Universität Zürich

Master's Thesis

Brownian Motion and Complex Analysis

A thesis submitted in partial fulfilment of the requirements
for the degree of Master of Science in Mathematics

Author:
Davide Dell'Angelo
20-706-057

Supervisor:
Prof. Dr. Jean Bertoin
Co-supervisor:
Dr. Alejandro Rosales Ortiz

Universität Zürich[UZH]

Institute of Mathematics
Universität Zürich
January 6, 2025

Acknowledgements

I would like to thank Prof. Dr. Jean Bertoin for the very instructive and interesting Probability 1 and 2 lectures. Thank you for being my supervisor and giving me the opportunity to write this thesis.

I would also like to thank my co-supervisor Dr. Alejandro Rosales Ortiz for always supporting and encouraging me in difficult moments. This interesting work was only possible because you encouraged me to attend the subject "Brownian motion and Stochastic Calculus" at ETHZ, thank you for believing in me. I was able to learn so much from you and your experience over the last few months, which has really shaped my mathematical thinking. I will miss our interesting discussions on your blackboard about difficult proofs and wish you all the best for your future.

I would also like to thank my parents Monika and Carlo, without you I would not be where I am now, thank you for loving me unconditionally. Not to forget my brother Daniele, you always showed so much understanding when my motivation was lacking and shared my suffering during difficult exam phases.

Many thanks to Vanessa Rey, for your exceptional empathy and willingness to listen, your support has meant so much to me. No matter how stupid the questions seemed, I was able to ask you all of them and we were able to discuss everything together, thank you very much for that. And of course at this point I don't want to refuse her the credits for the domains in my work (but only those for the domains).

I would like to thank the "Produktive Büro" with Michael Kaufmann and Ermira Shala. Although unfortunately we never got our own office in the maths institute, I had the opportunity to experience exciting semesters and many funny moments. Laughing together in lectures, discussing homework, creating stickers and rating exercise sessions made my studies much easier. I still don't know what I would have done without you, thank you from the bottom of my heart. I could count on you, your understanding and your tips in every exam phase. May we celebrate many more beautiful 4th of November together.

Not forgetting Rafaela Haltiner, Martina Häberling, Elia Stebler, Luana Kurmann, Salome Raas, Jana Binder, Oliver Bauer and everyone else I've had the pleasure of meeting along my journey, you were all an important part of my journey and kindly reminded me of the non-zero probability of finishing this thesis, thank you.

Finally, I would also like to thank Prof. Dr. Alexander Gorodnik. I still remember very well the first Analysis 1 lesson, in which I was so enthusiastic. Every semester I looked forward to your lectures, even if the homework were very difficult. Thanks to your analysis courses, I was able to learn so much about this topic, which I was also able to put into practice here. In that sense, "Es gibt keine grossen Überraschungen" in this thesis.

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Introduction

This work starts with a repetition of the main definitions and propositions of complex analysis used in the paper and continues with an introduction to complex Brownian motion. We then establish the basis for the proof of the conformal invariance theorem, which vaguely states that if you rotate and stretch a complex Brownian motion you get again a Brownian motion up to a time change. As a second main result of this work we will get to know the Dirichlet problem and thereby give a probabilistic representation of a harmonic function which depends only on a complex-valued Brownian motion and its exit time from a domain. Meanwhile, using the discovered theory on complex Brownian motion we prove as applications important results from complex analysis, for example the Fundamental theorem of algebra or the Liouville theorem. The advantage of this approach is that the proofs of these purely analytical theorems become relatively simple and short compared to the classical proofs, but as we will see, there is much more preparatory work to be done.

A complex-valued Brownian motion started from $x+iy \in \mathbb{C}$ is defined as $B_t = B_t^{(1)} + iB_t^{(2)}$ for $t \geq 0$ where $(B_t^{(1)} : t \geq 0)$ and $(B_t^{(2)} : t \geq 0)$ are real-valued independent Brownian motions, with respective starting points $B_0^{(1)} = x$ and $B_0^{(2)} = y$. By construction, complex-valued Brownian motion inherits several fundamental properties of real-valued Brownian motion, such as invariance under scaling. Roughly speaking, any multiple of a Brownian motion is, up to a time change, again a Brownian motion. A further geometric property which is easily derived from classic properties of Brownian motion is rotational invariance, which states that if a complex-valued Brownian motion is rotated by an angle the resulting process is still a complex-valued Brownian motion. Note that this property holds without applying a time change.

Locally, any holomorphic function f acts on the complex plane as a rotation and dilation. So now given the previous discussion, the natural question arises of what occurs if f is applied to any complex Brownian motion. This motivates the first main result in this work which in short states that for any holomorphic map f, the process $f(B)$ is a complex-valued time changed Brownian motion (possibly stopped at a random time). It is a crucial statement, because geometrically, it means that $f(B)$ inherits all the geometric properties of a complex Brownian motion, which are independent of time: for example the number of returns to a domain D, recurrence, or transience to name a few.

Theorem (Conformal invariance)**.** *Suppose $D \subset \mathbb{C}$ is a domain. Fix $z \in D$ and argue under the measure $\mathbb{P}_z$, which stands for the law of $B + z$ when B is a complex Brownian motion starting from zero. Let $f : D \to \mathbb{C}$ be a non-constant holomorphic function.*

Let $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$ *and*

$$\zeta = \int_0^{\tau_D} |f'(B_s)|^2 \, ds \in (0, \infty].$$

Define $\sigma(t) < \tau_D$ *as the inverse of the strictly increasing map* $t \mapsto \int_0^t |f'(B_s)|^2 \, ds$ *for* $t < \zeta$, *with* $\sigma(t) = \tau_D$ *if* $t \geq \zeta$.

(i) Then, $Y_t := f\left(B_{\sigma(t)}\right)$ *for* $0 \leq t < \zeta$ *is a complex Brownian motion under* $\mathbb{P}_{f(z)}$.

(ii) If $D = \mathbb{C}$ *then* $\zeta = \infty$ *and therefore* $(Y_t)_{t \geq 0}$ *is a complex Brownian motion.*

(iii) If we also assume that $f : D \to f(D)$ *is a bijective conformal mapping, then (i) holds with* $\zeta = \tau_{f(D)} = \inf\{t \geq 0 : Y_t \notin f(D)\}$.

As we mentioned above, immediate applications often concern the study of properties of complex Brownian motion which are independent of time. More precisely, a first application is that a complex-valued Brownian motion is neighbourhood recurrent but not almost surly pointwise recurrent. This means that a Brownian motion a.s. returns to every neighbourhood after having left, but on the other hand it never returns to a single point with probability one. Other remarkable applications of the conformal invariance theorem are that it allows to recover classic results from complex analysis. As an illustration, we shall recover the following two results.

Corollary (Liouville)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire, bounded function. Then* f *is constant.*

Corollary (Fundamental theorem of Algebra)**.** *Let* f *be a non-constant polynomial, then* $f = 0$ *has at least one solution.*

In order to further study planar properties of complex-valued Brownian motion we use the notion of the harmonic measure. Roughly speaking, given a domain, the harmonic measure is a measure on the boundary which encodes the probability that a complex Brownian motion exits the domain through a given subset of the boundary. In the special case where the domain is a ball and the complex Brownian motion starts from zero, the harmonic measure is the uniform measure on the boundary of the ball. This can be easily deduced by the rotational invariance of complex Brownian motion. However, the problem becomes more difficult as soon as the domain and the starting point change, but we shall see that in many examples of interest, thanks to the conformal invariance theorem, the harmonic measure can be made explicit.

Another class of functions, which are closely related to complex Brownian motion, are harmonic functions. These are functions which are real or imaginary parts of holomorphic functions. A harmonic function $g : D \to \mathbb{R}$ satisfies $\Delta g(z) := \frac{\partial^2 g}{\partial x^2}(z) + \frac{\partial^2 g}{\partial y^2}(z) = 0$ for all $z \in D$. Equivalently, g is called harmonic if it satisfies the Mean Value Property. This is, if the value of g at the centre of a disk is equal to the average of its values on the boundary. In particular, if g is known on the circle, we can recover the value of g at its centre. We exploit this equivalent characterisation to study the representation and existence of harmonic functions. We show that for any given bounded continuous function u on the boundary of a domain D, there exists a unique

harmonic extension h of u to the whole of $\overline{D}$, this is, it satisfies $\Delta h = 0$ on D and $h|_{\partial D} = u$. Furthermore, this theorem not only establishes existence, but also provides a representation in terms of complex Brownian motion stopped at the first exit time of the domain D.

Theorem. *(Solution to Dirichlet problem) Suppose $D \in \mathcal{D}$ and that u is a bounded continuous function on ∂D. Then, there exists a unique bounded continuous function $h : \overline{D} \to \mathbb{R}$ that extends u and is harmonic in D. Furthermore,*

$$h(z) = \mathbb{E}_z \left(u \left(B_{\tau_D} \right) \right), \ \forall z \in D.$$

In the final chapter, we first derive a representation of a complex-valued Brownian motion in polar coordinates, which offers a new perspective on its asymptotic properties. Let us be more precise.

By the well known Euler formula we know that any complex number z can be represented in polar coordinates, i.e. there exist $r > 0$ and $\theta \in (0, 2\pi]$ such that $z = re^{i\theta}$. Now let $B_t = B_t^{(1)} + iB_t^{(2)}$ for $t \geq 0$ be a complex Brownian motion with $B_0 = x + iy \in \mathbb{C} \setminus \{0\}$. We can construct two processes $(R_t)_{t \geq 0}$ and $(\theta_t)_{t \geq 0}$ by letting

$$\begin{aligned} R_t &:= \sqrt{\left(B_t^{(1)} \right)^2 + \left(B_t^{(2)} \right)^2}, \\ \theta_t &:= \arg(B_t), \end{aligned}$$

for all $t \geq 0$. Here θ_t, for $t \geq 0$, denotes a continuous determination of the argument. In particular, this means that $t \mapsto \theta_t$ is a continuous real-valued function such that $B_t = R_t \exp(i\theta_t)$. Informally, $(\theta_t)_{t \geq 0}$ keeps track of the number of windings of B around zero. Note that R is positive and that it inherits the self-similar property from B.

The question we are interested in is in finding two independent real-valued Brownian motions β and γ so that we can represent $(R_t)_{t \geq 0}$ and $(\theta_t)_{t \geq 0}$, and therefore B, in terms of those two Brownian motions. This problem is solved in the next theorem, the so called skew-product representation:

Theorem. *Let $z \in \mathbb{C} \setminus \{0\}$ and write $z = \exp(r + i\theta)$ where $r \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$. There exist two independent real-valued linear Brownian motions β and γ that start from r respectively from θ under $\mathbb{P}_z$, such that we have $\mathbb{P}_z$-a.s. for every $t \geq 0$,*

$$B_t = \exp(\beta_{\sigma(t)} + i\gamma_{\sigma(t)}),$$

where

$$\sigma(t) = \int_0^t \frac{1}{|B_s|^2} \, ds.$$

The representation of R, in terms of an exponential of a Brownian motion time changed, is a very particular case of a result due to Lamperti, which states that non-negative, self-similar Markov processes are exponentials of Lévy processes time changed.

In the remainder of Chapter 4 we discuss an important application of the skew-product representation, namely the so-called Spitzer theorem. One can see that $\limsup_{t \to \infty} \theta_t = \infty$ and

$\liminf_{t\to\infty} \theta_t = -\infty$ where $(\theta_t)_{t\geq 0}$ as above denotes a continuous determination of the argument of a complex-valued Brownian motion B. Because of this observation, the question arises what typical size θ_t has when t is large.

Theorem (Spitzer theorem)**.** *Let $(\theta_t)_{t\geq 0}$ be a continuous determination of the argument of the complex Brownian motion B started from $z \in \mathbb{C} \setminus \{0\}$. Then $\frac{2}{\log(t)}\theta_t$ converges in distribution to a standard symmetric Cauchy distribution as $t \to \infty$. In other words, for every $x \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}_z \left(\frac{2}{\log(t)}\theta_t \leq x \right) = \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)}\, dy.$$

Of course, θ naturally can be seen as the winding number of B around the starting point z. Therefore, Spitzer's Theorem states that the winding number is close to a Cauchy distribution.

In the last part of the thesis we talk about how this work could be taken further and we give some additional literature on Green functions that build on theorems and notations introduced in this thesis.

Chapter 1

Background

1.1 Background on Complex Analysis and Brownian Motion

1.1.1 Background on Complex Analysis

Let us start by defining the concept of a domain in the complex plane. This is a fundamental concept in complex analysis and provides the setting for most of the theory.

Definition 1. *A non-empty, open and connected subset $D \subset \mathbb{C}$ is called a domain.*

From now on we stick to the convention that a subset D of $\mathbb{C}$ is proper if $D \subsetneq \mathbb{C}$. Next, we introduce the notion of a simply connected domain. Simply connected domains have no "holes" and this property is crucial for many results in complex analysis, such as the existence of antiderivatives (see [Don19, Corollary 5.8]).

Definition 2. *A domain D is called simply connected if every continuous map $f : \{z \in \mathbb{C} : |z| = 1\} \to D$ is the restriction of some continuous map $g : \{z \in \mathbb{C} : |z| \leq 1\} \to D$ on $\{z \in \mathbb{C} : |z| = 1\}$, i.e. $f = g|_{\{z \in \mathbb{C} : |z| = 1\}}$.*

To simplify further discussions, we define the Riemann sphere to be $\mathbb{C} \cup \{\infty\}$. It is an essential concept in complex analysis because it provides a compactification of the complex plane, allowing us to handle infinity in a rigorous way.

The following important proposition relates simply connected domains to the Riemann sphere. It provides a useful criterion for determining whether a domain is simply connected.

Proposition 1. [BN16] *A domain D is simply connected if its complement is connected in the Riemann sphere.*

It is important to notice, that not every domain D is simply connected. For example, one can consider the infinite strip $\mathbb{S} := \{z \in \mathbb{C} : 0 < \Im(z) < 1\}$, and show that it is not a simply connected domain. Even more important examples of simply connected domains for this thesis are the open unit disk, which we denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$.

As we have already mentioned in the introduction, holomorphic functions are of great importance to us, since they respect angles and infinitesimally small figures, which will be a useful property later on when we talk about conformal invariance of complex Brownian motion.

Definition 3. *Let $U \subseteq \mathbb{C}$ be an open subset. A complex function $f : U \to \mathbb{C}$ is called holomorphic on U if for all $z \in U$,*

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z},$$

exists and is continuous.

We now give a more rigorous intuition as to what happens locally with the complex plane when a holomorphic function is applied. Let f be a holomorphic function with $f(z_0) = 0$ and $f'(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. By the Taylor expansion it follows that $f(z) = 0 + f'(z_0)z + O(z^2) = f'(z_0)z + O(z^2)$. Let us denote $r := |f'(z_0)|$ and $\theta := \arg(f'(z_0))$. By the Euler formula we have $f'(z_0) = re^{i\theta}$ and any $z \in \mathbb{C}$ can be represented as $z = r'e^{i\theta'}$ for some $r' > 0$ and $\theta' \in [0, 2\pi)$. Putting this together leads us to $f(z) = rr'e^{i(\theta+\theta')} + O(z^2)$ which graphically means that locally we have dilated z by the factor r and rotated it by the factor θ.

To expand on the concept of holomorphic functions, we introduce conformal maps. Later on, we will delve into a specific type of function known as Möbius transformations, which are particularly important when we speak about the harmonic measure.

Definition 4.

1. *A holomorphic function f defined on the whole of $\mathbb{C}$ is called an entire function.*
2. *Let $D \subseteq \mathbb{C}$ be a domain. A holomorphic function $f : D \to \mathbb{C}$ is called a conformal map if $f'(z) \neq 0$ for all $z \in D$.*
3. *A conformal isomorphism is a conformal bijection $f : D \to D'$ where $D' = f(D)$.*

The next proposition highlights some important properties of conformal isomorphisms, showing that they preserve domains and that their inverses are also conformal maps.

Proposition 2. [BN16, Page 4] *Let $f : D \to D'$ be a conformal isomorphism, then*

1. *$D' = f(D)$ is again a domain.*
2. *the inverse $f^{-1} : D' \to D$ is also a conformal map.*

We now state a fundamental result about existence of conformal isomorphisms.

Proposition 3 (Riemann mapping theorem)**.** [BN16, Theorem 1.1] *Let D be a proper simply connected domain. Then, there exists a conformal isomorphism $\phi : D \to \mathbb{D}$.*

To get a feeling for conformal maps, let us have a look at the following two examples. First, consider the complex-valued map $f(z) = e^z$, which is conformal on $\mathbb{C}$ since $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$. On the other hand, we can show that f is not a conformal isomorphism.

Indeed, let $z_1 = 2\pi i$ and $z_2 = 4\pi i$, then $z_1 \neq z_2$ in $\mathbb{C}$ but $f(z_1) = f(z_2) = 1$. Hence, f is not injective so in particular not bijective which shows the claim.

Secondly, consider $\psi(z) = \frac{i-z}{i+z}$ for $z \in \mathbb{H}$, which is claimed to be a conformal isomorphism on the upper half plane. Indeed, $\psi'(z) = -\frac{2i}{(z+i)^2} \neq 0$ for all $z \in \mathbb{H}$. To verify bijectivity one can check that the inverse is given by $\psi^{-1}(z) = \frac{i(1-z)}{1+z}$ for $z \in \mathbb{H}$, completing the proof.

Another important tool in complex analysis are the Cauchy-Riemann equations, which form a necessary and sufficient condition that a complex-valued function is differentiable.
Let us use the well-known notation $z = x + iy$ for a complex number z. If $f(z) = u(z) + iv(z)$, we can use the identification $z \cong (x, y)$ to write $f(x, y) = u(x, y) + iv(x, y)$. Assuming that f is holomorphic implies that

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x(x, y) = u_x(x, y) + iv_x(x, y) \\ &= \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{ik} = \frac{1}{i} f_y(x, y) = v_y(x, y) - iu_y(x, y). \end{aligned}$$

Comparing the real and imaginary part of the above equations yields the relations $u_x = v_y$ and $-v_x = u_y$.

Definition 5. *The above two relations are called the Cauchy-Riemann equations.*

After introducing the Cauchy-Riemann equations as fundamental conditions for the holomorphy of a function, we now establish their equivalence to the analyticity, as stated in the following proposition. For this let us first recall that a complex-valued function f is said to be analytic on a domain D, if for every $z_0 \in D$ there exists an $r > 0$ and a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ which converges in $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and satisfies

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Proposition 4. [Don19, Theorem 7.9] *Let u, v be continuous, differentiable functions on a domain D, then u, v satisfy the Cauchy-Riemann equations if and only if $f := u + iv$ is analytic on D.*

To understand the connection between the analyticity of a function and the Cauchy-Riemann equations more rigorously, we introduce the concept of complex derivatives in terms of partial derivatives with respect to z and $\overline{z}$.

Definition 6. *For $f : \mathbb{C} \to \mathbb{C}$ we define*

$$\begin{aligned} \partial f &:= \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(f_x - if_y), \\ \overline{\partial} f &:= \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(f_x + if_y). \end{aligned}$$

Using this definition, we can now provide an equivalent characterization of an analytic function f.

Lemma 1. [Fre15, Lemma 2.1.2] *Consider a domain D and a complex-valued map $f = u + iv$ on D. Then f is analytic if and only if $\overline{\partial} f = 0$ and in this case $f' = \partial f = u_x - iu_y$.*

Proof. We start by showing the first part of the lemma. Using the above definition, one can check that $2\overline{\partial} f = f_x + if_y = u_x - v_y + i(v_x + u_y)$. According to Proposition 4, $2\overline{\partial} f = 0$ if and only if f is analytic on D which proves the first part.

Assume now that f is analytic. Using again the above definition in connection with the Cauchy-Riemann equations gives us

$$2\partial f = 2(u_x + iv_x) = 2f_x = 2f',$$

therefore we deduce $\partial f = f'$. To conclude the proof we remark that combining the above definition and the Cauchy-Riemann equations, gives

$$\partial f = \frac{1}{2}\left(u_x + iv_x - i\left(u_y + iv_y\right)\right) = \frac{1}{2}\left(u_x + v_y + i\left(v_x - u_y\right)\right) = u_x - iu_y,$$

which concludes the proof. □

Let D be a domain in $\mathbb{C}$ and $S \subseteq D$. We recall from topology that $x \in S$ is called a limit point if for all $\epsilon > 0$ there exists a $y \in S \setminus \{x\}$ such that $d(x, y) < \epsilon$. Roughly speaking, this means that x can be "approximated" by points in S. So, we can introduce the following useful lemma on identifying analytic functions.

Lemma 2 (Identity lemma)**.** [Don19] *Given that f and g are analytic on a domain $D \subseteq \mathbb{C}$, if $f = g$ on $S \subseteq D$ where S has a limit point in D, then $f = g$ on the whole domain D.*

In the following section we introduce a very important class of conformal transformations, the so-called Möbius transformations, which are very useful due to their geometric properties. We have already seen an example of a Möbius transformation as an example of conformal isomorphism, but now we look at the theory behind it more generally.

A Möbius transformation is a non-constant rational function from the Riemann sphere to itself of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Additionally, we define $f(\infty) = \frac{a}{c}$ and $f\left(-\frac{d}{c}\right) = \infty$. From complex analysis, one knows that the set of Möbius transformations form a group under composition. Furthermore, the following lemma shows that they are all bijective.

Lemma 3. [Don19] *Every Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a bijection.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ and pick $w \in \mathbb{C}$ such that there exists $z \in \mathbb{C}$ with $w = \frac{az+b}{cz+d}$. Then $czw + dw = az + b$ which is equivalent to $z(cw - a) = b - dw$. Since $z \neq \infty$ it holds that $w \neq \frac{a}{c}$ and therefore $z = \frac{b-dw}{cw-a}$. Additionally, $f(\infty) = \frac{a}{c}$ and $f\left(-\frac{d}{c}\right) = \infty$ by definition. Hence, $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is invertible, which concludes the proof. □

At this point, it is worth noting that we can identify Möbius transformations with 2×2 matrices. If we denote by Möb the set of all Möbius transformations, then $\psi : \text{Möb} \to \{A \in \text{Mat}_{2\times 2}(\mathbb{C}) : \det(A) \neq 0\}$ is a bijection where

$$\psi\left(\frac{az+b}{cz+d}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This simplifies computations because, for example, to find the inverse of a Möbius transformation, you only need to invert its corresponding matrix.

Since Möbius transformations have a linear term in both the numerator and the denominator, it is natural that they can be uniquely determined if a certain number of equations is given. This idea is discussed in the following proposition.

Proposition 5. [Don19, Theorem 6.4.] *Let $(z_i)_{1\leq i\leq 3}$ and $(w_i)_{1\leq i\leq 3}$ be both three distinct points in the Riemann sphere $\mathbb{C}\cup\{\infty\}$. There is a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.*

Proof. First, let us take $w_1 = 0, w_2 = \infty, w_3 = 1$, then if we pick

$$f(z) = \left(\frac{z-z_1}{z-z_2}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right),$$

we see that $f(z_i) = w_i$ for $i = 1, 2, 3$.

Next, let us take $w_1, w_2, w_3 \in \mathbb{C} \cup \{\infty\}$ arbitrary. From the above computation we know that there exist two Möbius transformations g, h such that $g : \{z_1, z_2, z_3\} \mapsto \{0, \infty, 1\}$ and $h : \{w_1, w_2, w_3\} \mapsto \{0, \infty, 1\}$. So, let us chose g, h such that

$$\begin{aligned} g(z_1) &= h(w_1) = 0, \\ g(z_2) &= h(w_2) = \infty, \\ g(z_3) &= h(w_3) = 1. \end{aligned}$$

Then, define $f := h^{-1} \circ g$ which is map from $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$. Indeed, one can easily check that $f(z_i) = w_i$ for $i = 1, 2, 3$. In particular, since the set of Möbius transformations forms a group, we deduce that f is a Möbius transformation. This shows the existence part. For uniqueness, let us assume the existence of another Möbius transformation ϕ satisfying $\phi(z_i) = w_i$ for all $i = 1, 2, 3$. Then, $\psi := h \circ \phi \circ g^{-1}$ is a Möbius map satisfying

(i) $\psi(0) = h(\phi(z_1)) = h(w_1) = 0$

(ii) $\psi(\infty) = \infty$

(iii) $\psi(1) = 1$.

Now, we want to check that ψ is the identity. Since ψ is a Möbius transformation, we can represent it as $\psi(z) = \frac{az+b}{cz+d}$. From (i) we deduce that $\frac{b}{d} = 0$ which implies $b = 0$, (ii) is

equivalent to $\frac{a}{c} = \infty$ which means $c = 0$. So, we now have $\psi(z) = \frac{az}{d}$, using (iii), $\frac{a}{d} = 1$ i.e. $a = d$, hence we finally get $\psi(z) = \frac{az}{a} = z$. Therefore, we know have $f = h^{-1} \circ g = \phi$ which concludes the proof. □

To conclude the section on Möbius transformations, let us give two important examples which will be used in Subsection 3.1.

Lemma 4. [BN16, Page 4]

1. *For $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$ we define $\Phi_{\theta,w}$ on $\mathbb{D}$ by*

$$\Phi_{\theta,w}(z) := e^{i\theta} \frac{z - w}{1 - \overline{w}z}. \tag{1.1}$$

Then, $\Phi_{\theta,w}$ is a conformal automorphism of $\mathbb{D}$ and $\Phi_{\theta,w}|_{\mathbb{D}}$ is a Möbius transformation.

2. *For a fixed $w = x + iy \in \mathbb{H}$, we define $\Psi : \mathbb{H} \to \mathbb{D}$ by*

$$\Psi(z) := \frac{z - w}{z - \overline{w}}, \tag{1.2}$$

which is a conformal isomorphism taking w to 0.

Proof. 1. $\Phi_{\theta,w}$ is holomorphic on $\mathbb{D}$ since for all $z \in \mathbb{D}$

$$\Phi'_{\theta,w}(z) = e^{i\theta} \frac{1 - |w|^2}{(1 - z\overline{w})^2},$$

is defined and in particular we know that it is conformal since $1 - |w|^2 \neq 0$ for all $w \in \mathbb{D}$. Furthermore, we know from a previous remark that we can identify

$$\Phi_{\theta,w}(z) := e^{i\theta} \frac{z - w}{1 - \overline{w}z} \sim \begin{pmatrix} e^{i\theta} & -e^{i\theta}w \\ -\overline{w} & 1 \end{pmatrix} =: A.$$

This can be used to show that $\Phi_{\theta,w}$ is invertible, indeed $\det(A) = e^{i\theta} - e^{i\theta}w\overline{w}$ $= e^{i\theta} - e^{i\theta}|w|^2 = e^{i\theta}\left(1 - |w|^2\right) \neq 0$. Therefore, we know that $\Phi_{\theta,w}$ is a conformal isomorphism on $\mathbb{D}$. To finish this part, we check that $\Phi_{\theta,w}(\mathbb{D}) = \mathbb{D}$. Fix $w \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and pick $z \in \mathbb{D}$, then

$$\begin{aligned} |\Phi_{\theta,w}(z)|^2 &= \left| e^{i\theta} \frac{z - w}{1 - \overline{w}z} \right|^2 \\ &= \left| \frac{z - w}{1 - \overline{w}z} \right|^2. \end{aligned}$$

So, it is enough to show that $|z - w|^2 < |1 - \overline{w}z|^2$. Using that for any complex number $\xi \in \mathbb{C}$, $|\xi|^2 = \xi\overline{\xi}$ we can rewrite this inequality and get

$$|z|^2 - z\overline{w} - \overline{z}w + |w|^2 < 1 - z\overline{w} - \overline{z}w + |w|^2|z|^2.$$

Simplifying both sides leads to

$$|z|^2\left(1-|w|^2\right) < 1-|w|^2.$$

Since $|w|<1$ by assumption, $1-|w|^2>0$ hence we can divide both sides by $1-|w|^2$ and get $|z|^2<1$ which is indeed true. This ends the first part of the proof.

