $c, d \in Ob(C)$. Then $\mathrm{Map}_C(c, d)$ is a Kan complex.

Proof. Let

$$\Lambda^n_i \to \mathrm{Map}_C(c, d)$$

be a horn. By the definition of the mapping space and the internal Hom adjunction, check 2.7.1, this is the same as giving a map

$$\Lambda_i^n \times \Delta^1 \to C$$

whose two endpoint restrictions are c and d. A filler for the horn would be a map

$$\Delta^n \to \mathrm{Map}_C(c, d),$$

which is equivalently a map

$$\Delta^n \times \Delta^1 \to C$$

with the same fixed endpoint conditions. These two pieces agree on their overlap

$$\Lambda_i^n \times \partial\Delta^1,$$

and therefore glue to a map

$$\left(\Lambda_i^n \times \Delta^1\right) \cup \left(\Delta^n \times \partial\Delta^1\right) \longrightarrow C.$$

Hence the horn-filling problem for $\mathrm{Map}_C(c, d)$ is equivalent to extending this partially defined prism to the whole prism

$$\Delta^n \times \Delta^1.$$

The relevant inclusion

$$\left(\Lambda_i^n \times \Delta^1\right) \cup \left(\Delta^n \times \partial\Delta^1\right) \hookrightarrow \Delta^n \times \Delta^1$$

can be obtained by successively attaching inner horns. Since C is an ∞-category, all such inner horns admit fillers. Therefore the above map extends to

$$\Delta^n \times \Delta^1 \to C.$$

Thus every horn in $\mathrm{Map}_C(c, d)$ admits a filler, and so $\mathrm{Map}_C(c, d)$ is a Kan complex. □

Proposition 2.6.8. Let C be an ∞-category and let $c, d \in C_0$. Then

$$\pi_0 \mathrm{Map}_C(c, d) \cong \mathrm{Hom}_{hC}(c, d).$$

Proof. By construction, the 0-simplices of $\mathrm{Map}_C(c, d)$ are the morphisms $f : c \to d$ in C. Two such morphisms determine the same element of

$$\pi_0 \mathrm{Map}_C(c, d)$$

if and only if they lie in the same connected component, which is equivalent to being homotopic by 2.5.1. But the morphisms in the homotopy category hC are by definition homotopy classes of 1-simplices. Hence the connected components of $\mathrm{Map}_C(c, d)$ are exactly the morphisms from c to d in hC. □

Remark 2.6.9. Thus the mapping spaces of an ∞-category refine the Hom-sets of its homotopy category:

$$\mathrm{Hom}_{hC}(c,d) \cong \pi_0 \,\mathrm{Map}_C(c,d).$$

Passing from $\mathrm{Map}_C(c,d)$ to its set of connected components forgets the higher homotopical data and retains only the underlying 1-categorical information.

Example 2.6.10. If $\mathcal{C}$ is an ordinary category and $N(\mathcal{C})$ its nerve, then the mapping space

$$\mathrm{Map}_{N(\mathcal{C})}(c,d) \cong \mathrm{Hom}_{\mathcal{C}}(c,d).$$

is a discrete Kan complex.

Remark 2.6.11. Mapping spaces describe morphisms between two fixed objects, but composition is encoded by higher simplices of the ∞-category. For objects $a,b,c \in C$, one may consider the simplicial set of 2-simplices with vertices a,b,c, denoted informally by $\mathrm{Map}_C(a,b,c)$. There are natural restriction maps

$$\mathrm{Map}_C(a,b,c) \longrightarrow \mathrm{Map}_C(a,b) \times \mathrm{Map}_C(b,c)$$

and

$$\mathrm{Map}_C(a,b,c) \longrightarrow \mathrm{Map}_C(a,c)$$

where the first map is induced by restricting a 2-simplex along the inner horn inclusion

$$\Lambda^2_1 \hookrightarrow \Delta^2, \Delta^1 \hookrightarrow \Delta^2$$

and remembers the two composable edges. The second map is induced by restricting along the long edge and remembers the edge from a to c.

Thus a 2-simplex witnesses that the long edge behaves like a composite of the two shorter edges. In this sense, composition in an ∞-category is not given first as a strict operation on mapping spaces, but is encoded by fillers of inner horns.

Remark 2.6.12. Similarly, 3-simplices encode the coherence of composition. Given three composable morphisms, a 3-simplex relates the two possible ways of composing them. Thus associativity is not imposed as a strict equality of chosen composites, but is part of the higher simplicial structure.

2.7 Functors between ∞-Categories

We have already used the notation $\mathrm{Fun}(K,C)$ in the definition of mapping spaces. We now show their construction.

Definition 2.7.1. Let K and D be simplicial sets. The *internal Hom* simplicial set

$$\mathrm{Fun}(K,D)$$

is defined by

$$\mathrm{Fun}(K,D)_n := \mathrm{Hom}_{\mathbf{sSet}}(K \times \Delta^n, D).$$

The face and degeneracy maps are induced by the coface and codegeneracy maps between the standard simplices Δ^n.

Theorem 2.7.2 (Joyal)**.** Let S be a simplicial set. The following conditions are equivalent:

1. S is an ∞-category.
2. The restriction map induced by the inner horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$,

$$p : \mathrm{Fun}(\Delta^2, S) \longrightarrow \mathrm{Fun}(\Lambda^2_1, S),$$

   is a trivial Kan fibration.

Sketch. We recall the technical notions used in this proof in Appendix A.2. Let

$$p : \mathrm{Fun}(\Delta^2, S) \longrightarrow \mathrm{Fun}(\Lambda^2_1, S)$$

be the restriction map. By the lifting characterization of trivial Kan fibrations, p is a trivial Kan fibration if and only if every lifting problem

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & \mathrm{Fun}(\Delta^2, S) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^m & \longrightarrow & \mathrm{Fun}(\Lambda^2_1, S) \end{array}$$

admits a solution.

Using the exponential adjunction for simplicial sets, this is equivalent to requiring fillers for the lifting problems

$$\begin{array}{ccc} (\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \Delta^m \times \Delta^2 & \longrightarrow & \Delta^0. \end{array}$$

The technical input is that the inclusions on the left generate, under weak saturation, precisely the inner anodyne maps. Hence the above lifting condition is equivalent to the existence of fillers for all inner horns

$$\Lambda^n_i \hookrightarrow \Delta^n, \qquad 0 < i < n.$$

This is exactly the condition that S is an ∞-category. □

Remark 2.7.3. The theorem gives a precise way to formulate composition in an ∞-category. Given a composable pair of edges $x \xrightarrow{f} y \xrightarrow{g} z$ in an ∞-category S, this data defines a map

$$\Lambda^2_1 \to S.$$

A choice of composite is a filler $\Delta^2 \to S$. The theorem says more than existence: the restriction map

$$\mathrm{Fun}(\Delta^2, S) \to \mathrm{Fun}(\Lambda^2_1, S)$$

is a trivial Kan fibration. Hence the space of possible fillers over a fixed inner horn is contractible. Thus composites in an ∞-category are not strictly unique, but they are unique up to a contractible space of choices.

Proposition 2.7.4. Let K be a simplicial set and let D be an ∞-category. Then

$$\mathrm{Fun}(K, D)$$

is again an ∞-category.

Proof. By Theorem 2.7.2, it is enough to show that the restriction map

$$\mathrm{Fun}(\Delta^2, \mathrm{Fun}(K, D)) \longrightarrow \mathrm{Fun}(\Lambda^2_1, \mathrm{Fun}(K, D))$$

is a trivial Kan fibration.

Using the exponential law for simplicial sets, we identify this map with

$$\mathrm{Fun}(K, \mathrm{Fun}(\Delta^2, D)) \longrightarrow \mathrm{Fun}(K, \mathrm{Fun}(\Lambda^2_1, D)).$$

Since D is an ∞-category, Theorem 2.7.2 implies that

$$\mathrm{Fun}(\Delta^2, D) \longrightarrow \mathrm{Fun}(\Lambda^2_1, D)$$

is a trivial Kan fibration.

Finally, applying $\mathrm{Fun}(K, -)$ preserves trivial Kan fibrations: indeed, lifting problems against boundary inclusions correspond, by the exponential adjunction, to lifting problems against the monomorphisms

$$\partial\Delta^m \times K \hookrightarrow \Delta^m \times K.$$

Thus the restriction map above is a trivial Kan fibration, and therefore $\mathrm{Fun}(K, D)$ is an ∞-category. □

Remark 2.7.5. Equivalently, $\mathrm{Fun}(K, D)$ is characterized by the natural bijection

$$\mathrm{Hom}_{\mathbf{sSet}}\big(A, \mathrm{Fun}(K, D)\big) \cong \mathrm{Hom}_{\mathbf{sSet}}(A \times K, D)$$

for every simplicial set A. Meaning

$$- \times K \dashv \mathrm{Fun}(K, -).$$

Remark 2.7.6. The titel of this subsection is therefore a little missleading, since we do not require to K and D be both ∞-category.

Definition 2.7.7. Let K be a simplicial set and let D be an ∞-category. The objects of $\mathrm{Fun}(K, D)$ are *functors* from K to D. If $f, g : K \to D$ are two functors, then a morphism $\eta : f \to g$ in $\mathrm{Fun}(K, D)$ is a map

$$\eta : K \times \Delta^1 \to D$$

whose restriction to $K \times \{0\}$ is f and whose restriction to $K \times \{1\}$ is g. Such a morphism is a *natural transformation*.

Remark 2.7.8. Thus natural transformations are the 1-simplices of the functor ∞-category $\mathrm{Fun}(K, D)$. Higher simplices in $\mathrm{Fun}(K, D)$ encode higher homotopies between natural transformations.

Definition 2.7.9. Let $\eta : f \to g$ be a natural transformation between functors $f, g : K \to D$. For every vertex $k \in K_0$, restricting η to $\{k\} \times \Delta^1$ gives a morphism

$$\eta_k : f(k) \to g(k)$$

in D. We say that η is an *objectwise equivalence* if every

$$\eta_k : f(k) \to g(k)$$

is an equivalence in D, in the sense of Definition 2.5.6.

Proposition 2.7.10. A natural transformation $\eta : f \to g$ is an equivalence in the ∞-category $\mathrm{Fun}(K, D)$ if and only if it is an objectwise equivalence as in 2.7.9.

Remark 2.7.11. Here an edge in an ∞-category is called an equivalence if it becomes an isomorphism in the homotopy category. Hence the proposition says that equivalences between functors are detected pointwise.

3 Limits, Colimits and Stability

We now turn our attention to universal constructions in the ∞-categorical world. The idea is again to take a familiar classical notion and upgrade it: instead of universal properties expressed by Hom-sets, we formulate them using mapping spaces.

For our purposes, we do not need the full general theory of limits and colimits. The main goal is to understand pullbacks, pushouts, fibers and cofibers, since these are the constructions that enter the definition of stable ∞-categories. In particular, cofibers will be the ∞-categorical replacement for mapping cones.

For the general theory of limits and colimits in ∞-categories, we refer to Lurie's *Higher Topos Theory*, Section 1.2.13 [4]. The material on stable ∞-categories used below follows Lurie's *Higher Algebra*, Section 1.1 [1].

3.1 Limits, Colimits, Fibers and Cofibers

Definition 3.1.1. Let I be a simplicial set and let C be an ∞-category. A *diagram* of shape I in C is a functor

$$F : I \to C.$$

Definition 3.1.2. Let $F : I \to C$ be a diagram. A *cone* over F with vertex $x \in C$ is a natural transformation

$$\underline{x} \to F,$$

where $\underline{x} : I \to C$ denotes the constant diagram with value x. Explicitly $\underline{x}$ is the composite

$$I \longrightarrow \Delta^0 \xrightarrow{x} C,$$

where the first map is the unique map to the terminal simplicial set and the second map selects the object $x \in C$.

