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Master Thesis

Dynamical Stability of Ground States of the Gravitational Vlasov-Poisson System

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Abstract

This thesis aims to review proofs of the dynamical stability of ground states of the gravitational Vlasov-Poisson system. To do so, we introduce the Vlasov-Poisson system and define the Hamiltonian $\mathcal{H}$, consisting of a kinetic and a potential term representing the total energy of the system. Then we will prove the existence of minimizers under a mass-Casimir constraint, which act as ground states of the system. A specific type of solutions to the Vlasov-Poisson systems are polytropes, and we will see that these solutions are minimizers. With these intuitive definitions, it is already possible to show dynamical stability for minimizers of $\mathcal{H}$ over the constrained set. Motivated by this result, we pursue a stronger stability result in the whole energy space. Introducing a modified minimization problem, we use a concentration compactness argument to show relative compactness of minimizing sequences in the energy space. From this follows dynamical stability of ground states in the energy space.

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Preliminaries from Analysis

In this section we will provide essential definitions and principles from Analysis which will serve as a foundation for our work. For detailed proofs and further insights, we refer to literature such as [LL01] and [Eva22]. The main topic starts in Chapter 2.
For the generally applicable preliminaries, unless otherwise specified, we will consider $\Omega \subseteq \mathbb{R}^N$, with dimension N, and a Borel measure μ, while we typically use the Lebesgue measure λ in this work. Further, as our σ-Algebra $\mathcal{A}$ we will consider the Borel σ-Algebra.

1.1 Definitions

Definition 1.1.1 (L^p Spaces)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. Then the L^p space is defined as*

$$L^p(\Omega, \mu) := \{f : \Omega \to \mathbb{C} : f \text{ is measurable} \|f\|_p < \infty\}$$

where the norm is defined as

$$\|f\|_p := \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

Definition 1.1.2 (L^∞ Space)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then the L^∞ space is defined as*

$$L^\infty(\Omega, \mu) := \{f : \Omega \to \mathbb{C} : f \text{ is measurable}, \exists K \text{finite such that} |f(x)| \leq K \text{ for a.e. } x \in \Omega\}$$

The L^∞ norm is defined as

$$\|f\|_\infty := \|f\|_{L^\infty(\Omega)} := \inf\{K : |f(x)| \leq K \text{ for a.e. } x \in \Omega\}.$$

Remark 1.1.3. If not mentioned otherwise, we will usually consider $\Omega = \mathbb{R}^N$ and $\mu = \lambda$, where λ denotes the Lebesgue measure. For $1 \leq p \leq \infty$, we will often use the shorthand notations $L^p(\mathbb{R}^N)$ or simply L^p if the dimension is clear from the context.

Definition 1.1.4 (Convolution)**.** *Let f and g be two complex-valued functions on $\mathbb{R}^N$. Then we define their convolution as the function $f * g$ given by*

$$f * g(x) = \int_{\mathbb{R}^N} f(x - y) g(y) \, dy.$$

By change of variables, $f * g = g * f$.

Definition 1.1.5 (Strong Convergence)**.** *Let (f_n) be a sequence of functions in $L^p(\Omega)$. Then f_n converges strongly to $f \in L^p(\Omega)$ if*

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

In this case the strong convergence is denoted by $f_n \to f$.

Definition 1.1.6 (Weak Convergence)**.** *If (f_n) is a sequence of functions in $L^p(\Omega)$, then f_n converges weakly to f if*

$$\lim_{n\to\infty} L(f_n) = L(f)$$

for every L in the dual space of $L^p(\Omega)$, i.e. $L \in (L^p(\Omega))^*$. In this case the weak convergence is denoted by $f_n \rightharpoonup f$.

Definition 1.1.7 (Weak-$*$ Convergence)**.** *If (T_n) is a sequence in $(L^p(\Omega))^*$, then T_n converges weak-$*$ to T if*

$$\lim_{n\to\infty} T_n(f) = T(f)$$

for every $f \in L^p(\Omega)$.

Definition 1.1.8 (Weak Partial Derivative)**.** *Let $f, g \in L^1_{loc}(\Omega)$ and let α be a multiindex. Then g is the α^{th}-weak partial derivative of f, written as $g = D^\alpha f$, if for every $\varphi \in C_c^\infty(\Omega)$*

$$\int_\Omega f D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega g\varphi \, dx.$$

Definition 1.1.9 (Compact Operator)**.** *Let $\mathcal{B}$ be a Banach space. Then $T \in \mathcal{L}(\mathcal{B})$, is a compact operator, if for every bounded sequence φ_n, $T\varphi_n$ has a convergent subsequence.*
Here, $\mathcal{L}(\mathcal{B})$ is the set of bounded linear operators on $\mathcal{B}$.

Definition 1.1.10 (Sobolev Space)**.** *For $1 \le p \le \infty$ and $\kappa \in \mathbb{N}$, the Sobolev space $W^{\kappa,p}$ is defined as*

$$W^{\kappa,p}(\Omega) := \{f \in L^p(\Omega) \ : \ D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \le \kappa\}$$

and is equipped with the norm

$$\|f\|_{\kappa,p} = \left(\sum_{|\alpha|\le\kappa} \|D^\alpha f\|_p^2 \right)^{\frac{1}{2}}.$$

1.2 Inequalities

Lemma 1.2.1 (Fatou's Lemma)**.** *Let (f_n) be a sequence of non-negative and measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$. Then $f := \liminf_{n\to\infty} f_n$ is measurable and*

$$\liminf_{n\to\infty} \int f_n \, d\mu \ge \int f \, d\mu.$$

Theorem 1.2.2 (Hölder's Inequality). *Let p and q such that $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, where $\Omega \subset \mathbb{R}^n$. Then $(fg)(x) = f(x)g(x) \in L^1(\Omega)$ and*

$$\left| \int_\Omega fg \, d\mu \right| \leq \int_\Omega |f| \, |g| \, d\mu \leq \|f\|_p \|g\|_q.$$

Theorem 1.2.3 (Lower Semicontinuity of Norms). *Let (f_n) be a sequence of functions in $L^p(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$, then*

$$\liminf_{n\to\infty} \|f_n\|_p \geq \|f\|_p.$$

Here, the L^p norm is said to be weakly lower semicontinuous.

