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Master Thesis

Constructing bounds for the discrepancy of sequences

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Abstract

This master thesis provides a comprehensive exploration of discrepancy theory in one dimension. The research begins with a study of uniformly distributed sequences from a quantitative perspective, analyzing the methods for finding them and constructing examples. We proceed by defining the discrepancy of sequences and examining its properties. The main focus is on finding good general upper and lower bounds for the discrepancy in terms of exponential sums, which helps us understand how close a sequence is to having a uniform distribution. Finally, with the help of some basic concepts of Diophantine Approximation, we apply the results to delve deeper into some specific sequences.

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Introduction

Discrepancy Theory, also called *Theory of irregularities of distribution*, has its roots in number theory, and in particular in the study of Diophantine Approximation. The argument is linked to the older theory of uniformly distributed sequences. Interest in this topic arose when, in 1884, Kronecker [41] generalized Dirichlet Approximation Theorem in the following way. For each $\alpha, x, \delta \in \mathbb{R}$ with α irrational and $\delta > 0$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $|q\alpha - p - x| < \delta$.

The previous equation translates to $x-\delta < q\alpha-p < x+\delta$. Note that we can write $q\alpha-p = m+\{q\alpha\}$ for some $m \in \mathbb{Z}$, and therefore $x-\delta < m+\{q\alpha\} < x+\delta$. Choosing $x \in (0,1)$ and taking δ small enough, we can infer that $m = 0$. Hence we arrive at the equation $|\{q\alpha\} - x| < \delta$, which proves that the set $\mathrm{S}(\alpha) \coloneqq \{\{n\alpha\} \mid n \in \mathbb{N}\}$ is dense in the unit interval $(0,1)$. This simple result posed many other questions for mathematicians at the beginning of the twentieth century. In particular, they wondered how the elements of the set $\mathrm{S}(\alpha)$, which are indeed the elements of the sequence $(n\alpha)_{n=1}^{\infty}$ mod 1, are distributed in the unit interval.

Although some concepts concerning the uniformly distribution of sequences already appear in the papers of Bohl [9, 1909], Sierpinski [60, 1910] and Weyl [71, 1910], the first who treated the topic in a systematic way was Weyl with the paper [72, 1914], and then in a more complete way in the famous paper [73, 1916]. Over the years, the latter became a very important starting point for the theory of irregularities of distribution. The paper, entitled *Über die Gleichverteilung von Zahlen mod Eins*, reports a formal definition of the concept of uniformly distributed sequences mod 1, that is, sequences $(x_n)_{n=1}^{\infty}$ for which the empirical distribution of the fractional parts $(\{x_n\})_{n=1}^{\infty}$ approaches asymptotically the uniform distribution in the unit interval. In addition, Weyl introduced two criteria that help to find this type of sequence. These results will be later called Riemann-integral Criterion and Weyl Criterion, and with these it has been proved that the sequence $(n\alpha)_{n=1}^{\infty}$, where $\alpha \in \mathbb{R}$ is irrational, is uniformly distributed mod 1.

Many research articles have expanded Weyl's research in the following years. The initial goal was to find other sequences which would be uniformly distributed mod 1. The first to succeed were probably Ostrowski [54] and Behnke [4], who in 1922 published two articles in the journal *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*. Later Van der Corput [63, 1935] and Koksma [39, 1936], in addition to finding specific properties for some particular sequences, also collected the first results of the theory of uniformly distributed sequences.

In particular, Van der Corput introduced the concept of *discrepancy*, which mea-

sures how far the sequence is from having a uniform distribution. This has opened many more doors regarding research in this area. For example, sequences were extended to multiple dimensions, and the main goal became to study the discrepancy of particular sequences, and to find good bounds for this discrepancy. The main results were found in chronological order by Erdös and Turán [13, 14, 1948], Van Aardenne-Ehrenfest [62, 1949], Roth [56, 1954], Leveque [45, 1965], Schmidt [58, 1972], Beck and Chen [3, 1987].

The research was then extended to more abstract spaces, starting from compact spaces up to topological groups, as shown in Hlawka's papers [27, 1955] and [28, 1956]. In recent times the topic has been studied in depth thanks to the studies of Rindler, Niederreiter and Losert. In addition, monographs summarizing concepts regarding discrepancy theory have been written. Starting from the book of Cigler and Helmberg [11, 1960], we arrive at the complete monographs of Pòlya and Szegö [55, 1964] and of Kuipers and Niederreiter [44, 1974]. In more recent times, other monographs such as those by Hlawka [32, 1979] and [33, 1984] have been written, as well as the exhausting book of Drmota and Tichy [12, 1997] and the book of Bugeaud [10, 2012]. Many of the latter will be important sources for this work.

In this master's thesis, we will explore the theory of irregularities of distribution, starting from the more quantitative aspects concerning uniformly distributed sequences, up to analyzing the discrepancy. The main goal is to find good bounds for the discrepancy, which allow us to classify certain types of sequences. Since we will introduce the concepts starting from the most basic aspects, the results will all be presented for sequences in one dimension. However, many of the results can be extended to multiple dimensions, often in a natural way.

Chapter 1 introduces the quantitative theory of uniformly distributed sequences. We will analyze some of the main methods to understand whether a sequence is uniformly distributed or not, such as Riemann-integral Criterion (Theorem 1.9), Weyl Criterion (Theorem 1.13) and Fejér Theorem (1.21). Each result is accompanied by some examples. We will then focus on proving Theorem 1.28, which is a strong result and helps us to analyze other types of sequences. As last thing, we will state some difference Theorems and extend Fejér Theorem to get more examples.

In Chapter 2 we will introduce the concept of discrepancy and we will analyze the sequences from a more qualitative point of view. We will first see some properties of the discrepancy (such as Theorem 2.17), and then we will begin to construct upper and lower bounds for the discrepancy in different ways. In particular we will focus on Schmidt's Method (Theorem 2.30) and Leveque Inequality (Theorem 2.39).

The work will culminate in Chapter 3, where we will analyze in particular Erdös-Turán Theorem (Theorem 3.1), which will allow us to compute the discrepancy of many different sequences. We will recall some examples given in Chapter 1 and analyze them using Erdös-Turán's result. We will conclude with an application on the sequence $(n\alpha)_{n=1}^{\infty}$ (for α irrational), and we will see that the discrepancy of that sequence depends on the type of α (Theorem 3.21).

Chapter 1

Preliminaries

The goal of this chapter is to develop a basic theory, which will then be explored in more detail in the next chapter. We will begin with the definition of uniformly distributed sequences mod 1 given by Weyl [73], and we will analyze some methods for finding equidistributed sequences. A detailed survey on most of the topics of this chapter is given by Koksma [39, chap. 8-9]. For a generalization in higher dimensions see Drmota and Tichy [12, pp. 1-3, 14-15].

Our working space will be the standard Euclidean space $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the standard Euclidean norm, or absolute value. Since almost all the results we will present are general, we consider a given sequence $\omega = (x_n)_{n=1}^{\infty}$ of real numbers throughout the chapter.

1.1 Uniformly distributed sequences

Most of the results displayed in this section can be found in Kuipers and Niederreiter [44, pp. 1-4] and in Bugeaud [10, pp. 1-5]. Other elementary criteria about uniformly distributed sequences can be found in O'Neil [53] and Niederreiter [50]. Concerning Example 1.12 we refer to Knapowski [36].

Notation 1.1. Let $x \in \mathbb{R}$. Then we denote the integral part and the fractional part of x by $[x] := \max\{\, n \in \mathbb{Z} \mid n \leq x \,\}$, resp. $\{x\} := x - [x]$.

Note that for all $x \in \mathbb{R}$ we have: $\{x\} \in [0, 1)$.

Notation 1.2. Let $E \subset \mathbb{R}$ be a set and $N \in \mathbb{N}$. Then we denote

$$A(E, N) := A(E, N, \omega) := |\{\, x_n \in \omega \mid 1 \leq n \leq N, \{x_n\} \in E \,\}|. \tag{1.1}$$

The next definition refers to Weyl [73], but with a more recent notation.

Definition 1.3. The sequence $\omega = (x_n)_{n=1}^{\infty}$ is called *uniformly distributed* (or *equidistributed*) *modulo 1*, if for all $a, b \in \mathbb{R}$ with $0 \leq a < b \leq 1$ it holds

$$\lim_{N \to \infty} \frac{A([a, b), N, \omega)}{N} = b - a. \tag{1.2}$$

Let $\{\omega\} \coloneqq (\{x_n\})_{n=1}^{\infty}$ be the sequence of fractional parts. Informally, we can say that ω is uniformly distributed mod 1, if every sub-interval of $[0,1)$ contains the right number of elements of $\{\omega\}$ in proportion to its length.

Remark 1.4. Note that one can write (1.1) as $A(E,N,\omega) = \sum_{n=1}^{N} \chi_E(\{x_n\})$, so that (1.2) is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) = \int_0^1 \chi_{[a,b)}(x)\, dx. \tag{1.3}$$

This notation is more convenient when we have to do computations.

Remark 1.5. Note that the first terms of the sequence do not play a fundamental role for equidistribution. By removing the first k terms of an uniformly distributed sequence ω we get an other equidistributed sequence. Indeed

$$\left| \frac{1}{N} \sum_{n=1}^{k} \chi_{[a,b)}(\{x_n\}) \right| \leq \frac{k}{N} \xrightarrow{N\to\infty} 0.$$

Hence the sum in (1.3) can also start from $k+1$ instead of 1.

We will give some examples for uniformly distributed sequences. However, considering only the previous definition, it is difficult to understand whether a sequence is uniformly distributed or not. We first need to introduce some criteria that make it easier for us to give concrete examples.

First of all, we show how equidistribution is connected with density.

Lemma 1.6. *Assume that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Then the set $S \coloneqq \{\, \{x_n\} \mid n \in \mathbb{N} \,\}$ is dense in $[0,1]$.*

Proof. Assume that S is not dense in $[0,1]$. Then there exists $a, b \in \mathbb{R}$ with $0 \leq a < b \leq 1$ such that $S \cap (a,b) = \emptyset$. Let $c \in (a,b)$. Then

$$\lim_{N\to\infty} \frac{A([c,b),N,\omega)}{N} = 0 \neq b - c.$$

Hence $(x_n)_{n=1}^{\infty}$ is not uniformly distributed mod 1. ∎

Remark 1.7. The converse of Lemma 1.6 is not true. There are sequences $(x_n)_{n=1}^{\infty}$ for which $S \coloneqq \{\, \{x_n\} \mid n \in \mathbb{N} \,\}$ is dense in $[0,1]$, but which are not uniformly distributed mod 1. For example, consider the sequence $(\log(n))_{n=1}^{\infty}$ (see Example 1.15 and Remark 1.16).

From Lemma 1.6 it immediately follows that sequences of integers, convergent sequences, sequences with a finite number of accumulation points, and sequences such that $(\{x_n\})_{n=1}^{\infty}$ has at least one non-accumulation point are not uniformly distributed mod 1. As example consider the following.

Example 1.8. Let $r \in \mathbb{Q}$. Then the sequence given by $(nr)_{n=1}^{\infty}$ is not uniformly distributed mod 1.

Proof. Since $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $\gcd(p,q) = 1$ and such that $r = p/q$. For each term of the sequence, we can write out the fractional part:

$$\left\{\frac{p}{q}\right\}; \left\{\frac{2p}{q}\right\}; \left\{\frac{3p}{q}\right\}; \ldots; \left\{\frac{(q-1)p}{q}\right\}; 0; \left\{\frac{(q+1)p}{q}\right\}; \left\{\frac{(q+2)p}{q}\right\}; \ldots$$

But note that the $(q+1)$-th term of this sequence is equal to the first term, the $(q+2)$-th is equal to the second one, and so on. This means that the sequence $(\{nr\})_{n=1}^{\infty}$ is cyclic, and in particular finite. Hence $S \coloneqq \{\, \{nr\} \mid n \in \mathbb{N} \,\}$ can not be dense in $[0,1]$. So by Lemma 1.6 $(nr)_{n=1}^{\infty}$ is not uniformly distributed mod 1. ∎

Let's continue by introducing the first of the criteria found by Weyl [73].

Theorem 1.9 (Riemann integral criterion)**.** *The sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 if and only if for all continuous maps $f\colon [0,1] \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,dx. \tag{1.4}$$

Proof. $\Longrightarrow$: Assume that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

Step 1 Let $a_0, a_1, \ldots, a_k \in \mathbb{R}$ such that $0 = a_0 < a_1 < \cdots < a_k = 1$. First we consider a step function $g\colon [0,1] \to \mathbb{R}, x \mapsto \sum_{i=0}^{k-1} d_i \chi_{[a_i,a_{i+1})}(x)$, where $d_i \in \mathbb{R}$ for all $i \in \{0,\ldots,k-1\}$. We have

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\{x_n\}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{i=0}^{k-1} d_i \chi_{[a_i,a_{i+1})}(\{x_n\}) \\
&= \sum_{i=0}^{k-1} d_i \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a_i,a_{i+1})}(\{x_n\}) \overset{(1.3)}{=} \sum_{i=0}^{k-1} d_i \int_0^1 \chi_{[a_i,a_{i+1})}(x)\,dx \\
&= \int_0^1 \sum_{i=0}^{k-1} d_i \chi_{[a_i,a_{i+1})}(x)\,dx = \int_0^1 g(x)\,dx.
\end{aligned}$$

Hence (1.4) holds for step functions.

Step 2 Now we consider a general continuous function $f\colon [0,1] \to \mathbb{R}$. This is also Riemann-integrable (see Appendix C, Proposition C.10). Hence, given any real number $\epsilon > 0$, there exist two step functions $f_1, f_2\colon [0,1] \to \mathbb{R}$ such that $f_1 \le f \le f_2$ and $\int_0^1 (f_2(x) - f_1(x))\,dx \le \epsilon$ (see Appendix C, Proposition C.11). By *Step 1* (1.4) holds for f_1 and f_2. Hence

$$\begin{aligned}
&\int_0^1 f(x)\,dx - \epsilon \le \int_0^1 f_2(x)\,dx - \epsilon \le \int_0^1 f_1(x)\,dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(\{x_n\}) \\
&\le \underline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) \le \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) \le \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_2(\{x_n\}) \\
&= \int_0^1 f_2(x)\,dx \le \int_0^1 f_1(x)\,dx + \epsilon \le \int_0^1 f(x)\,dx + \epsilon.
\end{aligned}$$

Now taking $\epsilon \to 0$, using Theorem C.1 in Appendix C, we get that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\})$$

exists and (1.4) holds for f.

$\Longleftarrow$: Assume that (1.4) holds for all $f\colon [0,1] \to \mathbb{R}$ continuous. Let $[a,b) \subseteq [0,1]$ be a sub-interval. Given any real number $\epsilon > 0$, there exist two continuous functions $g_1, g_2\colon [0,1] \to \mathbb{R}$ such that $g_1(x) \leq \chi_{[a,b)}(x) \leq g_2(x)$ for all $x \in [0,1]$ and $\int_0^1 (g_2(x) - g_1(x))\, dx \leq \epsilon$. Indeed, if $a \neq 0$ and $b \neq 1$, the maps g_1 and g_2 can be constructed as in Figure 1.1, while if $a = 0$ and $b \neq 1$, one can take g_1 and g_2 as in Figure 1.2 (the case $a \neq 0$ and $b = 1$ is completely analogous). Finally, if $a = 0$ and $b = 1$, then we can set g_1 and g_2 as in Figure 1.3.

Figure 1.1: Example of g_1 and g_2 for $a \neq 0$ and $b \neq 1$

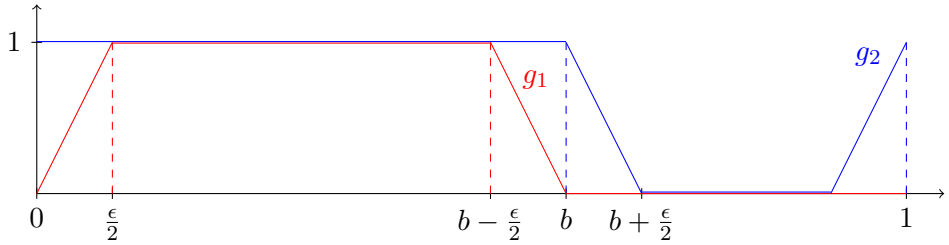

Figure 1.2: Example of g_1 and g_2 for $a = 0$ and $b \neq 1$

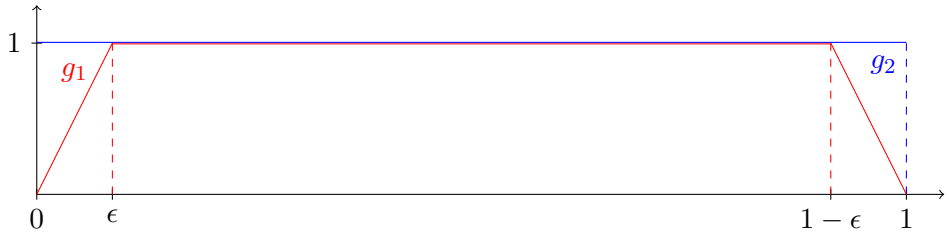

Figure 1.3: Example of g_1 and g_2 for $a = 0$ and $b = 1$

Then (1.4) holds for g_1 and g_2. Hence

$$\int_0^1 \chi_{[a,b)}(x)\,dx - \epsilon \le \int_0^1 g_2(x)\,dx - \epsilon \le \int_0^1 g_1(x)\,dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_1(\{x_n\})$$
$$\le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) \le \overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_2(\{x_n\})$$
$$= \int_0^1 g_2(x)\,dx \le \int_0^1 g_1(x)\,dx + \epsilon \le \int_0^1 \chi_{[a,b)}(x)\,dx + \epsilon.$$

Now taking $\epsilon \to 0$, using Theorem C.1 in Appendix C, we get that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)]}(\{x_n\})$$

exists and (1.3) holds, i.e. $(x_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1. ■

Corollary 1.10. *The sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 if and only if for all Riemann-integrable maps $f\colon [0,1] \to \mathbb{R}$, (1.4) holds.*

Proof. $\Longrightarrow$: Assume that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Consider a Riemann-integrable map $f\colon [0,1] \to \mathbb{R}$. Then we can proceed exactly as in the first part of the proof of Theorem 1.9.

$\Longleftarrow$: Assume that (1.4) holds for all Riemann-integrable maps $f\colon [0,1] \to \mathbb{R}$. Then it holds in particular for all continuous maps $f\colon [0,1] \to \mathbb{R}$ by Proposition C.10 in Appendix C. By Theorem 1.9 $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

One can extend Theorem 1.9 to complex-valued functions with period 1.

Corollary 1.11. *The sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 if and only if for all continuous maps $f\colon [0,1] \to \mathbb{C}$ with period 1 we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx. \tag{1.5}$$

Proof. $\Longrightarrow$: Assume that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Consider a general continuous function $f\colon [0,1] \to \mathbb{C}$ with period 1. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \lim_{N\to\infty} \frac{1}{N}\left(\sum_{n=1}^{N} \mathrm{Re}(f(\{x_n\})) + i\sum_{n=1}^{N} \mathrm{Im}(f(\{x_n\}))\right)$$
$$\overset{(1.4)}{=} \int_0^1 \mathrm{Re}(f(x))\,dx + i\int_0^1 \mathrm{Im}(f(x))\,dx = \int_0^1 f(x)\,dx.$$

Hence (1.4) holds for f. Since f is with period 1, we have $f(\{x_n\}) = f(x_n)$, and hence (1.5) holds for f.

$\Longleftarrow$: Assume that (1.5) holds for all $f\colon [0,1] \to \mathbb{C}$ continuous with period 1. This means (1.5) holds for real and imaginary part of all $f\colon [0,1] \to \mathbb{C}$ continuous. Then we can proceed as in the proof of Theorem 1.9, since the maps g_1 and g_2 are constructed such that $g_1(0) = g_1(1)$ and $g_2(0) = g_2(1)$ (see Figures 1.1, 1.2, 1.3). Doing the same computations we get that (1.3) holds. ■

Theorem 1.9 is very relevant as it reveals to us that the integral of a Riemann-integrable function f over $[0,1]$ can be approximated by the arithmetic mean of the images of f at the points $x \in [0,1]$ which form together a uniformly distributed sequence. Several studies have been done on this important application, and it has been discovered that the error can be calculated in terms of discrepancy (which we will define in Chapter 2). One of the most relevant results are the Koksma inequality [37, 1942/1943], then extended in more dimensions with the Koksma-Hlawka inequality [29, 1961]. Moreover, Korobov [40, 1959] and Hlawka [30, 31, 1962/1964] developed the method of good lattice points, which allows to find low-discrepancy sequences. For a complete overview about numerical applications of uniformly distributed sequences see Kuipers and Niederreiter [44, pp. 142-163].

We provide now a first example of uniformly distributed sequence.

Example 1.12. The sequence $(x_n)_{n=1}^{\infty}$ given by

$$\frac{0}{1};\quad \frac{0}{2},\quad \frac{1}{2};\quad \frac{0}{3},\quad \frac{1}{3},\quad \frac{2}{3};\quad \ldots;\quad \frac{0}{k},\quad \frac{1}{k},\quad \ldots,\quad \frac{k-1}{k};\quad \ldots$$

is uniformly distributed mod 1.

Proof. Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Note that $\{\, x_n \mid n \in \mathbb{N} \,\} = \mathbb{Q} \cap [0,1]$, which is dense in $[0,1]$. Moreover $\{x_n\} = x_n$ for all $n \in \mathbb{N}$. Thus $(x_n)_{n=1}^{\infty}$ is a good candidate for being uniformly distributed. Let $M \in \mathbb{N}$.

- Assume first that $M = \sum_{j=1}^{N} j$ for some $N \in \mathbb{N}$. By the convergence of the Riemann sum (Lemma C.12 in Appendix C), we have

$$\lim_{k\to\infty} \sum_{i=0}^{k-1} f\left(\frac{i}{k}\right)\frac{1}{k} = \int_0^1 f(x)\,dx.$$

Hence, given $\epsilon \in \mathbb{R}_{>0}$, there exists $\tilde{k} \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq \tilde{k}$

$$\left|\sum_{i=0}^{k-1} f\left(\frac{i}{k}\right)\frac{1}{k} - \int_0^1 f(x)\,dx\right| < \epsilon. \tag{1.6}$$

This implies

$$\begin{aligned}
\frac{1}{M}\sum_{n=1}^{M} f(\{x_n\}) &= \frac{1}{\sum_{j=1}^{N} j}\sum_{k=1}^{N}\sum_{i=0}^{k-1} f\left(\frac{i}{k}\right) = \frac{1}{\sum_{j=1}^{N} j}\sum_{k=1}^{N} k\sum_{i=0}^{k-1} f\left(\frac{i}{k}\right)\frac{1}{k} \\
&< \frac{1}{\sum_{j=1}^{N} j}\left[\sum_{k=1}^{\tilde{k}-1}\sum_{i=0}^{k-1} f\left(\frac{i}{k}\right) + \left(\int_0^1 f(x)\,dx + \epsilon\right)\sum_{k=\tilde{k}}^{N} k\right] \\
&= \frac{2}{N(N+1)}\left[\sum_{k=1}^{\tilde{k}-1}\sum_{i=0}^{k-1} f\left(\frac{i}{k}\right) + \left(\int_0^1 f(x)\,dx + \epsilon\right)\left(\frac{N(N+1)}{2} - \frac{(\tilde{k}-1)\tilde{k}}{2}\right)\right]
\end{aligned}$$

Note that in the third step we have used the upper bound given by (1.6). Taking $N \to \infty$ (which implies $M \to \infty$), we get

$$\lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} f(\{x_n\}) < \int_0^1 f(x)\,dx + \epsilon.$$

Now, doing the same computations but using the lower bound given by (1.6), we get also the inequality

$$\lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} f(\{x_n\}) > \int_0^1 f(x)\,dx - \epsilon.$$

In particular, (1.4) holds.

- Assume now that M is a general natural number. Then there exists some $N \in \mathbb{N}$ such that
$$\frac{(N-1)N}{2} \leq M < \frac{N(N+1)}{2}.$$
Therefore, if $k_1 \coloneqq (N-1)N/2$ and $k_2 \coloneqq N(N+1)/2$, we get

$$\begin{aligned}
&\left| \frac{1}{M} \sum_{n=1}^{M} f(\{x_n\}) - \int_0^1 f(x)\,dx \right| \\
&\leq \left| \frac{1}{M} \sum_{n=1}^{M} f(\{x_n\}) - \frac{1}{k_1} \sum_{n=1}^{k_1} f(\{x_n\}) \right| + \left| \frac{1}{k_1} \sum_{n=1}^{k_1} f(\{x_n\}) - \int_0^1 f(x)\,dx \right|.
\end{aligned}$$

The second term on the RHS tends to 0 as $k_1 \to \infty$ by the previous step. For the other term, we have

$$\begin{aligned}
&\left| \frac{1}{M} \sum_{n=1}^{M} f(\{x_n\}) - \frac{1}{k_1} \sum_{n=1}^{k_1} f(\{x_n\}) \right| \\
&= \left| \frac{1}{M} \sum_{n=k_1+1}^{M} f(\{x_n\}) + \frac{k_1 - M}{k_1 M} \sum_{n=1}^{k_1} f(\{x_n\}) \right| = \\
&\leq 2\|f\|_\infty \frac{M-k_1}{M} < 2\|f\|_\infty \frac{k_2 - k_1}{k_2} = \frac{4\|f\|_\infty}{N+1}.
\end{aligned}$$

where in the second step we have used that f is bounded as continuous function on a closed interval. The last expression tends to 0 as $N \to \infty$.

Hence by Theorem 1.9 $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

1.2 Weyl Criterion and Fejér Theorem

We proceed with the second criterion given by Weyl. As references, see Weyl [73] for Theorem 1.13, and Van der Corput [70] for Theorem 1.20. See also Kuipers and Niederreiter [44, pp. 7-9, 13-15] and Bugeaud [10, pp. 1-5].

Theorem 1.13 (Weyl Criterion)**.** *The sequence $(x_n)_{n=1}^{\infty}$ is u.d mod 1 if and only if for all $h \in \mathbb{Z} \setminus \{0\}$ it holds*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0. \tag{1.7}$$

Proof. $\Longrightarrow$: Assume that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Using (1.5) with the map $f(x) \coloneqq e^{2\pi i h x}$ we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x}\, dx = 0.$$

$\Longleftarrow$: Assume that (1.7) holds for all $h \in \mathbb{Z} \setminus \{0\}$. This means that, given $\epsilon \in \mathbb{R}_{>0}$, there exists $M \in \mathbb{N}$ such that for all integers $N \geq M$ we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| < \epsilon. \tag{1.8}$$

Let $f\colon [0,1] \to \mathbb{C}$ be continuous with period 1. By the Weierstrass Approximation Theorem (see Appendix C, Theorem C.2), there exists a trigonometric polynomial $\Psi\colon [0,1] \to \mathbb{C}, x \mapsto \sum_{h\in A} c_h e^{2\pi i h x}$, where $A \subseteq \mathbb{Z}$ is finite, $c_h \in \mathbb{R}$ for all $h \in \mathbb{Z}$, and such that

$$\|f - \Psi\|_\infty < \epsilon. \tag{1.9}$$

Now we have

$$\begin{aligned}
\left| \int_0^1 f(x)\, dx - \frac{1}{M} \sum_{n=1}^{M} f(x_n) \right| &\leq \left| \int_0^1 \left(f(x) - \Psi(x)\right)\, dx \right| \\
&\quad + \left| \int_0^1 \Psi(x)\, dx - \frac{1}{M} \sum_{n=1}^{M} \Psi(x_n) \right| + \left| \frac{1}{M} \sum_{n=1}^{M} \left(\Psi(x_n) - f(x_n)\right) \right| \\
&\leq \|f - \Psi\|_\infty + \left| \int_0^1 \Psi(x)\, dx - \frac{1}{M} \sum_{n=1}^{M} \Psi(x_n) \right| + \|f - \Psi\|_\infty \\
&\leq 2\|f - \Psi\|_\infty + \sum_{h\in A} |c_h| \left| \int_0^1 e^{2\pi i h x}\, dx - \frac{1}{M} \sum_{n=1}^{M} e^{2\pi i h x_n} \right| \\
&= 2\|f - \Psi\|_\infty + \sum_{h\in A} |c_h| \left| \frac{1}{M} \sum_{n=1}^{M} e^{2\pi i h x_n} \right| \overset{(1.8),(1.9)}{<} \left(2 + \sum_{h\in A} |c_h|\right) \epsilon.
\end{aligned}$$

Since $\sum_{h\in A} |c_h| < \infty$ and ϵ was arbitrary, this is equivalent to (1.5). By Corollary 1.11 $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. ∎

The first consequence proved by Weyl [73] using its criterion was the distribution of the sequence $(n\theta)_{n=1}^{\infty}$, where θ is irrational.

Example 1.14. Let $\theta \in \mathbb{R}$ be irrational. The sequence $(n\theta)_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. For all $h \in \mathbb{Z} \setminus \{0\}$ and $N \in \mathbb{N}$, we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h n\theta}\right| = \frac{1}{N}\left|\frac{1-e^{2\pi i h(N+1)\theta}}{1-e^{2\pi i h\theta}} - 1\right| = \frac{1}{N}\left|\frac{e^{2\pi i h\theta} - e^{2\pi i h(N+1)\theta}}{1-e^{2\pi i h\theta}}\right|$$
$$\leq \frac{1}{N}\frac{2}{|1-e^{2\pi i h\theta}|} = \frac{1}{N}\sqrt{\frac{2}{1-\cos(2\pi h\theta)}} = \frac{1}{N}\frac{1}{|\sin(\pi h\theta)|}.$$

Notice that $\sin(\pi h\theta) \neq 0$ for all $h \in \mathbb{Z} \setminus \{0\}$ since θ is irrational. Thus $1/|\sin(\pi h\theta)|$ is bounded. Hence taking $N \to \infty$ we get (1.7). ∎

Note that Example 1.14 has been already proven by Bohl [9], Sierpinski [60] and Weyl [71] using an elementary argument, without involving Theorem 1.13.

Now we will give an example of a sequence which is not uniformly distributed. To prove it using Theorem 1.13, we will need a results which helps us computing exponential sums (see Appendix C, Theorem C.15).

Example 1.15. The sequence $(\log(n))_{n=1}^{\infty}$ is not uniformly distributed mod 1.

Proof. Let $N \in \mathbb{N}$ and define $F\colon [1,N] \to \mathbb{C}, t \mapsto e^{2\pi i \log t}$. Then $F \in \mathrm{C}^1([1,N],\mathbb{C})$ as composition of $\mathrm{C}^1([1,N],\mathbb{C})$-functions. By the Euler Summation Formula (see Appendix C, Corollary C.16) we have

$$\sum_{n=1}^{N} F(n) = \int_1^N F(t)\,dt + \frac{1}{2}\left(F(1)+F(N)\right) + \int_1^N \left(\{t\} - \frac{1}{2}\right) F'(t)\,dt. \tag{1.10}$$

We divide both sizes of (1.10) by N, and consider each term on the RHS separately. We compute the first term by applying integration by parts:

$$\frac{1}{N}\int_1^N F(t)\,dt = \frac{1}{N}\int_1^N e^{2\pi i \log t}\,dt = \frac{1}{N}\left[te^{2\pi i\log t}\right]_{t=1}^{N} - \frac{1}{N}\int_1^N te^{2\pi i\log t}\frac{2\pi i}{t}\,dt$$
$$= \frac{1}{N}\left(Ne^{2\pi i\log N} - 1\right) - \frac{2\pi i}{N}\int_1^N e^{2\pi i\log t}\,dt = \left(e^{2\pi i\log N} - \frac{1}{N}\right) - \frac{2\pi i}{N}\int_1^N F(t)\,dt.$$

Hence

$$\frac{1}{N}\int_1^N F(t)\,dt = \frac{e^{2\pi i\log N} - 1/N}{2\pi i + 1}.$$

For the second term in absolute value we have

$$\left|\frac{1}{2N}\left(F(1)-F(N)\right)\right| = \left|\frac{1}{2N}\left(1-e^{2\pi i\log N}\right)\right| \leq \frac{2}{2N} = \frac{1}{N}.$$

For the last term in absolute value we have

$$\left|\frac{1}{N}\int_1^N \left(\{t\}-\frac{1}{2}\right)F'(t)\,dt\right| = \left|\frac{1}{N}\int_1^N \left(\{t\}-\frac{1}{2}\right)e^{2\pi i\log t}\frac{2\pi i}{t}\,dt\right|$$
$$\leq \frac{1}{N}\int_1^N \left|\{t\}-\frac{1}{2}\right|\left|e^{2\pi i\log t}\right|\left|\frac{2\pi i}{t}\right|\,dt \leq \frac{1}{N}\int_1^N \frac{1}{2}\cdot 1\cdot\frac{2\pi}{|t|}\,dt$$
$$= \frac{\pi}{N}\int_1^N \frac{1}{t}\,dt = \frac{\pi}{N}\left[\log(t)\right]_{t=1}^{N} = \frac{\pi\log(N)}{N}.$$

Putting all together we get

$$\left|\frac{1}{N}\sum_{n=1}^{N} F(n)\right| \leq \left|\frac{e^{2\pi i \log(N)} - 1/N}{2\pi i + 1}\right| + \frac{1}{N} + \frac{\pi \log(N)}{N}. \tag{1.11}$$

Note that the second and third term on the RHS of (1.11) tend to 0 as $N \to \infty$, while the first term does not converge since

$$e^{2\pi i \log(N)} = \cos(2\pi \log(N)) + i \sin(2\pi \log(N)).$$

Hence (1.7) with $h = 1$ and $x_n = \log(n)$ does not hold in this case. ■

Even if $(\log(n))_{n=1}^{\infty}$ is not uniformly distributed mod 1, the set of fractional parts of this sequence is dense in $[0, 1]$, as shown in the following Remark.

Remark 1.16. The set $S \coloneqq \{\, \{\log(n)\} \mid n \in \mathbb{N} \,\}$ is dense in $[0, 1]$.

Proof. The the sequence $(\log(2^n))_{n=1}^{\infty}$ is a sub-sequence of the sequence $(\log(n))_{n=1}^{\infty}$. But note that $\log(2^n) = n \log(2)$ for all $n \in N$. Since $\log(2)$ is irrational (see Appendix A, Example A.9), then $(\log(2^n))_{n=1}^{\infty}$ is uniformly distributed mod 1 by Example 1.14. Hence $\{\, \{\log(2^n)\} \mid n \in \mathbb{N} \,\} \subset S$ is dense in $[0, 1]$ by Lemma 1.6. Therefore also S is dense in $[0, 1]$. ■

After reading Example 1.15, a natural question we can ask ourselves is what's the distribution of a sequence of the type $(\log(n)^r)_{n=1}^{\infty}$ for $r \in \mathbb{R}$. For this we state another strong result that gives us other examples of uniformly distributed mod 1 sequences. For the proof we'll need two elementary lemmas.