Remark 3.1.3. The space of cones over F with vertex x is the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(I,C)}(\underline{x}, F).$$

A useful way to understand this space is through the functoriality of mapping spaces. The diagram $F : I \to C$ induces a diagram of spaces

$$\mathrm{Map}_C(x, F(-)) : I \to \mathcal{S},$$

sending an object i of I to the mapping space $\mathrm{Map}_C(x, F(i))$ in the ∞-category of spaces $\mathcal{S}$. Then, for every space $z \in \mathcal{S}$, one has a natural equivalence

$$\mathrm{Map}_{\mathcal{S}}\left(z, \mathrm{Map}_{\mathrm{Fun}(I,C)}(\underline{x}, F)\right) \simeq \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{S})}\left(\underline{z}, \mathrm{Map}_C(x, F(-))\right).$$

Thus a cone over F with vertex x may be understood as a system of maps

$$x \to F(i)$$

for all $i \in I$ compatible with the diagram F, where compatibility is understood in the homotopy-coherent sense.

Definition 3.1.4. Let $F : I \to C$ be a diagram, and let $\eta : \underline{x} \to F$ be a cone over F with vertex x. We say that η is a *limit cone* if, for every object $y \in C$, the induced map

$$\mathrm{Map}_C(y, x) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(I,C)}(\underline{y}, F)$$

is an equivalence of mapping spaces.

Remark 3.1.5. The map in the definition is induced by composition with the cone $\eta : \underline{x} \to F$. Indeed, a point of $\mathrm{Map}_C(y, x)$ is a morphism $u : y \to x$. This morphism determines a natural transformation between constant diagrams $\underline{u} : \underline{y} \to \underline{x}$. Composing with η gives a cone over F with vertex $y : \underline{y} \xrightarrow{\underline{u}} \underline{x} \xrightarrow{\eta} F$.

Thus every map $y \to x$ gives a cone from y to F, and this construction is functorial in higher simplices. Hence it defines the map of mapping spaces

$$\mathrm{Map}_C(y, x) \to \mathrm{Map}_{\mathrm{Fun}(I,C)}(\underline{y}, F).$$

Remark 3.1.6. This is the ∞-categorical version of the usual universal property of a limit. In ordinary category theory, maps $y \to x$ correspond bijectively to cones from y to F. In the ∞-categorical setting, Hom-sets are replaced by mapping spaces, so the universal property is expressed by an equivalence of spaces rather than a bijection of sets.

Example 3.1.7. Let I be a simplicial set and let X be a Kan complex. Consider the constant diagram

$$c_X : I \to \mathbf{sSet}$$

with value X. Informally, the limit of this constant diagram is the simplicial set of maps from I to X:

$$\lim_I c_X \simeq \mathrm{Fun}(I, X).$$

Thus the limit of a constant diagram is not necessarily just X; it also remembers the shape of the indexing simplicial set I. For example, if $I = \Delta^0$, then $\mathrm{Fun}(\Delta^0, X) \simeq X$. Generally if I is a discrete simplicial set it holds:

$$\mathrm{Fun}(I, X) \simeq \Pi_C X.$$

As a preview for later, if I is a simplicial model of the sphere S^n, then $\mathrm{Fun}(I, X)$ is the mapping space from S^n to X. In the pointed setting, its connected components recover the homotopy group $\pi_n(X)$.

Definition 3.1.8. Dually, a *cocone* under a diagram $F : I \to C$ with vertex x is a natural transformation

$$F \to \underline{x}.$$

A cocone is called a *colimit cocone* if, for every object $y \in C$, the induced map

$$\mathrm{Map}_C(x, y) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(I,C)}(F, \underline{y})$$

is an equivalence of mapping spaces.

3.2 Pullbacks and Pushouts

We now specialize the general language of limits and colimits to pullbacks and pushouts. These are the main finite limits and finite colimits needed for defining fibers, cofibers, and later stable ∞-categories.

Definition 3.2.1. Let C be an ∞-category. A diagram

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow{\scriptstyle p} \\ d & \xrightarrow[q]{} & c \end{array}$$

is called a *pullback square* if a is a limit of the cospan

$$b \xrightarrow{p} c \xleftarrow{q} d.$$

In this case we write

$$a \simeq b \times_c d.$$

Remark 3.2.2. The cospan

$$b \xrightarrow{p} c \xleftarrow{q} d$$

is the diagram we want to take the limit of. A cone over this diagram with vertex a consists of maps

$$a \to b, \qquad a \to d,$$

together with coherent data identifying the two composites

$$a \to b \xrightarrow{p} c \qquad \text{and} \qquad a \to d \xrightarrow{q} c.$$

If I_{pq} denotes the cospan shape and $F : I_{\mathrm{pq}} \to C$ is the diagram $b \to c \leftarrow d$, then the space of such cones is

$$\mathrm{Map}_{\mathrm{Fun}(I_{\mathrm{pq}},C)}(\underline{a}, F).$$

Using the description of spaces of cones, one obtains

$$\mathrm{Map}_{\mathrm{Fun}(I_{\mathrm{pb}},C)}(\underline{a}, F) \simeq \mathrm{Map}_C(a,b) \times_{\mathrm{Map}_C(a,c)} \mathrm{Map}_C(a,d).$$

Here the two maps to $\mathrm{Map}_C(a,c)$ are induced by postcomposition with p and q.

Remark 3.2.3. Equivalently, the square defining a is a pushout square if, for every object $x \in C$, the induced map

$$\mathrm{Map}_C(a,x) \longrightarrow \mathrm{Map}_C(b,x) \times_{\mathrm{Map}_C(c,x)} \mathrm{Map}_C(d,x)$$

is an equivalence of spaces. This is the ∞-categorical version of the ordinary universal property of pushouts. The compatibility is expressed inside mapping spaces, and therefore includes the higher coherent homotopy data.

Remark 3.2.4. If $C = N(\mathcal{C})$ is the nerve of an ordinary category, then the mapping spaces are discrete. Hence the formula above reduces to the usual ordinary description: a cone over the cospan

$$b \to c \leftarrow d$$

with vertex a is simply a commutative square

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow{\scriptstyle p} \\ d & \xrightarrow[q]{} & c \end{array}$$

that is, a pair of morphisms $a \to b$ and $a \to d$ such that the two composites to c are strictly equal.

Definition 3.2.5. Let C be an ∞-category. A diagram

$$\begin{array}{ccc} c & \xrightarrow{s} & b \\ {\scriptstyle r}\downarrow & & \downarrow \\ d & \longrightarrow & a \end{array}$$

is called a *pushout square* if a is a colimit of the span

$$d \xleftarrow{r} c \xrightarrow{s} b.$$

In this case we write

$$a \simeq b \sqcup_c d.$$

Remark 3.2.6. Equivalently, the square defining a is a pushout square if, for every object $x \in C$, the induced map

$$\mathrm{Map}_C(a, x) \longrightarrow \mathrm{Map}_C(b, x) \times_{\mathrm{Map}_C(a,x)} \mathrm{Map}_C(d, x)$$

is an equivalence of spaces. This is the dual universal property to the one for pullbacks: maps out of the pushout are the same as compatible pairs of maps out of b and d.

Remark 3.2.7. The formulas for pullbacks and pushouts are the main reason for introducing the mapping-space formulation of limits and colimits. They allow us to express universal properties in a way that remembers the higher homotopical coherence data. This will be crucial when defining fibers and cofibers.

3.3 Pointed ∞-Categories and Fibers and Cofibers

In order to define fibers and cofibers, we need the ∞-categorical analogue of a zero object.

Definition 3.3.1. Let C be an ∞-category. An object $0 \in C$ is called *initial* if for every object $x \in C$, the mapping space

$$\mathrm{Map}_C(0, x)$$

is contractible. It is called *terminal* if for every object $x \in C$, the mapping space

$$\mathrm{Map}_C(x, 0)$$

is contractible. An object is called a *zero object* if it is both initial and terminal. An ∞-category with a zero object is called *pointed*.

Remark 3.3.2. In a pointed ∞-category, there is a distinguished zero morphism between any two objects, obtained by factoring through the zero object.

Example 3.3.3. The ∞-category of spaces is denoted by $\mathcal{S}$. One may think of it as the ∞-category obtained from topological spaces, or simplicial sets, by remembering homotopy types. The ∞-category of pointed spaces is denoted by $\mathcal{S}_*$. Its objects are spaces equipped with a chosen base point. The object $* \in \mathcal{S}_*$, viewed as the one-point pointed space, is both initial and terminal. Hence $\mathcal{S}_*$ is a pointed ∞-category.

Definition 3.3.4. Let C be a pointed ∞-category, and let

$$f : X \to Y$$

be a morphism in C. The *fiber* of f denoted $\mathrm{fib}(f)$ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & Y. \end{array}$$

Equivalently,

$$\mathrm{fib}(f) \simeq X \times_Y 0.$$

Definition 3.3.5. Let $f : X \to Y$ be a morphism in a pointed ∞-category. The *cofiber* of f is defined by the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f). \end{array}$$

Equivalently,

$$\mathrm{cofib}(f) \simeq 0 \sqcup_X Y.$$

Remark 3.3.6. $\mathrm{cofib}(f)$ is characterized by the universal property of this pushout. Informally, it is the object obtained from Y by forcing the image of X to become zero. This is the point of writing

$$\mathrm{cofib}(f) \simeq 0 \sqcup_X Y.$$

The construction is not obtained by choosing a mapping cone only after passing to the homotopy category. (note for future: needs reference to verdier triangualted cat in classcial chapter which is not yet written) Rather, it is defined by an actual pushout diagram in the underlying ∞-category.

Definition 3.3.7. A sequence $X \to Y \to Z$ is called a *cofiber sequence* if

$$Z \simeq \mathrm{cofib}(X \to Y).$$

Dually, a sequence $W \to X \to Y$ is called a *fiber sequence* if

$$W \simeq \mathrm{fib}(X \to Y).$$

3.4 Stable ∞-Categories

We now have the language needed to define stable ∞-categories. The basic idea is that, in a stable ∞-category, fibers and cofibers are not independent constructions anymore, meaning pullback and pushout squares coincide. This is the higher-categorical structure behind exact triangles in triangulated categories.

Definition 3.4.1. Let C be a pointed ∞-category and let $X \in C$. The *suspension* of X is defined as the pushout

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

Equivalently,

$$\Sigma X \simeq 0 \sqcup_X 0.$$

Definition 3.4.2. Let C be a pointed ∞-category and let $X \in C$. The *loop object* of X is defined as the pullback

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X. \end{array}$$

Equivalently,

$$\Omega X \simeq 0 \times_X 0.$$

Definition 3.4.3. A pointed ∞-category C is called *stable* if it admits finite limits and finite colimits, and if a square in C is a pullback square if and only if it is a pushout square.

Remark 3.4.4. The last condition is the essential point. In a general pointed ∞-category, pullbacks and pushouts are different universal constructions. In a

stable ∞-category they coincide. Thus a square which is universal as a pushout is automatically universal as a pullback, and conversely.

This is why fiber and cofiber sequences become two sides of the same structure in a stable ∞-category. It is also the reason why cofiber sequences later give rise to exact triangles in the homotopy category.

Proposition 3.4.5. For any stable ∞-category C, the morphism sets in hC naturally carry abelian group structures.

