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Approximation of Local Kinetic Energy Evolution by Hartree Equation with Semiclassical Fermionic Initial Data

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Master's Thesis

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Matriculation Number	24-744-781
Submission Date	April 12, 2026

Abstract

This thesis studies, in the fermionic system, the evolution of local kinetic energy of an approximately mixed quasi-free states whose reduced density exhibits a local semiclassical structure, which is designed as an approximation of the related evolution of pseudo-relativistic Fermi gases in $\mathbb{R}^3$ with high density $\varrho \gg 1$ on an arbitrarily large domain at low but positive temperature. Building on the framework developed for the effective Hartree dynamics for extended fermionic systems, we revealed that the many-body evolution of the kinetic energy of initial data with local semiclassical structure converges to the kinetic energy under the Hartree evolution in a local sense. In order to capture the local kinetic energy instead of local density of particles as in the previous work, we introduced localized kinetic observables which is obtained by conjugating the pseudo-relativistic kinetic energy operator by the spatial weight used in the previous work. The proof relies on the propagation of the local semiclassical structure along the Hartree equation and on the analysis of the evolution of the kinetic energy of local excitations.

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1 Introduction

In quantum mechanical systems, a system consisting of one particles moving in $\mathbb{R}^3$ is described by the complex Hilbert space $L^2(\mathbb{R}^3;\mathbb{C})$. The possible *states* of the system are represented by vectors $\psi \in L^2(\mathbb{R}^3;\mathbb{C})$ with ψ normalized, $\|\psi\| = 1$. The normalization is designed for physical purpose so that $|\psi|^2$ represents the probability density for finding particles, i.e., $|\psi(x)|^2$ is interpreted as the probability for finding the particle at position x. An *observable*, a physical quantity that can be measured, is described by a self-adjoint operator on $L^2(\mathbb{R}^3;\mathbb{C})$. The *expectation* of an observable described by the self-adjoint operator A at a state ψ is calculated by the inner product $\langle \psi, A\psi \rangle$.

Similarly, a many-body quantum mechanical system consisting of N particles moving in $\mathbb{R}^3$ can be described by the complex Hilbert space $L^2(\mathbb{R}^{3N};\mathbb{C})$. In the following context, we use the shorthand notation $L^2(\mathbb{R}^{3N}) := L^2(\mathbb{R}^{3N};\mathbb{C})$. The indistinguishablility plays an important role in the many-body quantum mechanical systems. In classical mechanics, particles can be distinguished by their trajectories in principle. However this is not possible for a many-body quantum mechanical system even in principle. Therefore, when dealing with a N-particle quantum mechanical system of n indistinguishable particles, one cannot work on the full N-particle Hilbert space $L^2(\mathbb{R}^{3N};\mathbb{C})$ without taking particle statistics into account. This indistinguishability is expressed mathematically through a restriction on the class of admissible physical observables. Indeed, only those observables that are invariant under permutations of the N particles are physically meaningful and admissible.

Let S_N denote the permutation group of the set $\{1, 2, \ldots, N\}$. We define a unitary representation of S_N on the N-particle Hilbert space $L^2(\mathbb{R}^{3N})$ through the unitary operator U_π defined as follows: for $\pi \in S_N$

$$(U_\pi \psi)(x_1, \ldots, x_N) = \psi\left(x_{\pi(1)}, \ldots, x_{\pi(N)}\right) .$$

A physically meaningful observable for the N-particle system is therefore a self-adjoint operator A on $L^2(\mathbb{R}^{3N})$ satisfying $U_\pi^* A U_\pi = A$ for all $\pi \in S_N$. Now let $\psi \in L^2(\mathbb{R}^{3N})$ be a normalized wave function, describing a state of the N-particle system. Then, by the unity of U_π we have

$$\|U_\pi \psi\| = \|\psi\|$$

and

$$\langle U_\pi \psi, A U_\pi \psi \rangle = \langle \psi, U_\pi^* A U_\pi \psi \rangle = \langle \psi, A\psi \rangle \tag{1.1}$$

for each permutation $\pi \in S_N$ and for every admissible observable A. The identity (1.1) indicates that the wave function $U_\pi \psi$ describes exactly the same state as the wave function ψ, for every $\pi \in S_N$. Therefore, in order to avoid such redundancy, we restrict ourselves to classes of wave functions. One is the set of all wave functions ψ that are completely *symmetric* with respect to permutations, i.e., $U_\pi \psi = \psi$, for all $\pi \in S_N$. The other is the set of wave functions ψ that are completely *antisymmetric* with respect to permutations, i.e., $U_\pi \psi = \sigma_\pi \psi$, for all $\pi \in S_N$, where $\sigma_\pi \in \{-1, 1\}$ denotes the sign of the permutation π. For a given species, one considers either completely symmetric or completely antisymmetric wave functions, but not both simultaneously. Particles described by symmetric wave functions are called *bosons*, whereas particles described by antisymmetric wave functions are called *fermions*. In this work, we are looking at the behavior of fermions, so we restrict ourselves to the fermionic setting. The systems of n fermions in $\mathbb{R}^3$ are described on the antisymmetric Hilbert space

$$L^2_a(\mathbb{R}^{3N}) = L^2(\mathbb{R}^3)^{\wedge N} = \left\{ \psi \in L^2(\mathbb{R}^{3N}) \,:\, \psi\left(x_{\pi(1)}, \ldots, x_{\pi(N)}\right) = \sigma_\pi \psi(x_1, \ldots, x_N) \right\} .$$

The simplest example of fermionic wave functions is the *Slater determinant*. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis in $L^2(\mathbb{R}^3)$. We construct the Slater determinant as follows. Fix one particle in the state φ_{i_1}, one particle in the state φ_{i_2}, and so on, with all indices $i_1,\dots,i_N$ distinct, then here comes the fermionic wave function

$$\psi_N(x_1,\dots,x_N)=\frac{1}{\sqrt{N!}}\sum_{\pi\in S_N}\sigma_\pi\varphi_{i_1}(x_{\pi(1)})\cdots\varphi_{i_n}(x_{\pi(N)})$$

Notice that in the fermionic setting, each one-particle state φ_{i_j} can be "occupied" by at most one particle; otherwise, the anti-symmetrization vanishes. This is known as the *Pauli principle*. In this sense, Slater determinants play the role of the fermionic states that are are as independent as possible, subject only to anti-symmetry. This is why they are fundamental in fermionic many-body theory.

The time evolution of a quantum many-body system is governed by the time-dependent Schödinger equation

$$i\partial_t\psi_{N,t}=H_N\psi_{N,t} \tag{1.2}$$

Here, the operator H_N is a *Halmilton operator* which is the self-adjoint operator associated to the energy of the whole system. As one can see from (1.2), the Hamilton operator generates the time evolution of the system. According to fermionic statistics, the operator H_N acts on the Hilbert space $L^2_a(\mathbb{R}^{3N})$.

In this thesis, we are interested in the time evolution of kinetic energy for the fermionic many-body quantum systems in the so-called *high-density regime*. In the high-density regime, the system is confined in a torus and the density of the system is fixed to be large. When the total number of particles increases, the torus must expand in such a way that the density of the system remains a constant. Moreover, in this regime, each particle is assumed to only interact with particles within a distance of order one. Since the density of the system is fixed to be large, there are already sufficiently many particles involved in the interactions, and hence the interactions that we take into account are collective. The high-density regime plays a very similar role as the mean-field regime. Even though the mean-field regime allows all particles in the torus to interact weakly with each other, the similarity lies in the fact that each particle in both regime only experience interactions of order one. We will only focus on the high-density regime in this work. The N-particle Hamilton operator in the high-density regime that we are considering is given by

$$H_N=\sum_{j=1}^N\sqrt{1-\varepsilon^2\Delta_{x_j}}+\varepsilon^3\sum_{i<j}^N V(x_i-x_j)\,. \tag{1.3}$$

In particular, we are going to consider the high-density Schrödinger dynamics of the N-particle fermionic system with relativistic dispersion (pseudo-relativistic fermions) generated by the N-particle Hamilton operator H_N. In the equation (1.3), the operator $\sqrt{1-\varepsilon^2\Delta}$ is associated to the kinetic energy of the system which is a pseudo-differential operator with Fourier symbol

$$\xi\mapsto\sqrt{1+\varepsilon^2|\xi|^2}\,.$$

Two-particle multiplication operator $V:\mathbb{R}^3\to\mathbb{R}^3$ is associated to the interaction of two particles which is smooth and decay faster than any polynomial in the position space. In (1.3) and throughout the paper, we shall choose $\varepsilon=O(\varrho^{-1/3})$. This choice is designed to guarantee that both kinetic and potential energy per particle are of order one for physically relevant states, in the limit of large ϱ (or $N\to\infty$). In fact, when we are considering the limit of large ϱ, the smallness of ε is going to play an important role when proving the our main result for fix time.

From physical point of view, the scenario we are considering is that particles are initially trapped in a volume of order one by an external potential V_{ext} and are in equilibrium state, then we switch off the external trapping potential V_{ext} at time $t = 0$ to study the upcoming time evolution generated by the Hamilton operator (1.3).

Under the same scenario, the work in [5] studied the dynamics of pseudo-relativistic high-density Fermi gases which are initially trapped at a equilibrium state generated by the one-particle Hamilton operator

$$H^{\text{trap}} = \sqrt{1 - \varepsilon^2 \Delta} + V_{\text{ext}} \,.$$

Here, V_{ext} is the external potential which is a smooth, non-negative one-particle multiplication operator which grows at most quadratically at infinity. Additional to these, there is also an assumption of the trace norm bound [5, Assumption 2.1] being made to ensure that the number of particles with energies close to the Fermi surface is not too large, i.e., the initial data is locally close to a Slater determinant. The time evolution of such initial data is given by the solution of the many-body Schrödinger equation

$$i\varepsilon \partial_t \psi_t = H_N \psi_t = \Big[\sum_{j=1}^{N} \sqrt{1 - \varepsilon^2 \Delta_{x_j}} + \varepsilon^3 \sum_{i<j}^{N} V(\hat{x}_i - \hat{x}_j) \Big] \psi_t \,. \tag{1.4}$$

Let $\gamma_t^{(1)}$ be the one-particle reduced density of the solution ψ_t to (1.4). The main result shown in [5, Theorem 2.3] is that for large ϱ, the one-particle reduced density $\gamma_t^{(1)}$ is locally close to a Slater determinant, with reduced one-particle density matrix evolved through the time-dependent Hartree equation

$$\begin{aligned} i\varepsilon \partial_t \omega_t &= \big[\sqrt{1 - \varepsilon^2 \Delta} + \varepsilon^3 (V * \varrho_t), \omega_t \big] \\ &=: \big[h(t), \, \omega_t \big] \end{aligned} \tag{1.5}$$

with $\varrho_t(x) = \omega_t(x; x)$ and with initial datum $\omega_{t=0} = \omega_\mu$. The locality of the results around a point $z \in R^3$ is achieved by the multiplication operator

$$\mathcal{W}_z^{(n)}(\hat{x}) = \frac{1}{1 + |\hat{x} - z|^{4n}} \tag{1.6}$$

in the position space, for $2n \in \mathbb{N}$. More precisely, the bound

$$\Big| \operatorname{tr} \mathcal{O}_z \big(\gamma_t^{(1)} - \omega_t \big) \Big| \leq C \exp(c \exp(c|t|)) \varepsilon^{-3+\delta} \tag{1.7}$$

is obtained for some $0 \leq \delta \leq 1$, where $\mathcal{O}_z$ is a smooth fast decaying function in the position space.

The proof of the main theorem [5, Theorem 2.3] is based on an approximation argument of the zero-temperature pure state with reduced density ω_μ *by a mixed state* with reduced density given by the Fermi-Dirac distribution

$$\omega_{\mu,\beta} = \frac{1}{1 + e^{\beta(H-\mu)}} \tag{1.8}$$

at low temperature $T = 1/\beta = O(\varepsilon)$. The reason for introducing this approximation is that the kernel of the reduced one-particle density matrix $\omega_{\mu,\beta}$ has better decay properties at positive temperature. The preceding assumptions on H^{trap} allows them to show that $\omega_{\mu,\beta}$ exhibit a *local semiclassical structure*. Such a local semiclassical structure is preserved by the Hartree equation (1.5) which plays a crucial role to prove local convergence towards

the Hartree dynamics. Therefore, the main intermediate theorem they went to achieve is, by assuming the initial data $0 \leq \omega \leq 1$ has a nice local semiclassical structure,

$$\begin{aligned}
\left\| \mathcal{W}_z^{(n)} \omega \mathcal{W}_{z'}^{(n/2)} \right\|_{\mathrm{tr}} &\leq \frac{C\varepsilon^{-3}}{1+|z-z'|^{2n-4}} \,, \\
\left\| \mathcal{W}_z^{(n)} \big[e^{ip\cdot\hat{x}}, \omega \big] \mathcal{W}_{z'}^{(n/2)} \right\|_{\mathrm{tr}} &\leq \frac{C\varepsilon^{-2}(1+|p|)}{1+|z-z'|^{2n}} \,, \\
\left\| \mathcal{W}_z^{(n)} \big[\varepsilon\nabla, \omega \big] \mathcal{W}_{z'}^{(n/2)} \right\|_{\mathrm{tr}} &\leq \frac{C\varepsilon^{-2}}{1+|z-z'|^{2n}},
\end{aligned} \tag{1.9}$$

a bound of the same form as (1.7) is achieved with $\gamma_t^{(1)}$ be the one-particle reduced density of the time evolution Ψ_t. Here, Ψ_t is the time evolution of a quasi-free initial data whose one-particle reduced density is exactly ω. Based on this result, the goal of this work is to show that the *local kinetic energy* evolution of the same quasi-free states can be approximated by the local kinetic energy along the corresponding Hartree evolution.

A *quasi-free state* is a quantum state which is completely characterized by its one-particle reduced density and pairing density. The class of quasi-free states we are considering in this work – the quasi-free states whose one-particle reduced densities exhibit a proper semiclassical structure and the pairing density is vanishing. The physically relevant scenario for the initial data is with a positive low temperature. A simplest example of the quasi-free states are the Slater determinants. However, there is a significant difference between the Slater determinants and the quasi-free states which are relevant to the equilibrium state at positive temperature. Slater determinants are *pure* quasi-free states, while the states we are interested are *mixed* quasi-free states. In fact, the Slater determinant is the equilibrium state at zero temperature, while the states we are interested are the equilibrium state at positive temperature.

In order to establish the expected results, we introduce the local kinetic energy operator

$$\widetilde{\mathcal{W}}_z^{(n)} = \mathcal{W}_z^{(n)}(\hat{x}) \sqrt{1-\varepsilon^2\Delta}\, \mathcal{W}_z^{(n)}(\hat{x}) \tag{1.10}$$

based on the formulation of (1.6). Consider the initial data ω such that $0 \leq \omega \leq 1$ and that ω satisfies both the semiclassical structure (1.9) that is required in [5] and the analogous structure when tested by the local kinetic energy operator,

$$\begin{aligned}
\left\| \widetilde{\mathcal{W}}_z^{(n)} \omega \mathcal{W}_{z'}^{(n/2)} \right\|_{\mathrm{tr}} &\leq \frac{C\varepsilon^{-3}}{1+|z-z'|^{2n-4}} \,, \\
\left\| \widetilde{\mathcal{W}}_z^{(n)} \big[e^{ip\cdot\hat{x}}, \omega \big] \mathcal{W}_{z'}^{(n/2)} \right\|_{\mathrm{tr}} &\leq \frac{C\varepsilon^{-2}(1+|p|)}{1+|z-z'|^{2n}} \,, \\
\left\| \widetilde{\mathcal{W}}_z^{(n)} \big[\varepsilon\nabla, \omega \big] \mathcal{W}_{z'}^{(n/2)} \right\|_{\mathrm{tr}} &\leq \frac{C\varepsilon^{-2}}{1+|z-z'|^{2n}}.
\end{aligned} \tag{1.11}$$

These bounds reinforce the initial data behaves regular enough from a local kinetic energy point of view. Let $\gamma_t^{(1)}$ be the one-particle reduced density of the time evolution Ψ_t, where Ψ_t is the time evolution of a mixed quasi-free initial data whose one-particle reduced density is exactly ω. Then, we are going to establish the analogous convergence

$$\left| \mathrm{tr}\, \widetilde{\mathcal{O}}_z \big(\gamma_t^{(1)} - \omega_t \big) \right| \leq C \exp(c \exp(c|t|)) \varepsilon^{-3+\delta} \tag{1.12}$$

to (1.7), for $\widetilde{\mathcal{O}}_z = \mathcal{O}_z \sqrt{1-\varepsilon^2\Delta}\, \mathcal{O}_z$. Again, $\mathcal{O}_z$ is the smooth fast decaying function in the position space the same as in [5]. Let us give an explanation for the bad factor

$\exp(\exp(c|t|))$. Although the factor $\exp(\exp(c|t|))$ grows considerably rapidly in time, this does not affect the convergence result here, since the theorem is formulated for every fixed $t \in \mathbb{R}$. For fixed time, the the factor $\exp(\exp(c|t|))$ is simply a time-dependent constant. In other words, the double-exponential growth in time only indicates that the bound is not suitable for long-time analysis, but it does not interfere with the semiclassical limit at a fixed time period. Such a coefficient would become relevant only if one were interested in estimates uniform in time, or in a regime in which the time variable is itself allowed to depend on ε.

Let us briefly explain the non-triviality of the extension that we are going to achieve as mentioned in the above. Since the work in [5] saw that the Hartree equation well approximates the evolution of local density of such a high-density extended Fermi gas, one would expect that the Hartree equation also well approximates their *evolution of local kinetic energy*. This extension would not be trivial because, from physical point of view, the density alone is not enough to describe what is happening in a Fermi gas. Two states may have very similar local density but very different local kinetic energy content. On the other hand, the non-triviality of such an extension comes from the Pauli principle for fermions. For a bosonic system, bosons can occupy the same state as each other and they often exhibit a *Bose–Einstein condensate* at low temperature where most particles sit in the lowest momentum state and the kinetic energy for the equilibrium state would became a small correction. However, this is not possible for fermions. The Pauli exclusion principle prevents two fermions from occupying the same quantum state, which automatically drives the high-density fermionic system to a high momentum mode, and there will be a considerable growth of the the total kinetic energy with the growth of the density of the system.

From mathematical point of view, the non-triviality comes from the fact that the local kinetic energy operator $\widetilde{\mathcal{W}}_z^{(n)}$ (1.10) is not a multiplication operator as the spatial weight (1.6). For example, a more general local kinetic energy operator $\widetilde{\mathcal{O}}_z$, which the main result (1.7) concerns, *cannot* be simply bounded by a proportion of $\widetilde{\mathcal{W}}_z^{(n)}$. Instead, a more careful argument will see

$$\widetilde{\mathcal{O}}_z \leq C\widetilde{\mathcal{W}}_z^{(n)} + C\varepsilon \mathcal{W}_z^{(n)} \tag{1.13}$$

where the r.h.s. is in a ladder form. Moreover, when dealing with the kinetic energy associated with the Hartree evolution, we want to propagate $\widetilde{\mathcal{W}}_z^{(n)}$ by the Hartree generator. A Grönwall argument leads to the term

$$\big[\widetilde{\mathcal{W}}_z^{(n)},\, h(t)\big] = \big[\widetilde{\mathcal{W}}_z^{(n)},\, \sqrt{1-\varepsilon^2\Delta}\big] + \big[\widetilde{\mathcal{W}}_z^{(n)},\, \varepsilon^3(V * \varrho_t)\big]\,, \tag{1.14}$$

where the second term on the r.h.s. is a newly arising term which vanishes in the discussion in [5] when only local density (a multiplicative spatial weight) was considered. Moreover, a ladder form bound

$$\pm\big[\widetilde{\mathcal{W}}_z^{(n)},\, \sqrt{1-\varepsilon^2\Delta}\big] \leq C\varepsilon\widetilde{\mathcal{W}}_z^{(n)} + C\varepsilon^2\mathcal{W}_z^{(n)} \tag{1.15}$$

coming from the first term on the r.h.s. of (1.14) also makes the argument different from that appeared in [5]. The same problem will arise when we are going to find the kinetic energy associated to the Schrödinger evolution (See (4.43) and (4.46)). It is also worth mentioning that, in the inequalities (1.13) and (1.15), bounding an operator concerning $\widetilde{\mathcal{W}}_z^{(n)}$ requires both $\widetilde{\mathcal{W}}_z^{(n)}$ and $\varepsilon\mathcal{W}_z^{(n)}$, which we call a *ladder form* bound. Such ladder form bounds indicates that the kinetic observables depend on not only the local kinetic behavior but also the local number of particles. In particular, this forces the results for kinetic energy not only depending on the assumptions (1.9) (local semiclassical structure on the

local density) but also on (1.11) (local semiclassical structure tested by the local kinetic energy operator). However, the extra small coefficient ε in front of $\mathcal{W}_z^{(n)}$ than $\widetilde{\mathcal{W}}_z^{(n)}$ tells us that the dependence on the local number of particles is mild compared to the dependence on the local kinetic energy.

An natural extension of the present work is to derive the current work, the convergence to Hartree on kinetic energy at semiclassical scaling (approximating the positive temperature case), to a zero-temperature version by following an exactly similar approach as in [5]. One may approximate the zero-temperature equilibrium state (a pure state) described by the *Fermi projection*, $\omega_\mu = \chi(H^{\mathrm{trap}} \leq \mu)$ by a low-temperature mixed quasi-free state with Fermi–Dirac distribution (1.8). It is proven, in [5, Appendix B], that the Fermi–Dirac distribution (1.8) satisfies the local semiclassical structure (1.9) then the main intermediate theorem [5, Theorem 3.7] applies for the Fermi–Dirac distribution (1.8). Analogously, as in this work, one needs additionally show that the Fermi–Dirac distribution (1.8) satisfies not only the local semiclassical structure (1.9) on local density but also the structure (1.11) tested by the local kinetic energy, then the main theorem established in this work applies. In fact, the procedure to show this will be quite similar to the the methods we introduced in this thesis. The replacement of the spatial weight $\mathcal{W}_z^{(n)}$ by the local kinetic energy $\widetilde{\mathcal{W}}_z^{(n)}$ yields extra commutators and the newly arising terms can be dealt with by using similar approach to those in this work.

This thesis is organized as follows. In Section 2 we introduce the mathematical language that will be used, e.g., the fermionic Fock space, together with the formalism of second quantization and the notion of mixed quasi-free states which provide the natural language for describing the class of initial data considered in this work. In Section 3, we state our main result. Section 3.1 addresses all necessary assumptions on the initial data that will be required in the main theorem, i.e., Assumption 3.1 (bounded initial local density and bounded initial local kinetic energy); Assumption 3.3 (initial semiclassical structure on local density and initial semiclassical on local kinetic energy). The following subsection, Section 3.2, sees the statement of the main theorem, Theorem 3.5, which tells us that under the preceding assumptions on the initial data, the time evolution of the local kinetic energy converges to the local kinetic energy given by the Hartree equation. We also discuss the possible extension to the convergence to Hartree equation on kinetic energy at the zero-temperature setting in Section 3.3. Section 4 provides a complete proof of the main theorem. In Section 4.1, we show, in Proposition 4.1, the solution ω_t to the Hartree equation, with initial datum ω satisfying the preceding assumptions, again satisfies the structures in the preceding assumptions. The proof of this proposition relies on, Proposition 4.4, the propagation of the local kinetic energy operator along the Hartree evolution and we see the locality is preserved. In Section 4.2, by assuming the bound for the local kinetic energy of excitations, Proposition 4.7, we conclude the proof of the main theorem. The remaining subsections are devoted to prove Proposition 4.7. In Section 4.3, we state and prove the rough bound of order ε^{-3} for the local kinetic energy of excitations which is valid everywhere. In Section 4.4 and Section 4.5, we then improve the rough bound to be the required bound of order $\varepsilon^{-3+\delta}$ but only valid in a proper region, which completes the proof of Proposition 4.7.

Acknowledgements. I sincerely appreciate Professor Benjamin Schlein for motivating the interesting topic and for his guidance throughout the work of this thesis. I would like to thank Sara Lanzenberger for her helpful discussions during the this work. I would also like to express my gratitude to my family for their constant care, support, and encouragement throughout my studies.

2 Fock Space and Quasi-Free States

In this section we introduce the Fock space framework that will be used throughout the analysis. The fermionic Fock space provides a convenient representation of many-body fermionic quantum systems in which the number of particles is allowed to vary. In this formulation, many-body observables can be expressed through the formalism of second quantization, and the time evolution of the many-system can be studied in terms of fluctuations around suitable quasi-free reference states. Since the initial data considered in this work are close to quasi-free states, this representation allows us to describe the excitations around such states in a convenient way. We therefore begin by recalling the basic definitions and properties of the fermionic Fock space, the notion of quasi-free states and related notations.

2.1 Fock space formalism

Fock space. In the analysis of many-body evolution in this work, we switch to a representation of the system where the number of particles is not fixed. For the system of Fermi gases, we define the antisymmetric (or fermionic) Fock space $\mathcal{F}(\mathfrak{h})$ as

$$\mathcal{F}(\mathfrak{h}) := \mathbb{C} \oplus \bigoplus_{n\geq 1} \mathfrak{h}^{\wedge n} ,$$

where $\mathfrak{h} := L^2(\mathbb{R}^3)$ is the one-particle space and $\mathfrak{h}^{\wedge n} = L^2_a(\mathbb{R}^{3n})$ is the n-fold antisymmetric tensor product of $\mathfrak{h}$. Vectors in $\mathcal{F}(\mathfrak{h})$ are sequence of the form $\Psi = \{\psi^{(n)}\}_{n\geq 0}$ for each $\psi^{(n)} \in \mathfrak{h}^{\wedge n}$, where we use the convenient notation $\mathbb{C} =: \mathfrak{h}^{\wedge 0}$. On $\mathcal{F}(\mathfrak{h})$, we define the inner product

$$\langle \Psi, \Phi \rangle := \sum_{n\geq 0} \left\langle \psi^{(n)}, \phi^{(n)} \right\rangle_{\mathfrak{h}^{\wedge n}}$$

for any $\Psi = \{\psi^{(n)}\}_{n\geq 0}$ and $\Phi = \{\phi^{(n)}\}_{n\geq 0}$ in $\mathcal{F}(\mathfrak{h})$. Indeed, one can check this defines a valid scalar product on $\mathcal{F}(\mathfrak{h})$ which gives a complete induced normed space. Thus, the space $\mathcal{F}(\mathfrak{h})$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space. The state of a quantum system is described by a normalized sequence $\Psi = \{\psi^{(n)}\}_{n\geq 0} \in \mathcal{F}(\mathfrak{h})$, i.e.,

$$\langle \Psi, \Psi \rangle = \sum_{n\geq 1} \left\| \psi^{(n)} \right\|^2 = 1 ,$$

where each $\|\psi^{(n)}\|^2$ is the probability to find exactly n particles in the system. We note that the state of vacuum with no particles is described by the vacuum vector $\Omega = \{1, 0, 0, \cdots\} \in \mathcal{F}(\mathfrak{h})$.

Creation and annihilation operators. For any one-particle wave function $f \in \mathfrak{h}$, we define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on the space $\mathcal{F}(\mathfrak{h})$ as follows:

$$(a^*(f)\Psi)^{(n)}(x_1, \ldots, x_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^j f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

$$(a(f)\Psi)^{(n)}(x_1, \ldots, x_n) := \sqrt{n+1} \int \overline{f(x)}\, \psi^{(n+1)}(x, x_1, \ldots, x_n) dx$$

for each $\Psi = \{\psi^{(n)}\}_{n\geq 0} \in \mathcal{F}(\mathfrak{h})$ and each $n \in \mathbb{N}$. One can check that the creation operator $a^*(f)$ and the annihilation operator $a(f)$ are adjoint to each other and they are bounded operators with

$$\|a(f)\|, \|a^*(f)\| \leq \|f\|_{\mathfrak{h}} \,.$$

Moreover, they satisfy the following *canonical anti-commutation relations*:

$$\{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0 \,, \quad \{a^*(f), a(g)\} = \langle g, f\rangle_{\mathfrak{h}} \,, \quad \forall f, g \in \mathfrak{h} \,, \tag{2.1}$$

where we use the notation $\{A, B\} := AB + BA$.