Lemma 1.17. *Let $u, v \in \mathbb{R}$. Then*

$$\left|e^{2\pi i u} - e^{2\pi i v} - 2\pi i (u - v) e^{2\pi i v}\right| \leq 2\pi^2 (u - v)^2. \tag{1.12}$$

Proof. Let $u, v \in \mathbb{R}$. Using integration by parts we get

$$\begin{aligned}
&\left|e^{2\pi i u} - e^{2\pi i v} - 2\pi i (u - v) e^{2\pi i v}\right| = \left|e^{2\pi i (u-v)} - 1 - 2\pi i (u - v)\right| \\
&= 4\pi^2 \left|\int_0^{u-v} (u - v - w) e^{2\pi i w}\, dw\right| \leq 4\pi^2 \int_0^{u-v} (u - v - w)\, dw = 2\pi^2 (u - v)^2.
\end{aligned}$$

This concludes the proof. ■

Lemma 1.18. *Let $g \colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$ be a monotone map. Then for all $N \in \mathbb{N}$*

$$\sum_{n=1}^{N-1} \left|\frac{1}{g(n)} - \frac{1}{g(n+1)}\right| \leq \frac{1}{|g(1)|} + \frac{1}{|g(N)|}. \tag{1.13}$$

Proof. Let $N \in \mathbb{N}$. We consider the following two cases.

- Assume g is monotone increasing. Then

$$\sum_{n=1}^{N-1}\left|\frac{1}{g(n)}-\frac{1}{g(n+1)}\right| = \sum_{n=1}^{N-1}\frac{g(n+1)-g(n)}{g(n)g(n+1)} = \sum_{n=1}^{N-1}\left(\frac{1}{g(n)}-\frac{1}{g(n+1)}\right)$$
$$= \frac{1}{g(1)}-\frac{1}{g(N)} \le \left|\frac{1}{g(1)}\right|+\left|\frac{1}{g(N)}\right|.$$

- Assume g is monotone decreasing. Then

$$\sum_{n=1}^{N-1}\left|\frac{1}{g(n)}-\frac{1}{g(n+1)}\right| = \sum_{n=1}^{N-1}\frac{g(n)-g(n+1)}{g(n)g(n+1)} = \sum_{n=1}^{N-1}\left(\frac{1}{g(n+1)}-\frac{1}{g(n)}\right)$$
$$= -\frac{1}{g(1)}+\frac{1}{g(N)} \le \left|\frac{1}{g(1)}\right|+\left|\frac{1}{g(N)}\right|.$$

This concludes the proof. ■

Notation 1.19. For all $n \in \mathbb{N}$ we denote: $\Delta x_n \coloneqq x_{n+1} - x_n$.

Now we are able to state the main result, proven by Van der Corput [65].

Theorem 1.20. *Let $f\colon \mathbb{N} \to \mathbb{R}$ be such that $\Delta f(n)$ is monotone. Assume that*

$$\lim_{n\to\infty} \Delta f(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} n|\Delta f(n)| = \infty. \tag{1.14}$$

Then the sequence $(f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. Let $h \in \mathbb{Z}\setminus\{0\}$ and $n \in \mathbb{N}\setminus\{0\}$ and define $u = hf(n+1)$ and $v = hf(n)$. Thus the formula (1.12) divided by $|\Delta f(n)|$ becomes

$$\left|\frac{e^{2\pi i h f(n+1)}}{\Delta f(n)} - \frac{e^{2\pi i h f(n)}}{\Delta f(n)} - 2\pi i h e^{2\pi i h f(n)}\right| \le 2\pi^2 h^2 |\Delta f(n)|. \tag{1.15}$$

Let $N \in \mathbb{N}$. Then we have the following.

$$\begin{aligned}
&\left|2\pi i h \sum_{n=1}^{N-1} e^{2\pi i h f(n)}\right| \\
&= \left|\sum_{n=1}^{N-1}\left(2\pi i h e^{2\pi i h f(n)} + \frac{e^{2\pi i h f(n)}}{\Delta f(n)} - \frac{e^{2\pi i h f(n+1)}}{\Delta f(n+1)}\right) + \frac{e^{2\pi i h f(N)}}{\Delta f(N)} - \frac{e^{2\pi i h f(1)}}{\Delta f(1)}\right| \\
&\le \sum_{n=1}^{N-1}\left|2\pi i h e^{2\pi i h f(n)} + \frac{e^{2\pi i h f(n)}}{\Delta f(n)} - \frac{e^{2\pi i h f(n+1)}}{\Delta f(n)}\right| \\
&\quad + \sum_{n=1}^{N-1}\left|\frac{e^{2\pi i h f(n+1)}}{\Delta f(n)} - \frac{e^{2\pi i h f(n+1)}}{\Delta f(n+1)}\right| + \left|\frac{e^{2\pi i h f(N)}}{\Delta f(N)} - \frac{e^{2\pi i h f(1)}}{\Delta f(1)}\right| \\
&\le \sum_{n=1}^{N-1}\left(2\pi^2 h^2 |\Delta f(n)| + \left|\frac{1}{\Delta f(n)} - \frac{1}{\Delta f(n+1)}\right|\right) + \frac{1}{|\Delta f(N)|} + \frac{1}{|\Delta f(1)|} \\
&\le 2\pi^2 h^2 \sum_{n=1}^{N-1} |\Delta f(n)| + \frac{2}{|\Delta f(N)|} + \frac{2}{|\Delta f(1)|}
\end{aligned}$$

where in the third step we have used (1.15), while in the last step we have used Lemma 1.18 with $g(n) \coloneqq \Delta f(n)$. Therefore we conclude

$$\left|\frac{1}{N}\sum_{n=1}^{N-1} e^{2\pi i h f(n)}\right| \leq \frac{\pi|h|}{N}\sum_{n=1}^{N-1}|\Delta f(n)| + \frac{1}{\pi|h|N}\left(\frac{1}{|\Delta f(N)|}+\frac{1}{|\Delta f(1)|}\right). \tag{1.16}$$

Now look at the RHS of (1.16). The second term goes to 0 as $N \to \infty$ by the second equation of (1.14). For the first term, by the first equation of (1.14), for each $\epsilon \in \mathbb{R}_{>0}$ and for N big enough we can find $\tilde{n} \in \mathbb{N}_{\leq N}$ such that $|\Delta f(n)| < \epsilon$ for all $n \in \mathbb{N}$ with $n \geq \tilde{n}$. Therefore

$$\frac{\pi|h|}{N}\sum_{n=1}^{N-1}|\Delta f(n)| \leq \frac{\pi|h|}{N}\left(\sum_{n=1}^{\tilde{n}-1}|\Delta f(n)| + (N-\tilde{n})\epsilon\right).$$

The last term goes to 0 as $N \to \infty$ and $\epsilon \to 0$. This gives exactly (1.7). Hence by Theorem 1.13 $(f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

A direct extension of Theorem 1.20 to continuous functions is given by the following corollary, which is known as Fejér Theorem.

Corollary 1.21 (Fejér Theorem)**.** *Let $x_0 \in \mathbb{R}_{\geq 1}$. Let $f\colon [1,\infty] \to \mathbb{R}$ be differentiable on $[x_0,\infty]$. Assume that f' is monotone and*

$$\lim_{x\to\infty} f'(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} x|f'(x)| = \infty. \tag{1.17}$$

Then the sequence $(f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. Let $n \in \mathbb{N}_{\geq x_0}$. By the mean value Theorem (see Appendix C, Theorem C.3) there exists $x \in (n, n+1)$ such that

$$f'(x) = \frac{f(n+1)-f(n)}{n+1-n} = f(n+1) - f(n) = \Delta f(n).$$

Hence by (1.17) we get

$$\lim_{n\to\infty}\Delta f(n) = \lim_{x\to\infty} f'(x) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} n|\Delta f(n)| = \lim_{x\to\infty} x|f'(x)| = \infty.$$

Moreover $\Delta f(n)$ is monotone since f' is monotone. Therefore the assumptions of Theorem 1.20 are satisfied, and thus $(f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

A direct proof of Corollary 1.21 is given by Pólya and Szegö [55]. With this result we can verify the uniform distribution of three other classes of sequences.

Example 1.22. The following sequences are uniformly distributed mod 1:

- $(an^{\sigma}(\log n)^{\tau})_{n=2}^{\infty}$ where $a \in \mathbb{R}\setminus\{0\}$, $\sigma \in (0,1)$ and $\tau \in \mathbb{R}$.
- $(a(\log n)^{\tau})_{n=1}^{\infty}$ where $a \in \mathbb{R}\setminus\{0\}$ and $\tau \in \mathbb{R}_{>1}$.
- $(an(\log n)^{\tau})_{n=2}^{\infty}$ where $a \in \mathbb{R}\setminus\{0\}$ and $\tau \in \mathbb{R}_{<0}$.

Indeed one can check the conditions of Corollary 1.21 for each of those examples. Computations are not hard, but a bit technical.

1.3 Special sequences

In this section we are interested in the sequence $(n\log(n))_{n=1}^{\infty}$. We will see that it falls into a class of sequences that can be analysed using a certain result we will prove. For the results of this section, in particular Theorem 1.28 and the previous three lemmas which are used to prove it, we refer to Van der Corput [65] and Kuipers and Niederreiter [44, pp. 15-18].

The first lemma is an upper bound for the integral of the exponential of functions which have first derivative "far from zero".

Lemma 1.23. *Let $\lambda \in \mathbb{R}_{>0}$, $a, b \in \mathbb{R}$ with $a < b$, and $f \in \mathrm{C}^1([a,b],\mathbb{R})$ such that f' is monotone and for all $x \in [a,b]$ we have $f'(x) \geq \lambda$ or $f'(x) \leq -\lambda$. Then*

$$\left| \int_a^b e^{2\pi i f(x)}\, dx \right| < \frac{1}{\lambda}. \tag{1.18}$$

Proof. Let $\lambda \in \mathbb{R}_{>0}$ and f as in the Lemma. Then

$$\begin{aligned}
\left| \int_a^b e^{2\pi i f(x)}\, dx \right| &= \left| \frac{1}{2\pi i} \int_a^b \frac{1}{f'(x)} \frac{d}{dx}\left[e^{2\pi i f(x)}\right]\, dx \right| \\
&= \frac{1}{2\pi} \left| \frac{1}{f'(a)} \int_a^{x_0} \frac{d}{dx}\left[e^{2\pi i f(x)}\right]\, dx + \frac{1}{f'(b)} \int_{x_0}^b \frac{d}{dx}\left[e^{2\pi i f(x)}\right]\, dx \right| \\
&= \frac{1}{2\pi} \left| \frac{1}{f'(a)} \left(e^{2\pi i f(x_0)} - e^{2\pi i f(a)}\right) + \frac{1}{f'(b)} \left(e^{2\pi i f(b)} - e^{2\pi i f(x_0)}\right) \right| \\
&\leq \frac{1}{2\pi} \left(\frac{2}{|f'(a)|} + \frac{2}{|f'(b)|} \right) \leq \frac{1}{2\pi} \left(\frac{2}{\lambda} + \frac{2}{\lambda} \right) = \frac{2}{\pi\lambda} < \frac{1}{\lambda}.
\end{aligned}$$

Note that the second step we have used the second mean value Theorem (see Appendix C, Theorem C.4), which guarantees the existence of such $x_0 \in (a,b)$. ■

The second lemma, similarly to the first, provides a bound for the integral of the exponential of functions with second derivative "far from zero". First we show the following elementary Remark.

Remark 1.24. Let $\rho \in \mathbb{R}_{>0}$. Then

$$\min_{t \in \mathbb{R}_{>0}} \left\{ t + \frac{1}{\rho t} \right\} = \frac{2}{\sqrt{\rho}}. \tag{1.19}$$

Proof. Let $g(t) = t + 1/(\rho t)$ where $t \in \mathbb{R}_{>0}$. Then

$$g'(t) = 1 + \frac{-1}{\rho^2 t^2}\rho = 1 - \frac{1}{\rho t^2}.$$

Hence $g'(t) = 0$ iff $t = 1/\sqrt{\rho}$. Moreover $g''(1/\sqrt{\rho}) = \sqrt{\rho} > 0$. Thus

$$\min_{t \in \mathbb{R}_{>0}} \left\{ t + \frac{1}{\rho t} \right\} = \frac{1}{\sqrt{\rho}} + \frac{1}{\rho/\sqrt{\rho}} = \frac{2}{\sqrt{\rho}}.$$

This concludes the proof. ■

Lemma 1.25. *Let $\rho \in \mathbb{R}_{>0}$, $a, b \in \mathbb{R}$ with $a < b$, and $f \in \mathrm{C}^2([a,b], \mathbb{R})$ such that for all $x \in [a,b]$ we have $f''(x) \geq \rho$ or $f''(x) \leq -\rho$. Then*

$$\left| \int_a^b e^{2\pi i f(x)} \, dx \right| < \frac{4}{\sqrt{\rho}}. \tag{1.20}$$

Proof. We can assume WLOG that $f'' \geq \rho$. Indeed, if $f'' \leq -\rho$, then we consider $-f$ instead of f for the rest of the proof, and this would not change the result since

$$\left| \int_a^b e^{2\pi i(-f(x))} \, dx \right| = \left| \int_a^b \overline{e^{2\pi i f(x)}} \, dx \right| = \left| \overline{\int_a^b e^{2\pi i f(x)} \, dx} \right| = \left| \int_a^b e^{2\pi i f(x)} \, dx \right|.$$

Hence, assuming that $f''(x) \geq \rho$ for all $x \in [a,b]$, we infer that f' is increasing. Let $c \in (a,b)$ and consider the following cases:

- Assume $f' \geq 0$. For $x \in [c,b]$, by the mean value Theorem (see Appendix C, Theorem C.3) there exists $z \in (a,x)$ such that

  $$f''(z) = \frac{f'(x) - f'(a)}{x - a}.$$

  Hence for all $x \in [c,b]$

  $$f'(x) = f''(z)(x-a) + f'(a) \geq \rho(x-a) + f'(a) \geq \rho(c-a).$$

  Then we use Lemma 1.23 with $\lambda \coloneqq \rho(c-a)$ and we get

  $$\left| \int_a^b e^{2\pi i f(x)} \, dx \right| \leq \left| \int_a^c e^{2\pi i f(x)} \, dx \right| + \left| \int_c^b e^{2\pi i f(x)} \, dx \right| < (c-a) + \frac{1}{\rho(c-a)}.$$

  Since c was arbitrary, we have

  $$\left| \int_a^b e^{2\pi i f(x)} \, dx \right| < \min_{c \in (a,b)} \left\{ (c-a) + \frac{1}{\rho(c-a)} \right\} \overset{(1.19)}{=} \frac{2}{\sqrt{\rho}}.$$

- Assume $f' < 0$. For $x \in [a,c]$, by the mean value Theorem (see Appendix C, Theorem C.3) there exists $z \in (x,b)$ such that

  $$f''(z) = \frac{f'(b) - f'(x)}{b - x}.$$

  Hence for all $x \in [a,c]$

  $$f'(x) = f'(b) - f''(z)(b-x) \leq f'(b) - \rho(b-x) < -\rho(b-c).$$

  Then we use Lemma 1.23 with $\lambda \coloneqq \rho(b-c)$ and we get

  $$\left| \int_a^b e^{2\pi i f(x)} \, dx \right| \leq \left| \int_a^c e^{2\pi i f(x)} \, dx \right| + \left| \int_c^b e^{2\pi i f(x)} \, dx \right| < \frac{1}{\rho(b-c)} + (b-c).$$

  Since c was arbitrary, we have

  $$\left| \int_a^b e^{2\pi i f(x)} \, dx \right| < \min_{c \in (a,b)} \left\{ \frac{1}{\rho(b-c)} + (b-c) \right\} \overset{(1.19)}{=} \frac{2}{\sqrt{\rho}}.$$

- Assume there is $d \in [a,b]$ such that $f'(x) < 0$ on $[a,d)$ and $f'(x) \geq 0$ on $[d,b]$. Let moreover $c_1 \in (a,d)$ and $c_2 \in (d,b)$. Using the previous points we have

$$\begin{aligned}\left|\int_a^b e^{2\pi i f(x)}\,dx\right| &\leq \left|\int_a^d e^{2\pi i f(x)}\,dx\right| + \left|\int_d^b e^{2\pi i f(x)}\,dx\right| \\ &< \frac{1}{\rho(d-c_1)} + (d-c_1) + (c_2-d) + \frac{1}{\rho(c_2-d)} \leq \frac{2}{\sqrt{\rho}} + \frac{2}{\sqrt{\rho}} = \frac{4}{\sqrt{\rho}}.\end{aligned}$$

This concludes the proof. ■

The last and more important lemma provides a bound for a quantity involving a function with bounded and monotone derivative. To prove it, we first show the following identity for exponential sums.

Remark 1.26. Let $h \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R} \setminus \mathbb{Z}$. Then

$$\sum_{k=0}^{h} e^{2\pi i k x} = \frac{\sin(\pi(h+1)x)}{\sin(\pi x)} e^{ih\pi x}.$$

Proof. Using that the sum is geometric we get

$$\sum_{k=0}^{h} e^{2\pi i k x} = \frac{1-e^{2\pi i(h+1)x}}{1-e^{2\pi i x}} = \frac{e^{-i(h+1)\pi x} - e^{i(h+1)\pi x}}{e^{-i\pi x} - e^{i\pi x}} e^{ih\pi x} = \frac{\sin(\pi(h+1)x)}{\sin(\pi x)} e^{ih\pi x}$$

and the proof is completed. ■

Lemma 1.27. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f \in \mathrm{C}^1([a,b],\mathbb{R})$ such that f' is monotone and $|f'(x)| \leq 1/2$ for all $x \in [a,b]$. Then*

$$\left|\int_a^b \left(\{x\} - \frac{1}{2}\right) f'(x) e^{2\pi i f(x)}\,dx\right| < \frac{1}{\pi}. \tag{1.21}$$

Proof. Let f be as in the Lemma. For all $x \in \mathbb{R} \setminus \mathbb{Z}$ we have

$$\{x\} = \frac{1}{2} - \frac{1}{\pi} \sum_{h=1}^{\infty} \frac{\sin(2\pi h x)}{h}. \tag{1.22}$$

For a proof see Appendix B, Example B.10. Let $m \geq 1$ and $x \in [a,b] \setminus \mathbb{Z}$, and consider the partial sum given by

$$\chi_m(x) \coloneqq -\sum_{h=1}^{m} \frac{\sin(2\pi h x)}{\pi h}.$$

We start by showing the following claim.

Claim: It holds: $\left|\int_a^b \chi_m(x) f'(x) e^{2\pi i f(x)}\,dx\right| \leq \frac{1}{\pi^2} \sum_{h=1}^{m} \frac{1}{h(h-1/2)}$.

Proof. The integral can be computed as follows

$$\begin{aligned}
&\int_a^b \chi_m(x) f'(x) e^{2\pi i f(x)}\,dx = \frac{1}{i\pi}\sum_{h=1}^{m}\frac{1}{h}\int_a^b -i\sin(2\pi hx) f'(x) e^{2\pi i f(x)}\,dx \\
&= \frac{1}{i\pi}\sum_{h=1}^{m}\frac{1}{h}\int_a^b -\sinh(2\pi i hx) f'(x) e^{2\pi i f(x)}\,dx \\
&= \frac{1}{2\pi i}\sum_{h=1}^{m}\frac{1}{h}\int_a^b \left(e^{-2\pi i hx} - e^{2\pi i hx}\right) f'(x) e^{2\pi i f(x)}\,dx \\
&= \frac{-1}{4\pi^2}\sum_{h=1}^{m}\frac{1}{h}\int_a^b \frac{f'(x)}{f'(x)-h}\frac{d}{dx}\left[e^{2\pi i(f(x)-hx)}\right]dx \\
&\qquad - \frac{-1}{4\pi^2}\sum_{h=1}^{m}\frac{1}{h}\int_a^b \frac{f'(x)}{f'(x)+h}\frac{d}{dx}\left[e^{2\pi i(f(x)+hx)}\right]dx
\end{aligned}$$

By the second mean value Theorem (Appendix C, Theorem C.4) there exists some $z \in (a,b)$ such that

$$\begin{aligned}
&\left|\int_a^b \frac{f'(x)}{f'(x)\pm h}\frac{d}{dx}\left[e^{2\pi i(f(x)\pm hx)}\right]dx\right| \\
&= \left|\frac{f'(a)}{f'(a)\pm h}\int_a^z \frac{d}{dx}\left[e^{2\pi i(f(x)\pm hx)}\right]dx + \frac{f'(b)}{f'(b)\pm h}\int_z^b \frac{d}{dx}\left[e^{2\pi i(f(x)\pm hx)}\right]dx\right| \\
&\le \frac{1/2}{h-1/2}\left|\int_a^z \frac{d}{dx}\left[e^{2\pi i(f(x)\pm hx)}\right]dx\right| + \frac{1/2}{h-1/2}\left|\int_z^b \frac{d}{dx}\left[e^{2\pi i(f(x)\pm hx)}\right]dx\right| \\
&= \frac{1/2}{h-1/2}\left(\left|e^{2\pi i(f(z)\pm hz)} - e^{2\pi i(f(a)\pm ha)}\right| + \left|e^{2\pi i(f(b)\pm hb)} - e^{2\pi i(f(z)\pm hz)}\right|\right) \\
&\le \frac{1/2}{h-1/2}(1+1+1+1) = \frac{2}{h-1/2}
\end{aligned}$$

where in the second step we have used that $f'/(f'\pm h)$ is monotone. Hence

$$\left|\int_a^b \chi_m(x) f'(x) e^{2\pi i f(x)}\,dx\right| \le \frac{1}{4\pi^2}\sum_{h=1}^{m}\frac{1}{h}\cdot 2\cdot\frac{2}{h-1/2} = \frac{1}{\pi^2}\sum_{h=1}^{m}\frac{1}{h(h-1/2)}.$$

This concludes the proof. □

Notice that the map $g(x) = \{x\} - 1/2$ is bounded, and thus $g \in \mathrm{L}^2([a,b]\setminus\mathbb{Z},\mathbb{R})$. For all square-integrable functions $h \in \mathrm{L}^2([a,b]\setminus\mathbb{Z},\mathbb{C})$ we have

$$|\langle\chi_m,h\rangle - \langle g,h\rangle| = |\langle\chi_m - g,h\rangle| \le \|\chi_m - g\|_{\mathrm{L}^2([a,b]\setminus\mathbb{Z},\mathbb{R})}\|h\|_{\mathrm{L}^2([a,b]\setminus\mathbb{Z},\mathbb{R})} \tag{1.23}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in L^2. Note that in the second step we have used Cauchy-Schwarz inequality (Appendix C, Theorem C.14). Now using (1.22) and Riesz–Fischer Theorem (Theorem B.13 in Appendix B), we conclude that χ_m converges in $\mathrm{L}^2([a,b]\setminus\mathbb{Z},\mathbb{R})$ to g. In particular (1.23) implies

$$\lim_{m\to\infty}\langle\chi_m,h\rangle = \langle g,h\rangle \tag{1.24}$$

for all $h \in \mathrm{L}^2([a,b] \setminus \mathbb{Z}, \mathbb{C})$. Therefore

$$\left| \int_a^b \left(\{x\} - \frac{1}{2} \right) f'(x) e^{2\pi i f(x)} \, dx \right| = \lim_{m \to \infty} \left| \int_a^b \chi_m(x) f'(x) e^{2\pi i f(x)} \, dx \right|$$
$$\leq \lim_{m \to \infty} \frac{1}{\pi^2} \sum_{h=1}^{m} \frac{1}{h(h-1/2)} = \frac{1}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h(h-1/2)} = \frac{\log(16)}{\pi^2} < \frac{1}{\pi}.$$

where in the first step we have used (1.24) with $h(x) = f'(x)e^{2\pi i f(x)}$, and the second step we have used the Claim. ■

The trick now is to use the previous lemmas to find a bound for the exponential sum given by Weyl Criterion, which we can transform into integrals thanks to the Euler Summation Formula (see Appendix C, Corollary C.16).

Theorem 1.28. *Let $\rho \in \mathbb{R}_{>0}$, $a, b \in \mathbb{Z}$ with $a < b$, and $f \in \mathrm{C}^2([a,b], \mathbb{R})$ such that for all $x \in [a,b]$ we have $f''(x) \geq \rho$ or $f''(x) \leq -\rho$. Then*

$$\left| \sum_{n=a}^{b} e^{2\pi i f(n)} \right| \leq (|f'(b) - f'(a)| + 2) \left(\frac{4}{\sqrt{\rho}} + 3 \right). \tag{1.25}$$

Proof. For $p \in \mathbb{Z}$ let $A_p \coloneqq \{ n \in \mathbb{N} \mid n \in (a,b) \quad \text{and} \quad p - 1/2 \leq f'(n) < p + 1/2 \}$. Now we define the quantity

$$S_p \coloneqq \sum_{n \in A_p} e^{2\pi i f(n)}. \tag{1.26}$$

Let $p \in \mathbb{Z}$ such that $S_p \neq 0$. Since f'' is bounded, we infer that f' is monotone. Hence S_p is a sum over consecutive values of n, say from $n = a_p$ to $n = b_p$. Consider the map $F_p(x) = f(x) - px$. By the Euler Summation Formula (see Appendix C, Corollary C.16) we get

$$\begin{aligned} |S_p| = \left| \sum_{n=a_p}^{b_p} e^{2\pi i f(n)} \right| &= \left| \sum_{n=a_p}^{b_p} e^{2\pi i F_p(n)} \right| \leq \left| \int_{a_p}^{b_p} e^{2\pi i F_p(x)} \, dx \right| \\ &\quad + \left| \frac{1}{2} \left(e^{2\pi i F_p(a_p)} + e^{2\pi i F_p(b_p)} \right) \right| + \left| 2\pi i \int_{a_p}^{b_p} \left(\{x\} - \frac{1}{2} \right) F_p'(x) e^{2\pi i F_p(x)} \, dx \right| \\ &< \frac{4}{\sqrt{\rho}} + \frac{1}{2}(1+1) + 2\pi \frac{1}{\pi} = \frac{4}{\sqrt{\rho}} + 3 \end{aligned}$$

where in the last inequality we have used Lemma 1.25 and Lemma 1.27 (the assumption is satisfied since $|F_p'(x)| \leq 1/2$). Since there are at most $|f'(b) - f'(a)| + 2$ values of p for which S_p is a non-empty sum, we have

$$\left| \sum_{n=a}^{b} e^{2\pi i f(n)} \right| = \left| \sum_{p \in \mathbb{Z}} S_p \right| \leq \sum_{p \in \mathbb{Z}} |S_p| < (|f'(b) - f'(a)| + 2) \left(\frac{4}{\sqrt{\rho}} + 3 \right).$$

This concludes the proof. ■

The previous theorem has been generalized by Van der Corput himself in many of his other papers (see [66, 68, 67, 69]), and also by Koksma [39].

We can now answer our question at the beginning of this section.

Example 1.29. The sequence $(n\log(n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. Let $N \in \mathbb{N}$ and $h \in \mathbb{Z} \setminus \{0\}$. Define $f\colon [1,N] \to \mathbb{R}, x \mapsto hx\log(x)$. Then $f \in \mathrm{C}^2([1,N],\mathbb{R})$ and for all $x \in [1,N]$ we have

$$f'(x) = h\,(1+\log(x)) \qquad \text{and} \qquad f''(x) = \frac{h}{x}.$$

Hence $|f''(x)| \geq |h|/N$ for all $x \in [1,N]$. By Theorem 1.28 we get

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h n\log(n)}\right| &\leq \frac{1}{N}\left(|h(1+\log(N)) - h(1+\log(1))| + 2\right)\left(\frac{4}{\sqrt{|h|/N}} + 3\right)\\ &= \frac{1}{N}\left(|h|\log(N) + 2\right)\left(\frac{4\sqrt{N}}{\sqrt{|h|}} + 3\right).\end{aligned}$$

The last expression tends to 0 as $N \to \infty$. Hence by Theorem 1.13 the sequence $(n\log(n))_{n=1}^{\infty}$ is uniformly distributed mod 1. ∎

More in general we have

Example 1.30. Let $a \in \mathbb{R} \setminus \{0\}$ and $\tau \in \mathbb{R}_{>0}$. Then the sequence $(an\log(n)^{\tau})_{n=2}^{\infty}$ is uniformly distributed mod 1.

Other examples of sequence of this type are the following.

Example 1.31. The sequence $(n\log(\log(n)))_{n=2}^{\infty}$ is uniformly distributed mod 1.

Proof. Let $N \in \mathbb{N}_{\geq 3}$ and $h \in \mathbb{Z} \setminus \{0\}$. Define $f\colon [3,N] \to \mathbb{R}, x \mapsto hx\log(\log(x))$. Then $f \in \mathrm{C}^2([3,N],\mathbb{R})$ and for all $x \in [3,N]$ we have

$$f'(x) = h\left(\log(\log(x)) + \frac{1}{\log(x)}\right) \qquad \text{and} \qquad f''(x) = \frac{h}{x\log(x)^2}\left(\log(x) - 1\right).$$

Since $f''(x)$ is monotone decreasing for some sufficiently large x, if N is big enough we get that $|f''(x)| \geq |h|(\log(N)-1)/\left(N\log(N)^2\right)$ for all $x \in [3,N]$. Hence by Theorem 1.28 we get

$$\left|\frac{1}{N}\sum_{n=2}^{N} e^{2\pi i h n\log(\log(n))}\right| \leq \frac{1}{N} + \frac{|h|}{N}\mathcal{O}(\log(\log(N)))\left(\frac{4\sqrt{N\log(N)^2}}{\sqrt{|h|(\log(N)-1)}} + 3\right).$$

The last expression tends to 0 as $N \to \infty$. Hence by Theorem 1.13 the sequence $(n\log(\log(n)))_{n=2}^{\infty}$ is uniformly distributed mod 1. ∎

As last example, consider the following.

Example 1.32. Let $a \in \mathbb{R} \setminus \{0\}$ and $\sigma \in (1,2)$. Then the sequence $(an^\sigma)_{n=1}^\infty$ is uniformly distributed mod 1.

Proof. Let $N \in \mathbb{N}$ and $h \in \mathbb{Z} \setminus \{0\}$. Define $f\colon [1,N] \to \mathbb{R}, x \mapsto hax^\sigma$. Then $f \in \mathrm{C}^2([1,N],\mathbb{R})$ and for all $x \in [1,N]$ we have

$$f'(x) = ha\sigma x^{\sigma-1} \qquad \text{and} \qquad f''(x) = ha\sigma(\sigma-1)x^{\sigma-2}.$$

Hence $|f''(x)| \geq |ha|\sigma(\sigma-1)N^{\sigma-2}$ for all $x \in [1,N]$. By Theorem 1.28 we get

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i han^\sigma}\right| &\leq \frac{1}{N}\left(\left|ha\sigma N^{\sigma-1} - ha\sigma 1^{\sigma-1}\right| + 2\right)\left(\frac{4}{\sqrt{|ha|\sigma(\sigma-1)N^{\sigma-2}}} + 3\right) \\ &= \frac{1}{N}\left(|ha|\,\sigma\left(N^{\sigma-1}-1\right) + 2\right)\left(\frac{4\sqrt{N^{2-\sigma}}}{\sqrt{|ha|\sigma(\sigma-1)}} + 3\right).\end{aligned}$$

The last expression tends to 0 as $N \to \infty$ since $\sigma \in (1,2)$ and

$$\lim_{N\to\infty} \frac{N^{\sigma-1}\sqrt{N^{2-\sigma}}}{N} = \lim_{N\to\infty} \frac{\sqrt{N^{2-\sigma}}}{N^{2-\sigma}} = \lim_{N\to\infty} \frac{1}{\sqrt{N^{2-\sigma}}} = 0.$$

Hence by Theorem 1.13 the sequence $(an^\sigma)_{n=1}^\infty$ is uniformly distributed mod 1. ■

1.4 Difference Theorems

There are some notable results that help us extend the examples from the previous section into more complex sequences. For example, instead of $(n\log(n))_{n=1}^\infty$, we can consider the sequence $(n^2\log(n))_{n=1}^\infty$, or even $(n^k\log(n)^\tau)_{n=1}^\infty$ some some general k and τ. For this purpose we present some difference theorems in this section. All of them are proved by Van der Corput [69, 70] and are reported in the book of Kuipers and Niederreiter [44, pp. 25-30].

The first Theorem follows from the following famous inequality.

Lemma 1.33 (Van der Corput's Fundamental Inequality)**.** *Let $N, H \in \mathbb{N}$ such that $1 \leq H \leq N$ and $u_1, \ldots, u_N \in \mathbb{C}$. Then*

$$H^2\left|\sum_{n=1}^{N} u_n\right|^2 \leq (N+H-1)\left[H\sum_{n=1}^{N}|u_n|^2 + 2\sum_{h=1}^{H-1}(H-h)\,\mathrm{Re}\left(\sum_{n=1}^{N-h} u_n\overline{u}_{n+h}\right)\right] \quad (1.27)$$

Proof. First define $u_n \coloneqq 0$ for all $n \in \mathbb{Z} \setminus \{1,\ldots,N\}$. Then

$$H\sum_{n=1}^{N} u_n = \sum_{p=1}^{N+H-1}\sum_{h=0}^{H-1} u_{p-h}. \quad (1.28)$$

This can be seen more concretely by writing down each element of the sum on the RHS. By Cauchy-Schwarz inequality (see Appendix C, Theorem C.14) we have

$$H^2\left|\sum_{n=1}^{N} u_n\right|^2 \overset{(1.28)}{=} \left|\sum_{p=1}^{N+H-1}\sum_{h=0}^{H-1} u_{p-h}\right|^2 \leq \left(\sum_{p=1}^{N+H-1}\left|\sum_{h=0}^{H-1} u_{p-h}\right|^2\right)\left(\sum_{p=1}^{N+H-1}|1|^2\right)$$
$$= (N+H-1)\sum_{p=1}^{N+H-1}\left|\sum_{h=0}^{H-1} u_{p-h}\right|^2 = (N+H-1)\sum_{p=1}^{N+H-1}\left(\sum_{r=0}^{H-1} u_{p-r}\right)\left(\sum_{s=0}^{H-1}\overline{u}_{p-s}\right)$$
$$= (N+H-1)\left[\sum_{p=1}^{N+H-1}\sum_{h=0}^{H-1}|u_{p-h}|^2 + 2\,\mathrm{Re}\left(\sum_{p=1}^{N+H-1}\sum_{\substack{r,s=0\\ s<r}}^{H-1} u_{p-r}\overline{u}_{p-s}\right)\right].$$

For the last quantity, look at the two terms inside the square brackets. By (1.28) we can write the first term as

$$\sum_{p=1}^{N+H-1}\sum_{h=0}^{H-1}|u_{p-h}|^2 = H\sum_{n=1}^{N}|u_n|^2.$$

For the second term, instead, let $n \coloneqq p-r$. Then we are summing over terms of the form $u_n\overline{u}_{n+h}$, where $n \in \{1,\dots,N\}$ and $h \coloneqq r-s \in \{1,\dots,H-1\}$. Hence fixing n and h, we can count exactly $H-h$ possible choices for (r,s), namely: $(h,0),(h+1,1),\dots,(H-1,H-h-1)$. For each of those choices we have a different p, since p is uniquely determined by r. Hence

$$\sum_{p=1}^{N+H-1}\sum_{\substack{r,s=0\\ s<r}}^{H-1} u_{p-r}\overline{u}_{p-s} = \sum_{h=1}^{H-1}(H-h)\sum_{n=1}^{N} u_n\overline{u}_{n+h} = \sum_{h=1}^{H-1}(H-h)\sum_{n=1}^{N-h} u_n\overline{u}_{n+h}.$$

Putting all together we get exactly (1.27). ■

Theorem 1.34 (Van der Corput Difference Theorem)**.** *It holds*

$$(x_{n+h}-x_n)_{n=1}^{\infty} \quad u.d.\ mod\ 1 \quad \forall h\in\mathbb{N} \quad \Longrightarrow \quad (x_n)_{n=1}^{\infty} \quad u.d.\ mod\ 1. \tag{1.29}$$

Proof. Let $m \in \mathbb{Z}\backslash\{0\}$. We apply Lemma 1.33 to the complex number $u_n \coloneqq e^{2\pi i m x_n}$. Dividing (1.27) by H^2N^2 we get

$$\frac{1}{N^2}\left|\sum_{n=1}^{N} u_n\right|^2 \leq \frac{N+H-1}{H^2N^2}\left[HN + 2\sum_{h=1}^{H-1}(H-h)\,\mathrm{Re}\left(\sum_{n=1}^{N-h} e^{2\pi i m(x_n-x_{n+h})}\right)\right]$$
$$\leq \frac{N-H+1}{HN} + 2\sum_{h=1}^{H-1}\frac{(H-h)(N-H+1)(N-h)}{H^2N^2}\left|\frac{1}{N-h}\sum_{n=1}^{N-h} e^{2\pi i m(x_n-x_{n+h})}\right|$$

Since $(x_{n+h}-x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 for all $h \in \{1,\dots,H-1\}$, by Theorem 1.13 we have

$$\lim_{N\to\infty}\left|\frac{1}{N-h}\sum_{n=1}^{N-h} e^{2\pi i m(x_n-x_{n+h})}\right| = 0 \qquad \forall h \in \{1,\dots,H-1\}.$$

Since $(H-h)(N-H+1)(N-h)/(H^2N^2)$ tends to $(H-h)/H^2$ as $N \to \infty$ and the sum over h is finite, we get

$$\overline{\lim_{N\to\infty}} \frac{1}{N^2} \left| \sum_{n=1}^{N} u_n \right|^2 \leq \lim_{N\to\infty} \frac{N-H+1}{HN} = \frac{1}{H}.$$

But since this last inequality must holds for all integers H, we get exactly (1.7). Hence by Theorem 1.13 the sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

Remark 1.35. The converse of Theorem 1.34 is not true.