Theorem 1.2.4 (Hardy-Littlewood-Sobolev Inequality). *Let $p, r > 1$ and $0 < \lambda < N$ with $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{r} = 2$. Then there exists a constant C depending only on N, λ and p such that for all $g \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$,*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g(x) \, |x - y|^{-\lambda} \, h(y) \, dxdy \right| \leq C(N, \lambda, p) \|g\|_p \|h\|_r.$$

Theorem 1.2.5 (Young's Inequality). *Let $p, q, r \geq 1$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$. Then*

$$\left| \int_{\mathbb{R}^N} f(x)(g * h)(x) \, dx \right| = \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) g(x - y) h(y) \, dydx \right| \leq C \|f\|_p \|g\|_q \|h\|_r.$$

1.3 Important Theorems

Theorem 1.3.1 (Dominated Convergence Theorem). *Let (f_n) and f be measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$ such that $f_n \to f$ pointwise a.e. If there exists an integrable, non-negative function g on $(\Omega, \mathcal{A}, \mu)$ such that $|f_n(x)| \leq g(x)$ for a.e. $x \in \Omega$, then f_n for every n and f are integrable and*

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$

Theorem 1.3.2 (Divergence Theorem). *Let $\Omega \subset \mathbb{R}^N$ be a compact subset with piecewise smooth boundary $\partial\Omega$. If F is a continuously differentiable vector field on an open set U such that $\Omega \subset U$, then*

$$\int_\Omega \nabla \cdot F \, d\Omega = \int_{\partial\Omega} F \cdot n \, d(\partial\Omega),$$

where n denotes the outward pointing normal at each point on the boundary $\partial\Omega$.

Theorem 1.3.3 (Banach-Alaoglu Theorem). *Let $\Omega \subset \mathbb{R}^N$ be a measurable set and let $1 < p < \infty$. Let $(f_i) \subset L^p(\Omega)$ be a bounded sequence. Then there exists a subsequence (f_{i_k}) and $f \in L^p(\Omega)$ such that*

$$f_{i_k} \rightharpoonup f \text{ weakly in } L^p(\Omega).$$

Theorem 1.3.4 (Sobolev Embedding). *Let $\Omega \subset\subset \mathbb{R}^N$, i.e. $\Omega \subset \mathbb{R}^N$ and $\bar{\Omega} \subset \mathbb{R}^N$ compact, be a set of class C^1. Let $\kappa \in \mathbb{N}$, $1 \leq p \leq \infty$ and moreover, for $x \in \mathbb{R}$ denote $[x] = \max\{j \in \mathbb{N} \,:\, j \leq x\}$.*

(i) If $\kappa p < N$, then $W^{\kappa,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq q_0$ with $\frac{1}{q_0} = \frac{1}{p} - \frac{\kappa}{N}$. The embedding is compact, if $q < q_0$.

(ii) If $\kappa p = N$, then $W^{\kappa,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q < \infty$. This embedding is compact.

(iii) If $\kappa p > N$ with $\kappa - \frac{N}{p} \notin \mathbb{N}$, then $W^{\kappa,p}(\Omega) \hookrightarrow C^{l,\alpha}(\bar{\Omega})$ for $l = [\kappa - \frac{N}{p}] \in \mathbb{N}_0$ and $0 \leq \alpha \leq \alpha_0$ with $\alpha_0 = \kappa - l - \frac{N}{p}$. The embedding is compact, if $\alpha < \alpha_0$.

(iv) If $\kappa p > N$ with $\kappa - \frac{N}{p} \in \mathbb{N}$, then $W^{\kappa,p}(\Omega) \hookrightarrow C^{l,\alpha}(\bar{\Omega})$ for $l = \kappa - \frac{N}{p} - 1 \in \mathbb{N}_0$ and $0 \leq \alpha \leq 1$. This embedding is compact.

Remark 1.3.5. An open set $\Omega \subset \mathbb{R}^N$ is of class C^1 if for every point $x_0 \in \partial\Omega$ there exists $r > 0$ and a function $\gamma \in C^1(\mathbb{R}^{N-1})$ such that, after relabeling and reorienting the coordinate axes if needed, we obtain

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \,:\, x_N > \gamma(x_1, ..., x_{N-1})\}.$$

Remark 1.3.6. For the case where $\Omega = \mathbb{R}^N$, Theorem 1.3.4 holds true, preserving all statements except for the compact embedding properties.

The Vlasov-Poisson System

In order to study stable states of galaxies and globular clusters, we will make use of the so-called Vlasov-Poisson system in three dimensions. Let $t \in \mathbb{R}_+$ be the variable corresponding for time and let $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ vectors for position and velocity, respectively. All possible states of x and v constitute the phase space. To describe the system, one can think of a many body setting that is described by a distribution function $f := f(x, v, t)$, which gives the number of particles per unit volume in the phase space. We set $f \geq 0$. The gravitational Vlasov-Poisson system can be written as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \\ \Delta_x U = \rho, \quad \lim\limits_{|x| \to \infty} U(x, t) = 0, \\ \rho(x, t) = \displaystyle\int f(x, v, t)\, dv, \end{cases} \tag{2.1}$$

where ρ describes the spatial mass density of stars and $U(x, t)$ the gravitational potential [Gla96].