Proof. Since C is stable, the suspension functor $\Sigma : C \to C$ is an equivalence. Hence, for every object $A \in C$, there exists an object $A_2 \in C$ such that

$$A \simeq \Sigma^2 A_2.$$

Let $B \in C$. Then

$$\mathrm{Map}_C(A, B) \simeq \mathrm{Map}_C(\Sigma^2 A_2, B).$$

Now recall that $\Sigma^2 A_2 = \Sigma(\Sigma A_2)$ is defined as the pushout

$$\begin{array}{ccc} \Sigma A_2 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma^2 A_2. \end{array}$$

Since mapping out of a colimit gives a limit of mapping spaces, we obtain

$$\mathrm{Map}_C(\Sigma^2 A_2, B) \simeq \mathrm{Map}_C(0, B) \times_{\mathrm{Map}_C(\Sigma A_2, B)} \mathrm{Map}_C(0, B).$$

Because 0 is initial, $\mathrm{Map}_C(0, B) \simeq *$. Therefore

$$\mathrm{Map}_C(\Sigma^2 A_2, B) \simeq * \times_{\mathrm{Map}_C(\Sigma A_2, B)} * \simeq \Omega \,\mathrm{Map}_C(\Sigma A_2, B),$$

where the loop space is taken at the zero map. Applying the same argument once more to

$$\Sigma A_2 = 0 \sqcup_{A_2} 0$$

gives

$$\mathrm{Map}_C(A, B) \simeq \Omega^2 \,\mathrm{Map}_C(A_2, B).$$

Thus every mapping space in a stable ∞-category is equivalent to a double loop space. By the Eckmann–Hilton argument, two compatible monoid structures on the same set must coincide and be commutative. Applied to a double loop space, the two loop directions give such compatible multiplications. Consequently

$$\mathrm{Hom}_{hC}(A, B) = \pi_0 \,\mathrm{Map}_C(A, B)$$

is an abelian group. □

Proposition 3.4.6. Let C be a pointed ∞-category admitting cofibers, and let $f : X \to Y$ be a morphism in C. Then f gives rise to a triangle in the homotopy category

$$X \to Y \to \mathrm{cofib}(f) \to X[1],$$

where $X[1] = \Sigma X$.

Proof. By definition, the cofiber of f is given by the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f). \end{array}$$

naturally giving a sequence $X \to Y \to \mathrm{cofib}(f)$. It remains to construct the last morphism

$$\mathrm{cofib}(f) \to \Sigma X.$$

Recall that the suspension of X is defined by the pushout square

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

Equivalently,

$$\Sigma X \simeq 0 \sqcup_X 0.$$

On the other hand,

$$\mathrm{cofib}(f) \simeq 0 \sqcup_X Y.$$

Therefore, to construct a map

$$\mathrm{cofib}(f) = 0 \sqcup_X Y \longrightarrow 0 \sqcup_X 0 = \Sigma X,$$

it is enough to give a compatible map of diagrams

$$0 \longleftarrow X \xrightarrow{f} Y$$

to

$$0 \longleftarrow X \longrightarrow 0.$$

This is given by the identity on X, the identity on 0, and the zero map $Y \to 0$. By the universal property of the pushout, this induces a morphism

$$\mathrm{cofib}(f) \to \Sigma X = X[1].$$

Hence we obtain a triangle in hC:

$$X \to Y \to \mathrm{cofib}(f) \to X[1].$$

□

Theorem 3.4.7. Let C be a stable ∞-category. Then the homotopy category

$$hC$$

carries a natural triangulated structure. The translation functor is induced by the suspension functor

$$\Sigma : C \to C,$$

and the distinguished triangles are those induced by cofiber sequences in C.

Proof. This is Lurie's *Higher Algebra*, Theorem 1.1.2.14 [1]. □

Remark 3.4.8 (Idea of the proof)**.** The proof works by constructing the relevant universal cofiber and pushout diagrams in the stable ∞-category C. After passing to the homotopy category, these diagrams become the class triangles of Verdier's formalism.

For (TR1), one has to show that every morphism $f : X \to Y$ can be completed to a distinguished triangle. This follows from the existence of cofibers: the cofiber square of f gives a triangle

$$X \to Y \to \mathrm{cofib}(f) \to X[1].$$

For (TR2), one has to show that distinguished triangles are stable under rotation. This is where the suspension functor enters. In a stable ∞-category, the suspension and loop functors are inverse equivalences, so rotating a cofiber triangle again gives a triangle coming from the same stable structure.

For (TR3), one has to extend a morphism between the first two terms of two distinguished triangles to a morphism of the whole triangles. In Lurie's proof, this is done by lifting morphisms between the corresponding cofiber diagrams in C. Thus the required map on the third terms is not chosen artificially, but comes from functoriality of the higher-categorical cofiber construction.

Finally, (TR4), the octahedral axiom, is obtained from a larger diagram of pushout squares associated to two composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

The commutativity required in the homotopy category is the shadow of an actual homotopy-coherent diagram in C. This is one of the main advantages of the stable ∞-categorical setting: the octahedral axiom is encoded by coherent pushout data rather than imposed by hand.

Remark 3.4.9. The theorem explains how the triangulated structure on hC arises from the stable ∞-category C. Namely, the distinguished triangles are not added as extra data after passing to the homotopy category. They come from cofiber diagrams in C: A morphism $f : X \to Y$ gives a cofiber sequence $X \to Y \to \mathrm{cofib}(f)$ and hence a triangle $X \to Y \to \mathrm{cofib}(f) \to X[1]$ in hC. Lurie's theorem says that these triangles satisfy Verdier's axioms and therefore form the distinguished triangles of the triangulated category hC.

This is the precise sense in which stable ∞-categories refine Verdier's triangulated categories. The homotopy category hC retains the triangulated structure, while the stable ∞-category C remembers the mapping spaces and the homotopy-coherent pushout and pullback data from which this structure is induced.

4 The ∞-Category of Spectra

We now turn to the main example of a stable ∞-category: the ∞-category of spectra. We denote $\mathcal{S}_*$ the ∞-category of pointed spaces. The idea is that spectra are obtained from pointed spaces by forcing suspension and loop to become inverse operations. In this sense, spectra are stabilized pointed spaces. In this chapter, we look at spectra from several complementary viewpoints: first as stabilized pointed spaces, then as representing objects for generalized cohomology theories, and finally through compact generation and Brown representability. The material in this chapter follows Lurie's *Higher Algebra*, especially Section 1.4 [1].

4.1 Spectra as Stabilized Pointed Spaces

We denote by $\mathcal{S}_*$ the ∞-category of pointed spaces. The idea is that spectra are obtained from pointed spaces by forcing suspension and loop to become inverse operations. The following definition is the ∞-categorical model of spectra used by Lurie; see [1, Definition 1.4.1.6].

Definition 4.1.1. The ∞-category of spectra is defined as the limit

$$\mathrm{Sp} := \lim \left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right).$$

Remark 4.1.2. Unwinding the definition, an object $E \in \mathrm{Sp}$ consists of a sequence of pointed spaces $E = (E_0, E_1, E_2, \ldots)$ together with equivalences

$$E_n \simeq \Omega E_{n+1}$$

for all $n \geq 0$. E_{n+1} can be thought of as a delooping of E_n. Equivalently, by the adjunction

$$\Sigma \dashv \Omega,$$

this data may be described by structure maps

$$\Sigma E_n \to E_{n+1}.$$

A morphism of spectra $f : E \to F$ is given by maps of pointed spaces

$$f_n : E_n \to F_n$$

compatible with the equivalences

$$E_n \simeq \Omega E_{n+1}, \qquad F_n \simeq \Omega F_{n+1}.$$

In the ∞-categorical setting, this compatibility is understood homotopy-coherently.

Proposition 4.1.3. The ∞-category of spectra Sp is stable.

Proof idea. This is one of the basic properties of spectra; see [1, Corollary 1.4.1.10]. The zero object is the spectrum whose levels are all the one-point space. Moreover, by construction, the loop functor is an equivalence on Sp: a spectrum already consists of a sequence of pointed spaces related by equivalences

$$E_n \simeq \Omega E_{n+1}.$$

Thus looping a spectrum only shifts this sequence. Equivalently, suspension and loop are inverse operations in Sp. Hence spectra form a stable ∞-category. $\square$

Remark 4.1.4. This is the fundamental example of a stable ∞-category. Its stability expresses precisely that suspension has become invertible. Thus spectra are the natural objects in which suspension and loop are inverse operations.

Definition 4.1.5. The *infinite loop space functor* is the evaluation functor

$$\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*, \qquad E \mapsto E_0.$$

Thus $\Omega^\infty E$ is the zeroth space of the spectrum E.

Definition 4.1.6. The *suspension spectrum functor* is the functor

$$\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}.$$

It sends a pointed space X to the spectrum freely generated by X under suspension.

Remark 4.1.7. Informally, one may think of $\Sigma^\infty X$ as represented by the sequence

$$X,\ \Sigma X,\ \Sigma^2 X,\ \ldots.$$

More precisely, one model for its k-th space is given by

$$(\Sigma^\infty X)_k \simeq \operatorname*{colim}_i \Omega^i \Sigma^{k+i} X.$$

This formula reflects the fact that the naive suspension sequence has to be stabilized so that the equivalences

$$(\Sigma^\infty X)_k \simeq \Omega(\Sigma^\infty X)_{k+1}$$

hold.

Proposition 4.1.8. For $X \in \mathcal{S}_*$ and $Y \in \mathrm{Sp}$, there is a natural equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^\infty Y).$$

Equivalently,

$$\Sigma^\infty \dashv \Omega^\infty.$$

See [1, Remark 1.4.1.9].

Proof idea. The functor Σ^∞ is constructed as the free way of regarding a pointed space as a spectrum. Therefore a map

$$\Sigma^\infty X \to Y$$

of spectra is determined by a map from X to the zeroth space of Y, namely by a map

$$X \to Y_0 = \Omega^\infty Y.$$

The compatibility with the higher spaces of Y is exactly encoded by the spectrum structure. □

Remark 4.1.9. Let K be a finite pointed space and let $X \in \mathcal{S}_*$. By the adjunction $\Sigma^\infty \dashv \Omega^\infty$, we first obtain

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K, \Sigma^\infty X) \simeq \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^\infty \Sigma^\infty X).$$

Using the formula $\Omega^\infty \Sigma^\infty X \simeq \mathrm{colim}_i\, \Omega^i \Sigma^i X$ as in 4.1.7 this becomes

$$\mathrm{Map}_{\mathcal{S}_*}(K, \operatorname*{colim}_i \Omega^i \Sigma^i X).$$

Since K is finite, mapping out of K commutes with this filtered colimit and using the adjunction $\Sigma^i \dashv \Omega^i$ giving us:

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K, \Sigma^\infty X) \simeq \operatorname*{colim}_i \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^i \Sigma^i X). \simeq \operatorname*{colim}_i \mathrm{Map}_{\mathcal{S}_*}(\Sigma^i K, \Sigma^i X).$$

Passing to connected components gives

$$[\Sigma^\infty K, \Sigma^\infty X]_{h\mathrm{Sp}} \cong \operatorname*{colim}_i [\Sigma^i K, \Sigma^i X]_*.$$

Thus maps between suspension spectra recover the classical idea of stable maps: one suspends both source and target repeatedly and then passes to the stable colimit.

Definition 4.1.10. The *sphere spectrum* is the suspension spectrum of the pointed zero-sphere:

$$\mathbb{S} := \Sigma^\infty S^0.$$

It is the stable version of the spheres.

Remark 4.1.11. Taking $K = S^0$ in the previous formula gives

$$[\mathbb{S}, \Sigma^\infty X]_{h\mathrm{Sp}} \cong \operatorname*{colim}_i [S^i, \Sigma^i X]_*.$$

More generally, for $n \geq 0$,

$$\pi_n(\Sigma^\infty X) \cong \operatorname*{colim}_i \pi_{n+i}(\Sigma^i X).$$

In particular, for $X = S^0$, one obtains

$$\pi_n(\mathbb{S}) \cong \operatorname*{colim}_i \pi_{n+i}(S^i),$$

the stable homotopy groups of spheres.