Proof. Let $\theta \in \mathbb{R}$ be irrational. The sequence $(n\theta)_{n=1}^{\infty}$ is uniformly distributed by Example 1.14. However, for all $h \in \mathbb{N}$ we have $((n+h)\theta - n\theta)_{n=1}^{\infty} = (h\theta)_{n=1}^{\infty}$, which is a constant sequence, and thus not uniformly distributed. ■

As application, consider the following example.

Example 1.36. The sequence $(n^2 \log(n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. Let $h, N \in \mathbb{N}$. Define the functions $f\colon [1,N] \to \mathbb{R}, x \mapsto x^2 \log(x)$ and $g\colon [1,N] \to \mathbb{R}, x \mapsto f(x+h) - f(x)$. Then for all $x \in [1,N]$ it holds

$$g'(x) = 2x \log\left(\frac{x+h}{x}\right) + 2h \log(x+h) + h \qquad \text{and} \qquad g''(x) = 2\log\left(\frac{x+h}{x}\right).$$

Since g'' is monotone decreasing, $|g''(x)| \geq 2\log\left((N+h)/N\right)$ for all $x \in [1,N]$. Hence by Theorem 1.28 we have

$$\begin{aligned}
\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h g(n)} \right| &\leq \frac{1}{N} \left(|g'(N) - g'(1)| + 2 \right) \left(4 \left(2 \log\left(\frac{N+h}{N} \right) \right)^{-1/2} + 3 \right) \\
&= \frac{2}{N} \left(N \log\left(\frac{N+h}{N} \right) + h \log(N+h) + c_h \right) \left(\left(\frac{1}{8} \log\left(\frac{N+h}{N} \right) \right)^{-1/2} + 3 \right) \\
&= \frac{2}{\sqrt{N}} \left(N \log\left(\frac{N+h}{N} \right) + h \log(N+h) + c_h \right) \left(\left(\frac{N}{8} \log\left(\frac{N+h}{N} \right) \right)^{-1/2} + \frac{3}{\sqrt{N}} \right)
\end{aligned}$$

where $c_h \in \mathbb{R}$ is a constant that depends on h. Note that the last expression tends to 0 as $N \to \infty$ since

$$\lim_{N\to\infty} N \log\left(\frac{N+h}{N} \right) = \log\left(\lim_{N\to\infty} \left(1 + \frac{h}{N} \right)^N \right) = \log(e^h) = h.$$

Hence by Theorem 1.13 the sequence $(g(n))_{n=1}^{\infty}$ is uniformly distributed mod 1. By Theorem 1.34 we conclude that $(f(n))_{n=1}^{\infty}$ is also uniformly distributed mod 1. ■

With the same idea one can show the following more general result.

Example 1.37. Let $a \in \mathbb{R} \setminus \{0\}$ and $\tau \in (0,1]$. Then the sequence $\left(an^2 \log(n)^{\tau}\right)_{n=2}^{\infty}$ is uniformly distributed mod 1.

Moreover we have also

Example 1.38. The sequence $\left(n^2 \log(\log(n))\right)_{n=2}^{\infty}$ is uniformly distributed mod 1.

Another interesting result concerns sequences whose difference between two consecutive elements approaches an irrational number in the long run. For this, we first need to show the following elementary lemma.

Lemma 1.39. *For all $u, v \in \mathbb{R}$ it holds: $\left|e^{2\pi i u} - e^{2\pi i v}\right| \leq 2\pi|u - v|$.*

Proof. If $u = v$, the inequality is obvious (it is indeed an equality). Therefore we can assume WLOG that $v < u$. Then

$$\begin{aligned}\left|e^{2\pi i u} - e^{2\pi i v}\right| &= \left|e^{2\pi i(u-v)} - 1\right| = \left|2\pi i \int_0^{u-v} e^{2\pi i w}\, dx\right| \\ &\leq 2\pi \int_0^{u-v} \left|e^{2\pi i w}\right| \, dx = 2\pi \int_0^{u-v} dx = 2\pi(u - v).\end{aligned}$$

This concludes the proof. ∎

Theorem 1.40. *Assume that there exists $\theta \in \mathbb{R}$ irrational such that*

$$\lim_{n\to\infty} x_{n+1} - x_n = \theta.$$

Then $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. By assumption, for all $\epsilon \in \mathbb{R}_{>0}$ there exists some $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq m$ we have $|\Delta x_n - \theta| \leq \epsilon$. Choose $\epsilon := 1/q^2$ where $q \in \mathbb{N}$. Then for all $g, n \in \mathbb{N}$ with $n > g \geq m$, it holds

$$|x_n - x_g - (n - g)\theta| = \left|\sum_{j=g}^{n-1} (\Delta x_j - \theta)\right| \leq \sum_{j=g}^{n-1} |\Delta x_j - \theta| \leq \frac{n-g}{q^2}.$$

Let $h \in \mathbb{Z} \setminus \{0\}$. Using Lemma 1.39 we infer

$$\left|e^{2\pi i h x_n} - e^{2\pi i h(x_g + (n-g)\theta)}\right| \leq 2\pi|h| \, |x_n - x_g - (n-g)\theta| \leq 2\pi|h| \frac{n-g}{q^2}.$$

Hence

$$\begin{aligned}\left|\sum_{n=g}^{g+q-1} e^{2\pi i h x_n}\right| &\leq \left|\sum_{n=g}^{g+q-1} \left(e^{2\pi i h x_n} - e^{2\pi i h(x_g+(n-g)\theta)}\right)\right| + \left|\sum_{n=g}^{g+q-1} e^{2\pi i h(x_g+(n-g)\theta)}\right| \\ &\leq \frac{2\pi|h|}{q^2} \sum_{n=g}^{g+q-1} (n-g) + \left|\sum_{n=g}^{g+q-1} e^{2\pi i h(x_g+(n-g)\theta)}\right| \\ &= \frac{2\pi|h|}{q^2} \frac{(q-1)q}{2} + \left|\sum_{n=0}^{q-1} e^{2\pi i h(x_g+n\theta)}\right| \leq \pi|h| + \frac{1}{|\sin(\pi h\theta)|}\end{aligned}$$

where the last step follows by the proof of Example 1.14. Now rename the last quantity as $K \coloneqq \pi|h| + 1/|\sin(\pi h\theta)|$. Note that K is finite since θ is irrational. Therefore, for all $H \in \mathbb{N}$ we have

$$\left|\sum_{n=g}^{g-1+Hq} e^{2\pi i h x_n}\right| = \left|\sum_{k=0}^{H-1}\sum_{n=g+kq}^{g+(k+1)q-1} e^{2\pi i h x_n}\right| = \left|\sum_{k=0}^{H-1}\sum_{n=g}^{g+q-1} e^{2\pi i h x_{n+kq}}\right|$$
$$\leq \sum_{k=0}^{H-1}\left|\sum_{n=g}^{g+q-1} e^{2\pi i h x_{n+kq}}\right| \leq H \max_{k\in\{0,\dots,H-1\}}\left|\sum_{n=g}^{g+q-1} e^{2\pi i h x_{n+kq}}\right| \leq HK.$$

Let $N \in \mathbb{N}$ such that $N \geq g+q$. Then there is $H \in \mathbb{N}$ such that

$$g - 1 + Hq < N \leq g - 1 + (H+1)q.$$

Hence, putting together what we have found, we get

$$\left|\sum_{n=1}^{N} e^{2\pi i h x_n}\right| \leq \left|\sum_{n=1}^{g-1} e^{2\pi i h x_n}\right| + \left|\sum_{n=g}^{g-1+Hq} e^{2\pi i h x_n}\right| + \left|\sum_{n=g+Hq}^{N} e^{2\pi i h x_n}\right|$$
$$\leq (g-1) + HK + q \leq (g-1) + \frac{N-g}{q}K + q.$$

Therefore

$$\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right| \leq \frac{K}{q}.$$

Since $q \in \mathbb{N}$ was arbitrary, this is equivalent to (1.7). Hence by Theorem 1.13 the sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

As application of Theorem 1.40, consider the following two examples.

Example 1.41. Let $\theta \in \mathbb{R}$ be irrational. Then the sequence $(n\theta + \sin(2\pi\sqrt{n}))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. For $n \in \mathbb{N}$ let $x_n \coloneqq n\theta + \sin(2\pi\sqrt{n})$. Using the subtraction formula for the sine function we get

$$\begin{aligned}\lim_{n\to\infty} x_{n+1} - x_n &= \lim_{n\to\infty} (n+1)\theta + \sin\left(2\pi\sqrt{n+1}\right) - n\theta - \sin\left(2\pi\sqrt{n}\right)\\ &= \theta + \lim_{n\to\infty}\cos\left(\pi\left(\sqrt{n+1}+\sqrt{n}\right)\right)\sin\left(\pi\left(\sqrt{n+1}-\sqrt{n}\right)\right)\\ &= \theta + \lim_{n\to\infty}\cos\left(\pi\left(\sqrt{n+1}+\sqrt{n}\right)\right)\sin\left(\pi\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \theta + \lim_{n\to\infty}\cos\left(\pi\left(\sqrt{n+1}+\sqrt{n}\right)\right)\sin\left(\frac{\pi}{\sqrt{n+1}+\sqrt{n}}\right) = \theta.\end{aligned}$$

Hence $(n\theta + \sin(2\pi\sqrt{n}))_{n=1}^{\infty}$ is uniformly distributed mod 1 by Theorem 1.40. ■

Example 1.42. Consider the sequence of Fibonacci numbers, defined as

$$F_n \coloneqq \begin{cases} 1 & \text{for} \quad n \in \{1,2\} \\ F_{n-1} + F_{n-2} & \text{for} \quad n \in \mathbb{N}_{\geq 3} \end{cases} \tag{1.30}$$

Then the sequence $(\log(F_n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. For $n \in \mathbb{N}$ let $x_n \coloneqq F_{n+1}/F_n$. Moreover let $\phi \coloneqq (1+\sqrt{5})/2$ be the golden ratio. First we prove the following statement.

Claim: $|x_n - \phi| \leq 1/\phi^n$ for all $n \in \mathbb{N}$.

Proof. We will prove the statement proceeding by induction on n.

- *Base*: For $n = 1$, we have: $|x_1 - \phi| = |1 - \phi| = \phi - 1 = 1/\phi$.
  For $n = 2$, we have: $|x_2 - \phi| = |2 - \phi| = 2 - \phi = 1/\phi^2$.
- *Hypothesis*: Assume that $|x_n - \phi| \leq 1/\phi^n$ holds for some $n \in \mathbb{N}$.
- *Step*: Let $n \in \mathbb{N}$ be as in the hypothesis. Then

$$\begin{aligned} |x_{n+1} - \phi| &= \left| \frac{F_{n+2}}{F_{n+1}} - \phi \right| = \left| \frac{F_{n+1} + F_n}{F_{n+1}} - \phi \right| = \left| 1 + \frac{1}{x_n} - \phi \right| \\ &= \left| \frac{1}{x_n} - \frac{1}{\phi} \right| = \frac{|x_n - \phi|}{x_n \phi} \leq \frac{|x_n - \phi|}{\phi} \leq \frac{1}{\phi^n \phi} = \frac{1}{\phi^{n+1}} \end{aligned}$$

  where in the first inequality we have used that F_n is increasing in n, while in the second inequality we have used the induction hypothesis. This concludes the induction step. □

By this claim we infer that $\lim_{n\to\infty} F_{n+1}/F_n = \phi$. Therefore

$$\lim_{n\to\infty} \log(F_{n+1}) - \log(F_n) = \lim_{n\to\infty} \log\left(\frac{F_{n+1}}{F_n}\right) = \log(\phi).$$

Since $\log(\phi)$ is an irrational number (see Appendix A, Example A.9), we conclude that $(\log(F_n))_{n=1}^{\infty}$ is uniformly distributed mod 1 by Theorem 1.40. ■

Furthermore, Van der Corput extended Fejér Theorem (Theorem 1.21) to enlarge the set of uniformly distributed sequences in the following way.

Notation 1.43. For $k \in \mathbb{N}_{\geq 1}$ we define recursively the operator Δ^k by

$$\Delta^k x_n \coloneqq \begin{cases} \Delta x_n & \text{for} \quad k = 1 \\ \Delta\left(\Delta^{k-1} x_n\right) & \text{for} \quad k \geq 2 \end{cases} \tag{1.31}$$

where $\Delta x_n \coloneqq x_{n+1} - x_n$ as in Notation 1.19.

Theorem 1.44. *Let $k \in \mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{R}$ be such that $\Delta^k f(n)$ is monotone. Assume that*

$$\lim_{n\to\infty} \Delta^k f(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} n|\Delta^k f(n)| = \infty. \tag{1.32}$$

Then the sequence $(f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. We will prove the statement proceeding by induction on k.

- *Base*: For $k = 1$, the Theorem follows from Theorem 1.20.
- *Hypothesis*: Assume that the Theorem holds for some $k \in \mathbb{N}$.
- *Step*: Assume that $\Delta^{k+1} f(n)$ is monotone and

$$\lim_{n\to\infty} \Delta^{k+1} f(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} n|\Delta^{k+1} f(n)| = \infty. \tag{1.33}$$

  Let now $h \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we have

$$\Delta^k \left(f(n+h) - f(n)\right) = \Delta^k \left(\sum_{j=0}^{h-1} \Delta f(n+j)\right) = \sum_{j=0}^{h-1} \Delta^{k+1} f(n+j).$$

  Using this we can infer that

  - ▶ Since $\Delta^{k+1} f(n)$ is monotone, then also $\Delta^k \left(f(n+h) - f(n)\right)$ is monotone.
  - ▶ By (1.33) (first equation) we have that $\lim_{n\to\infty} \Delta^k \left(f(n+h) - f(n)\right) = 0$.
  - ▶ By (1.33) (first equation) and since $\Delta^{k+1} f(n)$ is monotone, $\Delta^{k+1} f(n)$ must be of constant sign. Hence by (1.33) (second equation) we have

$$\lim_{n\to\infty} n|\Delta^k \left(f(n+h) - f(n)\right)| = \lim_{n\to\infty} n \sum_{j=0}^{h-1} \left|\Delta^{k+1} f(n+j)\right| = \infty.$$

  Therefore, by the induction hypothesis $(f(n+h) - f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1. But then also $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 by Theorem 1.34. This concludes the induction step. ■

In a similar way we can extend Corollary 1.21.

Theorem 1.45 (Extended Fejér Theorem)**.** *Let $x_0 \in \mathbb{R}_{\geq 1}, k \in \mathbb{N}$ and $f\colon [1,\infty] \to \mathbb{R}$ such that $f \in \mathrm{C}^k([x_0,\infty],\mathbb{R})$. Assume that $f^{(k)}$ is monotone and*

$$\lim_{x\to\infty} f^{(k)}(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} x|f^{(k)}(x)| = \infty. \tag{1.34}$$

Then the sequence $(f(n))_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. We will prove the statement proceeding by induction on k.

- *Base*: For $k = 1$, the Theorem follows from Corollary 1.21.
- *Hypothesis*: Assume that the Theorem holds for some $k \in \mathbb{N}$.
- *Step*: Assume that $f^{(k+1)}$ is monotone and

$$\lim_{x\to\infty} f^{(k+1)}(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} x|f^{(k+1)}(x)| = \infty. \tag{1.35}$$

Let now $h \in \mathbb{N}$ and for all $x \in \mathbb{R}_{\geq 1}$ define the function $g_h(x) \coloneqq f(x+h) - f(x)$. By the main value Theorem (see Appendix C, Theorem C.3) there exists some $y \in (x, x+h)$ such that

$$g_h^{(k)}(x) = f^{(k)}(x+h) - f^{(k)}(x) = hf^{(k+1)}(y).$$

Hence $g_h^{(k)}$ is monotone, and by (1.35) we have

$$\lim_{x\to\infty} g_h^{(k)}(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} x|g_h^{(k)}(x)| = \infty.$$

Hence $(g_h)_{n=1}^{\infty}$ is uniformly distributed mod 1 by the induction hypothesis. But now, using Theorem 1.34, we infer that $(x_n)_{n=1}^{\infty}$ is also uniformly distributed mod 1. This concludes the induction step. ■

This allows us to find other classes of sequences that are uniformly distributed mod 1. In particular, we can generalize Example 1.22.

Example 1.46. The following sequences are uniformly distributed mod 1:

- $(an^{\sigma})_{n=2}^{\infty}$ where $a \in \mathbb{R} \setminus \{0\}$ and $\sigma \in \mathbb{R}_{>0} \setminus \mathbb{Z}$.
- $(an^{\sigma}(\log n)^{\tau})_{n=2}^{\infty}$ where $a \in \mathbb{R} \setminus \{0\}$, $\sigma \in \mathbb{R}_{>0} \setminus \mathbb{Z}$ and $\tau \in \mathbb{R}$.
- $(an^{k}(\log n)^{\tau})_{n=2}^{\infty}$ where $a \in \mathbb{R} \setminus \{0\}$, $k \in \mathbb{N}$ and $\tau \in \mathbb{R} \setminus [0,1]$.

Chapter 2

Discrepancy

In the study of sequences, in addition to understanding whether a sequence is uniformly distributed or not, one can also ask how close it actually is to perfect uniformity. In particular, in this chapter we will introduce the concept of *discrepancy*, i.e. the maximal difference between the proportion of points in an arbitrary sub-interval $[a, b)$ compared to the full unit interval and the length of $[a, b)$. Therefore, the lower the discrepancy, the closer the sequence will be to a uniform sequence.

The concept was first studied by Bergström [6, 1935], who used the term *Intensitätdispersion*. The actual discrepancy is cited first by Van der Corput, who developed an extensive study of discrepancy with Pisot in the paper [64, 1939]. Note that this topic also concerns uniformly distributed sequences, which can also surprisingly be far from an ideal sequence. It all depends on the discrepancy, which, in the case of uniformly distributed sequences, helps us understand how quickly the limit in (1.2) converges to the length of the sub-interval. In this way we can evaluate how irregular the elements of a sequence are, and this gives us the opportunity to qualitatively analyze the progress of a sequence.

The main goal of this chapter is to find good upper and lower bounds for the discrepancy, which allow us to then analyze individual sequences. For this chapter we fix a sequence of real numbers $\omega = (x_n)_{n=1}^{\infty}$ and an integer $N \in \mathbb{N}$.

2.1 Basic definitions

For this section, we refer to Weyl [73] for Theorem 2.3, and to Van der Corput and Pisot [64] for Theorem 2.4. See also Kuipers and Niederreiter [44, pp. 88-91]. Moreover, for Theorem 2.15 see Bergström [7].

The main idea is to truncate the sequence $(x_n)_{n=1}^{\infty}$ and analyze the position of the first N elements, in terms of how much they deviate from an ideal sequence.

Definition 2.1. The *discrepancy* of the sequence $\omega = (x_n)_{n=1}^{\infty}$ is the number

$$D_N \coloneqq D_N(\omega) \coloneqq \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta), N)}{N} - (\beta - \alpha) \right|. \tag{2.1}$$

In most cases, it is difficult to find an exact value for the discrepancy. Therefore, we will focus more on the behavior of D_N as function of N. For this reason, the following notation will be useful.

Notation 2.2. Let $E \subseteq \mathbb{R}_{>0}$ be unbounded. Consider two maps $f\colon E \to \mathbb{C}$ and $g\colon E \to \mathbb{R}$ such that $g > 0$. Then we denote

- $f(x) = \mathcal{O}(g(x)) \iff \exists c, x_0 > 0$ such that $\forall x \geq x_0 : |f(x)| \leq cg(x)$.
- $f(x) = \Omega(g(x)) \iff \exists c, x_0 > 0$ such that $\forall x \geq x_0 : |f(x)| \geq cg(x)$.
- $f(x) = \Theta(g(x)) \iff f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$.

For our computations we often consider $f\colon \mathbb{N} \to \mathbb{R}_{>0}, N \to D_N$.

A first result that we can intuit from Definition 2.1 is the following.

Theorem 2.3. *The sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 if and only if $\lim_{N\to\infty} D_N = 0$.*

Proof. $\Longleftarrow$: Assume that $\lim_{N\to\infty} D_N = 0$. Then for all $0 \leq \alpha < \beta \leq 1$ we have

$$\left| \frac{A([\alpha, \beta), N)}{N} - (\beta - \alpha) \right| \leq D_N.$$

If $N \to \infty$, we get that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

$\Longrightarrow$: Assume that $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. Let $m \in \mathbb{N}_{\geq 2}$ and define the partition of $[0, 1)$ given by

$$P \coloneqq \{ I_k \coloneqq [k/m, (k+1)/m) \mid k \in \{0, \dots, m-1\} \}.$$

By (1.2) there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}_{\geq M}$ and $I_k \in P$ we have

$$\frac{1}{m}\left(1 - \frac{1}{m}\right) \leq \frac{A(I_k, N)}{N} \leq \frac{1}{m}\left(1 + \frac{1}{m}\right) \tag{2.2}$$

Let $J \coloneqq [\alpha, \beta) \subseteq [0, 1)$ be an arbitrary sub-interval. Then define the numbers

$$\begin{aligned}
\alpha_1 &\coloneqq \min \{ k \in \{0, \dots, m-1\} \mid k/m \geq \alpha \} \\
\beta_1 &\coloneqq \max \{ k \in \{0, \dots, m-1\} \mid k/m \leq \beta \} \\
\alpha_2 &\coloneqq \max \{ k \in \{0, \dots, m-1\} \mid k/m \leq \alpha \} \\
\beta_2 &\coloneqq \min \{ k \in \{0, \dots, m-1\} \mid k/m \geq \beta \}
\end{aligned}$$

Now we construct two sub-intervals $J_1, J_2 \subseteq [0, 1)$ as follows:

$$J_1 \coloneqq \bigsqcup_{k=\alpha_1}^{\beta_1 - 1} I_k = \left[\frac{\alpha_1}{m}, \frac{\beta_1}{m}\right) \qquad J_2 \coloneqq \bigsqcup_{k=\alpha_2}^{\beta_2 - 1} I_k = \left[\frac{\alpha_2}{m}, \frac{\beta_2}{m}\right).$$

For instance, if $m = 5$, we have the following situation.

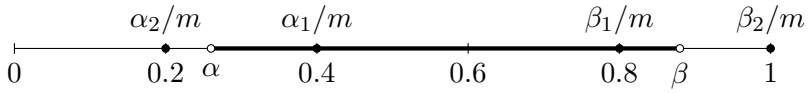

Figure 2.1: Example of J_1 and J_2 for the case $m = 5$.

Then $J_1 \subseteq J \subseteq J_2$ and

$$\lambda(J) - \lambda(J_1) = (\beta - \alpha) - \left(\frac{\beta_1}{m} - \frac{\alpha_1}{m}\right) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m}.$$
$$\lambda(J_2) - \lambda(J) = \left(\frac{\beta_2}{m} - \frac{\alpha_2}{m}\right) - (\beta - \alpha) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m}.$$

Hence using (2.2), for all $N \in \mathbb{N}_{\geq M}$ we have

$$\left(\lambda(J) - \frac{2}{m}\right)\left(1 - \frac{1}{m}\right) < \lambda(J_1)\left(1 - \frac{1}{m}\right) = \sum_{k=\alpha_1}^{\beta_1 - 1} \lambda(I_k)\left(1 - \frac{1}{m}\right)$$
$$\leq \sum_{k=\alpha_1}^{\beta_1 - 1} \frac{A(I_k, N)}{N} = \frac{A(J_1, N)}{N} \leq \frac{A(J, N)}{N} \leq \frac{A(J_2, N)}{N} = \sum_{k=\alpha_2}^{\beta_2 - 1} \frac{A(I_k, N)}{N}$$
$$\leq \sum_{k=\alpha_2}^{\beta_2 - 1} \lambda(I_k)\left(1 + \frac{1}{m}\right) = \lambda(J_2)\left(1 + \frac{1}{m}\right) < \left(\lambda(J) + \frac{2}{m}\right)\left(1 + \frac{1}{m}\right).$$

But since $\lambda(J) \leq 1$, we get

$$-\frac{3}{m} + \frac{2}{m^2} < \frac{A(J, N)}{N} - \lambda(J) < \frac{3}{m} + \frac{2}{m^2}.$$

Therefore $D_N < 3/m + 2/m^2$ for all $N \in \mathbb{N}_{\geq M}$. Since $m \in \mathbb{N}_{\geq 2}$ was arbitrary, we can choose $m := N + 1$. Then: $\lim_{N\to\infty} D_N = 0$. ■

The next result shows two simple first bounds for the discrepancy.

Theorem 2.4. *It holds:* $1/N \leq D_N \leq 1$.

Proof. For the second inequality let $[\alpha, \beta) \subseteq [0, 1)$ and note that

$$(a) \quad \frac{A([\alpha, \beta), N)}{N} - (\beta - \alpha) \leq \frac{A([\alpha, \beta), N)}{N} \leq \frac{N}{N} = 1.$$
$$(b) \quad (\beta - \alpha) - \frac{A([\alpha, \beta), N)}{N} \leq \beta - \alpha \leq 1.$$

Taking the supremum over all sub-intevals $[\alpha, \beta) \subseteq [0, 1)$ we get $D_N \leq 1$.

To prove the first inequality, consider a real number $\epsilon > 0$ and an arbitrary element x of the sequence $(x_n)_{n=1}^{\infty}$. Define $J := [\{x\}, \{x\} + \epsilon) \cap [0, 1)$. Then

$$D_N \geq \frac{A(J, N)}{N} - \lambda(J) \geq \frac{1}{N} - \lambda(J) \geq \frac{1}{N} - \epsilon.$$

Taking $\epsilon \to 0$, we get $D_N \geq 1/N$. ■

As mentioned, one of our main goals is to find good bounds for the discrepancy. Therefore we want to establish how good the bounds of Theorem 2.4 are. The following two examples show that the two bounds can not be improved.

Example 2.5. The upper bound 1 in Theorem 2.4 is the best possible.

Proof. Consider a sequence such that the first N terms are all zero. Then

$$\sup_{0\le\alpha<\beta\le 1}\left(\frac{A([\alpha,\beta),N)}{N}-(\beta-\alpha)\right)\ge\lim_{\epsilon\to 0}\left(\frac{A([0,\epsilon),N)}{N}-\epsilon\right)=\frac{N}{N}=1.$$

Hence $D_N\ge 1$. By Theorem 2.4 it holds $D_N=1$. ∎

Example 2.6. The lower bound $1/N$ in Theorem 2.4 is the best possible.

Proof. Consider a sequence such that the first N terms are $0,1/N,\dots,(N-1)/N$ in some arbitrary order. Let $[\alpha,\beta)\subseteq[0,1)$ be a sub-interval. Then there exists a unique $k\in\{0,1,\dots,N-1\}$ such that $k/N<\lambda([\alpha,\beta))\le(k+1)/N$. This means that $[\alpha,\beta)$ contains exactly k or $k+1$ elements of ω. Hence

$$\left|\frac{A([\alpha,\beta),N)}{N}-\lambda\left([\alpha,\beta)\right)\right|\le\frac{1}{N}.$$

Since $[\alpha,\beta)$ was arbitrary, we can also take the supremum on the LHS, and the inequality still holds. Thus $D_N\le 1/N$. By Theorem 2.4 it holds $D_N=1/N$. ∎

To count the elements within a sub-interval of $[0,1)$, it is useful to define the *star discrepancy*, for which we only consider sub-intervals of the type $J=[0,\alpha)$, and it will be easier to compute $A(J,N)$.

Definition 2.7. The *star-discrepancy* of the sequence $\omega=(x_n)_{n=1}^{\infty}$ is the number

$$D_N^\star\coloneqq D_N^\star(\omega)\coloneqq\sup_{0<\alpha\le 1}\left|\frac{A([0,\alpha),N)}{N}-\alpha\right|. \tag{2.3}$$

As application, consider the following example.

Example 2.8. Consider a sequence such that the first N terms are given by: $0,1^2/N^2,2^2/N^2,\dots,(N-1)^2/N^2$. Then: $\lim_{N\to\infty}D_N^\star=1/4$.

Proof. For $\alpha\in(0,1]$ we compute

$$\begin{aligned}A([0,\alpha),N)&=|\{\,n\in\{1,\dots,N\}\mid n^2/N^2<\alpha\,\}|\\&=|\{\,n\in\{1,\dots,N\}\mid n<N\sqrt{\alpha}\,\}|=\left[N\sqrt{\alpha}\,\right].\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{N\to\infty}D_N^\star&=\lim_{N\to\infty}\sup_{0<\alpha\le 1}\left|\frac{[N\sqrt{\alpha}\,]}{N}-\alpha\right|=\lim_{N\to\infty}\sup_{0<\alpha\le 1}\left|\frac{N\sqrt{\alpha}-\{N\sqrt{\alpha}\,\}}{N}-\alpha\right|\\&=\lim_{N\to\infty}\sup_{0<\alpha\le 1}\left|\sqrt{\alpha}-\alpha-\frac{\{N\sqrt{\alpha}\,\}}{N}\right|=\sup_{0<\alpha\le 1}\left|\sqrt{\alpha}-\alpha\right|\end{aligned}$$

where in the last step we have used that $f_N(\alpha) = \sqrt{\alpha} - \alpha - \{N\sqrt{\alpha}\}/N$ converges uniformly to $f(\alpha) = \sqrt{\alpha} - \alpha$, and therefore for all $\epsilon \in \mathbb{R}_{>0}$ there is $\tilde{n} \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq \tilde{n}$ it holds

$$\left| \sup_{\alpha\in(0,1]} f_n(\alpha) - \sup_{\alpha\in(0,1]} f(\alpha) \right| < \sup_{\alpha\in(0,1]} |f_n(\alpha) - f(\alpha)| < \epsilon.$$

Hence we can interchange limit and supremum. Now notice that

$$f'(\alpha) = 0 \quad \iff \quad \frac{1}{2\sqrt{\alpha}} - 1 = 0 \quad \iff \quad \alpha = \frac{1}{4}.$$

Moreover $f''(1/4) = -2 < 0$. Hence $\alpha = 1/4$ is a maximum of f. Therefore

$$\lim_{N\to\infty} D_N^\star = \sup_{0<\alpha\leq 1} \left|\sqrt{\alpha} - \alpha\right| = \sqrt{\frac{1}{4}} - \frac{1}{4} = \frac{1}{4}.$$

This concludes the proof. ■

With the same idea we can also prove the next example.

Example 2.9. Let $k \in \mathbb{N}$. Consider a sequence such that the first N terms are given by: $0, 1^k/N^k, 2^k/N^k, \ldots, (N-1)^k/N^k$. Then: $\lim_{k\to\infty} \lim_{N\to\infty} D_N^\star = 1$.

Proof. For $\alpha \in (0, 1]$ we compute

$$\begin{aligned} A([0,\alpha), N) &= |\{\, n \in \{1,\ldots,N\} \mid n^k/N^k < \alpha \,\}| \\ &= |\{\, n \in \{1,\ldots,N\} \mid n < N\sqrt[k]{\alpha} \,\}| = \left[N\sqrt[k]{\alpha}\right]. \end{aligned}$$

Therefore, as in Example 2.8, we have

$$\lim_{N\to\infty} D_N^\star = \sup_{0<\alpha\leq 1} \left|\sqrt[k]{\alpha} - \alpha\right|.$$

Now let $f(\alpha) \coloneqq \sqrt[k]{\alpha} - \alpha$. Then

$$f'(\alpha) = 0 \quad \iff \quad \frac{1}{k}\alpha^{(1-k)/k} - 1 = 0 \quad \iff \quad \alpha = k^{k/(1-k)}.$$

Moreover $f''(k^{k/(1-k)}) < 0$ if $k \geq 1$. Hence $\alpha = k^{k/(1-k)}$ is a maximum of f. Therefore

$$\lim_{k\to\infty} \lim_{N\to\infty} D_N^\star = \lim_{k\to\infty} \sup_{0<\alpha\leq 1} \left|\sqrt[k]{\alpha} - \alpha\right| = \lim_{k\to\infty} \left|k^{1/(1-k)} - k^{k/(1-k)}\right| = 1.$$

This concludes the proof. ■

The next Theorem shows how D_N and $D_N^\star$ are connected and therefore justifies that it is sufficient for us to work with $D_N^\star$ to compute discrepancies.

Theorem 2.10. *It holds:* $D_N^\star \leq D_N \leq 2D_N^\star$.