2. Clearly, for a fixed $w\in\mathbb{H}$, $\Psi(w)=\frac{0}{w-\overline{w}}=0$. Furthermore, for all $z\in\mathbb{H}$, Ψ is holomorphic because
$$\Psi'(z)=\frac{w-\overline{w}}{(z-\overline{w})^2}=\frac{2iy}{(z-\overline{w})^2},$$
exists. This follows from the fact that $\overline{w}$ has a strictly negative imaginary part and on the other hand $\Im(z)>0$, so in particular we can deduce that $z-\overline{w}\neq 0$. We also immediately deduce that Ψ is conformal since $y\neq 0$ and thus for all $z\in\mathbb{C}$, $\Psi'(z)\neq 0$. As above, we identify
$$\Psi(z)\sim\begin{pmatrix}1 & -w\\ 1 & -\overline{w}\end{pmatrix}=:B,$$
and since $\det(B)=-\overline{w}+w=2iy\neq 0$ we conclude that Ψ is bijective. It remains to check that $\Psi(z)\in\mathbb{D}$ for $z\in\mathbb{H}$, as in the first part it is again enough to show that $|z-w|^2<|z-\overline{w}|^2$, which is equivalent to $zw+\overline{z}\overline{w}<z\overline{w}+\overline{z}w$. If we denote $z=z_x+iz_y$ and $w=w_x+iw_y$, then

$$\begin{aligned}
zw &= (z_x+iz_y)(w_x+iw_y)=z_xw_x-z_yw_y+i(z_xw_y+z_yw_x),\\
\overline{z}\overline{w} &= z_xw_x-z_yw_y-i(z_xw_y+z_yw_x),\\
z\overline{w} &= (z_x+iz_y)(w_x-iw_y)=z_xw_x+z_yw_y+i(z_yw_x-z_xw_y),\\
\overline{z}w &= (z_x-iz_y)(w_x+iw_y)=z_xw_x+z_yw_y+i(z_xw_y-z_yw_x).
\end{aligned}$$

Using this we can rewrite the above inequality and get

$$-2z_yw_y<2z_yw_y,$$

dividing by 2 on both sides leads to $-z_yw_y<z_yw_y$ which is always true since $z_y,w_y>0$, by assumption. This ends the proof.

□

1.1.2 Background on Complex Brownian Motion

In the same way as we used the notation $z=x+iy$ for complex numbers, we denote a complex-valued stochastic process by $Z_t=X_t+iY_t$, for $t\geq 0$, where $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ are real-valued stochastic processes. Therefore, we can extend the definition of a martingale respectively a Brownian motion, into the complex case in a natural way.

For the rest of the thesis we will always work with a completed and right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ where necessary.

Definition 7. *1. A real-valued process $(X_t, Y_t)_{t\geq 0}$ is called a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ if X and Y are real-valued martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

2. A complex-valued process $Z_t = X_t + iY_t$, $t \geq 0$, is called a complex-valued martingale if $(X_t, Y_t)_{t\geq 0}$ is a $\mathbb{R}^2$-martingale.

In a similar way, we can now extend the definition of a real-valued Brownian motion to the complex case.

Definition 8. *We call a complex-valued process $B_t = X_t + iY_t$, for $t \geq 0$, to be a complex Brownian motion starting at $z = x + iy$, if $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent real-valued $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions starting at x, respectively y. If now B is a complex-valued Brownian motion starting from zero, we use $\mathbb{P}_z$ to define the law of $B + z$ for $z \in \mathbb{C}$.*

Many important theorems in stochastic calculus not only apply to martingales but to a larger class, the so-called local martingales. Let us first recall the notion of a real-valued local martingale. We will then extend this definition to the complex case.

Definition 9. *Let $(M_t)_{t\geq 0}$ be a real-valued process. M is called a real-valued local martingale if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that*

1. $T_n \nearrow \infty$ a.s. as $n \to \infty$

2. $(\mathbb{1}_{\{T_n>0\}} M_{t\wedge T_n})_{t\geq 0}$ is a martingale for all $n \in \mathbb{N}$

We call $(T_n)_{n\in\mathbb{N}}$ the localising sequence of M.

As mentioned earlier, every martingale is a local martingale. More precisely, we can choose the sequence of stopping times T_n to be $T_n = \infty$ and check that the definition is satisfied.
Now, we can define the class of complex-valued local martingales as follows.

Definition 10. *A complex-valued process $Z_t = X_t + iY_t$, $t \geq 0$, is called a complex-valued local martingale if $(X_t, Y_t)_{t\geq 0}$ is a $\mathbb{R}^2$-local martingale.*

For the next proposition recall first that if M, N are real-valued continuous local martingales then $(\langle M, N\rangle_t)_{t\geq 0}$ is the unique real-valued increasing process such that $(M_tN_t - \langle M, N\rangle_t)_{t\geq 0}$, is a continuous local martingale (see Definition 23). We can now extend this into the complex case by Proposition 6.

Proposition 6. [Fre15, Lemma 2.2.3] *If U, V are complex-valued continuous local martingales, then there exists a unique bilinear continuous complex-valued process denoted by $(\langle U, V\rangle_t)_{t\geq 0}$, vanishing at zero, such that $(U_tV_t - \langle U, V\rangle_t)_{t\geq 0}$ is a continuous complex-valued local martingale.*

Proof. By Definition 23 it is enough to define the process $\langle U, V\rangle$ using linearity in $\mathbb{C}$, i.e. for $U = X + iY$ and $V = X' + iY'$ we define

$$\langle U, V\rangle := \langle X, X'\rangle - \langle Y, Y'\rangle + i\left(\langle X, Y'\rangle + \langle Y, X'\rangle\right). \tag{1.3}$$

What remains to be checked is that $UV - \langle U, V\rangle$ is indeed a local martingale. For all fixed $t \geq 0$, we have

$$\begin{aligned} U_tV_t - \langle U, V\rangle_t &= X_tX_t' - Y_tY_t' + iX_tY_t' + iY_tX_t' - \langle X, X'\rangle_t + \langle Y, Y'\rangle_t - i\langle X, Y'\rangle_t - i\langle Y, X'\rangle_t \\ &= \left(X_tX_t' - \langle X, X'\rangle_t\right) - \left(Y_tY_t' - \langle Y, Y'\rangle_t\right) \\ &\quad + i\left(\left(X_tY_t' - \langle X, Y'\rangle_t\right) + \left(Y_tX_t' - \langle Y, X'\rangle_t\right)\right). \end{aligned}$$

We see that each of the terms above are real-valued continuous local martingales by Definition 23, hence $UV - \langle U, V\rangle$ is by definition a complex-valued continuous local martingale. To finish the argument recall that the real-valued quadratic covariation is bilinear, therefore by (1.3) we deduce that the quadratic covariation in the complex case is also bilinear. The fact that $\langle U, V\rangle$ vanishes at zero and is the unique such process follows immediately from Definition 23.

□

As in the real case, we call the process $\langle U\rangle := \langle U, U\rangle$ the quadratic variation of a complex-valued continuous local martingale $(U_t)_{t\geq 0}$.

A natural question to ask is whether there is a difference between the real and complex-valued quadratic variation. To show that there is indeed a difference, let us look at the case where we consider a real-valued Brownian motion on the one hand and a complex-valued one on the other. If B is a real-valued Brownian motion under $\mathbb{P}_0$, we know from Proposition 25 that $\langle B\rangle_t = t$ for any $t \geq 0$. More generally, one can show that for any $x \in \mathbb{R}$, if B is a Brownian motion under $\mathbb{P}_x$, then $B^2 - t$ is a local martingale which implies that $\langle B\rangle_t = t$ for all $t \geq 0$. On the other hand if we take $B = X + iY$ to be a complex-valued Brownian motion under $\mathbb{P}_z$, for any $z \in \mathbb{C}$, then using bilinearity of the complex-valued quadratic variation for all $t \geq 0$, we get

$$\langle B\rangle_t = \langle X + iY\rangle_t = \langle X\rangle_t - \langle Y\rangle_t + 2i\langle X, Y\rangle_t.$$

Using that X and Y are independent we immediately deduce that $\langle X, Y\rangle_t = 0$, hence we get that

$$\langle B\rangle_t = t - t + 2i \cdot 0 = 0,$$

for all $t \geq 0$. This indeed illustrates the difference between the real-valued quadratic variation and the complex-valued.

At this point, it is worth noting that we can also motivate the fact that $B^2 - t$ is a local martingale as a simple corollary of the conformal invariance theorem, which we look at in more detail in the next chapter. We will see that since $z \mapsto z^2$ is holomorphic, B^2 is a Brownian motion if we change its velocity. Therefore, it follows that B^2 is a local martingale and so is also $B^2 - t$.

To sum up, we have now shown that any complex Brownian motion has zero quadratic variation, which motivates the notion of a more general class of processes called conformal (local) martingales.

Definition 11. *Let Z be a complex-valued (local) martingale. Then Z is said to be conformal if its quadratic variation is zero, i.e. $\langle Z\rangle_t$=0 for all $t \geq 0$.*

Although we will rarely work with conformal martingales, other than stochastic processes B and $f(B)$ where B is a Brownian motion and f is holomorphic, we will briefly remark that the notion of a conformal martingale can be useful because important formulae, such as the Itô formula, are simplified.

In the following discussion, consider any conformal (local) martingale $Z = X + iY$. Since the complex quadratic variation is bilinear for all $t \geq 0$, it follows

$$\langle Z\rangle_t = \langle X\rangle_t - \langle Y\rangle_t + 2i\langle X, Y\rangle_t = 0 + 0i.$$

If we now compare the real and imaginary parts in the above equality, we must have $\langle X\rangle_t = \langle Y\rangle_t$ and $\langle X, Y\rangle_t = 0$ for all $t \geq 0$. This shows that for any conformal martingale Z,

$$\begin{aligned} \langle \Re(Z)\rangle_t &= \langle \Im(Z)\rangle_t, \\ \langle \Re(Z), \Im(Z)\rangle_t &= 0, \end{aligned} \tag{1.4}$$

for all $t \geq 0$.

Since we now know that every complex Brownian motion is a conformal martingale, we can ask, as in the real case (see Proposition 25 also known as Lévy's characterisation), whether we can find a characterisation so that a conformal martingale is a complex Brownian motion.

Lemma 5. [Fre15] *Let Z be a complex-valued continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted process. Then, Z is a complex Brownian motion under $\mathbb{P}_z$ if and only if Z is a conformal local martingale and $\left\langle Z, \overline{Z}\right\rangle_t = 2t$ for all $t \geq 0$.*

Proof. $\Rightarrow$, As we already discussed, if $Z := X + iY$ is a complex Brownian motion starting from $z \in \mathbb{C}$ then Z is a conformal local martingale. Additionally, for all $t \geq 0$

$$\left\langle Z, \overline{Z}\right\rangle_t = \langle X + iY, X - iY\rangle_t = \langle X, X\rangle_t + \langle Y, Y\rangle_t = 2t, \tag{1.5}$$

using from above that $\langle X\rangle_t = \langle Y\rangle_t = t$.
$\Leftarrow$, Since Z is conformal, we know from (1.4) that $\langle X, X\rangle_t = \langle Y, Y\rangle_t$. Combining this with Equation (1.5) we deduce that for any $t \geq 0$, $\langle X, X\rangle_t = t$ or equivalently, $\langle Y, Y\rangle_t = t$ and we again conclude by Lévy's characterisation theorem that X, Y are real-valued Brownian motions. Moreover, since Z is assumed to be conformal by (1.4) we also know that $\langle X, Y\rangle = 0$. So, implicitly we know that X, Y are independent real-valued Brownian motions. Putting those two properties together, by definition we deduce that Z is a complex Brownian motion. □

As in the real case, with Lévy's characterisation theorem, we can also construct important new complex Brownian motions with the above lemma.

Lemma 6. [Cas09, Corollary 39] *Let $B_t := X_t + iY_t$, for $t \geq 0$, be a complex Brownian motion starting from $x + iy \in \mathbb{C}$,*

1. *if $\theta \in [0, 2\pi)$, then $(W_t)_{t\geq 0} := \left(e^{i\theta} B_t\right)_{t\geq 0}$,*
2. *if $c \in [0, \infty)$, then $(Z_t)_{t\geq 0} := \left(\frac{1}{\sqrt{c}} B_{ct}\right)_{t\geq 0}$,*

are again complex Brownian motions starting from $e^{i\theta}(x + iy)$ respectively $\frac{1}{\sqrt{c}}(x + iy)$.

Proof. 1. Let $\theta \in [0, 2\pi)$, then clearly $(W_t)_{t\geq 0}$ is a complex local martingale and for each $t \geq 0$, $\langle W, W\rangle_t = e^{2i\theta}\langle B, B\rangle_t = 0$, which means W is conformal. Furthermore,

$$\left\langle W, \overline{W}\right\rangle_t = \left\langle e^{i\theta} B, \overline{e^{i\theta} B}\right\rangle = \underbrace{e^{i\theta}\overline{e^{i\theta}}}_{|e^{i\theta}|^2=1} \left\langle B, \overline{B}\right\rangle_t = \langle X, X\rangle_t + \langle Y, Y\rangle_t = 2t.$$

Since this holds for all $t \geq 0$, we can conclude by Lemma 5.

2. In the real case we already know that $(X'_t)_{t\geq 0} := \left(\frac{1}{\sqrt{c}} X_{ct}\right)_{t\geq 0}$ and $(Y'_t)_{t\geq 0} := \left(\frac{1}{\sqrt{c}} Y_{ct}\right)_{t\geq 0}$ are Brownian motions starting from $\frac{x}{\sqrt{c}}$, respectively $\frac{y}{\sqrt{c}}$. Since X and Y are independent real-valued Brownian motions, so are X' and Y'. Therefore, $Z = X' + iY'$ is a complex-valued Brownian motion starting from $\frac{1}{\sqrt{c}}(x + iy)$.

□

The second point can be viewed as a particular case of the more general statement called the conformal invariance theorem, which we will discuss in Chapter 2. In particular, we have that $(cB_t)_{t\geq 0}$ has the same law as $(B_{c^2 t})_{t\geq 0}$, so we can view $(c^2 t)_{t\geq 0}$ as a time change.

Now that we have defined both a local martingale and a Brownian motion in the complex plane, we can define the complex stochastic integral of a local martingale with respect to a Brownian motion. To do this, we first recall that if $z = x + iy$ and $w = u + iv$ are two complex numbers, then by definition $zw = (xu - yv) + i(xv + yu)$. For the definition of the complex stochastic integral, we use this multiplication rule for complex numbers. In the real case for any Brownian motion B and adapted process W we define the stochastic integral of W with respect to B as

$$\int_0^t W_s \, dB_s := \lim_{n\to\infty} \sum_{i=1}^{p_n} W_{t_i^{(n)}} \left(B_{t_i^{(n)}\wedge t} - B_{t_{i-1}^{(n)}\wedge t}\right),$$

for $t \geq 0$. Here $\left(t_i^{(n)}\right)_{i\in\mathbb{N}}$ is a sequence of partitions of $[0, t)$ in n with $0 =: t_0^{(n)} \leq t_1^{(n)} \leq \ldots \leq t_{p_n}^{(n)} = t$ such that $\lim_{n\to\infty} |\Delta_n| = 0$ where $|\Delta_n| := \max_{0\leq i\leq p_n} t_i^{(n)} - t_{i-1}^{(n)}$. Now, assume that $W = U + iV$, $B = X + iY$ are complex processes. Then by the above multiplication rule for complex numbers

$$W_{t_i^{(n)}} \left(B_{t_i^{(n)}\wedge t} - B_{t_{i-1}^{(n)}\wedge t}\right) = \left(U_{t_i^{(n)}} + iV_{t_i^{(n)}}\right) \left(X_{t_i^{(n)}\wedge t} - X_{t_{i-1}^{(n)}\wedge t} + i\left(Y_{t_i^{(n)}\wedge t} - Y_{t_{i-1}^{(n)}\wedge t}\right)\right)$$

$$= U_{t_i^{(n)}} \left(X_{t_i^{(n)} \wedge t} - X_{t_{i-1}^{(n)} \wedge t} \right) - V_{t_i^{(n)}} \left(Y_{t_i^{(n)} \wedge t} - Y_{t_{i-1}^{(n)} \wedge t} \right)$$
$$+ i \left(U_{t_i^{(n)}} \left(Y_{t_i^{(n)} \wedge t} - Y_{t_{i-1}^{(n)} \wedge t} \right) + V_{t_i^{(n)}} \left(X_{t_i^{(n)} \wedge t} - X_{t_{i-1}^{(n)} \wedge t} \right) \right).$$

Applying the above sum in front of all these terms gives us

$$\sum_{i=1}^{p_n} W_{t_i^{(n)}} \left(B_{t_i^{(n)} \wedge t} - B_{t_{i-1}^{(n)} \wedge t} \right) = \sum_{i=1}^{p_n} U_{t_i^{(n)}} \left(X_{t_i^{(n)} \wedge t} - X_{t_{i-1}^{(n)} \wedge t} \right) - \sum_{i=1}^{p_n} V_{t_i^{(n)}} \left(Y_{t_i^{(n)} \wedge t} - Y_{t_{i-1}^{(n)} \wedge t} \right)$$
$$+ i \left(\sum_{i=1}^{p_n} U_{t_i^{(n)}} \left(Y_{t_i^{(n)} \wedge t} - Y_{t_{i-1}^{(n)} \wedge t} \right) \right.$$
$$\left. + \sum_{i=1}^{p_n} V_{t_i^{(n)}} \left(X_{t_i^{(n)} \wedge t} - X_{t_{i-1}^{(n)} \wedge t} \right) \right).$$

Taking the limit in probability as $n \to \infty$ on both sides of the equation leads us to

$$\int_0^t W_s \, dB_s = \int_0^t U_s \, dX_s - \int_0^t V_s \, dY_s + i \left(\int_0^t U_s \, dY_s + \int_0^t V_s \, dX_s \right).$$

This motivates the following definition of the stochastic integral in the complex-valued case.

Definition 12. *For $t \geq 0$, let $W_t = U_t + iV_t$ be a complex stochastic process and $Z_t = X_t + iY_t$ a complex local martingale, then we define the stochastic integral of W with respect to Z by*

$$\int_0^t W_s \, dZ_s = \int_0^t U_s \, dX_s - \int_0^t V_s \, dY_s + i \left(\int_0^t U_s \, dY_s + \int_0^t V_s \, dX_s \right). \tag{1.6}$$

Note that the individual terms in the above equation are the already well-defined real-valued stochastic integrals with respect to $\mathbb{R}$-local martingales. However, this implicitly means that the processes U, V both need to be elements of

$$L^2_{\mathrm{loc}}(X) := \left\{ H \text{ progressively measurable} : \int_0^t H_s^2 \, d\langle X \rangle_s < \infty \ \forall t \geq 0 \right\},$$

respectively $L^2_{\mathrm{loc}}(Y)$.

With this definition, we prove a particular case of Itô's formula in $\mathbb{C}$. This is a well known theorem in stochastic calculus, as it has many practical applications, especially in the field of stochastic differential equations, which will not be discussed in this thesis.

Proposition 7 (Itô's formula in $\mathbb{C}$)**.** [Fre15, Theorem 2.2.9] *Suppose Z is a conformal complex continuous local martingale and $f : \mathbb{C} \to \mathbb{C}$ an analytic function. Then,*

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s) \, dZ_s. \tag{1.7}$$

Proof. Let us write $f = u + iv$ and $Z = X + iY$, then $f(Z_t) = u(Z_t) + iv(Z_t)$. Viewing u, v as functions from $\mathbb{R}^2$ to $\mathbb{R}$ we can apply the two-dimensional real-valued Itô formula.

$$\begin{aligned}
u(Z_t) &= u(Z_0) + \int_0^t u_x(Z_s)\ dX_s + \int_0^t u_y(Z_s)\ dY_s \\
&\quad + \frac{1}{2}\left(\int_0^t u_{xx}(Z_s)\ d\langle X\rangle_s + \int_0^t u_{yy}(Z_s)\ d\langle Y\rangle_s + \int_0^t u_{xy}(Z_s) + u_{yx}(Z_s)\ d\langle X, Y\rangle_s\right) \\
&\overset{(1)}{=} u(Z_0) + \int_0^t u_x(Z_s)\ dX_s + \int_0^t u_y(Z_s)\ dY_s + \frac{1}{2}\int_0^t u_{xx}(Z_s) + u_{yy}(Z_s)\ d\langle X\rangle_s \\
&\overset{(2)}{=} u(Z_0) + \int_0^t u_x(Z_s)\ dX_s + \int_0^t u_y(Z_s)\ dY_s,
\end{aligned}$$

where in (1) we use the fact that $\langle X\rangle = \langle Y\rangle$ and $\langle X, Y\rangle = 0$, since Z is assumed to be conformal (see (1.4)). In (2) we used the well known fact, that analyticity of f implies, that $u_{xx} + u_{yy} = 0$, which can be proven by deriving the first Cauchy-Riemann equation with respect to x and the second one with respect to y. Analogous, by replacing u with v in the above computation, we deduce that

$$v(Z_t) = v(Z_0) + \int_0^t v_x(Z_s)\ dX_s + \int_0^t v_y(Z_s)\ dY_s.$$

Putting these two parts together gives us

$$\begin{aligned}
f(Z_t) &= u(Z_t) + iv(Z_t) \\
&= f(Z_0) + \left(\int_0^t u_x(Z_s)\ dX_s + \int_0^t u_y(Z_s)\ dY_s\right) + i\left(\int_0^t v_x(Z_s)\ dX_s + \int_0^t v_y(Z_s)\ dY_s\right).
\end{aligned} \tag{1.8}$$

Let us now consider the right-hand side of (1.7). Using Definition 6 and Definition 12 we get

$$\begin{aligned}
f(Z_0) + \int_0^t \partial f(Z_s)\ dZ_s &= f(Z_0) + \frac{1}{2}\int_0^t f_x(Z_s) - if_y(Z_s)\ dZ_s \\
&= f(Z_0) + \frac{1}{2}\left(\int_0^t u_x(Z_s)\ dX_s - \int_0^t v_x(Z_s)\ dY_s + i\int_0^t u_x(Z_s)\ dY_s\right. \\
&\quad \left. + i\int_0^t v_x(Z_s)\ dX_s\right) + \frac{1}{2}\left(-i\int_0^t u_y(Z_s)\ dX_s + i\int_0^t v_y(Z_s)\ dY_s\right. \\
&\quad \left. + \int_0^t u_y(Z_s)\ dY_s + \int_0^t v_y(Z_s)\ dX_s\right).
\end{aligned} \tag{1.9}$$

By the Cauchy-Riemann equations we know that the second bracket term in (1.9) can be rewritten as

$$i\int_0^t v_x(Z_s)\ dX_s + i\int_0^t u_x(Z_s)\ dY_s - \int_0^t v_x(Z_s)\ dY_s + \int_0^t u_x(Z_s)\ dX_s,$$

after cancelling and reordering terms we see that (1.8) and (1.9) are equal, which concludes the proof. □

Let us remark at this point, that we can also apply Itô's formula, if f is only defined on an open set but Z takes a.s. its values in this set. However, we will not provide a proof of this statement.

An important consequence of the complex Itô formula is the following corollary.

Corollary 1. [Fre15, Page 32] *Let B be a complex-valued Brownian motion under $\mathbb{P}_z$ for $z \in \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ an analytic map, then $Z := f(B)$ is a conformal local martingale and in particular*

$$\frac{1}{2}\left\langle f(B), \overline{f(B)}\right\rangle_t = \langle \Re(f(B))\rangle_t = \langle \Im(f(B))\rangle_t = \int_0^t |f'(B_s)|^2 \, ds, \tag{1.10}$$

for all $t \geq 0$.

Proof. We can represent B as $B = B^{(1)} + iB^{(2)}$, and write $f = u + iv$ since f is analytic. Using the complex-valued Itô formula and (1.6) and Lemma 1, for all $t \geq 0$

$$\begin{aligned} Z_t &= f(B_0) + \int_0^t f'(B_s) \, dB_s \\ &= f(B_0) + \int_0^t u_x(B_s) - iu_y(B_s) \, dB_s \\ &= f(B_0) + \int_0^t u_x(B_s) \, dB_s^{(1)} + \int_0^t u_y(B_s) \, dB_s^{(2)} + i\left(\int_0^t u_x(B_s) \, dB_s^{(2)} - \int_0^t u_y(B_s) \, dB_s^{(1)}\right). \end{aligned}$$

Since all integral terms are local martingales we deduce that Z is a local martingale. Furthermore, using the above representation for Z_t we infer that

$$\langle Z\rangle_t = \left\langle f(B_0) + \int_0^{\cdot} f'(B_s) \, dB_s \right\rangle_t = \int_0^t |f'(B_s)|^2 \, d\langle B\rangle_t = 0,$$

since B is conformal. This shows that Z is a conformal local martingale. Defining $X := \Re\left(f(B)\right)$ and $Y := \Im\left(f(B)\right)$, and using the Cauchy-Riemann equations, we get for all $t \geq 0$

$$\begin{aligned} X_t &= \Re\left(f(B_0)\right) + \int_0^t u_x(B_s) \, dB_s^{(1)} - \int_0^t v_x(B_s) \, dB_s^{(2)}, \\ Y_t &= \Im\left(f(B_0)\right) + \int_0^t u_x(B_s) \, dB_s^{(2)} + \int_0^t v_x(B_s) \, dB_s^{(1)}. \end{aligned} \tag{1.11}$$

Since $f(B)$ is a conformal process we already know that $\langle X\rangle_t = \langle Y\rangle_t$ and $\langle X, Y\rangle_t = 0$ for all $t \geq 0$. Using the representation of X respectively Y found in (1.11) together with the fact that $\left\langle B^{(1)}_{\cdot}, B^{(2)}_{\cdot}\right\rangle_t = 0$ for any $t \geq 0$ we establish that

$$\begin{aligned} \langle Y\rangle_t = \langle X\rangle_t = &\left\langle \int_0^{\cdot} u_x(B_s) \, dB_s^{(1)} \right\rangle_t - 2\left\langle \int_0^{\cdot} u_x(B_s) \, dB_s^{(1)}, \int_0^{\cdot} v_x(B_s) \, dB_s^{(2)} \right\rangle_t \\ &+ \left\langle \int_0^{\cdot} v_x(B_s) \, dB_s^{(2)} \right\rangle_t \end{aligned}$$

$$= \int_0^t u_x(B_s)^2 \, ds + \int_0^t v_x(B_s)^2 \, ds \tag{1.12}$$
$$= \int_0^t |f'(B_s)|^2 \, ds.$$

To finish the argument we remark that for any $t \geq 0$

$$\begin{aligned}\frac{1}{2}\left\langle f(B), \overline{f(B)}\right\rangle_t &= \frac{1}{2}\langle X + iY, X - iY\rangle_t \\ &= \frac{1}{2}\left(\langle X\rangle_t - i\langle X, Y\rangle_t + i\langle X, Y\rangle_t + \langle Y\rangle_t\right) \\ &= \frac{1}{2}\left(\langle X\rangle_t + \langle Y\rangle_t\right) \\ &= \langle \Re(f(B))\rangle_t,\end{aligned}$$

where in the last equation we used that $\langle X\rangle_t = \langle Y\rangle_t$ for any $t \geq 0$. □

1.1.3 Harmonic Functions and Uniform Measure on $\partial B(x, r)$

In the following section, we will introduce the necessary basics, to which we refer to later when we talk about harmonic functions and the uniform measure.

Let $D \subset \mathbb{C}$ be an open set and $f : D \to \mathbb{R}$ a twice continuously differentiable function. Recall that the Laplacian of f is defined as

$$\Delta f(z) := \frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z), \tag{1.13}$$

for all $z \in D$.