Second quantization of operators. Under the Fock space formalism where the number of particles is not a fixed number, fundamental observables such as the number operator, kinetic energy, and interaction energy need to be reformulated which is called the second quantization of operators. We use creation and annihilation operators to define the second quantization of one-particle and two-particle operators which act on the Fock space $\mathcal{F}(\mathfrak{h})$ where the number of particles is not a fixed number. Given a one-particle operator $A : \mathfrak{h} \to \mathfrak{h}$, we define its second quantization $d\Gamma(A)$ on $\mathcal{F}(\mathfrak{h})$ through

$$\big(d\Gamma(A)\Psi\big)^{(n)} := \sum_{j=1}^{n} A_j \, \psi^{(n)},$$

where $A_j := 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1$ is the operator acting as A on the j-th particle and as the identity on the other $(n-1)$ particles. By considering a basis $\{f_j\}_{j\in\mathbb{N}}$ for the one-particle space $\mathfrak{h}$, we are able to write $d\Gamma(A)$ as a quadratic expression in creation and annihilation operators, as

$$d\Gamma(A) = \sum_{j,\ell\in\mathbb{N}} \langle f_j, Af_\ell\rangle \; a^*(f_j)\, a(f_\ell)\,. \tag{2.2}$$

Among the most important examples of the second quantization of one-particle operators is the *number of particles operator*

$$\mathcal{N} = d\Gamma(1),$$

which acts on the n-th sector as multiplication by n. Its expectation therefore measures the average number of particles contained in a vector.

As for a two-particle operator $B : \mathfrak{h} \wedge \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$, we define its second quantization $d\Gamma(B)$ on $\mathcal{F}(\mathfrak{h})$ by

$$\big(d\Gamma(B)\Psi\big)^{(n)} := \sum_{i<j}^{n} B_{ij} \, \psi^{(n)},$$

where B_{ij} is the operator acting as B on the i-th and j-th particles and as the identity on all other particles. Similarly, we can also write as a quartic expression in creation and annihilation operators as

$$d\Gamma(B) = \frac{1}{2} \sum_{j,\ell,k,m} \big\langle (f_j \otimes f_m),\, B(f_\ell \otimes f_k)\big\rangle \, a^*(f_j)a^*(f_m)a(f_\ell)a(f_k). \tag{2.3}$$

Notice that the second quantization of a two-particle operator if quartic in creation and annihilation operators.

Sometimes, it is convenient to use *operator-valued distributions* a_x^*, a_x for $x \in \mathbb{R}$ which are defined through

$$a(f) := \int dx\, a_x \overline{f(x)}\,, \qquad a^*(f) := \int dx\, a_x^* f(x)\,,$$

which also satisfy the canonical anti-commutation relations:

$$\{a_x, a_y^*\} = \delta(x-y); \qquad \{a_x^*, a_y^*\} = \{a_x, a_y\} = 0\,, \tag{2.4}$$

where δ is the Dirac delta function with $\delta(x) = 0$ if and only if $x = 0$. Now, if the one-particle operator A has integral kernel $A(x;y)$, we may rewrite (2.2) as

$$d\Gamma(A) = \int dxdy\, A(x;y) a_x^* a_y\,. \tag{2.5}$$

Here, we also introduce the notation

$$d\Gamma^+(A) := \int dxdy\, A(x;y)\, a_x^* a_y^*\,, \qquad d\Gamma^-(A) := \int dxdy\, A(x;y)\, a_x a_y\,.$$

If we consider the two-particle operator B being a multiplication operator. Then the operator B has diagonal integral kernel $B(x-y)$, then we may write (2.3) as

$$d\Gamma(B) = \frac{1}{2} \int dxdy\, B(x-y) a_x^* a_y^* a_y a_x\,.$$

This will be a convenient identity for us because the many-body interaction we are going to study is exactly in this form.

Second quantization of other many-body operators can be introduced in a similarly way. We only introduced the two most important kinds for us to use throught this thesis.

Second quantization of the Hamilton operator. On the Fock space $\mathcal{F}(\mathfrak{h})$, we define the second quantization of the Hamilton operator as

$$\mathcal{H} = 0 \oplus \bigoplus_{n \geq 1} H_n\,,$$

where H_n is the Hamilton operator on $\mathfrak{h}^{\wedge n}$ given by:

$$H_n = \sum_{i=1}^n \sqrt{1 - \varepsilon^2 \Delta_i} + \varepsilon^3 \sum_{i<j}^n V(\hat{x}_i - \hat{x}_j)\,.$$

Since we will consider initial data for which the density of particles ϱ is of order ε^{-3}, we get an equivalent expression for $\mathcal{H}$:

$$\mathcal{H} = d\Gamma(\sqrt{1 - \varepsilon^2 \Delta}) + \frac{\varepsilon^3}{2} \int dxdy\, V(x-y) a_x^* a_y^* a_y a_x\,. \tag{2.6}$$

To simplify our analysis, we are going to assume that the interaction V is a real-valued Schwartz function, i.e., $V \in \mathcal{S}(\mathbb{R}^3)$, with

$$\mathcal{S}(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3;\mathbb{R}) : \sup_{x \in \mathbb{R}^3} |x^\alpha \partial^\beta f(x)| < \infty \text{ for all multi-indices } \alpha, \beta \in \mathbb{N}^3\}\,.$$

By the definition of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, one can easily see that Schwartz functions are smooth (C^∞) functions and decay faster than any polynomial at infinity. We would also use the property that the Fourier transform of a Schwartz function is still a Schwartz function for the potential $V \in \mathcal{S}(\mathbb{R}^3)$.

2.2 Mixed states and quasi-free states

We follow similar discussion to [3]. In the description of a quantum mechanical system, we associate states in the system to normalized vectors $\psi \in \mathcal{F}(\mathfrak{h})$. Equivalently, we may describe the state ψ through a density matrix $\rho_\psi = |\psi\rangle\langle\psi|$ which is an orthogonal projection on $\mathcal{F}(\mathfrak{h})$. A density matrix ρ on $\mathcal{F}(\mathfrak{h})$ is a non-negative trace class operator with $\mathrm{tr}\rho = 1$. Any density matrix ρ on $\mathcal{F}(\mathfrak{h})$ can be written as

$$\rho = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n| \tag{2.7}$$

by using spectral theorem, where $\lambda_n \geq 0$, $\sum_n \lambda_n = 1$ and ψ_n is a proper orthonormal sequence on $\mathcal{F}(\mathfrak{h})$. For any bounded self-adjoint operator $\mathcal{O}$ on $\mathcal{F}(\mathfrak{h})$, the expectation of the observable described by $\mathcal{O}$ in the state ρ is given by

$$\mathrm{tr}\,\mathcal{O}\rho = \sum_n \lambda_n \langle\psi_n, \mathcal{O}\psi_n\rangle \ .$$

A significant advantage of using the representation of density matrices is that it can be easily extended to mixed states.

Mixed states. A density matrix ρ is called a *pure state* if there exists a normalized $\psi \in \mathcal{F}(\mathfrak{h})$ such that $\rho = |\psi\rangle\langle\psi|$. As mentioned before, ρ provides an equivalent representation of the state ψ in this case. If instead ρ is a convex combination of orthogonal projections, it describes a *mixed state* in the quantum system. From [1], we notice that the set $\mathcal{L}^2(\mathcal{F}(\mathfrak{h}))$ of Hilbert–Schmidt operators on $\mathcal{F}(\mathfrak{h})$ is isomorphic to $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ by simply considering the isomorphism defined by $|\psi\rangle\langle\phi| \mapsto \psi \otimes \overline{\phi}$, for $\psi, \phi \in \mathcal{F}(\mathfrak{h})$, and extended to the whole $\mathcal{L}^2(\mathcal{F}(\mathfrak{h}))$ by linearity. This allows us to describe each density matrix on $\mathcal{F}(\mathfrak{h})$, which automatically lives in $\mathcal{L}^2(\mathcal{F}(\mathfrak{h}))$, by a vector in the doubled Fock space $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$. More precisely, given a density matrix ρ (2.7), we first find an operator $\widetilde{\kappa}$ on $\mathcal{F}(\mathfrak{h})$ such that $\widetilde{\kappa}\widetilde{\kappa}^* = \rho$, say,

$$\widetilde{\kappa} = \sum_n \varepsilon_n |\psi_n\rangle\langle\phi_n| \ ,$$

with $|\varepsilon_n|^2 = \lambda_n$ and $\{\phi_n\}_n$ being an arbitrary orthonormal set. Then, we are able to describe the density matrix ρ by $\kappa \in \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ with

$$\kappa = \sum_n \varepsilon_n \psi_n \otimes \overline{\phi}_n \ . \tag{2.8}$$

Using such notations, the expectation of the operator $\mathcal{O}$ on $\mathcal{F}(\mathfrak{h})$ in state ρ, or equivalently in the state $\kappa \in \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$, can be rewritten as:

$$\mathrm{tr}\,\mathcal{O}\rho = \mathrm{tr}\,\mathcal{O}\widetilde{\kappa}\widetilde{\kappa}^* = \langle\kappa, (\mathcal{O}\otimes 1)\kappa\rangle_{\mathcal{F}(\mathfrak{h})\otimes\mathcal{F}(\mathfrak{h})} \ . \tag{2.9}$$

We also notice the fact that the doubled Fock space $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ is isomorphic to the Fock space $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$. The unitary map U that realizes the isomorphism is called the exponential law, and it is defined by the relations

$$U(\Omega_{\mathcal{F}(\mathfrak{h})} \otimes \Omega_{\mathcal{F}(\mathfrak{h})}) = \Omega_{\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})}$$

and

$$\begin{aligned} U(a(f)\otimes 1)U^* &= a(f\oplus 0) =: a_l(f) \\ U((-1)^{\mathcal{N}} \otimes a(f))U^* &= a(0\oplus f) =: a_r(f) \ , \end{aligned} \tag{2.10}$$

where $\mathcal{N} = d\Gamma(1)$ is the number of particles operator on $\mathcal{F}(\mathfrak{h})$. We call the operators $a_l(f)$ and $a_r(f)$ respectively the left and right representation of $a(f)$. By hermitian conjugation, we simply obtain

$$\begin{aligned} U(a^*(f)\otimes 1)U^* &= a^*(f\oplus 0) =: a_l^*(f) \\ U((-1)^{\mathcal{N}}\otimes a^*(f))U^* &= a^*(0\oplus f) =: a_r^*(f)\,. \end{aligned} \tag{2.11}$$

Notice that the presence of the operator $(-1)^{\mathcal{N}}$ on (2.10) and (2.11) guarantees that creation operators on the space $\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$ satisfy the canonical anti-commutation relations. It is convenient to introduce the left and right representations of the operator-valued distributions a_x, a_x^* by the relations

$$\begin{aligned} a_l(f) &= \int dx\, a_{x,l}\,\overline{f(x)}, \qquad & a_r(f) &= \int dx\, a_{x,r}\,\overline{f(x)}, \\ a_l^*(f) &= \int dx\, a_{x,l}^*\, f(x), \qquad & a_r^*(f) &= \int dx\, a_{x,r}^*\, f(x) \end{aligned}$$

for all $f\in\mathfrak{h}$. We also define the left and right representations of the second quantization of operators on $\mathfrak{h}$ by

$$\begin{aligned} U\big[d\Gamma(\mathcal{O})\otimes 1\big]U^* &= d\Gamma(O\oplus 0) =: d\Gamma_l(\mathcal{O}), \\ U\big[1\otimes d\Gamma(\mathcal{O})\big]U^* &= d\Gamma(0\oplus\mathcal{O}) =: d\Gamma_r(\mathcal{O}). \end{aligned}$$

The left and right representations of $d\Gamma(\mathcal{O})$ can be written in terms of the operator-valued distributions as

$$d\Gamma_l(\mathcal{O}) = \int dx\,dy\,\mathcal{O}(x;y)\,a_{x,l}^* a_{y,l}, \qquad d\Gamma_r(\mathcal{O}) = \int dx\,dy\,\mathcal{O}(x;y)\,a_{x,r}^* a_{y,r}\,. \tag{2.12}$$

Furthermore, it is convenient to introduce also the following notations

$$d\Gamma_{\sigma\sigma'}^+(\mathcal{O}) := \int dxdy\,\mathcal{O}(x;y) a_{\sigma,x}^* a_{\sigma',y}^*, \qquad d\Gamma_{\sigma\sigma'}^-(\mathcal{O}) := \int dxdy\,\mathcal{O}(x;y) a_{\sigma,x} a_{\sigma',y}\,, \tag{2.13}$$

for $\sigma,\sigma' = l,r$. If $\mathcal{O}$ is a trace-class operator over $\mathfrak{h}$, then $d\Gamma_\sigma\mathcal{O}, d\Gamma_{\sigma\sigma'}^+\mathcal{O}, d\Gamma_{\sigma\sigma'}^-\mathcal{O}$ are bounded operators which is the following lemma from [4].

Lemma 2.1. *Let J be a trace-class operator on $\mathfrak{h}$. Then, for any $\sigma,\sigma' = r,l$ the following bounds hold true:*

$$\big\|d\Gamma_\sigma(J)\big\| \le 2\|J\|_{\mathrm{tr}}, \qquad \big\|d\Gamma_{\sigma\sigma'}^{\pm}(J)\big\| \le 2\|J\|_{\mathrm{tr}}.$$

Time evolution of mixed states. Considering the Hamiltonian $\mathcal{H}$ (2.6) on the Fock space $\mathcal{F}(\mathfrak{h})$, the time evolution of the density matrix ρ is given by

$$\rho_t = e^{-i\mathcal{H}t/\varepsilon}\,\rho\, e^{i\mathcal{H}t/\varepsilon}.$$

Thus, we define the time evolution of $\kappa\in\mathcal{F}(\mathfrak{h})\otimes\mathcal{F}(\mathfrak{h})$ by

$$\kappa_t = e^{-i\mathcal{H}t/\varepsilon}\otimes e^{i\mathcal{H}t/\varepsilon}\,\kappa\,.$$

We denote by $\psi_t = U\kappa_t$ the vector in $\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$ describing the mixed state with density matrix ρ_t. Then

$$\psi_t = U\kappa_t = Ue^{-i(\mathcal{H}\otimes 1-1\otimes\mathcal{H})t/\varepsilon}\kappa = e^{-i\mathcal{L}t/\varepsilon}\psi$$

where the Liouvillian $\mathcal{L}$ is defined by

$$\mathcal{L} = U(\mathcal{H} \otimes 1 - 1 \otimes \mathcal{H})U^*.$$

In terms of the operator valued distributions, we find

$$\begin{aligned} \mathcal{L} = d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) + \frac{\varepsilon^3}{2}\int dxdy\, V(x-y)a^*_{l,x}a^*_{l,y}a_{l,y}a_{l,x} \\ - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) - \frac{\varepsilon^3}{2}\int dxdy\, V(x-y)a^*_{r,x}a^*_{r,y}a_{r,y}a_{r,x}\ . \end{aligned} \tag{2.14}$$

Hence, the expectation of an arbitrary operator $\mathcal{O}$ on $\mathcal{F}(\mathfrak{h})$ in the evolved mixed state is given by

$$\operatorname{tr} \mathcal{O}\rho_t = \langle \psi_t, (\mathcal{O}\otimes 1)\psi_t\rangle = \langle \psi, e^{i\mathcal{L}t/\varepsilon}(\mathcal{O}\otimes 1)e^{-i\mathcal{L}t/\varepsilon}\psi\rangle.$$

Reduced densities. By (2.9) and (2.12), the expectation of the observable $d\Gamma(\mathcal{O})$ in the mixed state described by $\psi \in \mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$ is given by

$$\langle \psi, d\Gamma_l(\mathcal{O})\psi\rangle = \int dxdy\, \mathcal{O}(x,y)\,\langle \psi, a^*_{x,l}a_{y,l}\psi\rangle = \operatorname{tr}_{\mathfrak{h}}\ \mathcal{O}\,\gamma^{(1)}_\psi, \tag{2.15}$$

where $\gamma^{(1)}_\psi$ is the one-particle reduced density associated with ψ, defined as the non-negative trace-class operator on $\mathfrak{h}$with the integral kernel

$$\gamma^{(1)}_\psi(x;y) := \langle \psi, a^*_{y,l}a_{x,l}\psi\rangle\,.$$

It is also useful to introduce the *pairing density* α_ψ associated with ψ. It is defined as the Hilbert–Schmidt operator on $\mathfrak{h}$ with integral kernel

$$\alpha_\psi(x;y) := \langle \psi, a_{y,l}a_{x,l}\psi\rangle\,,$$

with which we obtain, by (2.13),

$$\begin{aligned} \operatorname{tr}_{\mathcal{F}(\mathfrak{h})} d\Gamma^-(\mathcal{O})\rho &= \langle \psi, d\Gamma^-_{ll}(\mathcal{O})\psi\rangle_{\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})} = \operatorname{tr}_{\mathfrak{h}}\mathcal{O}\alpha_\psi \\ \operatorname{tr}_{\mathcal{F}(\mathfrak{h})} d\Gamma^+(\mathcal{O})\rho &= \langle \psi, d\Gamma^+_{ll}(\mathcal{O})\psi\rangle_{\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})} = \overline{\operatorname{tr}_{\mathfrak{h}}\mathcal{O}\alpha_\psi}\ . \end{aligned}$$

It is convenient to combine the reduced one-particle density $\gamma^{(1)}_\psi$ and the pairing density α_ψ in a single non-negative operator $\Gamma_\psi : \mathfrak{h}\oplus\mathfrak{h} \to \mathfrak{h}\oplus\mathfrak{h}$ acting as

$$\Gamma_\psi = \begin{pmatrix} \gamma^{(1)}_\psi & \alpha_\psi \\ \alpha^*_\psi & 1-\overline{\gamma}^{(1)}_\psi \end{pmatrix}. \tag{2.16}$$

Γ_ψ is known as the generalized one-particle reduced density associated with the mixed state described by $\psi \in \mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$. Also, it is easy to check that $0 \le \Gamma_\psi \le 1$ for any $\psi \in \mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$.

Bogoliubov transformation The *Bogoliubov transformation* is introduced in order to factor out the strong reference background from the full evolution of many-body systems. To define Bogoliubov transformations on $\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$, we first introduce the following notations for linear combinations of creation and annihilation operators on $\mathfrak{h}\oplus\mathfrak{h}$. For $f, g \in \mathfrak{h}\oplus\mathfrak{h}$, we define

$$A(f,g) := a(f) + a^*(\overline{g}), \qquad A^*(f,g) := (A(f,g))^* = a^*(f) + a(\overline{g}).$$

Notice that

$$A^*(f,g) = A(Jg, Jf) = A(\mathcal{J}(f,g)) \tag{2.17}$$

where $J : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ is the anti-linear operator defined by $Jf = \overline{f}$ and $\mathcal{J} : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \to (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$ is defined by

$$\mathcal{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

The anti-commutation relations (2.1) translate into

$$\{A(f_1, g_1),\, A^*(f_2, g_2)\} = \langle (f_1,\, g_1),\, (f_2, g_2) \rangle \tag{2.18}$$

for all $f_1, f_2, g_1, g_2 \in \mathfrak{h} \oplus \mathfrak{h}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $(\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$.

A *Bogoliubov map* is a bounded linear map $\nu : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \to (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$ which satisfies

$$\nu^* \nu = 1 \quad \text{and} \quad \nu \mathcal{J} = \mathcal{J} \nu.$$

These two conditions guarantee that the rotated field operators $B(f,g) := A(\nu(f,g))$ satisfy the same relations (2.17) and (2.18) as the original operators $A(f,g)$, for all $f, g \in \mathfrak{h} \oplus \mathfrak{h}$. One can check that every Bogoliubov map ν can be written in the block form

$$\nu = \begin{pmatrix} U & JVJ \\ V & JUJ \end{pmatrix} \tag{2.19}$$

where U, $V : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ are bounded linear maps with

$$U^*U + V^*V = 1, \quad \text{and} \quad U^*JVJ + V^*JUJ = 0 \tag{2.20}$$

A Bogoliubov map $\nu : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$ is said to be *implementable* if there exists a unitary operator $\mathbb{U} : \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h}) \to \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$, called the *Bogoliubov transformation*, such that

$$\mathbb{U}^* A(f,g) \mathbb{U} = A(\nu(f,g)) \tag{2.21}$$

for all f, $g \in \mathfrak{h} \oplus \mathfrak{h}$. A sufficient and necessary condition for a Bogoliubov map to be implementable is the *Shale-Stinespring condition* which thereby guarantees the existence of the Bogoliubov transformation for this map. This condition tells us that the Bogoliubov map (2.19) is implementable if and only if V is a Hilbert-Schmidt operator. The proof of this theorem and more information about Bogoliubov transformations can be found in the lecture notes [6].

Slater determinants. For bosonic many-body systems, the natural analogue of an uncorrelated many-body state is a *factorized* wave function, where all particles occupy the same one-particle state. For fermions, however, such a factorization is impossible because of the anti-symmetry imposed by the Pauli principle. In this sense, Slater determinants play the role of the fermionic states that are closest to factorized states: they describe particles that are as independent as possible, subject only to anti-symmetry. This is why they are fundamental in fermionic many-body theory. Precisely, the Slater determinants, on the n-particle space $\mathfrak{h}^{\wedge n}$, are of the form

$$(f_1 \wedge \cdots \wedge f_n)(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) f_1(x_{\pi(1)}) \cdots f_n(x_{\pi(n)}) \,, \tag{2.22}$$

where $\{f_i\}_{i=1}^n$ is an orthonormal system in $\mathfrak{h}$. They can be written using creation and annihilation operators as $a^*(f_1)\dots a^*(f_n)\Omega$. A useful fact for Slater determinants if that they can be generated by the action of unitary transformations on the vacuum. In particular, given an arbitrary orthogonal projection ω onto the n-dimensional subspace of the one-particle space $\mathfrak{h}$ spanned by $f_1, \cdots, f_n$, i.e.,

$$\omega = \sum_{i=1}^{n} |f_i\rangle\langle f_i|\,,$$

we can find a Bogoliubov transformation R_ω on $\mathcal{F}(\mathfrak{h})$ such that

$$R_\omega\Omega = a^*(f_1)\dots a^*(f_n)\Omega\,,$$

is the Slater determinant and the following holds:

$$R_\omega^* a(f) R_\omega = a(uf) + a^*(v\bar{f})$$

where $u = 1-\omega$ is the projection onto the orthogonal complement of the space spanned by $f_1,\dots,f_n$, and $v = \sum_{i=1}^n |\bar{f}_i\rangle\langle f_i|$.

Quasi-free states. After introducing Slater determinants, one may regard quasi-free states as their natural generalization. Also, quasi-free states provides a natural class of reference states for fermionic many-body systems in the semiclassical scaling. A vector $\psi \in \mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$ is said to describe a quasi-free state if for all $f_1,\dots,f_{2n+1} \in \mathfrak{h}$,

$$\langle\psi, a_l^{\sharp_1}(f_1)\dots a_l^{\sharp_{2n+1}}(f_{2n+1})\psi\rangle = 0,$$

and for all $f_1,\dots,f_{2n} \in \mathfrak{h}$, the Wick's rule

$$\langle\psi, a_l^{\sharp_1}(f_1)\dots a_l^{\sharp_{2n}}(f_{2n})\psi\rangle = \sum_{\pi}\sigma_\pi \prod_{j=1}^{n}\langle\psi, a_l^{\sharp_{i_j}}(f_{i_j}) a_l^{\sharp_{\ell_j}}(f_{\ell_j})\psi\rangle \tag{2.23}$$

holds, where $a^\sharp(f)$ denotes either $a(f)$ or $a^*(f)$ and where the sum on the r.h.s. of (2.23) runs over all pairings π, mapping the n indices $i_1,\dots,i_n$ into $\ell_1,\dots,\ell_n$.

Hence, as long as we are interested in expectations of the physically relevant left-observables (i.e., observables expressed through left creation and annihilation operators), quasi-free states are completely characterized by their generalized one-particle reduced density (2.16). Notice that for every $\Gamma : \mathfrak{h}\oplus\mathfrak{h} \to \mathfrak{h}\oplus\mathfrak{h}$ satisfying $0 \le \Gamma \le 1$, there exists a quasi-free state on $\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$ with generalized one-particle density matrix Γ; More precisely, for every hermitian operator ω and symmetric operator α on $\mathfrak{h}$, with $0 \le \omega \le 1$, $\operatorname{tr}\omega < \infty$ and $\alpha^*\alpha \le \omega(1-\omega)$, there exists a unique quasi-free state ψ on $\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})$ with ω as reduced density matrix and α as pairing density. In fact, the quasi-free state ψ is a pure state on $\mathcal{F}(\mathfrak{h})$ if and only if $\alpha^*\alpha = \omega(1-\omega)$ holds as an equality, rather than just as an inequality; see *e.g.* [2, 6] for mathematical reviews. Quasi-free states associated with an *orthogonal projection* ω and with $\alpha = 0$ are Slater determinants on $\mathcal{F}(\mathfrak{h})$, which is relevant to the equilibrium states of non-interacting Fermi gases at *zero temperature*. In this work, we will focus on the class of quasi-free states, describing equilibrium states at *positive temperature*, i.e., we restrict our attention to the quasi-free states with vanishing pairing density $\alpha = 0$ and one-particle reduced density $0 \le \omega \le 1$, which is not an orthogonal projection any more.

It turns out that quasi-free states can be conveniently written in terms of Bogoliubov transformations. In particular, the vector $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ describing a quasi-free state can always be written as $\psi = \mathcal{R}_\nu \Omega$, for a suitable Bogoliubov map ν on $(\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$; see [6]. For the class of quasi-free state $\Psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$, with vanishing pairing density $\alpha = 0$ and one-particle reduced density $0 \leq \omega \leq 1$, which we are going to study, their generalized one-particle reduced density is of the form

$$\Gamma^{(1)} = \begin{pmatrix} \omega & 0 \\ 0 & 1 - \overline{\omega} \end{pmatrix} . \tag{2.24}$$

It is then easy to check that the quasi-free state Ψ with generalized reduced density (2.24) can be written as $\Psi = \mathcal{R}_\nu \Omega$, with the Bogoliubov map $\nu : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \to (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$ defined by

$$\nu = \begin{pmatrix} U & \overline{V} \\ V & \overline{U} \end{pmatrix} , \tag{2.25}$$

where

$$U = \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix} , \qquad V = \begin{pmatrix} 0 & \overline{v} \\ -v & 0 \end{pmatrix} , \tag{2.26}$$

and $u = \sqrt{1 - \omega}$, $v = \sqrt{\omega}$. Notice that U, V satisfy the relations (2.20). Since $\operatorname{tr} V^* V = 2 \operatorname{tr} \omega = 2N < \infty$, one can simply check that the Bogoliubov map ν is implementable by the Shale-Stinespring condition.

Conversely, given the vanishing pairing density $\alpha = 0$ and the same one-particle reduced density ω, we can find the corresponding quasi-free state Ψ by lifting the vacuum Ω by the same Bogoliubov transformation $\mathcal{R}_\nu$, i.e., $\Psi = \mathcal{R}_\nu \Omega$. Since the notation $\mathcal{R}_\nu$ is not so involved in our context, we change to the notation $\mathcal{R}_\omega$ to denote the Bogoliubov transformation which lifts the vacuum Ω to the quasi-free state with vanishing pairing density and one-particle reduced density ω.