The Vlasov-Poisson system in this notation describes mechanical states of stellar systems if we neglect relativistic effects and collisions among the particles, i.e. stars in our case.

More precisely, the first equation is the so-called *Vlasov equation*, which describes the movement of particles in the system. The first term represents the time rate of change of the distribution function. It describes how the distribution of particles in the phase space evolves over time. The second term describes how the distribution in the position space changes as the particles move along their velocities. It takes into account the transport of particles in the phase space. In the last term, the gradient of the gravitational potential U represents the force caused by gravity, and as a result, this term corresponds to the change in the distribution in the velocity space caused by the gravitational force. The sum of these terms must be equal to zero, what means that the distribution function is conserved over time in the absence of external influences, i.e. there are no external sources for particles.

On the other hand, $\Delta_x U = \rho$ is the so-called *Poisson equation*. It tells us how the mass of particles in space generates the gravitational potential, which in turn influences the behavior of the other particles in the system as described by the *Vlasov equation*.

As the potential term in (2.1) is an attractive force, an equilibrium of the system is possible. We can expect that the system has stationary states and we want to prove this fact in Chapter 2.6.

For any nonnegative distribution function $f(x, v)$ we denote by ρ_f, U_f and G_f the corresponding

density, potential and Poisson force field, respectively. We define

$$\rho_f(x) := \int f(x,v,t)\, dv, \tag{2.2}$$

$$U_f(x) := -\frac{1}{4\pi |x|} * \rho_f = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \rho_f(y)\, dy, \tag{2.3}$$

$$G_f := \nabla_x U_f = \frac{1}{4\pi} \int \frac{(x-y)\,\rho(y,t)}{|x-y|^3} dy. \tag{2.4}$$

Remark that one can define the Vlasov-Poisson System in other dimensions. Let $N \in \mathbb{N}$. Then for $(x,v) \in \mathbb{R}^N \times \mathbb{R}^N$

$$U_f(x) = -\frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \rho_f(y)\, dy,$$

where ω_N is the volume of the unit N-ball.

In this chapter, the statements and proofs are mainly based on Chapter 1 and 2 of [GR01] and Chapter 1 of [LMR08].

2.1 Electrostatic Vlasov-Poisson System

Another area where the Vlasov-Poisson system is used is in a many body problem of charged particles. For example, one could consider a ionised gas with particles of charge of an electron inside. In contrast to the gravitational Vlasov-Poisson system, the electrons repel each other, while the gravity of stars is an attractive force. Hence, one has to change the sign of the potential term in de Vlasov-Poisson equation, i.e.

$$\partial_t f + v \cdot \nabla_x f + \nabla_x U \cdot \nabla_v f = 0.$$

In this system, f is the density of ionized particles in the system, ρ the spatial charge density of the system, the force field G describes the electrostatic potential and U is the corresponding electric potential. This electrostatic version of the Vlasov-Poisson system is a limit case of the electrodynamical Vlasov-Maxwell system. While the Vlasov-Maxwell system also considers magnetic fields and relativistic effects, the Vlasov-Poisson system sets the magnetic field equal to zero and takes the non-relativistic limit.

2.2 Invariances of Solutions of the Vlasov-Poisson System

A property of the solutions of the Vlasov-Poisson system is the invariance under certain transformations, namely space-time translation, Galilean transformation and scaling. The modified solutions will be useful in later on.

Proposition 2.2.1 (Space-time translation invariance)**.** *Let $f(x,v,t)$ be a solution to the Vlasov equation. Then for every $(x_0,t_0) \in \mathbb{R}^3 \times \mathbb{R}$, $f(x+x_0, v, t+t_0)$ is again a solution.*

Proof. Since the chain rule, applied to the translated function f, does not yield any constants, the terms in the Vlasov equation given in (2.1) remain without prefactors. As f is a solution for all $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in \mathbb{R}_+$, it is in particular a solution for $\tilde{x} = x + x_0$ and $\tilde{t} = t + t_0$. □

The proof for the next transformation, the Galilean transformation, can be conducted in a similar manner to the one about space-time translation.

Proposition 2.2.2 (Galilean invariance)**.** *Let $f(x, v, t)$ be a solution to the Vlasov equation. Then for every $v \in \mathbb{R}^3$, $f(x + v_0 t, v + v_0, t)$ is again a solution.*

Lastly, the scaling invariance is given under certain prefactors and therefore the proof needs a bit of effort.

Proposition 2.2.3 (Scaling invariance)**.** *Let $f(x, v, t)$ be a solution to the Vlasov equation. Then for every $\alpha, \beta \in \mathbb{R}_{>0}$,*

$$\frac{\beta}{\alpha^2} f\left(\frac{x}{\alpha}, \beta v, \frac{t}{\alpha\beta}\right)$$

is again a solution.

Proof. Let $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and define $F(x, v, t) := \frac{\beta}{\alpha^2} f\left(\tilde{x}, \tilde{v}, \tilde{t}\right)$ where we denote $\tilde{x} = \frac{x}{\alpha}, \tilde{v} = \beta v$ and $\tilde{t} = \frac{t}{\alpha\beta}$. Using the chain rule one computes

$$\begin{aligned}
\partial_t F(x, v, t) &= \frac{1}{\alpha^3} \left(\partial_t f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) \\
\nabla_x F(x, v, t) &= \frac{\beta}{\alpha^3} \left(\nabla_x f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) \\
\nabla_v F(x, v, t) &= \frac{\beta^2}{\alpha^2} \left(\nabla_v f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right)
\end{aligned}$$

and by a change of variables with $\tilde{y} = \frac{y}{\alpha}$ and $\tilde{v} = \beta v$,

$$\begin{aligned}
\nabla_x U_{F(x,v,t)} &= \frac{1}{4\pi} \iint \frac{(x - y)}{|x - y|^3} F(y, v, t)\, dv dy \\
&= \frac{\beta}{\alpha^2} \frac{1}{4\pi} \iint \frac{1}{\alpha^2} \frac{(\tilde{x} - \tilde{y})}{|\tilde{x} - \tilde{y}|^3} \left(\frac{\alpha}{\beta}\right)^3 f\left(\tilde{y}, \tilde{v}, \tilde{t}\right)\, d\tilde{v} d\tilde{y} \\
&= \frac{1}{\alpha\beta^2} \nabla_x U_{f\left(\tilde{x}, \tilde{v}, \tilde{t}\right)}. \qquad (2.5)
\end{aligned}$$