Proof. The first inequality follows by looking at the definitions:

$$\sup_{0<\alpha\leq 1}\left|\frac{A([0,\alpha),N)}{N}-\alpha\right|\leq\sup_{0\leq\alpha<\beta\leq 1}\left|\frac{A([\alpha,\beta),N)}{N}-(\beta-\alpha)\right|.$$

For the second inequality, let $[\alpha,\beta)\subseteq[0,1)$ be a sub-interval. Then

$$\begin{aligned}\left|\frac{A([\alpha,\beta),N)}{N}-(\beta-\alpha)\right|&=\left|\frac{A([0,\beta),N)-A([0,\alpha),N)}{N}-(\beta-\alpha)\right|\\&\leq\left|\frac{A([0,\beta),N)}{N}-\beta\right|+\left|\frac{A([0,\alpha),N)}{N}-\alpha\right|.\end{aligned}$$

Taking the supremum over all sub-intervals $[\alpha,\beta)\subseteq[0,1)$ on both sides we infer that $D_N\leq 2D_N^\star$. This concludes the proof. ■

Corollary 2.11. *The sequence $(x_n)_{n=1}^\infty$ is uniformly distributed mod 1 if and only if $\lim_{N\to\infty}D_N^\star=0$.*

Proof. $\Longrightarrow$: Assume that $(x_n)_{n=1}^\infty$ is uniformly distributed mod 1. Since $D_N^\star\leq D_N$ by Theorem 2.10, then $\lim_{N\to\infty}D_N^\star=0$ by Theorem 2.3.

$\Longleftarrow$: Assume that $\lim_{N\to\infty}D_N^\star=0$. Since $D_N\leq 2D_N^\star$ by Theorem 2.10, then $(x_n)_{n=1}^\infty$ is uniformly distributed mod 1 by Theorem 2.3. ■

Regarding the bounds for $D_N^\star$, we infer the following.

Corollary 2.12. *It holds: $1/(2N)\leq D_N^\star\leq 1$.*

Proof. This follows directly from Theorem 2.10 and Theorem 2.4. ■

As we made for Theorem 2.4 (see Examples 2.5 and 2.6), in a similar way we can show directly that the two bounds in Corollary 2.12 can not be improved. However, in the next section we will be able to show it in an easier way (see Example 2.19).

As last thing, we consider a sequence ω that can be divided into sub-sequences. This helps to compute D_N or $D_N^\star$ by computing the discrepancy of the sub-sequences.

Definition 2.13. Let $k\in\mathbb{N}$. Assume that there exist sub-sequences $\omega_1,\ldots,\omega_k$ of ω such that one can obtain the sequence ω by listing the elements of $\omega_1,\ldots,\omega_k$ in some order. Then ω is called a *superposition* of $\omega_1,\ldots,\omega_k$.

Notation 2.14. Let $k\in\mathbb{N}$ and ω be a superposition $\omega_1,\ldots,\omega_k$. Moreover we write $N=N_1+\cdots+N_k$ where $N_1,\ldots N_k\in\mathbb{N}$. Then we define the set

$$\mathrm{SP}_k\coloneqq\bigcup_{i=1}^k\left\{x_j^{(i)}\in\omega_i\mid j\leq N_i\right\}. \tag{2.4}$$

Theorem 2.15. *Let $k\in\mathbb{N}$ and $\omega_1,\ldots,\omega_k$ be sub-sequences of ω with discrepancies $D_{N_1},\ldots,D_{N_k}$. Set $N=N_1+\cdots+N_k$, and assume that ω is a superposition of $\omega_1,\ldots,\omega_k$ such that $x_n\in\mathrm{SP}_k$ for all $n\in\mathbb{N}_{\leq N}$. Then it holds*

$$D_N\leq\sum_{i=1}^k\frac{N_i}{N}D_{N_i}\qquad\text{and}\qquad D_N^\star\leq\sum_{i=1}^k\frac{N_i}{N}D_{N_i}^\star. \tag{2.5}$$

Proof. Let $[\alpha, \beta) \subseteq [0,1)$. Since $N = N_1 + \cdots + N_k$, we have

$$\begin{aligned} \left| \frac{A([\alpha,\beta), N, \omega)}{N} - (\beta - \alpha) \right| &= \left| \sum_{i=1}^{k} \frac{A([\alpha,\beta), N_i, \omega_i)}{N} - (\beta - \alpha) \right| \\ = \left| \sum_{i=1}^{k} \frac{N_i}{N} \left(\frac{A([\alpha,\beta), N_i, \omega_i)}{N_i} - (\beta - \alpha) \right) \right| &\leq \sum_{i=1}^{k} \frac{N_i}{N} D_{N_i}(\omega_i). \end{aligned} \tag{2.6}$$

Taking the supremum over all $[\alpha, \beta) \subseteq [0,1)$ on the LHS of (2.6) we get the first inequality of (2.5). The second inequality follows doing the same computations. ■

With Theorem 2.15 we can give a simpler proof of Example 1.12 by computing its discrepancy, which will vanish as $N \to \infty$.

Example 2.16. As in Example 1.12, let $(x_n)_{n=1}^{\infty}$ be the sequence

$$\frac{0}{1};\quad \frac{0}{2},\ \frac{1}{2};\quad \frac{0}{3},\ \frac{1}{3},\ \frac{2}{3};\quad \ldots;\quad \frac{0}{k},\ \frac{1}{k},\ \ldots,\ \frac{k-1}{k};\quad \ldots$$

Then $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. Let $k \in \mathbb{N}$ and $N := \sum_{i=1}^{k} i$. Let's cut $\omega = (x_n)_{n=1}^{\infty}$ and consider only its first N terms, that is up to $x_N = (k-1)/k$. For $i \in \{1, \ldots, k\}$ define ω_i as a sequence such that the first i terms are given by $0/i, 1/i, \ldots, (i-1)/i$. By Example 2.6, we have that $D_{N_i}(\omega_i) = 1/i$ for all $i \in \{1, \ldots, k\}$. Therefore, by Theorem 2.15 we have

$$D_N(\omega) \leq \sum_{i=1}^{k} \frac{N_i}{N} D_{N_i}(\omega_i) = \sum_{i=1}^{k} \frac{i}{N} \frac{1}{i} = \frac{k}{N}.$$

But note that $N = k(k+1)/2$, and thus $k = (\sqrt{1+8N} - 1)/2$. Hence

$$D_N(\omega) \leq \frac{\sqrt{1+8N} - 1}{2N} = \frac{4}{\sqrt{1+8N} + 1} \leq \frac{\sqrt{2}}{\sqrt{N}}.$$

In particular $D_N = \mathcal{O}(1/\sqrt{N})$. Hence, for $N \to \infty$, we get $D_N \to 0$. Therefore ω is uniformly distributed mod 1 by Theorem 2.3. ■

2.2 An explicit estimate for the discrepancy

As can be understood from the definition, the order of the first N elements of the sequence is not important for the discrepancy. Therefore, assuming that $\omega \subseteq [0,1)$, we can order the elements $x_1, \ldots, x_N$ from smallest to largest in order to obtain a subdivision of $[0,1)$. Using this we can obtain a nice representation of $D_N^{\star}$.

As references, for Theorem 2.17 and Corollary 2.18 see Niederreiter [48], in which it is also presented a multidimensional version. For Definition 2.21 see O'Neil [53], and for Theorem 2.22 see Niederreiter [47].

Theorem 2.17. *Assume that* $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N < 1$. *Then*

$$D_N^\star = \max_{i\in\{1,\dots,N\}} \max\left\{\left|x_i - \frac{i}{N}\right|, \left|x_i - \frac{i-1}{N}\right|\right\}. \tag{2.7}$$

Proof. Denote first $x_0 = 0$ and $x_{N+1} = 1$. Then $(x_i)_{i=0}^{N+1}$ is a subdivision of $[0,1]$. Therefore we can write

$$D_N^\star = \max_{\substack{i\in\{0,\dots,N\}\\ x_i<x_{i+1}}} \sup_{x_i<\alpha\leq x_{i+1}} \left|\frac{A([0,\alpha),N)}{N} - \alpha\right| = \max_{\substack{i\in\{0,\dots,N\}\\ x_i<x_{i+1}}} \sup_{x_i<\alpha\leq x_{i+1}} \left|\frac{i}{N} - \alpha\right|.$$

For $i \in \{0,\dots,N+1\}$ consider the function $g_i\colon (x_i, x_{i+1}] \to \mathbb{R}$, $\alpha \mapsto |i/N - \alpha|$. This map has a supremum at one of the two endpoints of $(x_i, x_{i+1}]$. Therefore

$$D_N^\star = \max_{\substack{i\in\{0,\dots,N\}\\ x_i<x_{i+1}}} \max\left\{\left|\frac{i}{N} - x_i\right|, \left|\frac{i}{N} - x_{i+1}\right|\right\}. \tag{2.8}$$

It remains to show that we can drop the restriction $x_i < x_{i+1}$ in the first maximum of (2.8). For this fix $i \in \{0,\dots,N+1\}$ and assume that there is $r \in \{2,\dots,N-i\}$ such that $x_i < x_{i+1} = x_{i+2} = \cdots = x_{i+r} < x_{i+r+1}$. Let $j \in \{1,\dots,r-1\}$. We need to prove that the numbers

$$\left|\frac{i+j}{N} - x_{i+j}\right| \quad \text{and} \quad \left|\frac{i+j}{N} - x_{i+j+1}\right|$$

are dominated by terms that appear inside the two maxima of (2.8). As above, the map $h_i\colon (i/N, (i+r)/N) \to \mathbb{R}$, $y \mapsto |y - x_{i+1}|$ has a supremum at one of the two endpoints of $(i/N, (i+r)/N)$. Hence

$$\begin{aligned}\left|\frac{i+j}{N} - x_{i+j}\right| = \left|\frac{i+j}{N} - x_{i+1}\right| &< \max\left\{\left|\frac{i}{N} - x_{i+1}\right|, \left|\frac{i+r}{N} - x_{i+1}\right|\right\}\\ &= \max\left\{\left|\frac{i}{N} - x_{i+1}\right|, \left|\frac{i+r}{N} - x_{i+r}\right|\right\}.\end{aligned}$$

Note that both numbers in the last maximum occur in (2.8). One can show the same for $|(i+j)/N - x_{i+j+1}|$ since $x_{i+j+1} = x_{i+1} = x_{i+r}$. Therefore (2.8) becomes

$$\begin{aligned}D_N^\star &= \max_{i\in\{0,\dots,N\}} \max\left\{\left|\frac{i}{N} - x_i\right|, \left|\frac{i}{N} - x_{i+1}\right|\right\}\\ &= \max_{i\in\{1,\dots,N\}} \max\left\{\left|\frac{i}{N} - x_i\right|, \left|\frac{i-1}{N} - x_i\right|\right\}.\end{aligned}$$

This concludes the proof. ∎

An other similar interpretation of $D_N^\star$, which involves only one maximum, is given in the following result.

Corollary 2.18. *Assume that* $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N < 1$. *Then*

$$D_N^\star = \frac{1}{2N} + \max_{i\in\{1,\dots,N\}} \left|x_i - \frac{2i-1}{2N}\right|. \tag{2.9}$$

Proof. For all $i \in \{1, \dots, N\}$ we have

$$\begin{aligned}\frac{1}{2N} + \left| x_i - \frac{2i-1}{2N} \right| &= \frac{1}{2N} + \max\left\{ -x_i + \frac{2i-1}{2N}\,,\, x_i - \frac{2i-1}{2N} \right\} \\ &= \max\left\{ -x_i + \frac{i}{N}\,,\, x_i - \frac{i-1}{N} \right\} = \max\left\{ \left| x_i - \frac{i}{N} \right|\,,\, \left| x_i - \frac{i-1}{N} \right| \right\}.\end{aligned}$$

Taking the maximum over all $i \in \{1, \dots, N\}$ on both sides and using Theorem 2.17, we get exactly the identity in (2.9). ∎

With Theorem 2.17 and Corollary 2.18, it's also easy to prove that the two bounds we have found for $D_N^\star$ can not be improved.

Example 2.19. The two bounds of $D_N^\star$ given in Corollary 2.12 are the best possible.

Proof. Consider sequences such that the first N terms are

- all zero. Using Theorem 2.17 we get

$$D_N^\star = \max_{i \in \{1,\dots,N\}} \max\left\{ \left| \frac{i}{N} \right|, \left| \frac{i-1}{N} \right| \right\} = \max_{i \in \{1,\dots,N\}} \left| \frac{i}{N} \right| = \frac{N}{N} = 1.$$

- given by $1/(2N), 3/(2N), \dots, (2N-1)/(2N)$. Using Corollary 2.18 we get

$$D_N^\star = \frac{1}{2N} + \max_{i \in \{1,\dots,N\}} \left| \frac{2i-1}{N} - \frac{2i-1}{2N} \right| = \frac{1}{2N}.$$

This concludes the proof. ∎

With Corollary 2.18 we can show the following example.

Example 2.20. Let $(x_n)_{n=1}^\infty$ be a sequence given by

$$\frac{1}{2};\quad \frac{1}{4},\quad \frac{3}{4};\quad \frac{1}{6},\quad \frac{3}{6},\quad \frac{5}{6};\quad \dots;\quad \frac{1}{2k},\quad \frac{3}{2k},\quad \dots,\quad \frac{2k-1}{2k};\quad \dots.$$

Then $D_N = \Theta(1/\sqrt{N})$.

Proof. Let $k \in \mathbb{N}$ and $N \coloneqq \sum_{i=1}^k i$. We have to prove two directions.

- To show: $D_N = \mathcal{O}(1/\sqrt{N})$. Let's cut ω and consider only its first N terms, that is up to $x_N = (2k-1)/(2k)$. Now, for each $i \in \{1, \dots, k\}$ define ω_i as a sequence such that the first i terms are given by $1/(2i), 3/(2i), \dots, (2i-1)/(2i)$. By Example 2.19, we have that $D_{N_i}^\star(\omega_i) = 1/(2N)$ for all $i \in \{1, \dots, k\}$. Therefore, by Theorem 2.15 we have

$$D_N^\star(\omega) \le \sum_{i=1}^k \frac{N_i}{N} D_{N_i}^\star(\omega_i) = \sum_{i=1}^k \frac{i}{N} \frac{1}{2i} = \frac{k}{2N}.$$

  But note that $N = k(k+1)/2$, and thus $k = (\sqrt{1+8N} - 1)/2$. Hence

$$D_N^\star(\omega) \le \frac{\sqrt{1+8N} - 1}{4N} \le \frac{1}{\sqrt{2N}}.$$

  By Theorem 2.10, it follows $D_N \le \sqrt{2}/\sqrt{N}$. In particular $D_N = \mathcal{O}(1/\sqrt{N})$.

- To show: $D_N = \Omega(1/\sqrt{N})$. By Corollary 2.18 we have

$$D_N^\star = \frac{1}{2N} + \max_{i\in\{1,\dots,N\}} \left| x_i - \frac{2i-1}{2N} \right| \geq \frac{1}{2N} + \left| x_1 - \frac{2\cdot 1 - 1}{2N} \right|$$
$$= \frac{1}{2N} + \left| \frac{1}{2k} - \frac{1}{2N} \right| = \frac{1}{2k} = \frac{1}{\sqrt{1+8N}-1} \geq \frac{1}{3\sqrt{N}}.$$

  By Theorem 2.10, it follows $D_N \geq 1/(3\sqrt{N})$. In particular $D_N = \Omega(1/\sqrt{N})$.

Summarizing, we get exactly: $D_N = \Theta(1/\sqrt{N})$. ■

In the next example we use Theorem 2.17 to compute the discrepancy of an "almost arithmetic" sequence, that is a sequence for which $x_{n+1} - x_n$ approaches a fixed value, but with a small perturbation. We will see that the discrepancy changes depending on how large the perturbation is.

Definition 2.21. Assume that $N \geq 2$ and $x_1 < \cdots < x_N$. Let $\delta \in [0,1)$ and $\epsilon > 0$ be real numbers such that there exists $\eta \in (0, \epsilon]$ with

(*a*) $0 \leq x_1 \leq \eta + \delta\eta$.

(*b*) For $n \in \{1, \dots, N-1\}$ we have: $\eta - \delta\eta \leq x_{n+1} - x_n \leq \eta + \delta\eta$.

(*c*) $1 - \eta - \delta\eta \leq x_N < 1$.

Then the sequence $(x_n)_{n=1}^\infty$ is called (δ, ϵ)-*almost arithmetic progression.*

Example 2.22. Assume that $(x_n)_{n=1}^\infty$ is an (δ, ϵ)-almost arithmetic progression, and let η be as in Definition 2.21. Then

$$D_N^\star \leq \min\left\{\eta, \frac{1}{N}\right\} \qquad \text{for} \quad \delta = 0 \tag{2.10}$$

$$D_N^\star \leq \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}} \qquad \text{for} \quad \delta \neq 0 \tag{2.11}$$

Proof. We prove the two inequalities separately.

- Assume first that $\delta = 0$. By Definition 2.21 we have $0 \leq x_1 \leq \eta$ and for all $i \in \{1, \dots, N-1\}$ it holds: $x_{i+1} = x_i + \eta$. This translates to $x_i = x_1 + (i-1)\eta$ for all $i \in \{1, \dots, N\}$. We consider now two cases.

  ▶ Assume that $\eta \geq 1/N$. Then for all $i \in \{1, \dots, N\}$ it holds

$$0 \leq x_1 = x_i - (i-1)\eta \leq x_i - \frac{i-1}{N} = x_1 + (i-1)\left(\eta - \frac{1}{N}\right)$$
$$\leq x_1 + (N-1)\left(\eta - \frac{1}{N}\right) = x_N - \frac{N-1}{N} < 1 - \frac{N-1}{N} = \frac{1}{N}.$$

  In particular we have

$$0 \leq x_i - (i-1)/N \leq 1/N \quad \text{and} \quad -1/N \leq x_i - i/N \leq 0 \tag{2.12}$$

  where the second inequality follows subtracting $1/N$ to the first inequality.

▶ Assume that $\eta < 1/N$. Then for all $i \in \{1, \dots, N\}$ it holds

$$\eta \geq x_1 = x_i - (i-1)\eta > x_i - \frac{i-1}{N} = x_1 + (i-1)\left(\eta - \frac{1}{N}\right)$$
$$\geq x_1 + (N-1)\left(\eta - \frac{1}{N}\right) = x_N - \frac{N-1}{N} \geq 1 - \eta - \frac{N-1}{N} = -\eta + \frac{1}{N}.$$

In particular we have

$$-\eta + 1/N \leq x_i - (i-1)/N \leq \eta \quad \text{and} \quad -\eta \leq x_i - i/N \leq \eta - 1/N \tag{2.13}$$

where the second inequality follows subtracting $1/N$ to the first inequality.

Combining (2.12) and (2.13) we get that for all $i \in \{1, \dots, N\}$

$$\max\left\{\left|x_i - \frac{i}{N}\right|, \left|x_i - \frac{i-1}{N}\right|\right\} \leq \min\left\{\eta, \frac{1}{N}\right\}. \tag{2.14}$$

Taking the maximum over all $i \in \{1, \dots, N\}$ on the LHS of (2.14) the inequality still holds. Therefore, using Theorem 2.17, we get exactly (2.10).

- Assume now that $\delta \neq 0$. For all $i \in \{1, \dots, N\}$ the following holds.

  Claim 1: $(i-1)(\eta - \delta\eta) \leq x_i \leq i(\eta + \delta\eta)$.

  Proof. We will prove the statement proceeding by induction on i.

  - *Base*: For $i = 1$, we have $0 \leq x_1 \leq \eta + \delta\eta$, which is true by (a) in Definition 2.21.
  - *Hypothesis*: Assume that Claim 1 holds for some $i \in \{1, \dots, N-1\}$.
  - *Step*: Using (b) in Definition 2.21 and the hypothesis we get

    $$x_{i+1} \geq \eta - \delta\eta + x_i \geq \eta - \delta\eta + (i-1)(\eta - \delta\eta) = i(\eta - \delta\eta).$$

    $$x_{i+1} \leq \eta + \delta\eta + x_i \leq \eta + \delta\eta + i(\eta + \delta\eta) = (i+1)(\eta + \delta\eta).$$

    This concludes the induction step. □

  Claim 2: $1 - (N-i+1)(\eta + \delta\eta) \leq x_i \leq 1 - (N-i)(\eta - \delta\eta)$.

  Proof. We will prove the statement proceeding by induction on i.

  - *Base*: For $i = N$, we have $1 - \eta - \delta\eta \leq x_N \leq 1$, which is true by (c) in Definition 2.21.
  - *Hypothesis*: Assume that Claim 2 holds for some $i \in \{2, \dots, N\}$.
  - *Step*: Using (b) in Definition 2.21 and the hypothesis we get

    $$x_{i-1} \geq x_i - \eta - \delta\eta \geq 1 - (N-i+1)(\eta + \delta\eta) - \eta - \delta\eta$$
    $$= 1 - (N-i+2)(\eta + \delta\eta) = 1 - (N - (i-1) + 1)(\eta + \delta\eta).$$

    $$x_{i-1} \leq x_i - \eta + \delta\eta \leq 1 - (N-i)(\eta - \delta\eta) - \eta + \delta\eta$$
    $$= 1 - (N-i+1)(\eta - \delta\eta) = 1 - (N - (i-1))(\eta - \delta\eta).$$

    This concludes the induction step. □

Fix $i \in \{1, \dots, N\}$. We consider again two different cases.

- Assume that $i(\eta + \delta\eta) \leq 1 - (N - i)(\eta - \delta\eta)$. Then

$$i \leq \frac{1 - N\eta(1-\delta)}{2\delta\eta} \tag{2.15}$$

Therefore $1 - N\eta(1-\delta) > 0$ since $i > 0$. Hence $N\eta < 1/(1-\delta)$ (∗). Now notice the following.

- If $\eta + \delta\eta - 1/N < 0$, using Claim 1 we get

$$x_i - \frac{i}{N} \leq i\left(\eta + \delta\eta - \frac{1}{N}\right) < 0 < \frac{\delta}{1 + \sqrt{1-\delta^2}}.$$

- If $\eta + \delta\eta - 1/N \geq 0$, then $N\eta \geq 1/(1+\delta)$. Adding the condition (∗) we get that $N\eta$ belongs to the interval

$$J \coloneqq \left[\frac{1}{1+\delta}, \frac{1}{1-\delta}\right].$$

Using Claim 1 and (2.15) we have

$$\begin{aligned} x_i - \frac{i}{N} &\leq i\left(\eta + \delta\eta - \frac{1}{N}\right) \leq \frac{1 - N\eta(1-\delta)}{2\delta\eta}\left(\eta + \delta\eta - \frac{1}{N}\right) \\ &= \frac{2N\eta - 1 - N^2\eta^2(1-\delta^2)}{2\delta N\eta}. \end{aligned} \tag{2.16}$$

Now we set $t \coloneqq N\eta$ and we define the function

$$h \colon J \to \mathbb{R},\ t \mapsto \frac{2t - 1 - t^2(1-\delta^2)}{2\delta t}.$$

Note that $h'(t) = 0$ iff $t = 1/\sqrt{1-\delta^2}$, and $h''(1/\sqrt{1-\delta^2}) < 0$, so that $t_0 \coloneqq 1/\sqrt{1-\delta^2}$ is a maximum of h. Hence by (2.16) we get

$$x_i - \frac{i}{N} \leq h(t_0) = \frac{\delta}{1 + \sqrt{1-\delta^2}}.$$

- Assume that $i(\eta + \delta\eta) > 1 - (N - i)(\eta - \delta\eta)$. Then

$$i > \frac{1 - N\eta(1-\delta)}{2\delta\eta} \tag{2.17}$$

Then $(1 - N\eta(1-\delta))/(2\delta\eta) < N$ since $i \leq N$. Hence $N\eta > 1/(1+\delta)$ (∗). Now notice the following.

- If $1/N + \delta\eta - \eta < 0$, using Claim 2 we get

$$\begin{aligned} x_i - \frac{i}{N} &\leq 1 - (N-i)(\eta - \delta\eta) - \frac{i}{N} \\ &= (N-i)\left(\frac{1}{N} + \delta\eta - \eta\right) < 0 < \frac{\delta}{1 + \sqrt{1-\delta^2}}. \end{aligned}$$

♦ If $1/N + \delta\eta - \eta \geq 0$, then $N\eta \leq 1/(1-\delta)$. Adding the condition $(*)$ we get that $N\eta$ belongs to J. Using Claim 2 and (2.17) we have

$$\begin{aligned} x_i - \frac{i}{N} &\leq (N-i)\left(\frac{1}{N} + \delta\eta - \eta\right) \\ &< \left(N - \frac{1 - N\eta(1-\delta)}{2\delta\eta}\right)\left(\frac{1}{N} + \delta\eta - \eta\right) \\ &= \frac{2N\eta - 1 - N^2\eta^2(1-\delta^2)}{2\delta N\eta} = h(N\eta) \leq \frac{\delta}{1+\sqrt{1-\delta^2}}. \end{aligned}$$

Summarizing, until now have shown that for all $i \in \{1, \dots, N\}$ it holds

$$x_i - \frac{i}{N} \leq \frac{\delta}{1+\sqrt{1-\delta^2}}. \tag{2.18}$$

Using the lower bounds of Claim 1 and Claim 2 and doing an analogous proof, instead of (2.18) we obtain that for all $i \in \{1, \dots, N\}$

$$\frac{i-1}{N} - x_i \leq \frac{\delta}{1+\sqrt{1-\delta^2}}. \tag{2.19}$$

Putting together (2.18) and (2.19) we get

$$\begin{aligned} -\frac{1}{N} - \frac{\delta}{1+\sqrt{1-\delta^2}} \leq x_i - \frac{i}{N} \leq \frac{\delta}{1+\sqrt{1-\delta^2}}. \\ -\frac{\delta}{1+\sqrt{1-\delta^2}} \leq x_i - \frac{i-1}{N} \leq \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}}. \end{aligned}$$

Therefore

$$\max\left\{\left|x_i - \frac{i}{N}\right|, \left|x_i - \frac{i-1}{N}\right|\right\} \leq \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}}. \tag{2.20}$$

Taking the maximum over all $i \in \{1, \dots, N\}$ on the LHS of (2.20) the inequality still holds. Therefore, using Theorem 2.17, we get exactly (2.11). ■

Remark 2.23. Adding the lower bound of Corollary 2.12 to the previous example we get a good description of the trend of $D_N^\star$ for almost arithmetic progressions:

$$\frac{1}{2N} \leq D_N^\star \leq \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}}.$$

Note that this holds also if $\delta = 0$.

2.3 Schmidt's Method

In this section we want to improve the lower bound of Corollary 2.12 with a method developed by Schmidt [58] (see Theorem 2.30 and the previous two lemmas). We follow Kuipers and Niederreiter [44, 107-109] for an improvement of the constants. We refer to Van der Corput [63] for Definition 2.32 and Example 2.33.

In the next notation we define the function R_N which is fundamental for this section, but which will also be useful later.

Notation 2.24. We define the function $R_N \coloneqq R_{N,\omega}$ by

$$R_{N,\omega}\colon \mathbb{R} \to (-N, N)\,,\; x \longmapsto \begin{cases} 0 & \text{for} \quad \{x\} = 0 \\ A([0,\{x\}), N, \omega) - N\{x\} & \text{for} \quad \{x\} > 0 \end{cases}.$$

Moreover, for $x, y \in \mathbb{R}$ we also define the quantity

$$R_N(x, y) \coloneqq R_N(y) - R_N(x).$$

Remark 2.25. Notice that $R_N(0) = R_N(1)$. Hence R_N has period 1. Moreover, R_N is connected to the discrepancy through the relation

$$D_N^\star = \sup_{x\in\mathbb{R}} \frac{|R_N(x)|}{N}. \tag{2.21}$$

In particular $D_N^\star \geq |R_N|/N$. Hence we can try to find an good lower bound of $|R_N|$ instead of $D_N^\star$. This is the fundamental idea behind Schmidt's Method.

Before stating the theorem with the lower bound, we must first prove two technical lemmas, which however will be necessary for the proof of the final theorem using the Schmidt's Method. The following notation helps us with the computations for the proofs of the two lemmas.

Notation 2.26. Let $K, L, L' \subset \mathbb{R}$ be intervals of the type $(a, b]$, where $0 \leq a < b$. For $x, y \in \mathbb{R}$ we denote the quantities

$$g^+(K, x, y) \coloneqq \max_{n\in K} R_n(x, y) \quad , \quad g^-(K, x, y) \coloneqq \min_{n\in K} R_n(x, y) \quad ,$$
$$h(L, L', x, y) \coloneqq \max\left\{g^-(L, x, y) - g^+(L', x, y), g^-(L', x, y) - g^+(L, x, y)\right\}.$$

Moreover, to simplify the notations, for the case $x = 0$, we denote

$$g^+(K, y) \coloneqq g^+(K, 0, y) \quad , \quad g^-(K, y) \coloneqq g^-(K, 0, y) \quad ,$$
$$h(K, y) \coloneqq g^+(K, y) - g^-(K, y).$$

Remark 2.27. Let $K \subset \mathbb{R}$ be an interval of the type $(a, b]$, where $0 \leq a < b$. For $x \in \mathbb{R}$, using the previous notation, we arrive at the explicit expression

$$h(K, x) = \max_{n\in K} R_n(x) - \min_{n\in K} R_n(x). \tag{2.22}$$

To find a good lower bound of $|R_N|$, we start by finding some properties of h. For a real number $x \in \mathbb{R}$, the next Lemma compare the quantity $h(K, x)$ to $h(L, x)$ and $h(L', x)$, where L, L' are two sub-intervals of an interval K.

Lemma 2.28. *Let $K, L, L' \subset \mathbb{R}$ be intervals of the type $(a, b]$, where $0 \leq a < b$, and such that $L, L' \subseteq K$. Then for all $x, y \in \mathbb{R}$ it holds*

$$h(K, x)+h(K, y) \geq h(L, L', x, y)+\frac{1}{2}\left(h(L, x) + h(L, y) + h(L', x) + h(L', y)\right) \tag{2.23}$$

Proof. We can assume WLOG that $h(L, L', x, y) = g^-(L, x, y) - g^+(L', x, y)$ (otherwise we interchange the roles of L and L' and we get the same result). Then for every $n \in L$ and $n' \in L'$ we have

$$\begin{aligned} &R_n(y) - R_n(x) - R_{n'}(y) + R_{n'}(x) = R_n(x, y) - R_{n'}(x, y) \\ &\geq g^-(L, x, y) - g^+(L', x, y) = h(L, L', x, y). \end{aligned} \tag{2.24}$$

Let $m(x), n(x), m(y), n(y) \in L$ such that $R_{m(x)}(x) = g^+(L, x)$, $R_{n(x)}(x) = g^-(L, x)$, $R_{m(y)}(y) = g^+(L, y)$ and $R_{n(y)}(y) = g^-(L, y)$. Then

$$R_{m(x)}(x) - R_{n(x)}(x) = h(L, x) \quad , \quad R_{m(y)}(y) - R_{n(y)}(y) = h(L, y). \tag{2.25}$$

Let $m'(x), n'(x), m'(y), n'(y) \in L'$ such that $R_{m'(x)}(x) = g^+(L', x)$, $R_{n'(x)}(x) = g^-(L', x)$, $R_{m'(y)}(y) = g^+(L', y)$ and $R_{n'(y)}(y) = g^-(L', y)$. Then

$$R_{m'(x)}(x) - R_{n'(x)}(x) = h(L', x) \quad , \quad R_{m'(y)}(y) - R_{n'(y)}(y) = h(L', y). \tag{2.26}$$

Let (A) be the inequality (2.24) with $n = m(x)$ and $n' = m'(y)$, and (B) be the inequality (2.24) with $n = n(y)$ and $n' = n'(x)$. Summing the four equations in (2.25) and (2.26) to $(A) + (B)$ we get

$$\begin{aligned} &\Big(R_{m'(x)}(x) - R_{n(x)}(x)\Big) + \Big(R_{m'(y)}(x) - R_{n(y)}(x)\Big) \\ &\quad + \Big(R_{m(y)}(y) - R_{n'(y)}(y)\Big) + \Big(R_{m(x)}(y) - R_{n'(x)}(y)\Big) \\ &\geq 2h(L, L', x, y) + h(L, x) + h(L, y) + h(L', x) + h(L', y). \end{aligned}$$

But now notice that, since $L, L' \subseteq K$, we have

$$\begin{aligned} \Big(R_{m'(x)}(x) - R_{n(x)}(x)\Big) + \Big(R_{m'(y)}(x) - R_{n(y)}(x)\Big) &\leq 2h(K, x), \\ \Big(R_{m(y)}(y) - R_{n'(y)}(y)\Big) + \Big(R_{m(x)}(y) - R_{n'(x)}(y)\Big) &\leq 2h(K, y). \end{aligned}$$

Therefore we get exactly (2.23). ∎

Using (2.23), we can find a lower bound for $h(K, x)$ by decomposing h into a sum up to some specific number. More precisely:

Lemma 2.29. *Let $t \in \mathbb{N}$ and $K \coloneqq (a, b]$, where $0 \leq a < b$, and such that $b - a \geq 4^t$. Then for all $y \in \mathbb{R}$ it holds*

$$\sum_{j=1}^{4^t} h\left(K, y + j4^{-t}\right) \geq 2^{2t-5}t. \tag{2.27}$$

Proof. We will prove the statement proceeding by induction on t. Fix $y \in \mathbb{R}$.

- *Base* ($t = 1$): Let $n \in \mathbb{N}$. Then

$$\begin{aligned} &R_{n+1}(y + 1/2) - R_n(y + 1/2) - (R_{n+1}(y) - R_n(y)) \\ &= A - (n+1)\{y + 1/2\} + n\{y + 1/2\} + (n+1)\{y\} - n\{y\} \\ &= A - \{y + 1/2\} + \{y\} = B - \{y\} - 1/2 + \{y\} = B - 1/2, \end{aligned}$$

where $A, B \in \mathbb{N}$ are some constants depending on n and y. Hence

$$\begin{aligned} &h\left(K, y+1/2\right) + h\left(K, y+1\right) = h\left(K, y+1/2\right) + h\left(K, y\right) \\ &\geq \left|R_{n+1}\left(y+1/2\right) - R_n\left(y+1/2\right)\right| + \left|R_{n+1}(y) - R_n(y)\right| \\ &\geq \left|R_{n+1}\left(y+1/2\right) - R_n\left(y+1/2\right) - \left(R_{n+1}(y) - R_n(y)\right)\right| \\ &= |B - 1/2| \geq 1/2. \end{aligned}$$

Since $h(K, z) \geq 0$ for all $z \in \mathbb{R}$, we get

$$\sum_{j=1}^{4} h\left(K, y + \frac{j}{4}\right) \geq h\left(K, y + \frac{1}{2}\right) + h\left(K, y+1\right) \geq \frac{1}{2} > \frac{1}{8}.$$

Hence (2.27) holds for $t = 1$.

