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MASTER THESIS

The Banach-Tarski Paradox in Euclidean and Hyperbolic Spaces

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Abstract

This thesis studies the Hausdorff and Banach-Tarski paradoxes, emphasizing the group-theoretic structures underlying paradoxical decompositions. After introducing the necessary background on group actions, free groups, and the Axiom of Choice, we establish the Hausdorff paradox and derive the Banach-Tarski paradox in $\mathbb{R}^3$, including its strong form, and discuss how these constructions extend to higher-dimensional Euclidean spaces. We also consider analogous constructions in hyperbolic space, where the resulting paradoxical decompositions admit a particularly transparent geometric illustration in contrast to the Euclidean case.

To understand the limitations of these phenomena, we study amenable groups and amenable group actions, showing that paradoxical decompositions cannot occur in dimensions one and two.

A central role throughout the thesis is played by the existence of free subgroups of rank 2. In the final chapter, we place this observation into a broader algebraic framework using Tits' Alternative, which describes when linear groups contain such free subgroups and relates this phenomenon to amenability and paradoxical decompositions.

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Chapter 1

Introduction

Paradoxes in mathematics often arise in situations where our intuitive expectations appear to fail. This is especially striking in geometry, where it seems self-evident that cutting an object into finitely many pieces cannot change its area or volume. This principle is captured by the geometric notion of congruence by dissection. Two polygons in the plane are called congruent by dissection if one can be partitioned into finitely many polygonal pieces which can be rearranged by *rigid motions* such as translations, rotations or reflections to form the other polygon. At first sight, such rearrangements seem entirely compatible with our geometric intuition. However, appearances can be misleading.

A well-known example is Mitsunobu Matsuyama's paradox [AN20], shown in Figure 1.1. Here, several congruent quadrilaterals together with a small square form a larger square. After rotating the quadrilaterals, the central square disappears and the pieces seem to fill the entire area, creating the impression that additional area has emerged merely by rearrangement. Upon closer inspection, the paradox dissolves: the outer square in the second arrangement is actually slightly smaller than in the first, so that no area has actually been created. In fact, we will later show that two polygons have the same area if and only if they are congruent by dissection (Thm 4.1.5). Thus, this construction only produces the illusion of obtaining something from nothing. It nevertheless leads to a natural question: can there exist mathematically correct decompositions in which such a phenomenon truly occurs?

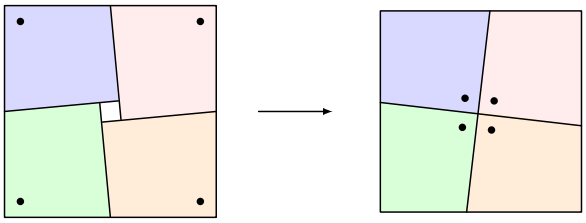

Figure 1.1: Mitsunobu Matsuyama's paradox: after rotating the pieces, the central square seems to disappear. Adapted from [AN20].

Such phenomena already occur naturally in the context of infinite sets. In set theory, infinite sets can be decomposed in ways that contradict our intuition about size. For instance, the set of natural numbers $\mathbb{N}$ can be partitioned into the even and odd numbers:

$$\mathbb{N} = 2\mathbb{N} \cup (2\mathbb{N} + 1).$$

Both subsets are proper parts of $\mathbb{N}$, yet each of them can be put into a one-to-one correspondence with the whole set. Indeed, the maps

$$k \mapsto 2k \quad \text{and} \quad k \mapsto 2k + 1$$

define bijections from $\mathbb{N}$ onto the even and odd numbers, respectively. Thus, each of these subsets is "as large" as the original set. This can even be interpreted as a form of duplication: by decomposing and rearranging a set, one can obtain two copies of the original. This illustrates the basic idea underlying a paradoxical decomposition. The Banach-Tarski paradox shows that, in three dimensions, such phenomenon can occur geometrically.

In its popular form, it is often described by statements such as "a ball can be partitioned into finitely many pieces and rearranged into two identical copies of itself" or "a pea can be cut up and reassembled into the Sun" (see Figure 1.2). The latter statement is stronger, since it asserts not only that a set can be duplicated, but that any two bounded sets with nonempty interior can be transformed into one another by such decompositions. We will refer to the latter formulation as the *strong form of the Banach-Tarski paradox*:

> **Strong Banach-Tarski Paradox [TW16, Theorem 3.12].** Any two bounded subsets of $\mathbb{R}^3$ with nonempty interior can be partitioned into finitely many pieces which can be rearranged by rigid motions to form each other.

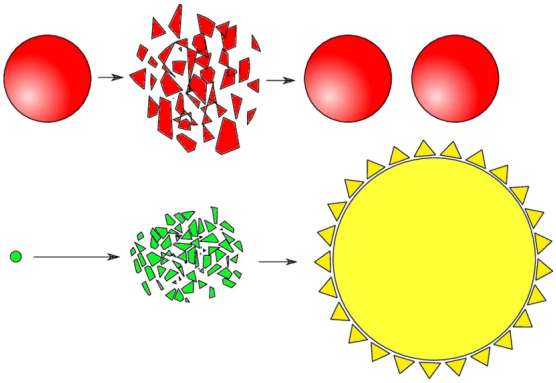

Figure 1.2: Illustrations of the Banach-Tarski paradox and its strong form [Wah22, Figure 1.1].

Both forms of the paradox are proved in chapter 4. There, we will also discuss how these phenomena extend to higher-dimensional Euclidean spaces.

The theorem above was proved by Stefan Banach and Alfred Tarski in 1924. Its origins, however, go back to Felix Hausdorff, who in 1914 discovered a paradoxical decomposition of the 3-dimensional sphere [Wah22]. A decade later Banach and Tarski extended these ideas and obtained the full paradox for solid balls in $\mathbb{R}^3$. Hausdorff's original construction already contains the essential ideas that later appear in the Banach-Tarski paradox. The intermediate statement arising from his work is now known as the *Hausdorff paradox*. In Chapter 3, we study Hausdorff's construction in detail before showing how it leads to the Banach-Tarski paradox.

Both paradoxes in the Euclidean space are a consequence of the Axiom of Choice, one of the fundamental principles of modern set theory. While the Axiom of Choice has many important applications throughout mathematics, it also leads to highly counterintuitive results, among which the Banach-Tarski paradox is perhaps the most striking. As we shall see, however, the existence of paradoxical decompositions is also closely related to the algebraic structure of the transformations involved. In particular, some transformation groups allow a high degree of flexibility in rearranging sets, while others impose strong restrictions that prevent such constructions.Understanding this distinction will be a central theme of this thesis. In particular, Chapter 7 discusses how *Tits' Alternative* relates the existence of paradoxical decompositions to the structure of the underlying transformation groups.

In the Euclidean setting, this difference becomes apparent when comparing low and high dimensions. In dimensions one and two, no analogue of the paradoxes exists: it is not possible to

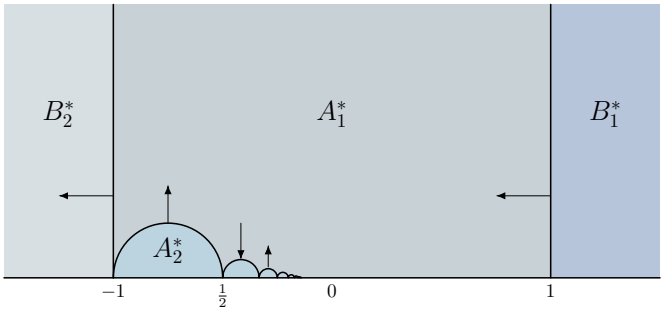

Figure 1.3: Paradoxical decomposition of the whole hyperbolic plane.

decompose a set into finitely many pieces and rearrange them to obtain two copies of the original. This phenomenon will be explained in Chapter 6, where we introduce the notion of amenability and show how it prevents such decompositions.

Another viewpoint is provided by considering analogous constructions in hyperbolic space, where the role of the geometry becomes more apparent. In this setting, certain paradoxical decompositions can even be carried out without invoking the Axiom of Choice. Simply stated, the following holds.

> **Banach-Tarski analogue in hyperbolic space [TW16, Theorem 4.5].** There exists a decomposition of the hyperbolic plane $\mathbb{H}^+$ into finitely many pieces which can be rearranged by suitable transformations to form two copies of the original set.

In contrast to the Euclidean case, these decompositions arise from a geometric tessellation of the hyperbolic plane. As a result, the pieces involved can be described and even visualized explicitly, providing a more intuitive picture of the construction. Figure 1.3 shows the tessellation underlying the construction. By cutting the hyperbolic plane along the lines, we obtain the four regions A_1^*, A_2^*, B_1^* and B_2^*, where the boundaries are included as indicated by the arrows. If we shift B_2^* two units to the right by the transformation $\alpha(z) = z + 2$, then B_1^* together with the shifted B_2^* covers the entire space. Similarly, the transformation $\beta(z) = \frac{1}{2z+1}$ moves A_2^* into position so that, together with A_1^*, it covers the whole space as well.

This result, together with a hyperbolic analogue of the Hausdorff paradox, will be discussed in Chapter 5.

These ideas provide the foundation for the developments in the chapters that follow. Although the Banach-Tarski paradox is closely connected with measure-theoretic questions concerning area, volume, and non-measurable sets, this thesis focuses primarily on the algebraic and geometric structures underlying paradoxical decompositions, particularly the role of group actions and free subgroups. Measure-theoretic ideas are introduced where they are needed.

Chapter 2

Preliminaries

This chapter collects the algebraic and set-theoretic background required for the study of paradoxical decompositions. We begin with a brief review of group actions and free groups before discussing the Axiom of Choice and its role in the constructions that follow. Most of the material is standard and is included primarily to establish notation and definitions. The main reference for the more specialized material is [TW16, Chapter 1]. Any additional sources are mentioned where needed.

2.1 Group actions

Group actions provide the bridge between algebra and geometry and serve as the natural framework for the paradoxical decompositions that will be studied throughout this thesis.

Definition 2.1.1. Let G be a group and X be any set. An *(left) action* of G on X is a map

$$\begin{aligned} \phi : G \times X &\longrightarrow X \\ (g, x) &\mapsto \phi(g, x) \end{aligned}$$

such that the following conditions hold for all elements $x \in X$:

- $\phi(e, x) = x$ where e denotes the identity element of G.
- $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$.

The group action $\phi(g, x)$ will be denoted as gx.

Definition 2.1.2. A group action of G on X is called *free* if for all $x \in X$,

$$gx = x \;\Rightarrow\; g = e.$$

Equivalently, the action is said to have no nontrivial fixed points.

Example 2.1.3 (Action of a group G on itself)**.** Every group G acts on itself by left multiplication. Define

$$\phi : G \times G \to G, \qquad \phi(g, h) = gh.$$

Then for all $g, h, k \in G$,

$$\phi(e, h) = eh = h, \qquad \phi(g, \phi(h, k)) = g(hk) = (gh)k = \phi(gh, k),$$

so this defines a group action. Moreover, this action is free, since $gh = h$ implies $g = e$.

Example 2.1.4 (Action of $SO_3(\mathbb{R})$ on $\mathbb{R}^3$)**.** The special orthogonal group

$$SO_3(\mathbb{R}) = \{A \in Mat_3(\mathbb{R}) : A^T A = I,\ \det(A) = 1\}$$

acts naturally on $\mathbb{R}^3$ by rotations via matrix multiplication:

$$(A, x) \mapsto Ax.$$

Indeed, for all $A, B \in SO_3(\mathbb{R})$ and $x \in \mathbb{R}^3$, we have

$$Ix = x, \qquad A(Bx) = (AB)x,$$

so this defines a group action. In particular, the elements of $SO_3(\mathbb{R})$ act as linear isometries of Euclidean space. This action is not free, since every nontrivial rotation fixes the axis of rotation (and in particular the origin). We will discuss the group $SO_3(\mathbb{R})$ in more detail in Chapter 3.

These examples illustrate two important perspectives: on the one hand, groups can act on themselves via multiplication, and on the other hand, they can act as transformations on geometric spaces. Both types of actions play a central role in the construction of the paradoxes considered later.

2.2 Free Groups

Free groups play a central role in the theory of paradoxical decompositions. Their importance lies in the fact that the generators satisfy no relations other than those required by the group axioms, which gives free groups a particularly flexible algebraic structure. In particular, the existence of free subgroups in transformation groups such as $SO_3(\mathbb{R})$ will be a key ingredient in the Hausdorff and Banach-Tarski paradox.

Definition 2.2.1. Let A be a set, called the *alphabet*, whose elements are called *letters*. A *word* over A is a finite sequence

$$w = a_1 a_2 \cdots a_n, \qquad a_i \in A,$$

where $n \geq 0$. The case $n = 0$ is called the *empty word* and is denoted by e.

Definition 2.2.2. Let A be a non-empty set. The *free semigroup on* A, denoted by $\tilde{F}(A)$, is the set of all non-empty words over A equipped with concatenation as operation.

Definition 2.2.3. Let A be an alphabet. Define

$$A^{-1} = \{a^{-1} : a \in A\},$$

where a^{-1} is a formal symbol called the *formal inverse* of a. A word

$$w = a_1 \cdots a_n \quad \text{with } a_i \in A \cup A^{-1}$$

is called *reduced* if $a_i \neq a_{i+1}^{-1}$ for all $1 \leq i < n$.

Definition 2.2.4. Let A be a non-empty set. The *free group generated by* A, denoted by $F(A)$, is the set of all reduced words over $A \cup A^{-1}$, with group operation given by concatenation followed by reduction. The identity element is the empty word e and the cardinality of A is called the *rank* of $F(A)$.

Example 2.2.5 (A free subgroup of $SO_2(\mathbb{R})$)**.** Let ρ be the rotation about the origin through the angle $\sqrt{2}\pi$, i.e.

$$\rho = \begin{pmatrix} \cos(\sqrt{2}\pi) & -\sin(\sqrt{2}\pi) \\ \sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi) \end{pmatrix}.$$

Consider the subgroup $F \subseteq SO_2(\mathbb{R})$ generated by ρ, that is,

$$F = \{\rho^n : n \in \mathbb{Z}\},$$

which is precisely the set of rotations about the origin through the angles $n\sqrt{2}\pi$, $n \in \mathbb{Z}$. Since $\sqrt{2}$ is irrational, no nonzero multiple of $\sqrt{2}\pi$ is a multiple of 2π. Hence the elements ρ^n are all distinct, and F is a free group of rank 1, that is, an infinite cyclic group.

Note that free groups of rank 1 are abelian, whereas free groups of rank at least 2 are all non-abelian.

2.3 The Axiom of Choice

The Axiom of Choice is a fundamental principle of set theory and will play an important role in the developments that follow. We therefore briefly introduce the axiom and discuss some of its basic consequences. The presentation is based primarily on [Cun12, Chapter 5.4.2] and [Kha18, Chapter 2].

Informally, the Axiom of Choice states that it is possible to select one element from each set in a collection of non-empty sets, even if there is no explicit rule for making these choices. A simple example helps to illustrate this idea. Suppose we are given infinitely many non-empty boxes, each containing at least one object. If there is a clear rule (for example, always choosing the smallest object), then selecting one object from each box is straightforward. However, if no such rule exists and the boxes contain unrelated objects, it is not clear how to make all these choices at once. The Axiom of Choice asserts that such a selection is still possible.

Despite its intuitive formulation, the Axiom of Choice is not without controversy. Its use leads to highly counterintuitive results, such as the Banach-Tarski paradox, and historically its acceptance was a subject of considerable debate.

To develop a better understanding of the axiom and its consequences, we will consider several examples throughout this section. We begin by introducing the notion of a choice function, which formalizes the idea of selecting elements from a collection of sets.

Definition 2.3.1. Let $\mathcal{A}$ be a collection of non-empty sets. A function $f : \mathcal{A} \to \bigcup_{A\in\mathcal{A}} A$ is called a *choice function* for $\mathcal{A}$ if

$$f(A) \in A \quad \text{for all } A \in \mathcal{A}.$$

Example 2.3.2. Let $\mathcal{A}$ be a collection of sports teams. Then $\bigcup_{C\in\mathcal{A}} C$ is the set of all the players. Suppose that each team has a designated captain. Assigning to each team its captain defines a choice function.

Example 2.3.3 (A choice function on $\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$)**.** Let

$$\mathcal{A} = \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\},$$

the family of all non-empty subsets of the natural numbers. Define a function

$$f : \mathcal{A} \longrightarrow \mathbb{N}, \qquad f(A) = \min(A).$$

Since every non-empty subset of $\mathbb{N}$ has a least element, the value $\min(A)$ is well defined, and clearly

$$f(A) \in A \qquad \text{for all } A \in \mathcal{A}.$$

Thus, f is a choice function for the family $\mathcal{A}$.

These examples work because a natural rule is available. In general, however, such a rule does not always exist. This leads us to the Axiom of Choice, which asserts that such selections are always possible.

> **Axiom of Choice (AC).** Let $\mathcal{A}$ be a non-empty set of non-empty sets. Then there exists a choice function for $\mathcal{A}$.

Example 2.3.4 (An application of the Axiom of Choice)**.** Let

$$\mathcal{A} = \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\},$$

the family of all non-empty subsets of the real numbers.

We would like to define a function

$$f : \mathcal{A} \longrightarrow \mathbb{R}$$

such that $f(A) \in A$ for every $A \in \mathcal{A}$.

Unlike the case of $\mathbb{N}$, there is no general rule that selects an element from every non-empty subset of $\mathbb{R}$. For example, many sets do not have a smallest element, such as the interval $(0,1)$. The Axiom of Choice asserts that a choice function

$$f : \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\} \to \mathbb{R}$$

exists, even though we cannot describe it concretely.

The Axiom of Choice becomes particularly important in the context of group actions.

Example 2.3.5 (Choosing orbit representatives)**.** [TW16, Proposition 1.10] Let G act on a set X. The set X can be partitioned into disjoint orbits of the form

$$Gx = \{gx : g \in G\}.$$

Our goal is to pick exactly one representative from each orbit.

If there are infinitely many orbits and no natural rule or additional structure for selecting representatives, such a choice cannot, in general, be made explicitly. The Axiom of Choice guarantees that such a selection nevertheless exists. This type of selection will be essential in the construction of the paradoxes. In particular, it will allow us to transfer paradoxical properties from groups to the sets on which they act.

2.4 Paradoxical decompositions

So far, we have used the term *paradoxical decomposition* informally to describe the process of cutting a set into finitely many pieces and rearranging them to obtain two copies of the original set. We now give a precise definition in terms of group actions.

Definition 2.4.1. Let G be a group acting on a set X. A non-empty subset $E \subseteq X$ is called *G-paradoxical* if there exist pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ and elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$E = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j).$$

We call the group G *paradoxical* if its action on itself by left multiplication is paradoxical.

The following theorem shows that free groups of rank 2 are paradoxical. These groups provide the fundamental examples of paradoxical groups and, as we shall see later, form the basis for the geometric paradoxes studied throughout this thesis.

Theorem 2.4.2. A free group F of rank 2 is paradoxical.

Proof. Let $A = \{\sigma, \tau\}$ be a generating set of F. Every element $w \in F \setminus \{e\}$ can be written as a reduced word beginning with one of the letters $\sigma, \tau, \sigma^{-1}, \tau^{-1}$. For each $x \in \{\sigma, \tau, \sigma^{-1}, \tau^{-1}\}$, let $W(x)$ denote the set of all words that begin with x.

Then the sets $W(\sigma), W(\sigma^{-1}), W(\tau), W(\tau^{-1})$ are pairwise disjoint, and we have the decomposition

$$F = \{e\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1}).$$

We claim that

$$F = W(\sigma) \cup \sigma W(\sigma^{-1}).$$

Indeed, if $w \notin W(\sigma)$, then $\sigma^{-1}w \in W(\sigma^{-1})$, so $w = \sigma(\sigma^{-1}w) \in \sigma W(\sigma^{-1})$. This shows that $F \subseteq W(\sigma) \cup \sigma W(\sigma^{-1})$, and the reverse inclusion is immediate.

Similarly, one shows that

$$F = W(\tau) \cup \tau W(\tau^{-1}).$$

Since the sets $W(\sigma), W(\sigma^{-1}), W(\tau), W(\tau^{-1})$ are pairwise disjoint, this yields a paradoxical decomposition of F. More precisely, this corresponds to a decomposition with pieces $A_1 = W(\sigma)$, $A_2 = W(\sigma^{-1})$, $B_1 = W(\tau)$, $B_2 = W(\tau^{-1})$ and group elements $g_1 = e, g_2 = \sigma, h_1 = e$ and $h_2 = \tau$. Hence F is paradoxical. □

See Figure 2.1 for a visualization of this construction.

The preceding proof can be modified so that the four sets in the paradoxical decomposition already cover all of F, rather than only $F \setminus \{e\}$. We will return to this refinement later in Example 4.2.6.

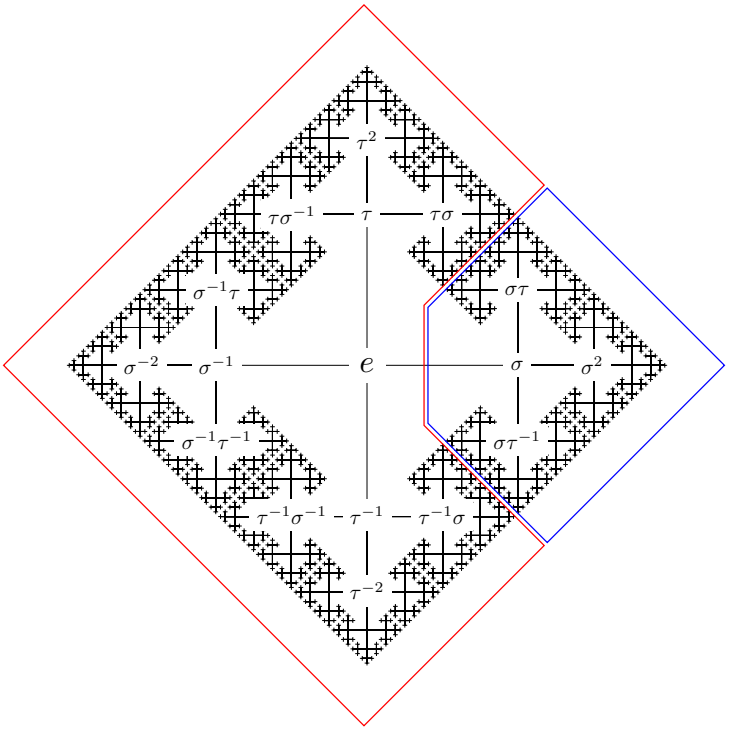

Figure 2.1: The blue region represents $W(\sigma)$, while the red region represents $\sigma W(\sigma^{-1})$. This illustrates the decomposition $F = W(\sigma) \cup \sigma W(\sigma^{-1})$. This illustrates half of the paradoxical decomposition of F. Source: [Wib].

We now show how paradoxicality can be transferred from a group to a set on which it acts on. This observation forms the crucial link between paradoxical groups and geometric paradoxes such as the Hausdorff and Banach-Tarski paradoxes. This transfer requires selecting a representative from each orbit and therefore depends on the Axiom of Choice.

Proposition 2.4.3 (AC). Suppose G is a paradoxical group acting freely on a set X. Then X is G-paradoxical.

Proof. Assume that G is paradoxical. Then there exist subsets $A_i, B_j \subseteq G$ and elements $g_i, h_j \in G$ for $1 \leq i \leq n$, $1 \leq j \leq m$ such that

$$G = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j). \tag{2.1}$$

For each $x \in X$, consider the orbit $Gx = \{gx : g \in G\}$. To transfer the paradoxical decomposition of G to X, we choose one representative from each orbit. By the Axiom of Choice, there exists a set $M \subseteq X$ containing exactly one representative from every orbit. The corresponding translates gM, $g \in G$, will allow us to transfer subsets of G to subsets of X.

We first verify that $\{g(M) : g \in G\}$ is a partition of X by showing disjointness and covering.

Disjointness Suppose $g(M) \cap h(M) \neq \varnothing$ for some $g, h \in G$. Then there exist $m_1, m_2 \in M$ such that $gm_1 = hm_2$. Hence,

$$m_1 = g^{-1}h\, m_2 \in Gm_2.$$

Since M contains exactly one representative from each orbit, we must have $m_1 = m_2$, and thus $g^{-1}h\, m_1 = m_1$. As the action is free, this implies $g^{-1}h = e$, hence $g = h$.

Covering Let $x \in X$. The orbit Gx contains a unique element $m \in M$, so there exists $g \in G$ such that $m = gx$. Hence $x = g^{-1}m \in g^{-1}M$, showing that $x \in \bigcup_{g \in G} g(M)$.

Thus $\{g(M) : g \in G\}$ is a partition of X.

We now use this partition to transfer subsets of G to subsets of X. For a subset $S \subseteq G$, define

$$S^* = \bigcup_{g \in S} g(M).$$

In particular, set

$$A_i^* = \bigcup_{g \in A_i} g(M), \qquad B_j^* = \bigcup_{h \in B_j} h(M).$$

These sets are pairwise disjoint since $\{gM : g \in G\}$ is a partition. Moreover,

$$\bigcup_{i=1}^{n} g_i(A_i^*) = \bigcup_{i=1}^{n} g_i \left(\bigcup_{g \in A_i} g(M) \right) = \bigcup_{i=1}^{n} \bigcup_{g \in A_i} g_i g(M) = \bigcup_{g \in G} g(M) = X,$$

where we used that $\bigcup_{i=1}^{n} g_i A_i = G$ by (2.1). Similarly,

$$\bigcup_{j=1}^{m} h_j(B_j^*) = X.$$

Hence X is G-paradoxical. □

Remark 1. The assumption that the action is free is essential. It ensures that the sets $g(M)$ are pairwise disjoint, allowing the paradoxical decomposition of G to transfer to a paradoxical decomposition of X. Without freeness, the transferred sets may overlap.