Inserting these results into the Vlasov equation given in (2.1) gives

$$\begin{aligned}
&\partial_t F(x, v, t) + v \cdot \nabla_x F(x, v, t) - \nabla_x U \cdot \nabla_v F(x, v, t) \\
&= \frac{1}{\alpha^3} \left(\partial_t f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) + \frac{1}{\beta} \tilde{v} \cdot \frac{\beta}{\alpha^3} \left(\nabla_x f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) - \frac{1}{\alpha\beta^2} \nabla_x U_{f\left(\tilde{x}, \tilde{v}, \tilde{t}\right)} \cdot \frac{\beta^2}{\alpha^2} \left(\nabla_v f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) \\
&= \frac{1}{\alpha^3} \left(\left(\partial_t f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) + \tilde{v} \cdot \left(\nabla_x f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right) - \nabla_x U_{f\left(\tilde{x}, \tilde{v}, \tilde{t}\right)} \cdot \left(\nabla_v f\right)\left(\tilde{x}, \tilde{v}, \tilde{t}\right)\right) = 0,
\end{aligned}$$

since f is a solution for all $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in \mathbb{R}_+$. □

2.3 Steady States

A class of interest are steady states of the Vlasov-Poisson system. These are f such that f fulfills (2.1) and f is time-independent, i.e. we have that $\partial_t f = 0$. In order to describe such solutions, we will define the particle energy.

Definition 2.3.1. *Let U be a time-independent potential. Then we define the particle energy by*

$$E := \frac{1}{2}|v|^2 + U_0(x).$$

Remark 2.3.2. The particle energy E is a conserved quantity if v and U_0 are parts of a Vlasov-Poisson system.

The idea to obtain steady states of the Vlasov-Poisson system is to express the particle distribution f_0 in terms of the particle energy E.

Lemma 2.3.3. *Let g be a differentiable function. Then any $f(x,v) = g(E(x,v))$ solves the Vlasov equation.*

Proof. As the energy E is time-independent, we obtain by the chain rule

$$\begin{aligned}\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f &= \partial_t E \cdot \partial_t g + v \cdot \nabla_x E \cdot \frac{dg}{dE} - \nabla_x U \cdot \nabla_v E \frac{dg}{dE} \\ &= v \cdot \nabla_x U \frac{dg}{dE} - \nabla_x U \cdot v \frac{dg}{dE} = 0.\end{aligned}$$

□

2.4 Energy Definitions and Hamiltonian Formulation

Let $f := f(x,v)$, where $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$, be a measurable function. An important quantity that describes the system is the total energy, which is given by a Hamiltonian defined as follows:

Definition 2.4.1. *Define the kinetic energy as*

$$E_{kin}(f) := \frac{1}{2}\iint |v|^2 f(x,v)\, dvdx,$$

and the potential energy as

$$E_{pot}(f) := -\frac{1}{2}\int |\nabla U_f(x)|^2 \, dx.$$

Then the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(f) := E_{kin}(f) + E_{pot}(f).$$

For later results, it is useful to express the potential energy in terms of the density function ρ or the distribution function f.

Lemma 2.4.2. *Let E_{pot} be the potential energy as defined in Definition 2.4.1. Then*

(i)

$$E_{pot}(f) = -\frac{1}{8\pi} \iint \frac{\rho_f(x)\rho_f(y)}{|x-y|}\, dxdy,$$

(ii)

$$E_{pot}(f) = \frac{1}{2} \iint f(x,v)U_f(x)\, dxdv.$$

Proof. To show *(i)*, we can integrate by parts and use the fact that $\lim_{|x|\to\infty} U(x,t) = 0$ to obtain

$$\begin{aligned} E_{pot}(f) &= -\frac{1}{2} \int |\nabla U_f(x)|^2 \, dx \\ &= \frac{1}{2} \int \Delta U_f(x) U_f(x)\, dx. \end{aligned} \tag{2.6}$$

Using the fact that $\Delta_x U_f = \rho_f$ and (2.3), we conclude

$$\begin{aligned} E_{pot}(f) &= \frac{1}{2} \int \rho_f(x) U_f(x)\, dx \\ &= -\frac{1}{8\pi} \iint \frac{\rho_f(x)\rho_f(y)}{|x-y|}\, dxdy. \end{aligned} \tag{2.7}$$

To establish *(ii)*, we can start from (2.7) and insert (2.2). Therefore

$$E_{pot}(f) = \frac{1}{2} \iint f(x,v)U_f(x)\, dxdv.$$

□

The kinetic and potential energies are correctly defined in the sense that their sum, expressed through the Hamiltonian, remains conserved over time. Thus the total energy is a conserved quantity.

Proposition 2.4.3. *Let $f(x,v,t)$ be a solution to the Vlasov-Poisson system. Then*

$$\frac{d}{dt}\mathcal{H}(f) = 0,$$

meaning that the Hamiltonian $\mathcal{H}$ is conserved over time.