Definition 13. *Let us consider f as above, then we say that $f : D \to \mathbb{R}$ is harmonic if $\Delta f = 0$ on D.*

At this point, it is worth noting that there is a connection between holomorphic and harmonic maps.

Lemma 7. [Fre15, Lemma 2.1.4] *If $f = u + iv$ is holomorphic, then u and v are harmonic.*

Proof. Since f is holomorphic, u and v satisfy the Cauchy-Riemann equations, more precisely $u_x = v_y$ and $u_y = -v_x$. Using Proposition 4 we know that every holomorphic map is analytic, i.e. it is infinitely differentiable. In particular, it is twice-continuously differentiable, therefore

$$\begin{aligned} u_{xx} &= v_{yx}, & u_{yy} &= -v_{xy}, \\ v_{xx} &= -u_{yx}, & v_{yy} &= u_{xy}. \end{aligned}$$

Using these equations we can compute the Laplacian of u and v:

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} = v_{yx} - v_{xy} = v_{xy} - v_{xy} = 0, \\ \Delta v &= v_{xx} + v_{yy} = -u_{yx} + u_{xy} = -u_{xy} + u_{xy} = 0,\end{aligned}$$

where in the last line we used the well-known Schwarz theorem (Proposition 32). By definition, we have shown that u, v are harmonic. □

An important characterisation of harmonic functions is the Mean Value Property. In short, it states that the value of f in the centre of a disk is equal to the average of its values on the boundary of this disk. In particular, this means that if f is known on the circle, we also know the values of f inside the circle, more precisely at its centre.

Before stating the Mean Value Property we introduce the notion of the uniform measure on $\partial B(z,r)$ for $z \in \mathbb{C}$ and $r > 0$ which denotes the boundary of the ball with radius r centred at z.

Definition 14. *The uniform measure $\mathcal{U}_{z,r}(\cdot)$ on $\partial B(z,r)$ for $z \in \mathbb{C}$ and $r > 0$ is defined by the pushforward measure of $\phi_{z,r}(\theta) = z + re^{i\theta}$, where $\phi_{z,r} : (\lambda, [0, 2\pi]) \to (\mathcal{U}_{z,r}, \partial B(z,r))$ and λ is the Lebesgue measure. More precisely, for $A \in \mathcal{B}(\partial B(z,r))$,*

$$\mathcal{U}_{z,r}(A) = \lambda\left(\phi_{z,r}^{-1}(A)\right).$$

As a short exercise, one can show that $\mathcal{U}_{z,r}(\cdot)$ is invariant under rotations, and hence by Theorem 7.4 in [LG22] it is the unique such measure. Now we can rigorously state the Mean Value Property. Let us first recall that if D is any domain and $z \in D$, then $\operatorname{dist}(z, \partial D)$ denotes the smallest distance between z and ∂D.

Proposition 8. *(Mean Value Property)* [BN16, Proposition 2.4] *Let $f : D \to [0, \infty]$ be a measurable function, and suppose that for all $z \in D$ and any $r \in (0, \operatorname{dist}(z, \partial D))$ we have*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta = \int_{\partial B(z,r)} f(y)\, \mathcal{U}_{z,r}(dy). \tag{1.14}$$

Then, either $f(z) = \infty$ for all $z \in D$ or f is harmonic on D.

Note that, in the second equation, we used the change of variable formula for a pushforward measure. The following proposition relates harmonicity with the Mean Value Property.

Proposition 9. *A function $f : D \to \mathbb{R}$ is harmonic if and only if f satisfies the Mean Value Property.*

We will postpone the proof because we will first prove the following two propositions.

Proposition 10. [LG22, Theorem 7.4] *Let $z \in \mathbb{C}$ and $r > 0$. Then, for any complex Brownian motion B starting from z, the measure $\mathbb{P}_z(B_{\tau_{B(z,r)}} \in dy)$ is the uniform measure on $\partial B(z,r)$, where $\tau_{B(z,r)} = \inf\{t \geq 0 : B_t \notin B(z,r)\}$.*

Proof. We show that $\mathbb{P}_z(B_{\tau_{B(z,r)}} \in dy)$ is invariant under rotations and has total mass 1. This is enough since by definition rotational invariance characterizes the uniform measure. Let $\theta \in [0, 2\pi)$ and $A \subset \partial B(z,r)$ be measurable, then also $e^{i\theta}A := \left\{e^{i\theta}a : \ a \in A\right\}$ is a measurable subset of $\partial B(z,r)$. Using that by Lemma 6, $B' := e^{-i\theta}B$ is still a complex Brownian motion we get,

$$\mathbb{P}_z\left(B_{\tau_{B(z,r)}} \in e^{i\theta}A\right) = \mathbb{P}_z\left(B'_{\tau_{B(z,r)}} \in A\right).$$

This shows that $\mathbb{P}_z(B_{\tau_{B(z,r)}} \in dy)$ is invariant under rotation. Furthermore, it holds that

$$\mathbb{P}_z\left(B_{\tau_{B(z,r)}} \in B(z,r)\right) = 1.$$

Therefore, $\mathbb{P}_z(B_{\tau_{B(z,r)}} \in dy)$ coincides with the uniform measure on $\partial B(z,r)$. □

Later in this paper, we will revisit this concept in a more general context, when studying the so-called harmonic measure.

To prove Proposition 9, we still need Kakutani's formula.

Proposition 11. *(Kakutani's formula)* [BN16, Theorem 2.3] *Let h be a harmonic function defined on a bounded domain D, and having a continuous extension $\overline{h}$ to the closure $\overline{D}$. Fix $z \in D$ and let B be a complex Brownian motion under $\mathbb{P}_z$. Set $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$, then*

$$h(z) = \mathbb{E}_z\left(\overline{h}\left(B_{\tau_D}\right)\right). \tag{1.15}$$

Later in Section 3, we will observe that there is a generalisation of Kakutani's formula, see Theorem 2.

Proof. Fix $z \in D$, in the following proof we argue under $\mathbb{P}_z$. First, suppose that h is the restriction to D of a C^2-function on $\mathbb{C}$ which we denote by H. For all $0 \leq t \leq \tau_D$ define the process $M_t := H(B_t)$ which can be represented by Proposition 7 as

$$M_t := H(B_0) + \int_0^t H'(B_s)\, dB_s = H(z) + \int_0^t \partial H(B_s)\, dB_s.$$

Remark that in the second equation we use Lemma 1. Then, since the stochastic integral with respect to a complex Brownian motion is a continuous local martingale, we deduce that also $(M_t)_{t\geq 0}$ is a continuous local martingale. Since H is bounded on D and B exits D at time τ_D,

$$|M^{\tau_D}| := |H(B_{t\wedge\tau_D})| \leq \max_{w\in D} |H(w)|,$$

which means that M^{τ_D} is bounded, and in particularly uniformly integrable. So, by the optional stopping theorem for all $z \in D$,

$$h(z) = H(z) = H(B_0) = M_0 \overset{\text{OST}}{=} \mathbb{E}_z(M_{\tau_D}) = \mathbb{E}_z(H(B_{\tau_D})) = \mathbb{E}_z(h(B_{\tau_D})) = \mathbb{E}_z\left(\overline{h}(B_{\tau_D})\right).$$

Now it remains to consider the general case where h is harmonic on D but not necessarily C^2. In the following argument we want to approximate h by smooth functions and use the first part of the proof. For each $n \in \mathbb{N}$ let us therefore define a sequence of domains $D_n \subseteq D$ by

$$D_n := \left\{z \in D : \operatorname{dist}(z, \partial D) > \frac{1}{n}\right\}.$$

We observe that $D_n \to D$ as $n \to \infty$. Since h is harmonic and continuous on D, there exists a sequence $h|_{D_n} =: h_n$ which has a C^2 extension to $\mathbb{C}$, denoted by H_n for all $n \in \mathbb{N}$, and that

converges to h uniformly on compact subsets of D as $n \to \infty$. From above, we deduce that $H_n(z) = \mathbb{E}_z(\overline{h}_n(B_{\tau_{D_n}}))$. As $n \to \infty$ we know that $\tau_{D_n} \uparrow \tau_D$ almost surely. Since $\overline{h_n}$ converges uniformly we deduce that $\overline{h}_n(B_{\tau_{D_n}}) \to \overline{h}(B_{\tau_D})$. Applying the dominated convergence theorem, we get for all $z \in D$,

$$h(z) = \lim_{n\to\infty} H_n(z) = \lim_{n\to\infty} \mathbb{E}_z\left(\overline{h}_n(B_{\tau_{D_n}})\right) = \mathbb{E}_z(\overline{h}(B_{\tau_D})),$$

which concludes the proof of Kakutani's formula. □

Now we will look at the proof of Proposition 9. We will only show that if f is harmonic, then f satisfies the Mean Value Property. References for the other direction can be found in [BN16].

Proof. (Proposition 9) Let us assume that f is harmonic on D. Since for any $z \in D$ and $r < \operatorname{dist}(z, \partial D)$ it holds that $B(z,r) \subseteq D$, we also know that f is harmonic on $B(z,r)$. By Kakutani's formula and the fact that f is defined on $\overline{B(z,r)} \subset D$, we can write $f(z) = \mathbb{E}_z(f(B_{\tau_{B(z,r)}}))$. Using the definition of the push forward measure leads to $\mathbb{E}_z(f(B_{\tau_{B(z,r)}})) = \int_{\partial B(z,r)} f(y)\ \mathbb{P}_z(B_{\tau_{B(z,r)}} \in dy)$. Hence,

$$f(z) = \int_{\partial B(z,r)} f(y)\ \mathbb{P}_z(B_{\tau_{B(z,r)}} \in dy),$$

which is exactly the Mean Value Property by Proposition 10. □

1.1.4 Time Changes

We now introduce the concept of time changes. The fundamental idea is that we change the clock at which our process moves, without touching the geometry of the process. Let $s \geq 0$ and $s \mapsto A_s$ be an increasing, continuous function. Define its right inverse as

$$A_s^{-1} := \inf\{t \geq 0 : A_t > s\} \text{ for } s \in [0, A_\infty), \tag{1.16}$$

with the convention that $\inf\{\emptyset\} = \infty$. By definition the map $s \mapsto A_s^{-1}$ is again non-decreasing, so we can define

$$A_{s-}^{-1} := \lim_{h\uparrow s} A_h^{-1}, \tag{1.17}$$

as the left limit at s. The following lemma gives us an alternative way to express A_{s-}^{-1}.

Lemma 8. *With the same notation as above, it holds that* $A_{s-}^{-1} = \inf\{t \geq 0 : A_t \geq s\}$.

Proof. Consider an increasing sequence $s_n \to s$ as $n \to \infty$. By definition,

$$A_{s_n}^{-1} = \inf\{t \geq 0 : A_t > s_n\}.$$

Since $t \mapsto A_t$ is right continuous and increasing we have

$$\{t \geq 0 : A_t > s_n\} \downarrow \{t \geq 0 : A_t \geq s\},$$

as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \inf\{t \geq 0 : A_t > s_n\} = \inf\left\{\bigcap_{n\in\mathbb{N}} \{t \geq 0 : A_t > s_n\}\right\},$$

using the fact that $\bigcap_{n\in\mathbb{N}}\{t \geq 0 : A_t > s_n\} = \{t \geq 0 : A_t \geq s\}$ we have

$$\inf\left\{\bigcap_{n\in\mathbb{N}} \{t \geq 0 : A_t > s_n\}\right\} = \inf\{t \geq 0 : A_t \geq s\},$$

thus in conclusion

$$A_{s-}^{-1} = \lim_{n\to\infty} A_{s_n}^{-1} = \inf\{t \geq 0 : A_t \geq s\}.$$

□

Now, let us introduce the notion of a time change.

Definition 15. *A time change with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a family $C = (C_s)_{s\geq 0}$ of $(\mathcal{F}_t)$-stopping times such that $s \mapsto C_s$ is a.s. right continuous and non-decreasing.*

According to this definition, it can be seen that the previously introduced process A_s^{-1} is a time change if A is adapted.

Definition 16. *Given a time change C and a stochastic process X we can define the time changed filtration $\widehat{\mathcal{F}}_t := \mathcal{F}_{C_t}$ for $t \geq 0$, and the time changed process $\widehat{X}_t := X_{C_t}$ for $0 \leq t < \inf\{s \geq 0 : C_s > \infty\}$.*

Now, we study briefly some basic connections between X and its time changed process $\widehat{X}$.

Lemma 9. *Let X be a stochastic process.*

1. *If X is non-decreasing, then $\widehat{X}$ is non-decreasing.*
2. *If X is right-continuous, then $\widehat{X}$ is right-continuous.*

Proof. 1. Let $s > t \geq 0$, then $\widehat{X}_s - \widehat{X}_t = X_{C_s} - X_{C_t}$. Since $s \mapsto C_s$ is non-decreasing, $C_s \geq C_t$. Using the assumption that X is non-decreasing proves the first point.

2. Since X and C are right continuous $\lim_{t\to s^+} \widehat{X}_t = \lim_{t\to s^+} X_{C_t} = X_{C_s} = \widehat{X}_s$

□

We can also see that if X is continuous, the time changed process $\widehat{X}$ does not necessarily need to be continuous, since C might have jumps. In particular, if M is any martingale and C a time change, M_C does not have to be a martingale. For this let us consider a real-valued Brownian motion B and define $C_s := \inf\{t \geq 0 : B_t > s\}$. Then, $B_{C_s} = s$, hence $\mathbb{E}(B_{C_s}) = s$,

which is not constant for all $s \geq 0$, therefore B_C cannot be a martingale. For this reason we need the following definition.

Definition 17. *Let $(X_t)_{t\geq 0}$ be a stochastic process and $(C_t)_{t\geq 0}$ a time change, then $(X_t)_{t\geq 0}$ is called C-continuous if for all $s \geq 0$ the process X is a.s. constant on the interval $[C_{s-}, C_s]$.*

The next result states that the martingale property is preserved when there is C-continuity and observes that if X is C-continuous, then $\widehat{X}$ is continuous. In particular, it shows that a Brownian motion under a suitable time change remains a local martingale.

Proposition 12. [RY99, Proposition 1.5] *Let C be an a.s. finite time change and X an $(\mathcal{F}_t)$-local martingale. If X is C-continuous, then $\widehat{X}$ is a continuous $(\widehat{\mathcal{F}}_t)$-local martingale and $\langle \widehat{X} \rangle = \widehat{\langle X \rangle}$.*

Proof. The proof can be found in [RY99, Proposition 1.5]. □

1.1.5 Regular Points for Brownian Motion

Let $D \subset \mathbb{C}$ be an open set and B a complex Brownian motion. We will use the notation τ_D^+ for the first exit time of D by B strictly after zero, i.e.

$$\tau_D^+ := \inf\{t > 0 : B_t \notin D\}.$$

Using this notation we can introduce the notion of a regular point for a Brownian motion.

Definition 18. *Let $D \subset \mathbb{C}$ be a domain and B a complex Brownian motion. We call $z \in \partial D$ a regular point if*

$$\mathbb{P}_z\left(\tau_D^+ = 0\right) = 1,$$

which, in other words, means that starting at z, B immediately hits the boundary of D.

With this notation we can restrict the set of all domains to the following two subsets $\mathcal{D}$ and $\mathcal{D}^*$.

Definition 19.

1. *Let us denote by $\mathcal{D}$ the set of all domains $D \subset \mathbb{C}$ such that every $z \in \partial D$ is regular.*
2. *By $\mathcal{D}^\star$ we denote the set of all domains $D \subset \mathbb{C}$ such that there exists $z \in \partial D$ which is regular.*

We immediately deduce from this definition that $\mathcal{D} \subset \mathcal{D}^\star$.
The following proposition now gives us some intuition on what it means to be in $\mathcal{D}$, since it is quite difficult to give exact criteria.

Proposition 13. [Law, Corollary 1.1.3] *Suppose $D \subset \mathbb{C}$ is a domain such that all connected components of $\mathbb{C} \setminus D$ are larger than one point. Then, $D \in \mathcal{D}$, in particular every simply connected $D \subset \mathbb{C}$ is in $\mathcal{D}$.*

Chapter 2

Conformal Invariance of Brownian Motion and Applications

Before stating the main result of this chapter, let us start by introducing some notation. Let $\mathcal{O} \subseteq \mathbb{C}$ be an open set and B a complex Brownian motion started at $B_0 = z$ for $z \in \mathbb{C}$. We denote by $\tau_{\mathcal{O}} := \inf\{t \geq 0 : B_t \notin \mathcal{O}\}$ the first exit time of B from $\mathcal{O}$. Note that if B does not start in $\mathcal{O}$, i.e. $B_0 \in \mathbb{C} \setminus \mathcal{O}$, then $\tau_{\mathcal{O}} = 0$.

In short, the following theorem states that if f is a holomorphic map, then $f(B)$ is a complex-valued Brownian motion up to a random time change. It is based on the idea that for $z_0 \in \mathbb{C}$ a holomorphic map f with $f(z_0) = 0,\ f'(z_0) \neq 0$ locally looks like a dilation by $|f'(z_0)|$ and a rotation by $\arg(f'(z_0))$ (this idea was discussed after Definition 3 in more detail). However, since a Brownian motion is invariant (up to a time change) under rotations and scaling, by Lemma 6, the following theorem should hold intuitively.

Theorem 1 (Conformal invariance)**.** *Suppose $D \subset \mathbb{C}$ is a domain. Fix $z \in D$ and argue under $\mathbb{P}_z$. Let $f : D \to \mathbb{C}$ be a non-constant holomorphic function. Let*

$$\zeta = \int_0^{\tau_D} |f'(B_s)|^2 \, ds \in (0, \infty].$$

Define $\sigma(t) < \tau_D$ as the inverse of the strictly increasing map $t \mapsto \int_0^t |f'(B_s)|^2 \, ds$ for $t < \zeta$, with $\sigma(t) = \tau_D$ if $t \geq \zeta$.

(i) [Law, Theorem 1.2.10] *Then, $Y_t := f\left(B_{\sigma(t)}\right)$ for $0 \leq t < \zeta$ is a complex Brownian motion under $\mathbb{P}_{f(z)}$.*

(ii) [Mar20] *If $D = \mathbb{C}$, then $\zeta = \infty$ and therefore $(Y_t)_{t \geq 0}$ is a complex Brownian motion.*

(iii) [Law, Theorem 1.2.12] *If we also assume that $f : D \to f(D)$ is a bijective conformal mapping, then (i) holds with $\zeta = \tau_{f(D)} = \inf\{t \geq 0 : Y_t \notin f(D)\}$.*

At this point it is necessary to remark that in (i) the statement is imprecise since the law of (Y, ζ) is not specified. Therefore we need to provide more details: If we let B' be the D.D.S. Brownian motion of $\Re(f(B))$ defined after Proposition 15, then Y is B' taken up to time $\zeta = \int_0^{\tau_D} |f'(B_s)|^2 \, ds$. Now we have clarified what we mean by $(Y_t)_{0 \leq t < \zeta}$ being a complex Brownian motion.

2.1 Proof of Theorem 1

The rest of this section is devoted to the proof of the conformal invariance theorem. We need certain results in advance and we will start with the Dambis, Dubins-Schwarz theorem.

Proposition 14 (Dambis, Dubins-Schwarz)**.** [RY99, Theorem 1.6] *Let M be a real-valued $(\mathcal{F}_t, \mathbb{P})$-continuous local martingale, $M_0 = 0$ such that $\langle M \rangle_\infty = \infty$ and set $T_t := \inf\{s \geq 0 : \langle M \rangle_s > t\}$, then for all $t \geq 0$, $B_t = M_{T_t}$ is a real-valued $(\mathcal{F}_{T_t})_{t \geq 0}$-Brownian motion.*

Before we start the proof, we should first note that $(T_t)_{t \geq 0}$ is a time change, which can be also denoted by $\left(\langle M \rangle_t^{-1}\right)_{t \geq 0}$ using the notation of Section 1.1.4.

Proof. To check that $(B_t)_{t \geq 0}$ is an $(\mathcal{F}_{T_t})$-Brownian motion, we shall rely on Lévy's characterisation theorem, which can be found in the appendix (Proposition 25). First, observe that $B_0 = M_{T_0} = 0$, where we use Proposition 24 telling that M is constant if and only if $\langle M \rangle$ is. Indeed, we know that on the interval $[0, T_0]$, the quadratic variation $t \mapsto \langle M \rangle_t$ is constant. By the above fact therefore also M is constant on the same interval. Using that $M_0 = 0$ we can conclude that $M_{T_0} = 0$.

By Lévy's characterisation theorem it remains to check that for all $t \geq 0$, $\langle B \rangle_t = t$ and B is an $(\mathcal{F}_{T_t})_{t \geq 0}$-local martingale. Since the quadratic variation by definition is a non-decreasing, continuous process and by assumption $\langle M \rangle_\infty = \infty$ we deduce $\mathbb{P}(T_t < \infty, \, \forall t \geq 0) = 1$, i.e. $(T_t)_{t \geq 0}$ is a sequence of a.s. finite stopping times. In order to use Proposition 12 to deduce that $(B_t)_{t \geq 0}$ is a $(\mathcal{F}_{T_t})_{t \geq 0}$-local martingale, we need to check that M is $(T_t)_{t \geq 0}$-continuous. This follows immediately using the fact that $\langle M \rangle_\cdot$ is a continuous process and that M is constant if and only if $\langle M \rangle_\cdot$ is. Applying Proposition 12 we get, $\langle B_\cdot \rangle_t = \langle M_{T_\cdot} \rangle_t = \langle M_\cdot \rangle_{T_t}$ for all $t \geq 0$. Since for each $t \geq 0$, $\langle M_\cdot \rangle_{T_t} = t$, by Lévy's characterisation theorem we infer that $(B_t)_{t \geq 0}$ is a $(\mathcal{F}_{T_t})$-Brownian motion, which concludes the proof. □

Let us now extend this statement and drop the assumption $\langle M \rangle_\infty = \infty$. First recall Proposition 30 which states that $\langle M \rangle_\infty < \infty$ if and only if $M_\infty := \lim_{t \to \infty} M_t$ exists. Using this fact, we can define a new process W by

$$W_t := \begin{cases} M_{T_t} & \text{for } t < \langle M \rangle_\infty \, , \\ M_\infty & \text{for } t \geq \langle M \rangle_\infty . \end{cases}$$

Proposition 15. [RY99, Theorem 1.7] *Let $(T_t)_{t\geq 0}$ be as above and consider the filtered probability space $\left(\Omega, (\mathcal{F}_{T_t})_{t\geq 0}, \mathbb{P}\right)$. There exists an enlargement $\left(\tilde{\Omega}, \left(\tilde{\mathcal{F}}_t\right)_{t\geq 0}, \tilde{\mathbb{P}}\right)$ of the original probability space with a Brownian motion $\tilde{\beta}$ on $\tilde{\Omega}$ independent of M such that*

$$B_t := \begin{cases} M_{T_t} & \text{for } t < \langle M\rangle_\infty \ , \\ M_\infty + \tilde{\beta}_{t-\langle M\rangle_\infty} & \text{for } t \geq \langle M\rangle_\infty, \end{cases}$$

is a Brownian motion. In particular, $W_t = B_{t\wedge\langle M\rangle_\infty}$ for $t \geq 0$.

Proof. We consider an arbitrary probability space $\left(\Omega', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}'\right)$ which supports a Brownian motion β. Define $\left(\tilde{\Omega}, \left(\tilde{\mathcal{F}}_t\right)_{t\geq 0}, \tilde{\mathbb{P}}\right) := \left(\Omega\times\Omega', (\mathcal{F}_{T_t}\otimes\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}\otimes\mathbb{P}'\right)$ and consider the process $\tilde{\beta}_t(\omega,\omega') := \beta_t(\omega')$. By definition of the product measure $\tilde{\mathbb{P}}$ we conclude that $\tilde{\beta}$ is independent of M. Remark that we can write

$$M_{T_t} + \int_0^t \mathbb{1}_{\{s>\langle M\rangle_\infty\}}\, d\tilde{\beta}_s = \begin{cases} M_{T_t} & \text{for } t < \langle M\rangle_\infty \ , \\ M_\infty + \tilde{\beta}_{t-\langle M\rangle_\infty} & \text{for } t \geq \langle M\rangle_\infty, \end{cases}$$

where the right-hand side is exactly the definition of B. Hence, $B_t = M_{T_t} + \int_0^t \mathbb{1}_{\{s>\langle M\rangle_\infty\}}\, d\tilde{\beta}_s$ for $t \geq 0$. In order to apply Lévy's characterisation theorem for a Brownian motion we again need to check that B is a local martingale. Let us start by verifying that W is a local martingale. By Proposition 12 it is enough to check that M is $(\langle M\rangle_t^{-1})_{t\geq 0}$-continuous which means that for all $t \geq 0$, M is constant on the intervals $[\langle M\rangle_{t-}^{-1}, \langle M\rangle_t^{-1}]$, but this directly follows from the fact that constancy intervals of a martingale and its quadratic variation process coincide and we get that W is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-local martingale. Furthermore, that $t \mapsto \mathbb{1}_{\{t>\langle M\rangle_\infty\}}$ is $(\mathcal{F}_{T_t})_{t\geq 0}$ adapted since $\{t > \langle M\rangle_\infty\} = \langle M\rangle_\infty^{-1}([0,t)) \in \mathcal{F}_{T_t}$. Here we use that T_t is $\mathcal{F}_{T_t}$-measurable. Therefore, by definiton of $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ we get that $\int_0^t \mathbb{1}_{\{s>\langle M\rangle_\infty\}}\, d\tilde{\beta}_s$ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-local martingale. Using the fact that the sum of local martingales is again a local martingale, we can conclude that B is a $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-local martingale.

Since M and $\tilde{\beta}$ are independent, $\langle M, \tilde{\beta}\rangle_t = 0$ for all $t \geq 0$. Using this together with the bilinearity of quadratic variation gives us

$$\langle B\rangle_t = \langle M_{T_\cdot}\rangle_t + \int_0^t \mathbb{1}_{\{s>\langle M\rangle_\infty\}}\, ds, \text{ for all } t \geq 0.$$

By Proposition 12, $\langle M_{T_\cdot}\rangle_t = \langle M\rangle_{T_t} = t\wedge\langle M\rangle_\infty$, which implies that

$$\langle B\rangle_t = (t\wedge\langle M\rangle_\infty) + \int_0^t \mathbb{1}_{\{s>\langle M\rangle_\infty\}}\, ds = t, \text{ for all } t \geq 0.$$

Now the statement of the proposition follows by Lévy's theorem (Proposition 25). □

Using this proposition, we can introduce the notation of a Dambis, Dubins-Schwarz Brownian motion, which will be abbreviated as D.D.S. Brownian motion in the following. Let M be any $\mathbb{R}$-valued, $\left((\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$- continuous local martingale starting at zero with possibly finite

total variation. By the above proposition, we know that for $0 \leq t < \langle M \rangle_\infty$, there exists a complex Brownian motion B' such that $B'_t = M_{T_t} = M_{\langle M \rangle_t^{-1}}$, so in particular $M_t = B'_{\langle M \rangle_t}$. We call B' the D.D.S. Brownian motion of M.

Let us now generalise the Dambis, Dubin-Schwarz theorem and Proposition 15 to higher dimensions. Since we have proven the one-dimensional cases of the following propositions, we will not give a proof for the multidimensional ones. The proofs can be found in [RY99].

Proposition 16 (Knight)**.** [RY99, Theorem 1.9] *Let $M := \left(M^{(1)}, ..., M^{(d)}\right)$ be a continuous d-dimensional local martingale, vanishing at zero, such that*

1. *for all $1 \leq k \leq d$, $\left\langle M^{(k)}, M^{(k)} \right\rangle_\infty = \infty$ and*
2. *for all $1 \leq k \neq l \leq d$, $\left\langle M^{(k)}, M^{(l)} \right\rangle = 0$.*

For $t \geq 0$ let us denote by $T_t^{(k)} := \inf\left\{s \geq 0 : \left\langle M^{(k)}, M^{(k)} \right\rangle_s > t\right\}$ a coordinate wise time change and define $B_t^{(k)} := M^{(k)}_{T_t^{(k)}}$. Then, $B := \left(B^{(1)}, ..., B^{(d)}\right)$ is a d-dimensional Brownian motion.