Combining this proposition with Theorem 2.4.2, it follows that every free action of a free group of rank 2 is paradoxical.

The converse of the previous proposition is valid as well.

Proposition 2.4.4. Let G be a group acting on a set X. If X is G-paradoxical, then G is paradoxical.

Proof. Assume that X is G-paradoxical. Then there exist pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq X$ and elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$X = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j).$$

Fix $x_0 \in X$ and let $\mathcal{O} = Gx_0$ be its orbit. Since $\mathcal{O}$ is invariant under the action of G, restricting the above decomposition to $\mathcal{O}$ gives

$$\mathcal{O} = \bigcup_{i=1}^{n} g_i(A_i \cap \mathcal{O}) = \bigcup_{j=1}^{m} h_j(B_j \cap \mathcal{O}).$$

Now consider the map

$$\varphi : G \to \mathcal{O}, \qquad \varphi(g) = gx_0.$$

This map is surjective and G-equivariant, meaning that

$$\varphi(hg) = h\varphi(g) \qquad \text{for all } g, h \in G.$$

Define the sets

$$A_i' = \varphi^{-1}(A_i \cap \mathcal{O}), \qquad \text{and} \qquad B_j' = \varphi^{-1}(B_j \cap \mathcal{O}).$$

Since preimages preserve disjointness, the sets A_i' and B_j' are pairwise disjoint subsets of G.

Moreover, by equivariance, for each i and j we have

$$\varphi^{-1}(g_i(A_i \cap \mathcal{O})) = g_i(A_i'), \qquad \text{and} \qquad \varphi^{-1}(h_j(B_j \cap \mathcal{O})) = h_j(B_j').$$

Taking preimages of the decomposition of $\mathcal{O}$, we obtain

$$G = \varphi^{-1}(\mathcal{O}) = \bigcup_{i=1}^{n} \varphi^{-1}(g_i(A_i \cap \mathcal{O})) = \bigcup_{i=1}^{n} g_i(A_i').$$

Similarly,

$$G = \bigcup_{j=1}^{m} h_j(B_j').$$

Hence G is paradoxical. □

Corollary 2.4.5. Let G be a group acting on X. If G is not paradoxical, then X is not G-paradoxical.

Together, these results establish the foundation needed to prove the Hausdorff paradox.

Chapter 3

Hausdorff Paradox

In this chapter, we prove the Hausdorff paradox, which gives a paradoxical decomposition of the unit sphere in $\mathbb{R}^3$ after removing a countable set. This chapter applies the theory of paradoxical group actions to a geometric setting and prepares the way for the proof of the Banach–Tarski paradox. The essential step is to identify a free subgroup of the rotation group $SO_3(\mathbb{R})$ and study its action on the sphere. The main reference for this chapter is [Wag85, Chapter 2].

We briefly recall some notation. For $n \geq 1$, let

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

denote the unit sphere in $\mathbb{R}^n$. Furthermore, let

$$SO_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^T A = I,\ \det(A) = 1\}$$

denote the special orthogonal group. In this chapter, we work with the group $SO_3(\mathbb{R})$. Every $A \in SO_3(\mathbb{R})$ represents a rotation in $\mathbb{R}^3$ about a line through the origin. The group $SO_3(\mathbb{R})$ acts naturally on $\mathbb{R}^3$ by matrix multiplication. Since rotations preserve lengths, this action restricts to an action on the unit sphere S^2.

Proposition 3.0.1. Let $A \in SO_3(\mathbb{R})$ with $A \neq I$. Then:

- The set of fixed points
$$\ell = \{x \in \mathbb{R}^3 : Ax = x\}$$
is a line through the origin, called the *axis* of A.
- The intersection $\ell \cap S^2$ consists of exactly two points, say $\{p, -p\}$, called the *poles* of A.

Proof. These statements are standard properties of rotations in $\mathbb{R}^3$ and follow from basic linear algebra. □

As a first step toward the Hausdorff paradox, we construct a free subgroup of rank 2 in $SO_3(\mathbb{R})$.

For this, we define the two rotations

$$\sigma = \begin{bmatrix} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \tau = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}.$$

σ corresponds to a rotation about the z–axis through the angle $\theta = \arccos\left(\frac{1}{3}\right)$, and τ corresponds to a rotation about the x–axis through the same angle. Note that both rotations have the same rotation angle but different axes. Moreover, the angle $\theta = \arccos\left(\frac{1}{3}\right)$ is not a rational multiple of π, so both σ and τ have infinite order.

The inverses of σ and τ are

$$\sigma^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ \frac{-2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tau^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & \frac{-2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}.$$

The following result shows that these rotations generate a free group of rank 2.

Theorem 3.0.2. Let F be the subgroup of $SO_3(\mathbb{R})$ generated by σ and τ. Then F is a free group of rank 2.

Proof. The idea of the proof is to show that every nontrivial reduced word moves the vector $(1,0,0)$, and therefore cannot represent the identity.

Let w be a nontrivial reduced word. Then w ends in either $\sigma^{\pm 1}$ or $\tau^{\pm 1}$. Since conjugation by τ does not affect whether a word equals the identity, we may assume that the rightmost term of w is $\sigma^{\pm 1}$. The key idea is to track the second coordinate of $w(1,0,0)$ and show that it never equals zero.

Claim 1. Let n be the length of w. Then $w(1,0,0) = \frac{1}{3^n}(a, b\sqrt{2}, c)$ where $a, b, c \in \mathbb{Z}$ and b is not divisible by 3.

Proof of Claim. We prove the claim by induction on n.

Base case. For $n = 1$, the word w consists only of its rightmost term, which by assumption is $\sigma^{\pm 1}$. Hence,

$$w(1,0,0) = \frac{1}{3}(1, \pm 2\sqrt{2}, 0),$$

so that $a = 1$, $b = \pm 2$, and $c = 0$. This verifies the statement for the base case $n = 1$.

Induction hypothesis. Assume that the statement holds for some fixed $n = k$, i.e.,

$$w(1,0,0) = \frac{1}{3^k}(a, b\sqrt{2}, c),$$

where $a, b, c \in \mathbb{Z}$ and b is not divisible by 3.

Induction step. Let $w = \gamma w'$, where $\gamma \in \{\sigma^{\pm 1}, \tau^{\pm 1}\}$ and w' has length k. There are four possible cases, depending on the initial letter of w. Since the cases are similar, we consider only the case where w starts with σ. We then have either $w = \sigma\sigma w''$ or $w = \sigma\tau^{\pm 1} w''$ where now w'' is a word of length $k - 1$.

If $w = \sigma\tau^{\pm 1} w''$, then

$$\begin{aligned} w(1,0,0) &= \sigma\tau^{\pm 1} w''(1,0,0) \\ &\overset{\text{I.H.}}{=} \frac{1}{3^{k-1}}\, \sigma\tau^{\pm 1}\big(a,\, b\sqrt{2},\, c\big) \\ &= \frac{1}{3^k}\, \sigma\big(3a,\, (b \mp 2c)\sqrt{2},\, c \pm 4b\big) \\ &= \frac{1}{3^{k+1}}(3a - 4b \pm 8c, (6a + b \mp 2c)\sqrt{2}, 3c \pm 12b) \\ &= \frac{1}{3^{k+1}}(a', b'\sqrt{2}, c') \end{aligned}$$

for $a' := 3a - 4b \pm 8c$, $b' := 6a + b \mp 2c$ and $c' := 3c \pm 12b$. It is immediate that $a', b', c' \in \mathbb{Z}$. It remains to show that 3 does not divide b'. Since w is reduced, the word $\tau^{\pm 1} w''$ is also reduced and has length k. Hence, we may apply the induction hypothesis, which implies that $b \mp 2c$ is not divisible by 3. So for $b' = 6a + b \mp 2c$ we have that $6a$ is divisible by 3 but $b \mp 2c$ is not. Therefore,

b' cannot be divisible by 3.
If now $w = \sigma\sigma w''$, then

$$\begin{aligned} w(1,0,0) &= \sigma\sigma w''(1,0,0) \\ &\overset{I.H.}{=} \frac{1}{3^{k-1}}\sigma\sigma(a, b\sqrt{2}, c) \\ &= \frac{1}{3^k}\sigma(a-4b, (b+2a)\sqrt{2}, 3c) \\ &= \frac{1}{3^{k+1}}(-7a-8b, (4a-7b)\sqrt{2}, 9c) \\ &= \frac{1}{3^{k+1}}(a'', b''\sqrt{2}, c'') \end{aligned}$$

for $a'' := -7a - 8b$, $b'' := 4a - 7b$ and $c'' := 9c$. Again, $a'', b'', c'' \in \mathbb{Z}$ and by the induction hypothesis $b + 2a$ is not divisible by 3. We can rewrite b'' as $b'' = 2(b + 2a) - 9b$ where $2(b + 2a)$ is not divisible by 3 but $9b$ is, so b'' cannot be divisible by 3. The remaining cases, where the initial letter is σ^{-1} or $\tau^{\pm 1}$, are treated in the same way. □

By the claim,

$$w(1,0,0) = \frac{1}{3^n}(a,\ b\sqrt{2},\ c),$$

where $a, b, c \in \mathbb{Z}$ and $3 \nmid b$. If $w(1,0,0) = (1,0,0)$, the second coordinate would satisfy $b\sqrt{2}/3^n = 0$, implying $b = 0$, which contradicts $3 \nmid b$. Therefore, $w(1,0,0) \neq (1,0,0)$, so w does not act as the identity on $\mathbb{R}^3$, and hence $w \neq e$. It follows that the group $F = \langle \sigma, \tau \rangle$ is free of rank 2. □

We now combine the results established so far. We have shown that the subgroup F generated by σ and τ is a free group of rank 2, and hence paradoxical. Moreover, paradoxicality can be transferred to any set on which F acts freely. By removing a suitable countable subset $D \subset S^2$, we obtain such a free action on the sphere. This brings us to the Hausdorff paradox.

Theorem 3.0.3 (The Hausdorff Paradox (AC)). There exists a countable subset $D \subset S^2$ such that $S^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical.

Proof. Let F be the free subgroup of $SO_3(\mathbb{R})$ constructed above. By Theorem 2.4.2, the group F is paradoxical. Define D to be the set of points in S^2 that are fixed by some nontrivial element of F, that is,

$$D := \{\, x \in S^2 : gx = x \text{ for some } g \in F \setminus \{e\} \,\}.$$

Every nontrivial rotation $g \in F$ has exactly two fixed points on S^2, namely the intersection of its rotation axis with the sphere. Since F is countable, it follows that D is countable.

To transfer the paradoxical decomposition of F to $S^2 \setminus D$, we first need to verify that F acts on $S^2 \setminus D$. Let $x \in S^2 \setminus D$ and $g \in F$. Suppose, for contradiction, that $gx \in D$. Then there exists $h \in F \setminus \{e\}$ such that

$$h(gx) = gx.$$

Hence,

$$g^{-1}hgx = x.$$

Since $g^{-1}hg \in F \setminus \{e\}$, this contradicts the assumption that $x \notin D$. Therefore $gx \in S^2 \setminus D$, so $S^2 \setminus D$ is invariant under the action of F. Moreover, by definition of D, the action is free on $S^2 \setminus D$.

We now apply Proposition 2.4.3, which uses the Axiom of Choice (AC). Since F is paradoxical and acts freely on $S^2 \setminus D$, it follows that $S^2 \setminus D$ is F-paradoxical. The paradoxicality of $S^2 \setminus D$ with respect to F immediately implies its paradoxicality with respect to $SO_3(\mathbb{R})$, since $F \subseteq SO_3(\mathbb{R})$. □

The proof above shows the Hausdorff paradox by using the paradoxicality of a free group of rank 2 acting freely on $S^2 \setminus D$. Historically, however, Hausdorff's original idea used a subgroup of $SO(3)$ isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. This algebraic structure gives a very intuitive paradoxical decomposition of the group itself, which then carries over to the sphere (minus a countable set D). Specifically, the remaining part can be split up so that one piece acts as both "one-half" and "one-third" of the whole space under the actions of the order-2 and order-3 generators.

In Chapter 5.2, we will look at a hyperbolic analogue of this classic setup. Instead of using F_2, we will show that $\mathbb{Z}_2 * \mathbb{Z}_3$ is paradoxical and transfer that decomposition via a nice action on the hyperbolic plane $\mathbb{H}^+$. This way, Hausdorff's original geometric idea comes back into play in a hyperbolic setting.

In any case, the Hausdorff Paradox shows that the unit sphere becomes paradoxical once a countable subset is removed. Although the set D is small compared to the whole sphere, removing it is essential to make sure the action is free. The next chapter begins by addressing this restriction.

Chapter 4

Banach-Tarski Paradox in the Euclidean Space

In the previous chapter, we proved Hausdorff's paradox, which shows that the sphere S^2, up to a countable subset, admits a paradoxical decomposition under rotations. In this chapter, we begin by extending the paradoxical decomposition from $S^2 \setminus D$ to the entire sphere. This will serve as the starting point for extending the construction to solid balls in $\mathbb{R}^3$. We also prove a stronger form of the Banach-Tarski paradox, which concerns all bounded subsets of $\mathbb{R}^3$ with nonempty interior. In the final section of the chapter, we also discuss the corresponding higher-dimensional versions of the paradoxes.

Before turning to these constructions, we introduce several notions and establish some general results that will be useful in the sequel. The main references for this chapter are Chapters 3 and 5 of [TW16]. All figures were created by the author based on illustrations from this reference. Any additional sources are mentioned where needed.

4.1 Congruence by dissection and equidecomposability

The Banach-Tarski paradox arises from the action of rigid motions on subsets of $\mathbb{R}^n$. We therefore begin by recalling the relevant class of transformations.

Definition 4.1.1. An *isometry* (or *rigid motion*) of $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distances, that is,

$$\|f(x) - f(y)\| = \|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Every isometry has the form $f(x) = Ax + b$, where $A \in O(n)$, the group of orthogonal matrices, and $b \in \mathbb{R}^n$. For each $n \in \mathbb{N}$, we denote by G_n the *group of all isometries* of $\mathbb{R}^n$.

Before introducing equidecomposability, we consider the classical notion of congruence by dissection for polygons. This is the geometric prototype of the idea: one cuts a figure into finitely many pieces and rearranges them by isometries.

Definition 4.1.2. Two polygons in the plane are said to be *congruent by dissection* if one can be decomposed into finitely many polygonal pieces that can be rearranged by isometries to form the other one, ignoring boundaries.

Remark 2. Congruence by dissection is an equivalence relation on the class of polygons.

Example 4.1.3. Figure 4.1 illustrates how a regular octagon can be dissected into five polygonal pieces and reassembled, using isometries of the plane, to form a square.

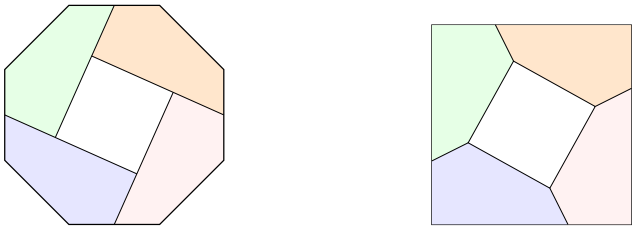

Figure 4.1: Congruence by dissection between a regular octagon and a square.

In this setting, congruence by dissection is closely related to area, since cutting a figure into finitely many pieces and rearranging them by isometries preserves its total area. To make this relationship precise, it is useful to decompose polygons into simpler pieces, namely triangles.

Lemma 4.1.4 (Triangulation Lemma)**.** Every simple polygon, that is, a polygon whose boundary does not intersect itself, can be decomposed into finitely many triangles whose union is the polygon.

Sketch of proof. The result follows by induction on the number of vertices n of the polygon. The base case $n = 3$ is immediate, since a triangle is already triangulated. For $n > 3$, every simple polygon admits a diagonal that lies entirely inside the polygon. Cutting along this diagonal and applying the induction hypothesis to the two resulting polygons yields the desired triangulation. For a complete proof, see [Ber08, Theorem 3.1]. □

The following theorem shows that, in the case of polygons, congruence by dissection behaves as one would expect from geometric intuition.

Theorem 4.1.5 (Bolyai-Gerwien)**.** Two polygons are congruent by dissection if and only if they have equal area.

Proof. Suppose two polygons are congruent by dissection. Then one can be partitioned into finitely many polygonal pieces which can be rearranged by isometries to form the other. Since isometries preserve area and the area of a polygon is additive over such partitions, the two polygons must have the same area. This uses that we are working with polygons, for which area behaves in the usual way.

Conversely, suppose that the two polygons have equal area. It suffices to show that every polygon is congruent by dissection to a square of the same area. The proof proceeds in three steps.

Step 1: From a polygon to rectangles. By the Triangulation Lemma, a polygon can be decomposed into finitely many triangles. Figure 4.2 shows that every triangle is congruent by dissection to a rectangle. Consequently, every polygon is congruent by dissection to a finite collection of rectangles whose total area equals that of the original polygon.

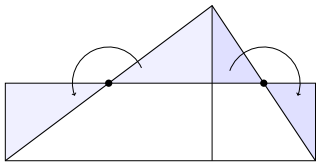

Figure 4.2: Dissection of a triangle into a rectangle of equal area.

Step 2: From rectangles to squares. If the rectangle has length at most four times its width, it can be transformed into a square by the dissection illustrated in Figure 4.3. If the length

exceeds four times the width, the rectangle can first be reduced to this case by repeatedly halving and stacking its parts. Once this reduction has been achieved, the same construction as in Figure 4.3 can be applied to obtain a square of equal area.

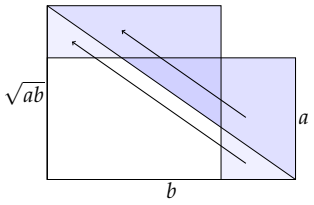

Figure 4.3: Dissection of a rectangle into a square of equal area.

Step 3: Combining the squares. It remains to show that a finite collection of squares can be dissected into a single larger square. We first show that two squares can be combined by dissection into one square of equal total area, as illustrated in Figure 4.4. This construction is based on the relation $a^2 + b^2 = c^2$ and corresponds to the classical dissection proof of the Pythagorean theorem. By repeatedly applying this procedure, any finite number of squares can be transformed into a single square whose area equals that of the original polygon.

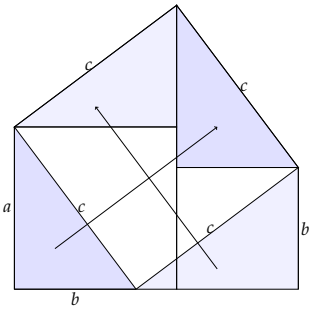

Figure 4.4: Dissection of two squares into one big square.

We have shown that every polygon is congruent by dissection to a square of equal area. Since congruence by dissection is transitive, it follows that any two polygons of equal area are congruent by dissection. □

The notion of congruence by dissection can be generalized to arbitrary sets equipped with a group action. In this setting, decompositions are no longer restricted to polygonal pieces, and rearrangements are performed by elements of the acting group.

Definition 4.1.6. Let G be a group acting on a set X, and let $A, B \subseteq X$. The sets A and B are said to be *G-equidecomposable* if each can be written as a union of the same number of pairwise disjoint subsets that correspond to one another under elements of G. Formally, there exist disjoint subsets $A_1, \dots, A_n \subseteq A$ and $B_1, \dots, B_n \subseteq B$ and elements $g_1, \dots, g_n \in G$ such that

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{i=1}^{n} B_i \qquad \text{and} \qquad g_i(A_i) = B_i \text{ for each } i = 1, \dots, n.$$

In this case, we write $A \sim_G B$. If we wish to specify the number of pieces used in the equidecomposition, we write $A \sim_n B$.

Moreover, we write $A \preceq_G B$ to indicate that A is G-equidecomposable with a subset of B.

The following examples, inspired by [Fre88], illustrate that equidecomposability can already occur in very simple settings. Although the examples arise in different settings, both rely on the same idea: shifting points along an infinite orbit of the group action.

Example 4.1.7 ($\mathbb{N}$ is equidecomposable with $\mathbb{N} \setminus \{p\}$)**.** Let $X = \mathbb{Z}$ and let $G = (\mathbb{Z}, +)$ act on X by translations. Fix $p \in \mathbb{N}$ and consider the sets

$$A = \mathbb{N} \qquad \text{and} \qquad B = \mathbb{N} \setminus \{p\}.$$

Define

$$A_1 = \{0, 1, \dots, p-1\} \qquad \text{and} \qquad A_2 = \{p, p+1, p+2, \dots\},$$

then $A = A_1 \cup A_2$. Applying the identity to A_1 and the translation by $+1$ to A_2 yields

$$A_1 \mapsto \{0, 1, \dots, p-1\}, \qquad A_2 \mapsto \{p+1, p+2, \dots\},$$

whose union yields exactly $B = \mathbb{N} \setminus \{p\}$. Thus, $A \sim_G B$.

Example 4.1.8 (S^1 is equidecomposable with $S^1 \setminus \{p\}$)**.** Let $X = S^1$ and let $G = SO_2(\mathbb{R})$ act on S^1 by matrix multiplication. Fix a point $p \in S^1$, and choose a rotation $R \in SO_2(\mathbb{R})$ by an angle α such that $\alpha/2\pi \notin \mathbb{Q}$. Then the orbit $\{R^n(p) : n \in \mathbb{Z}\}$ is infinite and consists of pairwise distinct points. Define the set

$$A_0 := \{R^n(p) : n \geq 0\}.$$

Then

$$S^1 = A_0 \cup (S^1 \setminus A_0)$$

is a partition of S^1. Moreover, since $R(A_0) = A_0 \setminus \{p\}$, we obtain

$$S^1 \setminus \{p\} = (A_0 \setminus \{p\}) \cup (S^1 \setminus A_0) = R(A_0) \cup (S^1 \setminus A_0) \sim_G A_0 \cup (S^1 \setminus A_0) = S^1.$$

This shows that S^1 is G-equidecomposable with $S^1 \setminus \{p\}$.

Equidecomposability naturally defines a relation on the class of subsets of X. In fact, this relation is an equivalence relation.

Lemma 4.1.9. $\sim_G$ is an equivalence relation.

Proof. The proof is motivated by the proof of [Joe16, Thm. 11.12]. We verify reflexivity, symmetry, and transitivity.

Reflexivity: Every set $A \subseteq X$ is G-equidecomposable with itself by taking a single piece and the identity element of G.

Symmetry: If $A \sim_G B$ via elements $g_1, \dots, g_n \in G$, then applying the inverses $g_1^{-1}, \dots, g_n^{-1}$ shows that $B \sim_G A$.

Transitivity: Suppose $A \sim_G B$ via $A = \bigcup_{i=1}^m A_i$, $B = \bigcup_{i=1}^m B_i$, $g_i \in G$ with $g_i(A_i) = B_i$, and $B \sim_G C$ via $B = \bigcup_{j=1}^k B'_j$, $C = \bigcup_{j=1}^k C_j$, $h_j \in G$ with $h_j(B'_j) = C_j$. To prove transitivity, we refine the given decompositions of B. We do this by intersecting the B-pieces:

$$B_{ij} := B_i \cap B'_j, \qquad A_{ij} := g_i^{-1}(B_{ij}) \subseteq A_i, \qquad C_{ij} := h_j(B_{ij}) \subseteq C_j.$$

The families $\{A_{ij}\}_{i,j}$ and $\{C_{ij}\}_{i,j}$ are pairwise disjoint, and

$$\bigcup_{i=1}^m \bigcup_{j=1}^k A_{ij} = \bigcup_{i=1}^m g_i^{-1}\left(B_i \cap \bigcup_{j=1}^k B'_j\right) = \bigcup_{i=1}^m g_i^{-1}(B_i \cap B) = \bigcup_{i=1}^m g_i^{-1}(B_i) = A.$$

Similarly,

$$C = \bigcup_{i=1}^m \bigcup_{j=1}^k C_{ij}.$$

Moreover,

$$(h_j g_i)(A_{ij}) \;=\; h_j\big(g_i(g_i^{-1}(B_{ij}))\big) \;=\; h_j(B_{ij}) \;=\; C_{ij}.$$

Thus, $A \sim_G C$ via the pieces A_{ij} and the elements $h_j g_i \in G$.

Hence, $\sim_G$ is an equivalence relation. □

Remark 3. From the proof, we see that if $A \sim_n B$ and $B \sim_m C$, then $A \sim C$ can be realized using at most nm pieces.

4.2 Banach–Schröder–Bernstein and paradoxicality

Having introduced equidecomposability, we now consider the relation $\preceq_G$ on subsets of X. Recall that $A \preceq_G B$ means that A is equidecomposable with a subset of B. This provides a way to compare the "size" of subsets of X.

Theorem 4.2.1 (Banach-Schröder-Bernstein)**.** Let G act on a set X, and let $A, B \subseteq X$. If $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$. Consequently, the relation $\preceq_G$ induces a partial order on the $\sim_G$-equivalence classes of subsets of X.

The proof of the theorem relies on the following two lemmas, whose proofs are inspired by [Hoo01, Lemma 3.3, 3.4]. The first lemma shows that equidecomposability is compatible with taking subsets, while the second one shows that it is preserved under unions of disjoint sets.

Lemma 4.2.2. If $A \sim_G B$ and $C \subseteq A$, then there is a bijection $g : A \longrightarrow B$ such that $C \sim_G g(C)$.