Proof. For this proof we will first describe a continuity equation which connects the relation between the change in time of the density ρ and the current density j which is defined by

$$j(x,t) := \int v f(x,v,t)\, dv.$$

The current density describes the flow of particles in space associated with f. Then the continuity equation can be stated as

$$\frac{\partial}{\partial t}\rho + \nabla \cdot j = 0. \tag{2.8}$$

Indeed, one can rewrite the sum using (2.1) and take the time derivative inside due to *Dominated Convergence Theorem* (Theorem 1.3.1) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\rho + \nabla \cdot j &= \int \frac{\partial}{\partial t} f + v f(x,v,t)\, dv \\ &= -\int \nabla_x U \cdot \nabla_v f \, dv,\end{aligned}$$

where the last step follows from integration by parts as seen in (2.6). Now the integrand can be expressed as a divergence, i.e. we have $\nabla_x U \cdot \nabla_v f = \nabla_v \cdot ((\nabla_x U) f)$. Moreover, since f is a distribution function, we assume it is compactly supported with respect to v, and we denote its support by $K \subset \mathbb{R}^3$. Let $B \subset \mathbb{R}^3$ be an open ball such that $K \subset B$. Hence from *Divergence Theorem* (Theorem 1.3.2) follows

$$\begin{aligned}\frac{\partial}{\partial t}\rho + \nabla \cdot j &= -\int_{B_R} \nabla_v \cdot ((\nabla_x U) f)\ dv \\ &= -\int_{\partial B_R} ((\nabla_x U) f) \cdot n \, dv \\ &= 0\end{aligned}$$

with the outward pointing normal n at each point of the boundary ∂B_R. Therefore (2.8) holds true. Next we compute the time derivative of the kinetic energy. We have

$$\begin{aligned}\frac{d}{dt} E_{kin} &= \frac{1}{2}\frac{\partial}{\partial t} \iint |v|^2 f(x,v,t)\, dv dx \\ &= -\frac{1}{2} \iint |v|^2 v \cdot \nabla_x f - |v|^2 \nabla_x U \cdot \nabla_v f \, dv dx,\end{aligned}$$

where the first term in the integral becomes zero after integration due to similar arguments as before using the compact support of f with respect to x. However for the second term we obtain by the chain rule

$$\begin{aligned}-\iint |v|^2 \nabla_x U \cdot \nabla_v f \, dv dx &= \iint \nabla_v \left(|v|^2 (\nabla_x U) f\right) - \nabla_v \left(|v|^2 \nabla_x U\right) f \, dv dx \\ &= \iint 2 v f \cdot \nabla_x U \, dv dx \\ &= 2 \int j \cdot \nabla_x U \, dx\end{aligned}$$

and thus

$$\frac{d}{dt} E_{kin} = -\int j \cdot \nabla_x U \, dx.$$

The time derivative of the potential energy is given by

$$\begin{aligned}\frac{d}{dt}E_{pot}(f) &= -\frac{1}{2}\frac{\partial}{\partial t}\int |\nabla U(x)|^2 \, dx \\ &= -\int \nabla U \cdot \nabla \left(\partial_t U\right) \, dx \\ &= \int U \Delta \left(\partial_t U\right) \, dx \\ &= \int U(\partial_t \rho) \, dx \\ &= -\int U \nabla \cdot j \, dx \\ &= \int j \cdot \nabla U \, dx,\end{aligned}$$

where we applied the Poisson equation from (2.1), (2.8) as well as *Divergence Theorem* (Theorem 1.3.2). Combining these computations we conclude

$$\frac{d}{dt}\mathcal{H}(f) = \frac{d}{dt}E_{kin}(f) + \frac{d}{dt}E_{pot}(f) = 0.$$

□

2.5 Mass-Casimir Constraint

Another quantity that we want to include in our studies is the mass of the system under certain circumstances.

Definition 2.5.1. *Let $Q \in C^1([0,\infty))$ be a non-negative function such that $Q(0) = 0$. Moreover,*

(i) there exist $C > 0$ and $0 < k < \frac{7}{2}$ such that

$$Q(f) \geq C\left(f + f^{1+\frac{1}{k}}\right)$$

for $f \geq 0$,

(ii) Q is convex, i.e. for every $0 \leq t \leq 1$ and for every f_1, f_2 measurable we have

$$Q(tf_1 + (1-t)f_2) \leq tQ(f_1) + (1-t)Q(f_2).$$

Then we define the Casimir constraint $\mathcal{C}$ by

$$\mathcal{C}(f) := \iint Q(f(x,v)) \, dxdv.$$

Our first goal minimize the total energy of the system within the constraints of a mass-Casimir condition. In general, Casimir constraints are used to describe systems with certain conditions on the mass or mass distribution. $\mathcal{C}$ is a suitable quantity to restrict to certain classes of solutions, as for f being a solution to the Vlasov-Poisson system, $\mathcal{C}(f)$ is time-independent.

Proposition 2.5.2. *Let $f(x,v,t)$ be a solution to the Vlasov-Poisson system. Then $\mathcal{C}(f)$ is time-independent, i.e. $\frac{d}{dt}\mathcal{C}(f) = 0$.*

Proof. As Q is differentiable and convex, we can dominate the function and therefore take the derivative inside the integral by *Dominated Convergence Theorem* (Theorem 1.3.1). Further, applying the chain rule gives us

$$\begin{aligned}\frac{d}{dt}\mathcal{C}(f) &= \iint Q(f(x,v,t))\,dxdv \\ &= \iint \frac{\partial}{\partial t}\left(Q(f(x,v,t))\right)\,dxdv \\ &= \iint \frac{d}{df}Q\ \frac{\partial}{\partial t}f(x,v,t)\,dxdv \\ &= \iint \frac{d}{df}Q\ \left(\nabla_x U\cdot\nabla_v f - v\cdot\nabla_x f\right)\,dxdv,\end{aligned}$$

where the last step follows from the Vlasov-equation in (2.1). Using the chain rule leads to

$$\frac{d}{dt}\mathcal{C}(f) = \iint \left(\nabla_x U\cdot\nabla_v - v\cdot\nabla_x\right)\,Q dxdv.$$