- *Hypothesis*: Assume that the Lemma holds for some $t \in \mathbb{N}$.
- *Step*: Let $K \coloneqq (a, b]$, where $0 \leq a < b$, and such that $b - a \geq 4^{t+1}$. Set

$$L \coloneqq \left(a, a + 4^t\right] \qquad \text{and} \qquad L' \coloneqq \left(a + 2 \cdot 4^t, a + 3 \cdot 4^t\right].$$

By the periodicity of $h(K, y)$, assume $0 \leq y < 4^{-(t+1)}$. For $j \in \{0, 1, \ldots, 4^{t+1}\}$, we define the number $z_j \coloneqq y + j4^{-(t+1)}$. Moreover, for $j \in \{1, \ldots, 4^{t+1}\}$, we define the number

$$d_j \coloneqq \begin{cases} \left|\{m \in \mathbb{N} \cap (a + 4^t, a + 2 \cdot 4^t] \mid x_m \in [0, z_0) \cup [z_{j-1}, 1)\}\right| & j = 4^{t+1} \\ \left|\{m \in \mathbb{N} \cap (a + 4^t, a + 2 \cdot 4^t] \mid x_m \in [z_{j-1}, z_j)\}\right| & j \neq 4^{t+1} \end{cases}.$$

Let $n \in L$ and $n' \in L'$. For $j \in \{1, \ldots, 4^{t+1}\}$, we define the number

$$e_j \coloneqq \begin{cases} \left|\{m \in \mathbb{N} \cap (n, n'] \mid x_m \in [0, z_0) \cup [z_{j-1}, 1)\}\right| & j = 4^{t+1} \\ \left|\{m \in \mathbb{N} \cap (n, n'] \mid x_m \in [z_{j-1}, z_j)\}\right| & j \neq 4^{t+1} \end{cases}.$$

Then, using Notations 2.24 and 2.26, we get

$$\begin{aligned} &h(L, L', z_{j-1}, z_j) \geq R_{n'}(z_{j-1}, z_j) - R_n(z_{j-1}, z_j) \\ &= e_j - (n' - n)(z_j - z_{j-1}) \geq d_j - 3 \cdot 4^t \cdot 4^{-(t+1)} = d_j - 3/4. \end{aligned}$$

Figure 2.2: Example of L and L'.

If $d_j \neq 0$, we infer: $h(L, L', z_{j-1}, z_j) \geq d_j/4$. By Lemma 2.28 we have

$$\begin{aligned} &h(K, z_{j-1}) + h(K, z_j) \\ &\geq \frac{1}{4} d_j + \frac{1}{2}\left(h(L, z_{j-1}) + h(L, z_j) + h(L', z_{j-1}) + h(L', z_j)\right). \end{aligned} \tag{2.28}$$

Now notice that, for $j \in \{0, \dots, 4^{t+1}\}$, we have $h(K, z_j) \geq h(L, z_j)$ and $h(K, z_j) \geq h(L', z_j)$. Therefore (2.28) holds also if $d_j = 0$. Moreover

$$\sum_{j=1}^{4^{t+1}} h(L, z_j) = \sum_{j=1}^{4^{t+1}} h\left(L, y + j4^{-(t+1)}\right) = \sum_{k=0}^{3} \sum_{j=1}^{4^t} h\left(L, y + (4j-k)4^{-(t+1)}\right)$$
$$= \sum_{k=0}^{3} \sum_{j=1}^{4^t} h\left(L, \left(y - k4^{-(t+1)}\right) + j4^{-t}\right) \geq \sum_{k=0}^{3} 2^{2t-5}t = 2^{2t-3}t$$

where in the last step we have used the induction hypothesis. Notice that one can replace $h(L, z_j)$ with $h(L, z_{j-1}), h(L', z_j)$ or $h(L', z_{j-1})$, and the result is the same. Now recall that $z_{4^{t+1}} = z_0 + 1$, so that $h(K, z_0) = h(K, z_{4^{t+1}})$. Hence, applying the sum over all $j \in \{1, \dots, 4^{t+1}\}$ to (2.28), we get

$$2 \sum_{j=1}^{4^{t+1}} h(K, z_j) \geq \frac{1}{4} \sum_{j=1}^{4^{t+1}} d_j + \frac{1}{2} \cdot 4 \cdot 2^{2t-3}t$$
$$= \frac{1}{4}\left(a + 2 \cdot 4^t - (a + 4^t)\right) + 2^{2t-2}t = 2^{2t-2}(t+1).$$

Therefore

$$\sum_{j=1}^{4^{t+1}} h(K, z_j) \geq 2^{2t-3}(t+1) = 2^{2(t+1)-5}(t+1).$$

This concludes the induction step. ■

Lemma 2.29 gives us a lower bound for $h(K, x)$. Now, using the relations given in (2.21) and (2.22), it remains to extract a lower bound for the discrepancy.

Theorem 2.30 (Schmidt)**.** *There exists $c \in \mathbb{R}_{>0}$ such that for infinitely many $N \in \mathbb{N}$ it holds*

$$D_N^\star > \frac{c \log(N)}{N}. \tag{2.29}$$

In particular we have: $D_N^\star = \Omega(\log(N)/N)$.

Proof. Let $m \geq 4^{32}$. Then there exists $t \in \mathbb{N}_{\geq 32}$ such that $4^t \leq m < 4^{t+1}$. Consider now $K \coloneqq (0, 4^t]$ and $y \in \mathbb{R}$. By Lemma 2.29 we have

$$2^{-5}t \leq 4^{-t} \sum_{j=1}^{4^t} h\left(K, y + j4^{-t}\right) \leq 4^{-t}4^t h(K, x) = h(K, x),$$

where $x \in \mathbb{R}$ is such that $h(K, x) \geq h\left(K, y + j4^{-t}\right)$ for all $j \in \{1, \dots, 4^t\}$. Therefore there exist $p, q \in K \cap \mathbb{N}$ such that

$$2^{-5}t \leq R_p(x) - R_q(x) \leq |R_p(x)| + |R_q(x)| \leq 2 \max\left\{|R_p(x)|, |R_q(x)|\right\}.$$

Assuming WLOG that $|R_p(x)| \geq |R_q(x)|$, we infer that $|R_p(x)| \geq 2^{-6}t$. Hence

$$\begin{aligned} pD_p^\star \geq D_p^\star \geq |R_p(x)| \geq 2^{-6}t \geq 2^{-6}\frac{32}{33}(t+1) \\ = \frac{1}{66}(t+1) > \frac{1}{66}\frac{\log(m)}{\log(4)} = \frac{\log(m)}{132\log(2)}. \end{aligned}$$

Therefore we have shown that for all $m \in \mathbb{N}_{\geq 32}$ there exists $N \leq m$ such that

$$ND_N^\star > c\log(m) > c\log(N).$$

where $c \coloneqq 1/(132\log(2))$. This is the same thing as saying that (2.29) holds for infinitely many $N \in \mathbb{N}$. ■

Similar results can be found for higher dimensional sequences in Roth [56]. See also Kuipers and Niederreiter [44, 100-107]. As an application of Schmidt's Method, let's take the example of a sequence with very low discrepancy.

Notation 2.31. Let $n \in \mathbb{N}$. Then we can expand n in its dyadic expansion, that is

$$n = \sum_{j=0}^{s} a_j 2^j \quad \text{where} \quad s \in \mathbb{N} \quad \text{and} \quad a_j \in \{0,1\} \quad \forall j \in \{0,\dots,s\}.$$

Definition 2.32. For $n \in \mathbb{N}$ consider the dyadic expansion: $n-1 = \sum_{j=0}^{s} a_j 2^j$. Set now $x_n \coloneqq \sum_{j=0}^{s} a_j 2^{-j-1}$. Then $(x_n)_{n=1}^\infty$ is called *Van der Corput sequence*.

Example 2.33. Let $(x_n)_{n=1}^\infty$ be the Van der Corput sequence. Then

$$D_N = \Theta(\log(N)/N). \tag{2.30}$$

Proof. We have to prove two directions.

- To show: $D_N = \mathcal{O}(\log(N)/N)$. The main idea of this part is to show that ω can be divided into sub-sequences that are almost arithmetic progressions. First, with the dyadic expansion, write $N = \sum_{j=1}^{s} 2^{h_j}$, where $s \in \mathbb{N}$ and $h_1,\dots,h_s \in \mathbb{N}\cup\{0\}$ with $h_1 > \cdots > h_s \geq 0$. With this notation we can construct the intervals of integers

$$M_j \coloneqq \begin{cases} \left[1, 2^{h_1}\right] \cap \mathbb{N} & \text{for} \quad j = 1 \\ \left[1 + 2^{h_1} + \cdots + 2^{h_{j-1}}, 2^{h_1} + \cdots + 2^{h_j}\right] \cap \mathbb{N} & \text{for} \quad j \in \{2,\dots,s\} \end{cases}.$$

  Now if $n \in M_j$ for some $j \in \{1,\dots,s\}$, we can write

$$n = \begin{cases} 1 + \sum_{i=0}^{h_1-1} a_i 2^i & \text{for} \quad j = 1 \\ 1 + 2^{h_1} + \cdots + 2^{h_{j-1}} + \sum_{i=0}^{h_j-1} a_i 2^i & \text{for} \quad j \in \{2,\dots,s\} \end{cases}.$$

  Note that running over all $a_i \in \{0,1\}$, n reaches exactly all 2^{h_j} values of M_j. Therefore, by construction of the Van der Corput sequence, we have

$$x_n \coloneqq \begin{cases} \sum_{i=0}^{h_1-1} a_i 2^{-i-1} & \text{for} \quad j = 1 \\ 2^{-h_1-1} + \cdots + 2^{-h_{j-1}-1} + \sum_{i=0}^{h_j-1} a_i 2^{-i-1} & \text{for} \quad j \in \{2,\dots,s\} \end{cases}.$$

  Let $y_j \coloneqq x_n - \sum_{i=0}^{h_j-1} a_i 2^{-i-1}$. Then note the following.

(1) Running over all $a_i \in \{0,1\}$, the sum $\sum_{i=0}^{h_j-1} a_i 2^{-i-1}$ reaches exactly all 2^{h_j} values given (in some order) by

$$0 \quad , \quad 2^{-h_j} \quad , \quad \dots \quad , \quad (2^{h_j}-1)2^{-h_j}.$$

(2) It holds $0 \leq y_j < 2^{-h_j}$, where the first inequality follows by the case $j = 1$, while the second holds since for all $j \in \{2, \dots, s\}$ we have

$$\begin{aligned} y_j &= \frac{1}{2}\sum_{i=1}^{j-1} 2^{-h_i} \leq \frac{1}{2}\sum_{i=1}^{j-1} 2^{-(h_j+j-i)} = 2^{-(h_j+j+1)}\sum_{i=1}^{j-1} 2^i \\ &= 2^{-(h_j+j+1)}\frac{1-2^j}{1-2} = 2^{-(h_j+1)}\left(1-2^{-j}\right) < 2^{-(h_j+1)} < 2^{-h_j}. \end{aligned}$$

where in the first inequality we have used the construction of the h_i's.

Let $\omega_j := (z_i^j)_{i=1}^{2^{h_j}}$ be a finite sequence which contains exactly the first 2^{h_j} elements of x_n with $n \in M_j$, such that the elements are ordered according to their magnitude. In terms of Definition 2.21 note the following.

(a) By (1) : $z_1^j = y_j$. Hence by (2) : $0 \leq z_1^j < 2^{-h_j}$.

(b) By (1), for $i \in \{1, \dots, 2^{h_j}-1\}$, we have: $z_{i+1}^j - z_i^j = 2^{-h_j}$.

(c) By (1) : $z_{2^{h_j}}^j = y_j + 1 - 2^{-h_j}$. Hence (2) : $1 - 2^{-h_j} \leq z_{2^{h_j}}^j < 1$.

This shows that ω_j is an almost arithmetic progression with parameters $\delta = 0$ and $\eta = 2^{-h_j}$. Therefore, by Example 2.22 we have $D_{2^{h_j}}(\omega_j) \leq 2^{-h_j}$. But since $(x_n)_{n=1}^N$ can be decomposed into $\omega_1, \dots, \omega_s$, using Theorem 2.15 we get

$$ND_N(\omega) \leq \sum_{j=1}^{s} 2^{h_j} D_{2^{h_j}}(\omega_j) \leq \sum_{j=1}^{s} 2^{h_j} 2^{-h_j} = s. \tag{2.31}$$

But note that $N = \sum_{j=1}^{s} 2^{h_j} \geq \sum_{j=0}^{s-1} 2^j = 2^s - 1$, so that $s \leq \log(N+1)/\log(2)$. Hence, by (2.31), for $N \geq 2$ we get

$$ND_N \leq \frac{\log(N+1)}{\log(2)} \leq \frac{2\log(N)}{\log(2)}.$$

In particular we have: $D_N = \mathcal{O}(\log(N)/N)$.

- To show: $D_N = \Omega(\log(N)/N)$. For this part, it's enough to apply Schmidt's Method. Indeed, using Theorem 2.30 and Theorem 2.10, we get

$$ND_N \geq 2ND_N^\star > 2c\log(N).$$

for some $c \in \mathbb{R}_{>0}$. In particular we have: $D_N = \Omega(\log(N)/N)$.

Summarizing, we get exactly: $D_N = \Theta(\log(N)/N)$. ■

An improvement of Example 2.33 can be found in Harber [19], while some generalized versions of the Van der Corput sequence to higher dimensions have been constructed by Hammersley [21] and Halton [20].

2.4 Leveque's Inequality

Our next goal is to improve the upper bound of Theorem 2.4. We have already seen that the trivial bound $D_N \leq 1$ is the best possible (see Example 2.5), but we can still construct a type of bound that is related to the individual sequences, and which will therefore improve the trivial bound depending on the sequence. This is the concept behind Leveque Inequality and Erdös Turán Theorem.

In this section we focus on the bound found by Leveque [45] in 1965. The proof of Lemma 2.37 is due first to Koksma but unpublished. Therefore, as reference for it and Theorem 2.39 see Kuipers [42] and Kuipers and Niederreiter [44, pp. 110-112].

We start by defining the follwoing important quantities.

Notation 2.34. Let $h \in \mathbb{Z}$. We denote $y_n \coloneqq \{x_n\}$ and we define

$$S_N \coloneqq \sum_{n=1}^{N} \left(y_n - \frac{1}{2} \right) \qquad \text{and} \qquad U_h \coloneqq U_{h,N} \coloneqq \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n}.$$

Remark 2.35. Let $h \in \mathbb{Z}$. Then: $U_{-h} = \overline{U_h}$.

Proof. For all $h \in \mathbb{Z}$ it holds

$$U_{-h} = \frac{1}{N} \sum_{n=1}^{N} e^{-2\pi i h x_n} = \frac{1}{N} \sum_{n=1}^{N} \overline{e^{2\pi i h x_n}} = \frac{1}{N} \overline{\sum_{n=1}^{N} e^{2\pi i h x_n}} = \overline{U_h}.$$

This concludes the proof. ∎

To state the inequality, we first prove a lemma in which we compute the integral of R_N^2 using the Fourier Theory (see also Appendix B). This is useful to us as we will find an upper bound for D_N in terms of the integrals of R_N and R_N^2.

Remark 2.36. To compute the integral of R_N and R_N^2, we have to switch from a "discrete expression" to a "continuous expression", so that it'll be easier to compute the integral. In particular, for all $x \in [0,1]$ it holds

$$\sum_{n=1}^{N} \chi_{[0,x)}(y_n) = \sum_{n=1}^{N} \chi_{(y_n,1]}(x). \tag{2.32}$$

This can be viewed with an example in Figure 2.3. Note that the LHS of (2.32) is the number of $y_n's$ that are smaller than x (in the example $y_1, y_2, y_3, y_6, y_8, y_{10}$). Instead, the RHS of (2.32) is the number of times there is an y_n smaller than x.

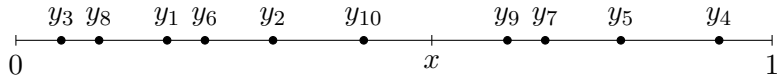

Figure 2.3: Example of sequence with $x = 5.5$ and $N = 10$.

Now we are able to prove the following identity.

Lemma 2.37. *It holds*

$$\int_0^1 R_N^2(x)\,dx = S_N^2 + \frac{N^2}{2\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\,|U_h|^2\,. \tag{2.33}$$

Proof. First note that R_N is a piece-wise linear function, where the linear pieces a have slope of $-N$. Moreover it is left-continuous, and discontinuous only at finitely many points. More precisely, R_N is discontinuous on the set

$$S \coloneqq \bigcup_{k\in\mathbb{Z}}\bigcup_{j=1}^{N}\{x_j + k\}.$$

This set has zero Lebesgue measure since it is countable. Hence R_N can be expanded into a Fourier series in the domain $\mathbb{R}\backslash S$ (see Appendix B, Theorem B.9). The Fourier coefficients are given by

$$a_h = \int_0^1 R_N(x)e^{-2\pi ihx}\,dx. \tag{2.34}$$

Hence, for $h = 0$, using (2.32) we have

$$\begin{aligned} a_0 = \int_0^1 R_N(x)\,dx &= \int_0^1 \sum_{n=1}^{N}\chi_{[0,x)}(y_n)\,dx - N\int_0^1 x\,dx \\ &= \int_0^1 \sum_{n=1}^{N}\chi_{(y_n,1]}(x)\,dx - \frac{N}{2} = \sum_{n=1}^{N}\int_{y_n}^1 dx - \sum_{n=1}^{N}\frac{1}{2} \\ &= \sum_{n=1}^{N}\left(1 - y_n - \frac{1}{2}\right) = -\sum_{n=1}^{N}\left(y_n - \frac{1}{2}\right) = -S_N. \end{aligned} \tag{2.35}$$

Instead, for $h \in \mathbb{Z}\setminus\{0\}$, using again (2.32) we get

$$\begin{aligned} a_h &= \int_0^1 R_N(x)e^{-2\pi ihx}\,dx = \sum_{n=1}^{N}\int_0^1 \chi_{(y_n,1]}(x)e^{-2\pi ihx}\,dx - N\int_0^1 xe^{-2\pi ihx}\,dx \\ &= \sum_{n=1}^{N}\int_{y_n}^1 e^{-2\pi ihx}\,dx + \frac{N}{2\pi ih} = \frac{1}{2\pi ih}\sum_{n=1}^{N}\left(e^{-2\pi ihy_n} - 1\right) + \sum_{n=1}^{N}\frac{1}{2\pi ih} \\ &= \frac{1}{2\pi ih}\sum_{n=1}^{N}e^{-2\pi ihx_n} = \frac{NU_{-h}}{2\pi ih} = \frac{N\overline{U_h}}{2\pi ih} \end{aligned} \tag{2.36}$$

where in the last step we have used Remark 2.35. Therefore, by Parseval identity (see Appendix B, Theorem B.11) we have

$$\int_0^1 R_N^2(x)\,dx = a_0^2 + 2\sum_{h=1}^{\infty}|a_h|^2.$$

Inserting the a_h's we found above, we get exactly (2.33). ∎

As last thing, the following remark shows a simple inequality that we will use during the proof of Leveque Inequality.

Remark 2.38. Let $r, s \in \mathbb{R}_{\geq 0}$. Then: $r^3 + s^3 \geq (r+s)^3/4$.

Proof. Define $t \coloneqq (r+s)/2$ and $u \coloneqq (r-s)/2$. Then

$$r^3 + s^3 = (t+u)^3 + (t-u)^3 = 2t^3 + 6tu^2 \geq 2t^3 = (r+s)^3/4.$$

This concludes the proof. ■

Now we have all the elements to prove the main theorem of this section.

Theorem 2.39 (Leveque's Inequality)**.** *It holds*

$$D_N \leq \left(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} |U_h|^2 \right)^{1/3}. \tag{2.37}$$

Proof. For $x \in \mathbb{R}$ we define the quantity

$$T_N(x) \coloneqq \frac{1}{N}(R_N(x) + S_N).$$

Notice that $T_N(0) = T_N(1)$. So since R_N is of period 1, so is T_N. Moreover T_N is piece-wise linear, where the linear pieces a have slope of -1. Furthermore it is left-continuous, and discontinuous only at finitely many points. Using (2.35) it holds

$$\int_0^1 T_N(x)\,dx = \frac{1}{N}\left(\int_0^1 R_N(x)\,dx + S_N \right) = \frac{1}{N}\left(-S_N + S_N\right) = 0.$$

But since $T_N \neq 0$ (indeed $T_N = 0$ iff $R_N(x) = -S_N$ for all $x \in \mathbb{R}$, which is impossible since R_N is not constant), there exists $\alpha, \beta \in [0,1]$ such that $T_N(\alpha) > 0$ and $T_N(\beta) < 0$. Now notice that in the interval $[\alpha, \alpha + T_N(\alpha)]$ the graph of the map T_N will not lie below the line segment from $A \coloneqq (\alpha, T_N(\alpha))$ to $B \coloneqq (\alpha + T_N(\alpha), 0)$.

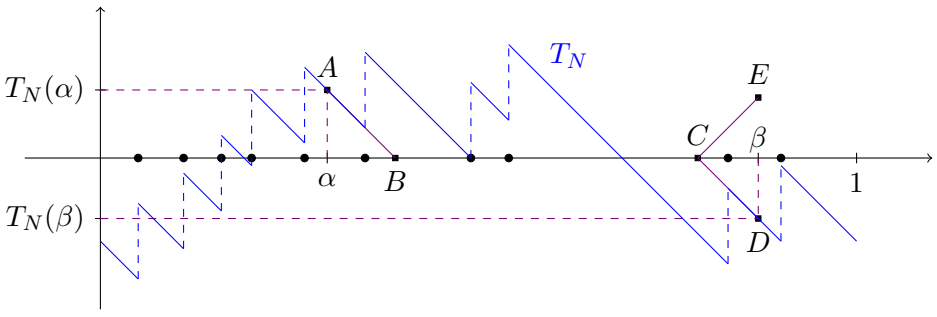

Figure 2.4: Example of graph of T_N with $\alpha = 0.3$, $\beta = 0.87$ and $N = 10$.

By the periodicity of T_N, there exists $\beta_1 \in (\alpha, \alpha+1)$ such that $T_N(\beta_1) = T_N(\beta)$ (more precisely $\beta_1 = \beta$ or $\beta_1 = \beta + 1$). Using the same idea as above, in the

interval $[\beta_1 + T_N(\beta_1), \beta_1]$ the graph of T_N will not lie above the line segment from $C \coloneqq (\beta_1 + T_N(\beta_1), 0)$ to $D \coloneqq (\beta_1, T_N(\beta_1))$. Therefore in the interval $[\beta_1 + T_N(\beta_1), \beta_1]$ the graph of $|T_N|$ will not lie below the line segment from C to $E \coloneqq (\beta_1, -T_N(\beta_1))$. Hence $|T_N|$ is bounded from below as follows:

$$\begin{aligned} &\forall x \in (\alpha, \alpha + T_N(\alpha)): \quad T_N(x) \geq \alpha + T_N(\alpha) - x \\ &\forall x \in (\beta_1 + T_N(\beta_1), \beta_1): \quad T_N(x) \geq -\beta_1 - T_N(\beta_1) + x \end{aligned}$$

Moreover note that the intervals $[\alpha, \alpha + T_N(\alpha)]$ and $[\beta_1 + T_N(\beta_1), \beta_1]$ can have at most one point in common (in particular we can have $B = C$). Hence

$$\begin{aligned} &\int_0^1 T_N^2(x)\,dx = \int_\alpha^{\alpha+1} T_N^2(x)\,dx \geq \int_\alpha^{\alpha+T_N(\alpha)} T_N^2(x)\,dx + \int_{\beta_1+T_N(\beta_1)}^{\beta_1} T_N^2(x)\,dx \\ &\geq \int_\alpha^{\alpha+T_N(\alpha)} (\alpha + T_N(\alpha) - x)^2\,dx + \int_{\beta_1+T_N(\beta_1)}^{\beta_1} (-\beta_1 - T_N(\beta_1) + x)^2\,dx \\ &= \left[-\frac{1}{3}(\alpha + T_N(\alpha) - x)^3\right]_{x=\alpha}^{\alpha+T_N(\alpha)} + \left[\frac{1}{3}(-\beta_1 - T_N(\beta_1) + x)^3\right]_{x=\beta_1+T_N(\beta_1)}^{\beta_1} \\ &= \frac{1}{3}\left(T_N^3(\alpha) + (-T_N(\beta_1))^3\right) = \frac{1}{3}\left(T_N^3(\alpha) + (-T_N(\beta))^3\right). \end{aligned}$$

Using Remark 2.38 with $r = T_N(\alpha)$ and $s = -T_N(\beta)$ we get

$$\int_0^1 T_N^2(x)\,dx \geq \frac{1}{12}\left(T_N(\alpha) - T_N(\beta)\right)^3 = \frac{1}{12}\left(\frac{R_N(\alpha) - R_N(\beta)}{N}\right)^3. \tag{2.38}$$

Note that we can choose α and β such that the RHS of (2.38) is maximal. This implies that (2.38) holds for all $\alpha, \beta \in [0, 1]$. Hence

$$\frac{1}{12} D_N^3 \leq \int_0^1 T_N^2(x)\,dx. \tag{2.39}$$

It remains to compute the RHS of (2.39). So using (2.35) and Lemma 2.37 we get

$$\begin{aligned} \int_0^1 T_N^2(x)\,dx &= \frac{1}{N^2}\left(\int_0^1 R_N^2(x)\,dx + 2S_N \int_0^1 R_N(x)\,dx + S_N^2\right) \\ &= \frac{1}{N^2}\left(\int_0^1 R_N^2(x)\,dx - S_N^2\right) = \frac{1}{2\pi^2}\sum_{h=1}^{\infty} \frac{1}{h^2}\left|U_h\right|^2. \end{aligned} \tag{2.40}$$

Putting together (2.39) and (2.40) we get (2.37). ■

Note that the Theorem 2.39 holds for any type of sequence $(x_n)_{n=1}^{\infty}$. In fact, the result does not contradict the trivial bound $D_N \leq 1$ of Theorem 2.4, as shown in the following example.

Example 2.40. The constant $6/\pi^2$ in Theorem 2.39 is the best possible.

Proof. Consider a sequence such that the first N terms are $x_1 = \cdots = x_N = 0$. Then $D_N = 1$ (see Example 2.5), and indeed

$$\left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h x_n}\right|^2\right)^{1/3} = \left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\right)^{1/3} = 1.$$

Hence the constant $6/\pi^2$ can not be improved. ■

Remark 2.41. One can also shown that the constant $1/3$ in Theorem 2.39 is the best possible. A proof can be found in Leveque [45]. The idea is that we can construct a sequence such that for all $\epsilon \in \mathbb{R}_{>0}$ the discrepancy satisfies

$$D_N > \left(\left(\frac{3}{2\pi^2} - \epsilon\right)\sum_{h=1}^{\infty}\frac{1}{h^2}\left|U_h\right|^2\right)^{1/3}$$

for infinitely many $N \in \mathbb{N}$.

A more general inequality than (2.37) can be found in Elliott [43]. Surprisingly, a higher-dimensional version of Leveque inequality has not yet been found.

Using what we have found we get an immediate proof of the sufficiency part of the Weyl Criterion (Theorem 1.13).

Example 2.42. Assume that $\lim_{N\to\infty} U_h = 0$ for all $h \in \mathbb{Z} \setminus \{0\}$. Then $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

Proof. By Theorem 2.39 it holds

$$\lim_{N\to\infty} D_N \leq \lim_{N\to\infty}\left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|U_h\right|^2\right)^{1/3} = 0.$$

By Theorem 2.3 $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. ■

The interesting thing about Leveque Inequality is that we can analyze the sequences in terms of exponential sums. Therefore, the problem reduces to establish how the quantity $|U_h|$ changes for different sequences. This also helps us find bounds for the discrepancy of specific sequences. However, the biggest problem with Leveque inequality is that the sum on the RHS of (2.37) is an infinite sum. Therefore, for many sequences the sum does not converge. This is not optimal for computing the bounds for D_N. As example see the following.

Example 2.43. Consider the sequence $(\log(n))_{n=1}^{\infty}$. We have already shown that this sequence is not uniformly distributed mod 1 (see Example 1.15). Thus D_N will not converge to zero for $N \to \infty$ (Theorem 2.3). However, we try to improve the trivial bound $D_N \leq 1$ given in Theorem 2.4 using the Leveque Inequality. By the

Euler Summation Formula (see Appendix C, Corollary C.16) we can proceed as in the proof of Example 1.15, and we get (1.11), but with a general $h \in \mathbb{Z} \setminus \{0\}$, that is

$$\begin{aligned} |U_h| &\leq \left| \frac{e^{2\pi i h \log(N)} - 1/N}{2\pi i h + 1} \right| + \frac{1}{N} + \frac{\pi |h| \log(N)}{N} \\ &\leq \frac{2}{4\pi^2 h^2 + 1} + \mathcal{O}\left(\frac{1}{N} + \frac{|h| \log(N)}{N} \right) \leq \frac{1}{2\pi^2 h^2} + \mathcal{O}\left(\frac{|h| \log(N)}{N} \right). \end{aligned} \tag{2.41}$$

It follows that

$$|U_h|^2 \leq \frac{1}{4\pi^4 h^4} + \mathcal{O}\left(\frac{\log(N)}{|h| N} + \frac{h^2 \log(N)^2}{N^2} \right).$$

Therefore, for all $m \in \mathbb{N}$ we have

$$\begin{aligned} S_m &\coloneqq \frac{6}{\pi^2} \sum_{h=1}^{m} \frac{1}{h^2} |U_h|^2 \leq \frac{6}{\pi^2} \sum_{h=1}^{m} \frac{1}{h^2} \left(\frac{1}{4\pi^4 h^4} + \mathcal{O}\left(\frac{\log(N)}{|h| N} + \frac{h^2 \log(N)^2}{N^2} \right) \right) \\ &= \frac{3}{2\pi^6} \sum_{h=1}^{m} \frac{1}{h^6} + \mathcal{O}\left(\frac{\log(N)}{N} \sum_{h=1}^{m} \frac{1}{h^2} + \frac{m \log(N)^2}{N^2} \right). \end{aligned}$$

Note that the last expression does not converge for $m \to \infty$. Using Theorem 2.39 we get $D_N \leq \infty$, which tell us nothing about the discrepancy.

The problem also appears for many other sequences that we analyzed in the first chapter, such as the sequence $(an^\sigma)_{n=1}^\infty$ (where $a \in \mathbb{R}_{>0}$, $\sigma \in (0,1)$) and the sequence $(a \log(n)^\tau)_{n=1}^\infty$ (where $a \in \mathbb{R}_{>0}$, $\tau \in \mathbb{R}_{>1}$), but also the sequence $(n\theta)_{n=1}^\infty$ (where $\theta \in \mathbb{R}$ is irrational), and the sequences $(n \log(n))_{n=1}^\infty$ and $(n \log(\log(n)))_{n=2}^\infty$. In the next chapter we will state another important result that will allow us to find good upper bounds for the sequences we have mentioned.

Chapter 3

Erdös-Turán Theorem

In this chapter we will focus on another upper bound of D_N, similar to that of Leveque inequality (see Theorem 2.39), as it relates the discrepancy to the exponential sum U_h. The result has been proven by Erdös and Turán [13, 14, 1948], and many improvements and generalizations have been made over the years. Van der Corput and Koksma [39, p. 101] had already found a weaker version of the theorem. Moreover, Van der Corput and Pisot [64] also proved a similar inequality.

The bound of Erdös and Turán has been used in many applications of Diophantine Approximation. In particular, in the last section we will use the inequality to compute the discrepancy of the particular sequence $(n\theta)_{n=1}^{\infty}$ for $\theta \in \mathbb{R}$ irrational.

For this chapter we consider again an arbitrary sequence $\omega = (x_n)_{n=1}^{\infty}$ of real numbers and a fixed random integer $N \in \mathbb{N}$.

3.1 Statement and examples

In this section we will state first the theorem and then give some examples. The proof will be given in the next section. As reference see Kuipers and Niederreiter [44, pp. 112-114] for Theorem 3.1 and Corollary 3.3.

Theorem 3.1 (Erdös-Turán)**.** *For every $m \in \mathbb{N}$ it holds*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1} \right) |U_h| \,. \tag{3.1}$$

Remark 3.2. As for Leveque Inequality (see Example 2.42), it's easy to show that Erdös Turán Theorem implies the sufficiency part of the Weyl Criterion

In the practice, equation (3.1) is a little difficult to use. Since the second term of the sum does not play a fundamental role, we can modify the bound obtaining a simpler one, which will be useful for the examples we will see later.

Corollary 3.3. *There exists $C \in \mathbb{R}_{>0}$ such that for every $m \in \mathbb{N}$ it holds*

$$D_N \leq C \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} |U_h| \right) . \tag{3.2}$$

Proof. Applying Theorem 3.1 we get

$$\begin{aligned} D_N &\leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|U_h| \\ &\leq \frac{6}{m} + \frac{4}{\pi}\sum_{h=1}^{m}\frac{1}{h}|U_h| \leq 6\left(\frac{1}{m} + \sum_{h=1}^{m}\frac{1}{h}|U_h|\right). \end{aligned} \tag{3.3}$$

This concludes the proof. ■

Compared to Leveque Inequality (Theorem 2.39), the advantage is that the sum in the RHS of (3.2) is finite, and the term $|U_h|/h$ is not squared. This makes it easier to evaluate the sum. We start by recalling Example 2.43, in which we have failed to find a bound for D_N using Leveque Inequality.

Example 3.4. Consider the sequence $(\log(n))_{n=1}^{\infty}$. Then

$$D_N \leq \frac{2\,\zeta(3)}{\pi^3} + \mathcal{O}\left(\sqrt{\frac{\log(N)}{N}}\right).$$

where ζ is the Riemann zeta function.

Proof. By (2.41), for all $h \in \mathbb{Z} \setminus \{0\}$ we have

$$|U_h| \leq \frac{1}{2\pi^2 h^2} + \mathcal{O}\left(\frac{|h|\log(N)}{N}\right).$$

Therefore, by (3.3), for all $m \in \mathbb{N}$ it holds

$$\begin{aligned} D_N &\leq \frac{6}{m} + \frac{4}{\pi}\sum_{h=1}^{m}\frac{1}{h}|U_h| \leq \frac{6}{m} + \frac{4}{\pi}\sum_{h=1}^{m}\frac{1}{h}\left(\frac{1}{2\pi^2h^2} + \mathcal{O}\left(\frac{h\log(N)}{N}\right)\right) \\ &= \frac{6}{m} + \frac{2}{\pi^3}\sum_{h=1}^{m}\frac{1}{h^3} + \mathcal{O}\left(\frac{m\log(N)}{N}\right) = \mathcal{O}\left(\frac{6}{m} + \frac{2\,\zeta(3)}{\pi^3} + \frac{m\log(N)}{N}\right). \end{aligned} \tag{3.4}$$

Note that we have used (3.3) instead of (3.2) since in this case we want to take the best constants. Indeed $(\log(n))_{n=1}^{\infty}$ is not uniformly distributed mod 1, and therefore the constants play a more relevant role. To minimize the discrepancy, let $f(x) \coloneqq 6/x + 2\zeta(3)/\pi^3 + x\log(N)/N$ and note that

$$f'(x) = 0 \quad \iff \quad x = \sqrt{\frac{6N}{\log(N)}}.$$

Moreover $f''(\sqrt{6N/\log(N)}) > 0$, and thus $k \coloneqq \sqrt{6N/\log(N)}$ is a minimum for f. Therefore, continuing from (3.4) and choosing $m = [k]$, we get

$$D_N = \mathcal{O}\left(6\sqrt{\frac{\log(N)}{6N}} + \frac{2\,\zeta(3)}{\pi^3} + \sqrt{\frac{6N}{\log(N)}}\frac{\log(N)}{N}\right) = \frac{2\,\zeta(3)}{\pi^3} + \mathcal{O}\left(\sqrt{\frac{\log(N)}{N}}\right).$$

This concludes the proof. ■

In the next examples we will apply Erdös Turán Theorem to find an approximation of the discrepancy. Depending on the sequence, we will use different methods to find a bound for the exponential sum U_h. Let's start with an example where the exponential sum U_h is already bounded.