Proof. Since $A \sim_G B$, there exist partitions

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{i=1}^{n} B_i,$$

and elements $g_i \in G$ such that $B_i = g_i(A_i)$ for each $i = 1, \dots, n$. We define a map $g : A \longrightarrow B$ by setting $g(x) = g_i x$ for $x \in A_i$. Since each element of G acts as a bijection on X, each restricted map

$$g|_{A_i} : A_i \to B_i, \quad x \mapsto g_i x,$$

is a bijection. Therefore, g is a bijection from A onto B. Next, we partition C according to the sets A_i:

$$C = \bigcup_{i=1}^{n} C_i, \qquad \text{where } C_i := A_i \cap C \subseteq A_i.$$

Then

$$g(C) = g\left(\bigcup_{i=1}^{n} C_i\right) = \bigcup_{i=1}^{n} g(C_i) = \bigcup_{i=1}^{n} g_i(C_i).$$

Hence, C and $g(C)$ admit corresponding decompositions into G-congruent pieces, and thus

$$C \sim_G g(C).$$

□

Lemma 4.2.3. Assume $A \sim_G B$ and $A' \sim_G B'$. If $A \cap A' = B \cap B' = \varnothing$, then $A \cup A' \sim_G B \cup B'$.

Proof. Since $A \sim_G B$, there exist disjoint subsets $A_1, \dots, A_n \subseteq A$ and $B_1, \dots, B_n \subseteq B$, together with elements $g_1, \dots, g_n \in G$, such that $g_i(A_i) = B_i$ for each $i = 1, \dots, n$. Similarly, since $A' \sim_G B'$, there exist disjoint subsets $A'_1, \dots, A'_m \subseteq A'$ and $B'_1, \dots, B'_m \subseteq B'$, with elements $h_1, \dots, h_m \in G$, such that $h_j(A'_j) = B'_j$ for each $j = 1, \dots, m$. Since A and A' (and likewise B and B') are disjoint, the families

$$\{A_i\}_{i=1}^{n} \cup \{A'_j\}_{j=1}^{m} \quad \text{and} \quad \{B_i\}_{i=1}^{n} \cup \{B'_j\}_{j=1}^{m}$$

form disjoint decompositions of $A \cup A'$ and $B \cup B'$, respectively. Each piece of the first family is mapped to the corresponding piece of the second by an element of G, namely g_i or h_j. Hence, by the definition,

$$A \cup A' \sim_G B \cup B'.$$

□

We are now ready to prove the Banach-Schröder-Bernstein theorem.

Proof. Suppose that $A \preceq_G B$ and $B \preceq_G A$, that is, there exist subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that

$$A \sim_G B_1 \quad \text{and} \quad A_1 \sim_G B.$$

As in Lemma 4.2.2, these relations induce bijections

$$f : A \longrightarrow B_1 \quad \text{and} \quad g : A_1 \longrightarrow B.$$

To construct a correspondence between A and B, we iteratively transfer elements of A through the maps f and g^{-1}. Define the sequence of subsets $(C_n)_{n\geq 0}$ of A by

$$C_0 \coloneqq A \setminus A_1, \qquad C_{n+1} \coloneqq g^{-1}\big(f(C_n)\big) \quad \text{for } n \geq 0,$$

and set

$$C \coloneqq \bigcup_{n=0}^{\infty} C_n.$$

This construction partitions A into

$$A = C \cup (A \setminus C),$$

where f maps C into B_1, yielding $C \sim_G f(C)$ using Lemma 4.2.2. It remains to understand the complement $A \setminus C$. The following claim shows that g maps this complement precisely onto the remaining part of B. We claim that

Claim 2. $g(A \setminus C) = B \setminus f(C)$

Proof of Claim. First, observe that since $C_0 = A \setminus A_1$, we have $A \setminus C \subseteq A_1$. Hence, the map g is defined on $A \setminus C$. We now prove the two inclusions required to get the desired equality.

To show that $g(A \setminus C) \subseteq B \setminus f(C)$, let $x \in A \setminus C$. Then $g(x) \in B$ by definition of g. Suppose, for the sake of contradiction, that $g(x) \in f(C)$. Then there exists some $n \geq 0$ such that $g(x) \in f(C_n)$. Consequently,

$$x \in g^{-1}\big(f(C_n)\big) = C_{n+1},$$

which contradicts the assumption that $x \in A \setminus C$. Therefore, $g(x) \notin f(C)$, and we conclude that

$$g(A \setminus C) \subseteq B \setminus f(C).$$

For the reverse inclusion $B \setminus f(C) \subseteq g(A \setminus C)$, let $y \in B \setminus f(C)$. Since $g : A_1 \longrightarrow B$ is a bijection, there exists $x \in A_1$ such that $g(x) = y$. Assume, for the sake of contradiction, that $x \in C$. Then $x \in C_n$ for some $n \geq 1$, and hence

$$x \in C_n = g^{-1}\big(f(C_{n-1})\big) \quad \text{so that} \quad y = g(x) \in f(C_{n-1}) \subseteq f(C),$$

which contradicts the assumption that $y \in B \setminus f(C)$. Therefore, $x \notin C$, and consequently $y = g(x) \in g(A \setminus C)$. We conclude that

$$B \setminus f(C) \subseteq g(A \setminus C).$$

□

Thus, the complement $A \setminus C$ is mapped by g precisely onto the part of B that is not already covered by $f(C)$. Using the claim and Lemma 4.2.2, we obtain

$$A \setminus C \sim_G B \setminus f(C).$$

Hence, by Lemma 4.2.3, we have

$$(A \setminus C) \cup C \sim_G (B \setminus f(C)) \cup f(C),$$

which yields

$$A \sim_G B.$$

□

The Banach-Schröder-Bernstein theorem provides a useful characterization of paradoxicality in terms of equidecomposability. The following corollary shows that a set is paradoxical precisely when it can be partitioned into two disjoint subsets, each equidecomposable with the whole set.

Corollary 4.2.4. Let G act on a set X and let $E \subseteq X$. The following are equivalent:

1. E is G-paradoxical.
2. There exist disjoint subsets $A, B \subseteq E$ with $A \cup B = E$ and

$$A \sim_G E \sim_G B.$$

Proof. (1) $\Rightarrow$ (2): Suppose that E is G-paradoxical. Then there exist pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq E$ and elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that

$$E = \bigcup_{i=1}^{n} g_i(A_i) \qquad \text{and} \qquad E = \bigcup_{j=1}^{m} h_j(B_j).$$

Let

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{j=1}^{m} B_j.$$

Then A and B are disjoint subsets of E satisfying

$$A \sim_G E \qquad \text{and} \qquad B \sim_G E.$$

If $A \cup B = E$, we are done. Otherwise, replace A by

$$A' := E \setminus B.$$

Then A' and B are disjoint and $A' \cup B = E$. Moreover, since $A \subseteq A'$, we have $A \preceq_G A'$, and hence $E \preceq_G A'$ because $E \sim_G A$. On the other hand, since $A' \subseteq E$, we have $A' \preceq_G E$. Thus,

$$E \preceq_G A' \preceq_G E,$$

and by Theorem 4.2.1, it follows that

$$A' \sim_G E \sim_G B.$$

(2) $\Rightarrow$ (1): Suppose that there exist disjoint subsets $A, B \subseteq E$ such that $A \cup B = E$ and

$$A \sim_G E \sim_G B.$$

Choose decompositions witnessing these relations. Then E can be written as a union of translates of pieces of A, and also as a union of translates of pieces of B. Since A and B are disjoint, these two decompositions together form a paradoxical decomposition of E.

□

This corollary makes the nature of paradoxicality particularly transparent. A G-paradoxical set E can be partitioned into two disjoint subsets, A and B, such that each subset is equidecomposable with the whole set. In contrast to the original definition of paradoxicality, the whole set E now appears explicitly in the equidecompositions. Informally, each of the two disjoint halves can be reassembled into a copy of E. This characterization will play an important role in the proof of the strong form of the Banach-Tarski paradox (Theorem 4.4.1).

The corollary also leads naturally to a quantitative refinement of paradoxicality, where one keeps track of the number of pieces used in the corresponding paradoxical decompositons.

Definition 4.2.5. Suppose G acts on a set X and let $E \subseteq X$. We say that E is *G-paradoxical using r pieces* if there exist disjoint subsets $A, B \subseteq E$ with $E = A \cup B$, such that

$$A \sim_m E \qquad \text{and} \qquad B \sim_n E$$

for some integers $m, n \in \mathbb{N}$ satisfying

$$m + n = r.$$

Example 4.2.6. Corollary 4.2.4 implies that the free group F admiants a paradoxical decomposition covering the entire group, including the identity. Figure 4.5 illustrates such a decomposition, where $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, and

$$A_1 \cup \sigma A_2 = F = B_1 \cup \tau B_2.$$

Hence, F is paradoxical using four pieces. This improves the earlier construction in Figure 2.1, where the sets used in the decomposition covered only $F \setminus \{e\}$.

Another useful consequence of the preceding characterization is that paradoxicality is preserved under equidecomposability. In particular, equidecomposable sets are either both G-paradoxical or both non-G-paradoxical.

Proposition 4.2.7. Assume G acts on X and $E \sim_G E'$ where $E, E' \subseteq X$. If E is G-paradoxical, then so is E'.

Proof. The argument follows [Sha15, Cor 5.7]. By Corollary 4.2.4, the assumption that E is G-paradoxical means that there exist disjoint subsets $A, B \subseteq E$ such that $E = A \cup B$ and $A \sim_G E \sim_G B$. Since $E \sim_G E'$, there exist partitions

$$E = \bigcup_{i=1}^{n} E_i, \qquad E' = \bigcup_{i=1}^{n} E_i',$$

together with elements $g_1, \ldots, g_n \in G$ such that

$$g_i(E_i) = E_i' \qquad \text{for each } i.$$

Our aim is to construct disjoint subsets $A', B' \subseteq E'$ satisfying $A' \cup B' = E'$ and $A' \sim_G E' \sim_G B'$. Define

$$A' \coloneqq \bigcup_{i=1}^{n} g_i(A \cap E_i), \qquad B' \coloneqq \bigcup_{i=1}^{n} g_i(B \cap E_i).$$

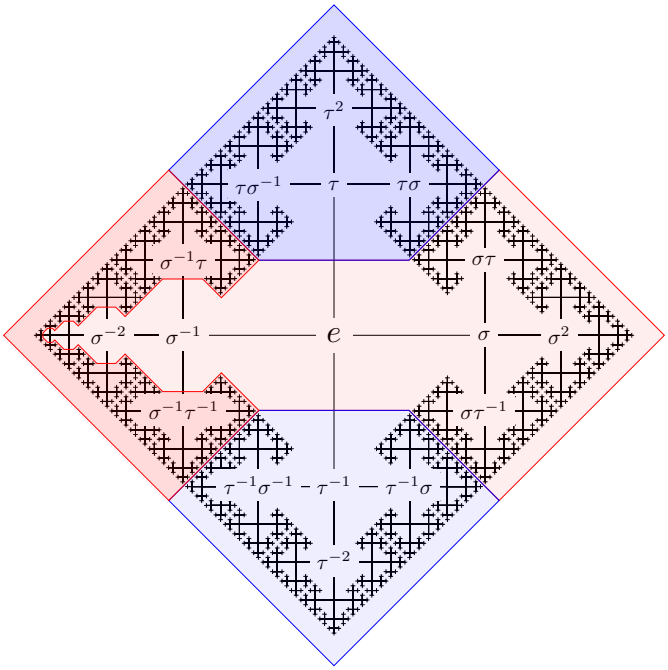

Figure 4.5: An improved paradoxical decomposition of F. The sets $A = A_1 \cup A_2$ (light red and medium red) and $B = B_1 \cup B_2$ (medium blue and light blue) are each equidecomposable with F.

For each i, the sets $A \cap E_i$ and $B \cap E_i$ are disjoint since A and B are. Hence, A' and B' are disjoint subsets of E'. Moreover,

$$A' \cup B' = \bigcup_{i=1}^{n} g_i\big((A \cap E_i) \cup (B \cap E_i)\big) = \bigcup_{i=1}^{n} g_i\big((A \cup B) \cap E_i\big) = \bigcup_{i=1}^{n} g_i\big(E \cap E_i\big) = \bigcup_{i=1}^{n} g_i(E_i) = E'.$$

Since we can partition A and B as

$$A = \bigcup_{i=1}^{n} (A \cap E_i), \qquad B = \bigcup_{i=1}^{n} (B \cap E_i),$$

the families $\{A \cap E_i\}$ and $\{g_i(A \cap E_i)\}$ witness that $A' \sim_G A$, and similarly $B' \sim_G B$. Using transitivity of $\sim_G$ together with $A' \sim_G A \sim_G E \sim_G E'$ and $B' \sim_G B \sim_G E \sim_G E'$, we obtain

$$A' \sim_G E' \sim_G B'.$$

By Corollary 4.2.4, this is equivalent to E' being G-paradoxical. □

We now return to the classical notion of congruence by dissection for polygons and relate it to equidecomposability. While equidecomposability was introduced as a generalization to arbitrary sets via group actions, it is natural to ask how the two notions compare when both are considered in the same setting. As an application of the Banach–Schröder–Bernstein theorem, the following result shows that congruence by dissection implies G_2-equidecomposability.

Theorem 4.2.8. If two polygons P_1 and P_2 are congruent by dissection, then they are G_2-equidecomposable.

Proof. Assume that P_1 and P_2 are congruent by dissection. Then there exist polygonal pieces

$$P_{1,1}, \ldots, P_{1,n} \subseteq P_1, \qquad P_{2,1}, \ldots, P_{2,n} \subseteq P_2,$$

and isometries $g_1, \dots, g_n \in G_2$ such that

$$P_1 = \bigcup_{i=1}^{n} P_{1,i}, \qquad P_2 = \bigcup_{i=1}^{n} P_{2,i}, \qquad g_i(P_{1,i}) = P_{2,i} \quad \text{for } i = 1, \dots, n.$$

Let

$$Q_1 \coloneqq \bigcup_{i=1}^{n} P_{1,i}^{\circ}, \qquad Q_2 \coloneqq \bigcup_{i=1}^{n} P_{2,i}^{\circ},$$

where $P_{1,i}^{\circ}$ and $P_{2,i}^{\circ}$ denote the interiors of the pieces. Then the sets Q_1 and Q_2 are unions of pairwise disjoint pieces, and since isometries map interiors to interiors, we have

$$g_i(P_{1,i}^{\circ}) = P_{2,i}^{\circ}.$$

Hence, $Q_1 \sim_{G_2} Q_2$.

It remains to show that removing the boundary pieces does not affect equidecomposability, that is, that $P_1 \sim_{G_2} Q_1$ and $P_2 \sim_{G_2} Q_2$. By symmetry, it suffices to prove the first statement.

Set $A \coloneqq Q_1$ and $T \coloneqq P_1 \setminus Q_1$. Then T is the union of the boundaries of finitely many polygonal pieces, and hence consists of finitely many line segments.

We claim that

$$A \sim_{G_2} A \cup T.$$

We prove this by first showing that adding a single line segment to a set with nonempty interior does not change its equidecomposability class, and then applying this argument to all segments in T.

Since A is a finite union of open sets, it is itself open. In particular, A has nonempty interior and therefore contains a disk D of radius $r > 0$. If a line segment in T has length $L \geq r$, then we dissect it into $\left\lceil \frac{L}{r} \right\rceil + 1$ equal segments so that all parts have length $< r$. Thus, we may assume that each line segment has length less than r.

Let θ be a rotation of D about its center of infinite order. Choose a radius segment R of D (excluding the center), and consider the set

$$\widetilde{R} \coloneqq \bigcup_{n \geq 0} \theta^n(R).$$

Then $\widetilde{R}$ is a union of pairwise disjoint lines in D.

Let s be one of the line segments in T. There exists an isometry $\sigma \in G_2$ that maps s into a subset of R. Since $\theta(\widetilde{R})$ is disjoint from both R and $D \setminus \widetilde{R}$, we obtain

$$D \cup s = (D \setminus \widetilde{R}) \cup \widetilde{R} \cup s \sim_{G_2} (D \setminus \widetilde{R}) \cup \theta(\widetilde{R}) \cup \sigma(s) \subseteq D.$$

Thus, $D \cup s \preceq_{G_2} D$ and $D \preceq_{G_2} D \cup s$ where the second relation follows from $D \subseteq D \cup s$. By the Banach-Schröder-Bernstein theorem, it follows that

$$D \sim_{G_2} D \cup s.$$

Applying this argument successively to each segment in T, we obtain

$$D \sim_{G_2} D \cup T.$$

By applying Lemma 4.2.3 to the relations $D \sim_{G_2} D \cup T$ and $A \setminus D \sim_{G_2} A \setminus D$, we get

$$D \cup (A \setminus D) \sim_{G_2} (D \cup T) \cup (A \setminus D).$$

Hence, $Q_1 = A \sim_{G_2} A \cup T = P_1$.

Similarly, one shows that $P_2 \sim_{G_2} Q_2$. By transitivity of $\sim_{G_2}$, we conclude

$$P_1 \sim_{G_2} Q_1 \sim_{G_2} Q_2 \sim_{G_2} P_2.$$

Thus, P_1 and P_2 are G_2-equidecomposable. □

Corollary 4.2.9. Any two polygons of the same area are equidecomposable.

This statement is a direct consequence of the Bolyai–Gerwien theorem together with previous theorem.

The converse also holds, that is, equidecomposable polygons have the same area (and hence are congruent by dissection), although its proof is considerably more involved and relies on deeper results from measure theory.

4.3 The Banach-Tarski paradox for balls

We now turn to the Banach-Tarski paradox. Recall that the goal is to show that a solid ball in $\mathbb{R}^3$ can be decomposed into finitely many pieces and reassembled by isometries into two copies of the original ball. The first step is to remove the countable exceptional set appearing in the Hausdorff paradox. We show that this set does not affect the equidecomposability class of the sphere, allowing the paradoxical decomposition to be extended to the entire sphere.

Theorem 4.3.1. If $D \subseteq S^2$ is countable, then S^2 and $S^2 \setminus D$ are $SO_3(\mathbb{R})$-equidecomposable using two pieces.

Proof. Since D is countable, there exists a line ℓ through the origin that avoids D. For $\theta \in [0, 2\pi)$, let ρ_θ denote the rotation about ℓ by angle θ.

Let A be the set of all angles θ for which there exist $n \in \mathbb{N}$ and $P \in D$ such that $\rho_\theta^n(P) \in D$. Since both $\mathbb{N}$ and D are countable, the set A is countable. Hence, we may choose $\varphi \in [0, 2\pi) \setminus A$ and set $\rho := \rho_\varphi$.

By construction, for every $n \geq 1$ we have

$$\rho^n(D) \cap D = \varnothing.$$

Consequently, for integers $m < n$,

$$\rho^n(D) \cap \rho^m(D) = \rho^m\big(\rho^{n-m}(D) \cap D\big) = \rho^m(\varnothing) = \varnothing,$$

and thus the sets $D, \rho(D), \rho^2(D), \dots$ are pairwise disjoint. Let

$$D^* := \bigcup_{n \geq 0} \rho^n(D).$$

Then $\rho(D^*)$ is disjoint from $S^2 \setminus D^*$, and therefore we obtain the equidecomposition

$$S^2 = D^* \cup (S^2 \setminus D^*) \sim_{SO_3(\mathbb{R})} \rho(D^*) \cup (S^2 \setminus D^*) = (D^* \setminus D) \cup (S^2 \setminus D^*) = S^2 \setminus D,$$

using exactly two pieces. Hence, S^2 and $S^2 \setminus D$ are $SO_3(\mathbb{R})$-equidecomposable. □

Combining the previous theorem with the Hausdorff paradox, we obtain a paradoxical decomposition of the entire sphere.

Proposition 4.3.2 (AC). The sphere S^2 is $SO_3(\mathbb{R})$-paradoxical.

Proof. By the Hausdorff Paradox, there exists a countable set $D \subseteq S^2$ such that $S^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical. By Theorem 4.3.1, we have $S^2 \setminus D \sim_{SO_3(\mathbb{R})} S^2$. Since paradoxicality is preserved under equidecomposability (Proposition 4.2.7), it follows that S^2 is $SO_3(\mathbb{R})$-paradoxical. □

Remark 4. By the Example 4.2.6, the free group F is paradoxical using four pieces. Consequently, the corresponding decomposition arising in the Hausdorff paradox yields

$$A' \sim_2 S^2 \setminus D \qquad \text{and} \qquad B' \sim_2 S^2 \setminus D.$$

Moreover, Theorem 4.3.1 shows that

$$S^2 \sim_2 S^2 \setminus D.$$

Composing these equidecompositions therefore gives A and B being equidecomposable with S^2 both using at most 4 pieces. Since $S^2 = A \cup B$, this construction duplicates the sphere using at most eight pieces in total.

Starting from the paradoxical decomposition of S^2, we can extend the construction to solid balls and ultimately to the entire space $\mathbb{R}^3$. This yields the classical Banach-Tarski paradox.

Theorem 4.3.3 (The Banach-Tarski Paradox (AC)). Any solid ball in $\mathbb{R}^3$ is G_3-paradoxical. Moreover, $\mathbb{R}^3$ itself is G_3-paradoxical.

Proof. We follow [Hoo01, Thm 10.8] for some of the details. It suffices to consider balls centered at the origin, since G_3 contains all translations. In particular, any ball can be translated to the origin by an element of G_3. We therefore fix the centered ball

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq r^2\}.$$

Subsets of S^2 can be transferred to subsets of $B \setminus \{0\}$ via the radial correspondence

$$\Pi(s) = \{\alpha s : 0 < \alpha \leq r\}, \qquad s \in S^2.$$

Note that every point of $B \setminus \{0\}$ lies on a unique radial segment of the form $\Pi(s)$ with $s \in S^2$. Moreover, this correspondence is compatible with rotations, allowing the paradoxical decomposition of S^2 to be transferred to $B \setminus \{0\}$. If S^2 is $SO_3(\mathbb{R})$-paradoxical, there exist disjoint families $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ in S^2, together with rotations $g_1, \ldots, g_n, h_1, \ldots, h_m \in SO_3(\mathbb{R})$, such that

$$S^2 = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j).$$

Applying Π to these pieces, we obtain two families $\Pi(A_1), \ldots, \Pi(A_n)$ and $\Pi(B_1), \ldots, \Pi(B_m) \subseteq B \setminus \{0\}$, where the same rotations provide a paradoxical decomposition of $B \setminus \{0\}$. To see this, observe that

$$\bigcup_{i=1}^{n} g_i(\Pi(A_i)) = \bigcup_{i=1}^{n} \Pi(g_i(A_i)) = \Pi\left(\bigcup_{i=1}^{n} g_i(A_i)\right) = \Pi(S^2) = B \setminus \{0\},$$

and similarly,

$$B \setminus \{0\} = \bigcup_{j=1}^{m} h_j(\Pi(B_j)).$$

Excluding the origin from the radial correspondence ensures that the sets $\Pi(A_i)$ and $\Pi(B_j)$ remain pairwise disjoint, since the families $\{A_i\}$ and $\{B_j\}$ are disjoint in S^2.

It remains to show that $B \sim_{G_3} B \setminus \{0\}$. To achieve this, we proceed similarly as in the proof of Theorem 4.3.1. Let $P = (0, \frac{1}{2}, 0)$ and let ℓ be the vertical line in the yz–plane passing through P.

Choose a rotation ρ about the axis ℓ of infinite order, for example by choosing the angle of rotation to be $\sqrt{2}\pi$. Explicitly, ρ is the rotation

$$\begin{bmatrix} \cos(\sqrt{2}\pi) & -\sin(\sqrt{2}\pi) & 0 \\ \sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

conjugated by the translation sending the z-axis to ℓ. Since ℓ does not pass through the origin, $\rho \notin SO_3(\mathbb{R})$ but $\rho \in G_3$. Define the set

$$D := \{\rho^n(0) : n \geq 0\}.$$

Then $\rho(D) = D \setminus \{0\}$. Consequently,

$$B = D \cup (B \setminus D) \sim_{G_3} \rho(D) \cup (B \setminus D) = (D \setminus \{0\}) \cup (B \setminus D) = B \setminus \{0\}.$$

This yields an equidecomposition of B with $B \setminus \{0\}$ with respect to G_3. Proposition 4.2.7 implies that B is G_3-paradoxical.

A similar argument applies to $\mathbb{R}^3$. To this end, we express the radial correspondence as

$$\Pi : S^2 \longrightarrow \mathbb{R}^3 \setminus \{0\}, \qquad \Pi(s) = \{\alpha s : \alpha > 0\}.$$

As in the case of $B \setminus \{0\}$, this correspondence transfers the decomposition of S^2 to a paradoxical decomposition of $\mathbb{R}^3 \setminus \{0\}$ using the same rotations. Moreover, by the same argument as above, we get

$$\mathbb{R}^3 \sim_{G_3} \mathbb{R}^3 \setminus \{0\}.$$

Hence, $\mathbb{R}^3$ is G_3-paradoxical.

This completes the proof of the Banach-Tarski paradox. □

Although the Banach–Tarski paradox appears to contradict the usual geometric intuition that volume should be preserved under isometries, no contradiction arises. The pieces appearing in the decomposition are highly non-constructive, arising from applications of the Axiom of Choice, and are not measurable in the sense of Lebesgue measure, the standard mathematical notion of volume in Euclidean space. Consequently, they cannot be assigned a classical volume. TThe existence of such paradoxical decompositions is closely related to the notion of amenability, which will be discussed in Chapter 6.

Remark 5. The radial correspondence transfers the eight-piece paradoxical decomposition of the sphere to $B \setminus \{0\}$, preserving the number of pieces. The additional absorption argument used to reinsert the origin doubles the number of pieces, yielding a paradoxical decomposition of a solid ball in $\mathbb{R}^3$ using at most sixteen pieces. Considerably more efficient constructions are known: the Banach-Tarski paradox for balls can in fact be realized using only five pieces. However, we do not pursue questions concerning the minimal number of pieces required any further, as they lie beyond the scope of this thesis.