Definition 4.1.12. Let $E \in \mathrm{Sp}$ be a spectrum. Its n-th homotopy group is defined by

$$\pi_n(E) := \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\mathbb{S}[n], E).$$

Equivalently,

$$\pi_n(E) = \operatorname{Hom}_{h\mathrm{Sp}}(\mathbb{S}[n], E).$$

Remark 4.1.13. The shifted sphere spectrum $\mathbb{S}[n]$ plays the role of the stable n-sphere. For $n \geq 0$, one has

$$\mathbb{S}[n] = \Sigma^n \mathbb{S} = \Sigma^n \Sigma^\infty S^0 \simeq \Sigma^\infty S^n.$$

For $n < 0$, the shift is defined using loops:

$$\mathbb{S}[n] = \Omega^{-n} \mathbb{S}.$$

Thus the definition of $\pi_n(E)$ is the stable analogue of the classical description of homotopy groups by maps out of spheres.

Remark 4.1.14. We end this subsection with one technical observation which will be used later in the discussion of compact generation. The next formula expresses the idea that a spectrum is built from the suspension spectra of its individual spaces. We will use it only as a structural input in the discussion of compact generation.

Proposition 4.1.15 (Reconstruction from suspension spectra)**.** Let Y be a spectrum. Then there is a canonical equivalence

$$Y \simeq \operatorname*{colim}_n \Sigma^{-n} \Sigma^\infty Y_n.$$

Here the transition maps are induced by the structure maps $\Sigma Y_n \to Y_{n+1}$ of the spectrum Y.

Proof idea. Set

$$A_n := \Sigma^{-n} \Sigma^\infty Y_n.$$

The structure map $\Sigma Y_n \to Y_{n+1}$ induces a map $A_n \to A_{n+1}$, so the objects A_n form diagrams in Sp.

We show that Y and $\operatorname{colim}_n A_n$ have the same mapping spaces into every spectrum Z. That is, we show

$$\operatorname{Map}_{\mathrm{Sp}}(Y, Z) \simeq \operatorname{Map}_{\mathrm{Sp}}\left(\operatorname*{colim}_n A_n, Z\right).$$

Since by the definition of spectra 4.1.1 mapping spaces in Sp are computed as the corresponding limit of mapping spaces in $\mathcal{S}_*$. Thus the left hand side

$$\operatorname{Map}_{\mathrm{Sp}}(Y, Z) \simeq \lim_n \operatorname{Map}_{\mathcal{S}_*}(Y_n, Z_n).$$

The right hand side, pulling out the colimit gives a limit of mapping spaces, we have

$$\operatorname{Map}_{\mathrm{Sp}}(\operatorname*{colim}_n A_n, Z) \simeq \lim_n \operatorname{Map}_{\mathrm{Sp}}(A_n, Z).$$

By definition of A_n, shifting and using the adjunction $\Sigma^\infty \dashv \Omega^\infty$ gives

$$\mathrm{Map}_{\mathrm{Sp}}(A_n, Z) = \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{-n}\Sigma^\infty Y_n, Z) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^n\Sigma^{-n}\Sigma^\infty Y_n, \Sigma^n Z)$$

$$\simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty Y_n, \Sigma^n Z) \simeq \mathrm{Map}_{\mathcal{S}_*}(Y_n, \Omega^\infty\Sigma^n Z) \simeq \mathrm{Map}_{\mathcal{S}_*}(Y_n, Z_n).$$

where the following was used in the last step $\Omega^\infty\Sigma^n Z \simeq Z_n$. Hence

$$\mathrm{Map}_{\mathrm{Sp}}\left(\operatorname*{colim}_n A_n, Z\right) \simeq \lim_n \mathrm{Map}_{\mathcal{S}_*}(Y_n, Z_n) \simeq \mathrm{Map}_{\mathrm{Sp}}(Y, Z).$$

The Yoneda lemma implies

$$Y \simeq \operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty Y_n.$$

□

4.2 Spectra as Representing Objects for Cohomology Theories

We have introduced spectra as stabilized pointed spaces. There is another important viewpoint explaining why spectra are central: they represent generalized cohomology theories. Thus spectra are not only objects of a stable ∞-category, but also serve as representing objects for invariants of spaces. For the relationship between spectra and reduced generalized cohomology theories, see [1, Section 1.4.1].

Definition 4.2.1. Let $E \in \mathrm{Sp}$ be a spectrum. For a pointed space X, we define the *reduced E-cohomology of X* by

$$\widetilde{E}^n(X) := \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, E[n]).$$

Equivalently,

$$\widetilde{E}^n(X) = \pi_0 \,\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E[n]).$$

Thus the value of the cohomology theory on X is obtained by mapping the suspension spectrum of X into a shift of the representing spectrum E.

Remark 4.2.2. The construction is contravariant in X. Indeed, a map of pointed spaces $X \to Y$ induces a map of spectra $\Sigma^\infty X \to \Sigma^\infty Y$. By precomposition, this gives a map $\mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty Y, E[n]) \to \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, E[n])$. Hence $X \mapsto \widetilde{E}^n(X)$ is a contravariant functor. This is exactly the contravariance one expects in cohomology.

Definition 4.2.3. Let A be an abelian group and let $n \geq 0$. An *Eilenberg–Mac Lane space* $K(A, n)$ is a space characterized by

$$\pi_i(K(A,n)) = \begin{cases} A, & i = n, \\ 0, & i \neq n. \end{cases}$$

Definition 4.2.4. Let A be an abelian group. The *Eilenberg–Mac Lane spectrum* associated to A is the spectrum denoted by

$$HA.$$

Informally, it is the spectrum whose n-th space is given by

$$(HA)_n \simeq K(A, n).$$

The compatibility condition comes from the equivalences

$$\Omega K(A, n+1) \simeq K(A, n).$$

Remark 4.2.5. From the stable point of view, the Eilenberg–Mac Lane spectrum HA packages the individual spaces

$$K(A,0),\ K(A,1),\ K(A,2),\dots$$

into one stable object. Hence ordinary cohomology with coefficients in A is represented by the spectrum HA:

$$\widetilde{H}^n(X; A) \cong \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, HA[n]).$$

This is the stable analogue of the classical formula

$$H^n(X; A) \cong [X, K(A, n)].$$

More generally, different choices of the representing spectrum E give different generalized cohomology theories.

Example 4.2.6. Recall from Example 1.4.17 that, for a smooth manifold M, the de Rham complex can be interpreted as a resolution computing the cohomology of the constant sheaf $\underline{\mathbb{R}}$. Concretely, applying global sections gives the complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots,$$

whose cohomology is the de Rham cohomology $H^n_{\mathrm{dR}}(M)$.

From the stable homotopy-theoretic point of view, the same ordinary cohomology theory with real coefficients is represented by the Eilenberg–Mac Lane spectrum $H\mathbb{R}$. Thus, using the de Rham theorem and the representability of ordinary cohomology, one obtains

$$H^n_{\mathrm{dR}}(M) \cong \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty M_+, H\mathbb{R}[n]).$$

Here M_+ denotes the pointed space obtained from M by adjoining a disjoint basepoint.

Thus the same invariant appears in two different languages: in the classical derived-functor picture it is computed by a complex of differential forms, while in the stable homotopy-theoretic picture it is represented by the spectrum $H\mathbb{R}$.

Definition 4.2.7. A *reduced generalized cohomology theory* on pointed spaces is a sequence of contravariant functors

$$\widetilde{h}^n : \mathcal{S}_*^{op} \to \mathbf{Ab}, \qquad n \in \mathbb{Z},$$

satisfying the following properties:

1. *Homotopy invariance:* if $X \to Y$ is an equivalence of pointed spaces, then
$$\widetilde{h}^n(Y) \cong \widetilde{h}^n(X).$$
2. *Suspension axiom:* there are natural isomorphisms
$$\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X).$$
3. *Exactness:* every cofiber sequence of pointed spaces
$$X' \to X \to X''$$
gives rise to a long exact sequence of cohomology groups.
4. *Wedge axiom:* for every family of pointed spaces $\{X_\alpha\}_{\alpha \in A}$, the canonical map
$$\widetilde{h}^n \left(\bigvee_\alpha X_\alpha \right) \longrightarrow \prod_\alpha \widetilde{h}^n(X_\alpha)$$
is an isomorphism.

Remark 4.2.8. The word *reduced* means in particular that $\widetilde{h}^n(*) = 0$ for the one-point space $*$.

Proposition 4.2.9. Let $E \in \mathrm{Sp}$ be a spectrum. Then the functors

$$\widetilde{E}^n : \mathcal{S}_*^{op} \to \mathbf{Ab}, \qquad X \longmapsto \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, E[n])$$

define a reduced generalized cohomology theory on pointed spaces.

Remark 4.2.10. This proposition gives one direction of the relation between spectra and cohomology theories:

$$\text{spectrum} \quad \Longrightarrow \quad \text{reduced generalized cohomology theory.}$$

The converse direction is Brown representability. To discuss this properly, we first introduce filtered colimits, compact objects, and compact generation.

4.3 Filtered Colimits and Compact Generation

Before stating Brown representability, we introduce the compactness condition which controls the size of the ∞-categories involved. This is the part of the general theory which will be relevant for us.

Definition 4.3.1. An ∞-category I is called *filtered* if every finite diagram in I admits a cocone. A *filtered colimit* in an ∞-category C is a colimit of a diagram

$$F : I \to C$$

where I is filtered.

Definition 4.3.2. Let C be an ∞-category admitting filtered colimits. An object $K \in C$ is called *compact* if the functor

$$\mathrm{Map}_C(K, -) : C \to \mathcal{S}$$

commutes with filtered colimits. Equivalently, for every filtered diagram

$$F : I \to C,$$

the natural map

$$\operatorname*{colim}_{i \in I} \mathrm{Map}_C(K, F(i)) \longrightarrow \mathrm{Map}_C(K, \operatorname*{colim}_{i \in I} F(i))$$

is an equivalence of spaces.

Remark 4.3.3. The compactness condition says that maps out of K into a filtered colimit can be computed before taking the colimit:

$$\mathrm{Map}_C(K, \operatorname*{colim}_{i \in I} F(i)) \simeq \operatorname*{colim}_{i \in I} \mathrm{Map}_C(K, F(i)).$$

Thus compact objects behave like finite test objects. Analogously to finitely presented modules, where maps out of a finitely presented module commute with filtered colimits.

Definition 4.3.4. Let C be an ∞-category admitting filtered colimits. We say that C is *compactly generated* if there is a small subcategory

$$C^\omega \subseteq C$$

consisting of compact objects such that every object of C can be obtained as a filtered colimit of objects in C^ω. Equivalently, C is obtained from its compact objects by freely adjoining filtered colimits:

$$C \simeq \mathrm{Ind}(C^\omega).$$

Remark 4.3.5. The word "small" is important: the compact generators should form a set, not a proper class. This smallness condition is the set-theoretic part of the formal machinery which allows one to control a large category by compact generators.

Remark 4.3.6. In Lurie's formulation, compact generation is often placed inside the broader language of presentable ∞-categories. We will not use the full theory of presentability here. For our purposes, the relevant point is compact generation: a large ∞-category is controlled by a small subcategory of compact objects.

Definition 4.3.7. The *finite spectra* are the spectra generated from finite pointed spaces by suspension and desuspension. More precisely, we denote by

$$\mathrm{Sp}^{\mathrm{fin}} \subseteq \mathrm{Sp}$$

the small subcategory generated by spectra of the form

$$\Sigma^{-n}\Sigma^{\infty}K,$$

where K is a finite pointed space and $n \geq 0$. These spectra of this form are the finite version of 4.1.15.

Proposition 4.3.8 (Compact generation of spectra)**.** The ∞-category of spectra is compactly generated. More precisely, there is an equivalence

$$\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}}).$$

Proof. This is Lurie's *Higher Algebra*, Corollary 1.4.4.6 [1]. □

Remark 4.3.9. The notation

$$\mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$$

means that one freely adds filtered colimits to the small ∞-category of finite spectra. Thus the proposition says that every spectrum can be built from finite spectra by allowing filtered colimits.