Example 3.5. Assume that $(x_n)_{n=1}^{\infty}$ has the property that there exists $\lambda \in \mathbb{R}_{>0}$ such that $|U_h| \leq h^{\lambda}/N$ for all $h \in \{1, \dots, [N^{1/(\lambda+1)}]\}$. Then

$$D_N = \mathcal{O}\left(N^{-1/(\lambda+1)}\right).$$

Proof. Let $m \in \mathbb{N}$ and $F(t) = t^{\lambda}$. Note that $F \in \mathrm{C}^1([1,m], \mathbb{R})$. Hence, using Euler summation Formula (see Appendix C, Theorem C.15), we have

$$\begin{aligned}\sum_{h=1}^{m} h^{\lambda-1} &= 1 + \int_1^m t^{\lambda-1}\,dt + (\lambda-1)\int_1^m \{t\}t^{\lambda-2}\,dt \\ &\leq 1 + \int_1^m t^{\lambda-1}\,dt + \lambda\int_1^m t^{\lambda-2}\,dt \\ &= 1 + \frac{m^{\lambda}-1}{\lambda} + \lambda\frac{m^{\lambda-1}-1}{\lambda-1}\mathbb{1}\{\lambda \neq 1\} + \log(m)\mathbb{1}\{\lambda = 1\}.\end{aligned}$$

In particular there exists $c_\lambda \in \mathbb{R}_{>0}$, which depends on λ, such that

$$\sum_{h=1}^{m} h^{\lambda-1} \leq c_\lambda m^{\lambda}. \tag{3.5}$$

Choose now $m \coloneqq [N^{1/(\lambda+1)}]$. If N big enough, using Corollary 3.3 we get

$$\begin{aligned}D_N &\leq C\left(\frac{1}{m} + \sum_{h=1}^{m}\frac{1}{h}\,|U_h|\right) \leq C\left(\frac{1}{m} + \frac{1}{N}\sum_{h=1}^{m} h^{\lambda-1}\right) \leq C\left(\frac{1}{m} + \frac{c_\lambda m^{\lambda}}{N}\right) \\ &\leq C\left(\frac{1}{N^{1/(\lambda+1)}-1} + c_\lambda N^{\lambda/(\lambda+1)-1}\right) = \mathcal{O}\left(N^{-1/(\lambda+1)}\right).\end{aligned}$$

This concludes the proof. ∎

The following two examples deal with sequences that are uniformly distributed mod 1 by Fejér Theorem (see Theorem 1.21 and Example 1.22). We will indeed use the idea of the proof of Theorem 1.20 to bound the exponential sum U_h.

Example 3.6. Let $a \in \mathbb{R}_{>0}$, $\sigma \in (0,1)$ and $\tau \coloneqq \min\{\sigma, (1-\sigma)/2\}$. Consider the sequence $(an^{\sigma})_{n=1}^{\infty}$. Then

$$D_N = \mathcal{O}(N^{-\tau}). \tag{3.6}$$

Proof. Let $f\colon \mathbb{R}_{\geq 1} \to \mathbb{R}, x \mapsto ax^{\sigma}$ and $n \in \mathbb{N}$. Note that f is differentiable on $[1,\infty)$. Hence, by mean value Theorem (see Theorem C.3 in Appendix C), there exists $c_n \in (n, n+1)$ such that

$$\Delta f(n) \coloneqq f(n+1) - f(n) = \frac{f(n+1)-f(n)}{(n+1)-n} = f'(c_n) = \frac{a\sigma}{c_n^{1-\sigma}}.$$

Therefore $\Delta f(n)$ is monotone decreasing. Since $c_n > n$, we have

$$\lim_{n\to\infty} \Delta f(n) = a\sigma \lim_{n\to\infty} \frac{1}{c_n^{1-\sigma}} \leq a\sigma \lim_{n\to\infty} \frac{1}{n^{1-\sigma}} = 0.$$

Moreover, since $c_n < n+1$, we have

$$\lim_{n\to\infty} n|\Delta f(n)| = a\sigma \lim_{n\to\infty} \frac{n}{c_n^{1-\sigma}} \geq a\sigma \lim_{n\to\infty} \frac{n}{(n+1)^{1-\sigma}} = \frac{a\sigma}{1-\sigma} \lim_{n\to\infty} (n+1)^{\sigma} = \infty$$

where in the third step we have used L'Hôpital rule. Therefore, the assumptions of Theorem 1.20 are satisfied. By (1.16), for all $h \in \mathbb{Z} \setminus \{0\}$ we get

$$\begin{aligned}
|U_h| &\leq \frac{\pi|h|a\sigma}{N} \sum_{n=1}^{N-1} \frac{1}{c_n^{1-\sigma}} + \frac{1}{\pi|h|a\sigma N} \left(c_N^{1-\sigma} + c_1^{1-\sigma}\right) \\
&\leq \frac{\pi|h|a\sigma}{N} \sum_{n=1}^{N-1} \frac{1}{n^{1-\sigma}} + \frac{1}{\pi|h|a\sigma N} \left((N+1)^{1-\sigma} + 2^{1-\sigma}\right) \\
&\leq \frac{\pi|h|a\sigma}{N} c_\sigma (N-1)^{\sigma} + \frac{1}{\pi|h|a\sigma N} \left((N+1)^{1-\sigma} + 2^{1-\sigma}\right) \\
&\leq \frac{\pi|h|a\sigma c_\sigma}{N} N^{\sigma} + \frac{1}{\pi|h|a\sigma N} \left((2N)^{1-\sigma} + 2^{1-\sigma}\right) \\
&= \frac{\pi|h|a\sigma c_\sigma}{N^{1-\sigma}} + \frac{2^{1-\sigma}}{\pi|h|a\sigma} \left(\frac{1}{N^{\sigma}} + \frac{1}{N}\right) = \mathcal{O}\left(\frac{|h|}{N^{1-\sigma}} + \frac{1}{|h|N^{\sigma}}\right)
\end{aligned}$$

where in the third step we have used (3.5) with $\lambda = \sigma$. Therefore, by Corollary 3.3, for all $m \in \mathbb{N}$ it holds

$$D_N = \mathcal{O}\left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left(\frac{h}{N^{1-\sigma}} + \frac{1}{hN^{\sigma}}\right)\right) = \mathcal{O}\left(\frac{1}{m} + \frac{m}{N^{1-\sigma}} + \frac{1}{N^{\sigma}}\right) \tag{3.7}$$

To minimize the discrepancy, let $f(x) \coloneqq 1/x + x/N^{1-\sigma} + 1/N^{\sigma}$ and note that

$$f'(x) = 0 \quad \iff \quad x = N^{(1-\sigma)/2}.$$

Moreover $f''(N^{(1-\sigma)/2}) > 0$, and thus $k \coloneqq N^{(1-\sigma)/2}$ is a minimum for f. Therefore, continuing from (3.7) and choosing $m = [k]$, we get

$$D_N = \mathcal{O}\left(\frac{1}{N^{(1-\sigma)/2}} + \frac{N^{(1-\sigma)/2}}{N^{1-\sigma}} + \frac{1}{N^{\sigma}}\right) = \mathcal{O}\left(\frac{1}{N^{(1-\sigma)/2}} + \frac{1}{N^{\sigma}}\right).$$

This is equivalent to (3.6). ■

Example 3.7. Let $a \in \mathbb{R}_{>0}$ and $\tau \in \mathbb{R}_{>1}$. The sequence $(a\log(n)^{\tau})_{n=1}^{\infty}$ satisfies

$$D_N = \mathcal{O}(\log(N)^{1-\tau}).$$

Proof. Let $f\colon \mathbb{R}_{\geq 1} \to \mathbb{R}, x \mapsto a\log(x)^\tau$ and $n \in \mathbb{N}$. Note that f is differentiable on $[1,\infty)$. Hence, by mean value Theorem (see Theorem C.3 in Appendix C), there exists $c_n \in (n, n+1)$ such that

$$\Delta f(n) \coloneqq f(n+1) - f(n) = \frac{f(n+1)-f(n)}{(n+1)-n} = f'(c_n) = \frac{a\tau\log(c_n)^{\tau-1}}{c_n}.$$

Therefore $\Delta f(n)$ is monotone decreasing. Since $n < c_n < n+1$, we have

$$\lim_{n\to\infty} \Delta f(n) = a\tau \lim_{n\to\infty} \frac{\log(c_n)^{\tau-1}}{c_n} \leq a\tau \lim_{n\to\infty} \frac{\log(n+1)^{\tau-1}}{n} = 0.$$

Moreover, again since $n < c_n < n+1$, we have

$$\lim_{n\to\infty} n|\Delta f(n)| = a\tau \lim_{n\to\infty} \frac{n\log(c_n)^{\tau-1}}{c_n} \geq a\tau \lim_{n\to\infty} \frac{n\log(n)^{\tau-1}}{n+1} = \infty.$$

Therefore, the assumptions of Theorem 1.20 are satisfied. Using (1.16), we get that for all $h \in \mathbb{Z}\setminus\{0\}$

$$\begin{aligned}
|U_h| &\leq \frac{\pi|h|a\tau}{N}\sum_{n=1}^{N-1}\frac{\log(c_n)^{\tau-1}}{c_n} + \frac{1}{\pi|h|a\tau N}\left(\frac{c_N}{\log(c_N)^{\tau-1}} + \frac{c_1}{\log(c_1)^{\tau-1}}\right)\\
&\leq \frac{\pi|h|a\tau}{N}\sum_{n=1}^{N-1}\frac{\log(n+1)^{\tau-1}}{n} + \frac{1}{\pi|h|a\tau N}\left(\frac{N+1}{\log(N)^{\tau-1}} + \frac{2}{\log(c_1)^{\tau-1}}\right)\\
&= \mathcal{O}\left(\frac{|h|}{N}\sum_{n=1}^{N}\frac{\log(n)^{\tau-1}}{n} + \frac{1}{|h|\log(N)^{\tau-1}}\right).
\end{aligned}$$

To estimate the last sum, we use the Euler summation formula (see Appendix C, Theorem C.15). We have

$$\begin{aligned}
\sum_{n=1}^{N}\frac{\log(n)^{\tau-1}}{n} &= \int_1^N \frac{\log(t)^{\tau-1}}{t}\,dt + \int_1^N \{t\}\frac{(\tau-1)\log(t)^{\tau-2} - \log(t)^{\tau-1}}{t^2}\,dt\\
&\leq \int_1^N \frac{\log(t)^{\tau-1}}{t}\,dt + \tau\int_1^N \frac{\log(t)^{\tau-2}}{t}\,dt\\
&= \log(N)^\tau + \tau\log(N)^{\tau-1} = \mathcal{O}\left(\log(N)^\tau\right).
\end{aligned}$$

Therefore

$$|U_h| = \mathcal{O}\left(\frac{|h|\log(N)^\tau}{N} + \frac{1}{|h|\log(N)^{\tau-1}}\right)$$

By Corollary 3.3 we conclude that for all $m \in \mathbb{N}$ we have

$$\begin{aligned}
D_N &= \mathcal{O}\left(\frac{1}{m} + \sum_{h=1}^{m}\frac{1}{h}\left(\frac{h\log(N)^\tau}{N} + \frac{1}{h\log(N)^{\tau-1}}\right)\right)\\
&= \mathcal{O}\left(\frac{1}{m} + \frac{m\log(N)^\tau}{N} + \frac{1}{\log(N)^{\tau-1}}\right).
\end{aligned} \tag{3.8}$$

To minimize the discrepancy, proceeding as in Example 3.6, we get that

$$k \coloneqq \left(\frac{N}{\log(N)^\tau}\right)^{1/2}$$

minimizes the RHS of (3.8). Therefore, choosing $m = [k]$, we get

$$D_N = \mathcal{O}\left(\frac{\log(N)^{\tau/2}}{N^{1/2}} + \frac{1}{\log(N)^{\tau-1}}\right) = \mathcal{O}\left(\frac{1}{\log(N)^{\tau-1}}\right).$$

This concludes the proof. ■

The next two examples take up special sequences that we saw in Section 1.3, which are uniformly distributed mod 1 thanks to Theorem 1.28, which gives a direct bound of U_h for this type of sequences.

Example 3.8. The sequence $(n\log(n))_{n=1}^{\infty}$ satisfies

$$D_N = \mathcal{O}\left(N^{-1/3}\log(N)^{2/3}\right).$$

Proof. In Example 1.29 we have already shown that for all $h \in \mathbb{Z} \setminus \{0\}$ we have

$$|U_h| \leq \frac{1}{N}\left(|h|\log(N) + 2\right)\left(\frac{4\sqrt{N}}{\sqrt{|h|}} + 3\right).$$

Therefore, if N is large enough, we have: $|U_h| = \mathcal{O}(\sqrt{|h|N}\log(N)/N)$. Now, using Corollary 3.3, for all $m \in \mathbb{N}$ we have

$$\begin{aligned} D_N &= \mathcal{O}\left(\frac{1}{m} + \sum_{h=1}^{m}\frac{\sqrt{hN}\log(N)}{Nh}\right) \\ &= \mathcal{O}\left(\frac{1}{m} + \frac{\log(N)}{\sqrt{N}}\sum_{h=1}^{m}h^{-1/2}\right) = \mathcal{O}\left(\frac{1}{m} + \frac{\log(N)}{\sqrt{N}}\sqrt{m}\right) \end{aligned} \tag{3.9}$$

where in the last step we have used (3.5) with $\lambda = 1/2$. To minimize the discrepancy, let $f(x) \coloneqq 1/x + \log(N)\sqrt{x}/\sqrt{N}$ and note that

$$f'(x) = 0 \quad \Longleftrightarrow \quad x = \left(\frac{4N}{\log(N)^2}\right)^{1/3}.$$

Moreover $f''((4N/\log(N)^2)^{1/3}) > 0$, and thus $k \coloneqq (4N/\log(N)^2)^{1/3}$ is a minimum for f. Therefore, continuing from (3.9) and choosing $m = [k]$, we get

$$D_N = \mathcal{O}\left(\left(\frac{\log(N)^2}{4N}\right)^{1/3} + \frac{\log(N)}{\sqrt{N}}\frac{(4N)^{1/6}}{\log(N)^{1/3}}\right) = \mathcal{O}\left(\frac{\log(N)^{2/3}}{N^{1/3}}\right).$$

This concludes the proof. ■

Example 3.9. The sequence $(n\log(\log(n)))_{n=2}^{\infty}$ satisfies

$$D_N = \mathcal{O}\left(N^{-1/3}\log(N)^{1/3}\log(\log(N))^{2/3}\right).$$

Proof. In Example 1.31 we have already shown that for all $h \in \mathbb{Z}\setminus\{0\}$ we have

$$|U_h| \leq \frac{1}{N} + \frac{|h|}{N}\mathcal{O}(\log(\log(N)))\left(\frac{4\sqrt{N\log(N)^2}}{\sqrt{|h|(\log(N)-1)}}+3\right)$$

where $N \geq 3$. Therefore, if N is large enough, we have

$$|U_h| = \mathcal{O}\left(\frac{\sqrt{|h|N\log(N)}\log(\log(N))}{N}\right).$$

Now, using Corollary 3.3, for all $m \in \mathbb{N}$ we have

$$\begin{aligned} D_N &= \mathcal{O}\left(\frac{1}{m} + \sum_{h=1}^{m}\frac{\sqrt{hN\log(N)}\log(\log(N))}{Nh}\right) \\ &= \mathcal{O}\left(\frac{1}{m} + \frac{\sqrt{\log(N)}\log(\log(N))}{\sqrt{N}}\sum_{h=1}^{m}h^{-1/2}\right) \\ &= \mathcal{O}\left(\frac{1}{m} + \frac{\sqrt{\log(N)}\log(\log(N))}{\sqrt{N}}\sqrt{m}\right) \end{aligned} \tag{3.10}$$

where in the last step we have used (3.5) with $\lambda = 1/2$. To minimize the discrepancy, proceeding as in Example 3.8, we get that

$$k \coloneqq \left(\frac{4N}{\log(N)\log(\log(N))^2}\right)^{1/3}$$

minimizes the RHS of (3.10). Therefore, choosing $m = [k]$, we get

$$D_N = \mathcal{O}\left(\frac{\log(N)^{1/3}\log(\log(N))^{2/3}}{N^{1/3}}\right).$$

This concludes the proof. ■

As last example, consider the following special case.

Example 3.10. Consider the sequence $(n\theta)_{n=1}^{\infty}$, where $\theta \in \mathbb{R}$ is irrational. To find the discrepancy of this sequence, we can not use the previous methods since $(n\theta)_{n=1}^{\infty}$ does not satisfies the assumptions of Theorem 1.20 (since $\Delta n\theta = \theta \nrightarrow 0$), and of Theorem 1.28 (since $(n\theta)'' = 0$). The only thing we can do is use the bound for U_h we have found in Example 1.14. In particular, for all $h \in \mathbb{Z}\setminus\{0\}$ it holds

$$|U_h| \leq \frac{1}{N}\frac{1}{|\sin(\pi h\theta)|}.$$

Therefore, by Corollary 3.3, for all $m \in \mathbb{N}$ we have

$$D_N \leq C\left(\frac{1}{m} + \frac{1}{N}\sum_{h=1}^{m}\frac{1}{h|\sin(\pi h\theta)|}\right). \tag{3.11}$$

Note that $F\colon [1,m] \to \mathbb{R}, x \mapsto 1/(x|\sin(\pi x\theta)|)$ is not defined in some points since for m large enough, there exists some some $x \in [1,m]$ with $\sin(\pi x\theta) = 0$. Therefore we can not use Euler summation Formula to bound the discrepancy. We have to proceeded another way.

We will return on the discrepancy of $(n\theta)_{n=1}^{\infty}$ in Section 3.3.

3.2 Proof of the Theorem

In this section we focus on the proof of Theorem 3.1. Since it is quite long, we divide it into several parts. We start with Remark 3.11, which tell us a property of the discrepancy we will use in the actual proof. Proceeding we will state an implicit bound for D_N (Lemma 3.13), which will be then developed in the last part. The actual proof is divided in two parts: a particular and the general case. The proof follows Niederreiter and Philipp [52] and Kuipers and Niederreiter [44, pp. 112-114].

Remark 3.11. Discrepancy is invariant under shifting the sequence mod 1.

Proof-sketch. Assume that $\omega = (x_n)_{n=1}^{\infty}$ is shifted by a constant $c \in \mathbb{R}_{>0}$, and define the shifted sequence as $\tau \coloneqq (x_n + c)_{n=1}^{\infty}$. For a set $E \subseteq [0,1)$ define the quantity

$$Q_\omega(E) \coloneqq \left|\frac{A(E,N)}{N} - \lambda(E)\right|.$$

First note that $Q_\omega(E) = Q_\omega([0,1)\setminus E)$ $(*)$. Indeed

$$\begin{aligned} Q_\omega([0,1)\setminus E) &= \left|\frac{A([0,1)\setminus E,N)}{N} - \lambda([0,1)\setminus E)\right| = \left|\frac{N - A(E,N)}{N} - (1-\lambda(E))\right| \\ &= \left|-\frac{A(E,N)}{N} + \lambda(E)\right| = Q_\omega(E). \end{aligned}$$

Let $\alpha, \beta \in [0,1]$ be such that $D_N(\omega) = Q_\omega([\alpha,\beta))$, and define $\tilde{\alpha} \coloneqq \alpha + c$ and $\tilde{\beta} \coloneqq \beta + c$. Then we have three possibilities.

- Assume that $c + \beta < 1$. In this case it's easy to see that the interval $[\alpha,\beta)$ is shifted to the interval $[\tilde{\alpha},\tilde{\beta})$ but its length is preserved. We can therefore convince ourselves that the discrepancy will be

  $$D_N(\tau) = Q_\tau([\tilde{\alpha},\tilde{\beta})) = Q_\omega([\alpha,\beta)) = D_N(\omega).$$

  This can be proved formally using the fact that, even if the sequence has a complicated distribution, using the $(*)$ condition, we can show that, by shifting the sequence, neither accumulations of points in a small space nor large spaces with a low density of points can appear, that give a larger Q than that given by the interval $[\tilde{\alpha},\tilde{\beta})$. Figure 3.1 presents an example (ω is represented in blue, while the shifted sequence τ is in red) where $c = 0.5$.

- Assume that $c+\alpha \geq 1$. As in the previous case, the length of the interval $[\alpha, \beta)$ is preserved, and so is the discrepancy, since we only consider the fractional parts of the sequence. Notice that Figure 3.1 can be viewed as an example of this situation with $c = 1.5$.

- Assume that $c+\alpha < 1$ and $c+\beta \geq 1$. In this case the interval $[\alpha, \beta)$ is split in two. Note however that the set $[0,1) \setminus [\alpha, \beta)$ is shifted in the interval $[\{\tilde{\beta}\}, \tilde{\alpha})$ mod 1. Therefore, adding the $(*)$ condition, we get the same discrepancy. For an example of this situation see Figure 3.2.

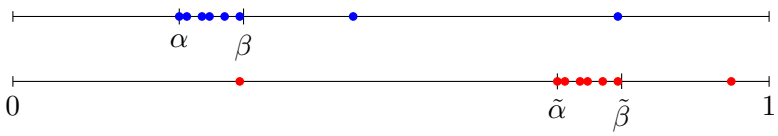

Figure 3.1: Example of shifted sequence with $N = 8$ and $c = 0.5$ mod 1.

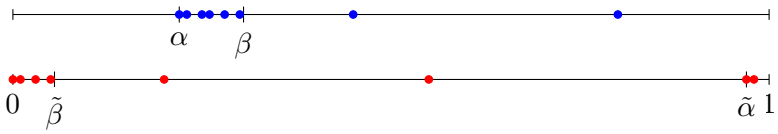

Figure 3.2: Example of shifted sequence with $N = 8$ and $c = 0.75$.

This concludes the proof-sketch. ■

As for the proof of Leveque inequality, also in this case we define a quantity that will be useful for the computations.

Notation 3.12. We define the map $\Delta_N \coloneqq \Delta_{N,\omega}$ by

$$\Delta_{N,\omega} \colon \mathbb{R} \to (-1,1)\,,\ x \mapsto R_{N,\omega}(x)/N.$$

As a first step, using the fact that $|\Delta_N(x+t)|$ can be bounded in terms of $D_N^\star$ for some particular value $t \in \mathbb{R}$, we can find an implicit bound for $D_N^\star$ which depends on an integral involving $\Delta_N(x)$.

Lemma 3.13. *Let $m \in \mathbb{N}$. Then there exists $t \in [0,1)$ such that*

$$\int_{-1/2}^{1/2} \Delta_N(x+t) \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\, dx \geq \frac{1}{2}\left((m+1)D_N^\star - 3\right). \tag{3.12}$$

Proof. By definition of Δ_N, there exists some $b \in [0,1)$ such that $\Delta_N(b) = D_N^\star$ or $\Delta_N(b) = -D_N^\star$ (note that $b \neq 1$ since $\Delta_N(1) = 0$, but this can not be a discrepancy

since $D_N^\star \geq 1/(2N)$ by Corollary 2.12). We consider only the first case (the second is completely similar). Therefore, for $t \in (b, b + D_N^\star]$, it holds

$$\begin{aligned}\Delta_N(t) &= D_N^\star + \Delta_N(t) - \Delta_N(b) \\ &= D_N^\star + \left(A([0,t),N) - A([0,b),N)\right)/N - t + b \geq D_N^\star + b - t.\end{aligned}$$

So choosing $t \coloneqq b + D_N^\star/2$, for $x \in (-D_N^\star/2, D_N^\star/2)$ we get

$$\Delta_N(x+t) \geq D_N^\star + b - (x+t) = D_N^\star/2 - x. \tag{3.13}$$

Instead, for $x \in [-1/2, 1/2] \setminus (-D_N^\star/2, D_N^\star/2)$, we have simply that

$$\Delta_N(x+t) \geq -\Delta_N(b) = -D_N^\star. \tag{3.14}$$

Now we can split the integral on the LHS of (3.12) into three integrals as follows:

$$\begin{aligned}&\int_{-1/2}^{1/2} \Delta_N(x+t) \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx \\ &\quad= \left(\int_{-1/2}^{-D_N^\star/2} + \int_{-D_N^\star/2}^{D_N^\star/2} + \int_{D_N^\star/2}^{1/2}\right) \Delta_N(x+t) \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx \\ &\quad\geq -D_N^\star \int_{-1/2}^{-D_N^\star/2} \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx \\ &\qquad + \int_{-D_N^\star/2}^{D_N^\star/2} \left(\frac{D_N^\star}{2} - x\right) \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx - D_N^\star \int_{D_N^\star/2}^{1/2} \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx \\ &\quad= D_N^\star \int_0^{D_N^\star/2} \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx - 2D_N^\star \int_{D_N^\star/2}^{1/2} \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx \\ &\quad= D_N^\star \int_0^{1/2} \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx - 3D_N^\star \int_{D_N^\star/2}^{1/2} \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}\,dx \\ &\quad\geq \frac{m+1}{2} D_N^\star - 3D_N^\star \int_{D_N^\star/2}^{1/2} \frac{1}{4x^2}\,dx = \frac{m+1}{2} D_N^\star + \frac{3}{2}(D_N^\star - 1) \\ &\quad\geq \frac{1}{2}\left((m+1)D_N^\star - 3\right)\end{aligned}$$

where in the second step we have used (3.13) and (3.14), in the third step we have used that $\sin((m+1)\pi x)^2/\sin(\pi x)^2$ is even, and in the fifth step we have computed the first integral (for explicit computations see Appendix B, Example B.12), while for the second integral we have used that for $x \in [0, 1/2]$ it holds

$$\sin((m+1)\pi x) \leq 1 \qquad \text{and} \qquad \sin(\pi x) \geq 2x.$$

This concludes the proof. ■

Now it remains only to find an upper bound for the integral on the LHS of (3.12) which depends on the exponential sum U_h.

Proof of Theorem 3.1. We divide the proof in two parts.

▶ Assume first that the sequence $(x_n)_{n=1}^{\infty}$ satisfies

$$\int_0^1 \Delta_N(x)\,dx = 0. \tag{3.15}$$

Using (2.36) and considering $-h$ instead of h we get that for all $h \in \mathbb{Z} \setminus \{0\}$

$$\int_0^1 \Delta_N(x) e^{2\pi i h x}\,dx = \frac{U_h}{-2\pi i h}. \tag{3.16}$$

Now choose $m \in \mathbb{N}$ and let $a \in \mathbb{R}$. By (3.16) we have

$$\begin{aligned}
&\sum_{\substack{h=-m \\ h\neq 0}}^{m} \left[(m+1-|h|)e^{-2\pi i h a}\frac{U_h}{-2\pi i h}\right] \\
&\qquad = \int_0^1 \Delta_N(x) \sum_{h=-m}^{m} \left[(m+1-|h|)e^{2\pi i h(x-a)}\right] dx \\
&\qquad = \int_{-a}^{1-a} \Delta_N(y+a) \sum_{h=-m}^{m} \left[(m+1-|h|)e^{2\pi i h y}\right] dy \\
&\qquad = \int_{-1/2}^{1/2} \Delta_N(y+a) \sum_{h=-m}^{m} \left[(m+1-|h|)e^{2\pi i h y}\right] dy \\
&\qquad = \int_{-1/2}^{1/2} \Delta_N(y+a)\frac{\sin((m+1)\pi y)^2}{\sin(\pi y)^2}\,dy
\end{aligned} \tag{3.17}$$

where in the first step we can drop the assumption $h \neq 0$ in the sum because of (3.15), in the second step we have made the substitution $y \coloneqq x - a$, in the third step we have used that $\Delta_N(y)$ and $e^{2\pi i h y}$ have period 1, and for the last step see Appendix B, Example B.8. The idea is now to bound the quantity

$$\left| \int_{-1/2}^{1/2} \Delta_N(y+a)\frac{\sin((m+1)\pi y)^2}{\sin(\pi y)^2}\,dy \right|. \tag{3.18}$$

A lower bound is given by Lemma 3.13. For the upper bound we have

$$\begin{aligned}
&\left| \int_{-1/2}^{1/2} \Delta_N(y+a)\frac{\sin((m+1)\pi y)^2}{\sin(\pi y)^2}\,dy \right| \overset{(3.17)}{\leq} \frac{1}{2\pi} \sum_{\substack{h=-m \\ h\neq 0}}^{m} \left[(m+1-|h|)\frac{|U_h|}{|h|}\right] \\
&= \frac{1}{\pi}\sum_{h=1}^{m} \left[(m+1-|h|)\frac{|U_h|}{|h|}\right] = \frac{1}{\pi}\sum_{h=1}^{m} \left[(m+1-h)\frac{|U_h|}{h}\right]
\end{aligned}$$

where in the second step we have used Remark 2.35. Hence putting together the two bounds for (3.18) we get

$$\frac{1}{2}\left((m+1)D_N^{\star} - 3\right) \leq \frac{1}{\pi}\sum_{h=1}^{m} \left[(m+1-h)\frac{|U_h|}{h}\right].$$

And therefore

$$D_N^\star \le \frac{3}{m+1} + \frac{2}{\pi}\sum_{h=1}^{m}\frac{(m+1-h)|U_h|}{(m+1)h} = \frac{3}{m+1} + \frac{2}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|U_h|.$$

But recalling that $D_N^\star \ge D_N/2$ by Theorem 2.10, we get (3.1).

▶ Assume now that $(x_n)_{n=1}^\infty$ does not satisfies (3.15). For $c \in [0,1)$ we have

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}(\{y_n + c\} - y_n) \\
&\quad= \frac{1}{N}\sum_{\substack{n=1\\ y_n+c<1}}^{N}(\{y_n+c\}-y_n) + \frac{1}{N}\sum_{\substack{n=1\\ y_n+c\ge 1}}^{N}(\{y_n+c\}-y_n) \\
&\quad= \frac{1}{N}\sum_{\substack{n=1\\ y_n+c<1}}^{N} c + \frac{1}{N}\sum_{\substack{n=1\\ y_n+c\ge 1}}^{N}(c-1) \\
&\quad= \frac{1}{N}\left[c\cdot A([0,1-c),N) + (c-1)(N - A([0,1-c),N))\right] \\
&\quad= \frac{1}{N}\left[A([0,1-c),N) - (1-c)N\right] = \Delta_N(1-c).
\end{aligned} \tag{3.19}$$

By (2.35) we have

$$\int_0^1 \Delta_N(t)\,dt = -\frac{S_N}{N} = \frac{1}{2} - \frac{1}{N}\sum_{n=1}^{N} y_n. \tag{3.20}$$

Denote by s the RHS of (3.20). Since $s \neq 0$, we have two possibilities:

- Assume that $s > 0$. By (3.20), there exists $x \in (0,1)$ with $\Delta_N(x) \ge s$. But since Δ_N is piece-wise linear with slope -1 and positive jumps of length $1/N$, and since $\Delta_N(1) = 0$, there must be an $y \in [x,1)$ such that $\Delta_N(y) = s$. An example can be seen in Figure 3.3.

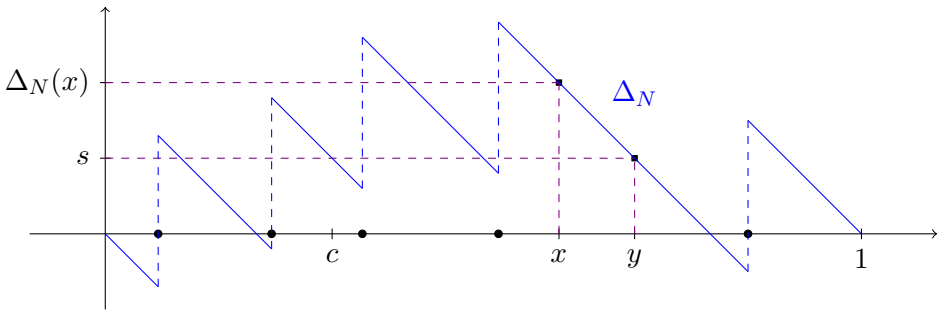

Figure 3.3: Example of graph of Δ_N with $s = 0.1$ and $N = 5$.

- Assume that $s < 0$. By (3.20), there exists $x \in (0,1)$ with $\Delta_N(x) \leq s$. But since Δ_N is piece-wise linear with slope -1 and positive jumps, and since $\Delta_N(0) = 0$, there must be an $y \in (0, x]$ such that $\Delta_N(y) = s$.

In each of the two cases we choose $c \coloneqq 1 - y$. Hence by (3.19) we get

$$\frac{1}{N}\sum_{n=1}^{N}(\{y_n + c\} - y_n) = \Delta_N(1-c) = s = \frac{1}{2} - \frac{1}{N}\sum_{n=1}^{N} y_n$$

where in the last step we have used (3.20). It follows that

$$\frac{1}{N}\sum_{n=1}^{N}\{y_n + c\} = \frac{1}{2}. \tag{3.21}$$

Therefore, if we let $\tilde{\omega} \coloneqq (y_n + c)_{n=1}^{\infty}$, we have

$$\begin{aligned}\int_0^1 \Delta_{N,\tilde{\omega}}(x)\,dx = \frac{1}{N}\int_0^1 R_{N,\tilde{\omega}}(x)\,dx &= -\frac{1}{N}\sum_{n=1}^{N}\left(\{y_n + c\} - \frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{N}\sum_{n=1}^{N}\{y_n + c\} = 0\end{aligned}$$

where the second step holds by (2.35), and the last step holds by (3.21). Therefore $\tilde{\omega}$ satisfies the assumption of the first part of the proof. Hence (3.1) holds for $\tilde{\omega}$, more precisely

$$D_N(\tilde{\omega}) \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h(y_n+c)}\right|. \tag{3.22}$$

But note that

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h(y_n+c)}\right| = \left|e^{2\pi i h(c-[x_n])}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right| = \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right|. \tag{3.23}$$

Moreover, using Remark 3.11, we have also that

$$D_N(\tilde{\omega}) = D_N(\omega). \tag{3.24}$$

Hence, substituting (3.23) and (3.24) into (3.22), we get (3.1). ■

There are several generalization of Erdös-Turán Theorem, such as those developed by Elliott [43], Fainleib [15] and Niederreiter and Philipp [51, 52]. For generalizations to more dimension, there exists a version of the theorem called *Theorem of Erdös-Turán-Koksma*. A proof can be found in Koksma [38] or Szüsz [61].

3.3 Application: the discrepancy of $(n\alpha)_{n=1}^{\infty}$

One of the most important applications of the Erdös Turán Theorem (Theorem 3.1) allows us to compute, with a good approximation, the discrepancy of the sequence $(n\alpha)_{n=1}^{\infty}$, where α is an irrational number.

For Theorem 3.21 we refer to Hecke [26] and Ostrowski [54]. See also Hardy and Littlewood [23, 24] and Behnke [4]. For Theorem 3.22 we refer to Behnke [5], while for Theorem 3.25 see Niederreiter [49]. A complete overview can be found in Kuipers and Niederreiter [44, pp. 122-126].