4.4 Strong form of the Banach-Tarski paradox

The Banach-Tarski paradox can be strengthened considerably. More precisely, any two bounded subsets of $\mathbb{R}^3$ with nonempty interior are G_3-equidecomposable. This result is sometimes referred to as the *pea-to-sun paradox*, since it implies that a ball of arbitrarily small volume (for example, the size of a pea) can be decomposed and reassembled into a ball of arbitrarily large volume (for example, the size of the sun).

Theorem 4.4.1 (Strong Form of the Banach-Tarski Paradox (AC)). Any two bounded subsets $A, B \subset \mathbb{R}^3$ with nonempty interior are G_3-equidecomposable.

Proof. The proof is inspired by [Hoo01, Thm 10.3]. By the Banach-Schröder-Bernstein theorem, it suffices to show that $A \preceq_{G_3} B$ and $B \preceq_{G_3} A$. Because A and B are bounded and have nonempty interior, there exist balls B_1 and B_2 such that

$$A \subseteq B_1 \qquad \text{and} \qquad B_2 \subseteq B.$$

Choose $n \in \mathbb{N}$ sufficiently large so that B_1 is contained in the union of n translates of B_2. To make this precise, let

$$T = \bigcup_{i=1}^{n} \tau_i(B_2),$$

where the maps τ_i are translations so that $B_1 \subseteq T$.

We also define a set of n disjoint copies of B_2,

$$S = \bigcup_{i=1}^{n} \sigma_i(B_2),$$

where the translations σ_i ensure that the copies are pairwise disjoint. The set T consists of finitely many possibly overlapping translates of B_2, whereas S consists of finitely many disjoint copies of B_2. As a first step, we use T to show that $B_1 \preceq_{G_3} S$.

Claim 3. $B_1 \preceq_{G_3} S$

Proof of Claim. Since $B_1 \subseteq T$, we have $B_1 \preceq_{G_3} T$. To eliminate overlaps among the translates $\tau_i(B_2)$, we partition T as follows. Set

$$C_1 = \tau_1(B_2) \qquad \text{and} \qquad C_i = \tau_i(B_2) \setminus \bigcup_{k=1}^{i-1} C_k \quad \text{for } i = 2, 3, \ldots, n.$$

Then $\{C_i\}_{i=1}^n$ are pairwise disjoint and partition T. Using this decomposition, we want to show that $T \preceq_{G_3} S$.

For this, define $\rho_i = \sigma_i \tau_i^{-1} \in G_3$. Since $C_i \subseteq \tau_i(B_2)$, we obtain

$$\rho_i(C_i) = \sigma_i \tau_i^{-1}(C_i) \subseteq \sigma_i \tau_i^{-1}\big(\tau_i(B_2)\big) = \sigma_i(B_2).$$

Consequently,

$$\bigcup_{i=1}^{n} \rho_i(C_i) \subseteq \bigcup_{i=1}^{n} \sigma_i(B_2) = S.$$

It follows that $B_1 \preceq_{G_3} T \preceq_{G_3} S$, and hence, by transitivity, $B_1 \preceq_{G_3} S$. □

We now embed the disjoint union S back into the single ball B_2. That is, we show that S is G_3-equidecompsable to a subset of B_2.

Claim 4. $S \preceq_{G_3} B_2$

Proof of the Claim. By the Banach-Tarski Paradox, the ball B_2 is G_3-paradoxical. Applying Corollary 4.2.4, there exist two disjoint subsets $X, Y \subseteq B_2$ such that $X \cup Y = B_2$ and

$$X \sim_{G_3} B_2 \sim_{G_3} Y.$$

We now repeatedly duplicate copies of B_2 inside S. Since equidecomposability is preserved under translations, we have

$$X \sim_{G_3} \sigma_1(B_2) \qquad \text{and} \qquad Y \sim_{G_3} \sigma_2(B_2),$$

where σ_1 and σ_2 are the ones appearing in the definition of S. By construction, the translates $\sigma_1(B_2)$ and $\sigma_2(B_2)$ are disjoint. Applying Lemma 4.2.3, we obtain

$$B_2 = X \cup Y \sim_{G_3} \sigma_1(B_2) \cup \sigma_2(B_2).$$

Next, we use the paradoxicality of $\sigma_2(B_2)$ to obtain an additional disjoint copy that we send to $\sigma_3(B_2)$ via the map $\sigma_3\sigma_2^{-1}$. Consequently,

$$B_2 \sim_{G_3} \sigma_1(B_2) \cup \sigma_2(B_2) \sim_{G_3} \sigma_1(B_2) \cup \sigma_2(B_2) \cup \sigma_3(B_2).$$

Repeating this construction a total of $n-1$ times, we arrive at

$$B_2 \sim_{G_3} \bigcup_{i=1}^{n} \sigma_i(B_2) = S.$$

This shows that $B_2 \sim_{G_3} S$, and in particular $S \preceq_{G_3} B_2$. □

Now, since $A \subseteq B_1$ and $B_1 \preceq_{G_3} S$, it follows that $A \preceq_{G_3} S$. Similarly, because $B_2 \subseteq B$ and $S \preceq_{G_3} B_2$, we obtain $S \preceq_{G_3} B$. Combining these relations, we get

$$A \preceq_{G_3} S \preceq_{G_3} B.$$

Repeating the entire construction with the roles of B_1 and B_2 reversed yields

$$B \preceq_{G_3} S' \preceq_{G_3} A,$$

where S' is defined analogously to S, but using disjoint copies of B_1 instead of B_2. Hence, $A \preceq_{G_3} B$ and $B \preceq_{G_3} A$. Therefore, by the Banach-Schröder-Bernstein theorem, we conclude that

$$A \sim_{G_3} B.$$

□

Remark 6. Unlike the paradoxical decompositions constructed earlier, the proof of the strong form does not yield a universal bound on the number of pieces involved. Indeed, the number of pieces produced by the construction depends on the particular sets A and B, for example through the number of translates required to cover one set by balls contained in the other.

4.5 Paradoxes in Higher Dimensions

After establishing the Banach-Tarski paradox in $\mathbb{R}^3$, it is natural to ask in which other dimensions such decompositions occur. The following proposition shows that for every $n \geq 3$, the group $SO_n(\mathbb{R})$ contains a free subgroup of rank 2. This suggests that the same mechanisms underlying the Hausdorff and Banach-Tarski paradoxes persist in higher dimensions.

The generalization is not entirely straightforward, however. The proof of the Hausdorff paradox relied on the fact that, after removing a countable set of fixed points, a free subgroup of $SO_3(\mathbb{R})$ acts freely on the sphere. In higher dimensions, the fixed-point sets of rotations are typically much larger, so this argument does not carry over directly. Fortunately, the three-dimensional case already provides the key ingredient needed to obtain the higher-dimensional results.

Therefore, using the results from the previous sections, one can show that S^{n-1} is $SO_n(\mathbb{R})$-paradoxical for every $n \geq 3$. The corresponding higher-dimensional versions of the Banach-Tarski

paradox then follow by arguments analogous to those used in dimension 3. The main reference for this section is [Hoo01, Chapter 4].

By contrast, the situation in dimensions 1 and 2 is fundamentally different. There, paradoxical decompositions do not occur. The explanation for this is closely related to amenability, which will be discussed in Chapter 6.

Proposition 4.5.1. For every $n \geq 3$, the group $SO_n(\mathbb{R})$ contains a free subgroup of rank 2.

Proof. By the construction in Chapter 3, there exist elements $\sigma, \tau \in SO_3(\mathbb{R})$ which freely generate a free subgroup of rank 2.

For $n \geq 3$, consider the natural embedding

$$\iota : SO_3(\mathbb{R}) \hookrightarrow SO_n(\mathbb{R}), \qquad A \mapsto \begin{bmatrix} A & 0 \\ 0 & I_{n-3} \end{bmatrix}.$$

This is an injective group homomorphism. Hence the elements

$$\iota(\sigma), \iota(\tau) \in SO_n(\mathbb{R})$$

satisfy exactly the same relations as σ and τ.

Since σ and τ generate a free subgroup of rank 2 in $SO_3(\mathbb{R})$, $\iota(\sigma)$ and $\iota(\tau)$ generate a free subgroup of rank 2 in $SO_n(\mathbb{R})$. □

Proposition 4.5.2 (AC). For every $n \geq 3$, the unit sphere S^{n-1} is $SO_n(\mathbb{R})$-paradoxical.

Proof. We proceed by induction on $n \geq 3$. The case $n = 3$ was established in Proposition 4.3.2, where we proved that the sphere S^2 is $SO_3(\mathbb{R})$-paradoxical.

Now suppose that $S^{n-1} \subseteq \mathbb{R}^n$ is $SO_n(\mathbb{R})$-paradoxical. We again consider the embedding

$$\iota : SO_n(\mathbb{R}) \hookrightarrow SO_{n+1}(\mathbb{R}), \qquad A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

By assumption, there exists a partition $A_1, ..., A_p, B_1, ..., B_q$ of S^{n-1} and elements $g_1, ..., g_p, h_1, ..., h_q \in SO_n(\mathbb{R})$ such that

$$S^{n-1} = \bigcup_{i=1}^{p} g_i(A_i) = \bigcup_{j=1}^{q} h_j(B_j).$$

We first construct a paradoxical decomposition of $S^n \setminus \{p^+, p^-\} \subseteq \mathbb{R}^{n+1}$, where

$$p^+ = (0, \ldots, 0, 1) \qquad \text{and} \qquad p^- = (0, \ldots, 0, -1)$$

are the two poles of S^n. For a point $x = (x_1, \ldots, x_n, x_{n+1}) \in S^n \setminus \{p^+, p^-\}$, the vector $(x_1, \ldots, x_n)$ is nonzero, and hence

$$\frac{(x_1, \ldots, x_n)}{\|(x_1, \ldots, x_n)\|} \in S^{n-1}.$$

Therefore, excluding p^+, p^- allows us to define subsets $A_i^*, B_j^* \subseteq S^n \setminus \{p^+, p^-\}$ by

$$A_i^* := \left\{ x \in S^n \setminus \{p^+, p^-\} : \frac{(x_1, \ldots, x_n)}{\|(x_1, \ldots, x_n)\|} \in A_i \right\},$$

and similarly

$$B_j^* := \left\{ x \in S^n \setminus \{p^+, p^-\} : \frac{(x_1, \ldots, x_n)}{\|(x_1, \ldots, x_n)\|} \in B_j \right\}.$$

The sets $A_1^*, ..., A_p^*, B_1^*, ..., B_q^*$ form a partition of $S^n \setminus \{p^+, p^-\}$. Indeed, their union covers $S^n \setminus \{p^+, p^-\}$ since for every $x \in S^n \setminus \{p^+, p^-\}$, $\frac{(x_1,\dots,x_n)}{\|(x_1,\dots,x_n)\|}$ belongs to some A_i or B_j and hence x is contained in the corresponding A_i^* or B_j^*. Moreover, they are pairwise disjoint since $A_1, ..., A_p, B_1, ..., B_q$ are.

Next, we extend the rotations $g_i, h_j \in SO_n(\mathbb{R})$ to rotations $g_i^*, h_j^* \in SO_{n+1}(\mathbb{R})$ by setting

$$g_i^* = \begin{bmatrix} g_i & 0 \\ 0 & 1 \end{bmatrix}, \qquad h_j^* = \begin{bmatrix} h_j & 0 \\ 0 & 1 \end{bmatrix}.$$

These rotations act on the first n coordinates exactly as g_i and h_j do, while fixing the last coordinate.

We verify that these sets and rotations give a paradoxical decomposition of $S^n \setminus \{p^+, p^-\}$. For $x = (x_1, \dots, x_n, x_{n+1}) \in A_i^*$ let

$$u := \frac{(x_1, \dots, x_n)}{\|(x_1, \dots, x_n)\|} \in A_i.$$

Since

$$S^{n-1} = \bigcup_{i=1}^{p} g_i(A_i),$$

the sets $g_i(A_i)$ cover S^{n-1}. Moreover, since the extended rotation g_i^* acts on the first n coordinates as g_i and fixes the last coordinate, we have

$$g_i^*(x) = (\|(x_1, \dots, x_n)\| \, g_i(u), x_{n+1}).$$

Therefore, as i ranges from 1 to p, the sets $g_i^*(A_i^*)$ cover all points of S^n whose last coordinate lies in $(-1, 1)$, that is, $S^n \setminus \{p^+, p^-\}$. It follows that

$$S^n \setminus \{p^+, p^-\} = \bigcup_{i=1}^{p} g_i^*(A_i^*).$$

The same argument applied to the sets B_j^* and the rotations h_j^* gives

$$S^n \setminus \{p^+, p^-\} = \bigcup_{j=1}^{q} h_j^*(B_j^*).$$

Thus, $S^n \setminus \{p^+, p^-\}$ is paradoxical with respect to $SO_{n+1}(\mathbb{R})$.

It remains to pass from $S^n \setminus \{p^+, p^-\}$ to all of S^n. We again proceed similarly as the proof of Theorem 4.3.1, where we showed that $S^2 \sim S^2 \setminus D$ using a rotation that disjointly rotates the set D. Thus, we require an element in $SO_{n+1}(\mathbb{R})$ of infinite order that moves the poles p^+ and p^-. For this, consider a rotation in the plane spanned by the last two coordinate axes while fixing the first $n-1$ coordinates. Define

$$\rho = \begin{bmatrix} I_{n-1} & & 0 \\ & \cos(\pi\sqrt{2}) & -\sin(\pi\sqrt{2}) \\ 0 & \sin(\pi\sqrt{2}) & \cos(\pi\sqrt{2}) \end{bmatrix} \in SO_{n+1}(\mathbb{R}).$$

Since ρ has infinite order (see Example 2.2.5),

$$\rho^n(\{p^+, p^-\}) \cap \{p^+, p^-\} = \varnothing \quad \text{for all } n \in \mathbb{N}.$$

Let

$$P := \bigcup_{k=0}^{\infty} \rho^k(\{p^+, p^-\}).$$

Then

$$S^n = P \cup (S^n \setminus P) \sim_{SO_{n+1}(\mathbb{R})} \rho(P) \cup (S^n \setminus P) = S^n \setminus \{p^+, p^-\}$$

and hence S^n is $SO_{n+1}(\mathbb{R})$-paradoxical by Proposition 4.2.7.

This completes the induction. □

Remark 7. In the inductive step, the maximal number of pieces doubles at each step, since every induction step requires an additional absorption argument to recover the two removed poles. Starting from the paradoxical decomposition of S^2 using at most 8 pieces, this yields a decomposition of S^{n-1} using at most 2^n pieces.

The preceding argument extends immediately to arbitrary spheres in $\mathbb{R}^n$. Indeed, for any sphere

$$S^n(x, r) = \{y \in \mathbb{R}^n : \|y - x\| = r\},$$

one may first translate the center x to the origin and then replace each piece in the paradoxical decomposition of S^{n-1} by its image under the map $s \mapsto rs$. Since we need to translate the center, such spheres are paradoxical with respect to G_n instead of only $SO_n(\mathbb{R})$.

Theorem 4.5.3 (Banach-Tarski Paradox for Higher Dimensions (AC)). Any solid ball in $\mathbb{R}^n$ with $n \geq 3$ is G_n-paradoxical. Moreover, $\mathbb{R}^n$ itself is G_n-paradoxical.

The proof proceeds exactly as in the case $n = 3$. Using the paradoxical decomposition of S^{n-1} established above, the radial correspondence transfers this decomposition to the punctured ball $B \setminus \{0\}$ and to $\mathbb{R}^n \setminus \{0\}$. The origin is then absorbed by the same infinite-order rotation argument used in dimension three. As in the three-dimensional case, the origin can be absorbed by means of an infinite-order rotation. Indeed, one may take the rotation ρ from the proof above and translate its fixed subspace away from the origin. Since no essentially new ideas are required, we omit the details.

Remark 8. By the previous remark, the decomposition of S^{n-1} uses at most 2^n pieces. Passing from the sphere to the ball requires one additional absorption step for the origin. Consequently, the construction above yields a paradoxical decomposition of a ball in $\mathbb{R}^n$ using at most 2^{n+1} pieces.

Theorem 4.5.4 (Strong Form of the Banach-Tarski Paradox (AC)). Any two bounded subsets $A, B \subset \mathbb{R}^n$, where $n \geq 3$, with nonempty interior are G_n-equidecomposable.

Again, the proof proceeds as in the three-dimensional case (Theorem 4.4.1). Using balls with nonempty interior, one reduces the problem to comparing finitely many translates of balls in $\mathbb{R}^n$. By replacing overlapping translates with disjoint copies, one obtains

$$A \preceq_{G_n} B \qquad \text{and} \qquad B \preceq_{G_n} A.$$

The conclusion

$$A \sim_{G_n} B$$

then again follows from the Banach-Schröder-Bernstein theorem.

Chapter 5

Paradoxes in the Hyperbolic Space

In the previous chapters, paradoxical decompositions were obtained through abstract arguments involving group actions and the Axiom of Choice. In this chapter, we consider analogous phenomena in hyperbolic space. While group actions remain the key ingredient, the hyperbolic setting offers a more geometric approach through explicit tessellations and fundamental polygons. This allows many constructions to be carried out concretely and avoids the use of the Axiom of Choice altogether. Our main reference for this chapter is [TW16, Chapter 4]. Unless stated otherwise, all figures were created by the author, inspired illustrations from this reference. Any additional sources are cited where needed.

5.1 The Hyperbolic Plane

The hyperbolic plane can be represented in several equivalent ways, two of which are used most frequently: the upper half-plane model and the Poincaré disk model. The *upper half-plane model* realizes the hyperbolic space as

$$\mathbb{H}^+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

In this model, hyperbolic lines appear as either vertical Euclidean lines or semicircles whose centers lie on the x-axis. The *Poincaré disk model* represents the hyperbolic space as the open unit disk

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

where hyperbolic lines are given by Euclidean diameters or by arcs of circles meeting the boundary circle orthogonally. Although these realizations differ visually, they describe the same underlying geometry and are related by explicit isometries. For most of the algebraic computations in this chapter, the upper half-plane model will be more convenient, while the disk model will mainly serve for visualization.

In the Euclidean setting, the paradoxes rely on the structure of the isometry group G_3. To study analogous phenomena in the hyperbolic plane, we first define the hyperbolic distance on the upper half-plane model and then investigate the corresponding group of isometries. For the discussion of hyperbolic space and hyperbolic isometries, we additionally use [Cha10].

Definition 5.1.1 ([Ive94])**.** The hyperbolic distance on the upper half-plane $\mathbb{H}^+$ is the function

$$d_{\mathbb{H}^+} : \mathbb{H}^+ \times \mathbb{H}^+ \to \mathbb{R}_{\geq 0}$$

defined by

$$\cosh d_{\mathbb{H}^+}(z, w) = 1 + \frac{|z - w|^2}{2 \operatorname{Im}(z) \operatorname{Im}(w)}.$$

Definition 5.1.2. An *isometry* of $\mathbb{H}^+$ is a map $f : \mathbb{H}^+ \to \mathbb{H}^+$ such that for all $x, y \in \mathbb{H}^+$,

$$d_{\mathbb{H}^+}(f(x), f(y)) = d_{\mathbb{H}^+}(x, y).$$

A fundamental role in the study of hyperbolic isometries is played by linear fractional transformations.

Definition 5.1.3. A *linear fractional transformation* is a map

$$f(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc \neq 0.$$

Such a map is represented by the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}),$$

and the composition of linear fractional transformations corresponds to the multiplication of their associated matrices, while the inverse transformation is represented by the inverse of the matrix.

Definition 5.1.4. Let T be an isometry of the upper half-plane $\mathbb{H}^+$. We say that T is *orientation-preserving* if for any three non-collinear points $a, b, c \in \mathbb{H}^+$, the oriented angle $\angle abc$ is equal to the oriented angle $\angle T(a)\, T(b)\, T(c)$.

We say that T is *orientation-reversing* if for any three non-collinear points $a, b, c \in \mathbb{H}^+$, the oriented angles $\angle abc$ and $\angle T(a)\, T(b)\, T(c)$ have opposite sign.

The following theorem characterizes all isometries of the upper half-plane.

Theorem 5.1.5.

- Every orientation-preserving isometry of the upper half-plane $\mathbb{H}^+$ is given by a linear fractional transformation of the form

$$z \mapsto \frac{az + b}{cz + d}, \qquad \text{with } a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

- Every orientation-reversing isometry of $\mathbb{H}^+$ is of the form

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}, \qquad \text{with } a, b, c, d \in \mathbb{R},\ ad - bc < 0.$$

A proof of this result can be found in [Cha10, Thm 3.4].

In the remainder of this chapter, we restrict our attention to orientation-preserving isometries. Every such isometry is represented by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$$

with $ad - bc > 0$. The following proposition determines when two such matrices represent the same transformation.

Proposition 5.1.6. Two matrices induce the same linear fractional transformation if and only if they differ by a nonzero scalar multiple.

Proof. Assume first that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

for some $\lambda \neq 0$. Then,

$$\frac{Az + B}{Cz + D} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az + b}{cz + d},$$

so both matrices give the same transformation.

Conversely, assume that the two transformations coincide, that is,

$$\frac{Az+B}{Cz+D} = \frac{az+b}{cz+d} \qquad \text{for all } z \in \mathbb{H}^+.$$

Cross-multiplication yields

$$(Az+B)(cz+d) = (az+b)(Cz+D).$$

Expanding both sides gives

$$Acz^2 + Adz + Bcz + Bd = aCz^2 + aDz + bCz + bD$$

and comparing coefficients gives

$$Ac = aC, \qquad Bd = bD \quad \text{and} \quad Ad + Bc = aD + bC.$$

Since $ad - bc \neq 0$, neither (a, c) nor (b, d) equals the zero vector. Hence the relation $Ac = aC$ implies that (A, C) is proportional to (a, c), so there exists $\mu_1 \in \mathbb{R}$ such that

$$A = \mu_1 a, \qquad C = \mu_1 c.$$

Similarly, the relation $Bd = bD$ implies that (B, D) is proportional to (b, d), so there exists $\mu_2 \in \mathbb{R}$ such that

$$B = \mu_2 b, \qquad D = \mu_2 d.$$

Substituting these expressions into the remaining relation $Ad + Bc = aD + bC$ gives

$$(\mu_1 - \mu_2)(ad - bc) = 0.$$

Since $ad - bc \neq 0$, it follows that $\mu_1 = \mu_2$. Setting $\lambda = \mu_1 = \mu_2$, we obtain

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Note that $\lambda \neq 0$, since otherwise the matrix on the left would be the zero matrix, which cannot represent a linear fractional transformation. □

As a consequence of the previous proposition, a linear fractional transformation is unchanged if the representing matrix is multiplied by a nonzero scalar. Hence, we may normalize the matrices so that $ad - bc = 1$. With this normalization, two matrices in $\mathrm{SL}_2(\mathbb{R})$ represent the same transformation if and only if they differ by multiplication by ± 1. Indeed, if $A, \lambda A \in \mathrm{SL}_2(\mathbb{R})$, then $\lambda^2 = 1$, so $\lambda = \pm 1$.

Therefore, every orientation-preserving hyperbolic isometry is represented not by a unique matrix in $\mathrm{SL}_2(\mathbb{R})$, but by a pair of matrices $\{A, -A\}$. It is thus natural to identify such matrices, which leads to the definition of the projective special linear group

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}.$$

To further classify orientation-preserving hyperbolic isometries, we consider the trace of their representing matrices. The trace determines the geometric behaviour of the corresponding isometries, in particular the number and location of their fixed points. As in the Euclidean case, fixed points will be crucial in constructing suitable group actions for paradoxical decompositions.

Definition 5.1.7. Let $T \in PSL_2(\mathbb{R})$ be represented by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We classify T according to the value of its trace defined as $|\operatorname{tr}(T)| = |a + d|$:

- T is *elliptic* if $|\operatorname{tr}(T)| < 2$;
- T is *parabolic* if $|\operatorname{tr}(T)| = 2$;
- T is *hyperbolic* if $|\operatorname{tr}(T)| > 2$.

Since an element of $PSL_2(\mathbb{R})$ is represented by a matrix only up to sign, the quantity $|\operatorname{tr}(T)|$ is well defined.

Proposition 5.1.8. Let T be a nonidentity element of $PSL2(\mathbb{R})$. Then:

- if $|\operatorname{tr}(T)| < 2$, then T has a unique fixed point in $\mathbb{H}^+$.
- if $|\operatorname{tr}(T)| = 2$, then T has exactly one fixed point on $\partial\mathbb{H}^+ = \mathbb{R} \cup \{\infty\}$.
- if $|\operatorname{tr}(T)| > 2$, then T has two distinct fixed points on $\partial\mathbb{H}^+$.

Proof. A fixed point of T is a solution of

$$\frac{az + b}{cz + d} = z.$$

We separate the argument into two cases depending on whether $c = 0$ or $c \neq 0$.
Case 1: $c \neq 0$**.** Solving the fixed-point equation yields the equation $cz^2 + (d - a)z - b = 0$, with solutions

$$z_{1,2} = \frac{(a - d) \pm \sqrt{(d - a)^2 - 4bc}}{2c}.$$

Using $ad - bc = 1$, we compute $(d - a)^2 - 4bc = (a + d)^2 - 4(ad - bc) = \operatorname{tr}(T)^2 - 4$. The nature and location of the fixed points is determined entirely by the sign of $\operatorname{tr}(T)^2 - 4$:

- If $|\operatorname{tr}(T)| < 2$, then $\operatorname{tr}(T)^2 - 4 < 0$ and the two solutions are non-real conjugates of which exactly one lies in $\mathbb{H}^+$.
- If $|\operatorname{tr}(T)| = 2$, then $\operatorname{tr}(T)^2 - 4 = 0$ and the solutions coincide, producing a single fixed point on $\partial\mathbb{H}^+$.
- If $|\operatorname{tr}(T)| > 2$, then $\operatorname{tr}(T)^2 - 4 > 0$, so both solutions lie in $\mathbb{R}$ and correspond to distinct boundary fixed points.