Also notice that, for any $f \in \mathfrak{h}$,

$$\begin{aligned} \mathcal{R}_\omega^* a_l(f) \mathcal{R}_\omega &= a_l(uf) - a_r^*(\overline{v}\overline{f}), \\ \mathcal{R}_\omega^* a_r(f) \mathcal{R}_\omega &= a_r(\overline{u}f) + a_l^*(v\overline{f}). \end{aligned}$$

Equivalently,

$$\begin{aligned} \mathcal{R}_\omega^* a_{x,l} \mathcal{R}_\omega &= a_l(u_x) - a_r^*(\overline{v}_x), \\ \mathcal{R}_\omega^* a_{x,r} \mathcal{R}_\omega &= a_r(\overline{u}_x) + a_l^*(v_x), \end{aligned} \tag{2.27}$$

where we used the notation $u_x(y) = u(y;x)$, $v_x(y) = v(y;x)$, and where $u(y;x)$ and $v(y;x)$ denote the kernels of the operators u, v. From (2.27), we also note the action of the Bogoliubov transformation $\mathcal{R}_\omega$ on operators of the form $d\Gamma_\sigma(\mathcal{O})$, which will play an important role in our analysis, [5, Lem. 3.2.].

Lemma 2.2. *Let $0 \leq \omega \leq 1$ and $\mathcal{R}_\omega$ denote the Bogoliubov transformation defined in (2.27), generating the quasi-free state with reduced density ω and with vanishing pairing density. Let $u = \sqrt{1 - \omega}$ and $v = \sqrt{\omega}$. Then, for any operator $\mathcal{O}$ on $\mathfrak{h}$, we have the identities*

$$\begin{aligned} \mathcal{R}_\omega^* d\Gamma_l(\mathcal{O}) \mathcal{R}_\omega &= \operatorname{tr}(\mathcal{O} v^* v) + d\Gamma_l(u\mathcal{O}u^*) - d\Gamma_r(\overline{v}\mathcal{O}\overline{v}^*) - d\Gamma_{lr}^+(u\mathcal{O}v^*) - d\Gamma_{rl}^-(v\mathcal{O}u^*) , \\ \mathcal{R}_\omega^* d\Gamma_r(\mathcal{O}) \mathcal{R}_\omega &= \operatorname{tr}(\mathcal{O}\overline{v}^*\overline{v}) + d\Gamma_l(\overline{u}\mathcal{O}\overline{u}^*) - d\Gamma_l(v\mathcal{O}v^*) + d\Gamma_{rl}^+(\overline{u}\mathcal{O}\overline{v}^*) + d\Gamma_{lr}^-(\overline{v}\mathcal{O}\overline{u}^*) . \end{aligned} \tag{2.28}$$

3 Main Result

The results of this work are mainly based on the framework developed in [5]. We are going to extend the main intermediate theorem [5, Theorem 3.7] in the following sense.

3.1 Assumptions on initial data

Recall that the main intermediate theorem in [5, Theorem 3.7] describes the many-body evolution, at low but positive temperature, of initial data with local semiclassical structure is close to the Hartree evolution in a local sense. In this work, we revealed that the time evolution of the *kinetic energy* of initial data with a stronger local semiclassical structure is also close to that under the Hartree evolution in a similar sense. In order to look at the local behavior in the position space, the spatial weight

$$\mathcal{W}_z^{(n)}(\hat{x}) := \frac{1}{1+|\hat{x}-z|^{4n}}\,, \tag{3.1}$$

localized around $z \in \mathbb{R}^3$, for $2n \in \mathbb{N}$, was introduced in [5, Eq. (2.11)]. This is a smooth localization with well controlled tail. For simplicity, we use the notation $\mathcal{W}_z^{(n)} := \mathcal{W}_z^{(n)}(\hat{x})$ to denote this multiplication operator. We notice two frequently used properties of the spatial weight $\mathcal{W}_z$:

$$\begin{aligned} \mathcal{W}_z^{(m)} &\leq 2\mathcal{W}_z^{(n)} \quad \forall\ m > n\,; \\ \mathcal{W}_{z_1}^{(n)}(x)\mathcal{W}_{z_2}^{(n)}(x) &\leq C_n \mathcal{W}_{z_1}^{(n)}(z_2)\,. \end{aligned} \tag{3.2}$$

We also introduce the *local kinetic energy operator* (or sometimes called local velocity operator) based on the spatial weight. Since the operator $\sqrt{1-\varepsilon^2\Delta}$ is associated to the kinetic energy of the pseudo-relativistic Fermi gases, for $2n \in \mathbb{N}$, we define the local kinetic energy operator as

$$\widetilde{\mathcal{W}}_z^{(n)} := \mathcal{W}_z^{(n)}\sqrt{1-\varepsilon^2\Delta}\mathcal{W}_z^{(n)} \tag{3.3}$$

to capture the local kinetic energy around the point $z \in \mathbb{R}^3$. The smooth localization on both sides is designed to guarantee the self-adjointness of the operator $\widetilde{\mathcal{W}}_z^{(n)}$. Notice that the local kinetic energy operator $\widetilde{\mathcal{W}}_z^{(n)}$ is no longer a multiplication operator, so similar properties to (3.2) are not immediate but can be expected with more effort. We will derive them in the later context.

We collect some statements of assumptions on the initial data. The first assumption is a uniform estimate for the local density and local kinetic energy of the quasi-free state.

Assumption 3.1 (Bounded density and bounded kinetic energy)**.** *Let $0 \leq \omega \leq 1$. Let $\mathcal{W}_z^{(1)}$ be the spatial weight and $\widetilde{\mathcal{W}}_z^{(1)}$ be the local kinetic energy operator, localizing around the point $z \in \mathbb{R}^3$ as introduced in (3.1) and (3.3) respectively. We assume that there exist some constants C such that both of the following hold:*

$$\operatorname{tr} \mathcal{W}_z^{(1)}\omega \leq C\varepsilon^{-3} \tag{3.4}$$

$$\operatorname{tr} \widetilde{\mathcal{W}}_z^{(1)}\omega \leq C\varepsilon^{-3}\,, \tag{3.5}$$

for all $z \in \mathbb{R}^3$. We say that ω has bounded density over $\mathbb{R}^3$ if (3.4) holds. We say ω has bounded kinetic energy over $\mathbb{R}^3$ if (3.5) holds.

Remark 3.2. *For the spatial weight $\mathcal{W}_z^{(n)}$, we use its monotonicity in n to get*

$$\operatorname{tr} \mathcal{W}_z^{(n)} \omega \leq C\varepsilon^{-3}$$

if (3.4) holds. As we mentioned before, $\widetilde{\mathcal{W}}_z^{(1)}$ is not a multiplication operator, we are not allowed to use such monotonicity immediately. In fact, a similar extension of (3.5) to all large $n \in \mathbb{N}$ requires further assumptions in the following context.

Indeed, the operator ω which will be defined as a one-particle reduced density on the one-particle space $\mathfrak{h}$ will play a role of the initial data for the non-linear Hartree equation and also the role of the one-particle reduced density of the quasi-free states we are going to study. We also assume that the operator ω exhibits a local semiclassical structure in a domain $\Lambda \subset \mathbb{R}^3$, and also exhibits a similar structure on local kinetic energy in the following sense.

Assumption 3.3 (Enhanced local semiclassical structure)**.** *Let $0 \leq \omega \leq 1$. Let π denotes either ω, $\sqrt{\omega}$ or $1-\sqrt{1-\omega}$. We assume that ω exhibits a local semiclassical structure in $\Lambda \subset \mathbb{R}^3$ in the sense that , for all $n \in \mathbb{N}$ large enough, there exist a constant $C_n > 0$, independent of Λ, such that all of the following hold:*

$$(1+|z-z'|^{2n-4})\left\|\mathcal{W}_z^{(n)}\omega\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\varepsilon^{-3}\,, \tag{3.6}$$

$$(1+|z-z'|^{2n})\left\|\mathcal{W}_z^{(n)}\left[\varepsilon\nabla,\pi\right]\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\varepsilon^{-2} \tag{3.7}$$

$$(1+|z-z'|^{2n})\left\|\mathcal{W}_z^{(n)}\left[e^{ip\cdot\hat{x}},\pi\right]\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\varepsilon^{-2}(1+|p|) \tag{3.8}$$

$$(1+|z-z'|^{2n})\left\|\mathcal{W}_z^{(n)}\sqrt{\omega}\sqrt{1-\omega}\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\varepsilon^{-2} \tag{3.9}$$

and

$$(1+|z-z'|^{2n-4})\left\|\widetilde{\mathcal{W}}_z^{(n)}\omega\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\varepsilon^{-3} \tag{3.10}$$

$$(1+|z-z'|^{2n})\left\|\widetilde{\mathcal{W}}_z^{(n)}\left[\varepsilon\nabla,\pi\right]\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\varepsilon^{-2} \tag{3.11}$$

$$(1+|z-z'|^{2n})\left\|\widetilde{\mathcal{W}}_z^{(n)}\left[e^{ip\cdot x},\pi\right]\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\,\varepsilon^{-2}(1+|p|) \tag{3.12}$$

$$(1+|z-z'|^{2n})\left\|\widetilde{\mathcal{W}}_z^{(n)}\,\sqrt{\omega}\sqrt{1-\omega}\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\,\varepsilon^{-2} \tag{3.13}$$

for all $p \in \mathbb{R}^3$, $z \in \Lambda$ and $z' \in \mathbb{R}^3$. Here $\mathcal{W}_z^{(n)}$ and $\widetilde{\mathcal{W}}_z^{(n)}$ are the spatial weight (3.1) and local kinetic energy operator (3.3) respectively.

Remark 3.4. *Assumptions (3.6)-(3.9) coincide with the semiclassical structure assumed in [5], while (3.10)-(3.13) which express a strengthened local semiclassical structure of the initial data are additional assumptions made in this thesis. The bound (3.7) means that, after localizing near z, the initial data $\pi \in \{\omega, \sqrt{\omega}, \sqrt{1-\omega}\}$ associated to the reduced density ω commutes "almost" with the momentum operator $\varepsilon\nabla$. The decay in $|z-z'|$ ensures that this property is local. Similarly, the bound (3.8) means that, after localizing around z, the initial π commutes "almost" with the phase $e^{ip\cdot x}$ in a local sense. The factor $(1+|p|)$ on the r.h.s. of (3.8) indicates a first-order control in momentum. For the last bound (3.9), we notice that the value of $\sqrt{\omega}\sqrt{1-\omega}$ measures the overlap between $\sqrt{\omega}$ and $\sqrt{1-\omega}$, which indicates how sharp the transition between $\sqrt{\omega}$ and $\sqrt{1-\omega}$ is. For example, in the case of a sharp Slater determinant, this term would vanish, while for positive temperature or mixed quasi-free states the transition between $\sqrt{\omega}$ and $\sqrt{1-\omega}$ is not sharp. Therefore, the last assumption (3.9) controls the overlap between $\sqrt{\omega}$ and $\sqrt{1-\omega}$ so that the mixing remains small enough in a local sense. Analogously, the assumptions (3.10)-(3.13) are in the sense when these initial data are tested against the local kinetic energy operator $\widetilde{\mathcal{W}}_z^{(n)}$.*

3.2 Main result

We are ready to introduce the main theorem of this work, which tells us that the many-body evolution of kinetic energy of the initial data satisfying the preceding assumptions converges to the kinetic energy given by the Hartree equation in a local sense.

Theorem 3.5 (Convergence to Hartree equation on local kinetic energy at low-temperature)**.** *Let $0 \leq \omega < 1$ be an operator over $\mathfrak{h}$, satisfying Assumption 3.1 (bounded local density and local kinetic energy) and Assumption 3.3 (enhanced local semiclassical structure) in a domain $\Lambda \subset \mathbb{R}^3$. Let $\Xi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ satisfy*

$$\sup_{z\in\mathbb{R}^3} \left\langle \Xi, d\Gamma_\sigma(\mathcal{W}_z^{(n)})\Xi \right\rangle \leq C\varepsilon^{-3}, \qquad \sup_{z\in\Lambda} \left\langle \Xi, d\Gamma_\sigma(\mathcal{W}_z^{(n)})\Xi \right\rangle \leq C\varepsilon^{-3+\delta}, \tag{3.14}$$

$$\sup_{z\in\mathbb{R}^3} \left\langle \Xi, d\Gamma_\sigma(\widetilde{\mathcal{W}}_z^{(n)})\Xi \right\rangle \leq C\varepsilon^{-3}, \qquad \sup_{z\in\Lambda} \left\langle \Xi, d\Gamma_\sigma(\widetilde{\mathcal{W}}_z^{(n)})\Xi \right\rangle \leq C\varepsilon^{-3+\delta}, \tag{3.15}$$

and

$$\begin{aligned} \sup_{z,z'\in\mathbb{R}^3} \left\langle \Xi, d\Gamma_\sigma(\mathcal{W}_z^{(n)}) d\Gamma_{\sigma'}(\mathcal{W}_{z'}^{(n)})\Xi \right\rangle \leq C\varepsilon^{-6}, \\ \sup_{z\in\Lambda} \left\langle \Xi, d\Gamma_\sigma(\mathcal{W}_z^{(n)}) d\Gamma_{\sigma'}(\mathcal{W}_z^{(n)})\Xi \right\rangle \leq C\varepsilon^{-6+2\delta} \ . \end{aligned} \tag{3.16}$$

for all $n \in \mathbb{N}$ large enough, for $\sigma, \sigma' = l, r$, for some $0 < \delta < 1$ and for some $C > 0$ independent of Λ. Let $\mathcal{L}$ denote the Liouvillian (2.14), with $V \in \mathcal{S}(\mathbb{R}^3)$. Consider the time evolution

$$\psi_t = e^{-i\mathcal{L}t/\varepsilon}\mathcal{R}_\omega \Xi,$$

of the initial datum $\mathcal{R}_\omega\Xi$ approximating the quasi-free state with reduced density ω and with vanishing pairing density (the Bogoliubov transformation which lifts the vacuum Ω to the quasi-free state with vanishing pairing density and one-particle reduced density ω.

Let $\gamma_t^{(1)}$ denote the one-particle reduced density matrix associated with ψ_t and let ω_t be the solution of the pseudo-relativistic Hartree equation

$$i\varepsilon\partial_t\omega_t = \left[\sqrt{1-\varepsilon^2\Delta} + \varepsilon^3(V * \varrho_t)\,,\, \omega_t\right],$$

with initial datum $\omega_{t=0} = \omega$. Let $\mathcal{O} \in L^1(\mathbb{R}^3)$ be such that

$$\left\|(1+|\cdot|)D^\alpha\hat{\mathcal{O}}\right\|_1 \leq C, \tag{3.17}$$

for $|\alpha| \leq \ell$ and with $\ell \in \mathbb{N}$ large enough, and let $\mathcal{O}_z(x) = \mathcal{O}(x-z)$. Denote $\mathcal{O}_z := \mathcal{O}_z(\hat{x})$ and let $\widetilde{\mathcal{O}}_z := \mathcal{O}_z\sqrt{1-\varepsilon^2\Delta}\mathcal{O}_z$. Then, there exist constants $C, c > 0$, independent of Λ, such that

$$\left|\mathrm{tr}\ \widetilde{\mathcal{O}}_z(\gamma_t^{(1)} - \omega_t)\right| \leq C\exp\big(c\exp(c|t|)\big)\,\varepsilon^{-3+\delta}, \tag{3.18}$$

for all $t \in \mathbb{R}$ and $z \in \Lambda$ with $B_{\varepsilon^{-\delta}|t|}(z) \subset \Lambda$.

Remark 3.6. *A particularly important special case of Theorem 3.5 is taking the vector $\Xi = \Omega$. In this case, the initial datum reduces to $\Psi_0 = \mathcal{R}_\omega\Omega$, which is exactly the quasi-free state with reduced density ω and vanishing pairing density. Thus, the theorem applies in particular to the genuine quasi-free reference state itself. The more general formulation in Theorem 3.5 with initial data of the form $\mathcal{R}_\omega\Xi$ allows for perturbations of this reference state by few excitations. Under assumptions from (3.14) to (3.16), the excitations Ξ remain sufficiently controlled, so that $\mathcal{R}_\omega\Xi$ can be interpreted as a state approximating the quasi-free state $\mathcal{R}_\omega\Omega$. Therefore, Theorem (3.5) does not only describe the evolution of the exact quasi-free initial datum, but also the class of initial states which are close to it in the sense of having only a sufficiently small number of excitations.*

Remark 3.7. *The main extension of this work from [5] is by additionally assume bounded local kinetic energy (3.5), local semiclassical structure on kinetic energy (3.10)-(3.13), (3.15) and (3.16) to obtain the local convergence (3.18) on local kinetic energy at low but positive temperature while the related theorem in [5, Theorem 3.7] saw the local convergence*

$$\left|\operatorname{tr} \mathcal{O}_z(\gamma_t^{(1)} - \omega_t)\right| \leq C \exp(c \exp(c|t|))\, \varepsilon^{-3+\delta}$$

on the number of particles without these additional assumptions. In other words, by proper assumptions on the initial data, the Hartree equation well approximates not only the number of local particles and also the local kinetic energy up to the given error.

The proof of 3.5 relies on similar argument as in [5]. Not only do we need to establish analogous preliminary results to those as in [5] but also use the existing results on the local density of particles from [5]. We are going to present the proof in the following context.

3.3 Possible extension

Theorem 3.5 is an extension of the main intermediate theorem in [5, Theorem 3.7] . The theorem in [5, Theorem 3.7] established the convergence towards the Hartree equation on local density of particles *at low but positive temperature*. In fact, they further established the convergence *at zero temperature* by approximating the equilibrium state at zero temperature (which is a pure state with reduced density ω_μ) by a mixed state of reduced density given by Fermi-Dirac distribution

$$\omega_{\mu,\beta} = \frac{1}{1 + e^{\beta(H-\mu)}} \tag{3.19}$$

at low temperature $T = 1/\beta = o(\varepsilon)$. [5, Theorem 2.3] By showing that the Fermi-Dirac distribution $\omega_{\mu,\beta}$ satisfies all the assumptions that the main intermediate theorem [5, Theorem 3.7] requires (i.e., local semiclassical structure), they applied the intermediate theorem[5, Theorem 3.7] directly to conclude the main theorem [5, Theorem 2.3]. The key gap from [5, Theorem 3.7] to [5, Theorem 2.3] is to show the Fermi-Dirac distribution $\omega_{\mu,\beta}$ exhibits the local semiclassical structure ((3.4) in Assumption 3.1 (bounded local density and bounded local kinetic energy) and (3.6)-(3.9) in Assumption 3.3 (enhanced local semiclassical structure)).

Analogously, it is highly feasible to follow the same approach to show that the Fermi-Dirac distribution $\omega_{\mu,\beta}$ satisfies additional semiclassical structure *on kinetic energy* ((3.5) in Assumption 3.1 (bounded local density and bounded local kinetic energy) and (3.10)-(3.13) in Assumption 3.3 (enhanced local semiclassical structure)) with an assumption

$$\sup_{z\in\Lambda} \left\| \frac{1}{(\beta(H-\mu))^{2m} + 1} \widetilde{\mathcal{W}}_z^{(n)} \right\|_{\mathrm{tr}} \leq C\varepsilon^{-2} \tag{3.20}$$

on the Hamiltonian H and chemical potential μ, additional to the assumption made in [5, Assumption 2.1]. Hence, an analogous result to [5, Theorem 2.3] can be expected by following the framework developed in [5]. In fact, the procedure to show this will be quite similar to the the methods we introduced in this thesis. The replacement of the spatial weight $\mathcal{W}_z^{(n)}$ by the local kinetic energy $\widetilde{\mathcal{W}}_z^{(n)}$ yields extra commutators and the newly arising terms can be dealt with by using similar approach to those in this work. Such an extension will provide an additional evidence that the Hartree equation well approximates the effective dynamics of not only the local number of particles but also the local kinetic energy at zero temperature.

4 Proof of Theorem 3.5

4.1 Propagation of local semiclassical structure along Hartree equation

We first propagate the bounds in Assumption 3.1 (bounded local density and bounded local kinetic energy) and the local semiclassical structure in Assumption 3.3 (enhanced local semiclassical structure) along the Hartree equation. We aims to show the solution of the Hartree equation still satisfies the preceding assumptions and exhibits the local semiclassical structure on kinetic energy.

Proposition 4.1 (Propagation of the Local Semiclassical Structure)**.** *Let $V \in \mathcal{S}(\mathbb{R}^3)$. Suppose that $0 \leq \omega \leq 1$ Assumption 3.1 (bounded local density and bounded local kinetic energy) and Assumption 3.3 (enhanced local semiclassical structure) in a domain $\Lambda \subset \mathbb{R}^3$. Let ω_t denote the solution of the Hartree equation*

$$i\varepsilon\partial_t\omega_t = \left[\sqrt{1-\varepsilon^2\Delta} + \varepsilon^3(V * \rho_t)\,,\, \omega_t\right] \tag{4.1}$$

with $\rho_t(x) = \omega_t(x;x)$ and initial datum $\omega_{t=0} = \omega$. Then, there exist $C, c > 0$ such that

$$\sup_{z\in\mathbb{R}^3} \operatorname{tr} \widetilde{\mathcal{W}}_z^{(1)}\omega_t \leq C\exp(c|t|)\varepsilon^{-3} \tag{4.2}$$

for all $t \in \mathbb{R}$. Moreover, for all $n \in \mathbb{N}$ large enough there are constants $C_n, c_n > 0$ independent of Λ, such that

$$\sup_{z\in\Lambda,\, z'\in\mathbb{R}^3} \left(1+|z-z'|^{2n-4}\right) \left\|\widetilde{\mathcal{W}}_z^{(n)}\,\omega_t\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\exp(c_n|t|)\varepsilon^{-3} \tag{4.3}$$

and, with π_t denoting either ω_t, $\sqrt{\omega_t}$ or $\sqrt{1-\omega_t}$,

$$\begin{aligned} \sup_{z\in\Lambda,\, z'\in\mathbb{R}^3,\, p\in\mathbb{R}^3} \frac{1+|z-z'|^{2n}}{1+|p|} \left\|\widetilde{\mathcal{W}}_z^{(n)}\,[\pi_t, e^{ip\cdot x}]\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} &\leq C_n\exp(c_n|t|)\varepsilon^{-2}, \\ \sup_{z\in\Lambda,\, z'\in\mathbb{R}^3} \left(1+|z-z'|^{2n}\right) \left\|\widetilde{\mathcal{W}}_z^{(n)}\,[\pi_t, \varepsilon\nabla]\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} &\leq C_n\exp(c_n|t|)\varepsilon^{-2}, \end{aligned} \tag{4.4}$$

for every $t \in \mathbb{R}$. Finally, for all $n \in \mathbb{N}$, there are constants $C_n, c_n > 0$ such that

$$\sup_{z\in\Lambda,\, z'\in\mathbb{R}^3} \left(1+|z-z'|^{2n}\right) \left\|\widetilde{\mathcal{W}}_z^{(n)}\,\sqrt{\omega_t}\sqrt{1-\omega_t}\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\exp(c_n|t|)\varepsilon^{-2} \tag{4.5}$$

for every $t \in \mathbb{R}$.

Remark 4.2 (Established results from Proposition 4.1 in [5])**.** *This is an extended version of [5, Proposition 4.1]. The results in this work in Proposition 4.1 are based on all assumptions from 3.1 (bounded local density and bounded local kinetic energy) and 3.3 (semiclassical structure of both local particle number and local kinetic energy). If one only admits assumptions on the number of particles ((3.4) and (3.6)-(3.9)), then the following*

hold, as proved in [5, Proposition 4.1]

$$\sup_{z\in\mathbb{R}^3} \operatorname{tr} \mathcal{W}_z^{(1)}\omega_t \leq C\exp(c|t|)\varepsilon^{-3}$$

$$\sup_{z\in\Lambda,\, z'\in\mathbb{R}^3} \left(1+|z-z'|^{2n-4}\right) \left\|\mathcal{W}_z^{(n)}\,\omega_t\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\exp(c_n|t|)\varepsilon^{-3}$$

$$\sup_{z\in\Lambda,\, z'\in\mathbb{R}^3,\, p\in\mathbb{R}^3} \frac{1+|z-z'|^{2n}}{1+|p|} \left\|\mathcal{W}_z^{(n)}\,[\pi_t, e^{ip\cdot x}]\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\exp(c_n|t|)\varepsilon^{-2}$$

$$\sup_{z\in\Lambda,\, z'\in\mathbb{R}^3} \left(1+|z-z'|^{2n}\right) \left\|\mathcal{W}_z^{(n)}\,[\pi_t, \varepsilon\nabla]\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\exp(c_n|t|)\varepsilon^{-2}$$

$$\sup_{z\in\Lambda,\, z'\in\mathbb{R}^3} \left(1+|z-z'|^{2n}\right) \left\|\mathcal{W}_z^{(n)}\,\sqrt{\omega_t}\sqrt{1-\omega_t}\,\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \leq C_n\exp(c_n|t|)\varepsilon^{-2}\,.$$

Remark 4.3. *It follows from Proposition 4.1 that*

$$\begin{aligned}
\left\|\widetilde{\mathcal{W}}_z^{(n)}\omega_t\right\|_{\mathrm{tr}} &\leq C\exp(c|t|)\varepsilon^{-3} \\
\left\|\widetilde{\mathcal{W}}_z^{(n)}\left[\pi_t,\, e^{ipx}\right]\right\|_{\mathrm{tr}} &\leq C\exp(c|t|)\varepsilon^{-2} \\
\left\|\widetilde{\mathcal{W}}_z^{(n)}\left[\pi_t,\, \varepsilon\nabla\right]\right\|_{\mathrm{tr}} &\leq C\exp(c|t|)\varepsilon^{-2} \\
\left\|\widetilde{\mathcal{W}}_z^{(n)}\sqrt{\omega_t}\sqrt{1-\omega_t}\right\|_{\mathrm{tr}} &\leq C\exp(c|t|)\varepsilon^{-2}
\end{aligned} \tag{4.6}$$

for $n\in\mathbb{N}$ large enough and for all $z\in\Lambda$. The justification is completely the same as [5, Remark 4.2]. For example, to show (4.6), we take $n\geq 2$ so that (4.3) holds true, and by using

$$\frac{1}{C}\int_{\mathbb{R}^3} \mathcal{W}_{z'}^{(n/2)}(x)\,dz' = 1$$

for some $C>0$, we can estimate

$$\begin{aligned}
\left\|\widetilde{\mathcal{W}}_z^{(n)}\omega_t\right\|_{\mathrm{tr}} &\leq \frac{1}{C}\int_{\mathbb{R}^3} dz' \left\|\widetilde{\mathcal{W}}_z^{(n)}\omega_t\mathcal{W}_{z'}^{(n/2)}\right\|_{\mathrm{tr}} \\
&\leq C\exp(c|t|)\varepsilon^{-3}\int_{\mathbb{R}^3} dz' \frac{1}{1+|z-z'|^{2n}} \\
&\leq C\exp(c|t|)\varepsilon^{-3}\,.
\end{aligned}$$

The proof of Proposition 4.1 relies on a bound for evolution of the local kinetic energy operator. We are going to see that the locality of the kinetic energy operator is preserved by the evolution of the Hartree generator in the following sense.