Proof idea. Let

$$\Phi : \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}}) \longrightarrow \mathrm{Sp}$$

be the functor induced by the inclusion

$$\mathrm{Sp}^{\mathrm{fin}} \hookrightarrow \mathrm{Sp}.$$

Since $\mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$ is obtained from $\mathrm{Sp}^{\mathrm{fin}}$ by freely adjoining filtered colimits, such a functor exists because Sp admits filtered colimits.

We first indicate why the objects of $\mathrm{Sp}^{\mathrm{fin}}$ are compact. Let K be a finite pointed space and consider an object of the form $\Sigma^{-n}\Sigma^{\infty}K$. Let $F : I \to \mathrm{Sp}$ be a filtered diagram of spectra. Using stability, shifting, and the adjunction $\Sigma^{\infty} \dashv \Omega^{\infty}$, we get

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{-n}\Sigma^{\infty}K, \operatorname*{colim}_{i\in I} F(i)) \simeq \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^{\infty}\Sigma^{n} \operatorname*{colim}_{i\in I} F(i)).$$

Filtered colimits in spectra are computed levelwise, and $\Omega^\infty\Sigma^n$ is evaluation at the n-th space. Hence

$$\Omega^\infty\Sigma^n \operatorname*{colim}_{i\in I} F(i) \simeq \operatorname*{colim}_{i\in I} \Omega^\infty\Sigma^n F(i).$$

Since K is finite, mapping out of K commutes with filtered colimits of spaces. Therefore

$$\mathrm{Map}_{\mathcal{S}_*}(K, \operatorname*{colim}_{i\in I} \Omega^\infty\Sigma^n F(i)) \simeq \operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^\infty\Sigma^n F(i)).$$

Using the adjunction again, this is equivalent to

$$\operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathrm{Sp}}\left(\Sigma^{-n}\Sigma^\infty K, F(i)\right).$$

Thus the finite spectra are compact. This compactness gives the fully faithful part of

$$\Phi : \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}}) \to \mathrm{Sp}.$$

It remains to explain why Φ is essentially surjective. Let

$$Y = (Y_0, Y_1, Y_2, \ldots)$$

be a spectrum. By the reconstruction formula (Proposition 4.1.15), one has

$$Y \simeq \operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty Y_n.$$

Thus every spectrum is built from shifted suspension spectra of its individual spaces. Each pointed space Y_n can be written as a filtered colimit of finite pointed spaces:

$$Y_n \simeq \operatorname*{colim}_\alpha K_{n,\alpha}.$$

For instance, one may represent Y_n by a pointed simplicial set and write it as the filtered colimit of its finite pointed subcomplexes. Since Σ^∞ preserves colimits, we obtain

$$\Sigma^{-n}\Sigma^\infty Y_n \simeq \operatorname*{colim}_\alpha \Sigma^{-n}\Sigma^\infty K_{n,\alpha}.$$

Each spectrum

$$\Sigma^{-n}\Sigma^\infty K_{n,\alpha}$$

belongs to $\mathrm{Sp}^{\mathrm{fin}}$. Combining this with the reconstruction formula, we see that Y is a filtered colimit of finite spectra.

Hence every spectrum lies in the essential image of Φ, and therefore

$$\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}}).$$

□

Remark 4.3.10. This compact generation statement is the main structural input needed for Brown representability. It says that functors on $h\mathrm{Sp}$ can be tested against a small collection of compact objects, namely the finite spectra.

4.4 Brown Representability and the Classification of Cohomology Theories

We now return to the converse direction. We have seen that every spectrum defines a reduced generalized cohomology theory. Brown representability gives a converse: under suitable compact generation hypotheses, a cohomology theory is represented by an object. In the case of pointed spaces, the representing objects assemble into a spectrum.

Definition 4.4.1. Let D be an ordinary category and let

$$F : D^{op} \to \mathbf{Set}$$

be a contravariant functor. We say that F is *representable* if there exists an object $R \in D$ such that, naturally in $X \in D$, one has a bijection

$$F(X) \cong \mathrm{Hom}_D(X, R).$$

In this case we say that R represents the functor F.

Remark 4.4.2. In the stable homotopy setting, the category D will often be the homotopy category $h\mathrm{Sp}$. Thus a spectrum E represents a functor if the values of the functor can be written as morphisms into E in $h\mathrm{Sp}$. For example, the definition

$$\widetilde{E}^n(X) = \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, E[n])$$

says precisely that the shifted spectrum $E[n]$ represents the n-th reduced E-cohomology functor.

Theorem 4.4.3 (Brown representability)**.** Let C be a pointed ∞-category admitting the relevant colimits, and suppose that C is compactly generated by a small set of compact cogroup objects

$$\{S_\alpha\}_{\alpha \in A}.$$

Let

$$F : hC^{op} \to \mathbf{Set}$$

be a contravariant functor. Then F is representable if it satisfies the following two conditions:

1. for every family of objects $\{C_\beta\}_{\beta \in B}$ in C, the natural map

$$F\left(\coprod_\beta C_\beta\right) \longrightarrow \prod_\beta F(C_\beta)$$

is a bijection;

2. for every pushout square

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow & & \downarrow \\ D & \longrightarrow & D' \end{array}$$

in C, the induced map

$$F(D') \longrightarrow F(B') \times_{F(B)} F(D)$$

is surjective.

Remark 4.4.4. The first condition says that F behaves like a representable functor with respect to coproducts. Indeed, if $F(X) \cong \mathrm{Hom}_{hC}(X, R)$, then maps out of a coproduct are products of maps:

$$\mathrm{Hom}_{hC}\left(\coprod_{\beta} C_\beta, R\right) \cong \prod_{\beta} \mathrm{Hom}_{hC}(C_\beta, R).$$

The second condition is an abstract gluing form of exactness. A pushout square heuristically is forming an object

$$D' \simeq B' \cup_C D$$

by gluing B' and D along B. Since F is contravariant, an element of $F(D')$ restricts to elements of $F(B')$ and $F(D)$, and these two restrictions agree in $F(B)$. Thus the map

$$F(D') \to F(B') \times_{F(B)} F(D)$$

records the two compatible restrictions. The surjectivity condition says that every compatible pair comes from an element on the pushout.

For the special pushout defining a cofiber,

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(A \to X), \end{array}$$

this recovers the familiar exactness statement: classes on X which vanish on A come from classes on the cofiber.

Remark 4.4.5. The assumption that the compact generators S_α are cogroup objects ensures that the sets

$$\mathrm{Hom}_{hC}(S_\alpha, X)$$

carry natural group structures. This is needed in the proof when one takes kernels of the maps

$$\mathrm{Hom}_{hC}(S_\alpha, X) \to F(S_\alpha).$$

Informally, a cogroup object is the dual notion of a group object. In the stable setting this assumption is natural, since hC is additive and Hom-sets are abelian groups.

Remark 4.4.6. In Lurie's formulation, this theorem is stated in the broader language of presentable ∞-categories; see *Higher Algebra*, Theorem 1.4.1.2 [1]. Here we emphasize the compact generation aspect, since this is the part that will be relevant for spectra and for the construction of representing objects.

Sketch of proof. This is Lurie's *Higher Algebra*, Theorem 1.4.1.2 [1]. We explain the construction of the representing object. We write 0 for the zero object of C. We may enlarge the chosen set of compact generators $\{S_\alpha\}_{\alpha\in A}$, if necessary, so that it is closed under suspension. This does not change the generated category.

We first show (*):

$(*)$ Suppose that $f : X \to Y$ is a morphism in C such that, for every compact generator S_α, the induced map

$$\mathrm{Hom}_{hC}(S_\alpha, X) \longrightarrow \mathrm{Hom}_{hC}(S_\alpha, Y)$$

is a bijection. Then f is an equivalence.

Indeed, the compact generators detect equivalences. More precisely, since the set of generators is closed under suspension, the Hom-sets

$$\mathrm{Hom}_{hC}(\Sigma^n S_\alpha, -)$$

detect the higher homotopy groups of the mapping spaces

$$\mathrm{Map}_C(S_\alpha, -).$$

Thus f induces equivalences on mapping spaces out of all compact generators. Since these objects generate C, it follows that f is an equivalence.

We now prove the main construction step.

$(*)'$ Let $X \in C$ and let $\eta \in F(X)$. Then there exists a morphism

$$X \to X'$$

and an element

$$\eta' \in F(X')$$

lifting η, such that for every compact generator S_α, the map induced by η',

$$\mathrm{Hom}_{hC}(S_\alpha, X') \longrightarrow F(S_\alpha), \qquad f \longmapsto F(f)(\eta'),$$

is a bijection.

We construct X' step by step. First, set

$$I = \{(\alpha, \gamma) \mid \alpha \in A,\ \gamma \in F(S_\alpha)\}.$$

For each pair $(\alpha, \gamma) \in I$, we take one copy of S_α. Define

$$X_0 := X \coprod \coprod_{(\alpha,\gamma)\in I} S_\alpha.$$

By the coproduct condition,

$$F(X_0) \cong F(X) \times \prod_{(\alpha,\gamma)\in I} F(S_\alpha).$$

Hence we may choose an element

$$\eta_0 \in F(X_0)$$

whose component in $F(X)$ is η, and whose component in the factor indexed by (α, γ) is precisely $\gamma \in F(S_\alpha)$. It follows that, for every α, the map

$$\mathrm{Hom}_{hC}(S_\alpha, X_0) \longrightarrow F(S_\alpha)$$

induced by η_0 is surjective.

We now kill the kernel. Suppose that X_n and $\eta_n \in F(X_n)$ have been constructed. Since S_α is a cogroup object, the set

$$\mathrm{Hom}_{hC}(S_\alpha, X_n)$$

has group structure, and the map

$$\mathrm{Hom}_{hC}(S_\alpha, X_n) \longrightarrow F(S_\alpha), \qquad f \longmapsto F(f)(\eta_n),$$

is a group homomorphism. Let

$$K_{\alpha,n} := \ker\left(\mathrm{Hom}_{hC}(S_\alpha, X_n) \longrightarrow F(S_\alpha)\right).$$

Define X_{n+1} by the pushout square

$$\begin{array}{ccc} \displaystyle\coprod_{\alpha\in A,\ \gamma\in K_{\alpha,n}} S_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n+1}. \end{array}$$

The left vertical map is given by the maps $\gamma : S_\alpha \to X_n$, and the upper horizontal map obvious one. Thus this pushout forces all kernel elements to become zero.

The pushout-surjectivity condition applied to this square shows that $\eta_n \in F(X_n)$ lifts to an element

$$\eta_{n+1} \in F(X_{n+1}).$$

Thus we obtain a sequence

$$X_0 \to X_1 \to X_2 \to \cdots$$

together with compatible elements $\eta_n \in F(X_n)$.

Let

$$X' := \operatorname*{colim}_n X_n.$$

Using the fact that a sequential colimit can be described by a pushout of coproducts, the coproduct condition and the pushout-surjectivity condition produce an element

$$\eta' \in F(X')$$

which restricts to the elements η_n.

We claim that η' has the required property. Fix α. Surjectivity of

$$\mathrm{Hom}_{hC}(S_\alpha, X') \longrightarrow F(S_\alpha)$$

follows from the construction of X_0. For injectivity, let

$$\gamma : S_\alpha \to X'$$

be an element in the kernel. Since S_α is compact, γ factors through some finite stage, lets say at m:

$$S_\alpha \to X_m \to X'.$$

Because γ lies in the kerne, this map belongs to $K_{\alpha,m}$. By construction it becomes zero after passing to X_{m+1}, and hence it is zero in X'. Thus the map is injective. This proves $(*)'$.

We now apply $(*)'$ to $X = 0$. $F(0)$ is a singleton. The unique element of $F(0)$ and $(*)'$ gives an object $X' := R \in C$ and an element

$$u \in F(R)$$

such that, for every compact generator S_α, the map

$$\mathrm{Hom}_{hC}(S_\alpha, R) \longrightarrow F(S_\alpha), \qquad f \longmapsto F(f)(u),$$

is a bijection.