Case 2: $c = 0$**.** In this case, $Tz = \frac{az+b}{d}$ and $ad = 1$, thus $d = \frac{1}{a}$. Hence, $Tz = a^2z + ab$ and $\operatorname{tr}(T) = a + \frac{1}{a}$. Since $a \in \mathbb{R}$, we have $|a + 1/a| \geq 2$, so T cannot be elliptic. We distinguish the remaining possibilities:

- If $|\operatorname{tr}(T)| = 2$, then $a + \frac{1}{a} = \pm 2$, which forces $a = \pm 1$. Thus, $Tz = z \pm b$, a Euclidean translation. Its unique fixed point is $\infty \in \partial\mathbb{H}^+$.
- If $|\operatorname{tr}(T)| > 2$, then $a^2 \neq 1$ and $Tz = a^2z + ab$ has one finite fixed point $z_1 = \frac{ab}{1-a^2} \in \mathbb{R}$, while ∞ is the second fixed point. Both lie on the boundary.

This proves the three cases. □

We see that among all the isometries in the hyperbolic plane, only the elliptic ones fix a point inside the hyperbolic plane $\mathbb{H}^+$. Parabolic and hyperbolic transformations fix points only on the boundary. For the paradoxical constructions later in this chapter, free subgroups of $PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R})$ will play an important role. While a free subgroup of $PSL_2(\mathbb{R})$ may contain elliptic elements, this cannot occur in $PSL_2(\mathbb{Z})$. This is because every elliptic element of $PSL_2(\mathbb{Z})$ has finite order.

Indeed, if $\operatorname{tr}(\sigma) = 0$, then

$$\sigma = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad \text{and} \qquad \sigma^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which represents the identity in $PSL_2(\mathbb{Z})$. Similarly, if $\mathrm{tr}(\sigma) = \pm 1$, then

$$\sigma = \begin{bmatrix} a & b \\ c & \pm 1 - a \end{bmatrix},$$

and in this case σ has order 3. Consequently, any subgroup generated by an independent pair of matrices in $PSL_2(\mathbb{Z})$ consists entirely of non-elliptic elements. In particular, such a subgroup acts on $\mathbb{H}^+$ without nontrivial fixed points.

This highlights an important difference from the Euclidean constructions discussed earlier. In the Hausdorff paradox on S^2, the free subgroup of $SO_3(\mathbb{R})$ necessarily consists of rotations which always have fixed points, forcing the removal of an exceptional set before the action becomes free. In the hyperbolic setting, however, parabolic and hyperbolic transformations have no fixed points inside $\mathbb{H}^+$. This makes it possible to transfer paradoxical decompositions directly to the whole hyperbolic plane.

Elliptic transformations, on the other hand, still possess fixed points in $\mathbb{H}^+$. The free product structure generated by such transformations nevertheless allows Hausdorff's original idea to be realized explicitly: certain regions of the hyperbolic plane can simultaneously be viewed as one half" and one third" of the space under suitable isometries. Thus, the hyperbolic setting admits both a realization of Hausdorff's original construction and paradoxical decompositions of the entire hyperbolic plane. Moreover, these constructions can be carried out explicitly and with geometrically simple pieces.

These two phenomena are realized by the following pairs of matrices:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

We will show that the first pair generates the free product $\mathbb{Z}_2 * \mathbb{Z}_3$, whereas the second generates a free group of rank 2. These groups provide the algebraic foundation for the paradoxical decompositions constructed in the following sections.

5.2 A Hyperbolic Analogue of the Hausdorff Paradox

We now develop a hyperbolic analogue of Hausdorff's original construction. The key role is played by the free product $\mathbb{Z}_2 * \mathbb{Z}_3$, realized as the subgroup generated by the first pair of matrices introduced above. Through its action on the hyperbolic plane, this group gives rise to a tessellation that makes Hausdorff's decomposition completely explicit. As in the Euclidean setting, points fixed by the action must be removed, so the resulting decomposition applies not to the entire hyperbolic plane, but only to the complement of a certain exceptional set.

Let

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad \tau = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

corresponding to the transformations

$$\sigma(z) = \frac{-1}{z} \qquad \text{and} \qquad \tau(z) = \frac{-1}{z+1}$$

of order 2 and 3. The next theorem shows that these are the only relations satisfied by σ and τ, and hence that they generate the free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

Proposition 5.2.1. The transformations σ and τ generate the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. More precisely, $\langle \sigma, \tau \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

Proof. Since $\sigma^2 = e$ and $\tau^3 = e$, we have $\sigma^{-1} = \sigma$ and $\tau^{-1} = \tau^2$. Consider a nonemprpty reduced word $w \in \langle \sigma, \tau \rangle$, written in the letters σ, τ, and τ^2. Conjugation does not affect whether a word

represents the identity, so we may assume, after conjugating if necessary, that w begins and ends with σ. Hence

$$w = \sigma y \sigma y \cdots \sigma y \sigma,$$

where each occurrence of y denotes either τ or τ^2.

The products $\sigma\tau$ and $\sigma\tau^2$ are given by

$$\sigma\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \sigma\tau^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We now evaluate w at the point $(-1, 0)$ and prove that $w(-1, 0) \neq (-1, 0)$. We have,

$$w \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \sigma y \sigma \cdots \sigma y \sigma \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \sigma y \sigma \cdots \sigma y \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the matrices $\sigma\tau$, $\sigma\tau^2$, and the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ all have nonnegative entries, every intermediate vector arising during the computation of $w \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ has nonnegative coordinates. In particular, its first coordinate can never equal -1. Therefore,

$$w \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

so w does not represent the identity transformation. Therefore, the only relations satisfied by σ and τ are $\sigma^2 = e$ and $\tau^3 = e$. Hence, the subgroup generated by σ and τ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. □

The group generated by σ and τ will be used to construct a tessellation of the hyperbolic plane using a fundamental domain. For this purpose, we introduce the notions of discreteness and fundamental domains.

Definition 5.2.2. A subgroup $G \subset PSL_2(\mathbb{R})$ is called *discrete* if G contains no convergent sequence of distinct elements. In this case, the orbit Gz is a discrete subset of $\mathbb{H}^+$ for every $z \in \mathbb{H}^+$.

Definition 5.2.3. Let $G \subseteq PSL_2(\mathbb{R})$ act on the hyperbolic plane $\mathbb{H}^+$. A *fundamental domain* for this action is an open set $P \subset \mathbb{H}^+$ such that:

- $\mathbb{H}^+$ is covered by the closure of G–translates of P, that is,

$$\mathbb{H}^+ = \bigcup_{g \in G} \overline{g(P)}.$$

- The translates $g(P)$ are pairwise disjoint.

Equivalently, every orbit intersects P in exactly one point, while points on the boundary of $\overline{P}$ may belong to several translates.

Theorem 5.2.4. If $G \subset PSL_2(\mathbb{R})$ is discrete, then the action of G on $\mathbb{H}^+$ admits a fundamental domain.

Proof. We follow the proof in [Kat92, Thm 3.2.2]. Let G be a discrete subgroup of $PSL_2(\mathbb{R})$, and choose a point $p \in \mathbb{H}^+$ that is not fixed by any nontrivial element of G. Define the Dirichlet region of G centered at p by

$$D_p(G) := \left\{ z \in \mathbb{H}^+ : d_{\mathbb{H}^+}(z, p) \leq d_{\mathbb{H}^+}(z, g(p)) \text{ for all } g \in G \right\}.$$

Geometrically, $D_p(G)$ consists of all points of $\mathbb{H}^+$ that are at least as close to p as to any other point in the orbit Gp. By definition, $D_p(G)$ is an intersection of closed sets, and is therefore closed. Since the hyperbolic distance is invariant under $PSL_2(\mathbb{R})$, the region can also be defined by

$$D_p(G) := \left\{ z \in \mathbb{H}^+ : d_{\mathbb{H}^+}(z,p) \leq d_{\mathbb{H}^+}(g(z),p) \text{ for all } g \in G \right\}.$$

We show that $D_p(G)$ is a fundamental domain for the action of G. Let $z \in \mathbb{H}^+$ and consider its orbit Gz. Since G is discrete, the orbit Gz is a discrete subset of $\mathbb{H}^+$. Hence, there exists a point $z_0 \in Gz$ minimizing the distance to p, that is,

$$d_{\mathbb{H}^+}(z_0,p) \leq d_{\mathbb{H}^+}(g(z_0),p) \quad \text{for all } g \in G.$$

By definition, this implies $z_0 \in D_p(G)$. Hence, every G-orbit intersects $D_p(G)$. Consequently,

$$\mathbb{H}^+ = \bigcup_{g \in G} g(D_p(G)).$$

Next, we show that no two distinct points in the interior $D_p(G)^\circ$ lie in the same G-orbit. Suppose, to the contrary, that $z_1, z_2 \in D_p(G)^\circ$ and $z_2 = g(z_1)$ for some $g \in G \setminus \{e\}$. Since $z_1 \in D_p(G)^\circ$, the defining inequality is strict for every nontrivial $g \in G$. Indeed, if equality held for some nontrivial $g \in G$, then z_1 would lie on the boundary of $D_p(G)$. Thus,

$$d_{\mathbb{H}^+}(z_1,p) < d_{\mathbb{H}^+}(g(z_1),p) = d_{\mathbb{H}^+}(z_2,p).$$

Similarly, since $z_2 \in D_p(G)^\circ$, we obtain

$$d_{\mathbb{H}^+}(z_2,p) < d_{\mathbb{H}^+}(g^{-1}(z_2),p) = d_{\mathbb{H}^+}(z_1,p).$$

Combining these inequalities yields

$$d_{\mathbb{H}^+}(z_1,p) < d_{\mathbb{H}^+}(z_2,p) < d_{\mathbb{H}^+}(z_1,p),$$

a contradiction. Therefore, no two distinct points in $D_p(G)^\circ$ lie in the same G-orbit, and in particular the translates $g(D_p(G)^\circ)$, $g \in G$, are pairwise disjoint.

Hence, $D_p(G)^\circ$ is a fundamental domain for the action of G on $\mathbb{H}^+$. □

The subgroup generated by σ and τ is in fact the modular group $PSL_2(\mathbb{Z})$. Although this observation is not essential for what follows, it identifies the resulting tessellation with the classical modular tessellation of the hyperbolic plane.

Proposition 5.2.5. Let σ and τ be as above, then $\langle \sigma, \tau \rangle = PSL_2(\mathbb{Z})$.

We only give an outline of the argument. For a full proof, see [Kur60, App. B].

Outline of proof. Recall that $PSL_2(\mathbb{Z})$ is the group of linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z},\ ad-bc=1.$$

Define the transformation

$$\delta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We first show that σ and δ generate the whole group $PSL_2(\mathbb{Z})$. Let

$$m = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{Z}).$$

Using left multiplication by δ and σ, one can perform a reduction process, analogous to the Euclidean algorithm, which decreases the size of the entries. After finitely many steps, m is reduced to a matrix of the form

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \delta^k,$$

showing that m can be expressed as a word in δ and σ. Thus,

$$PSL_2(\mathbb{Z}) = \langle \sigma, \delta \rangle.$$

Next, we can rewrite τ as $\tau = \sigma\delta$, and therefore

$$PSL_2(\mathbb{Z}) = \langle \sigma, \tau \rangle.$$

□

Combined with Proposition 5.2.1, this yields the relation

$$PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

Let $G = PSL_2(\mathbb{Z}) = \langle \sigma, \tau \rangle$. Since G is a discrete subgroup of $PSL_2(\mathbb{R})$, Theorem 5.2.4 implies that its action on $\mathbb{H}^+$ admits a fundamental domain. To describe this fundamental domain explicitly, we consider the Dirichlet domain of G centered at $z_0 = 2i$, namely

$$D(z_0) = \{z \in \mathbb{H}^+ : d(z, z_0) \leq d(z, gz_0) \text{ for all } g \in G\}.$$

Note that the center of a Dirichlet domain cannot be fixed by a nontrivial element of the acting group. Indeed, if

$$\frac{a(2i) + b}{c(2i) + d} = 2i$$

for some element of G, then comparing real and imaginary parts yields

$$b = -4c, \qquad a = d.$$

Since $ad - bc = 1$, we obtain

$$a^2 + 4c^2 = 1,$$

which forces $c = 0$ and $a = d = \pm 1$. Hence $b = 0$, so the corresponding element is the identity. Therefore, $2i$ is a valid choice of center.

The orbit points closest to z_0 are

$$\delta(z_0) = 1 + 2i, \qquad \delta^{-1}(z_0) = -1 + 2i, \qquad \sigma(z_0) = \frac{i}{2}.$$

Therefore, the Dirichlet domain is bounded by the three hyperbolic lines:

- $C_1 = \{z : d_{\mathbb{H}^+}(z, 2i) = d_{\mathbb{H}^+}(z, \sigma(2i))\} = \{z : d_{\mathbb{H}^+}(z, 2i) = d_{\mathbb{H}^+}(z, \frac{i}{2})\} = \{z : |z| = 1\}$. This is because, by definition of the hyperbolic metric (5.1.1), $d_{\mathbb{H}^+}(z, 2i) = d_{\mathbb{H}^+}(z, \frac{i}{2})$ is equivalent to

  $$\frac{|z - 2i|^2}{Im(2i)} = \frac{|z - \frac{i}{2}|^2}{Im(\frac{i}{2})}.$$

  Writing $z = x + iy$ and using $Im(2i) = 2$, $Im(\frac{i}{2}) = \frac{1}{2}$, we obtain

  $$x^2 + (y - 2)^2 = 4\Big[x^2 + (y - \frac{1}{2})^2\Big].$$

A straightforward simplification yields

$$1 = x^2 + y^2,$$

which is exactly the half circle $\{z : |z| = 1\}$.

- $C_2 = \{z : d_{\mathbb{H}^+}(z, 2i) = d_{\mathbb{H}^+}(z, \delta(2i))\} = \{z : Re(z) = \frac{1}{2}\}$. Since $Im(2i) = Im(\delta(2i)) = 2$, the condition $d_{\mathbb{H}^+}(z, 2i) = d_{\mathbb{H}^+}(z, \delta(2i))$ reduces to

$$|z - 2i|^2 = |z - (1 + 2i)|^2.$$

Writing $z = x + iy$, this becomes

$$x^2 + (y-2)^2 = (x-1)^2 + (y-2)^2,$$

hence

$$x^2 = (x-1)^2 \quad \Longleftrightarrow \quad x = \frac{1}{2}.$$

- $C_3 = \{z : d_{\mathbb{H}^+}(z, 2i) = d_{\mathbb{H}^+}(z, \delta^{-1}(2i))\} = \{z : Re(z) = -\frac{1}{2}\}$ by similar argument.

It follows that the Dirichlet domain for G is

$$\mathcal{F} = \left\{ z \in \mathbb{H}^+ : |z| \geq 1,\ |\Re(z)| \leq \frac{1}{2} \right\}$$

and therefore a fundamental domain is given by

$$\mathcal{F}^\circ = \left\{ z \in \mathbb{H}^+ : |z| > 1,\ |\Re(z)| < \frac{1}{2} \right\}$$

For later convenience, we apply σ to this domain, obtaining the hyperbolic triangle P with vertices

$$0, \qquad -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, \qquad \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,$$

which we will use below as a fundamental domain for the action of G on $\mathbb{H}^+$. Since this domain is a hyperbolic polygon, the resulting tessellation of the hyperbolic plane admits a particularly clear geometric visualization, see Figure 5.1.

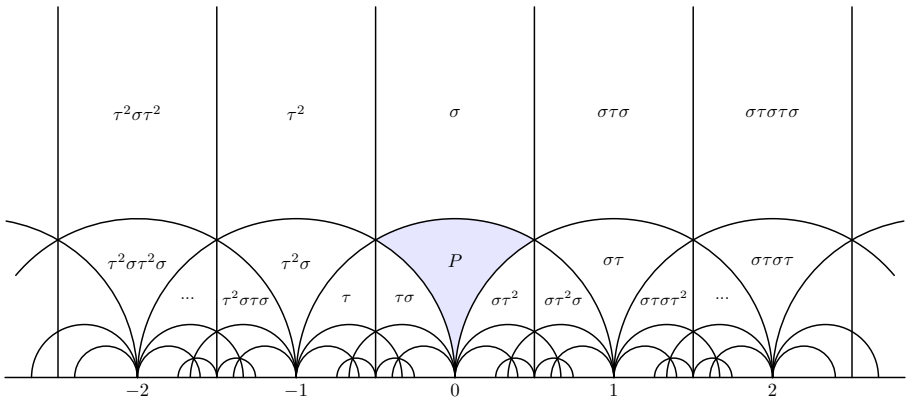

Figure 5.1: Tessellation of the hyperbolic plane for the action of discrete group G.

We now return to the algebraic structure underlying this tessellation. Recall from Proposition 5.2.1 that the transformations σ and τ generate the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. As first observed

by Hausdorff, this free product is paradoxical. The resulting decomposition provides the group-theoretic foundation of Hausdorff's paradox and can be transferred to the hyperbolic plane via the tessellation constructed above.

Proposition 5.2.6. The free product $\mathbb{Z}_2 * \mathbb{Z}_3$ is paradoxical.

Proof. For some details, we follow [TW14]. Let $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle \sigma, \tau : \sigma^1 = \tau^3 = e \rangle$ and denote by W_x the set of words having x as the leftmost letter where $x \in \{\sigma, \tau, \tau^2\}$. For the partition of G, we define three sets

$$\begin{aligned} A &:= \{(\tau\sigma)^j : j \geq 0\} \cup (W_\sigma \setminus \{\tau^2(\tau\sigma)^i : i \geq 0\}), \\ B &:= \{\tau(\tau\sigma)^j : j \geq 0\} \cup (W_\tau \setminus \{(\tau\sigma)^i : i \geq 0\}), \\ C &:= \{\tau^2(\tau\sigma)^j : j \geq 0\} \cup (W_{\tau^2} \setminus \{\tau(\tau\sigma)^i : i \geq 0\}). \end{aligned}$$

By construction, the three sets are pairwise disjoint, and since $e = (\tau\sigma)^0 \in A$, their union equals G.

Claim 5. The following relations hold:

$$\tau(A) = B, \qquad \tau^2(A) = C, \qquad \sigma(A) = B \cup C. \tag{5.1}$$

Proof of Claim. We verify only the identity $\tau(A) = B$, as the other two follow by analogous arguments. Using the definition of A, we compute

$$\begin{aligned} \tau(A) &= \tau\Big(\{(\tau\sigma)^j : j \geq 0\} \cup \big(W_\sigma \setminus \{\tau^2(\tau\sigma)^i : i \geq 0\}\big)\Big) \\ &= \{\tau(\tau\sigma)^j : j \geq 0\} \cup \big(\tau(W_\sigma) \setminus \{\tau(\tau^2(\tau\sigma)^i) : i \geq 0\}\big). \end{aligned}$$

Since $\tau\tau^2 = e$ and $\tau(W_\sigma) = W_\tau$, the expression simplifies to

$$\{\tau(\tau\sigma)^j : j \geq 0\} \cup \big(W_\tau \setminus \{(\tau\sigma)^i : i \geq 0\}\big) = B,$$

as required. □

The relations in (5.1) provide the key ingredient for proving paradoxicality. Indeed,

$$\tau(A) = B, \qquad \tau^2(A) = C$$

identify A with each of the sets B and C, while

$$\sigma(A) = B \cup C$$

identifies A with their union. Thus, A may simultaneously be viewed as "one half" and "one third" of the group. We now use this structure to obtain an explicit paradoxical decomposition of $\mathbb{Z}_2 * \mathbb{Z}_3$. To do this, we split A into two pieces. Let

$$A_1 := \sigma^{-1}(B) = \sigma(B), \qquad A_2 := \sigma^{-1}(C) = \sigma(C).$$

Then $A_1 \cup A_2 = \sigma^{-1}(B \cup C) = A$, and consequently $\mathbb{Z}_2 * \mathbb{Z}_3 = A_1 \cup A_2 \cup B \cup C$. By the definitions of A_1 and A_2, we obtain $A_1 \sim B$, $A_2 \sim C$. The relations established in the claim imply the following equidecompositions:

$$\begin{aligned} A &\sim_G B && \text{(since } \tau(A) = B\text{)}, \\ B &\sim C && \text{(since } \tau^2(A) = \tau(B) = C\text{)}, \\ A &\sim B \cup C && \text{(since } \sigma(A) = B \cup C\text{)}, \\ B \cup C &\sim A \cup C && \text{(since } \tau(B \cup C) = \tau(B) \cup \tau(C) = C \cup A\text{)}. \end{aligned}$$

Combining these equivalences, we obtain the chain

$$A_1 \sim B \sim A \sim B \cup C \sim A \cup C.$$

By Lemma 4.2.3, this implies $A_1 \cup B \sim A \cup C \cup B = \mathbb{Z}_2 * \mathbb{Z}_3$. A completely analogous argument shows that $A_2 \cup C \sim \mathbb{Z}_2 * \mathbb{Z}_3$.

Thus, we have found two disjoint subsets of $\mathbb{Z}_2 * \mathbb{Z}_3$, namely $A_1 \cup B$ and $A_2 \cup C$, each equidecomposable with $\mathbb{Z}_2 * \mathbb{Z}_3$. Therefore, $\mathbb{Z}_2 * \mathbb{Z}_3$ is paradoxical. □

Theorem 5.2.7 (Hyperbolic Hausdorff Paradox). Let N denote the union of the boundaries of the tessellation. Then $\mathbb{H}^+ \setminus N$ is G-paradoxical, where N is the set of all the boundaries of the tessellation.

Proof. Since $G \cong \mathbb{Z}_2 * \mathbb{Z}_3$, the group G is paradoxical. We now transfer the corresponding paradoxical decomposition to $\mathbb{H}^+ \setminus N$, proceeding similarly as in the proof of Proposition 2.4.3.

Because the translates of the fundamental domain P are pairwise disjoint and cover $\mathbb{H}^+ \setminus N$, the set P provides a set of representatives for the free action of G on $\mathbb{H}^+ \setminus N$.

Define

$$\tilde{A} := \bigcup_{a \in A} a(P), \quad \tilde{B} := \bigcup_{b \in B} b(P) \quad \text{and} \quad \tilde{C} := \bigcup_{c \in C} c(P).$$

Then $\tilde{A}, \tilde{B}, \tilde{C}$ are pairwise disjoint and satisfy

$$\mathbb{H}^+ \setminus N = \tilde{A} \cup \tilde{B} \cup \tilde{C}.$$

Moreover, the relations

$$\tau(A) = B, \qquad \tau^2(A) = C, \qquad \sigma(A) = B \cup C$$

transfer directly to the corresponding unions of tiles:

$$\tau(\tilde{A}) = \tilde{B}, \qquad \tau^2(\tilde{A}) = \tilde{C}, \qquad \sigma(\tilde{A}) = \tilde{B} \cup \tilde{C}.$$

These are precisely the relations used in the proof of the paradoxicality of $\mathbb{Z}_2 * \mathbb{Z}_3$. Hence, again the paradoxical decomposition of G induces a paradoxical decomposition of $\mathbb{H}^+ \setminus N$. The sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are illustrated in Figure 5.2. □

Unlike the Euclidean Hausdorff construction, no use of the Axiom of Choice is required here. Indeed, the fundamental domain already provides an explicit choice of one representative from each orbit.

Remark 9. The paradoxical decomposition of $\mathbb{H}^+ \setminus N$ inherits the four-piece structure of the decomposition of $\mathbb{Z}_2 * \mathbb{Z}_3$. Consequently, it uses the same number of pieces as the Hausdorff's paradox of the sphere.

Although the upper half-plane model is convenient for computations, the geometry becomes more transparent in the Poincaré disk model. By applying a suitable linear fractional transformation, the tessellation can be transferred to the disk, allowing for a clearer visualization of the sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$.

To be precise, we consider two Poincaré disk representations obtained from the upper half-plane model by linear fractional transformations sending the points $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and i to the center of the disk, respectively. The resulting tessellations are shown in Figures 5.3 and 5.4.

In the first picture, the interpretation of $\tilde{A}$ as "one third" of the hyperbolic plane becomes apparent: the disk is partitioned into three sets, each of which is mapped clockwise onto the next by the transformation τ.

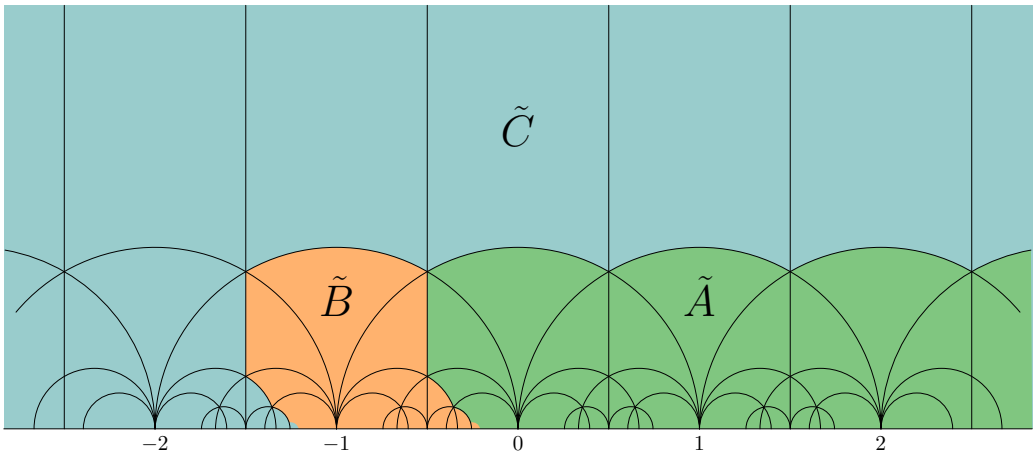

Figure 5.2: Visualization of the sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ as unions of tiles in the tessellation of $\mathbb{H}^+ \setminus N$.