Proposition 4.4 (Propagation of the local kinetic energy operator along Hartree evolution)**.** *Under the same assumption as Proposition 4.1. Consider the pseudo relativistic Hartree generator $U(t;s)$ defined by*

$$i\varepsilon\partial_t U(t;s) = h(t)U(t;s), \qquad U(t;t)=1\,,$$

for

$$h(t) = \sqrt{1-\varepsilon^2\Delta} + \varepsilon^3(V*\varrho_t)(x)\,.$$

Then for all $n\geq 1$, there exist constants $c, C>0$ such that for all $z\in\mathbb{R}^3, t\geq 0$,

$$U(t;s)^*\widetilde{\mathcal{W}}_z^{(n)}U(t;s) \leq Ce^{c|t-s|}\left(\widetilde{\mathcal{W}}_z^{(n)} + \varepsilon\mathcal{W}_z^{(n)}\right)\,.$$

Proof. We use Grönwall argument. Observe that

$$\begin{aligned} & i\varepsilon\partial_t U(t;s)^*\widetilde{\mathcal{W}}_z^{(n)}U(t;s) \\ & \quad = U(t;s)^*\big[\widetilde{\mathcal{W}}_z^{(n)}, h(t)\big]U(t;s) \\ & \quad = U(t;s)^*\big[\widetilde{\mathcal{W}}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]U(t;s) + U(t;s)^*\big[\widetilde{\mathcal{W}}_z^{(n)}, \varepsilon^3 V * \varrho_t\big]U(t;s) \\ & \quad =: \mathrm{I} + \mathrm{II}\,. \end{aligned} \tag{4.7}$$

For term I, we first claim that,

$$\pm\big[\widetilde{\mathcal{W}}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big] \leq C\varepsilon\widetilde{\mathcal{W}}_z^{(n)} + C\varepsilon^2\mathcal{W}_z^{(n)}\,. \tag{4.8}$$

In fact,

$$\begin{aligned} & \big[\widetilde{\mathcal{W}}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big] \\ & \quad = \big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\mathcal{W}_z^{(n)}\sqrt{1-\varepsilon^2\Delta} + \sqrt{1-\varepsilon^2\Delta}\mathcal{W}_z^{(n)}\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\,. \end{aligned}$$

For any $\xi \in L^2(\mathbb{R}^3)$, we bound

$$\Big|\Big\langle \xi, \big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\mathcal{W}_z^{(n)}\sqrt{1-\varepsilon^2\Delta}\;\xi\Big\rangle\Big| \tag{4.9}$$

by passing $\big(\mathcal{W}_z^{(n)}\big)^{1/2}$ through $\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]$, then passing $\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}$ through $\big(\mathcal{W}_z^{(n)}\big)^{1/2}\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\big(\mathcal{W}_z^{(n)}\big)^{1/2}$ to get the main contribution

$$\Big|\Big\langle \xi, \big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big(\mathcal{W}_z^{(n)}\big)^{1/2}\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\big(\mathcal{W}_z^{(n)}\big)^{1/2}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\;\xi\Big\rangle\Big|$$

to (4.9) and higher order errors coming from commutators. By using the bound

$$\Big\|\big(\mathcal{W}_z^{(n)}\big)^{-1/2}\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\big(\mathcal{W}_z^{(n)}\big)^{-1/2}\Big\|_{\mathrm{op}} \leq C\varepsilon \tag{4.10}$$

and the operator bound

$$i\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big] \leq C\varepsilon\mathcal{W}_z^{(n)} \tag{4.11}$$

from [5, Eq. (4.26), Eq. (4.27)] twice, we obtained

$$\begin{aligned} & \Big|\Big\langle \xi, \big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big(\mathcal{W}_z^{(n)}\big)^{1/2}\big[\mathcal{W}_z^{(n)}, \sqrt{1-\varepsilon^2\Delta}\big]\big(\mathcal{W}_z^{(n)}\big)^{1/2}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\;\xi\Big\rangle\Big| \\ & \quad \leq C\varepsilon\Big|\Big\langle \xi, \big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big(\mathcal{W}_z^{(n)}\big)^2\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\xi\Big\rangle\Big| \\ & \quad \leq C\Big\|\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\mathcal{W}_z^{(n)}\xi\Big\|^2 + C\Big\|\big[\mathcal{W}_z^{(n)}, \big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big]\xi\Big\|^2 \\ & \quad \leq C\Big\langle \xi, \widetilde{\mathcal{W}}_z^{(n)}\xi\Big\rangle + C\varepsilon^2\Big\langle \xi, \mathcal{W}_z^{(n)}\xi\Big\rangle\,. \end{aligned}$$

The remaining error terms coming from commutators can be dealt similarly. Therefore, we recall

$$U(t;s)^*\mathcal{W}_z^{(n)}U(t;s) \leq e^{c|t-s|}\mathcal{W}_z^{(n)} \tag{4.12}$$

from [4, Eq.(6.4)] and get

$$\begin{aligned} \mathrm{I} & \leq C\varepsilon U(t;s)^*\widetilde{\mathcal{W}}_z^{(n)}U(t;s) + C\varepsilon^2 e^{c|t-s|}\mathcal{W}_z^{(n)} \\ & \leq C\varepsilon\widetilde{\mathcal{W}}_z^{(n)} + C\varepsilon^2\mathcal{W}_z^{(n)} \end{aligned} \tag{4.13}$$

For term II, since $V \in \mathcal{S}(\mathbb{R}^3)$ and $\varrho_t \in L^1(\mathbb{R}^3)$, we have

$$\left\| \left[\sqrt{1-\varepsilon^2\Delta},\, V * \varrho_t \right] \right\|_{\mathrm{op}} \leq C\varepsilon \left\| \nabla (V * \varrho_t) \right\|_{L^\infty} \leq C\varepsilon \,, \tag{4.14}$$

which implies that

$$\left[\widetilde{\mathcal{W}}_z^{(n)},\, V * \varrho_t \right] = \mathcal{W}_z^{(n)} \left[\sqrt{1-\varepsilon^2\Delta},\, V * \varrho_t \right] \mathcal{W}_z^{(n)} \leq C \mathcal{W}_z^{(n)} \,,$$

where we used the fact that both $\mathcal{W}_z^{(n)}$ and $V * \varrho_t$ are multiplication operators and $\left(\mathcal{W}_z^{(n)}\right)^2 \leq C \mathcal{W}_z^{(n)}$. Therefore, we get a bound for the term II: .

$$\mathrm{II} = U(t;s)^* \left[\widetilde{\mathcal{W}}_z^{(n)},\, \varepsilon^3 V * \varrho_t \right] U(t;s) \leq C\varepsilon^3 e^{c|t-s|} \mathcal{W}_z^{(n)} \tag{4.15}$$

Now, we integrate both sides of (4.7) over τ and combine (4.13) and (4.15) to get, for any $\xi \in L^2(\mathbb{R}^3)$,

$$\begin{aligned}
&\left\langle \xi, U(t;s)^* \widetilde{\mathcal{W}}_z^{(n)} U(t;s) \xi \right\rangle \\
&\quad \leq \left\langle \xi, \widetilde{\mathcal{W}}_z^{(n)} \xi \right\rangle + \frac{C}{\varepsilon} \int_0^t d\tau \left\langle \xi, U(\tau;s)^* \left[\widetilde{\mathcal{W}}_z^{(n)},\, h(t) \right] U(\tau;s) \xi \right\rangle \\
&\quad \leq \left\langle \xi, \widetilde{\mathcal{W}}_z^{(n)} \xi \right\rangle + C\varepsilon e^{c|t-s|} \left\langle \xi, \mathcal{W}_z^{(n)} \xi \right\rangle + C \int_0^t d\tau \left\langle \xi, U(\tau;s)^* \widetilde{\mathcal{W}}_z^{(n)} U(\tau;s) \xi \right\rangle .
\end{aligned}$$

This concludes the proof by the Grönwall lemma. □

Proof of Proposition 4.1. We proceed analogously to [5, Proof of Proposition 4.1]. We consider the Hartree evolution U defined by

$$i\varepsilon \partial_t U(t;s) = h(t) U(t;s),$$

with $U(s;s) = 1$ for all $s \in \mathbb{R}$ and recall that $h(t) = \sqrt{1-\varepsilon^2\Delta} + \varepsilon^3 (V * \varrho_t)(x)$. By using the fact that $\omega_t = U(t;0)^* \omega U(t;0)$ and Proposition 4.4, we immediately get

$$\mathrm{tr}\ \widetilde{\mathcal{W}}_z^{(1)} \omega_t = \mathrm{tr}\ U(t;0) \widetilde{\mathcal{W}}_z^{(1)} U(t;0)^* \omega \leq C e^{ct}\, \mathrm{tr}\ \widetilde{\mathcal{W}}_z^{(1)} \omega + C e^{ct} \varepsilon^{-3} \leq C e^{ct} \varepsilon^{-3} \,.$$

Similarly, we can obtain (4.3) and (4.5) by using Proposition 4.4, the fact that

$$\sqrt{\omega_t} \sqrt{1-\omega_t} = U(t;0)^* \sqrt{\omega} \sqrt{1-\omega}\, U(t;0)$$

and the corresponding bounds at time $t = 0$. To obtain (4.4), we first consider the modified Hartree evolution. For each $p \in \mathbb{R}^3$, we define the modified Hartree evolution U_p by

$$i\varepsilon \partial_t U_p(t;s) = h_p(t) U_p(t;s),$$

with $U_p(s;s) = 1$ and

$$h_p(t) = e^{ip\cdot x} h(t) e^{-ip\cdot x} = \sqrt{1 + \varepsilon^2 (-i\nabla + p)^2} + \varepsilon^3 (V * \varrho_t)(x) \,.$$

Notice that, with the notation $U = U_{p=0}$, the results from Proposition 4.4 can be extended trivially to U_p, i.e.,

$$U_p(t;s)^* \widetilde{\mathcal{W}}_z^{(n)} U_p(t;s) \leq C e^{c|t-s|} \left(\widetilde{\mathcal{W}}_z^{(n)} + \varepsilon \mathcal{W}_z^{(n)} \right) . \tag{4.16}$$

By applying

$$i\varepsilon\partial_t\pi_t = \big[\pi_t, h(t)\big] \,,$$

we obtain

$$i\varepsilon\partial_\tau U(t;\tau)[e^{ip\cdot\hat{x}},\pi_\tau]U_p(t;\tau)^* = U(t;\tau)e^{ip\cdot\hat{x}}\big[\pi_\tau, \varepsilon p\cdot A(p)\big]U_p(t;\tau)^*$$

where

$$A(p) = \int_0^1 ds\, \frac{\varepsilon(-i\nabla + sp)}{\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}} \,.$$

Integrating over τ gives

$$\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[e^{ip\cdot\hat{x}},\pi_t\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad\le \Big\|\widetilde{\mathcal{W}}_z^{(n)}U(t;0)\big[e^{ip\cdot\hat{x}},\pi_0\big]U_p(t;0)^*\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\qquad\quad + |p|\int_0^t \Big\|\widetilde{\mathcal{W}}_z^{(n)}U(t;\tau)e^{ip\cdot\hat{x}}\big[\pi_\tau, A(p)\big]U_p(t;\tau)^*\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} d\tau \,.
\end{aligned}$$

With Proposition 4.4 and (4.16), we claim that

I. it is allowed to replace $[\pi_\tau, A(p)]$ with $[\pi, \varepsilon\nabla]$ up to a constant $C>0$;

II.
$$\begin{aligned}
&e^{-ct}\sup_{z\in\Lambda,z'\in\mathbb{R}^3}(1+|z-z'|^{2n})\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\varepsilon\nabla,\pi_t\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad\le \sup_{z\in\Lambda,z'\in\mathbb{R}^3}(1+|z-z'|^{2n})\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\varepsilon\nabla,\pi_0\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\qquad + C\int_0^t d\tau\, e^{-c\tau}\sup_{z\in\Lambda,z'\in\mathbb{R}^3,p\in\mathbb{R}^3}\frac{(1+|z-z'|^{2n})}{1+|p|}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau, e^{ip\cdot x}\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \,.
\end{aligned}$$

By assuming claim (I), dividing by $(1+|p|)$, multiplying by $(1+|z-z'|^{2n})$ and taking the supremum, we obtain

$$\begin{aligned}
&e^{-ct}\sup_{z\in\Lambda,z'\in\mathbb{R}^3,p\in\mathbb{R}^3}\frac{(1+|z-z'|^{2n})}{1+|p|}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[e^{ip\cdot\hat{x}},\pi_t\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad\le \sup_{z\in\Lambda,z'\in\mathbb{R}^3,p\in\mathbb{R}^3}\frac{(1+|z-z'|^{2n})}{1+|p|}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[e^{ip\cdot\hat{x}},\pi_0\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\qquad + C\int_0^t d\tau\, e^{-c\tau}\sup_{z\in\Lambda,z'\in\mathbb{R}^3}(1+|z-z'|^{2n})\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau,\varepsilon\nabla\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \,.
\end{aligned} \tag{4.17}$$

Then using Grönwall lemma, (4.17) and the claim (II) give the required (4.4).

Proof of claims: For the claim (I), we proceed as [4, Proof of Theorem 6.1, until Eq. (6.23)]. For simplicity, we denote $C_\tau = C\exp(C\tau)$, then by using Proposition 4.4, (4.16) and (4.12) we get

$$\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_z^{(n)}U(t;\tau)^*e^{ip\cdot\hat{x}}\big[\pi_\tau, A(p)\big]U_p(t;\tau)\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad\le C_{t-\tau}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau, A(p)\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} + C_{t-\tau}\varepsilon\Big\|\mathcal{W}_z^{(n)}\big[\pi_\tau, A(p)\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}}
\end{aligned}$$

where the second term on the r.h.s. was already done in [4] and we are going to discuss the first term on the r.h.s.. Observe that

$$\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_z\big[\pi_\tau,\, A(p)\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\quad\le \int_0^1 ds\,\Big\|\mathcal{W}_z^{(n)}\big[\pi_\tau,\,\frac{\varepsilon(-i\nabla+sp)}{\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}}\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\quad\le \int_0^1 ds\,\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau,\varepsilon(-i\nabla)\big]\frac{1}{\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\qquad\qquad+\int_0^1 ds\,\Big\|\widetilde{\mathcal{W}}_z^{(n)}\varepsilon(-i\nabla+sp)\Big[\pi_\tau,\,\frac{1}{\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}}\Big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\quad=:A+B\,.
\end{aligned}$$

For the term A, we notice that

$$\begin{aligned}
&\Big\|\Big(\mathcal{W}_{z'}^{(n/2)}\Big)^{-1}\big(\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}\big)^{-1}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm op}\\
&\quad\le\Big\|\big(\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}\big)^{-1}\Big\|_{\rm op}\\
&\qquad\qquad+\Big\|\Big[\Big(\mathcal{W}_{z'}^{(n/2)}\Big)^{-1},\big(\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}\big)^{-1}\Big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm op}\qquad(4.18)\\
&\quad\le 1+\Big\|\Big[|\hat x-z'|^{2n},\big(\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}\big)^{-1}\Big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm op}\\
&\quad\le 1+C\sum_{1\le|\alpha|\le 2n}\Big\|{\rm ad}_{\hat x}^{(\alpha)}\Big(\big(\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}\big)^{-1}\Big)\Big\|_{\rm op}\le C\,.
\end{aligned}$$

Then, we can control the term A as

$$A\le\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau,\,\varepsilon\nabla\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\,.$$

For the term B, considering the integral representation

$$\frac{1}{\sqrt{D}}=\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\sqrt{\lambda}}\frac{1}{D+\lambda}\,,\qquad(4.19)$$

valid for any self-adjoint $D>0$, we apply it for by substituting $D=1+\varepsilon^2(-i\nabla+sp)^2$:

$$\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_z^{(n)}\varepsilon(-i\nabla+sp)\Big[\pi_\tau,\,\frac{1}{\sqrt{1+\varepsilon^2(-i\nabla+sp)^2}}\Big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\le\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\sqrt{\lambda}}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\frac{\varepsilon(-i\nabla+sp)}{1+\varepsilon^2(-i\nabla+sp)^2+\lambda}\big[\pi_\tau,\varepsilon^2(-i\nabla+sp)^2\big]\frac{1}{1+\varepsilon^2(-i\nabla+sp)^2+\lambda}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\le\sum_j\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\sqrt{\lambda}}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\frac{\varepsilon(-i\nabla+sq)\,\varepsilon(-i\nabla+sp)_j}{1+\varepsilon^2(-i\nabla+sp)^2+\lambda}\big[\pi_\tau,\varepsilon\nabla_j\big]\frac{1}{1+\varepsilon^2(-i\nabla+sp)^2+\lambda}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\\
&\quad+\sum_j\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\sqrt{\lambda}}\Big\|\widetilde{\mathcal{W}}_z^{(n)}\frac{\varepsilon(-i\nabla+sq)}{1+\varepsilon^2(-i\nabla+sp)^2+\lambda}\big[\pi_\tau,\varepsilon\nabla_j\big]\frac{\varepsilon(-i\nabla+sp)_j}{1+\varepsilon^2(-i\nabla+sp)^2+\lambda}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\rm tr}\,,
\end{aligned}\qquad(4.20)$$

where we also use the fact that

$$\big[\pi_\tau,\,\frac{1}{D+\lambda}\big]=-\frac{1}{D+\lambda}\big[\pi_\tau,\,D\big]\frac{1}{D+\lambda}\,.$$

Now, using the notation that

$$Y := \frac{\varepsilon(-i\nabla + sq)\,\varepsilon(-i\nabla + sp)_j}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\,;$$
$$Z := \frac{1}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\,;$$

we can control the integrand

$$\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_z^{(n)} \frac{\varepsilon(-i\nabla + sq)\,\varepsilon(-i\nabla + sp)_j}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\big[\pi_\tau, \varepsilon\nabla_j\big] \frac{1}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad \le \Big\|\big(\mathcal{W}_z^{(n)}\big)^2 Y \big(\mathcal{W}_z^{(n)}\big)^{-2}\Big\|_{\mathrm{op}} \Big\|\big(\mathcal{W}_z^{(n)}\big)^2 \sqrt{1 - \varepsilon^2\Delta}\big[\pi_\tau, \varepsilon\nabla_j\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \Big\|\big(\mathcal{W}_{z'}^{(n/2)}\big)^{-1} Z \mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{op}}\,.
\end{aligned} \tag{4.21}$$

By using similar method as (4.18) and the following bounds, for $|\alpha| \le 4n$,

$$\begin{aligned}
\Big\|\mathrm{ad}_{\hat{x}}^{(\alpha)}\Big(\frac{\varepsilon(-i\nabla + sp)_j \varepsilon(-i\nabla + sp)_k}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\Big)\Big\|_{\mathrm{op}} &\le C(1+\lambda)^{-\frac{|\alpha|}{2}} \\
\Big\|\mathrm{ad}_{\hat{x}}^{(\alpha)}\Big(\frac{1}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\Big)\Big\|_{\mathrm{op}} &\le C(1+\lambda)^{-1-\frac{|\alpha|}{2}}
\end{aligned}$$

we get

$$\begin{aligned}
\Big\|\big(\mathcal{W}_z^{(n)}\big)^2 Y \big(\mathcal{W}_z^{(n)}\big)^{-2}\Big\|_{\mathrm{op}} &\le C \\
\Big\|\big(\mathcal{W}_{z'}^{(n/2)}\big)^{-1} Z \mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{op}} &\le C(1+\lambda)^{-1}\,.
\end{aligned} \tag{4.22}$$

Moreover, the equation (4.10) gives

$$\begin{aligned}
&\Big\|\big(\mathcal{W}_z^{(n)}\big)^2 \sqrt{1 - \varepsilon^2\Delta}\big[\pi_\tau, \varepsilon\nabla_j\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad \le \Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau, \varepsilon\nabla_j\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} + C\varepsilon\Big\|\mathcal{W}_z^{(n)}\big[\pi_\tau, \varepsilon\nabla_j\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}}\,.
\end{aligned} \tag{4.23}$$

Therefore, combining (4.21), (4.22) and (4.23), the first term on the r.h.s. of (4.20) can be bounded as

$$\begin{aligned}
&\sum_j \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \Big\|\widetilde{\mathcal{W}}_z^{(n)} \frac{\varepsilon(-i\nabla + sq)\,\varepsilon(-i\nabla + sp)_j}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\big[\pi_\tau, \varepsilon\nabla_j\big] \frac{1}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} \\
&\quad \le C\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[\pi_\tau, \varepsilon\nabla\big]\mathcal{W}_{z'}^{(n/2)}\Big\|_{\mathrm{tr}} + C\varepsilon^{-1}\exp(c\tau) \\
&\quad \le C\varepsilon^{-2}\exp(c\tau)\,,
\end{aligned}$$

by using Proposition 4.1. The second term on the r.h.s. of (4.20) can be bounded using similar method and the bound, for $|\alpha| \le 4n$,

$$\Big\|\mathrm{ad}_{\hat{x}}^{(\alpha)}\Big(\frac{\varepsilon(-i\nabla + sp)_j}{1 + \varepsilon^2(-i\nabla + sp)^2 + \lambda}\Big)\Big\|_{\mathrm{op}} \le C(1+\lambda)^{-\frac{1+|\alpha|}{2}}\,. \tag{4.24}$$

This concludes the proof of claim (I). The claim (II) can be treated analogously by following [4, Proof of Theorem 6.1, after Eq. (6.24)].

□

Based on Proposition 4.1, we are able to get another two bounds on commutators with π_t where the locality is achieved by sufficiently regular and decaying functions.

Corollary 4.5. *Let $V \in \mathcal{S}(\mathbb{R}^3)$. Suppose that $0 \leq \omega \leq 1$ satisfies Assumption 3.1 (bounded local density and bounded local kinetic energy) and Assumption 3.3 in a region $\Lambda \subset \mathbb{R}^3$. Let ω_t be the solution of the Hartree equation (4.1). Let*

$$F(x) = \int dp\, e^{ip\cdot x}\, \widehat{F}(p), \qquad \int dp\, (1+|p|)\, \left|\partial_p^\alpha \widehat{F}(p)\right| \leq C \tag{4.25}$$

for $|\alpha| \leq \ell$, with $\ell \in \mathbb{N}$ sufficiently large. For any $z \in \mathbb{R}^3$ let $F_z(x) := F(x-z)$, and denote $F_z := F_z(\hat{x})$ and let $\widetilde{F}_z := F_z\sqrt{1-\varepsilon^2\Delta}F_z$. Let π_t denote either ω_t, $\sqrt{\omega_t}$ or $\sqrt{1-\omega_t}$. Then, we have:

$$\sup_{\substack{z\in\Lambda \\ z'\in\mathbb{R}^3}} \left(1+|z-z'|^{2n}\right) \left\| \mathcal{W}_{z'}^{(n/2)}\, [\pi_t, \widetilde{F}_z] \right\|_{\mathrm{tr}} \leq C\, \exp(c|t|)\, \varepsilon^{-2} \tag{4.26}$$

$$\sup_{\substack{z\in\Lambda \\ z'\in\mathbb{R}^3}} \left(1+|z-z'|^{2n}\right) \left\| \widetilde{\mathcal{W}}_{z'}^{(n/2)}\, [\pi_t, F_z] \right\|_{\mathrm{tr}} \leq C\, \exp(c|t|)\, \varepsilon^{-2} \tag{4.27}$$

for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$ large enough.

Remark 4.6. *Corollary 4.5 is an extended version of [5, Corollary 4.4] where F satisfies the same condition as (4.25). In particular, without assuming the conditions on local kinetic energy, one get from [5, Corollary 4.4] that*

$$\sup_{\substack{z\in\Lambda \\ z'\in\mathbb{R}^3}} \left(1+|z-z'|^{2n}\right) \left\| \mathcal{W}_{z'}^{(n/2)}\, [\pi_t, F_z] \right\|_{\mathrm{tr}} \leq C\, \exp(c|t|)\, \varepsilon^{-2} \tag{4.28}$$

and also

$$\sup_{z\in\Lambda} \left\| \left[\pi_t,\, F_z\right] \right\| \leq C \exp(c|t|)\varepsilon^{-2}\,.$$

from [5, Corollary 4.5.]. Similarly, we are allowed to derive from the bound (4.26) also implies that

$$\sup_{z\in\Lambda} \left\| [\pi_t, \widetilde{F}_z(\hat{x})] \right\|_{\mathrm{tr}} \leq C\, \exp(c|t|)\, \varepsilon^{-2}.$$

This can be shown by proceeding similarly as we did in Remark (4.2).

In fact, the bound (4.27) is a more straightforward bound following from exactly similar approach of proof as [5, Corollary 4.5.] because the commutator part remains the same. However, the one in (4.27) changes the structure of the commutator that we are going to bound. Since $\widetilde{F}_z$ is not a multiplication operator as F_z and it contains the pseudo-relativistic kinetic energy operator $\sqrt{1-\varepsilon^2\Delta}$, we need to bound it with the help of some integral representation and other existing results in both Proposition 4.1 and Remark 4.2. Both the bound (4.26) and the bound (4.27) play an important role in the upcoming proofs.

Proof. To Prove (4.27), we proceed analogously as in [5, Proof of Corollary 4.4]. Let $G_z(x) = \left(\mathcal{W}_z^{(n)}(x)\right)^{-2} F_z(x)$ and denote $G_z := G_z(\hat{x})$. We rewrite

$$\begin{aligned} \left\| \widetilde{\mathcal{W}}_{z'}^{(n/2)} [\pi_t, F_z] \right\|_{\mathrm{tr}} &= \left\| \widetilde{\mathcal{W}}_{z'}^{(n/2)} [\pi_t, \left(\mathcal{W}_z^{(n)}\right)^2 G_z] \right\|_{\mathrm{tr}} \\ &\leq \left\| \widetilde{\mathcal{W}}_{z'}^{(n/2)} \left(\mathcal{W}_z^{(n)}\right)^2 [\pi_t, G_z] \right\|_{\mathrm{tr}} + \left\| \widetilde{\mathcal{W}}_{z'}^{(n/2)} [\pi_t, \left(\mathcal{W}_z^{(n)}\right)^2] G_z \right\|_{\mathrm{tr}}. \end{aligned} \tag{4.29}$$

The first term can be treated as

$$
\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_{z'}^{(n/2)}\big(\mathcal{W}_z^{(n)}\big)^2\big[\pi_t, G_z\big]\Big\|_{\mathrm{tr}} \\
&\quad= \Big\|\mathcal{W}_{z'}^{(n/2)}\sqrt{1-\varepsilon^2\Delta}\,\mathcal{W}_{z'}^{(n/2)}\big(\mathcal{W}_z^{(n)}\big)^2\big[\pi_t, G_z\big]\Big\|_{\mathrm{tr}} \\
&\quad\le \frac{C}{1+|z-z'|^{2n}}\Big\|\widetilde{\mathcal{W}}_z^{(n/2)}\big[\pi_t, G_z\big]\Big\|_{\mathrm{tr}} + \frac{C\varepsilon}{1+|z-z'|^{2n}}\Big\|\mathcal{W}_z^{(n)}\big[\pi_t, G_z\big]\Big\|_{\mathrm{tr}} \\
&\quad\le \frac{C}{1+|z-z'|^{2n}}\int dp\,|\hat{G}(p)|(1+|p|)\sup_{p\in\mathbb{R}^3}\frac{1}{1+|p|}\Big\|\widetilde{\mathcal{W}}_z^{(n/2)}\big[\pi_t, e^{ip\cdot\hat{x}}\big]\Big\|_{\mathrm{tr}} + \frac{C\exp(c|t|)\varepsilon^{-1}}{1+|z-z'|^{2n}} \\
&\quad\le \frac{C\exp(c|t|)\varepsilon^{-2}}{1+|z-z'|^{2n}}\,,
\end{aligned}
$$

where we use the fact that G_z satisfies (4.25). Now, consider the second term in (4.29):

$$
\begin{aligned}
&\Big\|\widetilde{\mathcal{W}}_{z'}^{(n/2)}\big[\pi_t, \big(\mathcal{W}_z^{(n)}\big)^2\big]G_z\Big\|_{\mathrm{tr}} \\
&\qquad\le C\Big\|\widetilde{\mathcal{W}}_{z'}^{(n/2)}\big[\pi_t, \mathcal{W}_z^{(n)}\big]\mathcal{W}_z^{(n)}\Big\|_{\mathrm{tr}} \\
&\qquad\le C\int dp\,|\hat{\mathcal{W}}_z^{(n)}(p)|(1+|p|)\sup_{p\in\mathbb{R}^3}\frac{1}{1+|p|}\Big\|\widetilde{\mathcal{W}}_{z'}^{(n/2)}\big[\pi_t, e^{ip\cdot\hat{x}}\big]\mathcal{W}_z^{(n)}\Big\|_{\mathrm{tr}} \\
&\qquad\le \frac{C\exp(c|t|)\varepsilon^{-2}}{1+|z-z'|^{2n}}\,,
\end{aligned}
$$

by Proposition 4.1 and by taking ℓ large enough in (4.25) to make sure that $|G|\le C\mathcal{W}^{(n)}$, which concludes the proof of (4.27).