As in Proposition 15, the assumption $\langle M^{(k)}, M^{(k)} \rangle_\infty = \infty$ can now be weakened again, but the probability space must be enlarged, as we see in the following theorem.

Proposition 17. [RY99, Theorem 1.10] *Let M be as in the Knight's theorem with the assumption that there exist some $0 \leq k \leq d$ such that $\left\langle M^{(k)}, M^{(k)} \right\rangle_\infty < \infty$. Then, there is a Brownian motion $\beta = \left(\beta^{(1)}, ..., \beta^{(d)}\right)$ independent of M, on an enlarged probability space, such that the process $B := \left(B^{(1)}, ..., B^{(d)}\right)$, where*

$$B_t^{(k)} := \begin{cases} M^{(k)}_{T_t^{(k)}} & \text{for } t < \langle M^{(k)}, M^{(k)} \rangle_\infty \ , \\ M^{(k)}_\infty + \beta^{(k)}_{t - \langle M^{(k)}, M^{(k)} \rangle_\infty} & \text{for } t \geq \langle M^{(k)}, M^{(k)} \rangle_\infty, \end{cases}$$

is a d-dimensional Brownian motion.

With these two generalisations in hand, we are now in a position to prove the first part of Theorem 1.

Proof. (Theorem 1) (i) Let us start by noticing that the map $t \mapsto \int_0^t |f'(B_s)|^2 \, ds$ is indeed strictly increasing for $0 \leq t < \zeta$. For this let $\Gamma = \{z \in D : f'(z) = 0\} \subseteq D$. If we assume Γ contains a limit point, then by the identity lemma (Lemma 2) $f' \equiv 0$ on D and hence f is constant on D, which contradicts our assumption. Hence, all zeros of f' are isolated and thus Γ is countable. Let us therefore write $\Gamma = \{z_k\}_{k \in \mathbb{N}}$. We know that for all $k \in \mathbb{N}$, $\mathbb{P}_z(B_t = z_k \ \forall t < \zeta) = 0$ which implies that $\mathbb{P}_z(B_t \in \Gamma \ \forall t < \zeta) = 0$. So, almost surely $B_t \notin \Gamma$ for all $t < \zeta$ and thus $f'(B_t) \neq 0$. Therefore, we conclude that $|f'(B_t)|^2 > 0$ a.s. for all $t < \zeta$ which proves the claim.

By Corollary 1, we know that $f(B)$ is a conformal local martingale and that in particular for all $t \geq 0$, $\langle X \rangle_t = \langle Y \rangle_t = \int_0^t |f'(B_s)|^2 \ ds < \infty$ where $X := \Re(f(B))$ and $Y := \Im(f(B))$. Using this representation we have already shown that $\sigma(t) = \langle X \rangle_t^{-1} = \langle Y \rangle_t^{-1}$.

Consider now the two-dimensional local martingale (X, Y). Using that $f(B) = X + iY$ is a conformal martingale we implicitly know that $\langle X, Y \rangle = 0$, hence by Proposition 17 there exists

a complex Brownian motion $\beta = \beta^{(1)} + i\beta^{(2)}$ on a possibly enlarged probability space such that $\beta_t = X_{\sigma(t)} + iY_{\sigma(t)}$ for all $t < \langle X\rangle_\infty = \zeta$, i.e. $\beta_t = f\left(B_{\sigma(t)}\right)$ for all $t < \zeta$. To finish let us shortly remark that for all $t \geq \tau_D$, $\langle X\rangle_t = \langle X\rangle_{\tau_D} = \zeta$, hence taking the limit as $t \to \infty$ on both sides indeed shows that $\langle X\rangle_\infty = \zeta$. □

In order to prove point (ii) of Theorem 1 we need two preliminary results, the first one on polar sets of Brownian motion and the second one on neighbourhood recurrence. For a set $A \subset \mathbb{C}$ and a complex-valued Brownian motion $(B_t)_{t\geq 0}$ starting from $z \in \mathbb{C}$ we denote by T_A the first hitting time of A by B, i.e. $T_A := \inf\{t > 0 : B_t \in A\}$. There is a well known fact that if A is a Borel set, T_A is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time, which in particular implies that $\{T_A < \infty\}$ is a measurable set (see [RY99, Theorem 2.17]).

Definition 20. *A Borel set A is called polar for Brownian motion if for every $z \in \mathbb{C}$*

$$\mathbb{P}_z(T_A < \infty) = 0.$$

We will now briefly discuss polar sets for complex-valued Brownian motions.

Proposition 18. [RY99, Proposition 2.7] *Let $B = B^{(1)} + iB^{(2)}$ be a complex Brownian motion under $\mathbb{P}_z$ for $z \in \mathbb{C}$. Then, every one-point set is a polar set.*

Proof. Since the law of B is invariant under rotation, scaling and translation up to a time change, it will be enough to show that under $\mathbb{P}_0$, B does not hit the one-point set $\{-1 + i \cdot 0\}$. A short argument on how to deduce the general statement is given at the end of the proof.

Consider the analytic map $f(z) = e^z$ for $z \in \mathbb{C}$ and assume B is a Brownian motion under $\mathbb{P}_0$. As in the assumption of the conformal invariance theorem, we set $\sigma(t) = A_t^{-1}$ where

$$A_t := \int_0^t \left|f'(B_s)\right|^2 \, ds = \int_0^t \left|e^{2B_s}\right| \, ds = \int_0^t \left|e^{2B_s^{(1)}} e^{2iB_s^{(2)}}\right| \, ds = \int_0^t e^{2B_s^{(1)}} \, ds.$$

At this point it is worth noting that $\sigma(t)$ for all $t \geq 0$ is in fact the inverse of $t \mapsto A_t$, because $t \mapsto A_t$ is a strictly increasing map. By part (i) of the conformal invariance theorem we also deduce that $Y_t := \exp\left(B_{\sigma(t)}\right)$ for $t < \zeta = A_\infty$ is a complex-valued Brownian motion starting from 1 up to time ζ. Hence, for all $t < A_\infty$, there exists a complex Brownian motion $\tilde{B}$ such that $M_t := \exp(B_t) - 1$ can be written as $\tilde{B}_{A_t}$ where $\tilde{B}_t = Y_t - Y_0$. Let us now show that in fact $A_\infty := \lim_{t\to\infty} A_t = \infty$, so that we actually do have a time changed Brownian motion and not one stopped at some point $\zeta < \infty$.

By contradiction, assume that $A_\infty < \infty$. Then, by Corollary 1, $e^B =: U + iV$ is a conformal martingale and we deduce that $\frac{1}{2}\left\langle e^B, \overline{e^B}\right\rangle_\infty = \langle U\rangle_\infty = \langle V\rangle_\infty = A_\infty < \infty$. So $\lim_{t\to\infty} U_t$ and $\lim_{t\to\infty} V_t$ exist by Proposition 30, which implies that $\lim_{t\to\infty} \exp(B_t) = \lim_{t\to\infty} U_t + i \cdot \lim_{t\to\infty} V_t$ exists in $\mathbb{C}$. But since $Y_t \to \infty$ as $t \to \infty$ we get a contradiction. We deduce that $A_\infty = \infty$ and therefore M is a time changed Brownian motion starting at zero. Since for all $t \geq 0$ we have $\exp(B_t) \neq 0$, M does not hit $-1 + i \cdot 0$. Using that M and B have the same distribution under $\mathbb{P}_0$ implies that also B does not hit $-1 + i \cdot 0$ under $\mathbb{P}_0$.

Now we want to show that starting from zero, B does not hit an arbitrary one-point set $A := \{x + iy\}$ for $x, y \in \mathbb{R}$. We can find a map Φ which is the composition of a rotation and a scaling so that $\Phi(x + iy) = -1 + i \cdot 0$ and $\Phi(0) = 0$. Using Lemma 6, $\Phi(B)$ is a time change of a possible scaled Brownian motion, so from the first part we deduce that $\mathbb{P}_0(T_A < \infty) = 0$.

To complete the proof, we still have to consider the case that B starts at any point $z \in \mathbb{C}$. Define the process $\tilde{B}_t := B_t - z$ for $t \geq 0$, which under $\mathbb{P}_z$ is a complex Brownian motion starting from zero. Then,

$$\begin{aligned}
\mathbb{P}_z(T_A < \infty) &= \mathbb{P}_z(\inf\{t > 0 : B_t \in A\} < \infty) \\
&= \mathbb{P}_z(\inf\{t > 0 : \tilde{B}_t = x + iy - z\} < \infty) \\
&= \mathbb{P}_0(\inf\{t > 0 : B_t = \tilde{z}\} < \infty) \\
&\overset{*}{=} 0,
\end{aligned}$$

where in $*$ we used that for any one point set A, $\mathbb{P}_0(T_A < \infty) = 0$. This concludes the proof. □

Corollary 2. *Denote by $S := \{z_1, z_2, z_3, ...\}$ a countable subset of $\mathbb{C}$. Then, S is polar for B under $\mathbb{P}_z$, for any $z \in \mathbb{C}$.*

Proof. Using Proposition 18 we already know that for all $i \geq 1$, $\{z_i\}$ are polar sets, i.e. $\mathbb{P}_z(\exists\, t > 0 : B_t = z_i) = 0$. Using countable additivity we get

$$\mathbb{P}_z(\exists\, t > 0 : B_t \in S) \leq \sum_{i \geq 1} \mathbb{P}_z(\exists\, t > 0 : B_t = z_i) = 0.$$

So, in particular $\mathbb{P}_z(T_S < \infty) = 0$ which by definition means that S is polar for B. □

We will now use Proposition 18 to prove that a planar Brownian motion is neighbourhood recurrent. This is the last missing result needed to proof (ii) of Theorem 1.

Proposition 19 (Recurrence of complex Brownian Motion)**.** [RY99, Theorem 2.8] *Let B be a complex Brownian motion starting from $z \in \mathbb{C}$. Then almost surely, $\{t \geq 0 : B_t \in U\}$ is unbounded for all open $U \subset \mathbb{C}$.*

As we begin the proof, we would like to point out that the statement says that almost surely for all U the above set is unbounded. This statement is stronger than saying that for all U, a.s. the set is unbounded.

Proof. As before it is enough to proof the statement for $z = 0$. Since open balls with rational radius and centred at $x + iy$ for $x, y \in \mathbb{Q}$ form a basis of the euclidean topology on $\mathbb{C}$ it is enough to prove the proposition for the open ball $B(z, r) := \{u \in \mathbb{C} : |u - z| < r\}$ where $z = x + iy$ with $x, y \in \mathbb{Q}$ and $r > 0$ is rational. Indeed, if we pick $U \subset \mathbb{C}$ open, then U can be written as a union of open balls. So, if we can show that almost surely, $\{t \geq 0 : B_t \in B(z, r))\}$ is unbounded for a general open ball with rational $r > 0$ and $x, y \in \mathbb{Q}$, we can deduce that $\{t \geq 0 : B_t \in U\}$ is unbounded almost surely. Here we use that the rational numbers form a dense subset of $\mathbb{R}$, which means that any point in $\mathbb{C}$ can be approximated arbitrarily close by rational points.

Therefore, the centers of the open balls with rational coordinates $x+iy$ and rational radii $r>0$ are dense in the space of all open balls in $\mathbb{C}$ which form a basis of the Euclidean topology in $\mathbb{C}$.

From Proposition 18 we know that one-point sets are polar. Let us now consider the $\mathbb{R}$-valued stochastic process $M_t := \log(|B_t - z|)$ for $t \geq 0$. We will now make use of the real Itô formula applied to $f(u) = \log(|u-z|)$ for $u \in \mathbb{C} \setminus \{z\}$. Note that f is not twice continuously differentiable in $\mathbb{C}$, but there is nevertheless a version of the Itô formula for which it is sufficient that $\mathbb{P}(B_t = z \text{ for some } t \geq 0) = 0$ (see Remark 2 after Theorem 3.3 in [RY99]). This can be applied here because we already know that one-point sets are polar.

Let us use the representation $u = a+ib$ and $z = x+iy$, then $f(u) = \log(|a-x+i(b-y)|) = \log\left(\sqrt{(a-x)^2+(b-y)^2}\right)$, so

$$\begin{aligned}
f_x(u) &= -(a-x)|u-z|^{-2},\\
f_y(u) &= -(b-y)|u-z|^{-2},\\
f_{xx}(u) &= |u-z|^{-2} - 2(a-x)^2|u-z|^{-4} = \frac{(b-y)^2-(a-x)^2}{|u-z|^4},\\
f_{yy}(u) &= |u-z|^{-2} - 2(b-y)^2|u-z|^{-4} = \frac{(a-x)^2-(b-y)^2}{|u-z|^4}.
\end{aligned}$$

Therefore on one hand, by Itô's formula we get

$$\begin{aligned}
M_t &= M_0 + \int_0^t f_x(B_s)\, dB_s^{(1)} + \int_0^t f_y(B_s)\, dB_s^{(2)}\\
&\quad + \frac{1}{2}\left(\int_0^t f_{xx}(B_s)\, d\langle B^{(1)}\rangle_s + \int_0^t f_{yy}(B_s)\, d\langle B^{(2)}\rangle_s\right)\\
&= M_0 - \int_0^t \left(B_s^{(1)} - x\right)|B_s - z|^{-2}\ \ dB_s^{(1)} - \int_0^t \left(B_s^{(2)} - y\right)|B_s - z|^{-2}\, dB_s^{(2)}\\
&\quad + \frac{1}{2}\int_0^t \frac{\left(B_s^{(2)}-y\right)^2 - \left(B_s^{(1)}-x\right)^2}{|B_s-z|^4} + \frac{\left(B_s^{(1)}-x\right)^2 - \left(B_s^{(2)}-y\right)^2}{|B_s-z|^4}\, ds\\
&= M_0 - \int_0^t \left(B_s^{(1)} - x\right)|B_s - z|^{-2}\ \ dB_s^{(1)} - \int_0^t \left(B_s^{(2)} - y\right)|B_s - z|^{-2}\, dB_s^{(2)},
\end{aligned}$$

so

$$\langle M\rangle_t = \int_0^t \left(B_s^{(1)} - x\right)^2 |B_s - z|^{-4} + \left(B_s^{(2)} - y\right)^2 |B_s - z|^{-4}\, ds = \int_0^t |B_s - z|^{-2}\, ds.$$

On the other hand for all $s \geq 0$

$$\begin{aligned}
\log(|B_s - z|) &= \log\left(\left|B_s^{(1)} - x + i\left(B_s^{(2)} - y\right)\right|\right)\\
&= \log\left(\sqrt{\left(B_s^{(1)} - x\right)^2 + \left(B_s^{(2)} - y\right)^2}\right)\\
&\geq \log\left(\sqrt{\left(B_s^{(1)} - x\right)^2}\right)\\
&= \log(|B_s^{(1)} - x|),
\end{aligned}$$

which implies that $\sup_{s\leq t} M_s \geq \sup_{s\leq t} \log\left(\left|B_s^{(1)} - x\right|\right) \to \infty$ as $t \to \infty$. This follows from the well known fact, that for any real-valued Brownian motion X, $\limsup_t X_t = \infty$, see Proposition 26. Since $\sup_t M_t = \infty$ a.s. we deduce from Proposition 30 that $\langle M\rangle_\infty = \infty$ and therefore

$$\infty = \langle M\rangle_\infty = \lim_{t\to\infty} \langle M\rangle_t = \lim_{t\to\infty} \int_0^t |B_s - z|^{-2}\, ds.$$

So, $\int_0^t |B_s - z|^{-2}\, ds \to \infty$ as $t \to \infty$. From Theorem 1 part (i) applied at the domain $D = \mathbb{C}\setminus\{z\}$ and for $f(u) = \log(|u-z|)$ we know that $Y_t := \log\left(\left|B_{\langle M\rangle_t^{-1}} - z\right|\right)$ is a complex-valued Brownian motion for $t < \zeta = \int_0^\infty |B_s - z|^{-2}\, ds = \infty$. Hence, M is a time changed Brownian motion in $\mathbb{C}$. As a result $\inf_t M_t = -\infty$, thus M takes values less than $\log(r)$ infinitely many times. Therefore, there are unbounded many t s.t. $B_t \in B(z, r)$ which concludes the proof. □

We are now in a position to prove part (ii) and (iii) of Theorem 1.

Proof. (Theorem 1) (*ii*) Let us show that if f is an entire, non-constant function and B a complex-valued Brownian motion, then $f(B)$ is a time changed Brownian motion. More precisely, there exists a complex Brownian motion β starting from zero such that

$$f(B_{\sigma(t)}) = f(B_0) + \beta_t,$$

for $t \geq 0$ where $\sigma(t) = \langle X\rangle_t^{-1}$ with $\langle X\rangle_t = \int_0^t |f'(B_s)|^2\, ds$. Here we use the notation $X = \Re(f(B))$.

We want to apply Proposition 16, and to do that we need $\langle X\rangle_\infty = \langle Y\rangle_\infty = \infty$. Since f is entire, f is analytic on the whole complex plane and therefore by Corollary 1

$$\langle X\rangle_t = \langle Y\rangle_t = \int_0^t |f'(B_s)|^2\, ds,\ \forall t \geq 0. \tag{2.1}$$

Remark that we already know that $t \mapsto \langle X\rangle_t$ is strictly increasing for $t < \langle X\rangle_\infty$. Let us show that $\langle X\rangle_\infty = \infty$. For this assume by contradiction that $\langle X\rangle_\infty < \infty$, by (2.1) we implicitly also assume that $\langle Y\rangle_\infty < \infty$. Then $\lim_{t\to\infty} X_t$ and $\lim_{t\to\infty} Y_t$ exist by Proposition 30, and thus $\lim_{t\to\infty} f(B_t)$ exists in $\mathbb{C}$. Since f is non-constant, we can find two open disjoint sets U_1, U_2 such that the images of their complements under f are disjoint, i.e. $f(U_1^c) \cap f(U_2^c) = \emptyset$. By Proposition 19, $\{t \geq 0 : B_t \in U_1\}, \{t \geq 0 : B_t \in U_2\}$ are unbounded. Additionally $f(U_2) \subset f(U_1^c)$, $f(U_1) \subset f(U_2^c)$. Since there are unbounded many t such that $B_t \in U_1$, respectively $B_t \in U_2$ there are unbounded many t such that $f(B_t) \in f(U_2^c)$, respectively $f(B_t) \in f(U_1^c)$. Hence, $f(B_t)$ cannot converge. Thus, we reached a contradiction which shows that $\langle X\rangle_\infty = \langle Y\rangle_\infty = \infty$.

We can now conclude by applying Proposition 16 to $M_t = f(B_t) - f(B_0)$ for $t \geq 0$, which is a two-dimensional local martingale vanishing at zero. Remark that we can indeed apply Proposition 16 since we can write $\sigma(t)$ as the inverse of its inverse i.e. for all $t \geq 0$, $\sigma(t) = \inf\{s \geq 0 : \sigma^{-1}(s) > t\} = \inf\{s \geq 0 : \langle X\rangle_s > t\}$.

(*iii*) Let us first remark that since $\sigma(t)$ is the inverse of $t \mapsto \int_0^t |f'(B_s)|^2 \, ds$, it holds that $\sigma(\zeta) = \tau_D$. Since by definition, B exits D at time τ_D, we know that $Y_t = f(B_{\sigma(t)})$ exits $f(D)$ at time ζ. Indeed, $Y_\zeta = f(B_{\sigma(\zeta)}) = f(B_{\tau_D}) \in \partial f(D)$. Since τ_D is the first such time, we also deduce that ζ is the first time $(Y_t)_{t\geq 0}$ exits $f(D)$. So, we indeed have that $\zeta = \tau_{f(D)} = \inf\{t \geq 0 : Y_t \notin f(D)\}$. This concludes the proof of the conformal invariance theorem. □

2.2 Some Applications of Theorem 1

2.2.1 Application to Brownian Motion

In the following sub-section we explore some applications of the conformal invariance theorem. We discuss the recurrence of a complex Brownian motion and show that a complex Brownian motion a.s. exits any domain D in finite time.

To discuss the recurrence properties of a complex Brownian motion, we introduce the notation of neighbourhood respectively pointwise recurrence. As the name suggests, we call a complex Brownian motion B, starting from $y \in \mathbb{C}$, neighbourhood recurrent if a.s. it returns to any neighbourhood after having left. More precisely, $\mathbb{P}_y$-a.s. for all $z \in \mathbb{C}, \epsilon > 0, T < \infty$ there exists a $t > T$ such that $|B_t - z| < \epsilon$. Further, B is called pointwise recurrent under $\mathbb{P}_y$ if $\mathbb{P}_y(T_y < \infty) = 1$, i.e. it returns to a single point with probability one. We will see later that even though two independent real-valued Brownian motions X, Y starting from zero return infinitely often to 0, it does not hold for the complex-valued Brownian motion $B = X + iY$, i.e. $\mathbb{P}_0(\inf\{t > 0 : B_t = 0\} < \infty) = 0$. Since we already know that real Brownian motions are recurrent, it is interesting to ask whether complex Brownian motions also fulfil such a property.

Corollary 3. [Law, Page 18] *A complex Brownian motion B starting from $z \in \mathbb{C}$ is neighbourhood recurrent but not pointwise recurrent under $\mathbb{P}_z$.*

Proof. We have already proved a stronger version of the fact that B is neighbourhood recurrent in Proposition 19. So, it remains to check that it is not pointwise recurrent.

For $z \in \mathbb{C}$ recall that we denote by T_z the first hitting time of z strictly after zero, i.e. $T_z := \inf\{t > 0 : B_t = z\}$. The idea is to first remark that starting from any $y \in \mathbb{C}$, $\mathbb{P}_y(T_z < \infty) = 0$ for all $z \in \mathbb{C}$. Then, by taking $y = z$ we deduce immediately that B is not pointwise recurrent.

By Proposition 18 we have already shown that $\mathbb{P}_y(T_z < \infty) = 0$ for all $y \neq z$. In particularly, this means that for every $z \neq 1$, we know that $\mathbb{P}_1(T_z < \infty) = 0$. Let us infer that this holds as well for $z = 1$. For this define $\tau_{B(1,\epsilon)} := \inf\{t \geq 0 : B_t \notin B(1,\epsilon)\}$ where $B(1,\epsilon)$ is the ball of radius $\epsilon > 0$ centred at 1. We now use the Strong Markov property. Define $\omega := (\omega(t))_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathbb{C})$ and $\Phi(\omega) := \mathbb{1}_{\{\exists\ t\geq 0:\ \omega(t)=1\}}$. Thus $\Phi(B_\cdot) := \mathbb{1}_{\{\exists\ t\geq 0:\ B_t=1\}}$. By Proposition 31,

$$\begin{aligned}\mathbb{E}_1\left(\mathbb{1}_{\left\{\tau_{B(1,\epsilon)}<\infty\right\}}\Phi\left(B_{\tau_{B(1,\epsilon)}+\cdot}\right)|\mathcal{F}_{\tau_{B(1,\epsilon)}}\right) &= \mathbb{1}_{\left\{\tau_{B(1,\epsilon)}<\infty\right\}}\mathbb{E}_{B_{\tau_{B(1,\epsilon)}}}\left(\Phi\left(B_\cdot\right)\right) \\ &= \mathbb{1}_{\left\{\tau_{B(1,\epsilon)}<\infty\right\}}\mathbb{P}_{B_{\tau_{B(1,\epsilon)}}}\left(\exists\ t\geq 0:\ B_t=1\right).\end{aligned}$$

Now, consider the term $\mathbb{P}_{B_{\tau_{B(1,\epsilon)}}}(\exists\, t \geq 0 :\ B_t = 1)$, since $B_{\tau_{B(1,\epsilon)}} \notin B(1,\epsilon)$ by definition of $\tau_{B(1,\epsilon)}$, it holds in particular that $B_{\tau_{B(1,\epsilon)}} \neq 1$. Using that $\mathbb{P}_y(T_1 < \infty) = 0$ for all $y \neq 1$, we deduce that $\mathbb{P}_{B_{\tau_{B(1,\epsilon)}}}(T_1 < \infty) = 0$. Hence,

$$\mathbb{E}_1\left(\mathbb{1}_{\{\tau_{B(1,\epsilon)}<\infty\}}\Phi\left(B_{\tau_{B(1,\epsilon)}+\cdot}\right)|\mathcal{F}_{\tau_{B(1,\epsilon)}}\right) = 0.$$

Using that $\tau_{B(1,\epsilon)} < \infty$ a.s. which we show in Corollary 4, and taking expectations on both sides we deduce that

$$\mathbb{P}_1\left(\exists\, t \geq \tau_{B(1,\epsilon)} :\ B_t = 1\right) = 0, \quad \forall\, \epsilon > 0.$$

Thus, we deduce that $\mathbb{P}_1(T_1 < \infty) = 0$.

In the remaining part, we show that $\mathbb{P}_z(T_z < \infty) = 0$ for all $z \in \mathbb{C}$. Consider the process $\tilde{B}_t := B_t - z + 1$ for $t \geq 0$, which under $\mathbb{P}_z$ is a complex Brownian motion starting from 1. Then,

$$\begin{aligned}\mathbb{P}_z(T_z < \infty) &= \mathbb{P}_z(\inf\{t > 0 : B_t = z\} < \infty)\\ &= \mathbb{P}_z(\inf\{t > 0 : \tilde{B}_t = 1\} < \infty)\\ &= \mathbb{P}_1(T_1 < \infty)\\ &= 0.\end{aligned}$$

To conclude, we have now shown that starting from y, $\mathbb{P}_y(T_z < \infty) = 0$ for all $z \in \mathbb{C} \setminus \{y\}$ and furthermore $\mathbb{P}_y(T_y < \infty) = 0$ for all $y \in \mathbb{C}$, putting these two together we get that under $\mathbb{P}_y$, $\mathbb{P}_y(T_z < \infty) = 0$ for all $z \in \mathbb{C}$. □

We already know that a complex Brownian motion B is neighbourhood recurrent for all $z \in \mathbb{C}$. This means that a.s. B is dense in $\mathbb{C}$. This now implies that if a Brownian motion starts in a domain $D \subset \mathbb{C}$, it will leave it sooner or later. More precisely:

Corollary 4. [BN16, Corollary 2.2] *Let D be a proper, simply connected domain, and $z_0 \in D$. Let B be a complex Brownian motion starting from z_0. Then,*

$$\mathbb{P}_{z_0}(\tau_D < \infty) = 1,$$

where $\tau_D := \inf\{t \geq 0 : B_t \notin D\}$.

Proof. Let $z \in \mathbb{C} \setminus \overline{D}$. Since $\overline{D} \subset \mathbb{C}$ is also a simply connected, strict subset, there exists a ball $B(z,r) \subset \overline{D}^c$ with some radius $r > 0$ in the complement of $\overline{D}$. By Corollary 3, B is neighbourhood recurrent, which means that a.s. $(B_t)_{t\geq 0}$ intersects $B(z,r)$ in finite time. Therefore, starting from $z_0 \in D$ a.s. B exits $\overline{D}$ in finite time. So, in particularly since $t \mapsto B_t$ is continuous, $\mathbb{P}_{z_0}$-a.s. B exits D in finite time, i.e. $\mathbb{P}_{z_0}(\tau_D < \infty) = 1$. □

2.2.2 Applications to Complex Analysis

In the following section, the interplay between Brownian motion and complex analysis becomes clearer. We prove two important propositions from complex analysis with the help of the conformal invariance theorem.