In the second representation, $\tilde{A}$ appears as exactly "one half" of the hyperbolic plane, since $\sigma(\tilde{A}) = \tilde{B} \cup \tilde{C}$. Together, the two pictures provide a geometric illustration of the relations in (5.1) and thus of the basic idea underlying Hausdorff's paradox, realized through explicitly described sets.

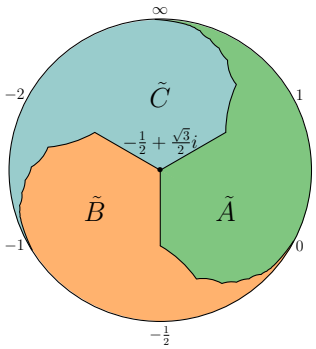

Figure 5.3: The sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ in the Poincaré disk centered at $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, where τ acts as a 120° rotation with $\tau(\tilde{A}) = \tilde{B}$ and $\tau^2(\tilde{A}) = \tilde{C}$.

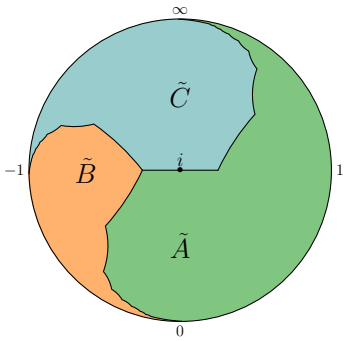

Figure 5.4: The sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ in the Poincaré disk centered at i, where σ acts as a 180° rotation and satisfies $\sigma(\tilde{A}) = \tilde{B} \cup \tilde{C}$.

Hyperbolic paradoxical decompositions also arise in connection with M. C. Escher's famous *Circle Limit IV* tessellation, also known as *Angels and Devils*. Following [Ben00], one considers the transformations

$$\tau = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \sigma = \begin{bmatrix} -1 & 2(1+\sqrt{2}) \\ -\sqrt{2} & 1 \end{bmatrix},$$

which generate another copy of the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. Using the corresponding fundamental domain D, one obtains a hyperbolic analogue of the Hausdorff paradox directly on the Escher tessellation.

The resulting coloring is illustrated in Figure 5.5. In one representation, the red region appears as one third of the hyperbolic plane, while after moving the viewpoint it appears as one half of the plane, reflecting the paradoxical nature of the decomposition.

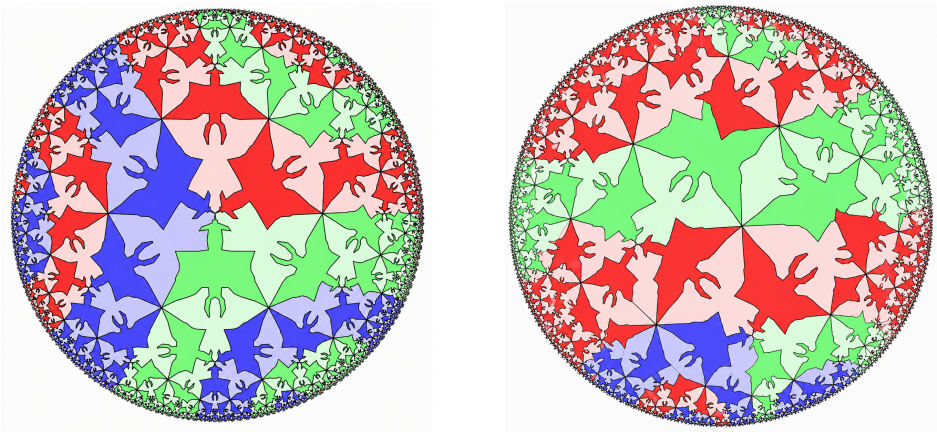

Figure 5.5: Hyperbolic colorings associated with Escher's Circle Limit IV. Source: [TW14, Figure 10].

5.3 A Hyperbolic Analogue of the Banach-Tarski Paradox

The construction of the previous section does not yield a paradoxical decomposition of the entire hyperbolic plane, since the boundaries of the tiles were not taken into account. More precisely, the decomposition applies only to $\mathbb{H}^+ \setminus N$, where N denotes the union of the tile boundaries arising from the tessellation. In this section, we show that the entire hyperbolic plane is paradoxical using a free group of rank 2. For this reason, the resulting construction may be regarded as a hyperbolic analogue of the Banach-Tarski paradox. To achieve this, we pass to a subgroup generated entirely by non-elliptic transformations.

Recall the two matrices

$$\alpha = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

and let F denote the subgroup they generate. Since these transformations have no fixed points in $\mathbb{H}^+$, the resulting action on $\mathbb{H}^+$ is free. We now investigate the structure of F and its action on $\mathbb{H}^+$.

We begin by proving that the two matrices are independent and therefore generate a free group of rank 2.

Proposition 5.3.1. The two transformations $\alpha, \beta \in PSL_2(\mathbb{Z})$ are independent.

Proof. Recall the matrices from the previous section

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \tau = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \tau^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},$$

which generate the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. A direct computation shows that

$$\alpha = \sigma\tau^2\sigma\tau^2, \qquad \beta = \sigma\tau\sigma\tau,$$

and likewise

$$\alpha^{-1} = \tau\sigma\tau\sigma, \qquad \beta^{-1} = \tau^2\sigma\tau^2\sigma.$$

Thus, any word in the alphabet $\{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}$ can be regarded as a word in $\{\sigma, \tau, \tau^2\}$, and therefore as a word in the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. To prove that α and β are independent, it suffices to show that every nontrivial reduced word in α, β, α^{-1}, β^{-1} corresponds to a nontrivial reduced word in $\mathbb{Z}_2 * \mathbb{Z}_3$. This means, we need to check that no cancellation can reduce a nontrivial word in α and β to the identity in $\mathbb{Z}_2 * \mathbb{Z}_3$. Since we only consider reduced words, adjacent inverse pairs such as $\alpha\alpha^{-1}$ and $\beta^{-1}\beta$ do not occur, so we examine the remaining pairings. The pairs $\alpha\alpha, \beta\beta, \alpha\beta, \beta\alpha, \alpha\beta^{-1}, \beta\alpha^{-1}$ produce no cancellations when written in σ, τ, τ^2. The leftover pairs reduce as follows:

$$\alpha^{-1}\beta = \tau\sigma\tau\sigma\,\sigma\tau\sigma\tau = \tau\sigma\tau^2\sigma\tau, \qquad \beta^{-1}\alpha = \tau^2\sigma\tau^2\sigma\,\sigma\tau^2\sigma\tau^2 = \tau^2\sigma\tau\sigma\tau^2.$$

Although these expressions simplify, they remain nontrivial and reduced in $\mathbb{Z}_2 * \mathbb{Z}_3$.

By Proposition 5.2.1, no nontrivial reduced word in $\mathbb{Z}_2 * \mathbb{Z}_3$ represents the identity. Hence every nontrivial reduced word in α and β remains nontrivial when rewritten in σ, τ, and τ^2. Therefore, α and β are independent. □

Since F acts on $\mathbb{H}^+$ without nontrivial fixed points, Proposition 2.4.3 implies, using the Axiom of Choice, that $\mathbb{H}^+$ is F-paradoxical. In the present setting, however, the paradoxical decomposition can again be described explicitly using a fundamental domain for the action of F. We now construct this domain and the associated tessellation.

Because F is a subgroup of $PSL_2(\mathbb{Z})$, F is discrete. Therefore, we can once more work with the Dirichlet domain. Since F acts on $\mathbb{H}^+$ without nontrivial fixed points, every point of $\mathbb{H}^+$ is a valid choice of center for the Dirichlet domain but. We choose the center $z_0 = i$. The orbit points nearest to z_0 are

$$\alpha(i) = \frac{2}{5} + \frac{1}{5}i, \quad \alpha^{-1}(i) = -\frac{2}{5} + \frac{1}{5}i, \quad \beta(i) = 2 + i \quad \text{and} \quad \beta^{-1}(i) = -2 + i.$$

Thus, the Dirichlet domain is bounded by the following four hyperbolic lines:

- $C_1 = \{z : d_{\mathbb{H}^+}(z, i) = d_{\mathbb{H}^+}(z, \alpha(i))\} = \{z : d_{\mathbb{H}^+}(z, i) = d_{\mathbb{H}^+}(z, \frac{2}{5} + \frac{1}{5}i)\} = \{z : |z - \frac{1}{2}| = \frac{1}{2}\}$. Indeed, since $Im(i) = 1$ and $Im(\alpha(i)) = \frac{1}{5}$, the condition $d_{\mathbb{H}^+}(z, i) = d_{\mathbb{H}^+}(z, \alpha(i))$ is equivalent to

  $$x^2 + (y-1)^2 = 5\Big[\big(x - \tfrac{2}{5}\big)^2 + \big(y - \tfrac{1}{5}\big)^2\Big].$$

  A straightforward expansion and simplification yields

  $$x^2 + y^2 - x = 0,$$

  which can be rewritten as

  $$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

- $C_2 = \{z : d_{\mathbb{H}^+}(z, i) = d_{\mathbb{H}^+}(z, \alpha^{-1}(i))\} = \{z : |z + \frac{1}{2}| = \frac{1}{2}\}$ by the same argument.

- $C_3 = \{z : d_{\mathbb{H}^+}(z,i) = d_{\mathbb{H}^+}(z,\beta(i))\} = \{z : d_{\mathbb{H}^+}(z,i) = d_{\mathbb{H}^+}(z,i+2)\} = \{z : Re(z) = 1\}$. Similarly, since $Im(i) = Im(\beta(i)) = 1$, the condition $d_{\mathbb{H}^+}(z,i) = d_{\mathbb{H}^+}(z,\beta(i))$ reduces to

$$x^2 + (y-1)^2 = (x-2)^2 + (y-1)^2,$$

hence

$$x^2 = (x-2)^2 \quad \Longleftrightarrow \quad x = 1.$$

- $C_4 = \{z : d_{\mathbb{H}^+}(z,i) = d_{\mathbb{H}^+}(z,\beta^{-1}(i))\} = \{z : Re(z) = -1\}$.

It follows that the Dirichlet domain for F is

$$\mathcal{F} = \left\{ z \in \mathbb{H}^+ : |z - \frac{1}{2}| \geq \frac{1}{2},\ |z + \frac{1}{2}| \geq \frac{1}{2},\ |\Re(z)| \leq 1 \right\}$$

and therefore, a fundamental domain is given by the polygon

$$P = \mathcal{F}^\circ = \left\{ z \in \mathbb{H}^+ : |z - \frac{1}{2}| > \frac{1}{2},\ |z + \frac{1}{2}| > \frac{1}{2},\ |\Re(z)| < 1 \right\}$$

This resulting tessellation is shown in Figure 5.6.

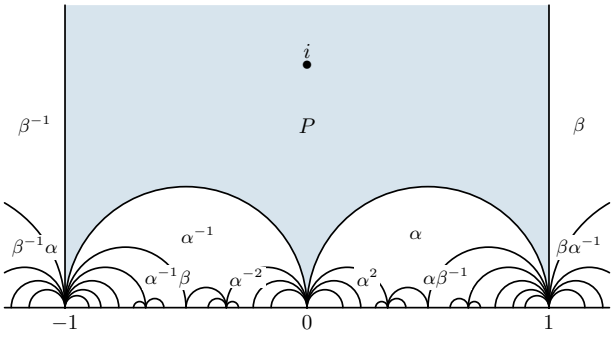

Figure 5.6: Tessellation of $\mathbb{H}^+$ for the action of the group $F = \langle \alpha, \beta \rangle$.

Theorem 5.3.2 (Banach-Tarski Paradox for Hyperbolic Space). The hyperbolic plane $\mathbb{H}^+$ is F-paradoxical.

Proof. To transfer the paradoxical decomposition of F to $\mathbb{H}^+$, we first have to construct a set M containing exactly one representative from each F-orbit in $\mathbb{H}^+$. Since we have an explicit fundamental polygon for the action of F, such a set can again be constructed directly and without invoking the Axiom of Choice.

Recall that the sides of the fundamental polygon are labeled $C_1, \ldots, C_4$, and that

$$\alpha(C_2) = C_1, \qquad \text{and} \qquad \beta(C_4) = C_3.$$

The action of G therefore identifies C_1 with C_2 and C_3 with C_4. Define the set

$$M = P \cup C_2 \cup C_4.$$

By excluding the sides C_1 and C_3, we make sure that the translates of M are disjoint. This way, no two points of M lie in the same F-orbit and every orbit intersects M, so the set M is indeed a choice set.

We may now once more apply the transfer argument of Proposition 2.4.3, using the choice set M.

Since F is a free group of rank 2, it is paradoxical and therefore admits the paradoxical decomposition

$$F = W(\alpha) \cup \alpha W(\alpha^{-1}) = W(\beta) \cup \beta W(\beta^{-1}).$$

Using the choice set M, define

$$A_1^* = \bigcup_{s \in W(\alpha)} s(M), \qquad A_2^* = \bigcup_{s \in W(\alpha^{-1})} s(M),$$

$$B_1^* = \bigcup_{t \in W(\beta)} t(M), \qquad B_2^* = \bigcup_{t \in W(\beta^{-1})} t(M).$$

Exactly as in Proposition 2.4.3, these sets satisfy

$$\mathbb{H}^+ = A_1^* \cup \alpha(A_2^*) = B_1^* \cup \beta(B_2^*).$$

Hence, $\mathbb{H}^+$ is F–paradoxical. □

Figure 5.7 illustrates the paradoxical decomposition of $\mathbb{H}^+$ obtained by transferring the paradoxical decomposition of the free group $F = \langle \alpha, \beta \rangle$ using the fundamental polygon P.

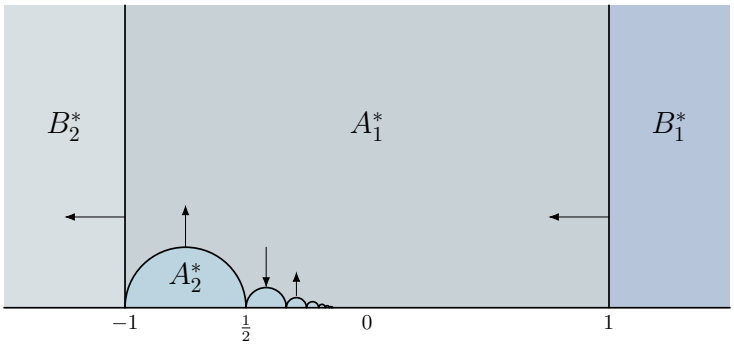

Figure 5.7: Visualization of the paradoxical decomposition of $\mathbb{H}^+$. The sets include boundaries as indicated by arrows.

Unlike the classical Banach-Tarski paradox for Euclidean balls, the paradoxical decompositions established in this chapter can be constructed explicitly from hyperbolic polygons and therefore do not require the Axiom of Choice. Since the pieces are geometrically well behaved, one may wonder why this does not contradict the existence of hyperbolic area. The key point is that the construction involves the entire hyperbolic plane, which already has infinite area. Intuitively, the paradox therefore corresponds only to the harmless identity

$$\infty = 2 \cdot \infty.$$

However, this observation alone does not explain why such constructions are possible in the hyperbolic setting. Indeed, Euclidean spaces also have infinite volume, but analogous paradoxical decompositions using such explicit pieces do not exist there. The reason for this distinction is measure-theoretic in nature and falls outside the scope of this thesis. We refer the reader to [TW16, Chapter 13.4] for further discussion.

Chapter 6

Paradoxical decompositions and amenable group actions

In the previous chapters, we saw that paradoxical decompositions of a space were obtained by first constructing paradoxical decompositions on the level of a group and then transferring them to the space on which the group acts. A similar transference principle appears in the study of invariant measures. Roughly speaking, if a group admits a finitely additive invariant probability measure, then such a measure can be transferred to spaces on which the group acts. The existence of such invariant measures prevents paradoxical decompositions and therefore provides a criterion for the absence of Banach-Tarski-type phenomena. It was von Neumann, in response to the Banach-Tarski paradox, who introduced the notion of amenability to formalize this idea.

In this chapter, we introduce amenable groups and amenable group actions, explain their connection to paradoxical decompositions, and apply these ideas to show that the Banach-Tarski-type paradoxes cannot occur in dimensions 1 and 2. The main references for this chapter are [Jus20, Chapter 2] and [Gar13, Chapters 1.2 and 2].

6.1 Amenability

To define amenability, we first introduce finitely additive invariant measures.

Definition 6.1.1. Let X be a set and $\mathcal{P}(X)$ denote its power set. A function

$$\nu : \mathcal{P}(X) \to [0, \infty]$$

is called a *finitely additive measure* if:

- $\nu(\varnothing) = 0$
- (finite additivity) for all disjoint sets $A, B \in \mathcal{P}(X)$,

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

The measure ν is called a *probability measure* if, in addition $\nu(X) = 1$.

Definition 6.1.2. Let G be a group acting on a set X.

1. The action $G \curvearrowright X$ is called *amenable* if there exists a finitely additive probability measure

$$\mu : \mathcal{P}(X) \to [0, 1]$$

such that

$$\mu(gA) = \mu(A) \qquad \text{for all } g \in G \text{ and } A \subseteq X.$$

2. The group G is called *amenable* if its left regular action on itself is amenable. Equivalently, G is amenable if there exists a left-invariant finitely additive probability measure on $\mathcal{P}(G)$.

We begin with two fundamental examples, one amenable and one non-amenable.

Example 6.1.3. Finite groups provide the simplest examples of amenable groups. Indeed, if G is finite, define for each subset $A \subseteq G$

$$\mu(A) := \frac{|A|}{|G|}.$$

Then μ is a finitely additive probability measure on G satisfying $\mu(gA) = \mu(A)$ for all $g \in G$ and all subsets $A \subseteq G$, since left translation preserves cardinality. Therefore, G is amenable.

Example 6.1.4. The free group F of rank 2 is the standard example of a non-amenable group. Let $\sigma, \tau \in F$ be free generators. We can decompose F as

$$F \;=\; \{e\} \;\cup\; W(\sigma) \;\cup\; W(\sigma^{-1}) \;\cup\; W(\tau) \;\cup\; W(\tau^{-1}).$$

Assume, for the sake of contradiction, that F admits a left-invariant finitely additive probability measure μ. Note that for every $x \in \{\sigma, \sigma^{-1}, \tau, \tau^{-1}\}$ we have the identity

$$W(x) = x\big(F \setminus W(x^{-1})\big).$$

Applying invariance of μ yields

$$\mu\big(W(x)\big) = \mu\Big(F \setminus W(x^{-1})\Big) = 1 - \mu\big(W(x^{-1})\big),$$

and thus

$$\mu\big(W(x)\big) + \mu\big(W(x^{-1})\big) = 1.$$

By left-invariance of μ, every singleton has the same measure. Since F is infinite, finite additivity forces this measure to be 0. Indeed, if $\mu(\{e\}) = c > 0$, then a finite set of n distinct elements would have measure nc, which exceeds 1 for sufficiently large n, contradicting $\mu(F) = 1$. Thus $\mu(\{e\}) = 0$.

Now compute $\mu(F)$ using the above identities:

$$\begin{aligned} 1 &= \mu(F) \\ &= \mu(\{e\}) + \mu(W(\sigma)) + \mu(W(\sigma^{-1})) + \mu(W(\tau)) + \mu(W(\tau^{-1})) \\ &= \mu(\{e\}) + 1 + 1 \\ &= 2 \end{aligned}$$

which is a contradiction. Therefore, the free group F cannot be amenable.

The next proposition shows that invariant measures can be transferred from groups to spaces on which the group acts, mirroring the transfer of paradoxical decompositions discussed in earlier chapters.

Proposition 6.1.5. Let G be an amenable group and let G act on a set X. Then the action of G on X is amenable.

Proof. The proof uses an idea from [TW16, Theorem 12.3]. Since G is amenable, there exists a left-invariant finitely additive probability measure $\mu_G : \mathcal{P}(G) \longrightarrow [0, 1]$. Fix a point $x_0 \in X$. For each subset $A \subseteq X$, define

$$\nu(A) := \mu_G\big(\{g \in G : g \cdot x_0 \in A\}\big).$$

Intuitively, ν is obtained by transferring the invariant measure μ_G from the group G to the orbit of x_0: a set $A \subseteq X$ is assigned the μ_G-measure of the elements of G that move x_0 into A.

We verify that ν is a G-invariant finitely additive probability measure on X.

Normalization. Since $\{g \in G : g \cdot x_0 \in X\} = G$, we obtain

$$\nu(X) = \mu_G(G) = 1.$$

Finite additivity. Let $A, B \subseteq X$ be disjoint. Then

$$\{g \in G : g \cdot x_0 \in A \cup B\} = \{g \in G : g \cdot x_0 \in A\} \cup \{g \in G : g \cdot x_0 \in B\}.$$

These sets are disjoint subsets of G, hence by finite additivity of μ_G,

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

G-invariance. Let $h \in G$ and $A \subseteq X$. Then

$$\begin{aligned}
\nu(hA) &= \mu_G\big(\{g \in G : g \cdot x_0 \in hA\}\big) \\
&= \mu_G\big(\{g \in G : h^{-1}g \cdot x_0 \in A\}\big) \\
&= \mu_G\big(\{hk \in G : k \cdot x_0 \in A\}\big) \\
&= \mu_G\big(h\{k \in G : k \cdot x_0 \in A\}\big) \\
&= \mu_G\big(\{k \in G : k \cdot x_0 \in A\}\big) \\
&= \nu(A),
\end{aligned}$$

by G-invariance of μ_G.

Thus, ν is a G-invariant finitely additive probability measure on X, and the action is amenable. □

Remark 10. The converse of the previous proposition does not hold. Indeed, every (non-amenable) group admits an amenable action on a one-point set. If $X = \{*\}$, the action defined by

$$g \cdot * = * \qquad \text{for all } g \in G$$

is amenable, since the measure $\mu(\{*\}) = 1$ is a G-invariant finitely additive probability measure on X.

6.2 Operations that preserve amenability

For later applications, we record several standard permanence properties of amenable groups. We begin by showing that amenability passes to subgroups.

Proposition 6.2.1 (AC). Let G be an amenable group. Then every subgroup $H \subseteq G$ is amenable.

Proof. Since G is amenable, there exists a left-invariant finitely additive probability measure

$$\mu : \mathcal{P}(G) \longrightarrow [0, 1].$$

Let $H \subseteq G$ be a subgroup. By the Axiom of Choice, there exists a set $M \subseteq G$ containing exactly one representative from each left coset of H in G, so that

$$G = \bigcup_{m \in M} Hm$$

is a partition of G. Thus, every element of G can be written uniquely in the form hm with $h \in H$ and $m \in M$. Define a function $\nu : \mathcal{P}(H) \to [0, 1]$ by

$$\nu(A) := \mu(AM), \quad \forall A \subseteq H.$$

where

$$AM = \bigcup_{m \in M} Am \subseteq G.$$

We verify that ν is a left-invariant finitely additive probability measure on H.
Normalization.

$$\nu(H) = \mu(HM) = \mu(G) = 1.$$

Finite additivity. If $A, B \subseteq H$ are disjoint, then the sets AM and BM are disjoint subsets of G. Hence,

$$\nu(A \cup B) = \mu((A \cup B)M) = \mu(AM \cup BM) = \mu(AM) + \mu(BM) = \nu(A) + \nu(B).$$

H-invariance. For any $h \in H$ and $A \subseteq H$, we have

$$\nu(hA) = \mu((hA)M) = \mu(h(AM)) = \mu(AM) = \nu(A),$$

where the third equality follows from left-invariance of μ.

Thus, ν is a left-invariant finitely additive probability measure on H, so H is amenable. □

Corollary 6.2.2 (AC). If G contains a non-amenable subgroup H, then G is non-amenable.

Since the free group F is non-amenable, the above corollary implies that every group containing a free subgroup of rank 2 is non-amenable. In particular, the rotation groups $SO_n(\mathbb{R})$ for $n \geq 3$ are non-amenable, and hence so are the Euclidean isometry groups G_n for $n \geq 3$.

Proposition 6.2.3. Let $N \triangleleft G$ be a normal subgroup. If N and G/N are amenable, then G is amenable.

Proof. Let μ_1 and μ_2 be left-invariant finitely additive probability measures on N and G/N, respectively. Note that here G/N denotes the quotient group of G by the normal subgroup N, that is, the group of left cosets gN with multiplication $(gN)(hN) = (gh)N$.

We use the invariant measures on N and G/N to construct an invariant measure on G. For a subset $A \subseteq G$, define a function $f_A : G \to [0,1]$ by

$$f_A(g) := \mu_1\big(N \cap g^{-1}A\big).$$

Intuitively, $f_A(g)$ measures how large the intersection of A with the coset gN is, using the invariant measure on N. Since μ_1 is left-invariant, the value of $f_A(g)$ depends only on the coset gN, so f_A may be viewed as a function on the quotient G/N.