To prove (4.26), we proceed differently. Observe that

$$
\begin{aligned}
&\Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t,\, \widetilde{F}_z\big]\Big\|_{\mathrm{tr}} \\
&\qquad= \Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t,\, F_z\sqrt{1-\varepsilon^2\Delta}F_z\big]\Big\|_{\mathrm{tr}} \\
&\qquad\le \Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t,\, F_z\big]\sqrt{1-\varepsilon^2\Delta}F_z\Big\|_{\mathrm{tr}} + \Big\|\mathcal{W}_{z'}^{(n/2)}F_z\big[\pi_t,\, \sqrt{1-\varepsilon^2\Delta}\big]F_z\Big\|_{\mathrm{tr}} \\
&\qquad\quad + \Big\|\mathcal{W}_{z'}^{(n/2)}F_z\sqrt{1-\varepsilon^2\Delta}\big[\pi_t,\, F_z\big]\Big\|_{\mathrm{tr}} \\
&\qquad=: \mathrm{I}+\mathrm{II}+\mathrm{III}\,.
\end{aligned}
\tag{4.30}
$$

For term I, we use the fact that F_z decays faster than any power in the position space and (4.10), then bound it as

$$
\begin{aligned}
\mathrm{I} &\le \Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t,\, F_z\big]\sqrt{1-\varepsilon^2\Delta}\big(\mathcal{W}_z^{(n)}\big)^2\Big\|_{\mathrm{tr}}\Big\|\big(\mathcal{W}_z^{(n)}\big)^{-2}F_z\Big\|_{\mathrm{op}} \\
&\le C\Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t,\, F_z\big]\widetilde{\mathcal{W}}_z^{(n)}\Big\|_{\mathrm{tr}} + C\varepsilon\Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t,\, F_z\big]\mathcal{W}_z^{(n)}\Big\|_{\mathrm{tr}} \\
&\le C\int dp\,|\hat{F}(p)|(1+|p|)\sup_{p\in\mathbb{R}^3}\frac{1}{1+|p|}\Big\|\mathcal{W}_{z'}^{(n/2)}\big[\pi_t, e^{ip\cdot\hat{x}}\big]\widetilde{\mathcal{W}}_z^{(n)}\Big\|_{\mathrm{tr}} \\
&\qquad + \frac{C\exp(c|t|)\varepsilon^{-1}}{1+|z-z'|^{2n}} \\
&\le \frac{C\exp(c|t|)\varepsilon^{-2}}{1+|z-z'|^{2n}}\,,
\end{aligned}
\tag{4.31}
$$

by (4.28) and Proposition 4.1. As for the term III, we notice that

$$\begin{aligned}\mathrm{III} &\leq \left\| \mathcal{W}_{z'}^{(n/2)} F_z \big(\mathcal{W}_z^{(n/2)}\big)^{-2} \right\|_{\mathrm{op}} \left\| \big(\mathcal{W}_z^{(n/2)}\big)^2 \sqrt{1-\varepsilon^2\Delta}\big[\pi_t,\, F_z\big] \right\|_{\mathrm{tr}} \\ &\leq \frac{C}{1+|z-z'|^{2n}} \left\| \widetilde{\mathcal{W}}_z^{(n/2)} \big[\pi_t,\, F_z\big] \right\|_{\mathrm{tr}} + \frac{C\varepsilon}{1+|z-z'|^{2n}} \left\| \mathcal{W}_z^{(n/2)} \big[\pi_t,\, F_z\big] \right\|_{\mathrm{tr}} ,\end{aligned}$$

which allows us to use (4.27) and (4.28) to get

$$\mathrm{III} \leq \frac{C\varepsilon^{-2}\exp(c|t|)}{1+|z-z'|^{2n}} \,. \tag{4.32}$$

In order to bound II, we first use the integral representation (4.19) to observe that

$$\begin{aligned}\big[\pi_t,\, P^{1/2}\big] &= \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \big[\pi_t,\, \frac{P}{P+\lambda}\big] \\ &= \frac{1}{\pi}\int_0^\infty d\lambda\, \sqrt{\lambda}(P+\lambda)^{-1}\big[\pi_t,\, P\big](P+\lambda)^{-1} \,,\end{aligned}$$

for any self-adjoint $P > 0$. Considering $P = 1-\varepsilon^2\Delta$ gives

$$\begin{aligned}\mathrm{II} &\leq \frac{1}{\pi}\int_0^\infty d\lambda\, \sqrt{\lambda} \left\| \mathcal{W}_{z'}^{(n/2)} F_z \frac{1}{1-\varepsilon^2\Delta+\lambda} \big[\pi_t,\, \varepsilon^2\Delta\big] \frac{1}{1-\varepsilon^2\Delta+\lambda} F_z \right\|_{\mathrm{tr}} \\ &\leq \sum_j \frac{1}{\pi}\int_0^\infty d\lambda\, \sqrt{\lambda} \left\| \mathcal{W}_{z'}^{(n/2)} F_z \frac{\varepsilon\nabla_j}{1-\varepsilon^2\Delta+\lambda} \big[\pi_t,\, \varepsilon\nabla_j\big] \frac{1}{1-\varepsilon^2\Delta+\lambda} F_z \right\|_{\mathrm{tr}} \\ &\quad + \sum_j \frac{1}{\pi}\int_0^\infty d\lambda\, \sqrt{\lambda} \left\| \mathcal{W}_{z'}^{(n/2)} F_z \frac{1}{1-\varepsilon^2\Delta+\lambda} \big[\pi_t,\, \varepsilon\nabla_j\big] \frac{\varepsilon\nabla_j}{1-\varepsilon^2\Delta+\lambda} F_z \right\|_{\mathrm{tr}} .\end{aligned} \tag{4.33}$$

For the first integrand on the r.h.s. of (4.33), we have

$$\begin{aligned}&\left\| \mathcal{W}_{z'}^{(n/2)} F_z \frac{\varepsilon\nabla_j}{1-\varepsilon^2\Delta+\lambda} \big[\pi_t,\, \varepsilon\nabla_j\big] \frac{1}{1-\varepsilon^2\Delta+\lambda} F_z \right\|_{\mathrm{tr}} \\ &\leq \left\| \mathcal{W}_{z'}^{(n/2)} F_z \frac{\varepsilon\nabla_j}{1-\varepsilon^2\Delta+\lambda} \big(\mathcal{W}_z^{(n/2)}\big)^{-1} \right\|_{\mathrm{op}} \left\| \mathcal{W}_z^{(n/2)} \big[\pi_t,\, \varepsilon\nabla_j\big] \right\|_{\mathrm{tr}} \left\| \frac{1}{1-\varepsilon^2\Delta+\lambda} F_z \right\|_{\mathrm{op}} \\ &\leq \frac{1}{(1+\lambda)^{3/2}} \frac{C\varepsilon^{-2}\exp(c|t|)}{1+|z-z|^{2n}} \,,\end{aligned}$$

by using (4.24), the bound

$$\left\| \frac{1}{1-\varepsilon^2\Delta+\lambda} \right\|_{\mathrm{op}} \leq \frac{1}{1+\lambda}$$

and the result from Proposition 4.1. The second integrand on the r.h.s. of (4.33) can be controlled similarly. Therefore, we can bound

$$\mathrm{II} \leq \frac{C\varepsilon^{-2}\exp(c|t|)}{1+|z-z|^{2n}} \,. \tag{4.34}$$

Combining (4.30), (4.31), (4.32) and (4.34), we conclude the proof of (4.26).

□

4.2 Evolution of local kinetic energy of excitations

Recall that in Theorem 3.5, we are studying the time evolution $\psi_t = e^{-i\mathcal{L}t/\varepsilon}\mathcal{R}_\omega\Xi$ of the initial datum $\mathcal{R}_\omega\Xi$ approximating the quasi-free state with reduced density ω and with vanishing pairing density where $\mathcal{R}_\omega$ denotes the Bogoliubov transformation introduced in (2.27). In order to study the kinetic energy evolution of excitations, we introduce the fluctuation dynamics

$$\mathcal{U}(t;s) := \mathcal{R}^*_{\omega_t} e^{-i\mathcal{L}(t-s)/\varepsilon}\mathcal{R}_{\omega_s}\,. \tag{4.35}$$

Notice that the fluctuation dynamics $\mathcal{U}(t;s)$ is a unitary evolution of excitations obtained by factoring out the time-dependent quasi-free background from the full evolution. It follows that

$$\psi_t = \mathcal{R}_{\omega_t}\mathcal{U}(t;0)\Xi = \mathcal{R}_{\omega_t}\Xi_t \tag{4.36}$$

with $\Xi_t = \mathcal{U}(t;0)\Xi$ describing the time evolution of excitations of the quasi-free state with reduced density ω_t. To show that the local kinetic energy of ψ_t is close to that of quasi-free state with reduced density ω_t and conclude the proof of Theorem 3.5, it is sufficient to prove that the local kinetic energy at the excitation Ξ_t of the quasi-free state is small (of order ε^{-3}), comparing to that of quasi-free state. Hence, here comes the following proposition.

Proposition 4.7 (Growth of local kinetic energy of excitations)**.** *Under the assumption of Theorem 3.5, we know that there exist constants $C, c > 0$ independent of Λ such that*

$$\left\langle \Xi_t, d\Gamma(\widetilde{\mathcal{W}}^{(n)}_z)\Xi_t \right\rangle \leq C\exp\big(c\exp(c|t|)\big)\,\varepsilon^{-3+\delta}$$

for all $n \in \mathbb{N}$ large enough, for all $t \in \mathbb{R}$ and all $z \in \Lambda$ with $B_{\varepsilon^{-\delta}|t|}(z) \subset \Lambda$.

With Proposition 4.7, we are able to prove Theorem 3.5.

Proof of Theorem 3.5. Using (2.15), recalling that $\psi_t = \mathcal{R}_{\omega_t}\Xi_t$ and applying Lemma 2.2, we get

$$\begin{aligned}
&\operatorname{tr}\widetilde{\mathcal{O}}_z(\gamma^{(1)}_t - \omega_t)\\
&\quad= \left\langle \Xi_t, \big(d\Gamma_l(u_t\widetilde{\mathcal{O}}_z u_t) - d\Gamma_r(\overline{v_t}\widetilde{\mathcal{O}}_z\overline{v_t}) - d\Gamma^+_{lr}(u_t\widetilde{\mathcal{O}}_z v_t) - d\Gamma^-_{rl}(v_t\widetilde{\mathcal{O}}_z u_t)\big)\Xi_t \right\rangle
\end{aligned} \tag{4.37}$$

with $u_t = \sqrt{1-\omega_t}, v_t = \sqrt{\omega_t}$. We first we claim that

$$\widetilde{\mathcal{O}}_z \leq C\widetilde{\mathcal{W}}^{(n)}_z + C\varepsilon\mathcal{W}^{(n)}_z \tag{4.38}$$

for some constants $C > 0$. In fact, for any $\xi \in L^2(\mathbb{R}^3)$, we have

$$\begin{aligned}
\left\langle \xi, \widetilde{\mathcal{O}}_z\xi \right\rangle &= \left\langle \xi, \mathcal{O}_z\sqrt{1-\varepsilon^2\Delta}\mathcal{O}_z\xi \right\rangle\\
&\leq 2\left\|\mathcal{O}_z\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\xi\right\|^2 + 2\left\|\big[\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}, \mathcal{O}_z\big]\xi\right\|^2\\
&\leq C\left\|\mathcal{W}^{(n)}_z\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\xi\right\|^2 + C\left\|\big[\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}, \mathcal{O}_z\big](\mathcal{O}_z)^{-1/2}\right\|^2_{\mathrm{op}}\left\|\big(\mathcal{W}^{(n)}_z\big)^{1/2}\xi\right\|^2\\
&\leq C\left\langle \xi, \widetilde{\mathcal{W}}^{(n)}_z\xi \right\rangle + C\left\|\big[\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}, \mathcal{O}_z\big](\mathcal{O}_z)^{-1/2}\right\|^2_{\mathrm{op}}\left\|\big(\mathcal{W}^{(n)}_z\big)^{1/2}\xi\right\|^2\\
&\qquad + C\left\|\big[\mathcal{W}^{(n)}_z, \big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big]\big(\mathcal{W}^{(n)}_z\big)^{-1/2}\right\|^2_{\mathrm{op}}\left\|\big(\mathcal{W}^{(n)}_z\big)^{1/2}\xi\right\|^2\\
&\leq C\left\langle \xi, \widetilde{\mathcal{W}}^{(n)}_z\xi \right\rangle + C\varepsilon\left\langle \xi, \mathcal{W}^{(n)}_z\xi \right\rangle\\
&\leq C\left\langle \xi, \left(\widetilde{\mathcal{W}}^{(n)}_z + \varepsilon\mathcal{W}^{(n)}_z\right)\xi \right\rangle
\end{aligned}$$

where we used $|\mathcal{O}_z| \leq C\mathcal{W}_z^{(n)}$ for $n \in \mathbb{N}$ large enough by the assumption (3.17) on $\mathcal{O}$, and two operator norms arose are bounded using similar method as in [5, Eq.(4.26)–Eq.(4.27)]. Now by considering

$$\mathcal{W}_z^{(n)}\sqrt{1-\varepsilon^2\Delta}\mathcal{W}_z^{(n)} = \Big(\mathcal{W}_z^{(n)}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\Big)\Big(\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\mathcal{W}_z^{(n)}\Big)\,,$$

we get

$$\begin{aligned} u_t\widetilde{\mathcal{W}}_z^{(n)}u_t &= \Big(\big(\mathcal{W}_z^{(n)}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big)u_t + \big[u_t,\, \mathcal{W}_z^{(n)}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big]\Big) \times \ldots \\ &\qquad \Big(u_t\big(\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\mathcal{W}_z^{(n)}\big) + \big[\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\mathcal{W}_z^{(n)},\, u_t\big]\Big) \\ &\leq 2\big(\mathcal{W}_z^{(n)}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big)u_t^2\big(\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\mathcal{W}_z^{(n)}\big) \\ &\qquad + 2\Big|\big[u_t,\, \mathcal{W}_z^{(n)}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big]\Big|^2 \\ &\leq 2\widetilde{\mathcal{W}}_z^{(n)} + 2\Big|\big[u_t,\, \mathcal{W}_z^{(n)}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big]\Big|^2\,, \end{aligned} \tag{4.39}$$

where we used Cauchy-Schwarz inequality and the fact that $u_t^2 \leq 1$. Therefore, we can bound the first term in (4.37) as:

$$\begin{aligned} d\Gamma_l(u_t\widetilde{\mathcal{O}}_z u_t) &\leq Cd\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}) + C\varepsilon d\Gamma_l(\mathcal{W}_z^{(n)}) + C\big\|\big[u_t, (\mathcal{W}_z^{(n)})^{1/2}\big(\sqrt{1-\varepsilon^2\Delta}\big)^{1/2}\big]\big\|_{\mathrm{tr}} \\ &\leq Cd\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}) + C\varepsilon d\Gamma_l(\mathcal{W}_z^{(n)}) + C\varepsilon^{-2}\exp(c|t|)\,, \end{aligned} \tag{4.40}$$

by applying Lemma 2.1 and Corollary 4.5. Similarly, the second term in (4.37) can be bounded as:

$$-d\Gamma_r(\overline{v_t}\widetilde{\mathcal{O}}_z\overline{v_t}) \leq Cd\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}) + C\varepsilon d\Gamma_l(\mathcal{W}_z^{(n)}) + C\varepsilon^{-2}\exp(c|t|)\,.$$

The third term in (4.37) can be bounded by Lemma 2.1, Remarks 4.3 and 4.6 and existing results from [5]:

$$\begin{aligned} \big|\langle\Xi_t, d\Gamma_{lr}^+(u_t\widetilde{\mathcal{O}}_z v_t)\Xi_t\rangle\big| &\leq C\big\|u_t\widetilde{\mathcal{O}}_z v_t\big\|_{\mathrm{tr}} \\ &\leq C\big\|\big[u_t, \widetilde{\mathcal{O}}_z\big]\big\|_{\mathrm{tr}} + C\big\|\widetilde{\mathcal{O}}_z u_t v_t\big\|_{\mathrm{tr}} \leq C\exp(c|t|)\varepsilon^{-2}\,. \end{aligned}$$

because $\widetilde{\mathcal{O}}_z$ fulfills the assumptions of Corollary 4.5, and we have the bound (4.38). The last term in (4.37) can be treated analogously. Finally, it comes to

$$\Big|\operatorname{tr}\,\widetilde{\mathcal{O}}_z(\gamma_t^{(1)} - \omega_t)\Big| \leq C\left\langle\Xi_t, d\Gamma(\widetilde{\mathcal{W}}_z^{(n)})\Xi_t\right\rangle + C\exp(c|t|)\varepsilon^{-2}\,,$$

for $n \in \mathbb{N}$ is large enough, which concludes the proof of Theorem 3.5 by Proposition 4.7. □

4.3 Rough bound on local kinetic energy of excitations

The remaining context is focused on the proof of Proposition 4.7. In order to bound the local kinetic energy of the excitation Ξ_t, we proceed as in [5] where the authors provided a rough priori bound of the order ε^{-3} [5, Lem.4.8.], which valid everywhere, then the improved bound of order $\varepsilon^{-3+\delta}$ for small δ, which only works in a subset Λ [5, Prop.4.6.]. In this subsection, We are going to establish the rough bound of the local kinetic energy which will be valid everywhere in $\mathbb{R}^3$.

Lemma 4.8 (Rough Bound on Kinetic Energy of Excitations)**.** *Let $0 \leq \omega \leq 1$ satisfy Assumption 3.1 (bounded density and bounded kinetic energy). Let $\Xi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ satisfy the first bound in (3.14), (3.15) and (3.16). Let Ξ_t be defined as in (4.36). Then, there exist constants $C, c > 0$ such that*

$$\left\langle \Xi_t, d\Gamma_\sigma(\widetilde{\mathcal{W}}_z^{(n)})\Xi_t \right\rangle \leq C \exp(c|t|)\, \varepsilon^{-3}$$

for all $n \in \mathbb{N}$ large enough, for all $z \in \mathbb{R}^3$ and for $\sigma = l, r$.

Remark 4.9. *Lemma 4.8 is an extension of [5, Lem.4.8.] which does not rely on assumptions on the initial kinetic energy, (3.15) and (3.16). Hence, the rough bound established in [5] is*

$$\left\langle \Xi_t, d\Gamma_\sigma(\mathcal{W}_z^{(n)})\Xi_t \right\rangle \leq C \exp(c|t|)\, \varepsilon^{-3} \tag{4.41}$$

for all $n \in \mathbb{N}$ large enough, for all $z \in \mathbb{R}^3$ and for $\sigma = l, r$, which is a rough estimate on the evolution of the local density of excitations. Following (4.41), the improved bound [5, Prop.4.6.] was built as

$$\left\langle \Xi_t, d\Gamma(\mathcal{W}_z^{(n)})\Xi_t \right\rangle \leq C \exp\big(c \exp(c|t|)\big)\, \varepsilon^{-3+\delta}$$

for all $n \in \mathbb{N}$ large enough, for all $t \in \mathbb{R}$ and all $z \in \Lambda$ with $B_{\varepsilon^{-\delta}|t|}(z) \subset \Lambda$. These two results also play an important role in the proof of the theory in this work.

Proof. We proceed analogously as in [5, proof of Lem.4.8.]. Here, we only consider the case $\sigma = l$ whereas the case $\sigma = r$ can be dealt similarly. Fix $n \geq 4$ large enough. Recall that $\Xi_t = \mathcal{R}^*_{\omega_t}\psi_t$ and we obtain

$$\begin{aligned}\left\langle \Xi_t, d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)})\Xi_t \right\rangle &= \operatorname{tr} \widetilde{\mathcal{W}}_z^{(n)}\omega_t \\ &+ \left\langle \psi_t, \big(d\Gamma_l(u_t\widetilde{\mathcal{W}}_z^{(n)}u_t) - d\Gamma_r(\overline{v_t}\widetilde{\mathcal{W}}_z^{(n)}\overline{v_t}) + d\Gamma^+_{lr}(u_t\widetilde{\mathcal{W}}_z^{(n)}v_t) + d\Gamma^-_{rl}(v_t\widetilde{\mathcal{W}}_z^{(n)}u_t)\big)\psi_t \right\rangle .\end{aligned} \tag{4.42}$$

by applying Lemma 2.1. We first look at the third term on the second line of (4.42). Recall that $\widetilde{\mathcal{W}}_z^{(n)} = \mathcal{W}_z^{(n)}\sqrt{1-\varepsilon^2\Delta}\mathcal{W}_z^{(n)}$ where $\mathcal{W}_z^{(n)}$ is a multiplication operator with the integral kernel being itself and $\sqrt{1-\varepsilon^2\Delta}$ is a pseudo-differential operator with Fourier symbol

$$\xi \mapsto \sqrt{1+\varepsilon^2|\xi|^2}\,.$$

Denote the kernel of the one particle kinetic energy operator $\sqrt{1-\varepsilon^2\Delta}$ in the position space as $K(x;y)$, and the notation

$$u_{t;x}(\cdot) := u_t(\cdot;x)\,, \qquad v_{t;x}(\cdot) := v_t(\cdot;x)\,,$$

we can write the integral kernel of $u_t\widetilde{\mathcal{W}}_z^{(n)}v_t$ as

$$\Big(u_t\widetilde{\mathcal{W}}_z^{(n)}v_t\Big)(x;x') = \int dy\; g_y(x)f_y(x')$$

where we denote

$$g_y(x) := \int ds\; \overline{u_{t;x}}(s)\mathcal{W}_z^{(n)}(s)K(s;y)^{1/2};$$

and

$$f_y(x') := \int ds'\; K(y;s')^{1/2}\mathcal{W}_z^{(n)}(s')v_{t;x'}(s').$$

Then from (2.13), we write

$$d\Gamma_{lr}^{+}(u_t\widetilde{\mathcal{W}}_z^{(n)}v_t) = \int dy\ a_l^*(g_y)\ a_r^*(f_y)\,,$$

and the third term on the second line of (4.42) can be bounded as

$$\begin{aligned}
\left|\left\langle \psi_t, d\Gamma_{lr}^{+}(u_t\widetilde{\mathcal{W}}_z^{(n)}v_t)\psi_t\right\rangle\right| &= \left|\int dy\ \langle \psi_t, a_l^*(g_y)a_r^*(f_y)\psi_t\rangle\right| \\
&\le \int dy\, \left\|a_l(g_y)\psi_t\right\|\left\|a_r^*(f_y)\psi_t\right\| \\
&\le \left(\int dy\, \|f_y\|_{\mathfrak{h}}^2\right)^{1/2}\left(\int dy\, \left\|a_l(\overline{g_y})\psi_t\right\|^2\right)^{1/2} \\
&\le \left(\operatorname{tr}\ \widetilde{\mathcal{W}}_z^{(n)}\omega_t\right)^{1/2}\left\langle \psi_t, d\Gamma_l(u_t\widetilde{\mathcal{W}}_z^{(n)}u_t)\psi_t\right\rangle^{1/2},
\end{aligned}$$

where we used Cauchy-Schwarz inequality twice, $v_t^2 = \omega_t$, and the definition of g_y and f_y in the last inequality. The term involving $d\Gamma_{rl}^{-}(v_t\widetilde{\mathcal{W}}_z^{(n)}u_t)$ on the second line of (4.42) is bounded by using the similar way. The second term on the second line of (4.42) is negligible for the purpose of an upper bound because it is negative. To control the term involving $d\Gamma_l\big(u_t\widetilde{\mathcal{W}}_z^{(n)}u_t\big)$, we use exactly the same way as in [5, Eq. (4.24) and (4.25)] to get

$$\left|\left\langle \psi_t, d\Gamma_l(u_t\widetilde{\mathcal{W}}_z^{(n)}u_t)\psi_t\right\rangle\right| \le C\left\langle \psi_t, d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)})\psi_t\right\rangle + C\varepsilon^{-3}\exp(c|t|)$$

for $n \in \mathbb{N}$ large enough, where we used Assumption 3.1 (bounded kinetic energy). Therefore, it suffices to control $\left\langle \psi_t, d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)})\psi_t\right\rangle$ by using Grönwall argument. Observe that

$$\begin{aligned}
&i\varepsilon\partial_t\left\langle \psi_t, d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)})\psi_t\right\rangle \\
&= \left\langle \psi_t, \left[d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}),\ d\Gamma_l(\sqrt{1-\varepsilon^2\Delta})\right]\psi_t\right\rangle \\
&\qquad + \frac{\varepsilon^3}{2}\int dxdy\, V(x-y)\left\langle \psi_t, \left[d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}),\ a_x^*a_y^*a_xa_y\right]\psi_t\right\rangle \\
&= : \mathrm{I} + \mathrm{II}\,.
\end{aligned} \tag{4.43}$$

For term I, we use (4.8), $\left[d\Gamma_l(A),\ d\Gamma_l(B)\right] = d\Gamma_l(\left[A,\ B\right])$ and the rough bound result

$$\left\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\psi_t\right\rangle \le C\varepsilon^{-3}\exp(c|t|) \tag{4.44}$$

in [5, Lem.4.8.], to get:

$$\begin{aligned}
|\mathrm{I}| &\le C\varepsilon\left\langle \psi_t, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\psi_t\right\rangle + C\varepsilon^2\left\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\psi_t\right\rangle \\
&\le C\varepsilon\left\langle \psi_t, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\psi_t\right\rangle + C\varepsilon^{-1}\exp(c|t|)
\end{aligned} \tag{4.45}$$

for some constants $C, c > 0$.