We claim that R represents F. The element $u \in F(R)$ defines, for every object $Y \in C$, a natural map

$$\theta_Y : \mathrm{Hom}_{hC}(Y, R) \longrightarrow F(Y), \qquad f \longmapsto F(f)(u).$$

We prove that θ_Y is bijective.

First, we prove surjectivity. Let $\xi \in F(Y)$. By the coproduct condition, the canonical map

$$F(R \coprod Y) \longrightarrow F(R) \times F(Y)$$

is a bijection. Therefore the pair

$$(u, \xi) \in F(R) \times F(Y)$$

comes from a unique element of $F(R \coprod Y)$.

Applying $(*)'$ to the object $R \coprod Y$ with this element, we obtain a morphism

$$R \coprod Y \to Z$$

and an element of $F(Z)$ which induces bijections

$$\mathrm{Hom}_{hC}(S_\alpha, Z) \longrightarrow F(S_\alpha)$$

for all α. The composite

$$R \to R \coprod Y \to Z$$

therefore induces bijections

$$\mathrm{Hom}_{hC}(S_\alpha, R) \longrightarrow \mathrm{Hom}_{hC}(S_\alpha, Z)$$

for all α, since both sides are identified with $F(S_\alpha)$. By $(*)$, the map

$$R \to Z$$

is an equivalence. Hence the map

$$Y \to R \coprod Y \to Z$$

can be viewed, using $Z \simeq R$, as a map

$$Y \to R.$$

By construction, this map is sent to $\xi \in F(Y)$. Thus θ_Y is surjective.

Now we prove injectivity. Suppose that two maps

$$f, g : Y \to R$$

have the same image in $F(Y)$. Write this common value as

$$\xi \in F(Y).$$

Form the pushout square

$$\begin{array}{ccc} Y \coprod Y & \xrightarrow{(f,g)} & R \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z. \end{array}$$

Here the left vertical map is the fold map $Y \coprod Y \to Y$.

The equality $F(f)(u) = F(g)(u) = \xi$ says precisely that the pair

$$(u, \xi) \in F(R) \times F(Y)$$

has the same image in $F(Y \coprod Y)$. Therefore it defines an element of the fiber product

$$F(R) \times_{F(Y \coprod Y)} F(Y).$$

By the pushout-surjectivity condition, this element lifts to an element of $F(Z)$.

Applying $(*)'$ to Z, we obtain a morphism

$$Z \to Z'$$

such that the induced maps

$$\mathrm{Hom}_{hC}(S_\alpha, Z') \longrightarrow F(S_\alpha)$$

are bijections for all α. As before, the composite

$$R \to Z \to Z'$$

induces bijections on Hom-sets out of all compact generators. Hence, by $(*)$, it is an equivalence.

But in the pushout defining Z, the two maps f and g become equal after composition with $R \to Z$. Therefore they also become equal after composition with the equivalence

$$R \to Z'.$$

Hence $f = g$ in hC. Thus θ_Y is injective.

We have shown that for every $Y \in C$, the map

$$\theta_Y : \mathrm{Hom}_{hC}(Y, R) \longrightarrow F(Y)$$

is a bijection, naturally in Y. Therefore F is represented by R. □

Corollary 4.4.7 (Degreewise representability of cohomology theories)**.** Let C be a pointed compactly generated ∞-category satisfying the hypotheses of Brown representability, and let

$$\{H^n : hC^{op} \to \mathbf{Set}\}_{n \in \mathbb{Z}}$$

be a cohomology theory on C. Then for every $n \in \mathbb{Z}$, the functor H^n is representable. Hence there exists an object

$$E(n) \in C$$

such that, naturally in $X \in C$,

$$H^n(X) \cong \mathrm{Hom}_{hC}(X, E(n)).$$

Idea. The axioms of a cohomology theory are designed so that each functor H^n satisfies the two conditions in Brown representability. The wedge axiom gives compatibility with coproducts:

$$H^n\left(\coprod_\beta C_\beta\right) \cong \prod_\beta H^n(C_\beta).$$

The exactness axiom for cofiber sequences gives the required surjectivity condition for pushout squares. Therefore Brown representability applies to each H^n, and each H^n is represented by some object $E(n) \in C$. □

Corollary 4.4.8 (Brown representability for reduced cohomology theories)**.** Let

$$\widetilde{h}^n : \mathcal{S}_*^{op} \to \mathbf{Ab}$$

be a reduced generalized cohomology theory on pointed spaces. Then there exists a spectrum $E \in \mathrm{Sp}$ such that, naturally in X,

$$\widetilde{h}^n(X) \cong \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, E[n]).$$

Thus every reduced generalized cohomology theory is represented by a spectrum.

Idea. By the degreewise representability statement, each functor $\widetilde{h}^n$ is represented by a pointed space $E(n)$:

$$\widetilde{h}^n(X) \cong [X, E(n)]_*.$$

The suspension axiom gives natural isomorphisms

$$\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X).$$

Using the representing spaces, this becomes

$$[X, E(n)]_* \cong [\Sigma X, E(n+1)]_*.$$

By the suspension-loop adjunction,

$$[\Sigma X, E(n+1)]_* \cong [X, \Omega E(n+1)]_*.$$

Thus

$$[X, E(n)]_* \cong [X, \Omega E(n+1)]_*$$

naturally in X. By Yoneda-type reasoning, the representing spaces are equivalent:

$$E(n) \simeq \Omega E(n+1).$$

Therefore the sequence

$$E = (E(0), E(1), E(2), \ldots)$$

satisfies the defining condition of a spectrum. This spectrum represents the whole cohomology theory. □

Remark 4.4.9. This corresponds to Lurie in *Higher Algebra*: see Definition 1.4.1.6, Remark 1.4.1.9, Corollary 1.4.1.10, and Remark 1.4.1.11 [1].

Remark 4.4.10. Let us summarize what has happened. Starting with a spectrum E, we constructed functors

$$\widetilde{E}^n(X) = \mathrm{Hom}_{h\mathrm{Sp}}(\Sigma^\infty X, E[n])$$

and checked that they satisfy the axioms of a reduced generalized cohomology theory. This gives one direction:

$$\text{spectrum} \quad \Longrightarrow \quad \text{generalized cohomology theory.}$$

Brown representability gives the converse. The theorem first implies that each degree of a cohomology theory is representable: for every n, there is a pointed space $E(n)$ such that

$$\widetilde{h}^n(X) \cong [X, E(n)]_*.$$

The suspension axiom then relates these representing spaces by equivalences

$$E(n) \simeq \Omega E(n+1).$$

Thus the spaces $E(n)$ are not independent; they assemble into a spectrum

$$E = (E(0), E(1), E(2), \ldots).$$

This is the sense in which spectra classify reduced generalized cohomology theories.

Remark 4.4.11. This completes the stable homotopy-theoretic part of the discussion. We have seen spectra both as stabilized pointed spaces and as representing objects for generalized cohomology theories.

The final goal is to relate this modern viewpoint back to the classical derived category. For this, one has to understand how algebraic objects, such as rings and modules, enter the stable world. The usual formulation uses Eilenberg–Mac Lane spectra and HR-module spectra. Since this requires additional monoidal ∞-categorical structure, we will approach the connection through simplicial modules and the Dold–Kan correspondence.

5 Dold–Kan and Connective Derived Modules

5.1 The Dold–Kan correspondence

The Dold–Kan correspondence is a classical result relating simplicial R-modules to connective chain complexes of R-modules. We recall the statement and the features needed below; for a standard reference see Weibel, Section 8.4 [5].

Definition 5.1.1. Let R be a commutative ring. A *simplicial R-module* is a simplicial object in Mod_R, i.e. a functor

$$M_\bullet : \Delta^{op} \to \mathrm{Mod}_R\,.$$

We denote the category of simplicial R-modules by

$$\mathbf{sMod}_R := \mathrm{Fun}(\Delta^{op}, \mathrm{Mod}_R).$$

Thus a simplicial R-module consists of R-modules $M_0, M_1, M_2, \ldots$ together with R-module face maps

$$d_i : M_n \to M_{n-1}$$

and R-module degeneracy maps

$$s_i : M_n \to M_{n+1}$$

satisfying the simplicial identities of 2.1.4.

Remark 5.1.2. For $R = \mathbb{Z}$, this recovers the category of simplicial abelian groups, since $\mathbf{Ab} = \mathrm{Mod}_{\mathbb{Z}}$. Thus

$$\mathbf{sAb} = \mathbf{sMod}_{\mathbb{Z}}.$$

Definition 5.1.3. A chain complex $C_\bullet$ is called *connective* if it is concentrated in nonnegative degrees, i.e. $C_n = 0 \qquad$ for all $n < 0$. We denote the category of connective chain complexes of R-modules by $\mathrm{Ch}_{\geq 0}(R)$.

Definition 5.1.4. Let $M_\bullet$ be a simplicial R-module. Its *normalized Moore complex*, also called the *normalized chain complex*, is the chain complex $N(M_\bullet)$ defined by

$$N(M)_0 = M_0$$

and, for $n \geq 1$,

$$N(M)_n = \bigcap_{i=1}^{n} \ker(d_i : M_n \to M_{n-1}).$$

The differential is given by the remaining face map

$$\partial = d_0 : N(M)_n \to N(M)_{n-1}.$$

Proposition 5.1.5. The normalized chain complex $N(M_\bullet)$ is a well-defined chain complex.

Proof. Let $x \in N(M)_n$. We first check that $d_0 x \in N(M)_{n-1}$. For $1 \leq j \leq n-1$, the simplicial identity gives

$$d_j d_0 x = d_0 d_{j+1} x.$$

Since $x \in N(M)_n$, we have $d_{j+1} x = 0$. Hence

$$d_j d_0 x = 0.$$

Thus $d_0 x \in N(M)_{n-1}$, so the differential is well-defined. It remains to check that $\partial^2 = 0$. By the simplicial identity

$$d_0 d_0 = d_0 d_1,$$

we get

$$\partial^2 x = d_0 d_0 x = d_0 d_1 x = 0,$$

because $d_1 x = 0$. Hence $N(M_\bullet)$ is a chain complex. □

Remark 5.1.6. The normalized complex removes the degenerate information in a simplicial module. For example, let M be an R-module regarded as a constant simplicial R-module. Then

$$M_0 = M_1 = M_2 = \cdots = M,$$

and all face maps are identities. Therefore, for $n \geq 1$,

$$N(M)_n = 0,$$

while

$$N(M)_0 = M.$$

Thus normalization recovers the chain complex concentrated in degree 0.

Theorem 5.1.7. The normalization functor

$$N : \mathbf{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$$

is an equivalence of ordinary categories.

Remark 5.1.8. We denote an inverse functor by

$$\Gamma : \mathrm{Ch}_{\geq 0}(R) \to \mathbf{sMod}_R.$$

Intuitively, N removes degenerate simplices, while Γ reconstructs a simplicial module from a chain complex by freely adding the degenerate simplices back in.

Proof of 5.1.7. We describe the inverse functor more explicitly. For a connective chain complex $C_\bullet$, define a simplicial R-module $\Gamma(C_\bullet)$ by

$$\Gamma(C_\bullet)_n = \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}\big(N(R[\Delta^n]), C_\bullet\big).$$

Here $R[\Delta^n]$ denotes the free simplicial R-module generated by the standard n-simplex, and $N(R[\Delta^n])$ is its normalized chain complex. The simplicial structure on $\Gamma(C_\bullet)$ is induced by the functoriality of the standard simplices Δ^n.