We now average these values over all cosets using the invariant measure μ_2 on G/N. Using the standard integration construction for finitely additive measures described in [KT12, Chapter 2], we define a measure

$$\mu : \mathcal{P}(G) \to [0,1]$$

by

$$\mu(A) := \int_{G/N} f_A \, d\mu_2.$$

We verify that μ is a left-invariant finitely additive probability measure on G.
Finite additivity. Let $A, B \in \mathcal{P}(G)$ be disjoint. Then

$$\mu(A \cup B) = \int_{G/N} f_{A \cup B} \, d\mu_2 = \int_{G/N} (f_A + f_B) \, d\mu_2 = \int_{G/N} f_A \, d\mu_2 + \int_{G/N} f_B \, d\mu_2,$$

since

$$\begin{aligned} f_{A\cup B}(g) &= \mu_1\big(N \cap g^{-1}(A \cup B)\big) \\ &= \mu_1\big((N \cap g^{-1}A) \cup (N \cap g^{-1}B)\big) \\ &= \mu_1\big(N \cap g^{-1}A\big) + \mu_1\big(N \cap g^{-1}B\big) \end{aligned}$$

by finite additivity of μ_1. Thus, μ is finitely additive.
Normalization. We have

$$\mu(G) = \int_{G/N} f_G \, d\mu_2 = \int_{G/N} 1 \, d\mu_2 = \mu_2(G/N) = 1,$$

since

$$f_G(g) = \mu_1(N \cap g^{-1}G) = \mu_1(N \cap G) = \mu_1(N) = 1.$$

Left-invariance. Let $x \in G$. Then

$$f_{xA}(g) = \mu_1(N \cap g^{-1}xA) = \mu_1(N \cap (x^{-1}g)^{-1}A) = f_A(x^{-1}g),$$

and therefore

$$\begin{aligned} \mu(xA) &= \int_{G/N} f_{xA}(gN) \, d\mu_2(gN) \\ &= \int_{G/N} f_A(x^{-1}gN) \, d\mu_2(gN) \\ &= \int_{G/N} f_A((xN)^{-1}gN) \, d\mu_2(gN). \end{aligned}$$

Since μ_2 is left-invariant, we conclude

$$\mu(xA) = \int_{G/N} f_A((xN)^{-1}gN) \, d\mu_2((xN)^{-1}gN) = \mu(A).$$

Hence, μ is a left-invariant finitely additive probability measure on G, and therefore G is amenable. □

The following finite-index criterion can be found in [Tho11a].

Corollary 6.2.4. Let $H \subseteq G$ be a subgroup of finite index. If H is amenable, then G is amenable.

Proof. Since H has finite index in G, the set of left cosets G/H is finite. The map

$$gH \mapsto gHg^{-1}$$

is surjective onto the set of conjugates of H, so H has only finitely many distinct conjugates in G. Define

$$N \coloneqq \bigcap_{g\in G} gHg^{-1}.$$

Equivalently, this is the intersection of the finitely many distinct conjugates of H. Hence N has finite index in G. Moreover, N is normal in G by construction, and since H itself occurs among the conjugates, we have $N \subseteq H$.

Since amenability passes to subgroups, N is amenable. Furthermore, G/N is finite, because N has finite index in G, and hence G/N is amenable. By Proposition 6.2.3, it follows that G is amenable. □

6.3 Connection to paradoxical decompositions

We now establish the fundamental connection between amenability and paradoxical decompositions. Informally, amenability expresses the existence of an invariant notion of size, whereas paradoxicality corresponds to the possibility of duplicating a set by decomposing and rearranging it. The following theorem shows that these two phenomena are incompatible: an amenable action cannot admit a paradoxical decomposition.

Theorem 6.3.1. If the action of a group G on a set X is amenable, then X is not G-paradoxical.

Proof. By amenability of the action, there exists a finitely additive, G-invariant probability measure μ on X. Assume for contradiction that the action is paradoxical. Then there exist two disjoint subsets $A, B \in X$ such that

$$X = A \cup B, \qquad A \sim_G X, \qquad B \sim_G X.$$

Let

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{j=1}^{m} B_j$$

be equidecompositions of A and B with X, so that

$$X = \bigcup_{i=1}^{n} g_i(A_i), \qquad X = \bigcup_{j=1}^{m} h_j(B_j)$$

for suitable elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$.

Using finite additivity and G-invariance of μ, we obtain

$$\begin{aligned} 1 = \mu(X) &= \mu(A) + \mu(B) \\ &= \sum_{i=1}^{n} \mu(A_i) + \sum_{j=1}^{m} \mu(B_j) \\ &= \sum_{i=1}^{n} \mu(g_i(A_i)) + \sum_{j=1}^{m} \mu(h_j(B_j)) \\ &= \mu\left(\bigcup_{i=1}^{n} g_i(A_i)\right) + \mu\left(\bigcup_{j=1}^{m} h_j(B_j)\right) \\ &= \mu(X) + \mu(X) \\ &= 2, \end{aligned}$$

a contradiction.

Hence, the action cannot be paradoxical. □

Remarkably, the converse of this theorem holds as well. This is the content of Tarski's theorem, which characterizes amenability precisely in terms of the absence of paradoxical decompositions. We state the theorem without proof and refer to [TW16, Corollary 11.2] for details.

Theorem 6.3.2 (Tarski's Theorem (AC)). Let G act on a set X. The following are equivalent:

1. The action of G on X is amenable.
2. X is not G-paradoxical.

Applying Tarski's theorem to the action of G on itself by left multiplication yields the classical group version: a group G is amenable if and only if it is not paradoxical.

Remark 11 (Von Neumann's conjecture [Tho11a])**.** As we have seen, every group containing a free subgroup of rank 2 is non-amenable. This led von Neumann to conjecture that the converse should also hold, namely that every non-amenable group contains a free subgroup of rank 2. By Tarski's theorem, this is equivalent to the statement that a group is paradoxical if and only if it contains a free subgroup.

This conjecture is false. The first counterexamples were constructed by Ol'shanskii in 1980. His so-called *Tarski monster groups* are infinite groups in which every proper nontrivial subgroup is cyclic of prime order. In particular, they do not contain any free subgroup, yet they are non-amenable and thus paradoxical.

Nevertheless, the conjecture does hold in important classes of groups. A fundamental result in this direction is given by a special case of the Tits Alternative (Thm 7.1.6), which states that every group $G \in GL_n(\mathbb{R})$ either contains a free subgroup of rank 2 or has a solvable subgroup of finite index. As we shall see in Chapter 7, the latter condition implies amenability. Thus, for linear groups, non-amenability is closely tied to the existence of free subgroups.

6.4 Amenability of Abelian Groups

We now turn to the most important class of amenable groups, namely abelian groups. Their amenability will later allow us to show that certain paradoxical decompositions that occur in higher dimensions do not occur in dimensions 1 and 2.

The proof proceeds by constructing invariant measures via compactness arguments [Wri94]. We therefore first recall several topological results that will be needed in the construction and state them without proof. For proofs of the following results, we refer to [Wri94].

Definition 6.4.1. Let X be a topological space. A collection C of subsets of X is said to have the *finite intersection property*, or short FIP, if for every finite subcollection $\{C_1, ..., C_n\}$ of C, the intersection $C_1 \cap ... \cap C_n$ is nonempty.

Theorem 6.4.2. A topological space X is compact if and only if every collection K of closed subsets of X with FIP has the property that the intersection of all the sets in K is nonempty.

Theorem 6.4.3 (Tychonoff's Theorem)**.** An arbitrary product of compact spaces is compact in the product topology.

To prove that every abelian group is amenable, we first show that amenability is preserved under directed unions and finite direct products. This reduces the problem to proving amenability of the infinite cyclic group $\mathbb{Z}$.

Lemma 6.4.4 (**AC**)**.** If all groups in a directed system $\{G_i\}_{i\in I}$ are amenable, then so is their directed union $G = \bigcup_{i\in I} G_i$.

Proof. Suppose that

$$G = \bigcup_{i\in I} G_i$$

is the directed union of subgroups G_i, meaning that for every $i, j \in I$ there exists $k \in I$ such that

$$G_i, G_j \subseteq G_k.$$

Note that G itself is a group. We denote by $[0,1]^{\mathcal{P}(G)}$ the set of all functions from $\mathcal{P}(G)$ to $[0,1]$, equipped with the product topology. For each $i \in I$, let μ_i be a finitely additive left-invariant probability measure on G_i. Define

$$M_i := \{\mu : \mathcal{P}(G) \to [0,1] : \mu \text{ is finitely additive, } \mu(G) = 1, \mu(gA) = \mu(A)\ \forall g \in G_i\}.$$

Each set M_i is a nonempty subset of $[0,1]^{\mathcal{P}(G)}$. Indeed, given μ_i we may define $\tilde{\mu}_i : \mathcal{P}(G) \to [0,1]$ by

$$\tilde{\mu}_i(A) := \mu_i(A \cap G_i),$$

which is clearly finitely additive and invariant under G_i since G_i is a subgroup of G, so $\tilde{\mu}_i \in M_i$.

Observe that $[0,1]^{\mathcal{P}(G)}$ is compact by Tychonoff's theorem. Moreover, each M_i is closed, as it is given by the intersection of the closed conditions expressing finite additivity, normalization $\mu(G) = 1$, and left invariance under elements of G_i. For example, the condition $\mu(G) = 1$ defines a closed subset: The evaluation map

$$\pi_G : [0,1]^{\mathcal{P}(G)} \to [0,1], \qquad \pi_G(\mu) = \mu(G),$$

is continuous in the product topology, and hence

$$\{\mu : \mu(G) = 1\} = \pi_G^{-1}(\{1\})$$

is closed. The other defining conditions are treated similarly.

Moreover, if $G_i, G_j \subseteq G_k$, then

$$M_k \subseteq M_i \cap M_j,$$

since invariance under G_k implies invariance under G_i and G_j. Hence, the family $\{M_i\}_{i \in I}$ of closed subsets of $[0,1]^{\mathcal{P}(G)}$ has the finite intersection property. By compactness of $[0,1]^{\mathcal{P}(G)}$,

$$\bigcap_{i \in I} M_i \neq \varnothing.$$

Therefore, there exists a finitely additive measure $\mu \in \bigcap_{i \in I} M_i$ which is invariant under all elements of G, showing that G is amenable. □

Lemma 6.4.5. Let G and H be amenable groups. Then their direct product $G \times H$ is amenable. More generally, any finite direct product of amenable groups is amenable.

Proof. Let G and H be amenable. Consider the subgroup

$$N := \{e_G\} \times H \subseteq G \times H,$$

which is clearly normal in $G \times H$ and isomorphic to H. Since H is amenable, N is an amenable normal subgroup of $G \times H$. The quotient group satisfies

$$(G \times H)/N \cong G,$$

which is amenable by assumption. Therefore, by Proposition 6.2.3, we conclude that $G \times H$ is amenable.

The general case for a finite direct product follows by induction on the number of factors. □

We can now prove the main result of this section.

Proposition 6.4.6. Every abelian group is amenable.

Proof. Let G be an abelian group. Every element $g \in G$ lies in the cyclic subgroup it generates, and hence G is the union of its finitely generated subgroups. Moreover, if H_1 and H_2 are finitely generated subgroups, say $H_i = \langle F_i \rangle$, then the subgroup $\langle F_1 \cup F \rangle$ is again finitely generated and contains both H_1 and H_2. Thus, the collection of finitely generated subgroups of G is a directed system under inclusion. By Lemma 6.4.4, it therefore suffices to consider finitely generated abelian groups. By the fundamental theorem of finitely generated abelian groups [Rom11, Theorem 13.4],

$$H \cong \mathbb{Z}^n \times \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z},$$

where $n \geq 0$ and $d_1 \mid d_2 \mid \cdots \mid d_k$. Let $T = \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z}$. Since T is finite, by Example 6.1.3, T is also amenable.

By Lemma 6.4.5, finite direct products of amenable groups are amenable. Hence, once $\mathbb{Z}$ is shown to be amenable, it follows that $\mathbb{Z}^n$ and therefore also H is amenable.

To show amenability of $\mathbb{Z}$, fix a generator a of $\mathbb{Z}$. We first construct measures that are approximately invariant under translation, and then obtain an invariant measure by compactness arguments. Given $\varepsilon > 0$, we construct a finitely additive probability measure μ_ε on $\mathcal{P}(\mathbb{Z})$ which is "almost invariant" under translation by a, meaning

$$\left|\mu_\varepsilon(A) - \mu_\varepsilon(aA)\right| \leq \varepsilon \quad \text{for all } A \subseteq \mathbb{Z}.$$

Set

$$M_\varepsilon := \left\{\mu : \mathcal{P}(\mathbb{Z}) \to [0,1] : \mu \text{ is finitely additive, } \mu(\mathbb{Z}) = 1 \text{ and } \left|\mu(A) - \mu(aA)\right| \leq \varepsilon \text{ for all } A \subseteq \mathbb{Z}\right\}$$

Our goal is to show that $M_\varepsilon \neq \emptyset$. For this, we choose $N \in \mathbb{N}$ such that $\frac{2}{N} \leq \varepsilon$, and define for any $A \subseteq \mathbb{Z}$ the quantity

$$\mu_\varepsilon(A) := \frac{|\{\, i \mid 1 \leq i \leq N,\ a^i \in A \,\}|}{N}.$$

This μ_ε is finitely additive and satisfies $\mu_\varepsilon(\mathbb{Z}) = 1$. Moreover,

$$\mu_\varepsilon(aA) = \frac{|\{\, i \mid 1 \leq i \leq N,\ a^i \in aA \,\}|}{N} = \frac{|\{\, i \mid 2 \leq i \leq N+1,\ a^i \in A \,\}|}{N}.$$

The two counting sets differ by at most two elements, namely at the endpoints of the interval $\{1, \ldots, N\}$. Therefore,

$$|\mu_\varepsilon(A) - \mu_\varepsilon(aA)| \leq \frac{2}{N} \leq \varepsilon,$$

so $\mu_\varepsilon \in M_\varepsilon$. Consequently, each M_ε is nonempty and closed in $[0,1]^{\mathcal{P}(\mathbb{Z})}$ by the same closure arguments as in Lemma 6.4.4. Moreover, the family $\{M_\varepsilon\}_{\varepsilon>0}$ has the finite intersection property, since

$$\bigcap_{i=1}^{n} M_{\varepsilon_i} = M_{\min\{\varepsilon_1,\ldots,\varepsilon_n\}}.$$

By compactness of $[0,1]^{\mathcal{P}(\mathbb{Z})}$, the intersection

$$\bigcap_{\varepsilon>0} M_\varepsilon \neq \varnothing.$$

Every μ in this intersection satisfies

$$\mu(A) = \mu(aA) \qquad \text{for all } A \subseteq \mathbb{Z}.$$

Hence, μ is a left-invariant finitely additive probability measure on $\mathbb{Z}$. We conclude that $\mathbb{Z}$ is amenable, and therefore also $\mathbb{Z}^n$ is amenable. Recall that

$$H \cong \mathbb{Z}^n \times T.$$

Lemma 6.4.5 implies that every finitely generated abelian group is amenable. Since amenability passes to directed unions, it follows that G is amenable. □

6.5 The failure of paradoxes in dimensions 1 and 2

We now apply the theory of amenability to the Euclidean isometry groups in lower dimensions. In Chapter 4, paradoxical decompositions in higher dimensions arose from the existence of a free subgroup in $SO_3(\mathbb{R})$. We will show that this mechanism breaks down in dimensions 1 and 2 because the Euclidean isometry groups of $\mathbb{R}$ and $\mathbb{R}^2$ are amenable.

The proofs of the following two propositions are inspired by [Tho11b].

Proposition 6.5.1. The group G_1 is amenable.

Proof. The isometry group of the real line consists of all maps of the form $x \mapsto \pm x + a$ for $a \in \mathbb{R}$. The translations form a normal subgroup of G_1 isomorphic to $\mathbb{R}$. Since every isometry has the form

$$x \mapsto \varepsilon x + a, \qquad \varepsilon \in \pm 1,$$

the quotient records the sign of the linear part. Hence,

$$G_1/\mathbb{R} \cong \mathbb{Z}_2.$$

The group $\mathbb{R}$ is abelian, hence amenable. The quotient $G_1/\mathbb{R} \cong \mathbb{Z}_2$ is finite and therefore amenable. By Proposition 6.2.3, it follows that G_1 is amenable. □

Proposition 6.5.2. The group G_2 is amenable.

Proof. Every Euclidean isometry of the plane can be written uniquely in the form

$$x \mapsto Ax + b,$$

where $A \in O_2(\mathbb{R})$ and $b \in \mathbb{R}^2$. The translations form a normal subgroup of G_2 isomorphic to $\mathbb{R}^2$. Two isometries lie in the same coset modulo translations precisely when they have the same linear part A. Hence, quotienting by translations records only the orthogonal part of the isometry, and therefore

$$G_2/\mathbb{R}^2 \cong O_2(\mathbb{R}).$$

We first show that $O_2(\mathbb{R})$ is amenable. The subgroup $SO_2(\mathbb{R})$ of rotations is abelian and therefore amenable. Moreover, the determinant map

$$\det : O_2(\mathbb{R}) \to \{\pm 1\} \cong \mathbb{Z}_2$$

is a surjective homomorphism with kernel $SO_2(\mathbb{R})$. Hence

$$O_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{Z}_2.$$

Since $SO_2(\mathbb{R})$ is normal in $O_2(\mathbb{R})$ and both $\mathbb{Z}_2$ and $SO_2(\mathbb{R})$ are amenable, it follows that $O_2(\mathbb{R})$ is amenable.

We now same argument to to G_2. The subgroup $\mathbb{R}^2$ of translations is abelian and therefore amenable and the quotient $G_2/\mathbb{R}^2$ is isomorphic to $O_2(\mathbb{R})$, which is amenable. Hence, G_2 is amenable. □

We have shown that both G_1 and G_2 are amenable. We now explain what this implies for paradoxical decompositions in low-dimensional Euclidean geometry.

Since amenability passes to subgroups, the relevant subgroups acting on simple geometric objects are amenable as well. For instance, the group $SO_2(\mathbb{R})$ acts naturally on all centered circles and disks. Hence these actions are amenable, and therefore neither the circle nor the disk is $SO_2(\mathbb{R})$-paradoxical. Similarly, in dimension 1, the Euclidean isometry groups acting on intervals are amenable, so intervals do not admit such decompositions with respect to these actions.

Moreover, the full Euclidean isometry groups G_1 and G_2 act naturally on the ambient spaces. Since G_1 is amenable, its action on $\mathbb{R}$ is amenable, and hence $\mathbb{R}$ is not G_1-paradoxical. Likewise, the plane $\mathbb{R}^2$ is not G_2-paradoxical. Thus, the usual Banach-Tarski type phenomena already fail in dimensions 1 and 2 when one restricts to Euclidean isometries.

There is, however, a stronger geometric statement one may ask for: namely, whether every bounded subset with nonempty interior is non-paradoxical. To understand why the amenability argument above does not immediately yield this stronger conclusion, we examine the invariant measures arising from amenability more closely.

For example, amenability of the action

$$G_2 \curvearrowright \mathbb{R}^2$$

gives a G_2-invariant finitely additive probability measure

$$\mu : \mathcal{P}(\mathbb{R}^2) \to [0, 1].$$

If a set $A \subseteq \mathbb{R}^2$ satisfies $\mu(A) > 0$, then A cannot be paradoxical, since finite additivity and invariance would imply

$$\mu(A) = 2\mu(A),$$

which is impossible.

Nevertheless, amenability alone does not guarantee that geometrically meaningful sets, such as bounded sets with nonempty interior, have positive measure with respect to μ. In principle, one could still have

$$\mu(A) = 0$$

for many such sets. In fact, a finitely additive probability measure on $\mathbb{R}^2$ necessarily vanishes on bounded sets. Indeed, if a bounded set had positive measure, then one could take arbitrarily many pairwise disjoint translates of it, contradicting the fact that the total measure of $\mathbb{R}^2$ is 1.

To rule out Banach-Tarski type decompositions for all bounded sets with nonempty interior, one needs a measure which is defined on all subsets and still agrees with geometric volume on ordinary sets. Since the pieces appearing in a Banach-Tarski decomposition are not Lebesgue measurable, the usual notion of volume is insufficient for this purpose. This role is instead played by finitely additive invariant extensions of geometric volume. Such measures assign positive finite measure to every bounded set with nonempty interior and remain invariant under Euclidean isometries. The same argument as above then shows that no such set can be paradoxical.

In dimensions 1 and 2, such invariant extensions exist and therefore rule out Banach-Tarski type phenomena for arbitrary bounded sets with nonempty interior. In dimensions at least 3, however, the Banach-Tarski paradox implies that no such invariant extensions can exist for the full Euclidean isometry group.

Since the construction of these invariant extensions lies beyond the scope of this thesis, we do not pursue this approach further. For a detailed treatment, see [TW16, Corollary 12.9].

It is important to note that the obstruction established in this section is not purely dimension but depends on the acting group. The results above show that paradoxical decompositions cannot occur when one restricts to Euclidean isometries in dimensions 1 and 2. They do not exclude paradoxical decompositions arising from different transformation groups.

Indeed, paradoxical decompositions do occur in two-dimensional hyperbolic geometry. In Chapter 5, we explicitly constructed paradoxical decompositions of the hyperbolic plane using isometries generated by a free subgroup of rank 2. Thus, unlike the Euclidean plane, the hyperbolic plane is paradoxical with respect to its group of hyperbolic isometries.

Chapter 7

Tits' Alternative and the Existence of Free Subgroups

In the previous chapters, free groups appeared repeatedly as the algebraic source of paradoxical decompositions. This naturally leads to the question which groups contain a free subgroup of rank 2. Tits' Alternative gives a remarkable answer to this question within the class of linear groups. It states that every linear group either contains a free subgroup of rank 2 or contains s solvable subgroup of finite index. In this way, it places the appearance of such groups into a broader algebraic context. This chapter is based on several sources, which are cited where they are used.

7.1 Solvable Groups and Statement of Tits' Alternative

The material in this section is based primarily on [Toi09, Chapter 0] and [TW16, Chapter 12]. Before stating Tits' Alternative, we recall the notions of solvable and virtually solvable groups and collect several basic properties of solvable groups.

Definition 7.1.1. A group G is called *solvable* if there exists a finite sequence of subgroups

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that each G_i is normal in G_{i+1} and each quotient G_{i+1}/G_i is abelian. We call G *virtually solvable* if it contains a solvable subgroup of finite index.

Proposition 7.1.2. Every subgroup of a solvable group is solvable.

Proof. The argument is inspired by the proof of Theorem C.9 in [New12]. Let G be solvable and $H \subseteq G$. Then there exists a chain

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each $G_i \triangleleft G_{i+1}$ and G_{i+1}/G_i abelian.

Define

$$H_i := H \cap G_i \quad \text{for } i = 0, \ldots, n.$$

Then

$$\{e\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H.$$

For H to be solvable, we need to show $H_i \triangleleft H_{i+1}$ and that H_{i+1}/H_i is abelian. First, let $h \in H_{i+1}$ and $x \in H_i$. Since $H_{i+1} \subseteq G_{i+1}$ and $G_i \triangleleft G_{i+1}$, we have

$$hxh^{-1} \in G_i.$$

Moreover, $h, x \in H$ and H is a subgroup, so

$$hxh^{-1} \in H.$$

Hence, $hxh^{-1} \in H \cap G_i = H_i$, showing that $H_i \triangleleft H_{i+1}$. Next, consider the restriction of the canonical projection

$$\varphi : H_{i+1} \to G_{i+1}/G_i, \quad h \mapsto hG_i.$$

This is a group homomorphism with the kernel

$$\ker(\varphi) = H_{i+1} \cap G_i = (H \cap G_{i+1}) \cap G_i = H \cap G_i = H_i,$$

since $G_i \subseteq G_{i+1}$.

By the First Isomorphism Theorem [Rom08, Theorem 3.5], we obtain

$$H_{i+1}/H_i \cong \varphi(H_{i+1}),$$

and $\varphi(H_{i+1})$ is a subgroup of G_{i+1}/G_i. Since G_{i+1}/G_i is abelian, it follows that H_{i+1}/H_i is abelian. Thus, H admits a subnormal series with abelian quotients and is therefore solvable. □

Proposition 7.1.3. Every subgroup of a virtually solvable group is virtually solvable.

Proof. Let G be virtually solvable and let $H \leq G$. Then there exists a solvable subgroup $S \leq G$ of finite index.

Since $H \cap S$ is a subgroup of the solvable group S, it is solvable. Moreover, distinct left cosets of $H \cap S$ in H are contained in distinct left cosets of S in G. Indeed, if

$$h_1 S = h_2 S$$

for $h_1, h_2 \in H$, then $h_2^{-1} h_1 \in S$. As $h_1, h_2 \in H$, we also have $h_2^{-1} h_1 \in H$, and therefore $h_2^{-1} h_1 \in H \cap S$. Hence

$$h_1(H \cap S) = h_2(H \cap S).$$

It follows that

$$[H : H \cap S] \leq [G : S] < \infty.$$

Therefore $H \cap S$ is a solvable subgroup of finite index in H, and thus H is virtually solvable. □

The significance of solvable groups for our purposes stems from their connection with amenability. The following proposition establishes this link.

Proposition 7.1.4. Every solvable group is amenable.

Proof. Let G be a solvable group with finite subnormal series

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G,$$

where each quotient G_{i+1}/G_i is abelian.

We proceed by induction along the series to show that each G_i is amenable. The trivial group G_0 is clearly amenable as it is finite.

Now suppose G_i is amenable. Since G_{i+1}/G_i is abelian, it is amenable by Proposition 6.4.6. Furthermore, G_i is a normal amenable subgroup of G_{i+1}, so the extension property (Proposition 6.2.3) implies that G_{i+1} is amenable.

By induction, it follows that $G_n = G$ is amenable. □

Corollary 7.1.5. Every virtually solvable group is amenable.

Proof. Let G be virtually solvable. Then G contains a solvable subgroup H of finite index. Since solvable groups are amenable, H is amenable. By Corollary 6.2.4, it follows that G is amenable. □

The free group of rank 2 provides a basic example of a non-solvable group. Indeed, F is not amenable (see Example 6.1.4), whereas every solvable group is amenable. Therefore, F cannot be solvable.

We now state the Tits Alternative, which establishes a fundamental dichotomy for linear groups.

Theorem 7.1.6 (Tits' Alternative)**.** Let G be a subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then either

- G contains a free subgroup of rank 2, or
- G is virtually solvable.

Remark 12. The two alternatives in Tits' Alternative are mutually exclusive. Indeed, suppose that a group G is virtually solvable and contains a free subgroup F of rank 2. By Proposition 7.1.3, the subgroup F is itself virtually solvable. It follows from Corollary 7.1.5 that F is amenable. However, a free group of rank 2 is not amenable (Example 6.1.4), a contradiction.