We fix an irrational number $\alpha \in \mathbb{R}$, and we denote $\omega = (n\alpha)_{n=1}^{\infty}$. We start by using Theorem 3.1 to find a first implicit bound for the discrepancy.

Notation 3.14. For $t \in \mathbb{R}$, we denote: $\langle t \rangle := \min_{n\in\mathbb{Z}}|t-n| = \min(\{t\}, 1-\{t\})$.

Lemma 3.15. *There exists a constant $C \in \mathbb{R}_{>0}$ such that for all $m \in \mathbb{N}$*

$$D_N \leq C\left(\frac{1}{m} + \frac{1}{N}\sum_{h=1}^{m}\frac{1}{h\langle h\alpha\rangle}\right). \tag{3.25}$$

Proof. By the Proof of Example 1.14, we have that $N|U_h| \leq 1/|\sin(\pi h\alpha)|$ for all $h \in \mathbb{Z}\setminus\{0\}$. Therefore, using Corollary 3.3 we get

$$D_N \leq C\left(\frac{1}{m} + \sum_{h=1}^{m}\frac{1}{h}|U_h|\right) \leq C\left(\frac{1}{m} + \frac{1}{N}\sum_{h=1}^{m}\frac{1}{h|\sin(\pi h\alpha)|}\right). \tag{3.26}$$

But now notice that for all $p \in \mathbb{N}$

$$\sin(\pi(h\alpha - p)) = \pm\sin(\pi h\alpha) \quad \text{and} \quad \sin(\pi(p - h\alpha)) = \pm\sin(\pi h\alpha).$$

Therefore, since $\langle h\alpha\rangle \in \{h\alpha - p, p - h\alpha\}$ for some $p \in \mathbb{N}$, we get that

$$\sin(\pi\langle h\alpha\rangle) = |\sin(\pi h\alpha)|.$$

Moreover recall that $\sin(\pi x) \geq 2x$ for $x \in [0, 1/2]$. Therefore, since $\langle h\alpha\rangle \in [0, 1/2]$, continuing from (3.26) we get

$$D_N \leq C\left(\frac{1}{m} + \frac{1}{N}\sum_{h=1}^{m}\frac{1}{2h\langle h\alpha\rangle}\right) \leq C\left(\frac{1}{m} + \frac{1}{N}\sum_{h=1}^{m}\frac{1}{h\langle h\alpha\rangle}\right).$$

This concludes the proof. ∎

The idea is now to rewrite the RHS of (3.25) another way. We can do it if α is of a particular *type*. Therefore we have to recall some properties of Diophantine Approximation (see also Appendix A).

Definition 3.16. Let $E \subseteq \mathbb{R}$ be a subset such that $\mathbb{N} \subseteq E$ and let $\psi\colon E \to \mathbb{R}$ be a non-decreasing and positive function. Then α is called *of type* $< \psi$, if for all $q \in \mathbb{N}$ it holds: $q\langle q\alpha\rangle \geq 1/\psi(q)$.

Remark 3.17. Assume that ψ is constant. If the irrational $\alpha \in \mathbb{R}$ is of type $< \psi$, then α is called *of constant type*.

Definition 3.18. Let $\alpha \in \mathbb{R}$ be irrational and $\eta \in \mathbb{R}_{>0} \cup \{\infty\}$. Then α is called *of type* η if η can be written as follows:

$$\eta = \sup \left\{ \gamma \in \mathbb{R}_{>0} \,\middle|\, \varliminf_{q \to \infty} q^{\gamma} \langle q\alpha \rangle = 0 \right\}.$$

The two definitions are connected in the following way.

Proposition 3.19. *Let $\alpha \in \mathbb{R}$ be irrational. Then α is of type η if and only if*

$$\eta = \inf \left\{ \tau \in \mathbb{R} \,\middle|\, \exists\, c \in \mathbb{R}_{>0} \,:\, \alpha \text{ is of type } < cq^{\tau - 1} \right\}. \tag{3.27}$$

The proof is covered in Appendix A. With this Proposition we are able to estimate the RHS of (3.25) for α of type $< \psi$.

Lemma 3.20. *Assume that α is of type $< \psi$. Then for all $m \in \mathbb{N}$*

$$\sum_{h=1}^{m} \frac{1}{h\langle h\alpha \rangle} = \mathcal{O}\left(\psi(2m) \log(m) + \sum_{h=1}^{m} \frac{\psi(2h) \log(h)}{h} \right). \tag{3.28}$$

Proof. Let $h \in \{1, \dots, m\}$. For $p, q \in \mathbb{Z}$ with $0 \leq p < q \leq h$ we have

$$|\langle q\alpha \rangle - \langle p\alpha \rangle| \geq \langle q\alpha \pm p\alpha \rangle = \langle (q \pm p)\alpha \rangle \geq \frac{1}{(q \pm p)\psi(q \pm p)} \geq \frac{1}{2h\psi(2h)} \tag{3.29}$$

where in the second inequality we have used that α is of type $< \psi$, and in the last inequality we have used that ψ is positive and non-decreasing. Now (3.29) tell us that in each of the intervals

$$\left[0, \frac{1}{2h\psi(2h)}\right) \quad , \quad \left[\frac{1}{2h\psi(2h)}, \frac{2}{2h\psi(2h)}\right) \quad , \quad \cdots \quad , \quad \left[\frac{h}{2h\psi(2h)}, \frac{h+1}{2h\psi(2h)}\right)$$

there is at most one number of the form $\langle j\alpha \rangle$ where $j \in \{1, \dots, h\}$, except for the first interval, which has no numbers of this type. Hence

$$s_h \coloneqq \sum_{j=1}^{h} \frac{1}{\langle j\alpha \rangle} \leq \sum_{j=1}^{h} \frac{2h\psi(2h)}{j} = \mathcal{O}\left(h\psi(2h) \log(h) \right) \tag{3.30}$$

where in the last step we have used that the harmonic series $\sum_{k=1}^{N} 1/k$ can be approximated by $\log(N) + \gamma$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Now, using summation by parts (see Appendix C, Theorem C.13) with $a_h = 1/\langle h\alpha \rangle$ and $b_h = 1/h$, we get

$$\sum_{h=1}^{m} \frac{1}{h\langle h\alpha \rangle} = \frac{s_m}{m} + \sum_{h=1}^{m} \frac{s_h}{h(h+1)}. \tag{3.31}$$

Inserting (3.30) in (3.31) we get exactly (3.28). ∎

Now, taking advantage of the connection between the Definitions 3.16 and 3.18, we can bound the discrepancy of $(n\alpha)_{n=1}^{\infty}$ in the case in which α is of type η.

Theorem 3.21. *Let α be of finite type η. Then, for all $\epsilon \in \mathbb{R}_{>0}$ it holds*

$$D_N = \mathcal{O}\left(N^{-1/\eta+\epsilon}\right). \tag{3.32}$$

Proof. Fix some $\epsilon \in \mathbb{R}_{>0}$. By Proposition 3.19 there exists $c \in \mathbb{R}_{>0}$ such that α is of type $< cq^{\eta-1+\epsilon/2}$. Hence, by Lemma 3.20, for all $m \in \mathbb{N}$

$$\begin{aligned}\sum_{h=1}^{m} \frac{1}{h\langle h\alpha\rangle} &= \mathcal{O}\left((2m)^{\eta-1+\epsilon/2}\log(m) + \sum_{h=1}^{m} \frac{(2h)^{\eta-1+\epsilon/2}\log(h)}{h}\right) \\ &= \mathcal{O}\left(m^{\eta-1+\epsilon} + \sum_{h=1}^{m} h^{\eta-2+\epsilon}\right) = \mathcal{O}\left(m^{\eta-1+\epsilon}\right)\end{aligned}$$

where in the second step we have used that $\log(x) = \mathcal{O}(x^{\epsilon/2})$, and in the last step we have used (3.5) with $\lambda = \eta - 1 + \epsilon$. Therefore, choosing $m = [N^{1/\eta}]$ and using Lemma 3.15, we infer that

$$D_N = \mathcal{O}\left(\frac{1}{m} + \frac{m^{\eta-1+\epsilon}}{N}\right) = \mathcal{O}\left(N^{-1/\eta} + N^{-1/\eta+\epsilon/\eta}\right) = \mathcal{O}\left(N^{-1/\eta+\epsilon}\right).$$

This concludes the proof. ∎

Regarding a lower bound for D_N, we have the following analogous result.

Theorem 3.22. *Let α be of finite type η. Then, for all $\epsilon \in \mathbb{R}_{>0}$ it holds*

$$D_N = \Omega\left(N^{-1/\eta-\epsilon}\right). \tag{3.33}$$

Proof. Fix some $\epsilon \in \mathbb{R}_{>0}$. Let $\delta \coloneqq \eta\epsilon/(1+\epsilon)$. Notice that $0 < \delta < \eta$ and

$$\frac{1}{\eta-\delta} = \frac{1}{\eta} + \epsilon. \tag{3.34}$$

Since α is of finite type η, we have

$$\varliminf_{q\to\infty} q^{\eta-\delta/2}\langle q\alpha\rangle = 0.$$

In particular for all $k \in \mathbb{N}$ there is $q_0 \in \mathbb{N}$ such that for all $q \in \mathbb{N}_{\geq q_0}$ we have that $q^{\eta-\delta/2}\langle q\alpha\rangle < 1/k$. Choosing $k = 1$, it follows $\langle q\alpha\rangle < q^{-\eta+\delta/2}$ for all $q \in \mathbb{N}_{\geq q_0}$. For each of those q's, let $p \in \mathbb{Z}$ be such that $\langle q\alpha\rangle = |q\alpha - p|$. Then

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{1+\eta-\delta/2}} \tag{3.35}$$

for all $q \in \mathbb{N}_{\geq q_0}$. Choose now some $\tilde{q} \in \mathbb{N}_{\geq q_0}$ such that $\tilde{q} > 4^{2/\delta}$ and set $N \coloneqq [\tilde{q}^{\,\eta-\delta}]$. By (3.35) we can write α explicitly as

$$\alpha = \frac{p}{\tilde{q}} + \frac{\theta}{\tilde{q}^{\,1+\eta-\delta/2}.}$$

where $\theta \in \mathbb{R}$ with $|\theta| < 1$. For $n \in \{1, \dots, N\}$, we infer that $n\alpha = np/\tilde{q} + \theta_n$ where

$$|\theta_n| = \left|n\theta\tilde{q}^{-1-\eta+\delta/2}\right| < \tilde{q}^{\eta-\delta}\tilde{q}^{-1-\eta+\delta/2} = \tilde{q}^{-1-\delta/2}. \tag{3.36}$$

Therefore, if $J \coloneqq [\tilde{q}^{-1-\delta/2}, \tilde{q}^{-1} - \tilde{q}^{-1-\delta/2})$, we have the following.

- If $p \neq 0$, then $\{np/\tilde{q}\} = k/\tilde{q}$ for some $k \in \mathbb{N}_{<\tilde{q}}$. Thus $\{np/\tilde{q}\} \notin [0, 1/\tilde{q})$. Hence, by (3.36) we conclude that $\{n\alpha\} \notin J$.
- If $p = 0$, then $n\alpha = \theta_n$. Thus $\{n\alpha\} \in [0, \tilde{q}^{-1-\delta/2}) \cup (1 - \tilde{q}^{-1-\delta/2}, 1)$ by (3.36). In particular $\{n\alpha\} \notin J$.

In both cases $\{n\alpha\} \notin J$ for all $n \in \{1, \dots, N\}$. Therefore

$$D_N \geq \left|\frac{A(J,N)}{N} - \lambda(J)\right| = \lambda(J) = \frac{1}{\tilde{q}} - \frac{2}{\tilde{q}^{1+\delta/2}} \geq \frac{1}{2\tilde{q}} \tag{3.37}$$

where the last step since $\tilde{q} > 4^{2/\delta}$. Moreover, by definition of N we have also that $\tilde{q}^{\eta-\delta} \leq N + 1 \leq 2N$, and thus $\tilde{q} \leq (2N)^{1/(\eta-\delta)}$ Combining this with (3.37) we get

$$D_N \geq \frac{1}{2}(2N)^{-1/(\eta-\delta)} = 2^{-1/\eta-\epsilon-1}N^{-1/\eta-\epsilon}$$

where in the last step we have used (3.34). ■

Summarizing, for an irrational number α of type η, we have shown that the discrepancy of $(n\alpha)_{n=1}^{\infty}$ satisfies

$$C_1 N^{-1/\eta-\epsilon} \leq D_N \leq C_2 N^{-1/\eta+\epsilon}$$

for all $\epsilon \in \mathbb{R}_{>0}$, where $C_1, C_2 \in \mathbb{R}_{>0}$ are constants. Taking $\epsilon \to 0$, D_N approaches the value $N^{-1/\eta}$ up to a constant. This means that, the smaller the type of α, the smaller the discrepancy. As one can understand reading Definition 3.18, the type of α tell us how good can α be approximated by rational numbers. Therefore, if α is badly approximable by rational numbers, then the sequence $(n\alpha)_{n=1}^{\infty}$ will have a small discrepancy. In the next example we discuss the limit case.

Example 3.23. Assume that α is of type $\eta = 1$. This is the smallest type for an irrational number. Therefore the discrepancy will be the smallest too. Now, using Theorem 3.21 and Theorem 3.22 we have

$$C_1 N^{-1-\epsilon} \leq D_N \leq C_2 N^{-1+\epsilon}$$

for all $\epsilon \in \mathbb{R}_{>0}$. Sequences of this type are for example those with α irrational and algebraic (for a proof see Appendix A, Corollary A.17).

There is also an explicit formula that gives the exactly discrepancy of the sequence $(n\alpha)_{n=1}^{\infty}$, which involves the continued fraction expansion of N. For a proof we refer to Drmota and Tichy [12, pp. 48-63].

As last thing, let's deal with the case in which α is of constant type. Using what we have done so far, we can prove the following.

Example 3.24. Let $\alpha \in \mathbb{R}$ be of constant type. Then $D_N = \mathcal{O}(\log(N)^2/N)$.

Proof. By Lemma 3.20. there is $C \in \mathbb{R}_{>0}$ with

$$\sum_{h=1}^{m} \frac{1}{h\langle h\alpha\rangle} \leq C\left(\log(m) + \sum_{h=1}^{m} \frac{\log(h)}{h}\right) \leq C\left(\log(m) + \log(m)\sum_{h=1}^{m} \frac{1}{h}\right).$$

As already said, the harmonic series $\sum_{k=1}^{m} 1/k$ can be approximated by $\log(m) + \gamma$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Therefore we can conclude that

$$\sum_{h=1}^{m} \frac{1}{h\langle h\alpha\rangle} = \mathcal{O}(\log(m)^2).$$

By Lemma 3.15, choosing $m = N$, we infer: $D_N = \mathcal{O}(\log(N)^2/N)$. ■

In this case, even if the discrepancy found in Example 3.24 is low, it can still be improved. By computing an upper bound with a method involving continued fractions, we can approximately describe the discrepancy of the sequence $(n\alpha)_{n=1}^{\infty}$ for α of constant type, which we will see is very low.

According to Definition A.18, we denote the continued fraction expansion of α with $[a_0, a_1, a_2, \dots, a_n, \dots]$.

Theorem 3.25. *Assume that α is of constant type. Then*

$$D_N = \mathcal{O}(\log(N)/N). \tag{3.38}$$

Proof. Assume that $\alpha = [a_0, a_1, a_2, \dots]$ is the continued fraction expansion of α. For all $n \in \mathbb{N} \cup \{0\}$ let $r_n = p_n/q_n$ be the n-th convergent to α, where p_n and q_n are defined in (A.5). Therefore there exists $r \in \mathbb{N} \cup \{0\}$ such that $q_r \leq N < q_{r+1}$. Now we implement the following algorithm, for which we assume that r is big enough.

- By the division algorithm, we can write $N = b_r q_r + N_{r-1}$, where $b_r \in \mathbb{N}$ and $N_{r-1} \in \{0, \dots, q_r - 1\}$. But recalling that q_n's are increasing, we get

  $$b_r q_r + N_{r-1} = N < q_{r+1} = a_{r+1}q_r + q_{r-1} \leq (a_{r+1} + 1)q_r.$$

  Therefore

  $$b_r < \frac{(a_{r+1}+1)q_r - N_{r-1}}{q_r} = a_{r+1} + 1 - \frac{N_{r-1}}{q_r} < a_{r+1} + 1.$$

  In particular $b_r \leq a_{r+1}$.

- By the division algorithm, write $N_{r-1} = b_{r-1}q_{r-1} + N_{r-2}$, where $b_{r-1} \in \mathbb{N}$ and $N_{r-2} \in \{0, \dots, q_{r-1} - 1\}$. Doing the same steps we get $b_{r-1} \leq a_r$.

- Repeating the algorithm up to N_0, we conclude that $1 \leq b_i \leq a_{i+1}$ (∗) for all $i \in \{0, \dots, r\}$, and we can represent N as $N = \sum_{i=0}^{r} b_i q_i$.

With this representation, we can decompose the finite sequence $(\{n\alpha\})_{n=0}^{N}$ into:

- b_0 sequences $(\{n\alpha\})$ where n runs through one (or q_0) integer
- b_1 sequences $(\{n\alpha\})$ where n runs through q_1 consecutive integers
- ...
- b_{r-1} sequences $(\{n\alpha\})$ where n runs through q_{r-1} consecutive integers
- b_r sequences $(\{n\alpha\})$ where n runs through q_r consecutive integers

Fix now some $i \in \{0, \dots, r\}$ and choose one of the b_i sequences where n runs through q_i consecutive integers. Let's denote it with $\omega_i = (\{n\alpha\}) = (\{(n_0 + j)\alpha\})_{j=0}^{q_i}$. Now, using (A.8) in Appendix A we get

$$\alpha = \frac{p_i}{q_i} + \frac{\theta}{q_i q_{i+1}}$$

where $\theta \in (-1, 1)$. Therefore

$$\{n\alpha\} = \left\{ n_0\alpha + \frac{jp_i}{q_i} + \frac{j\theta}{q_i q_{i+1}} \right\}.$$

Now the sequence $(n_0\alpha + jp_i/q_i)_{j=1}^{q_i}$ is a sequence of q_i equidistant points with distance p_i/q_i. But since $\gcd(p_i, q_i) = 1$, we have that $\{(n_0\alpha + jp_i/q_i)_{j=1}^{q_i}\}$ is a sequence of q_i equidistant points with distance $1/q_i$. But the left term $j\theta/q_i q_{i+1}$ has the property: $|j\theta/q_i q_{i+1}| < 1/q_{i+1}$ for all $j \in \{1, \dots, q_i\}$. Therefore, ω_i is obtained by shifting mod 1 (to the left or to the right depending on the sign of θ) the sequence $\{(n_0\alpha + jp_i/q_i)_{j=1}^{q_i}\}$ by a distance less than $1/q_{i+1}$. We can therefore conclude

$$D_{b_i q_i}(\omega_i) \leq \frac{1}{q_i} + \frac{1}{q_{i+1}}. \tag{3.39}$$

The RHS of (3.39) represents the length of the bigger sub-interval of $[0, 1)$ that can appear which does not contain any term of ω_i. Now, using Theorem 2.15, we have

$$\begin{aligned} ND_N(\omega) &\leq \sum_{i=0}^{r} b_i q_i \left(\frac{1}{q_i} + \frac{1}{q_{i+1}} \right) \leq \sum_{i=0}^{r} b_i + \sum_{i=0}^{r} a_{i+1} \frac{q_i}{a_{i+1}q_i + q_{i-1}} \\ &\leq \sum_{i=0}^{r} b_i + \sum_{i=0}^{r} a_{i+1} \frac{q_i}{a_{i+1}q_i} = r + 1 + \sum_{i=0}^{r} b_i \end{aligned} \tag{3.40}$$

where in the second step we have used the $(*)$ condition. The next step is to find an upper bound for the quantity

$$\sigma(N) := \sum_{i=0}^{r} b_i.$$

Note that σ is well-defined since every N corresponds exactly to an r. Recall that by assumption α is of constant type. Therefore, by Theorem A.23 there exists $K \in \mathbb{R}_{>0}$ such that $a_i \leq K$. We have the following Claim.

Claim 1: $\sigma(N) \leq 1 + K\log(N)/\log(K+1)$ for all $r \in \mathbb{N} \cup \{0\}$.

Proof. We will prove the statement proceeding by induction on r.

- *Base*: We have to distinguish two cases.
  - ▶ If $q_0 < q_1$, then $r = 0$ is the smallest possible r. In this case
  $$1 \leq \sigma(N) = b_0 = N < q_1 = a_1 \leq K \leq K + 1.$$
  - ▶ If $q_0 = q_1$, then $r = 1$ is the smallest possible r. In this case
  $$1 \leq \sigma(N) = b_1 = N < q_2 = a_2 q_1 + q_0 = a_2 + 1 \leq K + 1.$$

  Therefore we have to prove that for $N \in \{1, \ldots, K+1\}$ we have
  $$N \leq 1 + \frac{K\log(N)}{\log(K+1)}. \tag{3.41}$$
  Let $f\colon [1, K+1] \to \mathbb{R}$, $x \to x - K\log(x)/\log(K+1)$. Note that $f'' > 0$. This means f is concave upward, and therefore has its absolute maximum at the point $x = 1$ or $x = K + 1$. But $f(1) = f(K+1) = 1$, thus $f \leq 1$. Hence (3.41) follows.
- *Hypothesis*: Take an arbitrary $N \in \mathbb{N}$ with $1 < q_r \leq N < q_{r+1}$ and write $N = b_r q_r + N_{r-1}$ as before with $N_{r-1} \in \{0, \ldots, q_r - 1\}$. Assume that Claim 1 holds for $r - 1$, and therefore for the corresponding N_{r-1}.
- *Step*: We consider again two cases.
  - ▶ Assume that $N_{r-1} > 0$. Then
  $$\begin{aligned}\sigma(N) = b_r + \sigma(N_{r-1}) &\leq 1 + b_r + \frac{K\log(N_{r-1})}{\log(K+1)} \\ &\leq 1 + b_r + \frac{K\log(N/(b_r+1))}{\log(K+1)} \quad (*) \\ &\leq 1 + \frac{K}{\log(K+1)}\left(\log(b_r+1) + \log\left(\frac{N}{b_r+1}\right)\right) \\ &= 1 + \frac{K}{\log(K+1)}\log(N).\end{aligned}$$
  where the first inequality holds by the induction hypothesis, the second inequality holds since
  $$N = b_r q_r + N_{r-1} > b_r N_{r-1} + N_{r-1} = (b_r + 1)N_{r-1},$$
  and the third inequality holds since $1 \leq b_r \leq a_{r+1} \leq K$ (using the $(*)$ condition) and since the function $g\colon \mathbb{R}_{>0} \to \mathbb{R}$, $x \to x/\log(x+1)$ is increasing, which implies
  $$\frac{b_r}{\log(b_r+1)} \leq \frac{K}{\log(K+1)}.$$

▶ Assume that $N_{r-1} = 0$. Then $N = b_r q_r$. Therefore

$$\frac{N}{b_r + 1} = \frac{b_r q_r}{b_r + 1} \geq 1.$$

Hence

$$\sigma(N) = b_r \leq 1 + b_r + \frac{K \log(N/(b_r + 1))}{\log(K + 1)}.$$

Now we can proceed as for the case $N_{r-1} > 0$ starting from (∗).

This concludes the induction step. □

Now, putting together Claim 1 and (3.40) we get

$$N D_N(\omega) \leq r + 2 + \frac{K}{\log(K + 1)} \log(N). \tag{3.42}$$

It remains to bound r in terms of N. For this we have the following second Claim. Let $\phi = (1 + \sqrt{5})/2$ be the golden ratio.

Claim 2: $q_i \geq \phi^{i-1}$ for all $i \in \mathbb{N} \cup \{0\}$.

Proof. We will prove the statement proceeding by induction on i.

- *Base*: For $i = 0$, we have: $q_0 = 1 \geq \phi^{-1}$.
  For $i = 1$, we have: $q_1 = a_1 \geq 1 = \phi^0$.
- *Hypothesis*: Assume that Claim 2 holds for some $i \in \mathbb{N} \cup \{0\}$.
- *Step*: Let $i \in \mathbb{N} \cup \{0\}$ be as in the hypothesis. Then

$$\begin{aligned} q_{i+1} = a_{i+1} q_i + q_{i-1} \geq a_{i+1} \phi^{i-1} + \phi^{i-2} \geq b_i \phi^{i-1} + \phi^{i-2} \\ \geq \phi^{i-1} + \phi^{i-2} = \phi^i \left(\phi^{-1} + \phi^{-2} \right) = \phi^i. \end{aligned}$$

where in the first inequality we have used the induction hypothesis, while in the second inequality we have used the (∗) condition. This concludes the induction step. □

By Claim 2 we have: $N \geq q_r \geq \phi^{r-1}$. Therefore $r \leq 1 + \log(N)/\log(\phi)$. Adding this to (3.42) we get

$$N D_N(\omega) \leq 3 + \left(\frac{1}{\log(\phi)} + \frac{K}{\log(K + 1)} \right) \log(N). \tag{3.43}$$

This is equivalent to (3.38). ■

It follows immediately:

Corollary 3.26. *Assume that α is of constant type. Then*

$$D_N = \Theta(\log(N)/N). \tag{3.44}$$

Proof. By Theorem 3.25 we have that $D_N = \mathcal{O}(\log(N)/N)$. For the other direction we have simply to use Schmidt's Method. Indeed

$$D_N \geq D_N^\star > C \log(N)/N \tag{3.45}$$

where in the first step we have used Theorem 2.10, and in the second step we have used Theorem 2.30. ■

Over the years, equation (3.43) has been improved for the case $K = 1$. See in particular Zaremba [74] and Gillet [17, 18]. Regarding (3.45), Behnke [5] proved it first for all sequences $(n\alpha)_{n=1}^{\infty}$, when Schmidt's Method had not yet been developed.

Appendices

Appendix A

Introduction to Diophantine Approximation

Diophantine Approximation is a branch of number theory in which the approximations of real numbers by rational numbers are studied. In this appendix we will introduce some basic concepts about it, which will be useful in the main part, especially in Section 3.3. As references, see Kuipers and Niederreiter [44, pp. 121-122]. For extensive reports see Koksma [39], Cassels [46] and Hardy and Wright [25].

Definition A.1. A number $a \in \mathbb{R}$ is called *irrational*, if $a \notin \mathbb{Q}$.

One of the first important keystone is due to Dirichlet.

Theorem A.2 (Dirichlet Approximation Theorem)**.** *Let $\alpha \in \mathbb{R}$ be irrational. Then there exist infinitely many $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{A.1}$$

Proof. Let $n \in \mathbb{N}$ and divide $[0, 1)$ into n intervals of length $1/n$. Then there exists some $i, j \in \{0, 1, \ldots, n\}$ with $i < j$ such that $\{j\alpha\}$ and $\{i\alpha\}$ lie in the same subinterval (pigeonhole principle). Hence

$$|(j - i)\alpha - ([j\alpha] - [i\alpha])\}| = |\{j\alpha\} - \{i\alpha\}| < 1/n.$$

Dividing by $j - i$ we get

$$\left|\alpha - \frac{[j\alpha] - [i\alpha]}{j - i}\right| < \frac{1}{(j - i)n} \leq \frac{1}{(j - i)^2}.$$

This concludes the proof. ■

The theorem has been generalized by Kronecker [41] (cited in the introduction). From (A.1) then arises the question of which irrational numbers can be approximated well by rational numbers and which cannot. We introduce therefore the concept of algebraic and transcendental number.

Definition A.3. A number $a \in \mathbb{C}$ is called *algebraic*, if there exists $p(x) \in \mathbb{Z}[x]$ such that $p(a) = 0$. A non-algebraic number $b \in \mathbb{C}$ is called *transcendental*.

Remark A.4. Any complex number constructed as combination of algebraic numbers using the basic arithmetic operations is again algebraic (this can be proven algebraically for each operation).

Example A.5. Here there are some examples of algebraic numbers.

- All integers $k \in \mathbb{Z}$ are algebraic as roots of the polynomial $x^2 - k^2$.
- All rationals $r \in \mathbb{Q}$ are algebraic by Remark A.4.
- The number $\sqrt{2}$ is algebraic as root of the polynomial $x^2 - 2$.
- The golden ratio $(1 + \sqrt{5})/2$ is algebraic as root of the polynomial $x^2 - x - 1$.
- The number $2i$ is algebraic as root of the polynomial $x^2 + 4$.

Regarding trascendental numbers, the following theorem can be useful.

Theorem A.6 (Lindemann–Weierstrass)**.** *For $n \in \mathbb{N}$ let $a_1, \dots, a_n$ be algebraic numbers and $\alpha_1, \dots, \alpha_n$ be distinct algebraic numbers such that*

$$a_1 e^{\alpha_1} + a_2 e^{\alpha_2} + \cdots + a_n e^{\alpha_n} = 0. \tag{A.2}$$

Then: $a_1 = a_2 = \cdots = a_n = 0$.

For a proof see Baker [2, pp. 6-8].

Corollary A.7. *Let $\alpha \in \mathbb{C} \setminus \{0\}$ be algebraic. Then e^α is transcendental.*

Proof. Suppose that e^α is algebraic. Using Theorem A.6 with $n = 2$, $\alpha_1 = 0$ and $\alpha_2 = \alpha$ we get that the equation

$$a_1 + a_2 e^\alpha = 0$$

has a unique algebraic solution $(a_1, a_2) = (0, 0)$. But note that $(a_1, a_2) = (e^\alpha, -1)$ is an other algebraic solution. ↯ ■

In particular, Corollary A.7 shows that the Euler's number e is transcendental. The transcendence of π follows directly, too.

Example A.8. The number π is transcendental.

Proof. Suppose that π is algebraic. Then πi is also algebraic by Remark A.4. Now, using Corollary A.7 we conclude that $e^{\pi i}$ must be transcendental. But recall that by the Euler's identity it holds $e^{\pi i} = -1$, which is algebraic. ↯ ■

Moreover we can also prove the irrationality of the natural logarithm.

Example A.9. Let $\alpha \in \mathbb{R}_{>0} \setminus \{1\}$ be an algebraic number. Then $\log(\alpha) \notin \mathbb{Q}$.

Proof. Suppose that $\log(\alpha) \in \mathbb{Q}$. Then there exist $p \in \mathbb{Z} \setminus \{0\}$ and $q \in \mathbb{N}$ such that $\log(\alpha) = p/q$. Therefore: $e^p = \alpha^q$. But e^p is transcendental by Corollary A.7, while α^q is algebraic by Remark A.4 since $\alpha^q = \alpha \cdots \alpha$ (q times). ↯ ■

In addition to algebraic and transcendental, irrational numbers can also be classified by their *type*, in the following sense.

Notation A.10. For $t \in \mathbb{R}$, we denote: $\langle t \rangle \coloneqq \min_{n\in\mathbb{Z}}|t - n| = \min(\{t\}, 1 - \{t\})$.

Let $E \subseteq \mathbb{R}$ be a subset such that $\mathbb{N} \subseteq E$. For the sequel, we consider a non-decreasing and positive function $\psi\colon E \to \mathbb{R}$.

Definition A.11. Let $\alpha \in \mathbb{R}$ be irrational. Then α is called *of type* $< \psi$, if for all $q \in \mathbb{N}$ it holds: $q\langle q\alpha\rangle \geq 1/\psi(q)$.

Remark A.12. Assume that ψ is constant. If the irrational $\alpha \in \mathbb{R}$ is of type $< \psi$, then α is called *of constant type*.

Definition A.13. Let $\alpha \in \mathbb{R}$ be irrational and $\eta \in \mathbb{R}_{>0} \cup \{\infty\}$. Then α is called *of type* η if η can be written as follows:

$$\eta = \sup\left\{\gamma \in \mathbb{R}_{>0} \,\middle|\, \varliminf_{q\to\infty} q^\gamma\langle q\alpha\rangle = 0\right\}.$$

Remark A.14. Let $\alpha \in \mathbb{R}$ be irrational and $\gamma \in \mathbb{R}_{<1}$. By Theorem A.2 there exists infinitely many $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$q^\gamma\langle q\alpha\rangle = q^\gamma \min_{n\in\mathbb{Z}}|q\alpha - n| \leq q^\gamma|q\alpha - p| < q^\gamma/q = q^{\gamma-1}.$$

This implies: $\varliminf_{q\to\infty} q^\gamma\langle q\alpha\rangle = 0$. In particular $\eta \geq 1$.

The two definitions are connected by the following result.

Proposition A.15. *Let $\alpha \in \mathbb{R}$ be irrational. Then α is of type η if and only if*

$$\eta = \inf\left\{\tau \in \mathbb{R} \,\middle|\, \exists\, c \in \mathbb{R}_{>0} \,:\, \alpha \text{ is of type } < cq^{\tau-1}\right\}. \tag{A.3}$$

Proof. We consider two cases.

- Assume that $\eta < \infty$. Then α is of type η if and only if for all $\epsilon \in \mathbb{R}_{>0}$

$$\varliminf_{q\to\infty} q^{\eta-\epsilon}\langle q\alpha\rangle = 0 \qquad \text{and} \qquad \varliminf_{q\to\infty} q^{\eta+\epsilon}\langle q\alpha\rangle > 0 \tag{A.4}$$

  Now notice the following:

  ▶ The first equality of (A.4) is equivalent to saying that for all $c \in \mathbb{R}_{>0}$ there exists $q \in \mathbb{N}$ such that $q^{\eta-\epsilon}\langle q\alpha\rangle < 1/c$, which is equivalent to $q\langle q\alpha\rangle < 1/(cq^{\eta-1-\epsilon})$. This means, α is not of type $< cq^{\eta-1-\epsilon}$.

  - ▶ The second equality of (A.4) is equivalent to saying that there exists $d \in \mathbb{R}_{>0}$ depending on ϵ and α such that $q^{\eta+\epsilon}\langle q\alpha\rangle \geq d$ for all $q \in \mathbb{N}$, or equivalently $q\langle q\alpha\rangle \geq d/q^{\eta-1+\epsilon}$. This means, α is of type $< q^{\eta-1+\epsilon}/d$.

  Putting these two observation together we get exactly (A.3). Since the proof contains only equivalences, the other direction also holds.

- Assume that $\eta = \infty$. Then α is of type η if and only if $\varliminf_{q\to\infty} q^{\gamma}\langle q\alpha\rangle = 0$ for all $\gamma \in \mathbb{R}_{>0}$. This is equivalent to saying that for all $c \in \mathbb{R}_{>0}$ there exists $q \in \mathbb{N}$ such that $q^{\gamma}\langle q\alpha\rangle < 1/c$, or equivalently $q\langle q\alpha\rangle < 1/(cq^{\gamma-1})$. Hence for all $c \in \mathbb{R}_{>0}$ there exists $q \in \mathbb{N}$ such that α is not of type $< cq^{\gamma-1}$. Since $\eta = \infty$, this is equivalent to (A.3). ■

Recalling Remark A.14, an interesting question is which irrational numbers fall into the limit case $\eta = 1$. This is why the next theorem can be useful.