Corollary 5 (Liouville)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire, bounded function. Then, f is constant.*

Proof. Let B be a complex-valued Brownian motion under $\mathbb{P}_z$ for $z \in \mathbb{C}$. Assume by contradiction that f is non-constant. By the conformal invariance theorem point (ii) there exists a time change $(\sigma(t))_{t\geq 0}$ such that $W_t := f(B_{\sigma(t)})$ for $t \geq 0$, is a complex Brownian motion under $\mathbb{P}_{f(z)}$. Since W is neighbourhood recurrent by Corollary 3, W is dense in $\mathbb{C}$ and thus unbounded. This leads to a contradiction, since we assume f to be bounded, which ends the proof. □

Corollary 6 (Fundamental theorem of Algebra)**.** *Let f be a non-constant polynomial, then $f = 0$ has at least one solution.*

Proof. Let B be a complex Brownian motion starting from $z \in \mathbb{C}$. Using the conformal invariance theorem part (ii), and the fact that a polynomial is holomorphic, we deduce that there exists a time change $(\sigma(t))_{t\geq 0}$ such that for $t \geq 0$, $W_t := f\left(B_{\sigma(t)}\right)$ is a complex Brownian motion under $\mathbb{P}_{f(z)}$. Assume by contradiction that $f = 0$ has no solution. Since f is a polynomial, let us write $f(z) = \sum_{k=0}^{n} a_k z^k$. Then by the triangle inequality it holds that

$$|f(z)| \geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k = |z|^n \left(|a_n| - \sum_{k=0}^{n-1} |a_k||z|^{k-n} \right) \xrightarrow{|z|\to\infty} \infty.$$

Therefore, there exists $r > 0$ such that for all $z \in \mathbb{C}$, $f(z) \notin B(0, r)$. In particular, $W_t \notin B(0, r)$ for all $t \geq 0$. This leads to a contradiction, since W is neighbourhood recurrent. □

Chapter 3

Representation of Harmonic Functions and Applications

We have already seen a result concerning the representation of harmonic functions in terms of Brownian motion, namely the Kakutani formula (Proposition 11). In the following chapter we take a closer look at harmonic functions and prove a generalisation of this formula, which states that given a bounded domain D and a continuous function u defined on the boundary ∂D, we can find a harmonic function h inside D that matches u on ∂D. A big difference between these two results is that here we prove both the existence of a solution and the representation, whereas in Kakutani's formula we only prove the representation. Let us recall that we denote by $\mathcal{D}$ the set of all domains $D \subset \mathbb{C}$ such that every $z \in \partial D$ is regular.

Theorem 2. *(Solution to Dirichlet problem)* [Law, Proposition 1.2.4] *Suppose $D \in \mathcal{D}$ and that u is a bounded continuous function on ∂D. Then, there exists a unique bounded continuous function $h : \overline{D} \to \mathbb{R}$ that extends u and is harmonic in D. Furthermore,*

$$h(z) = \mathbb{E}_z \left(u \left(B_{\tau_D} \right) \right), \ \forall z \in D. \tag{3.1}$$

To prove this theorem, we need to build some background first.

3.1 The Harmonic Measure

Let us here introduce and discuss the notion of harmonic measures and its connection to complex analysis. As already remarked, we use this theory later in the proof of Theorem 2.

Harmonic measures are used to compute exit probabilities. To understand this intuitively, imagine a particle moving within a domain D according to the rules of a complex Brownian motion. We are particularly interested in identifying the point where this particle first strikes the boundary of D, or more specifically, we want to understand the probability distribution governing where this first exit occurs.

Before introducing the notation of the harmonic measure let us recall that we denote by $\mathcal{D}^\star$ the set of all domains $D \subset \mathbb{C}$ such that there exists $y \in \partial D$ which is regular.

Definition 21. *Let B be a complex Brownian motion. If $D \in \mathcal{D}^\star$ and $z \in D$, then the harmonic measure on ∂D denoted by $\mu_D(z, \cdot)$, is defined to be the distribution of B_{τ_D} under $\mathbb{P}_z$, i.e.*

$$\mu_D(z, dy) := \mathbb{P}_z \left(B_{\tau_D} \in dy \right).$$

So, in particular if f is a function defined on ∂D

$$\int_{\partial D} f(y)\ \mu_D(z, dy) = \mathbb{E}_z \left(f \left(B_{\tau_D} \right) \right), \tag{3.2}$$

and to simplify notation, we denote it by $\mu_D(z, f)$.

Using the conformal invariance theorem, we immediately deduce that the harmonic measure is "invariant under conformal transformations" in the following sense:

Lemma 10. [Law, Page 19] *Let $D \in \mathcal{D}^\star$. Assume $f : D \to f(D)$ is a conformal transformation that extends to a homeomorphism of $\overline{D} := D \cup \partial D$ and $V \subset \partial D$ is a Borel set, then*

$$\mu_{f(D)} \left(f(z), f(V) \right) = \mu_D(z, V). \tag{3.3}$$

Remark that we need the extension of f to $\overline{D}$ so that $f(B_{\tau_D})$ is defined.

Proof. Under the same assumptions as in the statement of the lemma,

$$\begin{aligned} \mu_D(z, V) &= \mathbb{P}_z \left(B_{\tau_D} \in V \right) \\ &= \mathbb{P}_z \left(f \left(B_{\tau_D} \right) \in f(V) \right) \\ &\overset{*}{=} \mathbb{P}_{f(z)} \left(B_{\tau_{f(D)}} \in f(V) \right) \\ &= \mu_{f(D)} \left(f(z), f(V) \right), \end{aligned}$$

where in $*$ we used Theorem 1 point (iii). It is also worth noticing that $f(D) \in \mathcal{D}^\star$. Indeed, first note that since D is non-empty, open and connected, also $f(D)$ is. So, by definition $f(D)$ is a domain. It remains to check that there exists $u \in \partial f(D)$ such that u is regular. Since $D \in \mathcal{D}^\star$, there exists a regular point $z \in \partial D$, i.e. $\mathbb{P}_z(\tau_D^+ = 0) = 1$. Let $u := f(z)$, then by conformal invariance $\mathbb{P}_{f(z)}(\tau_{f(D)}^+ = 0) = 1$, so u is regular. □

Now we consider a special case, namely when $D = \mathbb{D}$. In that case $\mu_{\mathbb{D}}$ is particularly simple.

Lemma 11. [LG22] *$\mu_{\mathbb{D}}(0, dy)$ is the uniform measure on $\partial \mathbb{D}$.*

We remark that this is a special case of Proposition 10, using that, by definition $\mu_{\mathbb{D}}(0, dy) = \mathbb{P}_0(B_{\tau_{\mathbb{D}}} \in dy)$. We next show that if the boundary of D is smooth, $\mu_D(z, dy)$ admits a density with respect to the arc-length measure. The following construction can be found in [Law, Page 19].

We start by recalling the arc-length integral. Let $D \in D^\star$ be a domain with a smooth boundary ∂D, i.e. there exists a smooth parametrisation $\gamma : [0, 1] \to \partial D$.

Let f be an integrable function on ∂D with respect to the arc-length measure. Therefore, we get

$$\int_{\partial D} f(y)\ |dy| := \int_0^1 f(\gamma(t))|\gamma'(t)|\ dt. \tag{3.4}$$

It is important to remark that this definition does not depend on the parametrisation γ, so the above integral is indeed well-defined. Now, we define the harmonic density $h_D(z, y)$ for $y \in \partial D$ and $z \in D$ as the density of $\mu_D(z, dy)$ with respect to the arc-length measure, i.e.

$$\int_{\partial D} f(y)\ \mu_D(z, dy) = \int_{\partial D} f(y)\ h_D(z, y)\ |dy|. \tag{3.5}$$

For this to be well-defined, we must check of course whether this density exists. Let $z \in D$ and as above choose $\gamma : [0, 1] \to \partial D$ to be a smooth parametrisation, then for $[a, b] \subseteq [0, 1]$, $\gamma([a, b]) =: I \subseteq \partial D$ is an arc of ∂D. Furthermore, let $\Phi : \overline{D} \to \mathbb{D}$ be a conformal isomorphism such that $\Phi(z) = 0$. Applying Lemma 10

$$\mu_D(z, I) = \mu_{\Phi(D)}(\Phi(z), \Phi(I)) = \mu_{\mathbb{D}}(0, \Phi(I)).$$

By Lemma 11, $\mu_{\mathbb{D}}(0, dy)$ is the uniform measure on $\partial\mathbb{D}$, so in particular $\mu_{\mathbb{D}}(0, dy) = \frac{1}{2\pi}|dy|$, hence

$$\mu_{\mathbb{D}}(0, \Phi(I)) = \int_{\Phi(I)} \mu_{\mathbb{D}}(0, dy) = \int_{\Phi(I)} \frac{1}{2\pi}\ |dy|.$$

Putting those facts together leads to

$$\begin{aligned}
\mu_D(z, I) &= \int_{\Phi(I)} \frac{1}{2\pi}\ |dy| \\
&= \int_{\Phi(\gamma([a,b]))} \frac{1}{2\pi}\ |dy| \\
&\overset{*}{=} \int_a^b \frac{1}{2\pi}|\Phi'(\gamma(t))| \cdot |\gamma'(t)|\ dt \\
&= \int_{I \cap \partial D} \frac{1}{2\pi}|\Phi'(y)|\ |dy|,
\end{aligned}$$

where in $*$ we have used (3.4) for $f \equiv 1$ and the parametrisation $\Phi \circ \gamma$. We therefore have the following lemma:

Lemma 12. [Law, Page 19] *Let D, Φ and I be as in the construction above, then for any $y \in \partial D$ the harmonic density $h_D(z, y)$ exists and is given by*

$$h_D(z, y) = \frac{1}{2\pi}|\Phi'(y)|. \tag{3.6}$$

To illustrate this result we now apply Lemma 12 to the following two examples where $D = \mathbb{D}$ and $D = \mathbb{H}$.

Example 1. [BN16, Example 3.1] *Let $D = \mathbb{D}$. Then,*

$$h_{\mathbb{D}}(z, y) = \frac{1}{2\pi} \frac{1 - |z|^2}{|y - z|^2},$$

for all $y \in \partial\mathbb{D}$ and $z \in \mathbb{D}$. In particular, $\mu_{\mathbb{D}}(z, dy)$ has a density with respect to $\mu_{\mathbb{D}}(0, dy)$, which is given by

$$\int_{\partial\mathbb{D}} f(y)\ \mu_{\mathbb{D}}(z, dy) = \int_{\partial\mathbb{D}} f(y) \frac{1 - |z|^2}{|y - z|^2}\ \mu_{\mathbb{D}}(0, dy).$$

Proof. Let $z \in \mathbb{D}$ arbitrary. Consider the conformal isomorphism from (1.1), $\Phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ defined by $\Phi(y) = \frac{y-z}{1-\overline{z}y}$ such that $\Phi(z) = 0$. Note that we used that Φ maps the unit circle into the unit circle. In order to use the above lemma we will compute $\Phi'(y)$ for $y \in \partial\mathbb{D}$,

$$\Phi'(y) = \frac{1(1 - \overline{z}y) + (y - z)\overline{z}}{(1 - \overline{z}y)^2} = \frac{1 - |z|^2}{(1 - \overline{z}y)^2}.$$

Since $|z| < 1$, in particular $1 - |z|^2 > 0$ hence

$$|\Phi'(y)| = \frac{1 - |z|^2}{|1 - \overline{z}y|^2}.$$

We now claim that since $y \in \partial\mathbb{D}$, $|1 - \overline{z}y|^2 = |y - z|^2$. Indeed,

$$\begin{aligned} |1 - \overline{z}y|^2 &= (1 - \overline{z}y)\overline{(1 - \overline{z}y)} \\ &= (1 - \overline{z}y)(1 - z\overline{y}) \\ &= 1 - z\overline{y} - \overline{z}y + |z|^2 \\ &= y\overline{y} - z\overline{y} - \overline{z}y + |z|^2 \\ &= (y - z)(\overline{y} - \overline{z}) \\ &= |y - z|^2. \end{aligned}$$

Using (3.6) we finally deduce that

$$h_{\mathbb{D}}(z, y) = \frac{1}{2\pi}|\Phi'(y)| = \frac{1}{2\pi} \frac{1 - |z|^2}{|y - z|^2}.$$

Therefore, using the above formula

$$\begin{aligned} \int_{\partial\mathbb{D}} f(y)\ \mu_{\mathbb{D}}(z, dy) &= \int_{\partial\mathbb{D}} f(y)\ h_{\mathbb{D}}(z, y)\ |dy| \\ &= \int_{\partial\mathbb{D}} \frac{1}{2\pi} f(y) \frac{1 - |z|^2}{|y - z|^2}\ |dy| \\ &= \int_{\partial\mathbb{D}} f(y) \frac{1 - |z|^2}{|y - z|^2}\ \mu_{\mathbb{D}}(0, dy). \end{aligned}$$

In the equation we used, that the uniform measure on $\partial\mathbb{D}$ is the arc length measure. □

If we consider $D = r\mathbb{D}$, i.e. the disk centred at zero with radius $r > 0$ we observe that, with a similar computation as in Exercise 1, we get

$$h_{r\mathbb{D}}(z, y) = \frac{1}{2\pi r} \frac{r^2 - |z|^2}{|y - z|^2}, \tag{3.7}$$

for $z \in D$ and $y \in \partial D$. This means that for any $A \subset \partial r\mathbb{D}$ measurable, $\mu_{r\mathbb{D}}(z, A) = \mathbb{P}_z \left(B_{\tau_{r\mathbb{D}}} \in A\right)$. For an illustration of this see Figure 1.

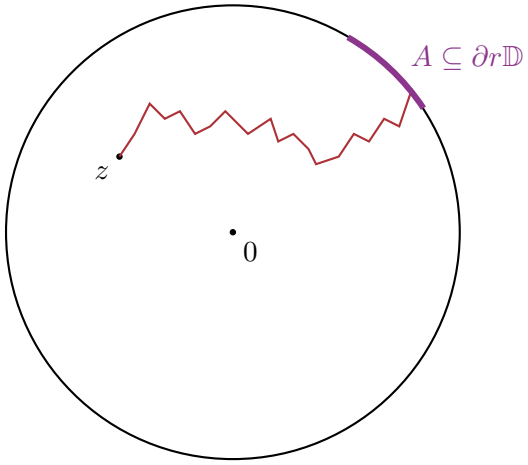

Figure 1: Path of a complex-valued Brownian motion B starting from z and exiting the boundary of $r\mathbb{D}$ through A.

Example 2. [BN16, Example 3.2] *Let $D = \mathbb{H}$. Then,*

$$h_{\mathbb{H}}(z, y) = \frac{1}{\pi} \Im \left(\frac{1}{y - z} \right),$$

for all $y \in \mathbb{R}$ and $z \in \mathbb{H}$. In particular, $\mu_{\mathbb{H}}(z, dy)$ has a density with respect to the arc-length on $\partial\mathbb{H}$. Hence,

$$\int_{\mathbb{R}} f(y)\ \mu_{\mathbb{H}}(z, dy) = \int_{\mathbb{R}} \frac{1}{\pi} f(y) \Im \left(\frac{1}{y - z} \right)\ |dy|.$$

Using this representation, we also have

$$\mu_{\mathbb{H}}(z, [-a, a]) = \frac{1}{\pi} \left(\tan^{-1} \left(\frac{u + a}{v} \right) - \tan^{-1} \left(\frac{u - a}{v} \right) \right),$$

for $a > 0$.

Proof. Let $z \in \mathbb{H}$ be arbitrary and remark that $\partial\mathbb{H} = \mathbb{R}$. Consider the conformal isomorphism from (1.2), $\Psi : \overline{\mathbb{H}} \to \overline{\mathbb{D}}$ defined by $\Psi(y) = \frac{y - z}{y - \overline{z}}$ mapping z to zero. Here we use a slightly generalised version of the map found in (1.2). This can be done because for $y \in \partial\mathbb{H}$, $|\Psi(y)|^2 = \frac{|y - z|^2}{|y - \overline{z}|^2} = 1$, i.e. $\Psi(\partial\mathbb{H}) = \partial\mathbb{D}$. As in the last example we need to compute its derivative,

$$\Psi'(y) = \frac{(y - \overline{z}) - (y - z)}{(y - \overline{z})^2} = \frac{z - \overline{z}}{(y - \overline{z})^2}.$$

If $y \in \partial\mathbb{H}$ taking the absolute value on both sides leads to

$$\begin{aligned}
|\Psi'(y)| &= \frac{|z-\overline{z}|}{|y-\overline{z}|^2} \\
&= \frac{|2i\Im(z)|}{|y-\overline{z}|^2} \\
&= \frac{2\Im(z)}{|y-\overline{z}|^2} \\
&= \frac{2\Im(z)}{(y-z)(y-\overline{z})} \\
&= 2\cdot\frac{z-\overline{z}}{2i}\cdot\frac{1}{(y-z)(y-\overline{z})} \\
&= 2\cdot\frac{(y-\overline{z})-(y-z)}{2i(y-z)(y-\overline{z})} \\
&= 2\cdot\frac{1}{2i}\left(\frac{1}{y-z}-\frac{1}{y-\overline{z}}\right) \\
&= 2\cdot\frac{1}{2i}\left(\frac{1}{y-z}-\overline{\frac{1}{y-z}}\right) \\
&= 2\Im\left(\frac{1}{y-z}\right).
\end{aligned}$$

Using (3.6), we finally deduce that

$$h_{\mathbb{H}}(z,y) = \frac{1}{2\pi}|\Psi'(y)| = \frac{2}{2\pi}\Im\left(\frac{1}{y-z}\right) = \frac{1}{\pi}\Im\left(\frac{1}{y-z}\right).$$

We can use this formula, as in the last example, in order to deduce the following integral equality:

$$\int_{\mathbb{R}} f(y)\ \mu_{\mathbb{H}}(z,dy) = \int_{\mathbb{R}} f(y)h_{\mathbb{H}}(z,y)\ |dy| = \int_{\mathbb{R}} \frac{1}{\pi}f(y)\Im\left(\frac{1}{y-z}\right)\ |dy|$$

To finish this example, we compute $\mu_{\mathbb{H}}(z,[-a,a])$ for $a>0$ explicitly. Let us therefore write $z = u+iv \in \mathbb{H}$, then

$$\begin{aligned}
\mu_{\mathbb{H}}(z,[-a,a]) &= \frac{1}{\pi}\int_{-a}^{a}\Im\left(\frac{1}{(y-u)-iv}\right)\ dy \\
&= \frac{1}{\pi}\int_{-a}^{a}\frac{v}{(y-u)^2+v^2}\ dy \\
&= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{y-u}{v}\right)\Big|_{y=-a}^{a}\right) \\
&= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{a-u}{v}\right)-\tan^{-1}\left(\frac{-a-u}{v}\right)\right) \\
&= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{u+a}{v}\right)-\tan^{-1}\left(\frac{u-a}{v}\right)\right),
\end{aligned}$$

□

To prove that the solution to the Dirichlet problem given in Theorem 2 is continuous, we first need to establish the following lemma. Informally, this lemma states, that if we start close to a regular boundary point, we will exit the domain quickly with high probability.

Lemma 13. [Law, Lemma 1.1.1] *Suppose $D \subset \mathbb{C}$ is an open subset and z_0 is a regular boundary point of ∂D. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that for $|z - z_0| < \delta$*

$$\mathbb{P}_z \left(\operatorname{diam} \left(B \left[0, \tau_D^+ \right] \right) \geq \epsilon \right) \leq \epsilon.$$

Before starting the proof, recall that $\tau_D^+ := \inf\{t > 0 : B_t \notin D\}$.

Proof. Without loss of generality, we can assume $z_0 = 0$ and define $\xi_s := \inf\{t \geq 0 : |B_t| = e^{-s}\}$ for $s \in \mathbb{N}$. If $z_0 \neq 0$ we can use a translation which maps z_0 to zero.

Suppose that 0 is a regular boundary point. By Definition 18 this means that $\mathbb{P}_0 \left(\tau_D^+ = 0 \right) = 1$ which implies that $\mathbb{P}_0 \left(B \left[0, \xi_s \right] \subset D \right) = 0$ since starting at $z_0 = 0$, B immediately hits the boundary ∂D and $\partial D \not\subseteq D$. Fix $s > 0$ and define $A_u := \{B[\xi_u, \xi_s] \subset D\}$ for any natural number u with $u > s$. Note that for all $s > 0$,

$$\mathbb{P}_z \left(\operatorname{diam} \left(B \left[0, \tau_D^+ \right] \right) \geq 2e^{-s} \right) \leq \mathbb{P}_z \left(B \left[\xi_u, \xi_s \right] \subset D \right).$$

Indeed, this is true, since if we choose $\omega \in \left\{ \operatorname{diam} \left(B \left[0, \tau_D^+ \right] \right) \geq 2e^{-s} \right\}$, then this implies that at some point in the path B must exit the circle $e^{-s}\mathbb{D}$ before exiting D. Therefore, $B_t(\omega)$ for $t \in [\xi_u, \xi_s]$ needs to be an element of D. Hence, $\left\{ \operatorname{diam} \left(B \left[0, \tau_D^+ \right] \right) \geq 2e^{-s} \right\} \subseteq \{B \left[\xi_u, \xi_s \right] \subset D\}$ which proves the claim. Informally speaking, this means that the green Brownian motion in Figure 2 cannot occur.

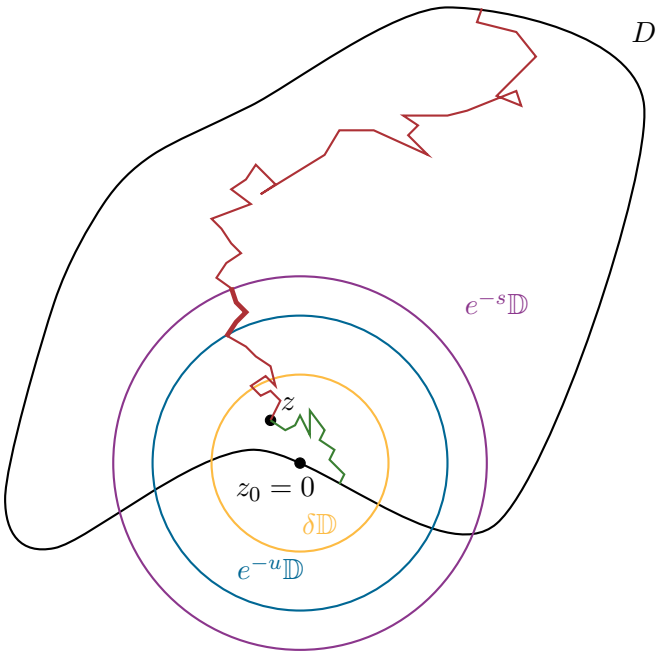

Figure 2: Two paths of a complex-valued Brownian motion starting from z and exiting at τ_D^+. The slightly thicker red section of the path describes the section between the two circles $e^{-u}\mathbb{D}$ and $e^{-s}\mathbb{D}$, i.e. $B[\xi_u, \xi_s]$, for our fixed path.

By construction $A_u \supset A_{u'}$ for any $u' > u$, hence the sequence $(A_u)_{u\in\mathbb{N}_{>s}}$ is decreasing. Thus,

$$\lim_{u\to\infty} \mathbb{P}_0\left(B\left[\xi_u,\xi_s\right]\subset D\right) = \mathbb{P}_0\left(\bigcap_{u=s+1}^{\infty}\{B[\xi_u,\xi_s]\subset D\}\right) = \mathbb{P}_0(B[0,\xi_s]\subset D) = 0.$$

Hence, there exists $u > s$ such that $\mathbb{P}_0(B[\xi_u,\xi_s]\subset D) \le e^{-s}$. In the following, we use the notation $\delta\mathbb{D} := \{\delta\cdot y : y\in\mathbb{D}\}$. Since $\xi_u > \tau_{\delta\mathbb{D}}$ for δ small enough and B is continuous, we can write $\xi_u = \xi_u\circ\theta_{\tau_{\delta\mathbb{D}}} + \tau_{\delta\mathbb{D}}$ and $\xi_s = \xi_s\circ\theta_{\tau_{\delta\mathbb{D}}} + \tau_{\delta\mathbb{D}}$, where θ_t is the shift operator. More precisely, let $\omega = (\omega_s)_{s\ge0}\in\Omega$ be a continuous function in the sample space, then $\theta_t : \Omega\to\Omega$ acts as $\theta_t(\omega) = (\omega_{t+s})_{s\ge0}$. Now, we compare $\mathbb{P}_0(B[\xi_u,\xi_s]\subset D)$ and $\mathbb{P}_z(B[\xi_u,\xi_s]\subset D)$ for $|z|<\delta$. By the Strong Markov property (Proposition 31) and the fact that $\tau_{\delta\mathbb{D}}$ is a.s. finite we deduce that

$$\begin{aligned}
\mathbb{P}_0(B[\xi_u,\xi_s]\subset D) &= \mathbb{E}_0\left(\mathbb{1}_{B[\xi_u,\xi_s]\subset D}\right)\\
&= \mathbb{E}_0\left(\mathbb{E}_0\left(\mathbb{1}_{B[\xi_u\circ\theta_{\tau_{\delta\mathbb{D}}}+\tau_{\delta\mathbb{D}},\xi_s\circ\theta_{\tau_{\delta\mathbb{D}}}+\tau_{\delta\mathbb{D}}]\subset D}\big|\mathcal{F}_{\tau_{\delta\mathbb{D}}}\right)\right)\\
&= \mathbb{E}_0\left(\mathbb{E}_{B_{\tau_{\delta\mathbb{D}}}}\left(\mathbb{1}_{B[\xi_u,\xi_s]\subset D}\right)\right)\\
&= \mathbb{E}_0\left(\mathbb{P}_{B_{\tau_{\delta\mathbb{D}}}}\left(B[\xi_u,\xi_s]\subset D\right)\right).
\end{aligned}$$

Similarly $\mathbb{P}_z(B[\xi_u,\xi_s]\subset D) = \mathbb{E}_z\left(\mathbb{P}_{B_{\tau_{\delta\mathbb{D}}}}(B[\xi_u,\xi_s]\subset D)\right)$, therefore using (3.2) and (3.5)

$$\left|\mathbb{P}_0(B[\xi_u,\xi_s]\subset D) - \mathbb{P}_z(B[\xi_u,\xi_s]\subset D)\right| \le \int_{\partial\delta\mathbb{D}} \left|h_{\delta\mathbb{D}}(0,y) - h_{\delta\mathbb{D}}(z,y)\right|\cdot\mathbb{P}_y(B[\xi_u,\xi_s]\subset D)\ |dy|, \tag{3.8}$$

where $h_{\delta\mathbb{D}}$ is the density of the harmonic measure with respect to the uniform measure in the circle. We know from (3.7) that

$$h_{\delta\mathbb{D}}(z,y) = \frac{1}{2\pi\delta}\frac{\delta^2-|z|^2}{|y-z|^2},$$

for some $y\in\partial\delta\mathbb{D}$ and $z\in\delta\mathbb{D}$. Since $z\mapsto h_{\delta\mathbb{D}}(z,y)$ is continuous for all $z\neq y$, we deduce that for any $\epsilon>0$ there exists $\delta_\epsilon>0$ such that for all z with $|z|<\delta_\epsilon$ it follows that $|h_{\delta\mathbb{D}}(0,y) - h_{\delta\mathbb{D}}(z,y)| < \epsilon$. Therefore, we conclude from (3.8) that for any fixed e^{-s},

$$\left|\mathbb{P}_0(B[\xi_u,\xi_s]\subset D) - \mathbb{P}_z(B[\xi_u,\xi_s]\subset D)\right| < e^{-s}.$$

Finally, using that $\mathbb{P}_0\left(B[\xi_u,\xi_s]\subset D\right)\le e^{-s}$ we deduce that for $|z|<\delta_\epsilon$

$$\mathbb{P}_z(B[\xi_u,\xi_s]\subset D)\le 2e^{-s},$$

and hence as remarked at the beginning

$$\mathbb{P}_z(\mathrm{diam}(B[0,\tau_D^+])\ge 2e^{-s}) \le \mathbb{P}_z(B[\xi_u,\xi_s]\subset D)\le 2e^{-s},$$

for all $s>0$ which concludes the proof. □

We can now start with the proof of Theorem 2, since we have done all the necessary preparatory work.