The next step is similar to the computation we did for the continuity equation in Proposition 2.4. We can rewrite the two terms inside the integral as divergences with respect to v and x. We find $\nabla_x U\cdot\nabla_v Q = \nabla_v\cdot((\nabla_x U)Q)$ and $v\cdot\nabla_x Q = \nabla_x\cdot(vQ)$. As f is a distribution function, we can assume that it has a compact support with respect to v given by $K_v\subset\mathbb{R}^3$ and, similarly, it has a compact support with respect to x given by $K_x\subset\mathbb{R}^3$. Choose open balls $B_v, B_x\subset\mathbb{R}^3$ such that $K_v\subset B_v$ and $K_x\subset B_x$, respectively. Then by *Divergence Theorem* (Theorem 1.3.2)

$$\begin{aligned}\iint \nabla_v\cdot((\nabla_x U)Q)\,dxdv &= \iint_{B_v}\nabla_v\cdot((\nabla_x U)Q)\,dxdv \\ &= \iint_{\partial B_v}((\nabla_x U)Q)\cdot n\,dxdv = 0,\end{aligned}$$

as U is independent of v and since $Q(f) = Q(0) = 0$ on ∂B_v, as $f(v) = 0$ for every $v\in\partial B_v$, and

$$\begin{aligned}\iint \nabla_x\cdot(vQ)\,dxdv &= \iint_{B_x}\nabla_x\cdot(vQ)\,dxdv \\ &= \iint_{\partial B_x}(vQ)\cdot n\,dxdv = 0,\end{aligned}$$

since $Q(f) = Q(0) = 0$ on ∂B_x, as $f(x) = 0$ for every $x\in\partial B_x$. Consequently, $\frac{d}{dt}\mathcal{C}(f) = 0$. □

Let $M > 0$. In our case we want to fix $\mathcal{C}$, meaning that we are looking for a minimizer f of $\mathcal{H}$ in the set

$$\mathcal{F}_M := \{f\in L^1\left(\mathbb{R}^6\right)\ :\ f\geq 0,\ \mathcal{C}(f) = M,\ E_{kin}(f) < \infty\}.$$

In Chapter 5, we will split $\mathcal{C}$ into two parts. To do so, let $j\in C^1\left([0,\infty)\right)$ be a convex non-negative function such that $j(0) = 0$. Furthermore, there exist $C > 0$ and $0 < k < \frac{7}{2}$ such that

$$j(f)\geq Cf^{1+\frac{1}{k}}$$

for all $f \geq 0$. Then we define the set

$$\mathcal{F}_{M_1,M_j} := \left\{ f \in L^1\left(\mathbb{R}^6\right) \ : \ f \geq 0, \iint f \, dxdv = M_1, \iint j(f) \, dxdv = M_j, \ E_{kin}(f) < \infty \right\}$$

with $M_1, M_j > 0$.

Remark 2.5.3. If f_0 is a minimizer of the system, it will satisfy

$$\lambda_0 Q'(f_0) = E = \frac{1}{2} |v|^2 + U_0(x), \tag{2.9}$$

where $\lambda_0 < 0$ is a Lagrange multiplier and E the particle energy in Definition 2.3.1. By the conditions on Q, we can solve for f_0 and find that f_0 is a function of the particle energy and hence f_0 is a steady state of the Vlasov-Poisson system by Lemma 2.3.3. This fact is discussed in Section 3.2.

There exist many functions that satisfy the assumptions on Q given in Definition 2.5.1. We want to discuss briefly some important examples.

Example 2.5.4. Let $0 < k < \frac{7}{2}$ and let

$$Q(f) = f + f^{1+\frac{1}{k}}, \quad f \geq 0.$$

Then Q satisfies the conditions given in Definition 2.5.1. Indeed:

(i) Clearly,

$$Q(f) = f + f^{1+\frac{1}{k}} \geq C\left(f + f^{1+\frac{1}{k}}\right)$$

for every $0 < C \leq 1$.

(ii) As f and $f^{1+\frac{1}{k}}$ are convex, Q is convex as well since the sum of convex functions is again convex.

Since Q is differentiable, we find

$$Q'(f) = 1 + \left(1 + \frac{1}{k}\right) f^{\frac{1}{k}}$$

and by (2.9), a minimizer f_0 in this case will therefore satisfy

$$f_0 = C(\lambda_0 - E)^k$$

for some $\lambda_0 < 0$ and $C > 0$. If f_0 vanished for $E > \lambda_0$, then f_0 is of the form of a so-called polytrope. These kind of states will be discussed in Section 2.6.
Since $k < \frac{7}{2}$, the potential energy is controlled by the kinetic energy, as we will prove in Lemma 2.7.1. If we set $k > \frac{7}{2}$, the potential energy becomes dominant and we cannot bound the Hamiltonian from below by the kinetic energy (c.f. Corollary 2.7.3). The case $k = \frac{7}{2}$ leads us to the limiting case, see Example 2.5.5.

Example 2.5.5. Let $k = \frac{7}{2}$. In this case,

$$Q(f) = f^{\frac{9}{7}}, \quad f \geq 0.$$

This limiting case corresponds to the description of globular clusters, which are strongly bound together by the gravitational forces and have a higher concentration of stars towards their centers. In Section 3.3.2 we will discuss this case in more detail.

2.6 Polytropes

An example of solutions to the Vlasov-Poisson system of great importance are polytropes. In Section 3.2, we will see that they satisfy the minimizing properties of the Hamiltonian $\mathcal{H}$. In general, polytropes satisfy a relation of the form $P = \rho^\gamma$ for some $\gamma > 1$, where P usually denotes the pressure and ρ the density of a system.