This construction is right adjoint to normalization. Thus, for every simplicial R-module $A_\bullet$ and every connective chain complex $C_\bullet$, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{sMod}_R}(A_\bullet, \Gamma(C_\bullet)) \cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}(N(A_\bullet), C_\bullet).$$

The proof of the Dold–Kan correspondence is then reduced to two standard facts. First, for every connective chain complex $C_\bullet$, the counit of the adjunction

$$N(\Gamma(C_\bullet)) \to C_\bullet$$

is an isomorphism. This says that if one first freely adds all simplicial degeneracies to a chain complex and then normalizes again, one recovers the original chain complex. Second, every simplicial R-module decomposes into normalized

and degenerate parts. More precisely, its n-simplices are built from the normalized pieces in lower degrees by all possible degeneracy maps. Equivalently, one has a canonical decomposition of the form

$$A_n \cong \bigoplus_{\alpha:[n]\twoheadrightarrow[k]} N(A)_k,$$

where the sum ranges over the surjective maps in Δ. The summand corresponding to the identity map $[n] \twoheadrightarrow [n]$ is the normalized part $N(A)_n$, while the other summands are degenerate.

We now show that Γ is fully faithful. Let $C_\bullet$ and $C'_\bullet$ be connective chain complexes. By the adjunction, a map $\Gamma(C_\bullet) \to \Gamma(C'_\bullet)$ is the same as a chain map $N(\Gamma(C_\bullet)) \to C'_\bullet$. Using the counit isomorphism $N(\Gamma(C_\bullet)) \cong C_\bullet$, this is the same as a chain map $C_\bullet \to C'_\bullet$. Hence the natural map

$$\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}(C_\bullet, C'_\bullet) \to \mathrm{Hom}_{\mathbf{sMod}_R}(\Gamma(C_\bullet), \Gamma(C'_\bullet))$$

is a bijection. Thus Γ is fully faithful. It remains to show essential surjectivity. Let $A_\bullet$ be a simplicial R-module. By the adjunction, the identity map on $N(A_\bullet)$ corresponds to a map

$$u : A_\bullet \to \Gamma(N(A_\bullet)).$$

After applying N, this map becomes the inverse of the counit isomorphism

$$N(\Gamma(N(A_\bullet))) \cong N(A_\bullet).$$

Hence $N(u)$ is an isomorphism. By the normalized–degenerate decomposition, a map of simplicial R-modules is an isomorphism if it is an isomorphism on normalized complexes. Therefore u is an isomorphism, and so

$$A_\bullet \cong \Gamma(N(A_\bullet)).$$

Thus Γ is essentially surjective. Hence the normalization functor

$$N : \mathbf{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$$

is an equivalence of categories. □

Remark 5.1.9. The Dold–Kan correspondence should be read as saying that simplicial R-modules and connective chain complexes are two different ways of encoding the same additive homological information.

Definition 5.1.10. A morphism $f : M_\bullet \to M'_\bullet$ of simplicial R-modules is called a *weak equivalence* if it induces isomorphisms on all homotopy groups:

$$\pi_n(M_\bullet) \xrightarrow{\sim} \pi_n(M'_\bullet)$$

for all $n \geq 0$.

Proposition 5.1.11. Let $M_\bullet$ be a simplicial R-module. Then there are natural isomorphisms

$$\pi_n(M_\bullet) \cong H_n(N(M_\bullet)).$$

Proof. We use the standard description of homotopy groups of a simplicial group by its Moore complex. Since $M_\bullet$ is a simplicial R-module, it is in particular a simplicial abelian group, with base point $0 \in M_0$.

An element of $\pi_n(M_\bullet)$ is represented by an n-simplex whose boundary is trivial. This means an element $x \in M_n$ such that

$$d_0 x = d_1 x = \cdots = d_n x = 0.$$

On the other hand, the normalized complex is given by

$$N(M)_n = \bigcap_{i=1}^{n} \ker(d_i : M_n \to M_{n-1})$$

with differential $\partial = d_0$. Hence the cycles in degree n are

$$\ker(\partial : N(M)_n \to N(M)_{n-1}) = \bigcap_{i=0}^{n} \ker(d_i : M_n \to M_{n-1}).$$

Thus the cycles of $N(M_\bullet)$ are exactly the n-simplices with trivial boundary.

It remains to identify the equivalence relation. In the Moore description of the homotopy groups of a simplicial group, two such n-simplices represent the same element of $\pi_n(M_\bullet)$ precisely when their difference is of the form

$$d_0 y$$

for some

$$y \in \bigcap_{i=1}^{n+1} \ker(d_i : M_{n+1} \to M_n) = N(M)_{n+1}.$$

But these are exactly the boundaries in the normalized complex:

$$\mathrm{im}(\partial : N(M)_{n+1} \to N(M)_n) = d_0 N(M)_{n+1}.$$

Therefore

$$\pi_n(M_\bullet) \cong \frac{\ker(\partial : N(M)_n \to N(M)_{n-1})}{\mathrm{im}(\partial : N(M)_{n+1} \to N(M)_n)} = H_n(N(M_\bullet)).$$

□

Corollary 5.1.12. A morphism $f : M_\bullet \to M'_\bullet$ of simplicial R-modules is a weak equivalence if and only if

$$N(f) : N(M_\bullet) \to N(M'_\bullet)$$

is a quasi-isomorphism.

Remark 5.1.13. The proposition above says that this identification also matches the relevant notions of equivalence:

weak equivalences of simplicial modules $\quad\leftrightarrow\quad$ quasi-isomorphisms of chain complexes.

Therefore the Dold–Kan correspondence descends to the localized categories. In this way, simplicial R-modules model the connective derived category $D_{\geq 0}(R)$.

Corollary 5.1.14. Let W_{simp} be the class of weak equivalences in $\mathbf{sMod}_R$, and let W_{qis} be the class of quasi-isomorphisms in $\text{Ch}_{\geq 0}(R)$. Then the Dold–Kan correspondence induces an equivalence after localization:

$$\mathbf{sMod}_R[W_{\text{simp}}^{-1}] \simeq \text{Ch}_{\geq 0}(R)[W_{\text{qis}}^{-1}].$$

In particular,

$$\mathbf{sMod}_R[W_{\text{simp}}^{-1}] \simeq D_{\geq 0}(R).$$

Localizations as such are understood ∞-categorical localizations. Their homotopy categories recover the corresponding ordinary localizations.

5.2 Sifted colimits and animation

Definition 5.2.1. A category I is called *sifted* if colimits indexed by I commute with finite products in **Set**. Important examples are filtered categories and the simplex category Δ^{op}, whose colimits correspond to geometric realizations of simplicial objects. For the general discussion of sifted colimits, see Lurie's *Higher Topos Theory*, Section 5.5.8 [4].

Remark 5.2.2. Sifted colimits are important for algebraic categories because ordinary algebraic structures are described by operations involving finite products. For example, the addition and multiplication maps of a ring have the form

$$+ : A \times A \to A$$

Thus sifted colimits are natural in this context because they are compatible with finite-product operations.

Definition 5.2.3. Let $\mathcal{C}$ be a category admitting the relevant colimits. If $X_\bullet : \Delta^{op} \to \mathcal{C}$ is a simplicial object, its *realization* is the colimit

$$|X_\bullet| := \text{colim}_{\Delta^{op}} X_\bullet.$$

Remark 5.2.4. The important point for us is that realization is itself a colimit. Thus, whenever an algebraic object is presented by a simplicial resolution, the process of passing from the resolution back to the object is a colimit construction. Since Δ^{op} is sifted, realizations of simplicial objects are examples of sifted colimits.

Definition 5.2.5. Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}$. An *augmented simplicial object over* A is a simplicial object $P_\bullet : \Delta^{op} \to \mathcal{C}$ together with a morphism $P_\bullet \to A$, where A is viewed as a constant simplicial object.

Remark 5.2.6. One should think of an augmented simplicial object $P_\bullet \to A$ as a simplicial diagram which is meant to present the object A. The levels

$$P_0, P_1, P_2, \ldots$$

encode successive layers of data: first approximations, relations, relations between relations, and so on.

Definition 5.2.7. An augmented simplicial object $P_\bullet \to A$ is called a *simplicial resolution* of A if the objects P_n are chosen from a class of free or projective objects, and if the induced morphism

$$|P_\bullet| \to A$$

is an equivalence.

Remark 5.2.8. Thus a simplicial resolution is analogous to a projective resolution. A projective resolution of an R-module M,

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

replaces M by a chain complex built from projective modules. A simplicial resolution replaces an object A by a simplicial diagram $P_\bullet$ built from simpler objects. The realization

$$|P_\bullet| = \mathrm{colim}_{\Delta^{op}} P_\bullet$$

is the operation which turns this simplicial diagram into a single object. Therefore the condition

$$|P_\bullet| \simeq A$$

says that the simplicial diagram really presents A.

Remark 5.2.9. This is the reason why realizations are relevant for animation. Animation allows such simplicial resolutions to be treated as genuine derived objects. Since realization is a colimit indexed by Δ^{op}, and since Δ^{op} is sifted, simplicial resolutions naturally lead to sifted colimits.

Definition 5.2.10 (Informal)**.** Let $\mathcal{C}$ be a cocomplete ordinary category generated by compact projective objects, and let

$$\mathcal{C}^{\mathrm{cp}} \subseteq \mathcal{C}$$

denote the full subcategory of compact projective objects. Informally, the animation of $\mathcal{C}$ is the ∞-category obtained from $\mathcal{C}^{\mathrm{cp}}$ by freely adjoining sifted colimits:

$$\mathrm{Ani}(\mathcal{C}) \simeq \mathrm{Ind}_{\mathrm{sift}}(\mathcal{C}^{\mathrm{cp}}).$$

Thus animated objects are built from compact projective objects by sifted colimit constructions.

Remark 5.2.11. This definition is meant as a guiding description rather than a construction from first principles. The precise definition of animation is an ∞-categorical sifted-colimit completion. The important point for us is that one does not simply add ordinary colimits to $\mathcal{C}$. Rather, one passes to a derived version in which simplicial resolutions by compact projective objects are allowed as genuine objects. In practice, this means that animated objects may be represented by simplicial resolutions whose terms are compact projective, for example degreewise free objects in the cases considered below. This is analogous to replacing an object by a projective resolution in classical homological algebra, except that the resolution is now simplicial rather than just a chain complex.

Proposition 5.2.12 (The additive case)**.** For R-modules, animation gives the connective derived category:

$$\mathrm{Ani}(\mathrm{Mod}_R) \simeq D_{\geq 0}(R).$$

Sketch of proof. The first input is the general theory of animation. For a cocomplete ordinary category generated by compact projective objects, animation is obtained by freely adjoining sifted colimits to the compact projective objects. In the case of R-modules, this theory identifies animated R-modules with simplicial R-modules localized at weak equivalences:

$$\mathrm{Ani}(\mathrm{Mod}_R) \simeq \mathbf{sMod}_R[W_{\mathrm{simp}}^{-1}].$$

We use this as a standard result from the theory of animation; see [3, Section 5.1.4 and Example 5.1.6], and for the module case over an ordinary ring see also [3, Section 5.1.7]. By the Dold–Kan correspondence, the normalization functor gives an equivalence

$$\mathbf{sMod}_R \simeq \mathrm{Ch}_{\geq 0}(R).$$

Moreover, under this equivalence, weak equivalences of simplicial R-modules correspond to quasi-isomorphisms of connective chain complexes. Therefore localization gives

$$\mathbf{sMod}_R[W_{\mathrm{simp}}^{-1}] \simeq \mathrm{Ch}_{\geq 0}(R)[W_{\mathrm{qis}}^{-1}] = D_{\geq 0}(R).$$

Combining the two equivalences yields

$$\mathrm{Ani}(\mathrm{Mod}_R) \simeq D_{\geq 0}(R).$$

□

5.3 Outlook: stabilization and animated rings

Remark 5.3.1. To obtain the full derived category $D(R)$, one has to allow shifts in both directions. Equivalently, one passes from the connective setting

to the stable setting. In stable ∞-categorical language, the full derived ∞-category of R-modules is identified with the ∞-category of modules over the Eilenberg–Mac Lane spectrum HR:

$$D(R) \simeq \mathrm{Mod}_{HR}.$$

Here HR is the ring spectrum associated to R, characterized by

$$\pi_0(HR) = R, \qquad \pi_i(HR) = 0 \quad \text{for } i \neq 0.$$

Thus the usual derived category of R-modules can be viewed as a stable ∞-category of module spectra.