Proof. (Theorem 2) Let us consider $h(z)$ as in (3.1). In a first step, we check that h is bounded. Since $u : \partial D \to \mathbb{R}$ is bounded, there exists $K \in \mathbb{R}$ such that for all $y \in \partial D$, $|u(y)| \leq K$. Let now $z \in D$ be an arbitrary element, then,

$$|h(z)| = |\mathbb{E}_z \left(u \left(B_{\tau_D}\right)\right)| \leq \mathbb{E}_z \left(|u \left(B_{\tau_D}\right)|\right) \leq K,$$

where in last inequality we used that $B_{\tau_D} \in \partial D$. So, we have shown that h is bounded. Now we need to prove that h extends u, this is, that u and h coincide on the boundary of D. This holds since for $y \in \partial D$,

$$h(y) = \mathbb{E}_y \left(u \left(B_{\tau_D}\right)\right) = \mathbb{E}_y \left(u(y)\right) = u(y).$$

Therefore, it remains to check that h is harmonic, continuous and unique.

For the harmonic part let us first recall that τ_D is finite almost surely, since by Corollary 3, B exits the domain D in finite time. Let $z \in D$ and define $B(z,r) := \{w \in D : |z - w| < r\}$ where $r \in (0, \operatorname{dist}(z, \partial D))$. Clearly by continuity of B, $\tau_D > \tau_{B(z,r)}$, therefore $\tau_D = \tau_D \circ \theta_{\tau_{B(z,r)}} + \tau_{B(z,r)}$, where θ_t is the shift operator. More precisely, let $\omega = (\omega_s)_{s \geq 0} \in \Omega$ be a continuous function in the sample space, then $\theta_t : \Omega \to \Omega$ acts as $\theta_t(\omega) = (\omega_{t+s})_{s \geq 0}$. Hence,

$$\begin{aligned}
h(z) &= \mathbb{E}_z \left(u \left(B_{\tau_D}\right)\right) \\
&= \mathbb{E}_z \left(u \left(B_{\tau_D \circ \theta_{\tau_{B(z,r)}} + \tau_{B(z,r)}}\right)\right) \\
&= \mathbb{E}_z \left(u \left(B_{\tau_D} \circ \theta_{\tau_{B(z,r)}}\right)\right) \\
&= \mathbb{E}_z \left(\mathbb{E}_z \left(u \left(B_{\tau_D} \circ \theta_{\tau_{B(z,r)}}\right)\right) | \mathcal{F}_{\tau_{B(z,r)}}\right) \\
&\overset{*}{=} \mathbb{E}_z \left(\mathbb{E}_{B_{\tau_{B(z,r)}}} \left(u(B_{\tau_D})\right)\right) \\
&= \mathbb{E}_z \left(h \left(B_{\tau_{B(z,r)}}\right)\right) \\
&= \int_{\partial B(z,r)} h(y) h_{B(z,r)}(z,y) \ |dy| \\
&= \frac{1}{2\pi r} \int_{\partial B(z,r)} h(y) \ |dy|,
\end{aligned}$$

where in $*$ we used the strong Markov property (Proposition 31). Notice that in the second last equation we have used the formula (3.7) together with the fact that the harmonic measure can be written as a harmonic density with respect to the arc length measure. In the last equation we do not only consider the domain $r\mathbb{D}$ but $r\mathbb{D} + z$, so we get the generalised formula $h_{B(z,r)}(v,y) = \frac{1}{2\pi r} \frac{r^2 - |v-z|^2}{|y-v+z|^2}$ for all $v \in B(z,r)$ and $y \in \partial B(z,r)$. In our case

$$h_{B(z,r)}(z,y) = \frac{1}{2\pi r} \frac{r^2}{|y|^2} = \frac{1}{2\pi r} \frac{r^2}{r^2} = \frac{1}{2\pi r}.$$

So, we have shown that h satisfies the Mean Value Property and thus by Proposition 9 we conclude that h is harmonic.

Since h is harmonic on D it is in particular continuous on D and it remains to show that it is as well continuous in $\overline{D}$. To have an intuitive graphical illustration of the following proof, consider Figure 3. Therefore, let $z \in \partial D$ and pick a sequence $(z_n)_{n\in\mathbb{N}} \subset D$ such that $\lim_{n\to\infty} z_n = z$. Since $D \in \mathcal{D}$ we know by definition, that z is regular. Fix $\epsilon > 0$ and choose $\delta > 0$ as in Lemma 13. We check that $|h(z_n) - h(z)| \leq \epsilon$ for n large enough. Since $z_n \to z$ there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $|z_n - z| < \delta$. Using Lemma 13 we deduce that

$$
\begin{aligned}
\left|h(z_n) - h(z)\right| &= \left|\mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\right) - \mathbb{E}_z\left(u\left(B_{\tau_D}\right)\right)\right| \\
&= \left|\mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) + \mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|\geq\epsilon}\right) - u(z)\right| \\
&\leq \left|\mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) - u(z)\right| + \left|\mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|\geq\epsilon}\right)\right| \\
&\leq \left|\mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) - u(z)\right| + |K| \cdot \underbrace{\mathbb{P}_{z_n}\left(|B_{\tau_D} - z| \geq \epsilon\right)}_{\leq\epsilon} \\
&\leq \left|\mathbb{E}_{z_n}\left(u\left(B_{\tau_D}\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) - \mathbb{E}_{z_n}\left(u\left(z\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) - \mathbb{E}_{z_n}\left(u\left(z\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|\geq\epsilon}\right)\right| \\
&\quad + |K| \cdot \epsilon \\
&\overset{*}{\leq} \left|\mathbb{E}_{z_n}\left(\left(u\left(B_{\tau_D}\right) - u(z)\right)\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right)\right| + 2|K| \cdot \epsilon \\
&\leq \mathbb{E}_{z_n}\left(\left|u\left(B_{\tau_D}\right) - u(z)\right|\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) + 2|K| \cdot \epsilon.
\end{aligned}
$$

Where in $*$ we used again Lemma 13 and the fact that for $n > N_0$, $|z_n - z| < \delta$ in order to bound $\left|\mathbb{E}_{z_n}\left(u(z)\mathbb{1}_{\left|B_{\tau_D}-z\right|\geq\epsilon}\right)\right| \leq |K| \cdot \epsilon$. After defining

$$\Gamma_\epsilon := \partial D \cap B(z,\epsilon) = \partial D \cap \{y \in \mathbb{C} : |y - z| < \epsilon\},$$

we can find another bound for the above inequalities:

$$\mathbb{E}_{z_n}\left(\left|u\left(B_{\tau_D}\right) - u(z)\right|\mathbb{1}_{\left|B_{\tau_D}-z\right|<\epsilon}\right) + 2|K| \cdot \epsilon \leq \sup_{y\in\Gamma_\epsilon} |u(y) - u(z)| + 2|K| \cdot \epsilon.$$

In conclusion, since ϵ can be taken arbitrary small and u is continuous on ∂D, $\sup_{y\in\Gamma_\epsilon} |u(y) - u(z)|$ can be made arbitrary small and hence we have shown that h is continuous on $\overline{\mathbb{D}}$.

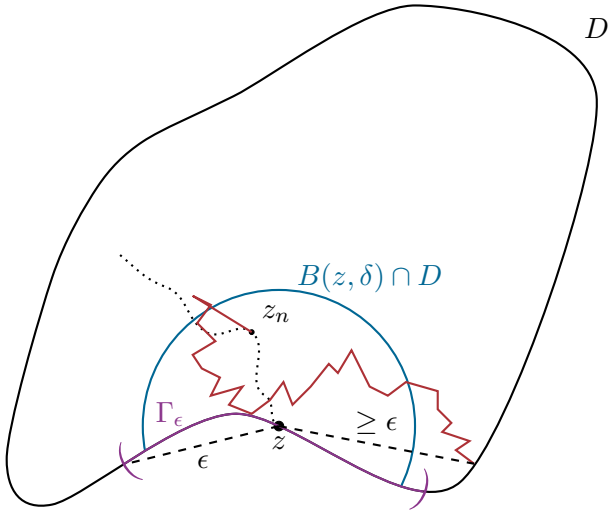

Figure 3: An illustration of a situation described in the proof of Theorem 2 showing $(z_n)_{n\in\mathbb{N}}$ and a Brownian motion.

To finish the proof of Theorem 2 we show that h is indeed unique. Assume there exists h' satisfying the same properties as h. Then, let us define for all $t \geq 0$, $M_t := h\left(B_{t\wedge\tau_D}\right)$ and $M'_t := h'\left(B_{t\wedge\tau_D}\right)$ which are both bounded martingales. Indeed, by the optional stopping theorem, the stopped process $\left(B_{t\wedge\tau_D}\right)_{t\geq 0}$ is a martingale and since h, h' are both harmonic we conclude with Itô's formula that $\left(h\left(B_{t\wedge\tau_D}\right)\right)_{t\geq 0}$ and $\left(h'\left(B_{t\wedge\tau_D}\right)\right)_{t\geq 0}$ are local martingales. Using the fact that any bounded local martingale is a true martingale we have shown that M and M' are both true martingales. By the optional sampling theorem

$$\begin{aligned}\mathbb{E}_z\left(M_{\tau_D}\right) =& \mathbb{E}_z\left(h\left(B_{\tau_D}\right)\right) = h(z),\\ \mathbb{E}_z\left(M'_{\tau_D}\right) =& \mathbb{E}_z\left(h'\left(B_{\tau_D}\right)\right) = h'(z).\end{aligned}$$

Since $B_{\tau_D} \in \partial D$, and on ∂D, h and h' coincide with u, we deduce that for all $z \in D$

$$h(z) = \mathbb{E}_z\left(h\left(B_{\tau_D}\right)\right) = \mathbb{E}_z\left(h'\left(B_{\tau_D}\right)\right) = h'(z),$$

which ends the proof. □

3.2 Some Applications of Theorem 2

In the following section, we look at important applications of Theorem 2. We prove the Schwarz reflection principle, maximum modulus principle and the Schwarz lemma, which are all important results from complex analysis.

Proposition 20 (Schwarz reflection principle)**.** [Law, Proposition 1.2.7] *Suppose h is a harmonic function on $\mathbb{H}\cap\mathbb{D}$ with boundary value 0 on the open interval $(-1,1)$. If h is extended to $\mathbb{D}$ by $h(\overline{z}) = -h(z)$, then h is harmonic on $\mathbb{D}$.*

Proof. We want to show that h satisfies the Mean Value Property on $\mathbb{D}$. In particular, we need to show that for any $z \in \mathbb{D}$ and $r < \operatorname{dist}(z, \partial\mathbb{D})$

$$\begin{aligned} h(z) &= \int_{\partial\mathcal{B}} h(y)\ \mathcal{U}_{z,r}(dy) \\ &= \int_{\partial\mathcal{B}} h(y)\ \mu_{\mathcal{B}}(z, dy) \\ &= \mathbb{E}_z\left(h\left(B_{\tau_{\mathcal{B}}}\right)\right), \end{aligned} \tag{3.9}$$

where $\mathcal{B} := \{v \in \mathbb{C} : |z - v| < r\}$. In the following argument we denote by $\overline{\mathbb{H}} := \{z \in \mathbb{C} : \overline{z} \in \mathbb{H}\}$. If we first assume $z \in \mathbb{R}$, then by assumption $h(z) = -h(z)$, which implies that $h(z) = 0$ so in particular

$$\int_{(\partial\mathcal{B})\cap\mathbb{R}} h(y)\ \mu_{\mathcal{B}}(z, dy) = 0. \tag{3.10}$$

On the other hand using equation (3.10) we get

$$\begin{aligned} \mathbb{E}_z\left(h\left(B_{\tau_{\mathcal{B}}}\right)\right) &= \int_{\partial\mathcal{B}} h(y)\ \mu_{\mathcal{B}}(z, dy) \\ &= \int_{(\partial\mathcal{B})\cap\mathbb{H}} h(y)\ \mu_{\mathcal{B}}(z, dy) + \int_{(\partial\mathcal{B})\cap\overline{\mathbb{H}}} h(y)\ \mu_{\mathcal{B}}(z, dy) \\ &= \int_{(\partial\mathcal{B})\cap\mathbb{H}} h(y)\ \mu_{\mathcal{B}}(z, dy) + \int_{(\partial\mathcal{B})\cap\mathbb{H}} h(\overline{y})\ \mu_{\mathcal{B}}(z, dy) \\ &= 0, \end{aligned}$$

using that $h(\overline{y}) = -h(y)$. Thus, we have shown the Mean Value Property in (3.9) if $z \in \mathbb{R}$.

Let us now assume $z \in \mathbb{D} \setminus \mathbb{R}$, since the cases $\Im(z) > 0$ and $\Im(z) < 0$ are identical, we will assume, without loss of generality, that $\Im(z) > 0$. If $\Im(z) > \operatorname{dist}(z, \partial\mathbb{D})$ we can pick $r < \operatorname{dist}(z, \partial\mathbb{D})$. In this case by construction $\mathcal{B} \subset \mathbb{H} \cap \mathbb{D}$ and since h is harmonic on $\mathbb{H} \cap \mathbb{D}$, (3.9) follows immediately by the Mean Value Property. Therefore, assume $\Im(z) < \operatorname{dist}(z, \partial\mathbb{D})$. Pick r such that $\Im(z) < r < \operatorname{dist}(z, \partial\mathbb{D})$. Let $V_- := (\partial\mathcal{B}) \cap \overline{\mathbb{H}}$, $V_+ := \{v : \overline{v} \in V_-\}$, $V := V_- \cup V_+$ and $\partial^* := \partial\left(\mathcal{B} \cap \mathbb{H}\right)$. Define a sequence of stopping times for $j \geq 1$ by

$$\begin{aligned} &\rho_0 = 0, \\ &\sigma_j := \tau_{\mathcal{B}} \wedge \inf\{t \geq \rho_{j-1} : B_t \in \partial^*\}, \\ &\rho_j := \tau_{\mathcal{B}} \wedge \inf\{t \geq \sigma_j : B_t \in V\}. \end{aligned}$$

A possible graphical illustration of this situation is given in Figure 4.

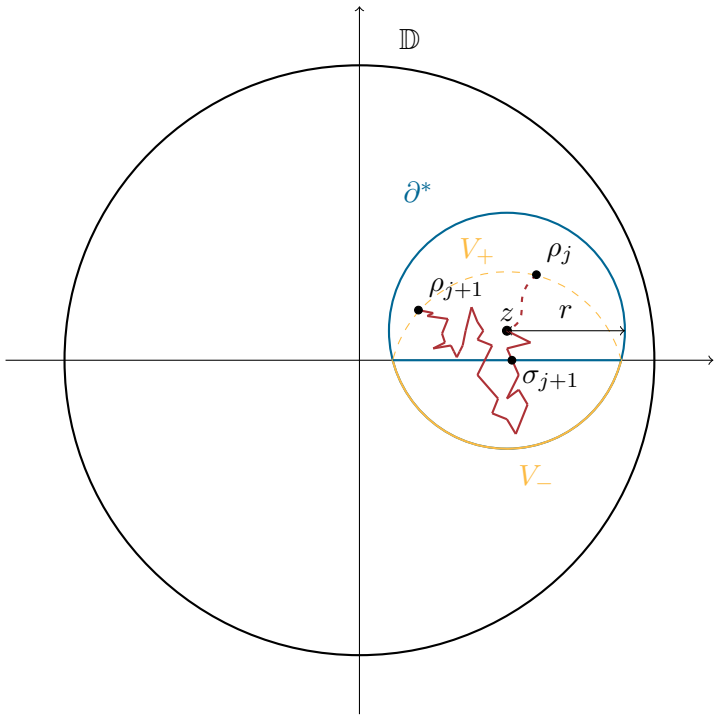

Figure 4: Graphical representation of a trajectory of a complex-valued Brownian motion in connection with the stopping times σ_i and ρ_i for $i \geq 1$ which were defined in this proof.

We now want to check the following relations between the stopping times, namely for any j

$$\mathbb{E}_z\left(h\left(B_{\sigma_{j+1}}\right)\right) = \mathbb{E}_z\left(h\left(B_{\rho_j}\right)\right), \tag{3.11}$$

$$\mathbb{E}_z\left(h\left(B_{\rho_j}\right)\right) = \mathbb{E}_z\left(h\left(B_{\sigma_j}\right)\right). \tag{3.12}$$

Indeed, let us show the first equality by using that h is harmonic on $\mathbb{D} \cap \mathbb{H}$:

$$\begin{aligned}
\mathbb{E}_z\left(h\left(B_{\sigma_{j+1}}\right)\right) &= \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\sigma_{j+1}}\right) + \mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} h\left(B_{\sigma_{j+1}}\right)\right) \\
&= \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\tau_{\mathcal{B}}}\right) + \mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} h\left(B_{\sigma_{j+1}}\right)\right) \\
&= \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\tau_{\mathcal{B}}}\right) + \mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} h\left(B_{\sigma_1} \circ \theta_{\rho_j}\right)\right) \\
&= \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\tau_{\mathcal{B}}}\right)\right) + \mathbb{E}_z\left(\mathbb{E}_z\left(\mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} h\left(B_{\sigma_1} \circ \theta_{\rho_j}\right) \middle| \mathcal{F}_{\rho_j}\right)\right) \\
&\overset{*}{=} \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\tau_{\mathcal{B}}}\right)\right) + \mathbb{E}_z\left(\mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} \mathbb{E}_z\left(h\left(B_{\sigma_1} \circ \theta_{\rho_j}\right) \middle| \mathcal{F}_{\rho_j}\right)\right) \\
&= \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\tau_{\mathcal{B}}}\right)\right) + \mathbb{E}_z\left(\mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} \mathbb{E}_{B_{\rho_j}}\left(h\left(B_{\sigma_1}\right)\right)\right) \\
&= \mathbb{E}_z\left(\mathbb{1}_{\rho_j \geq \tau_{\mathcal{B}}} h\left(B_{\rho_j}\right)\right) + \mathbb{E}_z\left(\mathbb{1}_{\rho_j < \tau_{\mathcal{B}}} h\left(B_{\rho_j}\right)\right) \\
&= \mathbb{E}_z\left(h\left(B_{\rho_j}\right)\right).
\end{aligned}$$

At this point notice that in the second line we used the fact that if $\rho_j \geq \tau_{\mathcal{B}}$, then by definition $\rho_j = \tau_{\mathcal{B}}$ and hence $\sigma_{j+1} = \tau_{\mathcal{B}}$ for all j. In $*$ we have used the well known fact that if S, T are two arbitrary stopping times, then $\{S < T\} \in \mathcal{F}_T \cap \mathcal{F}_S \subset \mathcal{F}_T$, so in particular $\mathbb{1}_{\{S<T\}}$ is $\mathcal{F}_T$-measurable. One line below we used the Strong Markov property (Proposition 31) and in the second last equality we could apply Kakutani's formula (Proposition 11) since h is harmonic

on $\mathbb{D} \cap \mathbb{H}$. This shows (3.11), then (3.12) follows by symmetry. Hence for all j,

$$\mathbb{E}_z \left(h \left(B_{\sigma_{j+1}} \right) \right) = \mathbb{E}_z \left(h \left(B_{\rho_j} \right) \right) = \mathbb{E}_z \left(h \left(B_{\sigma_j} \right) \right),$$

therefore,

$$\mathbb{E}_z \left(h \left(B_{\tau_{\mathcal{B}}} \right) \right) = \lim_{j \to \infty} \mathbb{E}_z \left(h \left(B_{\sigma_j} \right) \right) = \mathbb{E}_z (h(B_0)) = h(z).$$

□

As a next corollary, we study the maximum modulus principle, where Kakutani's formula immediately implies that a harmonic function u on a bounded domain D, which extends continuously to the closure $\overline{D}$, cannot exceed the supremum of its values on the boundary ∂D. Additionally, as we will show, a harmonic function cannot achieve its maximum value within the domain unless it is constant.

Proposition 21 (Maximum principle for harmonic functions)**.** [BN16, Theorem 2.5] *Let u be a harmonic function defined on a domain D. Suppose there exists a point $z_0 \in D$ such that $u(z) \leq u(z_0)$ for all $z \in D$, then u is constant.*

Proof. It is enough to consider the case $\sup_{z \in D} u(z) = m < \infty$ for some constant $m \in \mathbb{R}$ because if $\sup_{z \in D} u(z) = \infty$ then u cannot be constant. The idea of the proof is to show that $D_0 := \{z \in D : u(z) = m\} = D$ using the fact that D is connected.

Since a harmonic function on a domain is continuous, we deduce that $D_0 = u^{-1}(\{m\})$ is closed and nonempty in D. On the other hand, since D is open, for every $z \in D_0$ we can find $\epsilon > 0$ sufficiently small such that $B(z, \epsilon) \subset D$. Using that u is harmonic, by Kakutani's formula for every $z \in D_0$,

$$m = u(z) = \mathbb{E}_z \left(u \left(B_{\tau_{B(z,\epsilon)}} \right) \right) = \int_{\partial B(z,\epsilon)} u(y) \; \mu_{B(z,\epsilon)}(z, dy) \leq m.$$

As equality holds only if $u(\cdot) \equiv m$ on $\partial B(z, \epsilon)$ we see that $u(y) = m$ for all $y \in \partial B(z, \epsilon)$. Now we want to show that it also holds on the entire ball $B(z, \epsilon)$. This follows immediately from the fact that u is continuous, hence we can shrink $B(z, \epsilon)$ to any ball $B(z, \epsilon')$ for some $\epsilon' < \epsilon$. Hence, $B(z, \epsilon) \subset D_0$ and thus D_0 is open since for all $z \in D_0$ there exists a ball containing z and which is contained in D_0. We therefore conclude that D_0 is open and closed in D. Given that D is connected and in a connected space the only subsets that are both open and closed are the empty set and the entire space itself we conclude that D_0 is the whole of D and thus u is constant on D. □

In order to prove the Schwarz Lemma, we extend the maximum principle to holomorphic functions.

Proposition 22 (Maximum modulus principle)**.** [BN16, Theorem 2.6] *Let f be a holomorphic function defined on a domain D. Suppose there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Then f is constant.*

Proof. Since f is holomorphic, we can split it into its real and imaginary parts, say $f = u + iv$. Using that every holomorphic function is analytic, we immediately deduce by Proposition 4, that f satisfies the Cauchy-Riemann equations, specifically $u_x = v_y$ and $u_y = -v_x$.

By Lemma 7, u and v are harmonic and in particular C^2, so we can derive these equations with respect to x and y, respectively. Thus, we obtain $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Applying the H.A. Schwarz lemma (Proposition 32) to v_{yx} we conclude that

$$\Delta u(z) = u_{xx}(z) + u_{yy}(z) = 0, \ \forall \ z \in D.$$

Similarly, differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x, we find

$$\Delta v(z) = v_{xx}(z) + v_{yy}(z) = 0, \ \forall \ z \in D.$$

By definition, this means that u, v are harmonic on D. Consider the set $D_0 = \{z \in D : |f(z)| = |f(z_0)|\}$, we aim to show that $D_0 = D$, which would imply that $g(z) := |f(z)|$ is constant. The idea behind this is again that we show that D is open and closed, and conclude using connectedness of D.

Since $g(z)$ is a continuous function $D_0 = g^{-1}(\{|f(z_0)|\})$ is closed in D. On the other hand, suppose $z \in D_0$. Since D is open, there exists $\epsilon > 0$ small enough such that $B(z, \epsilon) \subset D$. Since u, v are harmonic, using Kakutani's formula we get that $u(z) = \mathbb{E}_z(u(B_{\tau_{B(z,\epsilon)}}))$ and $v(z) = \mathbb{E}_z(v(B_{\tau_{B(z,\epsilon)}}))$. Hence, by linearity of the expectation

$$f(z) = \mathbb{E}_z\left(u(B_{\tau_{B(z,\epsilon)}}) + iv(B_{\tau_{B(z,\epsilon)}})\right) = \mathbb{E}_z(f(B_{\tau_{B(z,\epsilon)}})).$$

Therefore,

$$\begin{aligned} |f(z_0)| &= |f(z)| \\ &= \left|\mathbb{E}_z(f(B_{\tau_{B(z,\epsilon)}}))\right| \\ &\leq \int_{\partial B(z,\epsilon)} |f(y)| \ \mu_{B(z,\epsilon)}(z, dy) \\ &\leq |f(z_0)|. \end{aligned}$$

This implies that $|f(y)| = |f(z_0)|$ for all $y \in \partial B(z, \epsilon)$. By the same argument as in the last proposition D_0 is also open. Given that D is connected and $D_0 \neq \emptyset$ we deduce that $D_0 = D$, thus $|f|$ is constant on D. Therefore, we can assume that $|f|^2 = c^2$ for some $c \in \mathbb{R}$. It remains to show that f itself is constant. Since $f = u + iv$, we know in particular that $|f|^2 = u^2 + v^2 = c^2$. By computing partial derivatives with respect to x and y we get

$$\begin{aligned} 2uu_x + 2vv_x &= 0, \\ 2uu_y + 2vv_y &= 0. \end{aligned} \tag{3.13}$$

Transforming these relations and applying the Cauchy-Riemann equations, we can rewrite (3.13) as

$$uu_x = -vv_x = vu_y,$$
$$uu_y = -vv_y = -vu_x.$$

Assume by contradiction that u and v are both non-zero. If u, v are both constant, we are done. If u is constant and v not then $u_x = u_y = 0$. Using the Cauchy-Riemann equations we deduce that $v_y = v_x = 0$ which implies that v also must be constant. Similarly, we can show that if v is constant, then u must also be constant. So, we only need to assume that u and v are both non-constant. Then, we can reorder the above equations to

$$\frac{u}{v} = \frac{u_y}{u_x}, \ -\frac{v}{u} = \frac{u_y}{u_x}.$$

By setting these two equations equal to each other and multiplying by u and v respectively, we get $u^2 = -v^2$ or equivalently $u^2 + v^2 = 0$. This only holds if and only if both u and v are constantly equal to zero, which contradicts the assumption. Hence, in order to satisfy (3.13) either u or v or both must be zero. If u and v are zero, then clearly f is constant. If $u = 0$, then $f(z) = iv(z)$ and $|f(z)| = |v(z)| = c$ for all $z \in D$. If $v(z)$ is constant for all $z \in D$ we are done. Else, since v is a real valued function, we can find $z' \in D$ such that $v(z') = \max\{c, -c\}$. Hence, $v(z) \leq v(z')$ for all $z \in D$ and by the Maximum principle for harmonic functions (Proposition 21) we deduce that v is constant. On the other hand if only $v = 0$, we can deduce similarly that u is constant and hence f is. Finally, we can conclude from these three cases that f must be constant, which ends the proof. □

The following corollary can be derived from the maximum modulus principle and will be used to prove the Schwarz lemma.

Corollary 7. *Let D be a bounded domain and denote by $\overline{D}$ its closure. Suppose that $f : \overline{D} \to \mathbb{C}$ is a continuous function that is holomorphic on D. Then, $|f(z)|$ attains a maximum at some point of the boundary of D.*

Proof. Since $\overline{D}$ is compact and non-empty, the continuous function $z \mapsto |f(z)|$ attains a maximum at some point $z_0 \in \overline{D}$. If $z_0 \in \partial D$ we are done. Else, if $z_0 \in \overline{D} \setminus \partial D$, then Proposition 22 implies that f is constant on D. Using that f is continuous on $\overline{D}$ and constant on D, we must have that f is constant on $\overline{D}$, hence $|f(z)| = |f(z_0)|$ for all $z \in \overline{D}$, so in particular $|f(z)|$ attains a maximum on the boundary. □

Now that we have everything together, we can state and prove the Schwarz lemma.