Definition 2.6.1. *Let $0 < k \leq \frac{7}{2}$ and let $E_0 \in \mathbb{R}$ be a cut-off energy or energy threshold, i.e. the upper bound of energies that are taken into account. Then the polytropes are given by*

$$\phi_k(x, v) = (E_0 - E)_+^k,$$

where we denote $\varphi_+ := \max(\varphi, 0)$.

Remark 2.6.2. As polytropes fulfill the conditions for Lemma 2.3.3, they are steady states of the Vlasov-Poisson system.

Another characteristic of polytropes is that their density can be expressed in the form of a potential with an exponent.

Lemma 2.6.3. *Let $0 < k \leq \frac{7}{2}$ and let $E_0 \in \mathbb{R}$ be the cut-off energy in Definition 2.6.1. If ϕ_k is a polytrope, then*

$$\rho_{\phi_k} = C \left(E_0 - U_{\phi_k}(x)\right)_+^{k+\frac{3}{2}}$$

for $C > 0$ depending on k.

Proof. By Definition 2.6.1, ϕ_k is given by

$$\phi_k(x, v) = \begin{cases} (E_0 - \frac{1}{2}|v|^2 - U_{\phi_k}(x))^k, & \frac{1}{2}|v|^2 + U_{\phi_k}(x) < E_0, \\ 0, & \text{else}, \end{cases}$$

With the use of spherical coordinates, integrating ϕ_k with respect to v yields

$$\begin{aligned}\rho_{\phi_k}(x) &= \int \phi_k(x,v)\,dv \\ &= \int_{|v|^2<2E_0-2U_{\phi_k}} \left(E_0 - \frac{1}{2}|v|^2 - U_\phi(x)\right)_+^k dv \\ &= 4\pi \int_{r^2<2E_0-2U_\phi} \left(E_0 - \frac{1}{2}r^2 - U_\phi(x)\right)_+^k r^2\,dr \\ &= 4\pi (E_0 - U_\phi(x))_+^k \int_{r^2<E_0-2U_\phi} \left(1 - \frac{r^2}{2(E_0-U_\phi(x))}\right)^k r^2\,dr.\end{aligned}$$

Substitute $t = \frac{r^2}{2(E_0-U_\phi(x))}$ with $\frac{dt}{dr} = \frac{r}{E_0-U_\phi(x)}$ to obtain

$$\int_{r^2<2E_0-2U_\phi} \left(1 - \frac{r^2}{2(E_0-U_\phi(x))}\right)^k r^2\,dr = \int_0^1 (1-t)^k \sqrt{2t}(E_0-U_\phi)^{\frac{3}{2}}\,dt$$

since $r = \sqrt{2(E_0-U_\phi(x))t}$ and therefore,

$$\begin{aligned}\rho_\phi(x) &= \left(4\pi \int_0^1 (2t)^{\frac{1}{2}}\,(1-t)^k\,dt\right)(E_0-U_\phi(x))_+^{k+\frac{3}{2}} \\ &=: C\,(E_0-U_\phi(x))_+^{k+\frac{3}{2}}\end{aligned}$$

□

2.7 Preliminary Results on the Energies

This section provides key preliminary lemmas that will be essential for our later studies. Let $M > 0$ and fix some $\mathcal{C}(f)$ with Q according to Definition 2.5.1. In the following Lemma, if $f \in \mathcal{F}_M$, we will find useful upper bounds for ρ_f and U_f in terms of the kinetic energy or constants $C > 0$, where C may depend on Q or M.

Lemma 2.7.1. *Let $f \in \mathcal{F}_M$ and $n = k + \frac{3}{2}$, where $k < \frac{7}{2}$. Then we have*

(i) $f \in L^{1+\frac{1}{k}}(\mathbb{R}^6)$ with

$$\iint f + f^{1+\frac{1}{k}}\,dvdx \le C.$$

(ii) $\rho_f \in L^{1+\frac{1}{n}}(\mathbb{R}^3)$ with

$$\int \rho_f^{1+\frac{1}{n}}\,dx \le C\left(\iint f^{1+\frac{1}{k}}\,dvdx\right)^{\frac{k}{n}}\left(\iint |v|^2 f\,dvdx\right)^{\frac{(n-k)}{n}} \le C E_{kin}(f)^{\frac{3}{2n}}.$$

(iii) $U_f \in L^6(\mathbb{R}^3)$ with $\nabla U_f \in L^2(\mathbb{R}^3)$ and

$$\int |\nabla U_f|^2\,dx \le C\|\rho_f\|_{6/5}^2 \le C E_{kin}(f)^{\frac{1}{2}}.$$

Proof. (i) As $f \in L^1(\mathbb{R}^6)$ and by the first assumption on Q for $C > 0$,

$$\iint f + f^{1+\frac{1}{k}} dvdx \leq \frac{1}{C} \iint Q(f(x,v))dvdx = \frac{M}{C} = \tilde{C}.$$

To show *(ii)*, we split the integral over v into $|v| \leq R$ and $|v| > R$ for some $R > 0$. Using spherical coordinates and *Hölder's Inequality* (Theorem 1.2.2) yields

$$\begin{aligned}\int_{R \geq |v|} f\, dv &= 4\pi \int_{\{R \geq r\}} R^2 f\, dr \\ &\leq 4\pi R^{\frac{3}{k+1}} \|f\|_{1+\frac{1}{k}}.\end{aligned}$$

For the other integral we obtain

$$\begin{aligned}\int_{R < |v|} f\, dv &= \int_{R<|v|} \frac{|v|^2}{R^2} f\, dr \\ &\leq 4\pi R^{-2} \int |v|^2 f\, dv.\end{aligned}$$