Remark 5.3.2. The notation Mod_{HR} hides some additional structure. The starting point is that spectra themselves form a symmetric monoidal stable ∞-category

$$(\mathrm{Sp}, \wedge, \mathbb{S}),$$

where $\wedge$ is the smash product and the sphere spectrum $\mathbb{S}$ is the monoidal unit. Thus $\mathbb{S}$ plays the role of the basic ground object in stable homotopy theory. The construction of the smash product is technically nontrivial and is not levelwise on spectra. We will only use the conceptual fact that $\mathbb{S}$ is the monoidal unit.

The Eilenberg–Mac Lane spectrum HR is then a ring object in Sp. Hence one can form the ∞-category

$$\mathrm{Mod}_{HR}$$

of spectra equipped with an action of HR. In this new category of HR-modules, the object HR itself plays the role of the free rank-one module, and, when R is commutative, it is the monoidal unit for the relative tensor product over HR.

Thus there are two levels to keep apart: $\mathbb{S}$ is the unit in the ambient symmetric monoidal ∞-category of spectra, while HR is the unit in the corresponding category of HR-modules. Stating and proving the equivalence

$$D(R) \simeq \mathrm{Mod}_{HR}$$

precisely requires this symmetric monoidal theory of spectra. In a more detailed treatment, one would therefore run into notions such as E_∞-algebras, ring spectra, and module objects in symmetric monoidal ∞-categories. We will not develop this monoidal machinery in detail here. The point is only that Mod_{HR} gives the stable ∞-categorical enhancement of the classical derived category of R-modules.

Remark 5.3.3. This gives the stable linear endpoint of the story. Starting from simplicial R-modules, Dold–Kan gives connective chain complexes; after localization one obtains $D_{\geq 0}(R)$; and after allowing all shifts one reaches the stable ∞-category

$$D_{\geq 0}(R) \subset D(R) \simeq \mathrm{Mod}_{HR}.$$

This connects the Dold–Kan correspondence back to the stable ∞-categorical framework discussed earlier in the thesis.

Remark 5.3.4. There is also a different direction one can take. The discussion above stays in the linear world of modules over a fixed ring. In derived algebraic geometry, however, one also wants to replace the commutative rings themselves by derived or homotopical objects. This leads to the theory of animated rings.

Definition 5.3.5. The ∞-category of *animated rings* is modeled by simplicial commutative rings localized at weak equivalences:

$$\mathrm{Ani}(\mathrm{Ring}) \simeq \mathbf{sRing}[W^{-1}].$$

More generally, for a commutative ring R, animated commutative R-algebras are modeled by simplicial commutative R-algebras

$$A_\bullet : \Delta^{op} \to \mathrm{CAlg}_R$$

localized at weak equivalences.

Remark 5.3.6. An ordinary commutative ring A gives an animated ring by being viewed as a constant simplicial ring. In that case,

$$\pi_0(A) = A, \qquad \pi_i(A) = 0 \quad \text{for } i > 0.$$

A genuinely animated ring may have nonzero higher homotopy groups. These higher homotopy groups record derived information which is invisible at the level of ordinary commutative rings.

Remark 5.3.7. The module case and the ring case should be compared with some care. For modules, the Dold–Kan correspondence identifies simplicial R-modules with connective chain complexes. For rings, one also has to remember multiplication. A simplicial commutative ring has multiplication in every simplicial degree, and this multiplication is compatible with all face and degeneracy maps. After passing to the derived or ∞-categorical setting, this multiplicative structure has to be kept coherently. This is one reason why animated rings naturally lead beyond ordinary homological algebra and into the language of ∞-categories and E_∞-algebras.

Remark 5.3.8. We do not develop the theory of animated rings here. The purpose of this final outlook is only to indicate that the simplicial and homotopical ideas discussed in this chapter are not limited to modules. For modules, Dold–Kan gives a linear model in terms of connective chain complexes. For commutative rings, the analogous derived objects are animated rings, where the multiplicative structure has to be treated homotopically. For this direction, see Lurie's *Spectral Algebraic Geometry*, Chapter VII [2], and the beginning of Section 5 of Cesnavicius and Scholze's *Purity for flat cohomology* [3].

A Appendix

A.1 Proof of Proposition 1.5.11

Proposition A.1.1. Let $f : X^\bullet \to Y^\bullet$ be a morphism of cochain complexes. Then f is a homotopy equivalence if and only if $\mathrm{Cone}(f)$ is contractible.

Proof. We use the convention

$$\mathrm{Cone}(f)^n = Y^n \oplus X^{n+1}$$

with differential

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^{n+1}_X \end{pmatrix}.$$

Assume first that f is a homotopy equivalence. We choose a homotopy inverse

$$g : Y^\bullet \to X^\bullet$$

together with homotopies

$$s : gf \simeq \mathrm{id}_{X^\bullet}, \qquad t : fg \simeq \mathrm{id}_{Y^\bullet},$$

so that

$$gf - \mathrm{id}_X = d_X s + s d_X, \qquad fg - \mathrm{id}_Y = d_Y t + t d_Y.$$

We choose the homotopy inverse data compatibly, so that

$$fs = tf.$$

With this choice, define maps

$$H^n : \mathrm{Cone}(f)^n \to \mathrm{Cone}(f)^{n-1}$$

by

$$H^n = \begin{pmatrix} -t^n & 0 \\ g^n & s^{n+1} \end{pmatrix}.$$

We claim that H is a contracting homotopy for $\mathrm{Cone}(f)$.

Indeed,

$$d^{n-1}_{\mathrm{Cone}(f)} H^n = \begin{pmatrix} d^{n-1}_Y & f^n \\ 0 & -d^n_X \end{pmatrix} \begin{pmatrix} -t^n & 0 \\ g^n & s^{n+1} \end{pmatrix} = \begin{pmatrix} -d_Y t + fg & fs \\ -d_X g & -d_X s \end{pmatrix},$$

and

$$H^{n+1} d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} -t^{n+1} & 0 \\ g^{n+1} & s^{n+2} \end{pmatrix} \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^{n+1}_X \end{pmatrix} = \begin{pmatrix} -t d_Y & -tf \\ g d_Y & gf - s d_X \end{pmatrix}.$$

Adding these two matrices gives

$$d^{n-1}_{\mathrm{Cone}(f)} H^n + H^{n+1} d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} fg - d_Y t - t d_Y & fs - tf \\ g d_Y - d_X g & gf - d_X s - s d_X \end{pmatrix}.$$

Since g is a morphism of complexes, $gd_Y = d_X g$. By the homotopy relations,

$$fg - d_Y t - td_Y = \mathrm{id}_Y, \qquad gf - d_X s - sd_X = \mathrm{id}_X .$$

Finally, by the compatibility condition $fs = tf$, the upper right entry is zero. Hence

$$d^{n-1}_{\mathrm{Cone}(f)} H^n + H^{n+1} d^n_{\mathrm{Cone}(f)} = \mathrm{id}_{\mathrm{Cone}(f)^n} .$$

Therefore $\mathrm{Cone}(f)$ is contractible.

Conversely, assume that $\mathrm{Cone}(f)$ is contractible. Let

$$H^n : \mathrm{Cone}(f)^n \to \mathrm{Cone}(f)^{n-1}$$

be a contracting homotopy. Write H^n in block form as

$$H^n = \begin{pmatrix} a^n & b^n \\ c^n & e^n \end{pmatrix},$$

where

$$a^n : Y^n \to Y^{n-1}, \qquad b^n : X^{n+1} \to Y^{n-1},$$
$$c^n : Y^n \to X^n, \qquad e^n : X^{n+1} \to X^n.$$

The equation

$$d_{\mathrm{Cone}(f)} H + H d_{\mathrm{Cone}(f)} = \mathrm{id}_{\mathrm{Cone}(f)}$$

then gives, by comparing the lower left entries,

$$-d_X c + cd_Y = 0.$$

Hence $c : Y^\bullet \to X^\bullet$ is a morphism of complexes. Set

$$g := c.$$

Comparing the upper left entries gives

$$d_Y a + fg + ad_Y = \mathrm{id}_Y .$$

Equivalently,

$$fg - \mathrm{id}_Y = -d_Y a - ad_Y.$$

Thus $fg \simeq \mathrm{id}_Y$, with homotopy $-a$.

Comparing the lower right entries gives

$$-d_X e + gf - ed_X = \mathrm{id}_X .$$

Equivalently,

$$gf - \mathrm{id}_X = d_X e + ed_X.$$

Thus $gf \simeq \mathrm{id}_X$, with homotopy e.

Therefore g is a homotopy inverse of f, and hence f is a homotopy equivalence. □

A.2 Technical Notions on Simplicial Sets

In this appendix subsection we collect a few technical notions used in the proof of Theorem 2.7.2. The aim is not to develop the full theory of inner anodyne maps, but only to record the terminology needed in the main text. The definitions of trivial Kan fibrations, weakly saturated classes, and inner anodyne morphisms follow Lurie, *Higher Topos Theory*, Section 2.1.

Definition A.2.1. Let $p : X \to Y$ be a morphism of simplicial sets. We say that p is a *trivial Kan fibration* if it has the right lifting property with respect to all boundary inclusions

$$\partial\Delta^n \hookrightarrow \Delta^n, \qquad n \geq 0.$$

Thus every lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array}$$

admits a solution.

Remark A.2.2. Equivalently, a trivial Kan fibration is a Kan fibration which is also a weak homotopy equivalence. In the proof of Joyal's 2.7.2, we only use the lifting characterization above.

Definition A.2.3. For simplicial sets K and L, the *internal Hom* or *function complex* is the simplicial set

$$\mathrm{Fun}(K, L)$$

defined by

$$\mathrm{Fun}(K, L)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times K, L).$$

It is characterized by the exponential adjunction

$$\mathrm{Hom}_{\mathbf{sSet}}(A \times K, L) \cong \mathrm{Hom}_{\mathbf{sSet}}(A, \mathrm{Fun}(K, L))$$

naturally in A, K, L.

Definition A.2.4. A class $\mathcal{T}$ of morphisms of simplicial sets is called *weakly saturated* if it is closed under the following operations:

1. *pushouts*;
2. *transfinite compositions*;
3. *retracts*.

Definition A.2.5. A morphism of simplicial sets is called *inner anodyne* if it belongs to the weakly saturated class generated by the inner horn inclusions

$$\Lambda^n_i \hookrightarrow \Delta^n, \qquad 0 < i < n.$$

Lemma A.2.6. Let S be a simplicial set. Then S is an ∞-category if and only if the projection

$$S \to \Delta^0$$

has the right lifting property with respect to all inner anodyne maps.

Idea. By definition, S is an ∞-category if and only if every inner horn

$$\Lambda^n_i \to S, \qquad 0 < i < n,$$

admits a filler. This says precisely that $S \to \Delta^0$ has the right lifting property with respect to all inner horn inclusions. Since inner anodyne maps are generated from inner horn inclusions by weak saturation, and right lifting properties are stable under weak saturation, this is equivalent to having the right lifting property with respect to all inner anodyne maps. □

Lemma A.2.7. For each $m \geq 0$, consider the inclusion

$$i_m : (\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \hookrightarrow \Delta^m \times \Delta^2.$$

The weakly saturated class generated by the maps i_m, for all $m \geq 0$, is the class of inner anodyne maps.

Idea. This is the combinatorial input in the proof of Joyal's theorem. The maps i_m appear after translating lifting problems for

$$\mathrm{Fun}(\Delta^2, S) \to \mathrm{Fun}(\Lambda^2_1, S)$$

through the exponential adjunction. The lemma says that these lifting problems generate the same lifting theory as the inner horn inclusions. □

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