Remark 13 (Relation to von Neumann's conjecture)**.** Recall that von Neumann conjectured that a group is non-amenable if and only if it contains a free subgroup of rank 2. Although this conjecture is false in general, Tits' Alternative shows that it holds for linear groups.

Indeed, let $G \subseteq GL_n(\mathbb{R})$. If G contains a free subgroup of rank 2, then G is non-amenable by Corollary 6.2.2. Conversely, if G does not contain a free subgroup of rank 2, then Tits' Alternative implies that G is virtually solvable. Hence, by Corollary 7.1.5, G is amenable.

Therefore, for every subgroup $G \subseteq GL_n(\mathbb{R})$,

$$G \text{ is non-amenable} \iff G \text{ contains a free subgroup of rank 2.}$$

7.2 Ping–pong lemma

To prove that certain elements generate a free group, one needs a criterion ensuring that no nontrivial reduced word represents the identity. The Ping-Pong Lemma provides such a criterion in terms of group actions on sets.

The basic idea is to consider two elements acting on disjoint regions in such a way that successive applications move points predictably between them. This prevents nontrivial reduced words from acting trivially and therefore guarantees that the subgroup generated by the elements is free.

The Ping-Pong Lemma is one of the principal tools for constructing free subgroups and plays a central role in proofs of Tits' Alternative.

The main references for this subsection are [Löh17, Chapter 4.4.3] and [Wit15, Chapter 4.9]. The following version of the Ping-Pong Lemma is taken from [Pha22, Lemma 3.1].

Theorem 7.2.1 (Abstract Ping-Pong Lemma)**.** Let G be a group acting on a set X. Suppose there exist elements $\sigma, \tau \in G$ and four nonempty, pairwise disjoint subsets $A^+, A^-, B^+, B^- \subset X$ such that

$$\sigma(X \setminus A^-) \subseteq A^+, \qquad \tau(X \setminus B^-) \subseteq B^+.$$

Then σ and τ freely generate a free subgroup $F \subseteq G$.

Proof. The proof follows the approach of [Wit15, Lemma 4.9.6]. By assumption, $\sigma(X \setminus A^-) \subseteq A^+$ and $\tau(X \setminus B^-) \subseteq B^+$. Since group elements act as bijections, taking complements and applying σ^{-1}, respectively τ^{-1}, give

$$\sigma^{-1}(X \setminus A^+) \subseteq A^- \quad \text{and} \quad \tau^{-1}(X \setminus B^+) \subseteq B^-.$$

Iterating these inclusions, we obtain

$$\sigma^n(B^- \cup B^+) \subseteq A^+, \quad \tau^n(A^- \cup A^+) \subseteq B^+ \quad \text{for all } n > 0$$

and

$$\sigma^n(B^- \cup B^+) \subseteq A^-, \quad \tau^n(A^- \cup A^+) \subseteq B^- \quad \text{for all } n < 0.$$

Let w be a nontrivial reduced word in the alphabet $\{\sigma^{\pm 1}, \tau^{\pm 1}\}$. Without loss of generality, it suffices to consider the case

$$w = \sigma^{m_1}\tau^{n_1} \cdots \sigma^{m_k}\tau^{n_k}, \qquad m_i, n_i \neq 0.$$

Applying w to $A^- \cup A^+$ and using the inclusions above, we obtain

$$\begin{aligned} w(A^- \cup A^+) &= \sigma^{m_1}\tau^{n_1} \cdots \sigma^{m_k}\tau^{n_k}(A^- \cup A^+) \\ &\subseteq \sigma^{m_1}\tau^{n_1} \cdots \sigma^{m_k}(B^- \cup B^+) \\ &\vdots \\ &\subseteq \sigma^{m_1}(B^- \cup B^+) \\ &\subseteq A^{\pm}, \end{aligned}$$

where the last inclusion depends on whether $m_1 < 0$ or $m_1 > 0$. It follows that

$$w(A^- \cup A^+) \subseteq A^{\pm} \subsetneq A^- \cup A^+.$$

Hence, w does not act trivially on X. Therefore no nontrivial reduced word in σ and τ represents the identity, and consequently the subgroup $\langle \sigma, \tau \rangle$ is free of rank 2. □

We conclude with an example illustrating the Ping-Pong Lemma.

Example 7.2.2. Let

$$\sigma = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \qquad \tau = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \in SL_2(\mathbb{R}),$$

with $\lambda \geq 2$, and set $G = \langle \sigma, \tau \rangle \subset SL_2(\mathbb{R})$. We show that G is a free group of rank 2 using the Abstract Ping-Pong Lemma.

Consider the natural action of G on $X = \mathbb{R}^2 \setminus \{0\}$ by matrix multiplication. We define four pairwise disjoint subsets of $\mathbb{R}^2$:

$$A^+ := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |y| < |x|,\, xy > 0 \right\}, \quad A^- := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |y| < |x|,\, xy < 0 \right\},$$

$$B^+ := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| < |y|,\, xy > 0 \right\}, \quad B^- := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| < |y|,\, xy < 0 \right\}.$$

The four regions are illustrated in Figure 7.1.

Action of σ: For any $v = \begin{bmatrix} x \\ y \end{bmatrix} \in X \setminus A^-$, we have $v \in A^+ \cup B^+ \cup B^-$. Then

$$\sigma(v) = \begin{bmatrix} x + \lambda y \\ y \end{bmatrix}.$$

We first consider points in $B^+ \cup B^-$. If $v \in B^+ \cup B^- = \{|x| < |y|\}$, then

$$|x + \lambda y| \geq \lambda|y| - |x| > (\lambda - 1)|y| \geq |y|.$$

for $\lambda \geq 2$. Moreover, if $y > 0$, then $x > -y$ since $|x| < |y|$, and hence

$$x + \lambda y > -y + \lambda y = (\lambda - 1)y \geq y > 0.$$

Thus, $y(x + \lambda y) > 0$. If $y < 0$, then $x < -y$, and therefore

$$x + \lambda y < -y + \lambda y = (\lambda - 1)y < 0.$$

Since the product of two negative numbers is positive, we have $y(x + \lambda y) > 0$. Hence, $\sigma(v) \in A^+$ for $v \in B^- \cup B^+$.
Now, let $v \in A^+$, then $|y| < |x|$ and $xy > 0$. Since x and y have the same sign, we have

$$|x + \lambda y| = |x| + \lambda|y| > |y| + \lambda|y| > |y| \quad \text{and} \quad (x + \lambda y)y = xy + \lambda y^2 > 0.$$

Thus,

$$\sigma(X \setminus A^-) \subseteq A^+.$$

Action of τ: Applying the same argument to τ, we obtain:

$$\tau(X \setminus B^-) \subseteq B^+.$$

Since the conditions of the Abstract Ping-Pong Lemma are satisfied, it follows that σ and τ freely generate a free subgroup of $SL_2(\mathbb{R})$.

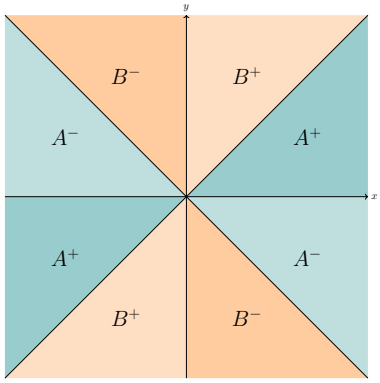

Figure 7.1: The transformation σ maps $X \setminus A^-$ into A^+ by a horizontal translation of magnitude λy, and τ maps $X \setminus B^-$ into B^+ by a vertical translation of magnitude λx.

Remark 14. For $\lambda = 2$, the matrices

$$\sigma = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \tau = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

coincide with the generators used in Chapter 5.3 to construct a free subgroup of rank 2 in $PSL_2(\mathbb{Z})$. The above argument therefore provides an alternative proof of the independence of those generators via the Ping-Pong Lemma.

To apply this idea more systematically in matrix groups, it is useful to classify matrices according to the behaviour determined by their eigenvalues. The following definitions are taken from Appendix A5 of [Wit15]. Although the names are similar, these notions should not be confused with the classification of isometries in $PSL_2(\mathbb{R})$.

Definition 7.2.3. An element $A \in GL_n(\mathbb{R})$ is called

- *hyperbolic* if it is diagonalizable over $\mathbb{C}$ and all its eigenvalues are real and positive,
- *elliptic* if it is diagonalizable over $\mathbb{C}$ and all its eigenvalues are on the unit circle,
- *parabolic* if 1 is the only eigenvalue of A.

Notice that in Example 7.2.2, we constructed a free subgroup of $SL_2(\mathbb{R})$ using two parabolic matrices. We now shift our attention to hyperbolic elements in $SL_2(\mathbb{R})$, where the geometric mechanism underlying the Ping-Pong Lemma becomes particularly transparent. If $\sigma \in SL_2(\mathbb{R})$ is hyperbolic, then its eigenvalues are of the form $\lambda_\sigma > 1$ and $\lambda_\sigma^{-1} < 1$. The following proposition shows how this spectral property gives rise to the regions required for the Ping-Pong argument.

Proposition 7.2.4. Let σ and τ be two hyperbolic elements of $SL_2(\mathbb{R})$ that have no common eigendirection. Then there exists $n \in \mathbb{N}$ such that $\langle \sigma^n, \tau^n \rangle$ is free of rank 2.

Proof. Let v_σ and w_σ be the eigenvectors of σ corresponding to the eigenvalues $\lambda_\sigma > 1$ and $\lambda_\sigma^{-1} < 1$, respectively. Similarly, let v_τ and w_τ be the eigenvectors of τ.

Since σ and τ have no common eigendirection, we may choose $\epsilon > 0$ sufficiently small and define four pairwise disjoint open cones in $\mathbb{R}^2 \setminus \{0\}$:

$$A^+ := \{x \neq 0 : \angle(x, \mathbb{R}v_\sigma) < \epsilon\}, \quad A^- := \{x \neq 0 : \angle(x, \mathbb{R}w_\sigma) < \epsilon\},$$

$$B^+ := \{x \neq 0 : \angle(x, \mathbb{R}v_\tau) < \epsilon\}, \quad B^- := \{x \neq 0 : \angle(x, \mathbb{R}w_\tau) < \epsilon\}.$$

Figure 7.2 illustrates this cone configuration.

Let $x \in \mathbb{R}^2 \setminus A^-$ and write x in the basis $\{v_\sigma, w_\sigma\}$, that is,

$$x = av_\sigma + bw_\sigma.$$

Since $x \notin A^-$, the angle between x and $\mathbb{R}w_\sigma$ is at least ϵ. Hence, there exists a constant $C > 0$ such that

$$|a| \geq C|b|$$

for all $x \notin A^-$. Applying σ^n gives

$$\sigma^n(x) = a\lambda_\sigma^n v_\sigma + b\lambda_\sigma^{-n} w_\sigma.$$

Therefore,

$$\frac{|b|\lambda_\sigma^{-n}}{|a|\lambda_\sigma^n} = \frac{|b|}{|a|}\lambda_\sigma^{-2n} \leq C^{-1}\lambda_\sigma^{-2n}.$$

Since $\lambda_\sigma > 1$, the right-hand side tends to 0 as $n \to \infty$. Hence, for sufficiently large n, the component of $\sigma^n(x)$ in the direction of w_σ is arbitrarily small compared to its component in the direction of v_σ. It follows that there exists $N \in \mathbb{N}$ such that

$$\sigma^n(\mathbb{R}^2 \setminus A^-) \subseteq A^+ \qquad \text{for all } n \geq N.$$

Similarly, there exists $M \in \mathbb{N}$ such that

$$\tau^m(\mathbb{R}^2 \setminus B^-) \subseteq B^+ \qquad \text{for all } m \geq M.$$

Taking $n = \max\{N, M\}$, the elements σ^n and τ^n satisfy the hypotheses of the Abstract Ping-Pong Lemma. Hence,

$$\langle \sigma^n, \tau^n \rangle$$

is a free group of rank 2. □

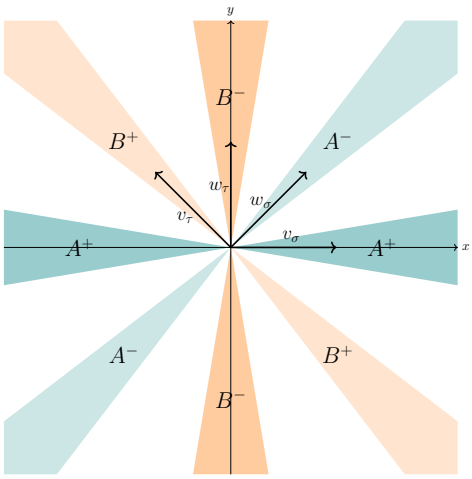

Figure 7.2: Schematic illustration of the cone configuration used in the Ping-Pong argument. The cones A^+ and A^- are centered around the eigendirections v_σ and w_σ, respectively, while B^+ and B^- are centered around v_τ and w_τ.

7.3 A special case in $SL_2(\mathbb{R})$

We now illustrate the strategy behind Tits' Alternative in a concrete special case inside $SL_2(\mathbb{R})$. The previous proposition showed that two hyperbolic elements of $SL_2(\mathbb{R})$ with no common eigenvector generate a free subgroup after passing to sufficiently large powers. We now explain why this situation arises naturally inside non-virtually solvable subgroups of $SL_2(\mathbb{R})$.

The key idea is that a non-virtually solvable subgroup of $SL_2(\mathbb{R})$ cannot preserve a common pair of eigendirections. Starting from a hyperbolic element, one can therefore find another hyperbolic element with no common eigendirection. The previous proposition then allows the Ping-Pong Lemma to be applied, producing a free subgroup of rank 2.

While the full proof of Tits' Alternative is considerably more involved, the argument presented here reflects its basic strategy.

Proposition 7.3.1. Let $G \subseteq \mathrm{SL}_2(\mathbb{R})$ be a non-virtually solvable group containing a hyperbolic element h. Then there exists a hyperbolic element $h' \in G$ that shares no eigendirection with h. Consequently, there exists $n \in \mathbb{N}$ such that $\langle h^n, (h')^n \rangle$ is a free group of rank 2.

Proof. Let $h \in G$ be a hyperbolic element with eigenspaces $\mathbb{R}v$ and $\mathbb{R}w$. We first show that there exists a hyperbolic element in G with no eigendirection in common with h. Assume, for contradiction, that every conjugate

$$g = khk^{-1}, \qquad k \in G,$$

shares at least one eigendirection with h. Since the eigenspaces of g are precisely $\mathbb{R}(kv)$ and $\mathbb{R}(kw)$, it follows that for every $k \in G$, at least one of the lines $\mathbb{R}(kv)$ or $\mathbb{R}(kw)$ coincides with $\mathbb{R}v$ or $\mathbb{R}w$.

Let

$$P := [\, v \mid w \,].$$

In the basis $\{v, w\}$, we have

$$PhP^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \qquad \lambda > 1.$$

If $g \in G$ shares the eigendirection v with h, then $PgP^{-1} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, and if $g \in G$ shares the eigendirection w with h, then $PgP^{-1} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$. Hence,

$$PGP^{-1} \subseteq B_1 \cup B_2,$$

where

$$B_1 = \left\{ \begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix} : x \in \mathbb{R} \setminus \{0\},\ y \in \mathbb{R} \right\} \quad \text{and} \quad B_2 = \left\{ \begin{bmatrix} x & 0 \\ z & x^{-1} \end{bmatrix} : x \in \mathbb{R} \setminus \{0\},\ z \in \mathbb{R} \right\}.$$

We now show that in fact PGP^{-1} must be contained entirely in one of these two subgroups. Suppose, for contradiction, that there exist

$$U = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in PGP^{-1} \cap B_1, \qquad L = \begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix} \in PGP^{-1} \cap B_2$$

with $b \neq 0$ and $d \neq 0$, then

$$UL = \begin{bmatrix} ac + bd & bc^{-1} \\ a^{-1}d & a^{-1}c^{-1} \end{bmatrix}.$$

Since $b \neq 0$ and $d \neq 0$, both off-diagonal entries of UL are nonzero. Thus, $UL \notin B_1 \cup B_2$, contradicting $PGP^{-1} \subseteq B_1 \cup B_2$.

Therefore, either every element of PGP^{-1} is upper triangular, or every element of PGP^{-1} is lower triangular. Consequently,

$$PGP^{-1} \subseteq B_1 \quad \text{or} \quad PGP^{-1} \subseteq B_2.$$

We now show that this implies that G is solvable, contradicting the hypothesis. For this, we first show the following.

Claim 6. The groups B_1 and B_2 are solvable.

Proof. Let

$$U_1 = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\} \subset B_1$$

be the subgroup of strictly upper unitriangular matrices. A direct calculation shows that $U_1 \triangleleft B_1$ and U_1 is abelian. Moreover, the quotient

$$B_1/U_1 \cong \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}$$

is abelian. Therefore, B_1 has a normal series

$$\{I\} \triangleleft U_1 \triangleleft B_1$$

with abelian successive quotients, and hence is solvable. By the same argument, B_2 is solvable. □

Since PGP^{-1} is a subgroup of either B_1 or B_2, both of which are solvable, it follows that PGP^{-1} is solvable. As PGP^{-1} is isomorphic to G, the group G is solvable as well, and therefore virtually solvable, contradicting the assumption.

Thus, there exists some $g \in G$ which shares no eigendirection with h. Set

$$h' := ghg^{-1}.$$

Then h' is again hyperbolic, and by construction shares no eigendirection with h.

Hence, by Proposition 7.2.4, there exists $n \in \mathbb{N}$ such that

$$\langle h^n, (h')^n \rangle$$

is a free group of rank 2. □

The argument above essentially proves Tits' Alternative for subgroups of $SL_2(\mathbb{R})$ that contain a hyperbolic element. Indeed, hyperbolic elements naturally provide the dynamical behaviour required for the Ping-Pong Lemma: under iteration, vectors increasingly align with the eigendirection corresponding to the eigenvalue of larger absolute value.

However, a general linear group need not contain such elements with respect to the usual absolute value. For example, if

$$G \subseteq SO(n),$$

then every element of G is elliptic. Indeed, all eigenvalues of elements of $SO(n)$ have absolute value 1. The main difficulty in the general proof is therefore to recover analogous behaviour in a much more general setting.

Tits' proof achieves this by passing to suitable field extensions and representations. After these changes, one can construct elements with an eigenvalue whose absolute value is strictly larger than that of all remaining eigenvalues. Large powers of such an element then behave similarly to the hyperbolic matrices considered above: vectors become increasingly aligned with a suitable eigenspace. After further work, one finds suitable conjugates whose associated ping-pong regions are disjoint, so that sufficiently large powers satisfy the hypotheses of the Ping-Pong Lemma and generate a free subgroup of rank 2.

The full proof requires substantially deeper methods from the theory of linear algebraic groups and normed fields and lies beyond the scope of this thesis. For a complete proof, see Tits' original paper [Tit72, Theorem 1].

7.4 Consequences for Paradoxical Decompositions

The discussion above reveals a close connection between free subgroups, amenability, and paradoxicality in linear groups.

By Tits' Alternative, every linear group is either virtually solvable or contains a free subgroup of rank 2. Equivalently, von Neumann's conjecture holds for linear groups:

$$G \text{ contains a free subgroup of rank 2} \iff G \text{ is non-amenable.}$$

Combining this with Tarski's theorem yields

$$G \text{ contains a free subgroup of rank 2} \iff G \text{ is paradoxical.}$$

We now revisit the main constructions of this thesis from this perspective.

The results of Chapter 2 explain how paradoxicality passes between groups and the spaces on which they act. On the one hand, if a paradoxical group acts freely on a set X, then the paradoxical decomposition of the group can be transferred to a paradoxical decomposition of X. This is precisely the mechanism underlying the constructions in Chapters 3 and 4. In Chapter 3, we constructed a free subgroup of rank 2 inside $SO_3(\mathbb{R})$ and used its paradoxicality to obtain the Hausdorff paradox on $S^2 \setminus D$. The Banach-Tarski paradox of Chapter 4 was then obtained by extending this construction from the sphere to the three-dimensional ball. Thus, both Euclidean constructions ultimately rely on a free subgroup of rank 2 inside the relevant rotation group.

A similar phenomenon appeared in Chapter 5. There we embedded $\mathbb{Z}_2 * \mathbb{Z}_3$ into the isometry group of the hyperbolic plane and constructed paradoxical decompositions using its action. For

the paradox on the whole hyperbolic plane, we identified a free subgroup of rank 2 in the group $\mathbb{Z}_2 * \mathbb{Z}_3$. The results of the present chapter show that the appearance of such a subgroup is not accidental. Since $\mathbb{Z}_2 * \mathbb{Z}_3 \cong PSL_2(\mathbb{Z})$ is paradoxical and linear, Tits' Alternative implies that it must contain a free subgroup of rank 2.

On the other hand, if a group action on a set X is paradoxical, then the acting group itself must already be paradoxical (Proposition 2.4.4). Thus, within the class of linear groups, the absence of free subgroups prevents paradoxical decompositions for every action of the group. Indeed, we proved that the Euclidean isometry groups G_1 and G_2 are amenable. Consequently, they contain no free subgroup of rank 2. This explains why Banach-Tarski type phenomena do not occur in dimensions 1 and 2 when one restricts to Euclidean isometries.

Thus, the existence of a free subgroup of rank 2 is what makes paradoxical decompositions possible in the examples considered throughout this thesis.

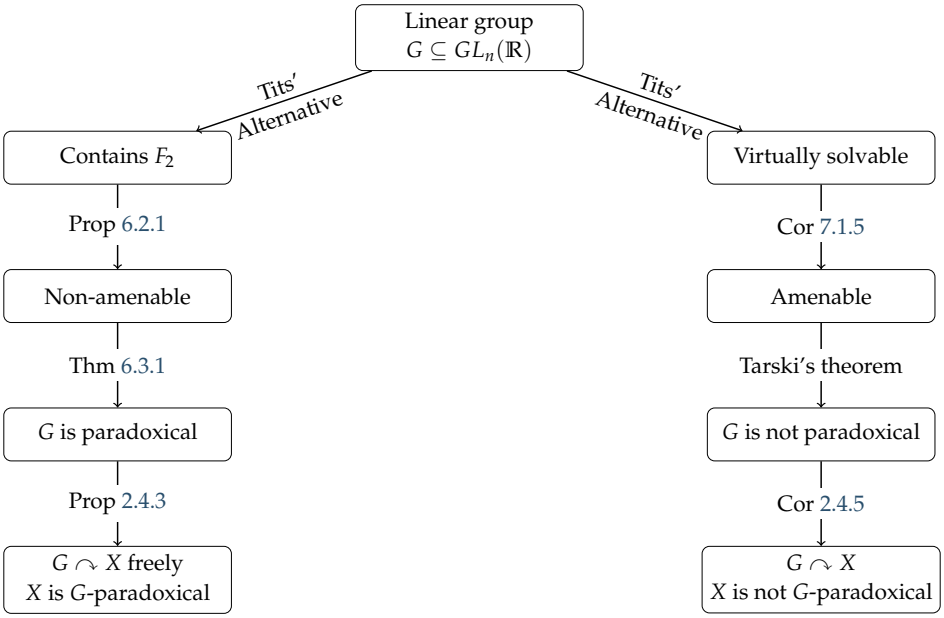

Figure 7.3: Relation between Tits' Alternative, amenability, and paradoxical decompositions for linear groups.

Conclusion

The aim of this thesis was to study the Hausdorff and Banach-Tarski paradoxes and to understand the mechanisms underlying such decompositions. Starting from geometric questions about cutting and rearranging sets, we developed the tools needed to construct paradoxical decompositions and relate them to the algebraic structure of the acting groups.

A central theme throughout this thesis is that paradoxical decompositions are not solely a consequence of the Axiom of Choice, but are closely connected to the structure of the groups acting on the spaces. Through the Hausdorff paradox, the Banach-Tarski paradox, and their hyperbolic analogue, we saw that paradoxical decompositions arise from actions of groups containing free subgroups of rank 2. Conversely, amenability explained their absence in dimensions one and two: the Euclidean isometry groups are amenable and therefore admit no paradoxical decompositions.

The topics discussed in this thesis naturally lead to several further questions. One question that I find particularly interesting concerns the manner in which the pieces are rearranged. Since the classical formulation of the Banach-Tarski paradox only specifies the initial and final positions of the pieces, it is natural to ask whether the rearrangement can be carried out continuously. This leads to the following problem.

Problem 1 (de Groot)**.** Does there exist a paradoxical decomposition of the unit ball into two such that the pieces can be moved continuously in $\mathbb{R}^3$ from their initial to their final positions while remaining pairwise disjoint at all times?

This question was answered affirmatively by Trevor Wilson in 2005 ([Wil05]), showing that paradoxical behaviour persists even under these additional constraints.

Another interesting direction concerns further equidecomposability problems. A famous example is Tarski's circle-squaring problem.

Problem 2 (Tarski's circle-squaring problem)**.** Is the unit disk finitely G_2-equidecomposable with a square of the same area?

After remaining open for more than sixty years, this problem was solved affirmatively by Laczkovich in 1990 ([Lac90]). It illustrates that many of the ideas around paradoxical decompositions extend beyond the Banach-Tarski paradox itself and continue to motivate further research.

While this thesis has focused primarily on the geometric and algebraic aspects of paradoxical decompositions, the Banach-Tarski paradox is equally rooted in measure theory through questions of measurability and invariant measures. We have explored only a small part of the theory surrounding these phenomena while many deeper questions remain to be investigated.

Overall, I chose to focus on some of the results that I found most interesting and surprising. What I especially enjoyed is seeing how the Banach-Tarski paradox connects different areas of mathematics. For me, this is what makes it such an interesting topic to study.

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