For II, we first recall that $V \in \mathcal{S}(\mathbb{R}^3)$ decays faster than any power in position space to bound

$$\begin{aligned}
|V(x-y)| &\le \frac{C}{1+|x-y|^{4n}} \\
&\le C\int dz'\, \frac{1}{1+|x-z'|^{4n}}\cdot\frac{1}{1+|y-z'|^{4n}} \\
&= C\int dz'\, \mathcal{W}_{z'}^{(n)}(x)\mathcal{W}_{z'}^{(n)}(y)\,,
\end{aligned}$$

for all $x, y \in \mathbb{R}^3$. Then, we can estimate

$$\begin{aligned}\int dxdy\, V(x-y) \Big\langle \psi_t, \big[d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}),\, a_x^* a_y^* a_x a_y\big]\psi_t\Big\rangle \\ &\leq C\int dxdydz'\, \mathcal{W}_{z'}^{(n)}(x)\mathcal{W}_{z'}^{(n)}(y)\Big\langle \psi_t, \big[d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)}),\, a_x^* a_y^* a_x a_y\big]\psi_t\Big\rangle \\ &\leq C\Big\langle \psi_t, \big[d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma(\mathcal{W}_{z'}^{(n)})^2\big]\psi_t\Big\rangle\end{aligned}$$

by using the representation of $d\Gamma(\,\cdot\,)$ in (2.5) and the fact that the integral kernel of the multiplication operator $\mathcal{W}_{z'}^{(n)}$ is diagonal. Also, Notice that

$$d\Gamma\big(\mathcal{W}_{z'}^{(n)}\big)^2 = d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2 + d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)^2 + 2d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\,.$$

Now, we can estimate

$$\begin{aligned}|\mathrm{II}| &\leq C\varepsilon^3\int dz'\ \Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma(\mathcal{W}_{z'}^{(n)})^2\big]\psi_t\Big\rangle\Big| \\ &\leq C\varepsilon^3\int dz'\ \Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l(\mathcal{W}_{z'}^{(n)})\big]d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\Big\rangle\psi_t\Big| \\ &\qquad + C\varepsilon^3\int dz'\ \Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l(\mathcal{W}_{z'}^{(n)})\big]d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle\Big|\,,\end{aligned} \tag{4.46}$$

where we used the fact that $\big[d\Gamma_l(\,\cdot\,),\, d\Gamma_r(\,\cdot\,)\big] = 0$. We first control the first term on the r.h.s. of (4.46). Observe that

$$\begin{aligned}\pm i\big[d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\big] &= \pm i\ d\Gamma_l\big(\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta},\, \mathcal{W}_{z'}^{(n)}\big]\mathcal{W}_z^{(n)}\big) \\ &\leq C\varepsilon\ d\Gamma_l\big(\mathcal{W}_z^{(n)}\mathcal{W}_{z'}^{(n)}\mathcal{W}_z^{(n)}\big) \\ &\leq \frac{C\varepsilon}{1+|z-z'|^8}\ d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,,\end{aligned} \tag{4.47}$$

where we used the operator bound (4.11) for the second line. Then we expand $d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)$ outside the commutator in terms of creation and annihilation operators by using (2.12) to make use of the operator bound (4.47):

$$\begin{aligned}&C\varepsilon^3\int dz'\ \Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\big]d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle\Big| \\ &\leq C\varepsilon^3\int dz'\int dx\ \mathcal{W}_{z'}^{(n)}\Big|\Big\langle \psi_t, a_{x,l}^* d\Gamma_l\big(\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta},\, \mathcal{W}_{z'}^{(n)}\big]\mathcal{W}_z^{(n)}\big)a_{x,l}\psi_t\Big\rangle\Big| \\ &\qquad + C\varepsilon^3\int dz'\int dx\ \mathcal{W}_{z'}^{(n)}\Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta},\, \mathcal{W}_{z'}^{(n)}\big]\mathcal{W}_z^{(n)}\big),\, a_{x,l}^*\big]a_{x,l}\psi_t\Big\rangle\Big| \\ &=: \mathrm{II}_a + \mathrm{II}_b.\end{aligned}$$

We first bound the term II_b. By using the canonical anti-commutation relation (2.4), we have

$$\begin{aligned}\big[a_{y,l}^* a_{s,l},\, a_{x,l}^*\big] &= a_{y,l}^* a_{s,l} a_{x,l}^* - a_{x,l}^* a_{y,l}^* a_{s,l} \\ &= a_{y,l}^* a_{s,l} a_{x,l}^* + a_{y,l}^* a_{x,l}^* a_{s,l} - a_{y,l}^* a_{x,l}^* a_{s,l} - a_{x,l}^* a_{y,l}^* a_{s,l} \\ &= a_{y,l}^*\{a_{s,l}, a_{x,l}^*\} - \{a_{y,l}^*, a_{x,l}^*\}a_{s,l} \\ &= a_{y,l}^*\delta(s-x)\,.\end{aligned}$$

Based on this, we find the following bound, for any one-particle operator A with integral kernel $A(y;s)$,

$$\begin{aligned}\int dx\ \mathcal{W}_{z'}^{(n)}(x)\big[d\Gamma_l(A),\, a_{x,l}^*\big]a_{x,l} &= \int dx\ \mathcal{W}_{z'}^{(n)}(x)\int dyds\ A(y;s)\big[a_{y,l}^*a_{s,l},\, a_{x,l}^*\big]a_{x,l}\\ &= \int dxdy\ \mathcal{W}_{z'}^{(n)}(x)A(y;x)a_{y,l}^*a_{x,l}\\ &\leq \big\|\mathcal{W}_{z'}^{(n)}\big\|_{\mathrm{op}} d\Gamma_l(A),\end{aligned} \tag{4.48}$$

where we used the canonical anti-commutation relation (2.1). Combing (4.47) and (4.48), we can control II_b as

$$\mathrm{II}_b \leq C\varepsilon^4 \int \frac{dz'}{1+|z-z'|^8}\ \Big|\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\psi_t\Big\rangle\Big| \leq C\varepsilon \exp(c|t|)$$

where we used (4.44) for the last inequality. By applying (4.47), (4.48) and $\big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\psi_t\big\rangle \leq C\varepsilon^{-3}\exp(c|t|)$ again, we bound II_a as

$$\mathrm{II}_a \leq C\varepsilon^4 \int \frac{dz'}{1+|z-z'|^8}\ \Big|\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle\Big| + C\varepsilon\exp(c|t|).$$

Now, it suffices to control the first term on the r.h.s. by Grönwall argument again. Observe that

$$\begin{aligned}&\Big|i\varepsilon\partial_t\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle\Big|\\ &\quad= \Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\psi_t\Big\rangle\Big|\\ &\quad\leq \Big|\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\psi_t\Big\rangle\Big|\\ &\qquad\quad+\Big|\Big\langle \psi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle\Big|\end{aligned} \tag{4.49}$$

where we used the fact that the operator $d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)$ is a multiplication operator. The first term on the r.h.s. of (4.49) can be written as

$$\begin{aligned}&\Big|\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\psi_t\Big\rangle\Big|\\ &\leq \int dx\,\mathcal{W}_z^{(n)}(x)\Big|\Big\langle \psi_t, a_{x,l}^*\big[d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]a_{x,l}\psi_t\Big\rangle\Big|\\ &\quad+\int dx\,\mathcal{W}_z^{(n)}(x)\Big|\Big\langle \psi_t, \Big[a_{x,l}^*,\, \big[d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\Big]a_{x,l}\psi_t\Big\rangle\Big|\ ,\end{aligned} \tag{4.50}$$

where we are going to use (4.10) and (4.48) to bound the first term on the r.h.s. of (4.50) as:

$$\begin{aligned}&\int dx\,\mathcal{W}_z^{(n)}(x)\Big|\Big\langle \psi_t, a_{x,l}^*\big[d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]a_{x,l}\psi_t\Big\rangle\Big|\\ &\qquad\leq C\varepsilon\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\psi_t\Big\rangle + C\varepsilon^2\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle\end{aligned} \tag{4.51}$$

and use (4.48) again to bound the second term on the r.h.s. of (4.50). The analogous result can be obtained for the second term on the r.h.s. of (4.49) and therefore we get the bound based on (4.49):

$$\partial_t\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t\Big\rangle \leq C\Big\langle \psi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\psi_t\Big\rangle + C\varepsilon^{-3}\exp(c|t|)\,.$$

Then Grönwall lemma gives

$$\left\langle \psi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big) d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi_t \right\rangle \leq \left\langle \psi, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big) d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi \right\rangle + C\varepsilon^{-3}\exp(c|t|).$$

We claim that

$$\left\langle \psi, d\Gamma_\sigma\big(\mathcal{W}_z^{(n)}\big) d\Gamma_{\sigma'}\big(\mathcal{W}_{z'}^{(n)}\big)\psi \right\rangle \leq C\varepsilon^{-6} \tag{4.52}$$

for all $z, z' \in \mathbb{R}^3$, and $\sigma, \sigma' = l, r$. By assuming the claim, we get $\mathrm{II}_a \leq C\varepsilon^{-2}\exp(c|t|)$. The second term of (4.46) can be bounded similarly and we drop the commutators of the form $\big[d\Gamma_l(\,\cdot\,),\, d\Gamma_r(\,\cdot\,)\big]$. Then we get $|\mathrm{II}| \leq C\varepsilon^{-2}\exp(c|t|)$. Combining this with (4.43) and (4.45), we get

$$\partial_t \left\langle \psi_t, d\Gamma_l(\widetilde{\mathcal{W}}_z^{(n)})\psi_t \right\rangle \leq C\left\langle \psi_t, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\psi_t \right\rangle + C\varepsilon^{-3}\exp(c|t|)\ .$$

which completes the proof by Grönwall inequality.

Proof of the claim (4.52)**:** We only show the proof for $\sigma, \sigma' = l$, and the other cases can be dealt using the same way. Recall that $\psi = \mathcal{R}_\omega \Xi$ and $\mathcal{R}_\omega$ is unitary. Lemma 2.2 gives

$$\begin{aligned}
&\left\langle \psi, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big) d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\psi \right\rangle \\
&\quad = \left\langle \Xi, \mathcal{R}_\omega^* d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\mathcal{R}_\omega\, \mathcal{R}_\omega^* d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\mathcal{R}_\omega \Xi \right\rangle \\
&\quad = \Big\langle \Xi, \Big[\Big(\operatorname{tr} \mathcal{W}_z^{(n)}\omega + d\Gamma_l(u\mathcal{W}_z^{(n)}u) - d\Gamma_r(\overline{v}\mathcal{W}_z^{(n)}\overline{v}) - d\Gamma_{lr}^+(u\mathcal{W}_z^{(n)}v) - d\Gamma_{rl}^-(v\mathcal{W}_z^{(n)}u)\Big) \times \dots \\
&\qquad \Big(\operatorname{tr} \mathcal{W}_{z'}^{(n)}\omega + d\Gamma_l(u\mathcal{W}_{z'}^{(n)}u) - d\Gamma_r(\overline{v}\mathcal{W}_{z'}^{(n)}\overline{v}) - d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v) - d\Gamma_{rl}^-(v\mathcal{W}_{z'}^{(n)}u)\Big)\Big] \Xi \Big\rangle\ .
\end{aligned}$$

We expand the terms and control them accordingly.

(i) By using Assumption 3.1 (bounded density), we get

$$\left\langle \Xi, \big(\operatorname{tr} \mathcal{W}_z^{(n)}\omega\big)\big(\operatorname{tr} \mathcal{W}_{z'}^{(n)}\omega\big)\Xi \right\rangle \leq C\varepsilon^{-6}\ ;$$

and

$$\begin{aligned}
&\Big\langle \Xi, \big(\operatorname{tr} \mathcal{W}_z^{(n)}\omega\big)\big(\operatorname{tr} \mathcal{W}_{z'}^{(n)}\omega + d\Gamma_l(u\mathcal{W}_{z'}^{(n)}u) - d\Gamma_r(\overline{v}\mathcal{W}_{z'}^{(n)}\overline{v}) - d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v) - d\Gamma_{rl}^-(v\mathcal{W}_{z'}^{(n)}u)\big)\Xi \Big\rangle \\
&\leq C\varepsilon^{-3}\Big\langle \Xi, \big(\operatorname{tr} \mathcal{W}_{z'}^{(n)}\omega + d\Gamma_l(u\mathcal{W}_{z'}^{(n)}u) - d\Gamma_r(\overline{v}\mathcal{W}_{z'}^{(n)}\overline{v}) - d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v) - d\Gamma_{rl}^-(v\mathcal{W}_{z'}^{(n)}u)\big)\Xi \Big\rangle \\
&\leq C\varepsilon^{-6}\,,
\end{aligned}$$

where we used result in [5, last step in the proof of Lem.4.8.] for the last inequality.

(ii) Forget about the negative terms,

$$\Big\langle \Xi, \big(d\Gamma_l(u\mathcal{W}_z^{(n)}u)\big)\big(-d\Gamma_r(\overline{v}\mathcal{W}_{z'}^{(n)}\overline{v}) - d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v) - d\Gamma_{rl}^-(v\mathcal{W}_{z'}^{(n)}u)\big)\Xi \Big\rangle\ ,$$

for the purpose of upper bound.

(iii) Next, we introduce the similar notation

$$u_x(\cdot) := u(\cdot; x)\ , \qquad v_x(\cdot) := v(\cdot; x)\ .$$

Since we are looking at $z, z' \in \mathbb{R}$, we observe that

$$\begin{aligned}
&\Big\langle \Xi, d\Gamma_r\big(\overline{v}\mathcal{W}_z^{(n)}\overline{v}\big) d\Gamma_r\big(\overline{v}\mathcal{W}_{z'}^{(n)}\overline{v}\big)\Xi \Big\rangle \\
&= \int dxdy\ \mathcal{W}_z^{(n)}(x)\mathcal{W}_{z'}^{(n)}(y)\, \langle \Xi, a_r^*(\overline{v_x})a_r(\overline{v_x})a_r^*(\overline{v_y})a_r(\overline{v_y})\Xi\rangle \\
&\leq \Big(\int dx\, \mathcal{W}_z^{(n)}(x)\, \|a_r(\overline{v_x})a_r^*(\overline{v_x})\Xi\| \Big)\Big(\int dy\, \mathcal{W}_{z'}^{(n)}(y)\, \|a_r^*(\overline{v_y})a_r(\overline{v_y})\Xi\| \Big) \\
&\leq \Big(\int dx\, \mathcal{W}_z^{(n)}(x)\, \|\overline{v_x}\|_{\mathfrak{h}}^2 \Big)\Big(\int dy\, \mathcal{W}_{z'}^{(n)}(y)\, \|\overline{v_y}\|_{\mathfrak{h}}^2 \Big) \\
&\leq \big(\operatorname{tr} \mathcal{W}_z^{(n)}\omega \big)\big(\operatorname{tr} \mathcal{W}_{z'}^{(n)}\omega \big) \leq C\varepsilon^{-6}\,,
\end{aligned} \tag{4.53}$$

where we use (2.12) to expand

$$d\Gamma_r\big(v\mathcal{W}^{(n)}v\big) = \int dx\, \mathcal{W}^{(n)}(x)a_r^*(v_x)a_r(v_x)$$

for the first equality on the second line of (4.53), and, for the inequalities on the r.h.s. of (4.53), we used Cauchy-Schwarz inequality and Assumption 3.1 (bounded density).

(iv) To control

$$\Big\langle \Xi, d\Gamma_r\big(\overline{v}\mathcal{W}_z^{(n)}\overline{v}\big) d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v)\Xi \Big\rangle\,,$$

we also use (2.13) to expand

$$d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v) = \int dx\, \mathcal{W}^{(n)}(x)a_l^*(u_x)a_r^*(v_x)\,.$$

Also, by using (2.1), we obtain

$$\begin{aligned}
&[a_r^*(v_x)a_r(v_x), a_l^*(u_y)] \\
&= a_r^*(v_x)a_r(v_x)a_l^*(u_y) - a_l^*(u_y)a_r^*(v_x)a_r(v_x) \\
&= a_r^*(v_x)a_r(v_x)a_l^*(u_y) + a_r^*(v_x)a_l^*(u_y)a_r(v_x) \\
&\qquad\qquad - a_r^*(v_x)a_l^*(u_y)a_r(v_x) - a_l^*(u_y)a_r^*(v_x)a_r(v_x) \\
&= a_r^*(v_x)\,\{a_r(v_x), a_l^*(u_y)\} - \{a_r^*(v_x), a_l^*(u_y)\}\, a_r(v_x) \\
&= 0\,.
\end{aligned}$$

Now, we can control

$$\begin{aligned}
&\Big\langle \Xi, d\Gamma_r\big(\overline{v}\mathcal{W}_z^{(n)}\overline{v}\big) d\Gamma_{lr}^+(u\mathcal{W}_{z'}^{(n)}v)\Xi \Big\rangle \\
&\leq \int dxdy\, \mathcal{W}_z^{(n)}(x)\mathcal{W}_{z'}^{(n)}(y)\, \|v_x\|_{\mathfrak{h}}^2\, \|a_l(u_y)\Xi\|\, \|a_r^*(\overline{v_y})\Xi\| \\
&\leq \big(\operatorname{tr} \mathcal{W}_z^{(n)}\omega \big)\big(\operatorname{tr} \mathcal{W}_{z'}^{(n)}\omega \big)^{1/2} \Big\langle \Xi, d\Gamma_l\big(u\mathcal{W}_{z'}^{(n)}u\big)\Xi \Big\rangle^{1/2} \leq C\varepsilon^{-6}\,,
\end{aligned}$$

where Cauchy-Schwarz inequality twice and the result $\langle \Xi, d\Gamma_l\big(u\mathcal{W}_{z'}^{(n)}u\big)\Xi\rangle \leq C\varepsilon^{-3}$ in [5, Proof of rough bound, Lem.4.8.].

(v) Next we use similar expanding and control

$$\begin{aligned}
&\Big\langle \Xi, d\Gamma^+_{lr}(u\mathcal{W}^{(n)}_z v) d\Gamma^+_{lr}(u\mathcal{W}^{(n)}_{z'} v)\Xi\Big\rangle \\
&\leq \int dxdy\, \mathcal{W}^{(n)}_z(x)\mathcal{W}^{(n)}_{z'}(y)\, \|a_l(u_y)a_l(u_x)\Xi\|\, \|a^*_r(\overline{v_x})a^*_r(\overline{v_y})\Xi\| \\
&\leq \bigg(\int dx\, \mathcal{W}^{(n)}_z \Big\langle \Xi, a^*_l(u_x) d\Gamma_l\big(u\mathcal{W}^{(n)}_{z'}u\big)a_l(u_x)\Xi\Big\rangle\bigg)^{1/2} \big(\operatorname{tr}\mathcal{W}^{(n)}_z\omega\big)^{1/2}\big(\operatorname{tr}\mathcal{W}^{(n)}_{z'}\omega\big)^{1/2} \\
&\leq C\varepsilon^{-3}\bigg(\int dx\, \mathcal{W}^{(n)}_z \Big\langle \Xi, a^*_l(u_x) d\Gamma_l\big(u\mathcal{W}^{(n)}_{z'}u\big)a_l(u_x)\Xi\Big\rangle\bigg)^{1/2}.
\end{aligned}$$

From [5, Eq. (4.24)], one can control

$$u\mathcal{W}^{(n)}_{z'}u \leq 2(u-1)\mathcal{W}^{(n)}_{z'}(u-1) + 4\mathcal{W}^{(n)}_{z'}\,, \tag{4.54}$$

which allows us to we get

$$\begin{aligned}
&\int dx\, \mathcal{W}^{(n)}_z \Big\langle \Xi, a^*_l(u_x) d\Gamma_l\big(u\mathcal{W}^{(n)}_{z'}u\big)a_l(u_x)\Xi\Big\rangle \\
&\leq 4\int dx\, \mathcal{W}^{(n)}_z \Big\langle \Xi, a^*_l(u_x) d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)a_l(u_x)\Xi\Big\rangle + 2\big(\operatorname{tr}\mathcal{W}^{(n)}_{z'}\omega\big)\Big\langle \Xi, d\Gamma_l\big(u\mathcal{W}^{(n)}_z u\big)\Xi\Big\rangle \\
&\leq 4\int dx\, \mathcal{W}^{(n)}_z \Big\langle \Xi, a^*_l(u_x) d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)a_l(u_x)\Xi\Big\rangle + C\varepsilon^{-6}\,.
\end{aligned}$$

Now it remains to control the first term on the r.h.s. on the last line by passing $a^*_l(u_x)$ though $d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)$ and paying the commutator. We get

$$\begin{aligned}
&\int dx\, \mathcal{W}^{(n)}_z \Big\langle \Xi, a^*_l(u_x) d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)a_l(u_x)\Xi\Big\rangle \\
&= \Big\langle \Xi, d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big) d\Gamma_l\big(\mathcal{W}^{(n)}_z\big)\Xi\Big\rangle + C\Big\langle \Xi, d\Gamma_l\big(\mathcal{W}^{(n)}_z\big)\Xi\Big\rangle \leq C\varepsilon^{-6}\,,
\end{aligned}$$

where we used the first bound in (3.16) for the last inequality.

The remaining terms can be dealt analogously by first expanding and then using Cauchy-Schwarz several times as the previous terms, which completes the proof of claim (4.52). □

Remark 4.10. *From this proof we get the following rough bound*

$$\Big\langle \Xi_t, d\Gamma_\sigma\big(\mathcal{W}^{(n)}_z\big) d\Gamma_{\sigma'}\big(\mathcal{W}^{(n)}_{z'}\big)\Xi_t\Big\rangle \leq C\exp(c|t|)\,\varepsilon^{-6} \tag{4.55}$$

for all $z, z' \in \mathbb{R}^3$, and $\sigma, \sigma' = l, r$, which will be useful for the following proofs. In fact, we recall that $\Xi_t = \mathcal{R}^*_{\omega_t}\psi_t$, and notice

$$\Big\langle \Xi_t, d\Gamma_\sigma\big(\mathcal{W}^{(n)}_z\big) d\Gamma_{\sigma'}\big(\mathcal{W}^{(n)}_{z'}\big)\Xi_t\Big\rangle = \Big\langle \psi_t, \mathcal{R}_{\omega_t} d\Gamma_\sigma\big(\mathcal{W}^{(n)}_z\big)\mathcal{R}^*_{\omega_t}\mathcal{R}_{\omega_t} d\Gamma_{\sigma'}\big(\mathcal{W}^{(n)}_{z'}\big)\mathcal{R}^*_{\omega_t}\psi_t\Big\rangle$$

Then the rough bound (4.55) follows by proceeding as the proof of the claim (4.52), and using the result

$$\Big\langle \psi_t, d\Gamma_\sigma\big(\mathcal{W}^{(n)}_z\big) d\Gamma_{\sigma'}\big(\mathcal{W}^{(n)}_{z'}\big)\psi_t\Big\rangle \leq C\exp(c|t|)\,\varepsilon^{-6}\,,$$

from the proof of Lemma 4.8.

4.4 A priori improved bound of excitations

Due to the fact that the local kinetic energy operator $\widetilde{\mathcal{W}}_z^{(n)}$ is not a multiplication operator, we need an extra priori bound (Proposition 4.11) before we reach to the proof of the improved bound of the local kinetic energy of excitations.

Proposition 4.11. *Under the assumption of Theorem 3.5, there exist constants $C, c > 0$ independent of Λ such that*

$$\langle \Xi_t, d\Gamma(\mathcal{W}_z^{(n)})^2 \Xi_t \rangle \leq C \exp\big(c \exp(c|t|)\big)\, \varepsilon^{-6+2\delta} \tag{4.56}$$

for all $n \in \mathbb{N}$ large enough and for all $t \in \mathbb{R}$ and all $z \in \Lambda$ with $B_{\varepsilon^{-2\delta}|t|}(z) \subset \Lambda$.

The proof of Proposition 4.11 and proof of Proposition 4.7 are based on analyzing the generator of the fluctuation dynamics $\mathcal{U}(t;s)$ defined in (4.35). The generator $\mathcal{G}(t)$ is expected so that

$$i\varepsilon \partial_t \mathcal{U}(t;s) = \mathcal{G}(t)\mathcal{U}(t;s)\,,$$

with $\mathcal{U}(s;s) = 1$ for all $s \in \mathbb{R}$. This motivates the definition

$$\mathcal{G}(t) := \big(i\varepsilon\partial_t \mathcal{U}(t;s)\big)\mathcal{U}(t;s)^* = \big(i\varepsilon\partial_t\, \mathcal{R}^*_{\omega_t}\big)\mathcal{R}_{\omega_t} + \mathcal{R}^*_{\omega_t} \mathcal{L}\, \mathcal{R}_{\omega_t}\,, \tag{4.57}$$

where $\mathcal{R}_\omega$ denotes the Bogoliubov transformation introduced in (2.27) and $\mathcal{L}$ is the Liouvillian introduced in (2.14). Let us recall computations of the generator $\mathcal{G}(t)$ from [5, Prop.4.9. and Lem.4.10.].

Proposition 4.12 (Generator of the Fluctuation Dynamics)**.** *The generator $\mathcal{G}(t)$, defined in (4.57), is a self-adjoint operator on $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ and can be written as*

$$\mathcal{G}(t) = d\Gamma_l(h_{\mathrm{H}}(t)) - d\Gamma_r(\overline{h_{\mathrm{H}}(t)}) + \mathcal{C}(t) + \mathcal{Q}(t) \tag{4.58}$$

where via the same the notation

$$u_{t;x}(\cdot) := u_t(\cdot;x)\,, \qquad v_{t;x}(\cdot) := v_t(\cdot;x)\,,$$

we have (collecting in $\mathcal{C}(t)$ contributions that commute with the number of particles operator)

$$\begin{aligned}
\mathcal{C}(t) := \frac{\varepsilon^3}{2} \int dx\, dy\, V(x-y) \Big(& a_l^*(u_{t;x}) a_l^*(u_{t;y}) a_l(u_{t;y}) a_l(u_{t;x}) + 2a_l^*(u_{t;x}) a_r^*(\overline{v_{t;x}}) a_r(\overline{v_{t;y}}) a_l(u_{t;y}) \\
& - 2a_l^*(u_{t;x}) a_r^*(\overline{v_{t;y}}) a_r(\overline{v_{t;y}}) a_l(u_{t;x}) + a_r^*(\overline{v_{t;y}}) a_r^*(\overline{v_{t;x}}) a_r(\overline{v_{t;x}}) a_r(\overline{v_{t;y}}) \\
& - a_r^*(\overline{u_{t;x}}) a_r^*(\overline{u_{t;y}}) a_r(\overline{u_{t;y}}) a_r(\overline{u_{t;y}}) - 2a_r^*(\overline{u_{t;x}}) a_l^*(v_{t;x}) a_l(v_{t;y}) a_r(\overline{u_{t;y}}) \\
& + 2a_r^*(\overline{u_{t;x}}) a_l^*(v_{t;y}) a_l(v_{t;y}) a_r(\overline{u_{t;x}}) - a_l^*(\overline{v_{t;y}}) a_l^*(\overline{v_{t;x}}) a_l(\overline{v_{t;x}}) a_l(\overline{v_{t;y}}) \\
& + 2a_r^*(\overline{v_{t;y}}) a_r(\overline{v_{t;x}}) \omega_t(x;y) - 2a_l^*(u_{t;x}) a_l(u_{t;y}) \omega_t(x;y) \Big)
\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}
\mathcal{Q}(t) := \frac{\varepsilon^3}{2} \int dx\, dy\, V(x-y) \Big(& a_l^*(u_{t;x}) a_l^*(u_{t;y}) a_r^*(\overline{v_{t;y}}) a_r^*(\overline{v_{t;x}}) + 2a_l^*(u_{t;x}) a_l^*(u_{t;y}) a_r^*(\overline{v_{t;x}}) a_l(u_{t;y}) \\
& - 2a_l^*(u_{t;x}) a_r^*(\overline{v_{t;y}}) a_r^*(\overline{v_{t;x}}) a_r(\overline{v_{t;y}}) - a_r^*(\overline{u_{t;x}}) a_r^*(\overline{u_{t;y}}) a_l^*(v_{t;y}) a_l^*(v_{t;x}) \\
& + 2a_r^*(\overline{u_{t;x}}) a_r^*(\overline{u_{t;y}}) a_l^*(v_{t;x}) a_r(\overline{u_{t;y}}) - 2a_r^*(\overline{u_{t;x}}) a_l^*(v_{t;y}) a_l^*(v_{t;x}) a_l(v_{t;y}) \\
& + 2a_l^*(u_{t;x}) a_r^*(\overline{v_{t;y}}) \omega_t(x;y) + h.c. \Big)\,.
\end{aligned} \tag{4.60}$$

Lemma 4.13. *If A is a bounded operator on $\mathfrak{h}$, we set $A_x(\cdot) := A(\cdot; x)$. Let V satisfy the assumptions of Theorem 3.5 and set*

$$V^{(1)}(x) := \int_{\mathbb{R}^3} dp\, \frac{e^{ip\cdot x}}{1+|p|^6}\,, \qquad V^{(2)}(x) := \int_{\mathbb{R}^3} dp\, e^{ip\cdot x}\,(1+|p|^6)\widehat{V}(p)\,. \tag{4.61}$$

Then, for any bounded operators J, A, B, C, D on $\mathfrak{h}$ and any $\rho, \sigma, \sigma', \bar\sigma, \bar\sigma' = r, l$ the following identities hold true:

$$\begin{aligned}
&\Big[d\Gamma_\rho(J), \int dxdyV(x-y)a^*_\sigma(A_x)a^*_{\sigma'}(B_y)a_{\sigma'}(C_y)a_\sigma(D_x)\Big] \\
&\quad = \delta_{\rho,\sigma}\int dz\int dy V^{(1)}_z(y)a^*_{\sigma'}(B_y)d\Gamma_\sigma\big([J, AV^{(2)}_z D^*]\big)a_{\sigma'}(C_y) \\
&\qquad + \delta_{\rho,\sigma'}\int dz\int dx V^{(1)}_z(x)a^*_\sigma(A_x)d\Gamma_{\sigma'}\big([J, BV^{(2)}_z C^*]\big)a_\sigma(D_x)\,,
\end{aligned} \tag{4.62}$$

$$\begin{aligned}
&\Big[d\Gamma_\rho(J), \int dxdyV(x-y)a^*_\sigma(A_x)a^*_{\sigma'}(B_x)a^*_{\bar\sigma}(C_y)a_{\bar\sigma}(D_y)\Big] \\
&\quad = \int dz\int dy V^{(2)}_z(y)a^*_{\bar\sigma}(C_y)d\Gamma^+_{\sigma\sigma'}\big(\delta_{\rho,\sigma}JAV^{(1)}_z B^T + \delta_{\rho,\sigma'}AV^{(1)}_z B^T J^T\big)a_{\bar\sigma}(D_y) \\
&\qquad + \delta_{\rho,\bar\sigma}\int dz d\Gamma^+_{\sigma\sigma'}\big(AV^{(1)}_z B^T\big)d\Gamma_{\bar\sigma}\big([J, CV^{(2)}_z D^*]\big)\,.
\end{aligned} \tag{4.63}$$

$$\begin{aligned}
&\Big[d\Gamma_\rho(J), \int dxdyV(x-y)a^*_\sigma(A_x)a^*_{\sigma'}(B_x)a_{\sigma'}(C_y)a_\sigma(D_y)\Big] \\
&\quad = \int dz d\Gamma^+_{\sigma\sigma'}\big(\delta_{\rho,\sigma}JAV^{(1)}_z B^T + \delta_{\rho,\sigma'}AV^{(1)}_z B^T J^T\big)d\Gamma^-_{\sigma'\sigma}\big(\overline{C}V^{(2)}_z D^*\big) \\
&\qquad - \int dz d\Gamma^+_{\sigma\sigma'}\big(AV^{(1)}_z B^T\big)d\Gamma^-_{\sigma'\sigma}\big(\delta_{\rho,\sigma'}J^T\overline{C}V^{(2)}_z D^* + \delta_{\rho,\sigma}\overline{C}V^{(2)}_z D^* J\big)\,,
\end{aligned} \tag{4.64}$$

$$\begin{aligned}
&\Big[d\Gamma_\rho(J), \int dxdyV(x-y)a^*_\sigma(A_x)a^*_{\sigma'}(B_x)a^*_{\bar\sigma}(C_y)a^*_{\bar\sigma'}(D_y) \\
&\quad = \int dz d\Gamma^+_{\sigma\sigma'}\big(\delta_{\rho,\sigma}JAV^{(1)}_z B^T + \delta_{\rho,\sigma'}AV^{(1)}_z B^T J^T\big)d\Gamma^+_{\bar\sigma\bar\sigma'}\big(CV^{(2)}_z D^T\big) \\
&\qquad + \int dz d\Gamma^+_{\sigma\sigma'}\big(AV^{(1)}_z B^T\big)d\Gamma^+_{\bar\sigma\bar\sigma'}\big(\delta_{\rho,\bar\sigma}JCV^{(2)}_z D^T + \delta_{\rho,\bar\sigma'}CV^{(2)}_z D^T J^T\big)\,.
\end{aligned} \tag{4.65}$$

Lemma 4.13 is an immediate result of Proposition 4.12 by using the canonical anti-commutation relation (2.1). The detailed proof of Proposition 4.12 and Lemma 4.13 can be found in [5].