Theorem A.16 (Thue-Siegel-Roth)**.** *Let $\alpha \in \mathbb{R}$ be irrational and algebraic. Then, for every $\epsilon \in \mathbb{R}_{>0}$ there exists $c \in \mathbb{R}_{>0}$ such that for all $p \in \mathbb{Z}$ and all $q \in \mathbb{N}$ it holds*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{2+\epsilon}}.$$

For a proof see Roth [57]. For us, the interesting thing is the following result.

Corollary A.17. *Let $\alpha \in \mathbb{R}$ be irrational and algebraic. Then α is of type $\eta = 1$.*

Proof. Let $\epsilon \in \mathbb{R}_{>0}$. By Theorem A.16 we have $q^{1+\epsilon}|q\alpha - p| \geq c$ for all $p \in \mathbb{Z}$ and all $q \in \mathbb{N}$. Choose p such that $\langle q\alpha\rangle = |q\alpha - p|$. Then $q^{1+\epsilon}\langle q\alpha\rangle \geq c$ for all $q \in \mathbb{N}$. Taking $\epsilon \to 0$, we get that α is of type $\eta = 1$ by Propostion A.15. ■

When we talk about irrational numbers and Diophantine Approximation, it is also useful to introduce continued fractions.

Definition A.18. A *continued fraction* is an expression x of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. This terms are called *partial quotients* of x.

Remark A.19. Usually we denote $x = [a_0, a_1, a_2, a_3, \dots]$. Moreover, for $n \in \mathbb{N}\cup\{0\}$, we denote $r_n := [a_0, \dots, a_n]$, which is called the n-th *convergent* to α.

Theorem A.20. *Let $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. Then there exists an irrational number $\alpha \in \mathbb{R}$ such that*

$$\alpha = [a_0, a_1, a_2, a_3, \dots].$$

Proof-sketch. Define recursively the following numbers:

$$\begin{aligned} &p_0 = a_0 \quad , \quad p_1 = a_1 a_0 + 1 \quad , \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \in \mathbb{N}_{\geq 2}) \\ &q_0 = 1 \quad , \quad q_1 = a_1 \quad , \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \in \mathbb{N}_{\geq 2}) \end{aligned} \tag{A.5}$$

With an induction argument we can show that $r_n = p_n/q_n$ for all $n \in \mathbb{N} \cup \{0\}$ and that $\gcd(p_n, q_n) = 1$. Moreover, we have also that for all $n \in \mathbb{N}$

$$r_{n+1} - r_n = \frac{(-1)^{n+1}}{q_{n+1} q_n}. \tag{A.6}$$

For more details about these steps see Hammond [22, pp. 6-11]. By (A.6) we can order the sequence $(r_n)_{n=0}^{\infty}$ in the following way

$$r_0 < r_2 < r_4 < \cdots < r_5 < r_3 < r_1. \tag{A.7}$$

This shows that the limits $\alpha := \lim_{n\to\infty} r_{2n}$ and $\beta := \lim_{n\to\infty} r_{2n+1}$ both exist and satisfy $r_{2n} \leq \alpha \leq \beta \leq r_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. But $\alpha = \beta$ since

$$|\alpha - \beta| \leq |\alpha - r_{2n}| + |r_{2n} - r_{2n+1}| + |r_{2n+1} - \beta| \leq \frac{3}{q_{2n+1} q_{2n}} \xrightarrow{n\to\infty} 0.$$

where in the second step we have used (A.6) and the convergence of r_n, and in the last step we have used that $q_n \to \infty$ as $N \to \infty$. Therefore $\alpha = [a_0, a_1, a_2, a_3, \dots]$.

It remains to show that α is irrational. Using again (A.6) and the convergence of r_n, for n big enough we get that for all $\epsilon \in \mathbb{R}_{>0}$

$$|\alpha - r_n| \leq |\alpha - r_{n+1}| + |r_{n+1} - r_n| \leq \epsilon + \frac{1}{q_{n+1} q_n}. \tag{A.8}$$

Suppose that $\alpha = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then (A.8) becomes

$$|pq_n - qp_n| \leq \frac{q}{q_{n+1}} \xrightarrow{n\to\infty} 0.$$

This means that $pq_n - qp_n = 0$ for n big enough. Thus: $r_n = p/q$ for n big enough. This is a contradiction to (A.6). ■

It's also easy to prove the converse of Theorem A.20.

Theorem A.21. *Let $\alpha \in \mathbb{R}$ be irrational. Then there exists $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \in \mathbb{N}$ such that*

$$\alpha = [a_0, a_1, a_2, a_3, \dots].$$

Proof. We define recursively $x_0 = \alpha$, $a_0 = [\alpha]$, and for $n \in \mathbb{N}$

$$x_{n+1} = \frac{1}{x_n - a_n} \quad \text{and} \quad a_{n+1} = [x_{n+1}]. \tag{A.9}$$

Note that $a_0 \in \mathbb{Z}$, while $a_i \in \mathbb{N}$ for all $i \in \mathbb{N}$ since $x_{i+1} = 1/\{x_i\} > 0$ (this because x_i is always irrational by construction). This gives us a continued fraction for α. ■

As application, we consider the following example.

Example A.22. Assume that $x = [1, 1, 1, \dots]$. Then $x = (1 + \sqrt{5})/2$.

Proof. By Theorem A.20 the number x exists and is irrational. Notice that

$$r_n = 1 + \frac{1}{r_{n-1}}. \tag{A.10}$$

Taking $n \to \infty$, we get $r_n \to x$, and therefore (A.10) becomes $x = 1 + 1/x$, for which the positive solution is $x = (1 + \sqrt{5})/2$. ■

Example A.22 is also very interesting for another reason. We have seen that we can approximate an irrational number α with the n-th convergent r_n. The larger n is, the better the approximation will be. However, if we analyze the problem from a qualitative point of view, we notice that the approximation of α also depends on the magnitude of the partial quotients $a_1, \dots, a_n$. In fact, the larger these are, the better the approximation. As example consider

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots].$$

In this case $r_3 = 355/113$ gives already a good approximation of π, but we have a big jump with $r_4 = 103993/33102$, which significantly improves the approximation of r_3 since 292 is very large. Instead, $r_5 = 104348/33215$ will not improve so much the approximation of r_4 since 1 is the smallest value for a partial quotient.

Back to Example A.22, since the partial quotients of the golden ratio are all 1, the convergence of r_n is very slow. Indeed the golden ratio is an irrational number which can be approximated very bad by rational numbers.

As last thing, we see how the continuous fractions are connected with the irrational numbers of constant type.

Theorem A.23. *Let $\alpha = [a_0, a_1, \dots]$ be irrational. Then*

$$\alpha \text{ is of constant type} \iff \exists K \in \mathbb{R}_{>0} \text{ such that } a_i \leq K \text{ for all } i \in \mathbb{N}.$$

For a proof see Baker [1, 46-48].

Appendix B

Basic concepts of Fourier Series

Since in many steps of some proofs in the main part there is a need to compute specific series, we review the basic concepts of the Fourier Analysis. We refer entirely to Felder [16]. Throughout the chapter we fix a number $L \in \mathbb{R}_{>0}$.

Notation B.1. Define the set $\mathbb{C}_\infty := \{a + bi \mid a, b \in \mathbb{R} \cup \{\infty\}\}$.

Definition B.2. A *Fourier series* is defined as a map

$$F_L \colon \mathbb{R} \to \mathbb{C}_\infty \,,\; x \mapsto \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n}{L} x}$$

where for all $n \in \mathbb{Z}$ we have $c_n \in \mathbb{C}$.

Definition B.3. Let $f \colon \mathbb{R} \to \mathbb{C}$ be Riemann-integrable and of period L. For $n \in \mathbb{Z}$, the n-th *Fourier coefficient* of f is defined as

$$f_n := \frac{1}{L} \int_0^L e^{-\frac{2\pi i n}{L} x} f(x)\, dx.$$

Theorem B.4 (Riemann-Lebesgue Lemma)**.** *Let $f \colon \mathbb{R} \to \mathbb{C}$ be continuous and of period L. Then $\lim_{n \to \pm\infty} f_n = 0$.*

Proof. We have

$$\begin{aligned} f_n &= \frac{1}{L} \int_0^L e^{-\frac{2\pi i n}{L} x} f(x)\, dx = \frac{1}{L} \int_{L/(2n)}^{L+L/(2n)} e^{-\frac{2\pi i n x}{L} x} f(x)\, dx \\ &= \frac{1}{L} \int_0^L e^{-\frac{2\pi i n}{L}(y + L/(2n))} f\left(y + \frac{L}{2n}\right) dy = -\frac{1}{L} \int_0^L e^{-\frac{2\pi i n}{L} y} f\left(y + \frac{L}{2n}\right) dy. \end{aligned}$$

where in the second step we have used periodicity of f, while in the third step we made the substitution $y = x - L/(2n)$. Therefore

$$\begin{aligned} f_n &= \frac{1}{2}(f_n + f_n) = \frac{1}{2} \left(\frac{1}{L} \int_0^L e^{-\frac{2\pi i n}{L} x} f(x)\, dx - \frac{1}{L} \int_0^L e^{-\frac{2\pi i n}{L} x} f\left(x + \frac{L}{2n}\right) dx \right) \\ &= \frac{1}{2L} \int_0^L e^{-\frac{2\pi i n}{L} x} \left(f(x) - f\left(x + \frac{L}{2n}\right) \right) dx. \end{aligned}$$

Since f is continuous on the compact interval $[-L/(2|x|), L+L/(2|x|)]$, it is uniform continuous (for details see Jossen [35, pp. 97-100]). Therefore, for all $\epsilon \in \mathbb{R}_{>0}$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N} \setminus (-N, N)$ we have $|f(x) - f(x + L/(2n))| < \epsilon$ for all $x \in [0, L]$. It follows that, for all $n \geq N$

$$|f_n| \leq \frac{1}{2L} \int_0^L \left| e^{-\frac{2\pi i n}{L} x} \left(f(x) - f\left(x + \frac{L}{2n} \right) \right) \right| dx < \frac{1}{2L} \int_0^L \epsilon \, dx = \frac{\epsilon}{2}.$$

This is equivalent to $\lim_{n \to \pm\infty} f_n = 0$. ■

Definition B.5. Let $N \in \mathbb{N}$ and $x \in \mathbb{R}$. The quantity $D_N(x) \coloneqq \sum_{n=-N}^{N} e^{2\pi i n x}$ is called *Dirichlet kernel*.

Lemma B.6. *The Dirichlet kernel has the following properties:*

(a) $\int_0^1 D_N(x) \, dx = 1$.

(b) *For all* $x \in \mathbb{R}$ *:* $L^{-1} \int_0^L D_N((x-y)/L) \, dy = 1$.

(c) *For all* $x \in \mathbb{R} \setminus \mathbb{Z}$ *:* $D_N(x) = \sin((2N+1)\pi x) / \sin(\pi x)$.

Proof. (a) We have

$$\int_0^1 D_N(x) \, dx = \sum_{n=-N}^{N} \int_0^1 e^{2\pi i n x} \, dx = 1 + \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{e^{2\pi i n} - 1}{2\pi i n} = 1.$$

(b) For all $x \in \mathbb{R}$ it holds

$$\frac{1}{L} \int_0^L D_N \left(\frac{x-y}{L} \right) dy = \int_{x/L - 1}^{x/L} D_N(t) \, dt = \int_0^1 D_N(t) \, dt = 1$$

where in the first step we made the substitution $t = (x-y)/L$, the second step holds since D_N has period 1, and in the last step we have used (a).

(c) For all $x \in \mathbb{R} \setminus \mathbb{Z}$ it holds

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{2\pi i n x} = \sum_{n=0}^{2N} e^{2\pi i (n-N) x} = e^{-2\pi i N x} \sum_{n=0}^{2N} e^{2\pi i n x} \\ &= e^{-2\pi i N x} \frac{1 - e^{2\pi i (2N+1) x}}{1 - e^{2\pi i x}} = \frac{e^{-\pi i (2N+1) x}}{e^{-\pi i x}} \frac{1 - e^{2\pi i (2N+1) x}}{1 - e^{2\pi i x}} \\ &= \frac{e^{-\pi i (2N+1) x} - e^{\pi i (2N+1) x}}{e^{-\pi i x} - e^{\pi i x}} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}. \end{aligned}$$

This concludes the proof. ■

Regarding the convergence of the Fourier series, we have the following result.

Theorem B.7. *Let $f \in \mathrm{C}^1(\mathbb{R}, \mathbb{C})$ be of period L. Then for all $x \in \mathbb{R}$ it holds*

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2\pi i n}{L} x}. \tag{B.1}$$

Proof. Fix some $x \in \mathbb{R}$ and define the function

$$F_x \colon \mathbb{R} \to \mathbb{C}\,,\, y \longmapsto \begin{cases} (f(y) - f(x))/\sin(\pi(x-y)/L) & \text{for} \quad y \notin x + L\mathbb{Z} \\ -(-1)^{(x-y)/L} L f'(x)/\pi & \text{for} \quad y \in x + L\mathbb{Z} \end{cases}.$$

Note that if $y \notin x + L\mathbb{Z}$ the map F_x is continuous as composition of continuous functions. Instead, if $y \in x + L\mathbb{Z}$, we have $y = x + kL$ for some $k \in \mathbb{Z}$. Using L'Hôpital rule and the periodicity of f' we have

$$\begin{aligned} \lim_{y \to x+kL} F_x(y) &= \lim_{y \to x+kL} \frac{f(y) - f(x)}{\sin(\pi(x-y)/L)} = \lim_{y \to x+kL} \frac{-L f'(y)}{\pi \cos(\pi(x-y)/L)} \\ &= \frac{-L f'(x+kL)}{\pi \cos(\pi k)} = -(-1)^k \frac{L f'(x)}{\pi} = F_x(x+kL). \end{aligned}$$

Hence $F_x(y)$ is continuous for all $y \in \mathbb{R}$. Moreover it is periodic with period $2L$. Therefore for $N \in \mathbb{N}$ we have

$$\begin{aligned} &\sum_{n=-N}^{N} f_n e^{\frac{2\pi i n}{L} x} - f(x) = \frac{1}{L} \int_0^L f(y) D_N\left(\frac{x-y}{L}\right) dy - f(x) \\ &= \frac{1}{L} \int_0^L (f(y) - f(x)) D_N\left(\frac{x-y}{L}\right) dy = \frac{1}{2L} \int_0^{2L} (f(y) - f(x)) D_N\left(\frac{x-y}{L}\right) dy \\ &= \frac{1}{2L} \int_0^{2L} (f(y) - f(x)) \frac{\sin((2N+1)\pi(x-y)/L)}{\sin(\pi(x-y)/L)}\, dy \\ &= \frac{1}{2i} \frac{1}{2L} \int_0^{2L} F_x(y) \left(e^{i(2N+1)\pi(x-y)/L} - e^{-i(2N+1)\pi(x-y)/L}\right) dy \\ &= \frac{1}{2i} \left[e^{i(2N+1)\pi x/L} (F_x)_{2N+1} - e^{i(2N+1)\pi x/L} (F_x)_{-(2N+1)}\right] \end{aligned}$$

where in the second step we have used Lemma B.6 (b), in the third step we have used periodicity of f and D_N, and in the fourth step we have used Lemma B.6 (c). Notice that here $(F_x)_n$ denotes the n-th Fourier coefficient of F_x. The last quantity tends to 0 as $N \to \infty$ by Theorem B.4. This implies (B.1). ∎

As application, consider the following example.

Example B.8. Let $x \in \mathbb{R} \setminus \mathbb{Z}$ and $m \in \mathbb{N}$. Then

$$\sum_{n=-m}^{m} \left[(m + 1 - |n|) e^{2\pi i n x}\right] = \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2}. \tag{B.2}$$

Proof. By Remark 1.26, for all $x \in \mathbb{R} \setminus \mathbb{Z}$ and $l \in \mathbb{N}$ we have

$$\frac{\sin(l\pi x)}{\sin(\pi x)} = e^{-i(l-1)\pi x} \sum_{k=0}^{l-1} e^{2\pi i k x}. \tag{B.3}$$

Consider now the function of period 1 given by

$$f\colon \mathbb{R} \to \mathbb{R}\,,\ x \longmapsto \begin{cases} \sin((m+1)\pi x)^2 / \sin(\pi x)^2 & \text{for} \quad x \notin \mathbb{Z} \\ (m+1)^2 & \text{for} \quad x \in \mathbb{Z} \end{cases}.$$

Notice that f is continuous (just apply L'Hôpital rule two times). Moreover, using (B.3), we can compute the Fourier coefficients of f. Indeed, for $n \in \mathbb{Z}$ we have

$$\begin{aligned} f_n &= \int_0^1 \frac{\sin((m+1)\pi x)^2}{\sin(\pi x)^2} e^{-2\pi i n x}\, dx = \int_0^1 \left(e^{-im\pi x} \sum_{k=0}^{m} e^{2\pi i k x} \right)^2 e^{-2\pi i n x}\, dx \\ &= \sum_{k=0}^{m} \sum_{j=0}^{m} \int_0^1 e^{2\pi i (k+j-(n+m))x}\, dx = \sum_{k=0}^{m} \sum_{j=0}^{m} \mathbb{1}\{k+j = n+m\}. \end{aligned}$$

Hence, if $n \in \{-m, \dots, m\}$, we get $f_n = m+1-|n|$, else if $|n| \in \mathbb{N}_{>m}$, we get $f_n = 0$. Therefore, by Theorem B.7 we get (B.2). ∎

One can also extend Theorem B.7 to functions of bounded variation.

Theorem B.9 (Dirichlet–Jordan Criterion)**.** *Let $f\colon \mathbb{R} \to \mathbb{C}$ be of period L and of bounded variation on $[0, L]$. Then* (B.1) *holds for all points of continuity of f.*

For a proof see Zygmund [75, pp. 57-60]. As application, see the following.

Example B.10. For all $x \in \mathbb{R} \setminus \mathbb{Z}$ it holds

$$\{x\} = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n}. \tag{B.4}$$

Proof. The goal here is to find the Fourier series of the map $f\colon \mathbb{R} \to [0,1)\,,\ x \mapsto \{x\}$. This map has period 1, is of bounded variation, and is discontinuous on $\mathbb{Z}$. For $n = 0$, the Fourier coefficient is given by

$$f_0 = \int_0^1 \{x\}\, dx = \int_0^1 x\, dx = \frac{1}{2}.$$

Instead, for $n \in \mathbb{Z} \setminus \{0\}$, the Fourier coefficients are given by

$$f_n = \int_0^1 e^{-2\pi i n x} \{x\}\, dx = \int_0^1 e^{-2\pi i n x} x\, dx = \frac{i}{2\pi n}$$

where in the last step we have used integration by parts. By Theorem B.9, for all $x \in \mathbb{R} \setminus \mathbb{Z}$ we have

$$f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{2\pi n} e^{2\pi i n x} = \frac{1}{2} + \sum_{n=-\infty}^{-1} \frac{i}{2\pi n} e^{2\pi i n x} + \sum_{n=1}^{\infty} \frac{i}{2\pi n} e^{2\pi i n x}$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{i}{2\pi n} \left(e^{2\pi i n x} - e^{-2\pi i n x} \right) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n}.$$

This concludes the proof. ■

An important result concerns the sum of the squares of the Fourier coefficients.

Theorem B.11 (Parseval's identity)**.** *Let $f\colon \mathbb{R} \to \mathbb{C}$ be of period L and assume that $f \in \mathrm{L}^2([0, L], \mathbb{C})$. Then*

$$\frac{1}{L} \int_0^L |f(x)|^2 \, dx = \sum_{n\in\mathbb{Z}} |f_n|^2. \tag{B.5}$$

Proof. We only consider the case for which f is continuous (for the general case we refer to Zygmund [75, pp. 35-37]). By Theorem B.7, for all $x \in [0, L]$ we have

$$|f(x)|^2 = f(x)\overline{f(x)} = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2\pi i n}{L} x} \overline{\sum_{m=-\infty}^{\infty} f_m e^{\frac{2\pi i m}{L} x}} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_n \overline{f_m} e^{\frac{2\pi i (n-m)}{L} x}.$$

Integrating both sides over $[0, L]$ we get

$$\int_0^L |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_n \overline{f_m} \int_0^L e^{\frac{2\pi i (n-m)}{L} x} \, dx$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_n \overline{f_m} L \delta_{n,m} = L \sum_{n=-\infty}^{\infty} |f_n|^2.$$

This is equivalent to (B.5). ■

Parseval identity is useful if we want to compute some special integrals.

Example B.12. Let $n \in \mathbb{N}$. Then

$$\int_0^{1/2} \frac{\sin(n\pi x)^2}{\sin(\pi x)^2} \, dx = \frac{n}{2}. \tag{B.6}$$

Proof. Consider the function of period 2 given by

$$f\colon \mathbb{R} \to \mathbb{R}\,,\ x \longmapsto \begin{cases} \sin(n\pi x)/\sin(\pi x) & \text{for} \quad x \notin \mathbb{Z} \\ n & \text{for} \quad x \in \mathbb{Z} \end{cases}.$$

Notice that f is continuous (just apply L'Hôpital rule), although this is not crucial for our purpose. To use Parseval identity we have to show that $f \in \mathrm{L}^2([0, 2], \mathbb{C})$. For this we first state the following result. Let $x \in \mathbb{R}$.

Claim: $|\sin(n\pi x)| \leq n\,|\sin(\pi x)|$ for all $n \in \mathbb{N}$.

Proof. We will prove the statement proceeding by induction on n.

- *Base*: For $n = 1$, we have: $|\sin(n\pi x)| = |\sin(\pi x)|$.
- *Hypothesis*: Assume that $|\sin(n\pi x)| \leq n\,|\sin(\pi x)|$ holds for some $n \in \mathbb{N}$.
- *Step*: Let $n \in \mathbb{N}$ be as in the hypothesis. Then

$$\begin{aligned} |\sin((n+1)\pi x)| &\leq |\sin(n\pi x)|\,|\cos(\pi x)| + |\cos(n\pi x)|\,|\sin(\pi x)| \\ &\leq n\,|\sin(\pi x)| + |\sin(\pi x)| = (n+1)\,|\sin(\pi x)|\,. \end{aligned}$$

  where in the first step we have used the addition formula for sine and the triangle inequality, while in the second step we have used the induction hypothesis. This concludes the induction step. □

Using this result it follows that

$$\|f\|^2_{\mathrm{L}^2([0,2],\mathbb{C})} = \int_0^2 \left|\frac{\sin(n\pi x)}{\sin(\pi x)}\right|^2 dx \leq n^2 \int_0^2 dx = 2n^2 < \infty.$$

Using Remark 1.26, for all $x \in \mathbb{R} \setminus \mathbb{Z}$ we get

$$f(x) = e^{-i(n-1)\pi x} \sum_{k=0}^{n-1} e^{2\pi i k x}.$$

Therefore the Fourier coefficients are given by

$$f_k = \begin{cases} e^{-i(n-1)\pi x} & \text{for} \quad k \in \{0, \ldots, n-1\} \\ 0 & \text{for} \quad k \in \mathbb{Z} \setminus \{0, \ldots, n-1\} \end{cases}.$$

Hence

$$\int_0^{1/2} |f(x)|^2\,dx = \frac{1}{4}\int_0^2 |f(x)|^2\,dx = \frac{1}{2}\sum_{k=0}^{n-1} |f_k|^2 = \frac{n}{2}.$$

where in the first step we have used that $|f|^2$ is even and has period 1, and in the second step we have used Theorem B.11. ■

As last result, the following criterion concerns square-integrable functions.

Theorem B.13 (Riesz-Fischer Theorem)**.** *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a measurable function with period L. Then f is square-integrable on $[0, L]$ if and only if its Fourier series converges in $\mathrm{L}^2([0, L], \mathbb{C})$. In other words:*

$$f \in \mathrm{L}^2([0,L],\mathbb{C}) \quad \Longleftrightarrow \quad \lim_{N\to\infty} \left\| \sum_{n=-N}^{N} f_n e^{\frac{2\pi i n}{L} x} - f(x) \right\|_{\mathrm{L}^2([0,L],\mathbb{C})} = 0.$$

For a proof see Zygmund [75, p. 127].

Appendix C

Some useful results of real Analysis

In this appendix we will make a list of all the theorems we used in the main part. In this way we avoid different formulations or different notations that are often used in the literature. Fix $N \in \mathbb{N}$ and a closed interval $[a, b] \subset \mathbb{R}$ where $a < b$. For most of the results we refer to Jossen [35]. We start with some approximation theorems.

Theorem C.1 (Squeeze)**.** *Let $x_0 \in [a, b]$. Let $f, g, h \colon [a, b] \to \mathbb{R}$ be functions such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a, b] \setminus \{x_0\}$. Assume additionally that*

$$L \coloneqq \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x).$$

Then also $\lim_{x \to x_0} f(x) = L$.

Proof. By assumption, for all $x \in [a, b] \setminus \{x_0\}$ we have

$$L = \lim_{x \to x_0} g(x) \leq \underline{\lim_{x \to x_0}} f(x) \leq \overline{\lim_{x \to x_0}} f(x) \leq \lim_{x \to x_0} h(x) = L.$$

Hence all the inequalities are equalities, and therefore $\lim_{x \to x_0} f(x) = L$. ∎

Theorem C.2 (Weierstrass Approximation Theorem)**.** *Let $f \in \mathrm{C}^0([a, b], \mathbb{C})$. Then, for all $\epsilon \in \mathbb{R}_{>0}$ there exists $p \in \mathbb{C}[x]$ such that for all $x \in [a, b]$ it holds*

$$|f(x) - p(x)| < \epsilon.$$

For a proof of the real version see Bernstein [8]. The complex version follows directly by *Stone-Weierstrass Theorem*. We state now some theorems of Differential Calculus. One of the most important results is the following.

Theorem C.3 (Mean value Theorem)**.** *Let $f \in \mathrm{C}^0([a, b], \mathbb{R})$ such that f is differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For a proof see Jossen [35, p. 212]. On the other hand, we have also a version of the mean value Theorem for Riemann-integrable functions.

Theorem C.4 (Second Mean value Theorem)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Let $G\colon [a,b] \to \mathbb{R}$ be monotone and $\varphi\colon [a,b] \to \mathbb{R}$ be Riemann-integrable. Then there exists $c \in (a,b)$ such that*

$$\int_a^b G(t)\varphi(t)\,dt = G(a)\int_a^c \varphi(t)\,dt + G(b)\int_c^b \varphi(t)\,dt.$$

For a proof see Hobson [34]. We proceed with a quick review of the Riemann integral (as reference see Jossen [35, pp. 101-117]).

Definition C.5. Let $a = x_0 < x_1 < \cdots < x_N = b$. A *step function* is a map

$$g\colon [a,b] \to \mathbb{R}, x \mapsto \sum_{i=0}^{N-1} d_i \chi_{[x_i, x_{i+1})}(x) \tag{C.1}$$

where $d_i \in \mathbb{R}$ for all $i \in \{0, \ldots, N-1\}$.

Notation C.6. We denote the set of all step functions with

$$\mathrm{SF} := \{g\colon [a,b] \to \mathbb{R} \mid g \text{ step function}\}.$$

Definition C.7. Let $a = x_0 < x_1 < \cdots < x_N = b$ and g a step function with parameters given in (C.1). The *Riemann-integral* of g is the real number

$$\int_a^b g(x)\,dx = \sum_{i=0}^{N-1} d_i(x_{i+1} - x_i).$$

Definition C.8. Let $f\colon [a,b] \to \mathbb{R}$. We denote the *lower sums* of f by

$$\mathcal{L}(f) := \left\{ \int_a^b l(x)\,dx \;\middle|\; l \in \mathrm{SF}\,,\, l \le f \right\}$$

and the *upper sums* of f by

$$\mathcal{U}(f) := \left\{ \int_a^b u(x)\,dx \;\middle|\; u \in \mathrm{SF}\,,\, u \ge f \right\}.$$

Definition C.9. A bounded map $f\colon [a,b] \to \mathbb{R}$ is called *Riemann-integrable* if

$$\sup \mathcal{L}(f) = \inf \mathcal{U}(f).$$

In this case, we define the *Riemann-integral* of f as

$$\int_a^b f(x)\,dx = \sup \mathcal{L}(f) = \inf \mathcal{U}(f).$$

Continuous maps are Riemann-integrable.

Proposition C.10. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann-integrable.*

For a proof see Jossen [35, p. 116]. One other intuitive property of Riemann-integrable functions is the following.

Proposition C.11. *Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann-integrable if and only if for all $\epsilon \in \mathbb{R}_{>0}$ there exists $l, u \in \mathrm{SF}$ such that*

$$l \leq f \leq u \qquad \text{and} \qquad \int_a^b (u(x) - l(x))\, dx < \epsilon.$$

Proof. We have

$$\begin{aligned}
& f \text{ Riemann-integrable} \\
\iff\ & \sup \mathcal{L}(f) = \inf \mathcal{U}(f) \\
\iff\ & \forall \epsilon \in \mathbb{R}_{>0}\ \exists l \in \mathcal{L}(f)\,, \exists u \in \mathcal{U}(f) \text{ such that } u - l < \epsilon \\
\iff\ & \forall \epsilon \in \mathbb{R}_{>0}\ \exists l, u \in \mathrm{SF} \text{ such that } l \leq f \leq u \text{ and } \textstyle\int_a^b (u(x) - l(x))\, dx < \epsilon.
\end{aligned}$$

This concludes the proof. ∎

The concept of Riemann-integral justifies the convergence of the Riemann-sum.

Lemma C.12. *Let $a = x_0 < x_1 < \cdots < x_N = b$ and for all $i \in \{0, \ldots, N-1\}$ let $c_i \in [x_i, x_{i+1})$. Then*

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(c_i)\,(x_{i+1} - x_i) = \int_a^b f(x)\, dx.$$

Proof. This follows from Definition C.9. ∎

As last thing, we state some results which help us to compute sums.

Theorem C.13 (Abel's summation by parts)**.** *Let $m \in \mathbb{N}$. For $h \in \{1, \ldots, m\}$ let $a_h, b_h \in \mathbb{R}$, and define $A_h \coloneqq \sum_{k=1}^{h} a_k$ and $B_h \coloneqq \sum_{k=1}^{h} b_k$. Then*

$$\sum_{h=1}^{m} a_h b_h = A_m b_m + \sum_{h=1}^{m-1} A_h (b_h - b_{h+1}).$$

Proof. Let $m \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{h=1}^{m} a_h b_h &= a_1 b_1 + \sum_{h=2}^{m} (A_h - A_{h-1}) b_h = a_1 b_1 + \sum_{h=2}^{m} A_h b_h - \sum_{h=1}^{m-1} A_h b_{h+1} \\
&= a_1 b_1 + A_m b_m + \sum_{h=2}^{m-1} A_h b_h - \sum_{h=2}^{m-1} A_h b_{h+1} - A_1 b_2 \\
&= a_1 b_1 + A_m b_m - a_1 b_2 + \sum_{h=2}^{m-1} A_h (b_h - b_{h+1}) \\
&= A_m b_m + \sum_{h=1}^{m-1} A_h (b_h - b_{h+1}).
\end{aligned}$$

This concludes the proof. ∎

Theorem C.14 (Cauchy-Schwarz inequality)**.** *Let $(V, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be an inner product space and $u, v \in V$. Then*

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \, .$$

For a proof see Schwarz [59].

Theorem C.15 (Euler summation formula)**.** *Assume that $a, b \in \mathbb{N}$ and consider a function $F \in \mathrm{C}^1([a,b], \mathbb{C})$. Then*

$$\sum_{n=a}^{b} F(n) = F(a) + \int_a^b F(t)\,dt + \int_a^b \{t\}F'(t)\,dt. \tag{C.2}$$

Proof. Let $n \in \{a+1, \ldots, b\}$. Then, using integration by parts, we get

$$\begin{aligned}
&\int_{n-1}^{n} \{t\}F'(t)\,dt = \int_{n-1}^{n} tF'(t)\,dt - \int_{n-1}^{n} [t]F'(t)\,dt \\
&= [tF(t)]_{n-1}^{n} - \int_{n-1}^{n} F(t)\,dt - (n-1)\int_{n-1}^{n} F'(t)\,dt \\
&= nF(n) - (n-1)F(n-1) - \int_{n-1}^{n} F(t)\,dt - (n-1)\left(F(n) - F(n-1)\right) \\
&= F(n) - \int_{n-1}^{n} F(t)\,dt.
\end{aligned}$$

Summing over all $n \in \{a+1, \ldots, b\}$, we get

$$\int_a^b \{t\}F'(t)\,dt = \sum_{n=a+1}^{b} F(n) - \int_a^b F(t)\,dt$$

This is equivalent to (C.2). ∎

An equivalent formulation is the following.

Corollary C.16. *Assume that $a, b \in \mathbb{N}$ and let $F \in \mathrm{C}^1([a,b], \mathbb{C})$. Then*

$$\sum_{n=a}^{b} F(n) = \int_a^b F(t)\,dt + \frac{1}{2}\left(F(a) + F(b)\right) + \int_a^b \left(\{t\} - \frac{1}{2}\right) F'(t)\,dt.$$

Proof. This follows directly from (C.2). ∎

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List of symbols

Let x be a number, $k \in \mathbb{N}$ and A, B, M be sets. Let $f\colon E \to \mathbb{C}$ and $g\colon E \to \mathbb{R}_{>0}$ be maps, where $E \subseteq \mathbb{R}_{>0}$ is unbounded.

$\mathbb{N} = \{1, 2, 3, \dots\}$	Natural numbers
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$	Integer numbers
$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}^k$	k-dimensional Euclidean space
$\mathbb{R}[x]$	One-variable polynomials with coefficients in $\mathbb{R}$
$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$	Complex numbers
$\mathbb{C}[x]$	One-variable polynomials with coefficients in $\mathbb{C}$
$\lvert x \rvert$	Absolute value or modulus of x
$[x] = \max\{\, n \in \mathbb{Z} \mid n \leq x \,\}$	Integral part of x
$\{x\} = x - [x]$	Fractional part of x
$\langle x \rangle = \min_{p \in \mathbb{Z}} \lvert x - p \rvert$	Distance from x to the nearest integer
$\lvert M \rvert$	Cardinality of the set M
χ_M	Characteristic function of the set M
$\mathbb{1}\{\text{"condition"}\}$	1 if "condition" holds, otherwise 0
$\delta_{ij} = \mathbb{1}\{i = j\}$	Kronecker delta
λ	1-dimensional Lebesgue measure
gcd	Greatest common divisor
$\mathrm{C}^0(A, B)$	Set of continuous functions $f\colon A \to B$
$\mathrm{C}^k(A, B)$	Set of k-differentiable functions $f\colon A \to B$ with continuous k-derivatives
$\mathrm{L}^1(A, B)$	Set of integrable functions $f\colon A \to B$
$\mathrm{L}^2(A, B)$	Set of square-integrable functions $f\colon A \to B$
$\mathrm{L}^\infty(A, B)$	Set of bounded functions $f\colon A \to B$
$f(x) = \mathcal{O}(g(x))$	$\exists c, x_0 > 0$ such that $\forall x \geq x_0 : \lvert f(x) \rvert \leq c g(x)$
$f(x) = \Omega(g(x))$	$\exists c, x_0 > 0$ such that $\forall x \geq x_0 : \lvert f(x) \rvert \geq c g(x)$
$f(x) = \Theta(g(x))$	$f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$