Proposition 23 (Schwarz lemma)**.** [BN16, Lemma 1.2] *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then, $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$ and some $\theta \in [0, 2\pi)$.*

Proof. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Consider the map $g(z) := \frac{f(z)}{z}$ for $z \in \mathbb{D} \setminus \{0\}$. From complex analysis, we know that every holomorphic function is analytic and vice versa. Assume that 0 is a zero of order $m \in \mathbb{N}$ for f. Since f is analytic, it can be written as $f(z) = z^m f_1(z)$ where $f_1(z)$ is analytic and $f_1(0) \neq 0$, hence

$$g(z) = \frac{z^m f_1(z)}{z} = z^{m-1} f_1(z),$$

which is again analytic and thus holomorphic on $\mathbb{D}$. For any $z \in \mathbb{D}$ and $r > 0$ such that $|z| < r < 1$,

$$|g(z)| \leq \sup_{|y|=r} |g(y)| = \sup_{|y|=r} \frac{|f(y)|}{|y|} = \sup_{|y|=r} \frac{|f(y)|}{r} \leq \frac{1}{r},$$

where we used Corollary 7 in the first inequality. Taking $r \to 1$ leads to

$$|g(z)| \leq \lim_{r \to 1} \frac{1}{r} = 1,$$

hence $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, which proves the first part.

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = \frac{|f(z_0)|}{|z_0|} = 1$, so we can find $\theta \in [0, 2\pi)$ such that $g(z_0) = e^{i\theta}$. As we already know, $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, and since there exists $z_0 \in \mathbb{D}$ with $|g(z_0)| = 1$ we conclude that $|g(z)| \leq |g(z_0)|$ for all $z \in \mathbb{D}$. Thus by Proposition 22, g is constant on $\mathbb{D}$, in particular $g(z) = g(z_0) = e^{i\theta}$ for all $z \in \mathbb{D}$, so $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$ which concludes the proof.

□

Chapter 4

Asymptotic Laws of Complex Brownian Motion

This chapter shows another application of the conformal invariance property for complex-valued Brownian motion, more precisely we prove the so-called skew-product representation. Let us start by introducing some notation. We know that for every $y \in \mathbb{C} \setminus \{0\}$ there exists an $r > 0$ and $\theta \in (0, 2\pi]$ such that $y = re^{i\theta}$, where θ is called the argument of y. Of course, we can also do this for a complex Brownian motion $B = B^{(1)} + iB^{(2)}$ starting form $z \in \mathbb{C} \setminus \{0\}$. For this let us first introduce the notion of a continuous determination of the argument of a complex Brownian motion B.

Definition 22. *Let $(B_t)_{t\geq 0}$ be a complex-valued Brownian motion starting from $z \in \mathbb{C}$. Then, for $t \geq 0$ we call $\theta_t := \arg(B_t)$ a continuous determination of the argument of B, if $t \mapsto \theta_t$ is a continuous real-valued function such that for any $t \geq 0$,*

$$B_t = |B_t| \exp(i\theta_t).$$

Roughly speaking, θ_t keeps track of the number of windings of B around zero. In the remaining part of this thesis we denote by $(\arg(B_t))_{t\geq 0}$ a continuous determination of the argument of B.

Now consider two real-valued stochastic processes $(R_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$ such that for all $t \geq 0$

$$B_t = R_t \exp(i\theta_t). \tag{4.1}$$

More precisely, for all $t \geq 0$

$$\begin{aligned} R_t &:= \sqrt{\left(B_t^{(1)}\right)^2 + \left(B_t^{(2)}\right)^2}, \\ \theta_t &:= \arg(B_t), \end{aligned}$$

where as above $\arg(\cdot)$ denotes a continuous determination of the argument. At this point it is worth noting, that $(R_t)_{t\geq 0}, (\theta_t)_{t\geq 0}$ do not need to be real valued Brownian motions. In particular, $R_t \geq 0$ for all $t \geq 0$, so it cannot be a real-valued Brownian motion.

The skew-product representation exactly states that we can find two independent linear real-valued Brownian motions that fulfil the above representation (4.1), meaning $(R_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$ are time change of independent Brownian motions. We later apply the skew-product representation to prove asymptotic results for complex Brownian motion such as the celebrated Spitzer theorem on the winding number of planar Brownian motion.

Theorem 3 (Skew-product representation)**.** [LG16, Theorem 7.19] *Let $z \in \mathbb{C}\setminus\{0\}$ and write $z = \exp(r+i\theta)$ where $r \in \mathbb{R}$ and $\theta \in (-\pi,\pi]$. There exist two independent real-valued linear Brownian motions β and γ that start from r respectively from θ under $\mathbb{P}_z$, such that we have $\mathbb{P}_z$-a.s. for every $t \geq 0$,*

$$B_t = \exp(\beta_{\sigma(t)} + i\gamma_{\sigma(t)}), \tag{4.2}$$

where

$$\sigma(t) = \int_0^t \frac{1}{|B_s|^2}\, ds.$$

Proof. First, assume that $z = 1$, so in particular $r = \theta = 0$. A short argument on how to deduce the general statement from this is given at the end of the proof. For $t \geq 0$ let $\Gamma_t = \Gamma_t^{(1)} + i\Gamma_t^{(2)}$ be a complex Brownian motion started from 0 and consider $f(y) = \exp(y)$ for all $y \in \mathbb{C}$. Then, f is holomorphic and in particular entire by Definition 4. Using the conformal invariance theorem part (ii) there exists a complex Brownian motion Z under $\mathbb{P}_{f(0)} = \mathbb{P}_1$ such that $\mathbb{P}_1$-a.s. for all $t \geq 0$,

$$Z_t = f(\Gamma_{\sigma_Z(t)}) = \exp\left(\Gamma^{(1)}_{\sigma_Z(t)} + i\Gamma^{(2)}_{\sigma_Z(t)}\right),$$

where $\sigma_Z(t)$ is the inverse of $t \mapsto \int_0^t |f'(\Gamma_s)|^2\, ds$. Let us prove the result for the complex-valued Brownian motion Z first, we later check it for B. We provide an explicit representation of $(\sigma_Z(t))_{t\geq 0}$, for this notice that

$$\begin{aligned}
\sigma_Z^{-1}(t) &= \int_0^t |f'(\Gamma_s)|^2\, ds \\
&= \int_0^t \left|\exp\left(2\Gamma_s^{(1)} + 2i\Gamma_s^{(2)}\right)\right|\, ds \\
&= \int_0^t \exp\left(2\Gamma_s^{(1)}\right)\, ds,
\end{aligned}$$

for all $t \geq 0$. Thus, we have found a representation for σ_Z^{-1}. To calculate σ_Z we use the formula for the derivative of an inverse function, namely for all $t \geq 0$,

$$\sigma_Z'(t) = \frac{1}{\left(\sigma_Z^{-1}\right)'(\sigma_Z(t))} = \frac{1}{\exp\left(2\Gamma^{(1)}_{\sigma_Z(t)}\right)} = \frac{1}{\exp\left(\Gamma^{(1)}_{\sigma_Z(t)}\right)^2}.$$

Integrating both sides and using the fact that $\sigma_Z(0) = 0$ gives us

$$\sigma_Z(t) = \int_0^t \frac{1}{\exp\left(\Gamma^{(1)}_{\sigma_Z(s)}\right)^2}\, ds = \int_0^t \frac{1}{|Z_s|^2}\, ds,$$

where we used that $\exp\left(\Gamma^{(1)}_{\sigma_Z(s)}\right) = |f\left(\Gamma_{\sigma_Z(s)}\right)| = |Z_s|$. If we now define for all $t \geq 0$, $\beta'_t := \Gamma^{(1)}_t$ and $\gamma'_t := \Gamma^{(2)}_t$ then by definition of $\Gamma^{(i)}$ for $i = 1, 2$ we deduce that γ' and β' are independent real-valued linear Brownian motions starting from zero, thus we have proven the desired result for Z.

In order to prove the result for B let us first recall that for all $t \geq 0$, we defined $\arg(B_t)$ to be a continuous determination of the argument of B_t, where we here assume that $\arg(B_0) > 0$ which can be done since $B_0 \neq 0$. We can rewrite $\sigma(t)$ as the inverse of its inverse, i.e.

$$\sigma(t) = \inf\left\{s \geq 0 : \sigma^{-1}(s) > t\right\} = \inf\left\{s \geq 0 : \int_0^s \exp(2\beta_u)\ du > t\right\},$$

and use this to reformulate Theorem 3. The statement with $z = 1$ is equivalent to saying that if we set

$$\beta_t := \log\left(\left|B_{\inf\{s\geq 0:\int_0^s |B_u|^{-2}\ du>t\}}\right|\right),$$
$$\gamma_t := \arg\left(B_{\inf\{s\geq 0:\int_0^s |B_u|^{-2}\ du>t\}}\right),$$

for all $t \geq 0$, then β and γ are two independent real-valued Brownian motions started from zero and we have

$$\beta_{\inf\left\{s\geq 0:\int_0^s \exp(2\beta_u)\ du>t\right\}} = \log(|B_t|),$$

respectively,

$$\gamma_{\inf\left\{s\geq 0:\int_0^s \exp(2\beta_u)\ du>t\right\}} = \arg(B_t).$$

Indeed, if this holds we get that $B_t = \exp\left(\beta_{\sigma(t)} + i\gamma_{\sigma(t)}\right)$ for all $t \geq 0$, which proves the result. It therefore remains to check that β, γ are indeed two real-valued independent linear Brownian motions.

Since β and γ are deterministic functions of B and $B \overset{d}{=} Z$ we can replace B by Z without changing the law of β and γ. So we get

$$\beta_t \overset{d}{=} \log\left(\left|Z_{\inf\{s\geq 0:\int_0^s |Z_u|^{-2}\ du>t\}}\right|\right) = \beta'_t,$$
$$\gamma_t \overset{d}{=} \arg\left(Z_{\inf\{s\geq 0:\int_0^s |Z_u|^{-2}\ du>t\}}\right) = \gamma'_t.$$

Note that we used the representation $\sigma_Z^{-1}(t) = \inf\{s \geq 0 : \int_0^s |Z_u|^{-2}\ du > t\}$ together with the fact that $\beta'_{\sigma_Z(t)} = \log(|Z_t|)$ and $\gamma'_{\sigma_Z(t)} = \arg(Z_t)$. We have already proven for Z, that there exist such two independent real-valued linear Brownian motions β' and γ' starting from zero under $\mathbb{P}_1$ and such that $\mathbb{P}_1$-a.s. for every $t \geq 0$, $Z_t = \exp(\beta'_{\sigma_Z(t)} + i\gamma'_{\sigma_Z(t)})$. Plugging this into the representation for β respectively γ from above we get for all $t \geq 0$,

$$\beta_{\sigma(t)} \overset{d}{=} \log(|Z_t|) = \log(\exp(\beta'_{\sigma_Z(t)})) = \beta'_{\sigma_Z(t)},$$
$$\gamma_{\sigma(t)} \overset{d}{=} \arg(Z_t) = \gamma'_{\sigma_Z(t)}.$$

All these observations imply that β, γ are real-valued independent linear Brownian motions

starting from zero under $\mathbb{P}_1$ and satisfying (4.2).

Let us now discuss the general case. First, let us assume $z = \exp(i\theta)$ for some $\theta \in (-\pi, \pi]$. Then, we define for $t \geq 0$ the process $B''_t := \frac{1}{e^{i\theta}} B_t$. By Lemma 6 we know that B'' is a complex Brownian motion and furthermore $B''_0 = 1$. By the above argument, we can find a skew-product representation for B''. Specifically, under $\mathbb{P}_z = \mathbb{P}_{e^{i\theta}}$, almost surely, for every $t \geq 0$, there exists a time change $(\sigma_{B''}(t))_{t\geq 0}$ such that

$$B''_t = \exp\left(\beta''_{\sigma_{B''}(t)} + i\gamma''_{\sigma_{B''}(t)}\right),$$

for all $t \geq 0$. It follows that for all $t \geq 0$,

$$\begin{aligned} B_t &= e^{i\theta} \exp\left(\beta''_{\sigma_{B''}(t)} + i\gamma''_{\sigma_{B''}(t)}\right) \\ &= \exp\left(\beta''_{\sigma_{B''}(t)} + i\left(\gamma''_{\sigma_{B''}(t)} + \theta\right)\right) \\ &= \exp\left(\beta_{\sigma_{B''}(t)} + i\gamma_{\sigma_{B''}(t)}\right), \end{aligned}$$

where we defined $\beta_t := \beta''_t$ and $\gamma_t := \gamma''_t + \theta$. We now observe that since β'' and γ'' are independent real-valued Brownian motions, both β and γ are also independent real-valued Brownian motions. Furthermore, under $\mathbb{P}_z = \mathbb{P}_{e^{i\theta}}$, $\beta_0 = 0$, $\gamma_0 = \theta$ and $\sigma_{B''}(t) = \int_0^t \frac{1}{|B''_s|^2}\, ds = \int_0^t \frac{1}{|B_s|^2}\, ds = \sigma(t)$ for all $t \geq 0$, which concludes this case.

For the last case, assume $z = \exp(r + i\theta)$ for some $r > 0$ and $\theta \in (-\pi, \pi]$. Then, define $\tilde{B}_t := \frac{1}{\exp(r)} B_{\exp(r)^2 t}$ for $t \geq 0$. Again by Lemma 6 we deduce that $\tilde{B}$ is a complex-valued Brownian motion with $\tilde{B}_0 = \frac{1}{\exp(r)} \exp(r + i\theta) = e^{i\theta}$. So, by the above case we know that there exists a skew-product representation for $\tilde{B}$. This means that under $\mathbb{P}_z = \mathbb{P}_{e^{r+i\theta}}$, almost surely, for every $t \geq 0$, there exists a time change

$$\sigma_{\tilde{B}}(t) = \int_0^t \frac{1}{|\tilde{B}_s|^2}\, ds = \exp(2r) \int_0^t \frac{1}{|B_{\exp(r)^2 s}|^2}\, ds = \int_0^{e^{2r}t} \frac{1}{|B_u|^2}\, du = \sigma\left(e^{2r}t\right),$$

such that

$$\tilde{B}_t = \exp\left(\tilde{\beta}_{\sigma_{\tilde{B}}(t)} + i\tilde{\gamma}_{\sigma_{\tilde{B}}(t)}\right).$$

Therefore, we get for all $t \geq 0$

$$\begin{aligned} B_{\exp(2r)t} &= \exp(r)\tilde{B}_t \\ &= \exp(r) \exp\left(\tilde{\beta}_{\sigma_{\tilde{B}}(t)} + i\tilde{\gamma}_{\sigma_{\tilde{B}}(t)}\right) \\ &= \exp\left(\left(\tilde{\beta}_{\sigma_{\tilde{B}}(t)} + r\right) + i\tilde{\gamma}_{\sigma_{\tilde{B}}(t)}\right) \\ &= \exp\left(\beta_{\sigma_{\tilde{B}}(t)} + i\gamma_{\sigma_{\tilde{B}}(t)}\right) \\ &= \exp\left(\beta_{\sigma(\exp(2r)t)} + i\gamma_{\sigma(\exp(2r)t)}\right), \end{aligned}$$

where $\beta_t := \tilde{\beta}_t + r$ and $\gamma_t := \tilde{\gamma}_t$ for all $t \geq 0$. Therefore, $B_t = \exp\left(\beta_{\sigma(t)} + i\gamma_{\sigma(t)}\right)$ for all $t \geq 0$. Since again $\tilde{\beta}$ and $\tilde{\gamma}$ are independent real-valued Brownian motions, also β and γ are, and in particular, under $\mathbb{P}_z = \mathbb{P}_{e^{r+i\theta}}$, $\beta_0 = r$ and $\gamma_0 = \theta$ which concludes the proof. □

4.1 Some Applications of Theorem 3

In this section we apply the skew-product representation to prove different asymptotic results for complex Brownian motion such as the well known Spitzer theorem. Let us again use the notation $\theta_t = \arg(B_t)$ for $t \geq 0$ to represent a continuous determination of the argument of a complex Brownian motion B starting from $z \in \mathbb{C}$. By the skew-product representation we already know that the process $(\theta_t)_{t\geq 0}$ can be written as a linear real-valued Brownian motion time changed, and thus we can derive a lot of information about its path properties. Before stating the first motivating property for the Spitzer theorem, let us state the following purely analytic fact.

Lemma 14. *If $f : [0,\infty) \to f([0,\infty))$ is a strictly increasing continuous function such that $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{y\to\infty} f^{-1}(y) = \infty$.*

With the help of this fact, we can easily prove the following lemma about the asymptotic behaviour of θ_t.

Lemma 15. [LG16, Page 201] *Let $(\sigma(t))_{t\geq 0}$ be as in Theorem 3 and denote by $\theta_t = \arg(B_t)$ for $t \geq 0$ a continuous determination of the argument of a complex Brownian motion B starting from $z \in \mathbb{C}$. Then*

$$\limsup_{t\to\infty} \theta_t = \infty,$$
$$\liminf_{t\to\infty} \theta_t = -\infty.$$

Proof. In the proof of Theorem 3 we have already found that for all $t \geq 0$,

$$\sigma^{-1}(t) = \int_0^t |f'(B_s)|^2 \, ds = \langle \Re(f(B_\cdot)) \rangle_t,$$

where $f(z) = \exp(z)$ and in the second equation we used Corollary 1. By the conformal invariance theorem $\sigma^{-1}(t)$ is strictly increasing and continuous, and in particular considering the proof of point (ii) of the conformal invariance theorem we have already shown that $\langle \Re(f(B)) \rangle_\infty = \infty$. Thus, we can now apply Lemma 14 to σ^{-1} which implies that $\lim_{t\to\infty} \sigma(t) = \infty$. From the skew-decomposition theorem we know an explicit representation of θ_t, namely $\theta_t = \gamma_{\sigma(t)}$ for all $t \geq 0$. Since γ_t is a real-valued Brownian motion we know from Proposition 25 that $\langle \gamma \rangle_t = t$ for all $t \geq 0$, hence in particular $\langle \gamma \rangle_\infty = \infty$. Using Proposition 30 we conclude that $\liminf_{t\to\infty} \gamma_t = -\infty$ and $\limsup_{t\to\infty} \gamma_t = \infty$ thus also

$$\liminf_{t\to\infty} \theta_t = \liminf_{t\to\infty} \gamma_{\sigma(t)} = -\infty,$$
$$\limsup_{t\to\infty} \theta_t = \limsup_{t\to\infty} \gamma_{\sigma(t)} = \infty,$$

where it is important that $\sigma(t) \to \infty$ as $t \to \infty$.

□

This motivates the question of how the growth rate of θ_t evolves as t increases. It is answered by the Spitzer theorem on the winding number of a complex Brownian motion.

Theorem 4 (Spitzer theorem)**.** [LG16, Theorem 7.20] *Let $(\theta_t)_{t\geq 0}$ be a continuous determination of the argument of the complex Brownian motion B started from $z \in \mathbb{C}\setminus\{0\}$. Then $\frac{2}{\log(t)}\theta_t$ converges in distribution to a standard symmetric Cauchy distribution as $t \to \infty$. In other words, for every $x \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}_z\left(\frac{2}{\log(t)}\theta_t \leq x\right) = \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)}\, dy.$$

Before proving Spitzer theorem, we need to establish the following result. In the next lemma we use the same notation as in Theorem 3 but with $z = 1$, i.e. B is assumed to be a complex Brownian motion started from 1 which implicitly implies that β, γ are two independent linear Brownian motions started from 0.

Lemma 16. [LG16, Lemma 7.21] *For every $\lambda > 0$ consider the scaled Brownian motion*

$$\beta_t^{(\lambda)} := \frac{1}{\lambda}\beta_{\lambda^2 t},\ \forall t \geq 0,$$

and set $T_1^{(\lambda)} := \inf\left\{t \geq 0 : \beta_t^{(\lambda)} = 1\right\}$. Then under $\mathbb{P}_1$,

$$\frac{4}{(\log(t))^2}\sigma(t) - T_1^{\left(\frac{\log(t)}{2}\right)} \to 0,$$

in probability as $t \to \infty$.

At this point, we briefly note that $\left(\beta_t^{(\lambda)}\right)_{t\geq 0}$ is actually a complex Brownian motion thanks to Lemma 6.

Proof. Since $B_0 = 1$, β is a complex Brownian motion under $\mathbb{P}_1$ starting from zero as in Theorem 3. For every $a > 0$ define $T_a := \inf\{t \geq 0 : \beta_t = a\}$ and for all $\lambda > 0$, $T_a^{(\lambda)} := \inf\left\{t \geq 0 : \beta_t^{(\lambda)} = a\right\}$. Throughout the proof we will always assume $t > 1$ and use the notation $\lambda_t := \frac{\log(t)}{2}$. We want to show that for any $\delta > 0$,

$$\mathbb{P}_1\left(\left|\lambda_t^{-2}\sigma(t) - T_1^{(\lambda_t)}\right| > \delta\right) \xrightarrow{t\to\infty} 0.$$

Let us remark that for any $\epsilon > 0$ fixed we get the following upper bound,

$$\begin{aligned}
\mathbb{P}_1\left(\left|\lambda_t^{-2}\sigma(t) - T_1^{(\lambda_t)}\right| > \delta\right) &= \mathbb{P}_1\left(\lambda_t^{-2}\sigma(t) - T_1^{(\lambda_t)} > \delta\right) + \mathbb{P}_1\left(T_1^{(\lambda_t)} - \lambda_t^{-2}\sigma(t) > \delta\right) \\
&\leq \mathbb{P}_1\left(\lambda_t^{-2}\sigma(t) - T_1^{(\lambda_t)} > T_{1+\epsilon}^{(\lambda_t)} - T_1^{(\lambda_t)}\right) + \mathbb{P}_1\left(\left|T_{1+\epsilon}^{(\lambda_t)} - T_1^{(\lambda_t)}\right| > \delta\right) \\
&\quad + \mathbb{P}_1\left(T_1^{(\lambda_t)} - \lambda_t^{-2}\sigma(t) > T_1^{(\lambda_t)} - T_{1-\epsilon}^{(\lambda_t)}\right) + \mathbb{P}_1\left(\left|T_1^{(\lambda_t)} - T_{1-\epsilon}^{(\lambda_t)}\right| > \delta\right) \\
&= \mathbb{P}_1\left(\lambda_t^{-2}\sigma(t) > T_{1+\epsilon}^{(\lambda_t)}\right) + \mathbb{P}_1\left(|T_{1+\epsilon} - T_1| > \delta\right) \\
&\quad + \mathbb{P}_1\left(T_{1-\epsilon}^{(\lambda_t)} > \lambda_t^{-2}\sigma(t)\right) + \mathbb{P}_1\left(|T_1 - T_{1-\epsilon}| > \delta\right). \qquad (4.3)
\end{aligned}$$

In the last equality we used that for any $\lambda > 0$ and any a, by the scaling property of Brownian motion, $T_a^{(\lambda)}$ has the same distribution as T_a. Using this upper bound, it is enough to show

that the right-hand side converges to zero as $t \to \infty$. We start by proving that for every $\epsilon > 0$,

$$\mathbb{P}_1\left(\frac{1}{\lambda_t^2}\sigma(t) > T_{1+\epsilon}^{(\lambda_t)}\right) \to 0,$$

as $t \to \infty$. From the proof of Theorem 3 we know that $\sigma^{-1}(t) = \int_0^t \exp(2\beta_u)\ du$ and hence writing $\sigma(t)$ as the inverse of its inverse we get

$$\sigma(t) = \inf\left\{s \geq 0 : \int_0^s \exp(2\beta_u)\ du > t\right\}.$$

This shows that

$$\begin{aligned}\left\{\frac{1}{\lambda_t^2}\sigma(t) > T_{1+\epsilon}^{(\lambda_t)}\right\} &= \left\{\sigma(t) > \lambda_t^2 T_{1+\epsilon}^{(\lambda_t)}\right\} \\ &= \left\{\int_0^{\lambda_t^2 T_{1+\epsilon}^{(\lambda_t)}} \exp(2\beta_u)\ du < t\right\} \\ &= \left\{\frac{1}{2\lambda_t}\log\left(\int_0^{\lambda_t^2 T_{1+\epsilon}^{(\lambda_t)}} \exp(2\beta_u)\ du\right) < 1\right\},\end{aligned}$$

where in the last equation we used that $\log(t) = 2\lambda_t$. By doing a change of variable with $u = \lambda_t^2 v$ we get,

$$\begin{aligned}\frac{1}{2\lambda_t}\log\left(\int_0^{\lambda_t^2 T_{1+\epsilon}^{(\lambda_t)}} \exp(2\beta_u)\ du\right) &= \frac{1}{2\lambda_t}\log\left(\int_0^{T_{1+\epsilon}^{(\lambda_t)}} \lambda_t^2 \exp(2\lambda_t \beta_v^{(\lambda_t)})\ dv\right) \\ &= \frac{1}{2\lambda_t}\left(2\log(\lambda_t) + \log\left(\int_0^{T_{1+\epsilon}^{(\lambda_t)}} \exp\left(2\lambda_t\beta_v^{(\lambda_t)}\right)\ dv\right)\right) \\ &= \frac{\log(\lambda_t)}{\lambda_t} + \frac{1}{2\lambda_t}\log\left(\int_0^{T_{1+\epsilon}^{(\lambda_t)}} \exp\left(2\lambda_t\beta_v^{(\lambda_t)}\right)\ dv\right).\end{aligned}$$

If we fix $t > 1$ we have already remarked that $T_{1+\epsilon}^{(\lambda)} \overset{d}{=} T_{1+\epsilon}$ and $\beta^{(\lambda)} \overset{d}{=} \beta$ for all $\lambda > 0$. Hence,

$$\frac{1}{2\lambda_t}\log\left(\int_0^{\lambda_t^2 T_{1+\epsilon}^{(\lambda_t)}} \exp(2\beta_u)\ du\right) \overset{d}{=} \frac{\log(\lambda_t)}{\lambda_t} + \frac{1}{2\lambda_t}\log\left(\int_0^{T_{1+\epsilon}} \exp\left(2\lambda_t\beta_v\right)\ dv\right).$$

We will now use the analytic fact, that for all continuous maps $g : \mathbb{R}_+ \to \mathbb{R}$ and for any $s > 0$,

$$\frac{1}{2\lambda}\log\left(\int_0^s \exp(2\lambda g(v))\ dv\right) \xrightarrow{\lambda\to\infty} \sup_{0\leq r\leq s} g(r).$$

A proof of this will be given in Lemma 17. From this fact, we deduce that

$$\frac{1}{2\lambda_t}\log\left(\int_0^{T_{1+\epsilon}} \exp\left(2\lambda_t\beta_v\right)\ dv\right) \xrightarrow{t\to\infty} \sup_{0\leq r\leq T_{1+\epsilon}} (\beta_r) = 1 + \epsilon, \tag{4.4}$$

almost surely. Since $\lambda_t \to \infty$ as $t \to \infty$,

$$\lim_{t\to\infty} \frac{\log(\lambda_t)}{\lambda_t} = \lim_{t\to\infty} \frac{1}{\lambda_t} = 0, \tag{4.5}$$

where we used l'hôpital. Therefore, combining (4.4) and (4.5) we get

$$\frac{\log(\lambda_t)}{\lambda_t} + \frac{1}{2\lambda_t} \log\left(\int_0^{T_{1+\epsilon}} \exp\left(2\lambda_t \beta_v\right)\, dv\right) \xrightarrow{t\to\infty} 1+\epsilon,$$

almost surely and hence in particular

$$\frac{1}{2\lambda_t} \log\left(\int_0^{\lambda_t^2 T_{1+\epsilon}^{(\lambda_t)}} \exp(2\beta_u)\, du\right) \xrightarrow{t\to\infty} 1+\epsilon,$$

in probability, thus $\mathbb{P}_1\left(\frac{1}{\lambda_t^2}\sigma(t) > T_{1+\epsilon}^{(\lambda_t)}\right) \xrightarrow{t\to\infty} 0$. By the same argument we have for all $1 > \epsilon > 0$ that $\mathbb{P}_1\left(\frac{1}{\lambda_t^2}\sigma(t) < T_{1-\epsilon}^{(\lambda_t)}\right) \xrightarrow{t\to\infty} 0$.