Proof of Proposition 4.11. We use the notation $C_t \equiv C\exp(c|t|)$ in this proof for some constants $C, c > 0$ independent of Λ which may vary from line to line. The value of n is fixed to be $n \geq 4$ large enough. Proceed analogously as in [5, proof of Prop. 4.6]. By using Proposition 4.12, and recalling the notation $d\Gamma = d\Gamma_l + d\Gamma_r$, we have

$$\begin{aligned}
&i\varepsilon\partial_t\Big\langle \Xi_t, d\Gamma(\mathcal{W}^{(n)}_z)^2\Xi_t\Big\rangle \\
&\quad = \Big\langle \Xi_t, \Big[d\Gamma_l(\mathcal{W}^{(n)}_z)^2, d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t)\Big]\Xi_t\Big\rangle \\
&\qquad + \Big\langle \Xi_t, \Big[d\Gamma_r(\mathcal{W}^{(n)}_z)^2, d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t)\Big]\Xi_t\Big\rangle \\
&\qquad + 2\Big\langle \Xi_t, \Big[d\Gamma_l(\mathcal{W}^{(n)}_z)d\Gamma_r(\mathcal{W}^{(n)}_z), d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t)\Big]\Xi_t\Big\rangle\,,
\end{aligned} \tag{4.66}$$

where $\mathcal{C}(t)$ contains the terms which are cubic in creation and annihilation operators as defined in (4.59) and $\mathcal{Q}(t)$ contains the terms which are quartic in creation and annihilation operators as defined in (4.60), and we used the fact that $[d\Gamma_\sigma(\mathcal{W}_z^{(n)})^2, d\Gamma_{\sigma'}(V * \varrho_t)] = 0$ for all $\sigma, \sigma' = l, r$ because both $d\Gamma_\sigma\big(\mathcal{W}_z^{(n)}\big)^2$ and $d\Gamma_{\sigma'}(V * \varrho_t)$ are multiplication operators and commute. By using the commutator identity

$$[A^2, B] = A[A, B] + [A, B]A\,,$$

for arbitrary operators A and B, we are allowed to deal with all terms on the r.h.s. of (4.66). We only show the detailed method for the first line on the r.h.s. of (4.66) whereas the other terms can be dealt with analogously. Despite the fact that the third line on the r.h.s. of (4.66) looks a little bit different from the first two lines, it will even easier to bound because $\big[d\Gamma_l(\,\cdot\,),\, d\Gamma_r(\,\cdot\,)\big] = 0$. For example,

$$\begin{aligned}
&\Big\langle \Xi_t, \Big[d\Gamma_l(\mathcal{W}_z^{(n)}) d\Gamma_r(\mathcal{W}_z^{(n)}), d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t)\Big]\Xi_t\Big\rangle \\
&= \Big\langle \Xi_t, d\Gamma_l(\mathcal{W}_z^{(n)})\Big[d\Gamma_r(\mathcal{W}_z^{(n)}),\, -d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t)\Big]\Xi_t\Big\rangle \\
&\quad + \Big\langle \Xi_t, \Big[d\Gamma_l(\mathcal{W}_z^{(n)}),\, d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t)\Big] d\Gamma_r(\mathcal{W}_z^{(n)})\Xi_t\Big\rangle\,,
\end{aligned}$$

Then, the remaining terms can be treated as in the following context.

Next we control each term in the first line on the r.h.s. of (4.66) accordingly.

Control of $\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]$. We start by using the identity mentioned before to get

$$\begin{aligned}
&\Big|\Big\langle \Xi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2,\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\Xi_t\Big\rangle\Big| \\
&\qquad \le 2\,\Big|\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\Xi_t\Big\rangle\Big|
\end{aligned}$$

By proceeding similarly as (4.50) and (4.51), we obtain

$$\Big|\Big\langle \Xi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2,\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\Xi_t\Big\rangle\Big| \le C\varepsilon\,\Big|\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2\Xi_t\Big\rangle\Big| + C_t\varepsilon^{-2}\,. \tag{4.67}$$

Control of $\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{C}(t)\big]$. Since

$$\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2,\, \mathcal{C}(t)\big] = d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big),\, \mathcal{C}(t)\big] + \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big),\, \mathcal{C}(t)\big] d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,,$$

we may consider different contributions to $\mathcal{C}(t)$ accordingly by Lemma 4.13.

(i) From (4.59) and (4.62), we consider the term

$$\begin{aligned}
\mathrm{I} :&= \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big),\, \int dxdy\, V(x-y) a_l^*(u_{t;x}) a_l^*(u_{t;y}) a_l(u_{t;x}) a_l(u_{t;y})\big] \\
&= 2\int dz' \int dx\, V_{z'}^{(1)}(x) a_l^*(u_{t;x}) d\Gamma_l\big(\big[\mathcal{W}_z^{(n)}, u_t V_{z'}^{(2)} u_t\big]\big) a_l(u_{t;x})\,,
\end{aligned}$$

where $V^{(1)}$ and $V^{(2)}$ are defined as (4.61). It is useful to notice that $V^{(1)}$ and $V^{(2)}$ decay

faster than any power in position space. We observe that

$$\begin{aligned}
&\mathrm{I}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big) \\
&\le 2\int dz' \int dxdy \left|V_{z'}^{(1)}(x)\right| \mathcal{W}_z^{(n)}(y) a_{y,l}^* a_l^*(u_{t;x}) d\Gamma_l\big(\big[\mathcal{W}_z^{(n)},\, u_t V_{z'}^{(2)} u_t\big]\big) a_l(u_{t;x}) a_{y,l} \\
&\quad + C\int dz' \int dxdy \left|V_{z'}^{(1)}(x)\right| \mathcal{W}_z^{(n)}(y) a_l^*(u_{t;x}) d\Gamma_l\big(\big[\mathcal{W}_z^{(n)},\, u_t V_{z'}^{(2)} u_t\big]\big) a_{y,l} \\
&\quad + C\int dz' \int dx\, V_{z'}^{(1)}(x) a_l^*(u_{t;x}) d\Gamma_l\big(\big[\mathcal{W}_z^{(n)}, u_t V_{z'}^{(2)} u_t\big]\big) a_l(u_{t;x})\,,
\end{aligned} \tag{4.68}$$

where we expand $d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)$ by (2.12) and bound the commutator arose by (4.48). Notice that we have got, from [5, Eq. (4.46)], that

$$\left\| d\Gamma_l\big(\big[\mathcal{W}_z^{(n)},\, u_t V_{z'}^{(2)} u_t\big]\big)\right\|_{\mathrm{op}} \le 2\left\|\big[\mathcal{W}_z^{(n)},\, u_t V_{z'}^{(2)} u_t\big]\right\|_{\mathrm{tr}} \le \frac{C_t\varepsilon^{-2}}{1+|z-z'|^8}\,,$$

and also recall the rough bound result

$$\langle \Xi_t, \mathrm{I}\, \Xi_t\rangle \le C_t\varepsilon^{-3}$$

from [5, Control of $\mathcal{C}(t)$ in the proof of Prop.4.6.]. Then we may bound

$$\begin{aligned}
&\left|\left\langle \Xi_t, \mathrm{I}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\right\rangle\right| \\
&\le C_t\varepsilon^{-2}\int \frac{dz'}{1+|z-z'|^8}\int dy\, \mathcal{W}_z^{(n)}(y)\left|\left\langle \Xi_t, a_{y,l}^* d\Gamma_l\big(u_t V_{z'}^{(1)} u_t\big) a_{y,l}\Xi_t\right\rangle\right| \\
&\quad + C_t\varepsilon^{-2}\int \frac{dz'}{1+|z-z'|^8}\int dxdy\, \left|V_{z'}^{(1)}(x)\right| \mathcal{W}_z^{(n)}(y)\, \|a_l(u_{t;x})\Xi_t\|\, \|a_{y,l}\Xi_t\| + C_t\varepsilon^{-3} \\
&\le C_t\varepsilon^{-2}\int \frac{dz'}{1+|z-z'|^8}\int dy\, \mathcal{W}_z^{(n)}(y)\left|\left\langle \Xi_t, a_{y,l}^* d\Gamma_l\big(u_t V_{z'}^{(1)} u_t\big) a_{y,l}\Xi_t\right\rangle\right| + C_t\varepsilon^{-5} + C_t\varepsilon^{-3}\,.
\end{aligned}$$

To control the first term on the r.h.s., we choose $R>0$ so that $B_R(z)\subset\Lambda$. For $z'\in B_R(z)$, we proceed as from (4.39) to (4.40) to bound

$$d\Gamma_l\big(u_t \mathcal{W}_{z'}^{(n)} u_t\big) \le C d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) + C\left\|\big[u_t,\, \big(\mathcal{W}_{z'}^{(n)}\big)\big]\right\|_{\mathrm{tr}} \le C d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) + C_t\varepsilon^{-2}\,. \tag{4.69}$$

For $z'\in B_R^c(z)$, we apply (4.54) and get

$$d\Gamma_l\big(u_t \mathcal{W}_{z'}^{(n)} u_t\big) \le 4 d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) + C_t\varepsilon^{-3}\,. \tag{4.70}$$

Then by using

$$\int_{B_R^c(z)} \frac{1}{1+|z-z'|^8} dz' = (1+R)^{-5}$$

and the rough bound from Remark 4.10 and (4.41), we have

$$\begin{aligned}
&\int_{B_R^c(z)} \frac{dz'}{1+|z-z'|^8}\int dy\, \mathcal{W}_z^{(n)}(y)\left|\left\langle \Xi_t, a_{y,l}^* d\Gamma_l\big(u_t V_{z'}^{(1)} u_t\big) a_{y,l}\Xi_t\right\rangle\right| \\
&\qquad \le C_t\varepsilon^{-3} + C_t\varepsilon^{-5} + C_t\varepsilon^{-6}(1+R)^{-5}\,.
\end{aligned} \tag{4.71}$$

We then arrive at

$$\begin{aligned}
&\left|\left\langle \Xi_t, \mathrm{I}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\right\rangle\right| \\
&\qquad \le C_t\big(\varepsilon^{-7} + \varepsilon^{-8}(1+R)^{-5}\big) + C_t\varepsilon^{-2}\sup_{z'\in B_R(z)}\left\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\right\rangle\,.
\end{aligned} \tag{4.72}$$

(ii) By (4.64), we write the next contribution to $\mathcal{C}(t)$ as

$$\begin{aligned}\mathrm{II} :&= \Big[d\Gamma_l(\mathcal{W}_z^{(n)}), \int dxdyV(x-y)a_l^*(u_{t;x})a_r^*(\overline{v_{t;x}})a_r(\overline{v_{t;y}})a_l(u_{t;y})\Big] \\ &= \int dz' d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)d\Gamma_{rl}^-\big(v_tV_{z'}^{(2)}u_t\big) - \int dz' d\Gamma_{lr}^+\big(u_tV_{z'}^{(1)}v_t\big)d\Gamma_{rl}^-\big(v_tV_{z'}^{(2)}u_t\mathcal{W}_z^{(n)}\big)\,.\end{aligned} \tag{4.73}$$

We only show the control of the first term on the r.h.s. and the second can be dealt with similarly. For $z' \in B_R(z)$, we have

$$\begin{aligned}&\int_{B_R(z)} dz' \,\Big|\Big\langle \Xi_t, d\Gamma_{rl}^-(v_tV_{z'}^{(2)}u_t)d\Gamma_{lr}^+(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t)d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \\ &\qquad \le \int_{B_R(z)} dz' \,\Big\|v_tV_{z'}^{(2)}u_t\Big\|_{\mathrm{tr}}\Big\|\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\Big\|_{\mathrm{tr}}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2\Xi_t\Big\rangle^{1/2} \\ &\qquad \le C_t\varepsilon^{-7}\,,\end{aligned} \tag{4.74}$$

where we used the result

$$\int_{B_R(z)} dz' \,\Big\|v_tV_{z'}^{(2)}u_t\Big\|_{\mathrm{tr}}\Big\|\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\Big\|_{\mathrm{tr}} \le C_t\varepsilon^{-4}$$

from [5, Eq. (4.56)] and the rough bound from Remark 4.10 for the last inequality. For $z' \in B_R^c(z)$, we proceed as [5, Eq. (4.57)-(4.60)] to separate

$$\begin{aligned}&\int_{B_R^c(z)} dz' d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)d\Gamma_{rl}^-\big(v_tV_{z'}^{(2)}u_t\big) \\ &\qquad = \int_{B_R^c(z)} dz' \int dy\, V_{z'}^{(2)}(y)\, a_r(\overline{v_{t;y}})d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)a_l(u_{t;y}) \\ &\qquad\quad - \int_{B_R^c(z)} dz'\, d\Gamma_l\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}\omega_tV_{z'}^{(2)}u_t\big) \\ &\qquad =: A + B\,.\end{aligned}$$

For B, we proceed as (4.74) and use the trace norm bound

$$\Big\|\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\Big\|_{\mathrm{tr}} \le \frac{C_t\varepsilon^{-2}}{1+|z-z'|^8} \tag{4.75}$$

form [5, Eq. (4.54)] to get

$$\Big|\Big\langle \Xi_t, B\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,\Xi_t\Big\rangle\Big| \le C_t\varepsilon^{-2}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2\Xi_t\Big\rangle^{1/2} \le C_t\varepsilon^{-5}\,. \tag{4.76}$$

To control the term $\langle \Xi_t, A\, d\Gamma_l(\mathcal{W}_z^{(n)})\Xi_t\rangle$, we proceed as from (4.68) to (4.71) ($z' \in B_R^c(z)$ part) and apply the result in (4.75) and the rough bounds from Remark 4.10 and (4.41) to get

$$\Big|\Big\langle \Xi_t, A\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,\Xi_t\Big\rangle\Big| \le C_t\varepsilon^{-8}(1+R)^{-5}\,. \tag{4.77}$$

Combining (4.74), (4.76) and (4.77), we arrive at

$$\Big|\Big\langle \Xi_t, \mathrm{II}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,\Xi_t\Big\rangle\Big| \le C_t\varepsilon^{-7} + C_t\varepsilon^{-8}(1+R)^{-5}\,.$$

(iii) By (4.62), we write the third contribution from $\mathcal{C}(t)$ as

$$\begin{aligned}\mathrm{III} &:= \Big[d\Gamma_l(\mathcal{W}_z), \int dxdy\, V(x-y) a_l^*(u_{t;x}) a_r^*(\overline{v_{t;y}}) a_r(\overline{v_{t;y}}) a_l(u_{t;x})\Big] \\ &= \int dz'dy\, V_{z'}^{(1)}(y) a_r^*(\overline{v_{t;y}}) d\Gamma_l\big(\big[\mathcal{W}_z^{(n)}, u_t V_{z'}^{(2)} u_t\big]\big) a_r(\overline{v_{t;y}})\,.\end{aligned}$$

This case has the same structure as in (i) and can be controlled analogously to get

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \mathrm{III}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \\ &\quad\leq C_t\big(\varepsilon^{-7} + \varepsilon^{-8}(1+R)^{-5}\big) + C_t\varepsilon^{-2} \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\,.\end{aligned} \tag{4.78}$$

(iv)-(ix) The remaining contributions to $\mathcal{C}(t)$ arising from [5, Eq. (4.31)] can be controlled completely analogously as the previous ones by expanding $d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)$ on the right of the contribution and then by using the methods introduced in [5, Proof of Prop. 4.6, Control of $\mathcal{C}(t)$].

Therefore, we combine the results from (i) to (ix) to obtain

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{C}(t)\big]\Xi_t\Big\rangle\Big| \\ &\quad\leq 2\Big|\Big\langle \Xi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big), \mathcal{C}(t)\big] d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \\ &\quad\leq C_t\big(\varepsilon^{-4} + \varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\Big\rangle \\ &\qquad\qquad + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\end{aligned} \tag{4.79}$$

The symmetric result

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \big[d\Gamma_r\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{C}(t)\big]\Xi_t\Big\rangle\Big| \\ &\quad\leq C_t\big(\varepsilon^{-4} + \varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\Big\rangle \\ &\qquad\qquad + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\end{aligned} \tag{4.80}$$

and

$$\begin{aligned}&\Big|\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_r\big(\mathcal{W}_z^{(n)}\big), \mathcal{C}(t)\big]\Xi_t\Big\rangle\Big| \\ &\quad\leq C_t\big(\varepsilon^{-4} + \varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\end{aligned} \tag{4.81}$$

can also be obtained using similar approach.

Control of $\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{Q}(t)\big]$. Similarly, we start by noticing

$$\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{Q}(t)\big] = d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big), \mathcal{Q}(t)\big] + \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big), \mathcal{Q}(t)\big] d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,,$$

which suggests us to consider different contributions to $\mathcal{Q}(t)$ accordingly by Lemma 4.13.

(i) From (4.65), we first define

$$\begin{aligned}\widetilde{\mathrm{I}} :&= \Big[d\Gamma_l(\mathcal{W}_z^{(n)}), \int dxdyV(x-y)a_l^*(u_{t;x})a_l^*(u_{t;y})a_r^*(\overline{v_{t;y}})a_r^*(\overline{v_{t;x}})\Big] \\ &= \int dz' d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)d\Gamma_{lr}^+\big(u_tV_{z'}^{(2)}v_t\big) \\ &\qquad + \int dz' d\Gamma_{lr}^+\big(u_tV_{z'}^{(1)}v_t\big)d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(2)}v_t\big)\ ,\end{aligned} \tag{4.82}$$

which is of the same structure as the term II, therefore can be controlled using the same method. We control the first term on the r.h.s. of (4.82). For $z' \in B_R(z)$, similarly as (4.74), we get

$$\begin{aligned}&\int_{B_R(z)} dz' \ \Big|\Big\langle \Xi_t, d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)d\Gamma_{lr}^+\big(u_tV_{z'}^{(2)}v_t\big)d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \\ &\qquad \le \int_{B_R(z)} dz' \ \Big\|u_tV_{z'}^{(2)}v_t\Big\|_{\mathrm{tr}}\Big\|\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\Big\|_{\mathrm{tr}}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2\Xi_t\Big\rangle^{1/2} \\ &\qquad \le C_t\varepsilon^{-7}\,.\end{aligned}$$

For $z' \in B_R^c(z)$, we first expand $d\Gamma_{lr}^+(u_tV_{z'}^{(2)}v_t)$ and then proceed as from (4.104) to (4.77) to get

$$\int_{B_R^c(z)} dz' \ \Big|\Big\langle \Xi_t, d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)d\Gamma_{lr}^+\big(u_tV_{z'}^{(2)}v_t\big)d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \le C_t\varepsilon^{-8}(1+R)^{-5}\,.$$

The second term on the r.h.s. of (4.82) can be handled analogously. Thus we get

$$\Big|\Big\langle \Xi_t, \widetilde{\mathrm{I}}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \le C_t\varepsilon^{-7} + C_t\varepsilon^{-8}(1+R)^{-5}\,.$$

(ii) From (4.63), we define

$$\begin{aligned}\widetilde{\mathrm{II}} &:= \Big[d\Gamma_l(\mathcal{W}_z^{(n)}), \int dxdyV(x-y)a_l^*(u_{t;x})a_l^*(u_{t;y})a_r^*(\overline{v_{t;x}})a_l(u_{t;y})\Big] \\ &= -\int dz' \int dyV_{z'}^{(2)}(y)a_l^*(u_{t;y})d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)a_l(u_{t;y}) \\ &\qquad - \int dz' d\Gamma_{lr}^+\big(u_tV_{z'}^{(1)}v_t\big)d\Gamma_l\big(\big[\mathcal{W}_z^{(n)}, u_tV_{z'}^{(2)}u_t\big]\big) \\ &=: \widetilde{\mathrm{II}}_1 + \widetilde{\mathrm{II}}_2\ .\end{aligned}$$

The term $\langle \Xi_t, \widetilde{\mathrm{II}}_1\, d\Gamma_l(\mathcal{W}_z^{(n)})\Xi_t\rangle$ can be controlled similarly as the term $\langle \Xi_t, \mathrm{I}\, d\Gamma_l(\mathcal{W}_z^{(n)})\Xi_t\rangle$ by using the trace norm bound (4.75). We get

$$\Big|\Big\langle \Xi_t, \widetilde{\mathrm{II}}_1\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \le C_t\big(\varepsilon^{-7} + \varepsilon^{-8}(1+R)^{-5}\big) + C_t\varepsilon^{-2}\sup_{z'\in B_R(z)}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\Big\rangle\ .$$

For $\widetilde{\mathrm{II}}_2$, it is of the same form as the term $\widetilde{\mathrm{I}}$, with the replacement $\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t$ by $[\mathcal{W}_z^{(n)}, u_tV_{z'}^{(2)}u_t]$, whose trace norm can be estimated with Remark 4.6. Thus we obtain the bound

$$\big|\big\langle \Xi_t, \widetilde{\mathrm{II}}_2\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,\Xi_t\big\rangle\big| \le C_t\big(\varepsilon^{-7} + \varepsilon^{-8}(1+R)^{-5}\big)$$

and thus

$$\Big|\Big\langle \Xi_t, \widetilde{\mathrm{II}}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big| \leq C_t\big(\varepsilon^{-7}+\varepsilon^{-8}(1+R)^{-5}\big) + C_t\varepsilon^{-2}\sup_{z'\in B_R(z)}\big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\big\rangle\,,$$

which is of the same form as (4.72).

(iii) From (4.63), we define the third contribution to $\mathcal{Q}(t)$ as

$$\begin{aligned}\widetilde{\mathrm{III}} &:= \Big[d\Gamma_l(\mathcal{W}_z^{(n)}), \int dxdyV(x-y)a_l^*(u_{t;x})a_r^*(\overline{v_{t;y}})a_r^*(\overline{v_{t;x}})a_r(\overline{v_{t;y}})\Big]\\ &= -\int dz'\int dyV_{z'}^{(2)}(y)a_r^*(\overline{v_{t;y}})d\Gamma_{lr}^+\big(\mathcal{W}_z^{(n)}u_tV_{z'}^{(1)}v_t\big)a_r(\overline{v_{t;y}})\,,\end{aligned}$$

which is of the same structure as $\widetilde{\mathrm{II}}_1$ and can be treated similarly to give

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \widetilde{\mathrm{III}}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big|\\ &\quad\leq C_t\big(\varepsilon^{-7}+\varepsilon^{-8}(1+R)^{-5}\big) + C_t\varepsilon^{-2}\sup_{z'\in B_R(z)}\big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\big\rangle\,.\end{aligned}$$

which is of the same form as (4.78).

(iv)-(vii) The remaining terms involving contributions from [5, Eq. (4.32)] can be controlled completely analogously as the previous terms by expanding $d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)$ in the right and then by using the methods introduced in [5, Proof of Prop. 4.6, Control of $\mathcal{Q}(t)$].