Now note that $T_{1-\epsilon}^{(\lambda_t)} < T_1^{(\lambda_t)} < T_{1+\epsilon}^{(\lambda_t)}$ and that $T_{1+\epsilon}^{(\lambda_t)} - T_{1-\epsilon}^{(\lambda_t)} \overset{d}{=} T_{1+\epsilon} - T_{1-\epsilon}$. Therefore, it remains to check that $T_{1+\epsilon} - T_{1-\epsilon} \xrightarrow{\epsilon\to 0} 0$ in probability. We check that $T_{1+\epsilon} \downarrow T_1$ and $T_{1-\epsilon} \uparrow T_1$ in probability as $\epsilon \to 0$. We start by showing that $T_{1+\epsilon} \xrightarrow{\epsilon\to 0} T_1$ in probability. Let us define $\psi(\omega) := \inf\{t \geq 0 : \omega_t - \omega_0 \geq \epsilon\}$, then

$$\begin{aligned}
\psi\left((\beta_{T_1+t})_{t\geq 0}\right) &= \inf\{t \geq 0 : \beta_{T_1+t} - \beta_{T_1} \geq \epsilon\} \\
&= \inf\{t \geq T_1 : \beta_t \geq 1+\epsilon\} - T_1 \\
&= T_{1+\epsilon} - T_1.
\end{aligned}$$

For any $\delta > 0$ we define $\Phi(\omega) := \mathbb{1}_{\psi(\omega)\geq\delta}$ then by the Strong Markov property (Proposition 31) we get

$$\mathbb{E}_0\left(\mathbb{1}_{T_1<\infty}\Phi\left((\beta_{T_1+t})_{t\geq 0}\right)\middle|\mathcal{F}_{T_1}\right) = \mathbb{1}_{T_1<\infty}\mathbb{E}_{\beta_{T_1}}\left(\Phi((\beta_t)_{t\geq 0})\right). \tag{4.6}$$

Using the well known fact about real-valued Brownian motions, namely that $T_1 < \infty$ a.s. we can rewrite (4.6) as

$$\begin{aligned}
\mathbb{P}_0\left(T_{1+\epsilon} - T_1 \geq \delta | \mathcal{F}_{T_1}\right) &= \mathbb{P}_1\left(\inf\{t \geq 0 : \beta_t \geq \epsilon\} \geq \delta\right) \\
&= \mathbb{P}_0(T_\epsilon \geq \delta).
\end{aligned}$$

Taking the expectation on both sides leads to

$$\mathbb{P}_0(T_{1+\epsilon} - T_1 \geq \delta) = \mathbb{P}_0(T_\epsilon \geq \delta) = \mathbb{P}_0\left(\epsilon \geq \sup_{0\leq t\leq\delta} \beta_t \geq 0\right).$$

Since $\mathbb{1}_{\epsilon \geq \sup_{0\leq t\leq \delta}\beta_t}$ is bounded we can apply Dominated Convergence theorem and get

$$\begin{aligned}\lim_{\epsilon\to 0}\mathbb{P}_0\left(T_{1+\epsilon}-T_1\geq\delta\right) &= \lim_{\epsilon\to 0}\mathbb{P}_0\left(\epsilon\geq \sup_{0\leq t\leq\delta}\beta_t\geq 0\right)\\ &= \mathbb{P}_0\left(\lim_{\epsilon\to 0}\epsilon\geq \sup_{0\leq t\leq\delta}\beta_t\geq 0\right)\\ &= \mathbb{P}_0\left(0\geq \sup_{0\leq t\leq\delta}\beta_t\geq 0\right)\\ &= 0,\end{aligned}$$

where in the last equality we used that for any real-valued Brownian motion $(B_t)_{t\geq 0}$ and any $\delta\geq 0$ it holds that $\sup_{0\leq t\leq\delta}B_t>0$ (see [LG16, Proposition 2.14]). We have therefore shown that for all $\delta>0$, $\mathbb{P}_0(T_{1+\epsilon}-T_1\geq\delta)\to 0$ as $\epsilon\to 0$. Using the fact that $\{T_{1+\epsilon}-T_1>\delta\}\subset\{T_{1+\epsilon}-T_1\geq\delta\}$, we deduce that $\mathbb{P}_0(T_{1+\epsilon}-T_1>\delta)\leq\mathbb{P}_0(T_{1+\epsilon}-T_1\geq\delta)\xrightarrow{\epsilon\to 0}0$, which proves that $T_{1+\epsilon}\downarrow T_1$ in probability as $\epsilon\to 0$. In order to show that $T_{1-\epsilon}\uparrow T_1$ as $\epsilon\to 0$ in probability we can redo the same argument with $\psi(\omega)=\inf\{t\geq 0:\omega_0-\omega_t\leq\epsilon\}$.

Therefore, we indeed have that $T^{(\lambda_t)}_{1+\epsilon}-T^{(\lambda_t)}_{1-\epsilon}\overset{d}{=}T_{1+\epsilon}-T_{1-\epsilon}\xrightarrow{\epsilon\to 0}0$ in probability. Hence, for all $\delta>0$

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}_1\left(\left|\lambda_t^{-2}\sigma(t)-T_1^{(\lambda_t)}\right|>\delta\right) &= \lim_{\epsilon\to 0}\lim_{t\to\infty}\mathbb{P}_1\left(\left|\lambda_t^{-2}\sigma(t)-T_1^{(\lambda_t)}\right|>\delta\right)\\ &\overset{(4.3)}{\leq} \lim_{\epsilon\to 0}\lim_{t\to\infty}\mathbb{P}_1\left(\lambda_t^{-2}\sigma(t)>T^{(\lambda_t)}_{1+\epsilon}\right)+\lim_{\epsilon\to 0}\mathbb{P}_1\left(|T_{1+\epsilon}-T_1|>\delta\right)\\ &\quad+\lim_{\epsilon\to 0}\lim_{t\to\infty}\mathbb{P}_1\left(T^{(\lambda_t)}_{1-\epsilon}>\lambda_t^{-2}\sigma(t)\right)+\lim_{\epsilon\to 0}\mathbb{P}_1\left(|T_1-T_{1-\epsilon}|>\delta\right)\\ &= 0.\end{aligned}$$

□

To complete the above argument, we need to give a proof for the analytic fact we used.

Lemma 17. [LG16, Proof of Lemma 7.21] *For all continuous functions $g:\mathbb{R}_+\to\mathbb{R}$ and for any $s>0$,*

$$\frac{1}{2\lambda}\log\left(\int_0^s\exp(2\lambda g(v))\,dv\right)\xrightarrow{\lambda\to\infty}\sup_{0\leq r\leq s}g(r).$$

Proof. Define $M:=\sup_{0\leq r\leq s}g(r)$, then

$$\begin{aligned}\frac{1}{2\lambda}\log\left(\int_0^s\exp(2\lambda g(v))\,dv\right) &= \log\left(\left(\int_0^s\exp(2\lambda g(v))\,dv\right)^{\frac{1}{2\lambda}}\right)\\ &\leq \log\left(\left(\int_0^s\exp(2M\lambda)\,dv\right)^{\frac{1}{2\lambda}}\right)\\ &= \log\left((\exp(2\lambda M)s)^{\frac{1}{2\lambda}}\right)\\ &= M+\frac{1}{2\lambda}\log(s)\xrightarrow{\lambda\to\infty}M.\end{aligned}$$

On the other hand, since g is continuous there exists $x_0 \in [0,s]$ such that $g(x_0) = M$ and for any $0 < \epsilon < M$ there exists $(c,d) \subset [0,s]$ such that $g(x) > M - \epsilon$ for all $x \in (c,d)$. Hence,

$$\begin{aligned}\frac{1}{2\lambda}\log\left(\int_0^s \exp(2\lambda g(v))\ dv\right) &\geq \log\left(\left(\int_c^d \exp(2\lambda g(v))\ dv\right)^{\frac{1}{2\lambda}}\right)\\ &\geq \log\left(\left(\int_c^d \exp(2\lambda(M-\epsilon))\ dv\right)^{\frac{1}{2\lambda}}\right)\\ &= M - \epsilon + \frac{1}{2\lambda}\log(d-c) \xrightarrow{\lambda\to\infty} M - \epsilon.\end{aligned}$$

Since this holds for any $\epsilon > 0$ we deduce that $\frac{1}{2\lambda}\log\left(\int_0^s \exp(2\lambda g(v))\ dv\right) \xrightarrow{\lambda\to\infty} M$, which concludes the proof. □

Now we have proved all the necessary lemmas to prove the Spitzer theorem.

Proof. (Theorem 4) Let us keep the notation as in the proof of Lemma 16. We first consider the case $z = 1$ and will give a short argument on how to deduce the general statement at the end. For all $\lambda > 0$ consider the scaled Brownian motion $\gamma_t^{(\lambda)} := \frac{1}{\lambda}\gamma_{\lambda^2 t}$ for $t \geq 0$ where $(\gamma_t)_{t\geq 0}$ is the Brownian motion found in the skew-product representation. We again use the notation $\lambda_t := \frac{\log(t)}{2}$ for $t > 1$, then from the skew-product representation it follows that for $t > 1$,

$$\frac{2}{\log(t)}\theta_t = \frac{1}{\lambda_t}\gamma_{\sigma(t)} = \gamma^{(\lambda_t)}_{(\lambda_t)^{-2}\sigma(t)},$$

where $(\sigma(t))_{t\geq 0}$ is the process defined in Theorem 3. For any Brownian motion $\tilde{B}$ we use the notation $T_1(\tilde{B}) := \inf\{t \geq 0 : \tilde{B}_t = 1\}$ to emphasise which exit time we are considering. Let $\delta > 0$ then for any $\epsilon > 0$ and $M > 0$,

$$\begin{aligned}\mathbb{P}_1\left(\left|\frac{2}{\log(t)}\theta_t - \gamma^{(\lambda_t)}_{T_1^{(\lambda_t)}}\right| > \delta\right) &= \mathbb{P}_1\left(\left|\gamma^{(\lambda_t)}_{(\lambda_t)^{-2}\sigma(t)} - \gamma^{(\lambda_t)}_{T_1^{(\lambda_t)}}\right| > \delta\right)\\ &\leq \mathbb{P}_1\left(\left|\gamma^{(\lambda_t)}_{(\lambda_t)^{-2}\sigma(t)} - \gamma^{(\lambda_t)}_{T_1^{(\lambda_t)}}\right| > \delta, \left|(\lambda_t)^{-2}\sigma(t) - T_1^{(\lambda_t)}\right| \leq \epsilon\right)\\ &\quad + \mathbb{P}_1\left(\left|(\lambda_t)^{-2}\sigma(t) - T_1^{(\lambda_t)}\right| > \epsilon\right)\\ &\leq \mathbb{P}_1\left(\sup_{\substack{|s-r|\leq\epsilon\\ s,r\leq M}}\left|\gamma_s^{(\lambda_t)} - \gamma_r^{(\lambda_t)}\right| > \delta, T_1^{(\lambda_t)} < M\right)\\ &\quad + \mathbb{P}_1\left(T_1^{(\lambda_t)} \geq M\right) + \mathbb{P}_1\left(\left|(\lambda_t)^{-2}\sigma(t) - T_1^{(\lambda_t)}\right| > \epsilon\right)\\ &\leq \mathbb{P}_1\left(\sup_{\substack{|s-r|\leq\epsilon\\ s,r\leq M}}\left|\gamma_s^{(\lambda_t)} - \gamma_r^{(\lambda_t)}\right| > \delta\right)\\ &\quad + \mathbb{P}_1\left(T_1^{(\lambda_t)} \geq M\right) + \mathbb{P}_1\left(\left|(\lambda_t)^{-2}\sigma(t) - T_1^{(\lambda_t)}\right| > \epsilon\right)\end{aligned}$$

$$= \mathbb{P}_0\left(\sup_{\substack{|s-r|\leq\epsilon\\ s,r\leq M}}\left|\tilde{B}_s-\tilde{B}_r\right|>\delta\right) + \mathbb{P}_0\left(T_1(\tilde{B})\geq M\right)+\mathbb{P}_0\left(\left|(\lambda_t)^{-2}\sigma(t)-T_1(\tilde{B})\right|>\epsilon\right),$$

where in the last line we used that under $\mathbb{P}_1$, $\gamma^{(\lambda_t)}$ has the same distribution as some real-valued Brownian motion $\tilde{B}$ starting from zero. Since $\mathbb{P}_0(T_1(\tilde{B})\geq M)$ does not depend on t, and in addition $T_1(\tilde{B})$ is an a.s. finite stopping time, for any $\eta>0$ we can choose $M>0$ large enough such that $\mathbb{P}_0(T_1(\tilde{B})\geq M)<\eta$. Using that every Brownian motion is a uniformly continuous process, we can also choose $\epsilon>0$ small enough such that $\mathbb{P}_0\left(\sup_{\substack{|s-r|\leq\epsilon\\ s,r\leq M}}\left|\tilde{B}_s-\tilde{B}_r\right|>\delta\right)<\eta$. In the proof of Lemma 16 we have already seen that $\mathbb{P}_0\left(\left|(\lambda_t)^{-2}\sigma(t)-T_1(\tilde{B})\right|>\epsilon\right)\to 0$ as $t\to\infty$. Therefore, we deduce that for any $\delta>0$ and for every $\eta>0$ there exists $N>0$ such that if $t>N$, then

$$\begin{aligned}\mathbb{P}_1\left(\left|\frac{2}{\log(t)}\theta_t-\gamma^{(\lambda_t)}_{T_1^{(\lambda_t)}}\right|>\delta\right) &\leq 2\eta+\mathbb{P}_0\left(\left|(\lambda_t)^{-2}\sigma(t)-T_1(\tilde{B})\right|>\epsilon\right)\\ &\leq 3\eta. \end{aligned} \tag{4.7}$$

To complete the proof note that for every fixed $\lambda>0$, $\gamma^{(\lambda)}_{T_1^{(\lambda)}}\sim$ Cauchy$(0,1)$. Indeed, from the skew-product representation we know that $\left(\beta^{(\lambda)}_\cdot,\gamma^{(\lambda)}_\cdot\right)$ is a pair of linear real-valued independent Brownian motions starting both from zero under $\mathbb{P}_1$. Thus, $\mathbb{P}_{\left(\beta^{(\lambda)}_\cdot,\gamma^{(\lambda)}_\cdot\right)}=\mathbb{P}_{\beta^{(\lambda)}_\cdot}\otimes\mathbb{P}_{\gamma^{(\lambda)}_\cdot}$ which implies that

$$\gamma^{(\lambda)}_{T_1^{(\lambda)}}\overset{d}{=}\gamma_{T_1}. \tag{4.8}$$

A classic result from probability theory states that if (Z,Y) is a pair of independent random variables with values in E_Z respectively E_Y, then for any measurable bounded function $f:E_Z\times E_Y\to\mathbb{R}$,

$$\mathbb{E}(f(Z,Y)|\sigma(Y))=\phi(Y),$$

where $\phi(y)=\mathbb{E}(f(Z,y))$ and $\sigma(Y)$ denoted the sigma algebra generated by Y.

Combining the tower property, the above probabilistic fact and (4.8) we deduce that for any $t\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}_1\left(\exp\left(it\gamma^{(\lambda)}_{T_1^{(\lambda)}(\beta)}\right)\right) &= \mathbb{E}_1\left(\exp\left(it\gamma_{T_1(\beta)}\right)\right)\\ &= \mathbb{E}_1\left(\mathbb{E}_1\left(\exp\left(it\gamma_{T_1(\beta)}\right)\mid(\beta_t)_{t\geq 0}\right)\right)\\ &= \mathbb{E}_1(\phi(\beta_\cdot))\\ &\overset{*}{=} \mathbb{E}_1\left(\exp\left(-\frac{t^2}{2}T_1(\beta)\right)\right)\\ &= \exp(-|t|).\end{aligned}$$

Here for any continuous function $\omega_\cdot : \mathbb{R}_+ \to \mathbb{R}$ with $t \mapsto \omega_t$, we defined

$$\phi(\omega_\cdot) = \mathbb{E}\left(\exp\left(it\gamma_{T_1(\omega_\cdot)}\right)\right).$$

In $*$, we used the Laplace transform for real-valued normal distributed random variables, and in the final equation, we applied the Laplace transform to $T_1(\beta)$, using the well-known result stated after Corollary 3.24 in [LG16]. Since $\exp(-|t|)$ is the characteristic function of a Cauchy$(0,1)$ distribution we have shown that for all $\lambda > 0$, $\gamma^{(\lambda)}_{T_1^{(\lambda)}} \sim \text{Cauchy}(0,1)$.

Applying again a well-known result from probability theory, which states that if for two stochastic processes $(Y_t)_{t\geq 0}$, $(Z_t)_{t\geq 0}$ and any $q \in \mathbb{R}, s > 0$, both of the following conditions hold:

$$\begin{aligned} Y_t &\xrightarrow[t\to\infty]{d} \text{Cauchy(q, s)}, \\ |Z_t - Y_t| &\xrightarrow[t\to\infty]{\mathbb{P}} 0, \end{aligned}$$

then it follows that $Z_t \xrightarrow[t\to\infty]{d} \text{Cauchy(q, s)}$. In our case, we define $Y_t := \gamma^{(\lambda_t)}_{T_1^{(\lambda_t)}}$ and $Z_t := \frac{2}{\log(t)}\theta_t$ for $t > 1$. Since, for all $\lambda > 0$, $\gamma^{(\lambda)}_{T_1^{(\lambda)}}$ is constantly distributed as Cauchy$(0,1)$, it follows that $Y_t \xrightarrow[t\to\infty]{d} \text{Cauchy}(0,1)$. Additionally, we have shown in Equation (4.7) that $|Z_t - Y_t| \xrightarrow[t\to\infty]{\mathbb{P}_1} 0$. Thus, we conclude by the above fact that $\frac{2}{\log(t)}\theta_t \xrightarrow[t\to\infty]{d} \text{Cauchy}(0,1)$. This ends the case $z = 1$.

Let us now generalise this argument to an arbitrary $z \in \mathbb{C}$. First, assume that $z = B_0 = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, i.e. B starts at some point on the unit circle. We can naturally define the process

$$B'_t := \frac{1}{e^{i\theta}} B_t, \ \forall t \geq 0,$$

which, by Lemma 6, is a complex Brownian motion such that $B'_0 = \frac{1}{e^{i\theta}} e^{i\theta} = 1$ under $\mathbb{P}_{e^{i\theta}}$. Defining a continuous determination of the argument of $(B'_t)_{t\geq 0}$ by $(\theta'_t)_{t\geq 0}$ leads to,

$$\begin{aligned} \theta'_t &= \arg\left(\frac{1}{e^{i\theta}} B_t\right) \\ &= \arg\left(\frac{1}{e^{i\theta}}\right) + \theta_t \\ &= \theta_t - \theta, \end{aligned}$$

for all $t \geq 0$. Using the first part of the proof we have already shown that

$$\frac{2}{\log(t)}\theta_t - \frac{2}{\log(t)}\theta = \frac{2}{\log(t)}\theta'_t \xrightarrow[t\to\infty]{d} \text{Cauchy}(0,1),$$

but since $\frac{2}{\log(t)}\theta \to 0$ as $t \to \infty$ we conclude that $\frac{2}{\log(t)}\theta_t \xrightarrow[t\to\infty]{d} \text{Cauchy}(0,1)$, which proves the case when $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$.

For the last case assume $z = B_0 = re^{i\theta}$ for some $r > 0$ and $\theta \in [0, 2\pi)$. Define the process

$$B''_t = \frac{1}{r} B_{r^2 t}, \ \forall t \geq 0,$$

again by Lemma 6, B'' is a complex-valued Brownian motion such that $B''_0 = \frac{1}{r}re^{i\theta} = e^{i\theta}$ under $\mathbb{P}_{re^{i\theta}}$. Let us denote by $(\theta''_t)_{t\geq 0}$ a continuous determination of the argument of B'', then for all $t \geq 0$,

$$\begin{aligned}\theta''_t &= \arg\left(\frac{1}{r}B_{r^2t}\right)\\ &= \arg(B_{r^2t})\\ &\overset{d}{=} \arg(rB_t)\\ &= \arg(B_t)\\ &= \theta_t.\end{aligned}$$

From the previous case we know that

$$\frac{2}{\log(t)}\theta_t = \frac{2}{\log(t)}\theta''_t \xrightarrow[t\to\infty]{d} \text{Cauchy}(0,1),$$

which proves the claim when $z = re^{i\theta}$ for some $r > 0$ and $\theta \in [0, 2\pi)$. Since every complex number $z \in \mathbb{C}$ can be represented as $z = re^{i\theta}$ for some r, θ as above we have proven the Spitzer Theorem in full generality. □

Continuation for the Interested

A natural continuation of this work would be the topic of Green functions, which is closely related to what we have done so far. After reading this work, one should be able to understand the following sources, also because a large part of this work uses similar notations. For interested readers I would recommend the construction of Green functions in [Law], more precisely one can find it in Proposition 1.3.1. After that one can find some important properties on the subsequent pages in [Law]. However, I would like to mention at this point that [Law] is not always the easiest source to read, as Lawler sometimes takes important steps and ideas in the proofs as given and the reader has to take some initiative to understand everything exactly. For this reason, I would also recommend reading [BN16]. There one can find a slightly different but equivalent introduction to the topic of Green functions which might be helpful for understanding.

Appendix

In this part of the appendix we would like to mention some important notations, definitions and theorems which are known from the stochastic calculus course and which are used in this thesis.

We take the following as a definition of quadratic variation, the proof of it can be found in [LG16] in Chapter 4.3, in [RY99] or in [Pos23].

Definition 23. [Pos23, Theorem 3.1.28] *Let M, N be two continuous $(\mathcal{F}_t)_{t\geq 0}$-local martingales. Then, there exists a unique bilinear continuous increasing process $(\langle M, N\rangle_t)_{t\geq 0}$ vanishing at zero, such that $(M_tN_t - \langle M, N\rangle_t)_{t\geq 0}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-local martingale. This process $\langle M, N\rangle$ is called the quadratic covariation of M and N.*

If we now choose $M = N$ in the above definition, we get the process $\langle M, M\rangle =: \langle M\rangle$ which is called quadratic variation and has the following important property.

Proposition 24. *Let M be an $\mathbb{R}$-valued $(\mathcal{F}_t)_{t\geq 0}$-continuous local martingale, then $\langle M\rangle$ is constant if and only if M itself is constant.*

An important process, which can be characterised by the quadratic variation, is the so-called Brownian motion.

Definition 24. *A real-valued process $(B_t)_{t\geq 0}$ is called an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion if B is $(\mathcal{F}_t)_{t\geq 0}$-adapted, has continuous paths, satisfies $B_0 = 0$, and is such that for any $0 \leq s \leq t$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$. Furthermore, it is distributed as a normal random variable with expectation zero and variance $t - s$.*

Proposition 25 (Lévy's characterisation of real-valued Brownian motion)**.** [Pos23] *Let M be a continuous $(\mathcal{F}_t)_{t\geq 0}$-local martingale vanishing at zero. Then, M is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion if and only if $\langle M\rangle_t = t$ for all $t \geq 0$.*

We now shortly want to discuss the recurrence property of a one-dimensional real-valued Brownian motion.

Definition 25. *Let X be a one-dimensional real-valued process. A state $x \in \mathbb{R}$ is called recurrent if $\mathbb{P}_x(X_t = x \text{ for infinitely many } t) = 1$. In other words, we say that x is recurrent if X returns to x infinitely often when starting from x. We say that X is recurrent if it is recurrent for all $x \in \mathbb{R}$.*

To say something about the recurrence property of a Brownian motion B, we give the following theorem, which treats the behaviour of B for t towards infinity.

Proposition 26. [Pos23] *Let B be a one-dimensional real-valued Brownian motion starting at 0, then $\mathbb{P}$-a.s.*

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty,$$
$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty.$$

Using that a Brownian motion has continuous sample paths, we can intuitively think about why the following theorem should be true.

Proposition 27. [Pos23] *Let B be a one-dimensional Brownian motion. Then, for any $x \in \mathbb{R}$,*

$$\mathbb{P}_x\left(\bigcap_{n\in\mathbb{N}} \{B_t = 0 \text{ for some } t \geq n\}\right) = 1.$$

In other words for any starting point x, the Brownian motion will return infinitely often to 0. Using that a Brownian motion is invariant under translation we deduce that a real-valued Brownian motion is recurrent.

Proposition 28 (Real-valued Itô formula in two dimensions)**.** [LG16, Theorem 5.10] *Let $M^{(1)}, M^{(2)}$ be two continuous martingales and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function. Then, for all $t \geq 0$,*

$$\begin{aligned} F\left(M_t^{(1)}, M_t^{(2)}\right) &= F\left(M_0^{(1)}, M_0^{(2)}\right) \\ &\quad + \left(\int_0^t F_{x^{(1)}}\left(M_s^{(1)}, M_s^{(2)}\right)\, dM_s^{(1)} + \int_0^t F_{x^{(2)}}\left(M_s^{(1)}, M_s^{(2)}\right)\, dM_s^{(2)}\right) \\ &\quad + \frac{1}{2}\left(\int_0^t F_{x^{(1)}x^{(1)}}\left(M_s^{(1)}, M_s^{(2)}\right)\, d\left\langle M^{(1)}\right\rangle_s \right. \\ &\quad + \int_0^t F_{x^{(1)}x^{(2)}}\left(M_s^{(1)}, M_s^{(2)}\right)\, d\left\langle M^{(1)}, M^{(2)}\right\rangle_s \\ &\quad \left. + \int_0^t F_{x^{(2)}x^{(2)}}\left(M_s^{(1)}, M_s^{(2)}\right)\, d\left\langle M^{(2)}\right\rangle_s\right). \end{aligned} \tag{4.9}$$

Proposition 29 (Measurability of a stopped process)**.** [Pos23, Proposition 1.3.17] *Let $(X_t)_{t\geq 0}$ be an adapted process to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let T be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$, then X_T is $\mathcal{F}_T$ measurable.*

Proposition 30. [RY99, Proposition 1.8] *For a continuous local martingale M the two sets $\{\langle M\rangle_\infty < \infty\}$ and $\{\lim_{t\to\infty} M_t \text{ exists}\}$ are a.s. equal.*

Furthermore, $\limsup_{t\to\infty} M_t = \infty$ and $\liminf_{t\to\infty} M_t = -\infty$ a.s. on the set $\{\langle M\rangle_\infty = \infty\}$. For a continuous local martingale the following are equivalent

1. $\sup_t M_t = \infty$ *a.s.*
2. $\inf_t M_t = -\infty$ *a.s.*
3. $\langle M\rangle_\infty = \infty$ *a.s.*

Proposition 31 (Strong Markov property)**.** *Let τ be a stopping time and Φ a measurable bounded or non-negative function on $\mathcal{C}(\mathbb{R}_+, \mathbb{C})$. Then, if B is a complex Brownian motion starting from any $z \in \mathbb{C}$,*

$$\mathbb{E}_z\left(\mathbb{1}_{\tau<\infty}\Phi\left((B_{\tau+t})_{t\geq 0}\right)|\mathcal{F}_\tau\right) = \mathbb{1}_{\tau<\infty}\mathbb{E}_{B_\tau}\left(\Phi\left((B_t)_{t\geq 0}\right)\right). \tag{4.10}$$

Proposition 32 (Theorem of H.A. Schwarz)**.** [AE06, Corollary 5.5] *Let $f : E \subset \mathbb{R}^2 \to \mathbb{R}$. If $f \in C^2(E, \mathbb{R})$ then,*

$$\partial_x\partial_y f(e) = \partial_y\partial_x f(e),$$

for all $e \in E$.

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