Finally, combine the results from (i) to (vii) in the control with respect to $\mathcal{Q}(t)$ to obtain

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{Q}(t)\big]\Xi_t\Big\rangle\Big|\\ &\quad\leq 2\Big|\Big\langle \Xi_t, \big[d\Gamma_l\big(\mathcal{W}_z^{(n)}\big), \mathcal{Q}(t)\big]d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\Big|\\ &\quad\leq C_t\big(\varepsilon^{-4}+\varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon\sup_{z'\in B_R(z)}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\Big\rangle\\ &\qquad\qquad + C_t\varepsilon\sup_{z'\in B_R(z)}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\,.\end{aligned}\tag{4.83}$$

Similarly, we may also get the following bound

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \big[d\Gamma_r\big(\mathcal{W}_z^{(n)}\big)^2, \mathcal{Q}(t)\big]\Xi_t\Big\rangle\Big|\\ &\quad\leq C_t\big(\varepsilon^{-4}+\varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon\sup_{z'\in B_R(z)}\Big\langle \Xi_t, d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)^2\Xi_t\Big\rangle\\ &\qquad\qquad + C_t\varepsilon\sup_{z'\in B_R(z)}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\,.\end{aligned}\tag{4.84}$$

and

$$\begin{aligned}&\Big|\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\big[d\Gamma_r\big(\mathcal{W}_z^{(n)}\big), \mathcal{Q}(t)\big]\Xi_t\Big\rangle\Big|\\ &\quad\leq C_t\big(\varepsilon^{-4}+\varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon\sup_{z'\in B_R(z)}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\,.\end{aligned}\tag{4.85}$$

Conclusion. Combining (4.67), (4.79), (4.80), (4.81), (4.83), (4.84) and (4.85), we get a bound

$$\begin{aligned}\Big\langle \Xi_t, \Big[d\Gamma(\mathcal{W}_z^{(n)})^2, d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) + \mathcal{C}(t) + \mathcal{Q}(t) \Big] \Xi_t \Big\rangle \\ \leq C_t\big(\varepsilon^{-4} + \varepsilon^{-5}(1+R)^{-5}\big) + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)^2 \Xi_t \Big\rangle \\ + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big)^2 \Xi_t \Big\rangle + C_t\varepsilon \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big) d\Gamma_r\big(\mathcal{W}_{z'}^{(n)}\big) \Xi_t \Big\rangle \,.\end{aligned} \tag{4.86}$$

Hence, we are arriving at

$$\partial_t \Big\langle \Xi_t, d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2 \Xi_t \Big\rangle \leq C_t\big(\varepsilon^{-5} + \varepsilon^{-6}(1+R)^{-5}\big) + C_t \sup_{z'\in B_R(z)} \Big\langle \Xi_t, d\Gamma\big(\mathcal{W}_{z'}^{(n)}\big)^2 \Xi_t \Big\rangle \tag{4.87}$$

for $n \in \mathbb{N}$ large enough and for all $t \in \mathbb{R}$. Finally, we proceed using exactly the same way as [5, Proof of Prop. 4.6, Conclusion part] (with all $d\Gamma\big(\mathcal{W}_z^{(n)}\big)$ replaced by $d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2$. We choose the parameter R as a function of time, denoted by R_t. Integrating both sides of (4.87) over time s gives

$$\begin{aligned}\Big\langle \Xi_t, d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2 \Xi_t \Big\rangle \leq \Big\langle \Xi_0, d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2 \Xi_0 \Big\rangle \\ + C_t\varepsilon^{-5} + C_t\varepsilon^{-6} \int_0^t \frac{ds}{(1+R_s)^5} + C_t \int_0^t ds \sup_{z'\in B_{R_s}(z)} \Big\langle \Xi_s, d\Gamma\big(\mathcal{W}_{z'}^{(n)}\big)^2 \Xi_s \Big\rangle \,.\end{aligned} \tag{4.88}$$

Consider $R_s = r(t-s)$ for $r > 0$ to be chosen later. We can rewrite (4.88) as

$$\begin{aligned}\Big\langle \Xi_t, d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2 \Xi_t \Big\rangle \leq \Big\langle \Xi_0, d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2 \Xi_0 \Big\rangle \\ + C_t\left(\varepsilon^{-5} + \frac{\varepsilon^{-6}}{r}\right) + C_t \int_0^t ds \sup_{z'\in B_{r(t-s)}(z)} \Big\langle \Xi_s, d\Gamma\big(\mathcal{W}_{z'}^{(n)}\big)^2 \Xi_s \Big\rangle \,,\end{aligned} \tag{4.89}$$

for all $t > 0$. Then, for fixed $T > 0$ and $z_0 \in \Lambda$, we define

$$F(t) := \sup_{z\in B_{r(T-t)}(z_0)} \Big\langle \Xi_t, d\Gamma\big(\mathcal{W}_z^{(n)}\big)^2 \Xi_t \Big\rangle$$

for all $t \in [0;T]$. By the observation that

$$\begin{aligned}\sup_{z\in B_{r(T-t)}(z_0)} \Big\langle \Xi_0, d\Gamma(\mathcal{W}_z^{(n)})^2 \Xi_0 \Big\rangle \leq F(0), \\ \sup_{z\in B_{r(T-t)}(z_0)} \sup_{z'\in B_{r(t-s)}(z)} \Big\langle \Xi_s, d\Gamma\big(\mathcal{W}_{z'}^{(n)}\big)^2 \Xi_s \Big\rangle \leq F(s),\end{aligned}$$

we get

$$F(t) \leq F(0) + C_T\Big(\varepsilon^{-5} + \frac{\varepsilon^{-6}}{r}\Big) + C_T \int_0^t ds\, F(s)$$

for all $t \in [0;T]$ from (4.88). Then Grönwall lemma gives

$$F(t) \leq e^{C_T}\left(F(0) + C_T\varepsilon^{-5} + C_T\,\frac{\varepsilon^{-6}}{r}\right)$$

for all $t \in [0;T]$. Finally, we take $t = T$ and $r = \varepsilon^{-2\delta}$, and conclude that

$$\begin{aligned}\Big\langle \Xi_T, d\Gamma(\mathcal{W}_{z_0}^{(n)})^2 \Xi_T \Big\rangle &\leq C\exp(c\exp(c|T|)) \left[\sup_{z\in B_{\varepsilon^{-2\delta}T}(z_0)} \Big\langle \Xi, d\Gamma(\mathcal{W}_z^{(n)})^2 \Xi \Big\rangle + \varepsilon^{-6+2\delta} \right] \\ &\leq C\exp(c\exp(c|T|))\varepsilon^{-6+2\delta}\end{aligned} \tag{4.90}$$

for all $z_0 \in \Lambda$, with $B_{\varepsilon^{-2\delta}T}(z_0) \subset \Lambda$, by the assumption (3.16). This concludes the proof of Proposition 4.11.

□

Remark 4.14. *It is worth mentioning that this priori bound (4.56) in Proposition 4.11 holds for all points $z \in \Lambda$ with $B_{\varepsilon^{-2\delta|t|}}(z) \subset \Lambda$, which is a larger set than the set of $z \in \Lambda$ with $B_{\varepsilon^{-\delta|t|}}(z) \subset \Lambda$. This tells us that the priori bound (4.56) holds in a weaker condition than that of (4.42) and of the prior bound in Proposition 4.7. In fact, this is because the second assumption in (3.16) allow us to choose a finer r such that $r = \varepsilon^{-2\delta}$ for the last step (4.90). However, the extra space is not crucial for the proof priori bound in Proposition 4.7 as we will see in the next subsection. Therefore, we are completely allowed to make a weaker assumption*

$$\sup_{z\in\Lambda} \Big\langle \Xi, d\Gamma_\sigma\big(\mathcal{W}_z^{(n)}\big) d\Gamma_{\sigma'}\big(\mathcal{W}_z^{(n)}\big)\Xi \Big\rangle \leq C\varepsilon^{-6+\delta}$$

and take $r = \varepsilon^{-\delta}$ in the last step (4.90) which leads to the same set of points on which a weaker, but sufficient, priori bound

$$\langle \Xi_t, d\Gamma(\mathcal{W}_z^{(n)})^2 \Xi_t \rangle \leq C\exp\big(c\exp(c|t|)\big)\,\varepsilon^{-6+\delta}$$

and the priori bound in Proposition 4.7 hold simultaneously, though the assumption in (3.16) and the priori bound introduced in Proposition 4.11 might be more physically meaningful.

4.5 Proof of Proposition 4.7

We use the notation $C_t \equiv C\exp(c|t|)$ in this proof for some constants $C, c > 0$ independent of Λ which may vary from line to line. The value of n is fixed to be $n \geq 4$ large enough. Let $z \in \Lambda$ with $B_{\varepsilon^{-\delta|t|}}(z) \subset \Lambda$. Proceed analogously as in [5, proof of Prop. 4.6]. By using Proposition 4.12, we have, for $\sigma = l, r$,

$$\begin{aligned}&i\varepsilon\partial_t \Big\langle \Xi_t, d\Gamma_\sigma(\widetilde{\mathcal{W}}_z^{(n)})\Xi_t \Big\rangle \\ &\quad= \Big\langle \Xi_t, \Big[d\Gamma_\sigma(\widetilde{\mathcal{W}}_z^{(n)}),\, d\Gamma_l(\sqrt{1-\varepsilon^2\Delta}) - d\Gamma_r(\sqrt{1-\varepsilon^2\Delta}) \\ &\qquad\qquad + d\Gamma_l\big(\varepsilon^3 V * \varrho_t\big) - d\Gamma_r\big(\varepsilon^3 V * \varrho_t\big) + \mathcal{C}(t) + \mathcal{Q}(t) \Big] \Xi_t \Big\rangle\end{aligned} \tag{4.91}$$

where $\mathcal{C}(t)$ and $\mathcal{Q}(t)$ have been defined in (4.59) and (4.60). The key difference here from the proof of Proposition 4.11 lies in the fact that $d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big)$ is not a multiplication operator which does not commute with the potential $V * \varrho_t$. This is the reason why we need an extra priori bound (Proposition 4.11) in this work comparing with the method used in [5]. Next we control each term in (4.91) accordingly.

Control of $\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big), d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]$. From (4.8), we have

$$\begin{aligned}\pm\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big] &= \pm\delta_{\sigma,l}\, d\Gamma_l\big(\big[\widetilde{\mathcal{W}}_z^{(n)},\, \sqrt{1-\varepsilon^2\Delta}\big]\big)\\ &\leq C\varepsilon\delta_{\sigma,l}\, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big) + C\varepsilon^2\delta_{\sigma,l}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,,\end{aligned}$$

which implies that

$$\begin{aligned}&\Big|\Big\langle \Xi_t, \big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]\Xi_t\Big\rangle\Big|\\ &\quad\leq C\varepsilon\delta_{\sigma,l}\Big\langle \Xi_t, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\Xi_t\Big\rangle + C\varepsilon^2\delta_{\sigma,l}\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle\\ &\quad\leq C\varepsilon\delta_{\sigma,l}\Big\langle \Xi_t, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\Xi_t\Big\rangle + C\varepsilon^{-1}\,.\end{aligned} \tag{4.92}$$

The same method can be applied to control the term involving $\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big), d\Gamma_r\big(\sqrt{1-\varepsilon^2\Delta}\big)\big]$.

Control of $\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big), d\Gamma_l\big(\varepsilon^3 V * \varrho_t\big)\big]$. By using (4.14), we get

$$\begin{aligned}\pm i\,\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l\big(\varepsilon^3 V*\varrho_t\big)\big] &= \pm i\,\varepsilon^3\delta_{\sigma,l}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta},\, V*\varrho_t\big]\mathcal{W}_z^{(n)}\big)\\ &\leq C\varepsilon^4\delta_{\sigma,l}\, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\,,\end{aligned}$$

and thus

$$\Big|\Big\langle \Xi_t, \big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, d\Gamma_l\big(\varepsilon^3 V*\varrho_t\big)\big]\Xi_t\Big\rangle\Big| \leq C\varepsilon^4\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_z^{(n)}\big)\Xi_t\Big\rangle \leq C\varepsilon\,, \tag{4.93}$$

where we used the rough bound from (4.41) for the last inequality. The same method can be applied to control the term involving $\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big), d\Gamma_r\big(\varepsilon^3 V*\varrho_t\big)\big]$.

Control of $\big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big), \mathcal{C}(t)\big]$. From Proposition 4.12 and Lemma 4.13, we consider different contributions to $\mathcal{C}(t)$ accordingly.

(i) From (4.59) and (4.60), we consider the term

$$\mathrm{I} := 2\delta_{\sigma,l}\int dz'\int dx\, V_{z'}^{(1)}(x) a_l^*(u_{t;x}) d\Gamma_l\big(\big[\widetilde{\mathcal{W}}_z^{(n)}, u_t V_{z'}^{(2)} u_t\big]\big) a_l(u_{t;x})\,,$$

where $V^{(1)}$ and $V^{(2)}$ are defined as in (4.61) and they decay faster than any power in position space. Observe that

$$\begin{aligned}&d\Gamma_l\big(\big[\widetilde{\mathcal{W}}_z^{(n)}, u_t V_{z'}^{(2)} u_t\big]\big)\\ &\quad\leq 2\,\Big\|V_{z'}^{(2)}\big[\widetilde{\mathcal{W}}_z^{(n)}, u_t\big]\Big\|_{\mathrm{tr}} + d\Gamma_l\big(u_t\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta},\, V_{z'}^{(2)}\big]\mathcal{W}_z^{(n)} u_t\big)\\ &\quad\leq \frac{C_t\varepsilon^{-2}}{1+|z-z'|^8} + \frac{C\varepsilon}{1+|z-z'|^8} d\Gamma_l\big(u_t\mathcal{W}_{z'}^{(n)} u_t\big)\,,\end{aligned}$$

where we applied Corollary 4.5 and (4.10) for the last inequality because we are looking at $z\in\Lambda$ and $z'\in\mathbb{R}^3$. We obtain

$$\begin{aligned}&|\langle \Xi_t, \mathrm{I}\,\Xi_t\rangle|\\ &\leq C_t\varepsilon^{-2}\int\frac{dz'}{1+|z-z'|^8}\Big|\Big\langle \Xi_t, d\Gamma_l\big(u_t|V_{z'}^{(1)}|u_t\big)\Xi_t\Big\rangle\Big|\\ &\quad + C\varepsilon\int\frac{dz'}{1+|z-z'|^8}\int dx\,\Big|V_{z'}^{(1)}(x)\Big|\,\Big|\Big\langle \Xi_t, a_l^*(u_{t,x}) d\Gamma_l\big(u_t\mathcal{W}_{z'}^{(n)} u_t\big) a_l(u_{t,x})\Xi_t\Big\rangle\Big|\\ &=: \mathrm{I}_a + \mathrm{I}_b\,.\end{aligned}$$

Now fix $\delta > 0$. Let us choose R such that $R = \varepsilon^{-\delta}|t|$ as a function of time t as defined in [5, Conclusion part of the proof of Prop. 4.6], then it follows from [5, proof of Prop. 4.6, Eq. (4.47)-(4.49)] that

$$\left\langle \Xi_t, d\Gamma_l\big(u_t|V^{(1)}_{z'}|u_t\big)\Xi_t \right\rangle \leq C\exp(C_t)\varepsilon^{-3+\delta} \qquad \forall\ z' \in B_R(z)\ ; \tag{4.94}$$

$$\int_{B^c_R(z)} \frac{dz'}{1+|z-z'|^8} \left| \left\langle \Xi_t, d\Gamma_l\big(u_t|V^{(1)}_{z'}|u_t\big)\Xi_t \right\rangle \right| \leq C\exp(C_t)\varepsilon^{-3+\delta}\ . \tag{4.95}$$

Then we simply get $\mathrm{I}_a \leq C\exp(C_t)\varepsilon^{-5+\delta}$. For I_b, we first recall that

$$\begin{aligned} d\Gamma_l\big(u_t\mathcal{W}^{(n)}_{z'}u_t\big) &\leq Cd\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big) + C_t\varepsilon^{-2} \qquad \forall z' \in B_R(z) \\ d\Gamma_l\big(u_t\mathcal{W}^{(n)}_{z'}u_t\big) &\leq 4d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big) + C_t\varepsilon^{-3} \qquad \forall z' \in B^c_R(z) \end{aligned} \tag{4.96}$$

from (4.69) and (4.70) respectively, which allow us to rewrite I_b as

$$\begin{aligned} \mathrm{I}_b \leq C\varepsilon \int \frac{dz'}{1+|z-z'|^8} \int dx\ \left|V^{(1)}_{z'}(x)\right| \left| \left\langle \Xi_t, a^*_l(u_{t,x})d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)a_l(u_{t,x})\Xi_t \right\rangle \right| \\ + C_t\varepsilon^{-1} \int_{B_R(z)} \frac{dz'}{1+|z-z'|^8} \left| \left\langle \Xi_t, d\Gamma_l\big(u_t|V^{(1)}_{z'}|u_t\big)\Xi_t \right\rangle \right| \\ + C_t\varepsilon^{-2} \int_{B^c_R(z)} \frac{dz'}{1+|z-z'|^8} \left| \left\langle \Xi_t, d\Gamma_l\big(u_t|V^{(1)}_{z'}|u_t\big)\Xi_t \right\rangle \right|\ . \end{aligned} \tag{4.97}$$

For the second and third terms on the r.h.s. of (4.97), we apply (4.94) and (4.95) respectively. For the second and third terms on the r.h.s. of (4.97), we expand $d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)$ and rewrite it as follows:

$$\begin{aligned} &\int \frac{dz'}{1+|z-z'|^8} \int dx\ \left|V^{(1)}_{z'}(x)\right| \left| \left\langle \Xi_t, a^*_l(u_{t,x})d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)a_l(u_{t,x})\Xi_t \right\rangle \right| \\ &\quad = \int \frac{dz'}{1+|z-z'|^8} \int dy\, \mathcal{W}^{(n)}_{z'}(y) \left| \left\langle \Xi_t, a^*_{y,l}d\Gamma_l\big(u_t|V^{(1)}_{z'}|u_t\big)a_{y,l}\Xi_t \right\rangle \right|\ , \end{aligned}$$

which allows us to use equations from (4.94) to (4.96) again to get

$$\begin{aligned} \mathrm{I}_b &\leq C\varepsilon \int \frac{dz'}{1+|z-z'|^8} \left\langle \Xi_t, d\Gamma_l\big(\mathcal{W}^{(n)}_{z'}\big)^2\Xi_t \right\rangle + C\exp(C_t)\varepsilon^{-5+\delta} \\ &\leq C\exp(C_t)\varepsilon^{-5+\delta}\ , \end{aligned}$$

where for the last inequality we used Proposition 4.11 and (4.71). Therefore, combining the results of I_a and I_b, we arrive at

$$|\langle \Xi_t, \mathrm{I}\,\Xi_t\rangle| \leq C\exp(C_t)\varepsilon^{-5+\delta}\ .$$

(ii) From [5, Eq. (4.36)]we write the next term as

$$\begin{aligned} \mathrm{II} = &\int dz' d\Gamma^+_{lr}\big(\delta_{\sigma,l}\widetilde{\mathcal{W}}^{(n)}_z u_t V^{(1)}_{z'} v_t + \delta_{\sigma,r} u_t V^{(1)}_{z'} v_t \widetilde{\mathcal{W}}^{(n)}_z\big) d\Gamma^-_{rl}\big(v_t V^{(2)}_{z'} u_t\big) \\ &- \int dz' d\Gamma^+_{lr}\big(u_t V^{(1)}_{z'} v_t\big) d\Gamma^-_{rl}\big(\delta_{\sigma,r}\widetilde{\mathcal{W}}^{(n)}_z v_t V^{(2)}_{z'} u_t + \delta_{\sigma,l} v_t V^{(2)}_{z'} u_t \widetilde{\mathcal{W}}^{(n)}_z\big)\ . \end{aligned}$$

Let us consider the first term on the r.h.s. of (4.73), the other can be dealt with similarly. Now, observe that

$$\begin{aligned}\Big|\big\langle \Xi_t, \int_{B_R(z)} dz' d\Gamma_{lr}^{+}\big(\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\big) d\Gamma_{rl}^{-}\big(v_t V_{z'}^{(2)} u_t\big)\Xi_t\big\rangle\Big| \\ \leq \int_{B_R(z)} dz' \big\|\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\big\|_{\mathrm{tr}} \big\|v_t V_{z'}^{(2)} u_t\big\|_{\mathrm{tr}} .\end{aligned} \tag{4.98}$$

For $z' \in B_R(z)$, we have

$$\begin{aligned}&\Big\|\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\Big\|_{\mathrm{tr}} \\ &\leq \Big\|\widetilde{\mathcal{W}}_z^{(n)} V_{z'}^{(1)} u_t v_t\Big\|_{\mathrm{tr}} + \Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[u_t, V_{z'}^{(1)}\big] v_t\Big\|_{\mathrm{tr}} \\ &\leq \Big\|V_{z'}^{(1)}\widetilde{\mathcal{W}}_z^{(n)} u_t v_t\Big\|_{\mathrm{tr}} + \Big\|\big[\widetilde{\mathcal{W}}_z^{(n)}, V_{z'}^{(1)}\big] u_t v_t\Big\|_{\mathrm{tr}} + \Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[u_t, V_{z'}^{(1)}\big]\Big\|_{\mathrm{tr}} ,\end{aligned} \tag{4.99}$$

where the third term on the r.h.s. can be bounded as

$$\Big\|\widetilde{\mathcal{W}}_z^{(n)}\big[u_t, V_{z'}^{(1)}\big]\Big\|_{\mathrm{tr}} \leq \frac{C_t \varepsilon^{-2}}{1+|z-z'|^8} \tag{4.100}$$

by using Corollary 4.5 directly. The first term on the r.h.s. of (4.99) can be treated as follows:

$$\begin{aligned}\Big\|V_{z'}^{(1)}\widetilde{\mathcal{W}}_z^{(n)} u_t v_t\Big\|_{\mathrm{tr}} &\leq \Big\|V_{z'}^{(1)}\mathcal{W}_z^{(n)}\Big\|_{\mathrm{op}} \Big\|\sqrt{1-\varepsilon^2\Delta}\,\mathcal{W}_z^{(n)} u_t v_t\Big\|_{\mathrm{tr}} \\ &\leq \frac{C}{1+|z-z'|^8}\Big(\Big\|\widetilde{\mathcal{W}}_z^{(n/2)} u_t v_t\Big\|_{\mathrm{tr}} + \Big\|\big[\sqrt{1-\varepsilon^2\Delta}, \mathcal{W}_z^{(n/2)}\big]\mathcal{W}_z^{(n/2)} u_t v_t\Big\|_{\mathrm{tr}}\Big) \\ &\leq \frac{C_t\varepsilon^{-2}}{1+|z-z'|^8} ,\end{aligned}$$

where we used Corollary 4.5 and (4.10) for the last inequality. For the second term on the r.h.s. of (4.99), we observe that

$$\begin{aligned}\Big\|\big[\widetilde{\mathcal{W}}_z^{(n)}, V_{z'}^{(1)}\big] u_t v_t\Big\|_{\mathrm{tr}} &= \Big\|\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta}, V_{z'}^{(1)}\big]\mathcal{W}_z^{(n)} u_t v_t\Big\|_{\mathrm{tr}} \\ &\leq \Big\|\mathcal{W}_z^{(n)}\big[\sqrt{1-\varepsilon^2\Delta}, V_{z'}^{(1)}\big]\mathcal{W}_z^{(n)}\big(\mathcal{W}_{z'}^{(n/2)}\big)^{-1}\Big\|_{\mathrm{op}} \Big\|\mathcal{W}_{z'}^{(n/2)} u_t v_t\Big\|_{\mathrm{tr}} \\ &\leq \frac{C_t\varepsilon^{-1}}{1+|z-z'|^8} ,\end{aligned}$$

where Corollary 4.5 and (4.10) are used again for the last inequality. Therefore, we arrive at

$$\Big\|\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\Big\|_{\mathrm{tr}} \leq \frac{C_t\varepsilon^{-2}}{1+|z-z'|^8} . \tag{4.101}$$

It is worth mentioning that (4.101) holds for all $z' \in \mathbb{R}$ because we are assuming that $z \in \Lambda$, so we are allowed to use this bound for the case when $z' \in B_R(z)^c$. The second norm on the r.h.s. of (4.98) is exactly the same as [5, Eq. (4.55)], which gives

$$\Big\|v_t V_{z'}^{(2)} u_t\Big\|_{\mathrm{tr}} \leq \Big\|v_t u_t V_{z'}^{(2)}\Big\|_{\mathrm{tr}} + \Big\|v_t\big[V_{z'}^{(2)}, u_t\big]\Big\|_{\mathrm{tr}} \leq C_t\varepsilon^{-2} . \tag{4.102}$$

Combining (4.101) with (4.102), we obtain

$$\int_{B_R(z)} dz' \big\|\mathcal{W}_z^{(n)} u_t V_{z'}^{(1)} v_t\big\|_{\mathrm{tr}} \big\|v_t V_{z'}^{(2)} u_t\big\|_{\mathrm{tr}} \leq C_t\varepsilon^{-4} \tag{4.103}$$

for $n \in \mathbb{N}$ large enough. For $z' \in B_R^c(z)$ we proceed as [5, Eq. (4.57)] to get

$$\begin{aligned}
&\int_{B_R^c(z)} dz' d\Gamma_{lr}^+\big(\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\big) d\Gamma_{rl}^-\big(v_t V_{z'}^{(2)} u_t\big) \\
&\quad = \int_{B_R^c(z)} dz' \int dy\, V_{z'}^{(2)}(y)\, a_r(\overline{v_{t;y}}) d\Gamma_{lr}^+\big(\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\big) a_l(u_{t;y}) \\
&\qquad - \int_{B_R^c(z)} dz'\, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} \omega_t V_{z'}^{(2)} u_t\big) \\
&\quad =: A + B\,.
\end{aligned} \tag{4.104}$$

For the term B, Lemma 2.1 and (4.101) gives

$$\begin{aligned}
\Big|\big\langle \Xi_t, B\,\Xi_t\big\rangle\Big| &\leq \int dz' \Big|\big\langle \Xi_t, d\Gamma_l\big(\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} \omega_t V_{z'}^{(2)} u_t\big)\Xi_t\big\rangle\Big| \\
&\leq \int_{B_R^c(z)} dz' \big\|\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} \omega_t V_{z'}^{(2)} u_t\big\|_{\mathrm{tr}} \\
&\leq \int_{B_R^c(z)} dz' \big\|\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\big\|_{\mathrm{tr}} \leq C_t \varepsilon^{-2}\,.
\end{aligned} \tag{4.105}$$

For the term A, we apply (4.101) to get

$$\begin{aligned}
\Big|\big\langle \Xi_t, A\Xi_t\big\rangle\Big| &\leq \int_{B_R^c(z)} dz' \int dy \,\big|V_{z'}^{(2)}(y)\big| \,\Big|\big\langle a_r^*(\overline{v_{t;y}})\Xi_t, d\Gamma_{lr}^+\big(\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\big) a_l(u_{t;y})\Xi_t\big\rangle\Big| \\
&\leq \int_{B_R^c(z)} dz' \Big\|\widetilde{\mathcal{W}}_z^{(n)} u_t V_{z'}^{(1)} v_t\Big\|_{\mathrm{tr}} \int dy \,\big|V_{z'}^{(2)}(y)\big|\, \|v_{t;y}\|_{\mathfrak{h}}\, \big\|a_l(u_{t;y})\Xi_t\big\| \\
&\leq C \exp(C_t)\varepsilon^{-5+\delta}\,,
\end{aligned} \tag{4.106}$$

because the choice of R gives (4.95) and

$$\int_{B_R^c(z)} \frac{dz'}{1+|z-z'|^8} \Big|\Big\langle \Xi_t, d\Gamma_l\big(\mathcal{W}_{z'}^{(n)}\big)\Xi_t\Big\rangle\Big| \leq C \exp(C_t)\varepsilon^{-3+\delta}\,.$$

Combining (4.103), (4.105) and (4.106), we arrive at

$$|\langle \Xi_t, \mathrm{II}\,\Xi_t\rangle| \leq C \exp(C_t)\varepsilon^{-5+\delta}\,.$$

The remaining terms coming from $\mathcal{C}(t)$ and $\mathcal{Q}(t)$ can be bounded analogously by following the procedure in [5, Proof of Prop.4.6, control of $\mathcal{C}(t)$ and control of $\mathcal{Q}(t)$] with $\mathcal{W}_z^{(n)}$ replaced by $\widetilde{\mathcal{W}}_z^{(n)}$. Though some steps may vary because $\widetilde{\mathcal{W}}_z^{(n)}$ does not commute with $V_{z'}^{(1)}$ of $V_{z'}^{(2)}$ anymore, they can be treated analogously as from (4.99) to (4.101). Hence, we will obtain

$$\Big|\Big\langle \Xi_t, \big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, \mathcal{C}(t)\big]\Xi_t\Big\rangle\Big| \leq C \exp(C_t)\varepsilon^{-2+\delta}\,; \tag{4.107}$$

$$\Big|\Big\langle \Xi_t, \big[d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big),\, \mathcal{Q}(t)\big]\Xi_t\Big\rangle\Big| \leq C \exp(C_t)\varepsilon^{-2+\delta}\,. \tag{4.108}$$

Conclusion. Combining (4.91), (4.92), (4.93), (4.107) and (4.108), and recalling the notation $d\Gamma = d\Gamma_l + d\Gamma_r$, we arrive at

$$\partial_t \left\langle \Xi_t, d\Gamma_\sigma\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\Xi_t \right\rangle \leq C \left\langle \Xi_t, d\Gamma\big(\widetilde{\mathcal{W}}_z^{(n)}\big)\Xi_t \right\rangle + C\exp(C_t)\varepsilon^{-3+\delta}\,. \tag{4.109}$$

Finally, we conclude the proof of Proposition 4.7 by Grönwall inequality and assumption (3.15).

□

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