We now want to optimize in R. To do so, we set

$$R^{\frac{3}{k+1}} \|f\|_{1+\frac{1}{k}} = R^{-2} \||v|^2 f\|_1$$

and solving for R yields

$$R = \left(\frac{\||v|^2 f\|_1}{\|f\|_{1+\frac{1}{k}}} \right)^{\frac{k+1}{2k+5}}.$$

Therefore, using $n = k + \frac{3}{2}$ gives us

$$\begin{aligned}\rho_f^{1+\frac{1}{n}} &\leq \left[8\pi \left(\frac{\||v|^2 f\|_1}{\|f\|_{1+\frac{1}{k}}} \right)^{-\frac{2k+2}{2k+5}} \||v|^2 f\|_1 \right]^{1+\frac{1}{n}} \\ &\leq C \left(\int f^{1+\frac{1}{k}}\, dv \right)^{\frac{2k}{2k+5} \cdot \frac{n+1}{n}} \left(\int |v|^2 f\, dv \right)^{\frac{3}{2k+5} \cdot \frac{n+1}{n}} \\ &= C \left(\int f^{1+\frac{1}{k}}\, dv \right)^{\frac{k}{n}} \left(\int |v|^2 f\, dv \right)^{\frac{n-k}{n}} \qquad (2.10)\end{aligned}$$

with $C > 0$. The first integral in (2.10) is bounded by *(i)* and the second integral corresponds to the kinetic energy in Definition 2.4.1. Therefore,

$$\int \rho_f^{1+\frac{1}{n}} \leq C \left(E_{kin}\right)^{\frac{n-k}{n}} = C \left(E_{kin}\right)^{\frac{3}{2n}},$$

as $k = n - \frac{3}{2}$ by definition.

(iii) Using Lemma 2.4.2 and applying *Hardy-Littlewood-Sobolev Inequality* (Theorem 1.2.4) with $\lambda = 1$, $N = 3$, $p = \frac{6}{5}$ and $g(x) = h(x) = \rho_f(x)$ yields

$$\begin{aligned}\int |\nabla U_f(x)|^2\, dx &= \frac{1}{4\pi} \iint \frac{\rho_f(x)\rho_f(y)}{|x-y|}\, dxdy \\ &\leq C \|\rho_f\|_{\frac{6}{5}}^2. \qquad (2.11)\end{aligned}$$

By *(ii)*, $\rho_f \in L^1 \cap L^{1+\frac{1}{n}}(\mathbb{R}^3)$ and since $1 < \frac{6}{5} < 1 + \frac{1}{n}$, the interpolation of L^p norms (c.f. Proposition B.1) generates

$$\begin{aligned}\int \rho_f^{\frac{6}{5}} &= \left(\|\rho_f\|_1^{\frac{5-n}{6}} \|\rho_f\|_{1+\frac{1}{n}}^{\frac{n+1}{6}}\right)^{\frac{6}{5}} \\ &\leq C\left((E_{kin})^{\frac{n+1}{6}\frac{3}{2(n+1)}}\right)^{\frac{6}{5}} \\ &\leq C\,(E_{kin})^{\frac{3}{10}}\,.\end{aligned}$$

Next, to prove that $U_f \in L^6(\mathbb{R}^3)$, consider the map

$$\|\nabla U_f\|_2 \longmapsto \|\Delta^{-\frac{1}{2}} \nabla U_f\|_2 = \|U_f\|_2.$$

We want to show that this map is an embedding of $W^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$. To this end, we will make use of the *Sobolev Embedding* (Theorem 1.3.4 *(i)*) and Remark 1.3.6 with $\kappa = 1$, $p = 2$ and $N = 3$. Then $\kappa p = 2 < 3 = N$ and $W^{\kappa,p}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for every $1 \leq q \leq 6$ as $\frac{1}{p} - \frac{\kappa}{N} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Clearly, as $\Delta U_f = \rho_f$ by (2.1), we have $\nabla U_f \in W^{1,2}(\mathbb{R}^3)$ and therefore $U_f \in L^6(\mathbb{R}^3)$. □

Remark 2.7.2. In the limiting case, i.e. in the case $k = \frac{7}{2}$ and $n = 5$, the statements in *(ii)* and *(iii)* are still valid.

As a consequence of this lemma, we can find a lower bound for the total energy $\mathcal{H}$ on the set $\mathcal{F}_M$. This bound will be only given in terms of the kinetic energy.

Corollary 2.7.3. *Let $f \in \mathcal{F}_M$. Then there exists a constant $C > 0$ such that*

$$\mathcal{H} \geq E_{kin}(f) - CE_{kin}(f)^{\frac{1}{2}}.$$

In particular we have

$$h_M := \inf_{f \in \mathcal{F}_M} \mathcal{H}(f) > -\infty$$

and moreover, E_{kin} is bounded along minimizing sequences of $\mathcal{H}$ in $\mathcal{F}_M$.

Proof. By Definition 2.4.1 and Lemma 2.7.1,

$$\begin{aligned}\mathcal{H}(f) &= E_{kin}(f) + E_{pot}(f) \\ &= E_{kin}(f) - \frac{1}{2}\int |\nabla U_f(x)|^2 \, dx \\ &\geq E_{kin}(f) - CE_{kin}(f)^{\frac{1}{2}}.\end{aligned}$$

□

Next, we will demonstrate that h_M is negative, and furthermore, we can establish a relationship between h_M and other $h_{\widetilde{M}}$ values with different values $\widetilde{M}$ instead of M.

Lemma 2.7.4. *(i) If $M > 0$, then $-\infty < h_M < 0$.*

(ii) If $M, \widetilde{M} > 0$, then

$$h_{\widetilde{M}} = \left(\frac{\widetilde{M}}{M}\right)^{\frac{7}{3}} h_M.$$

Proof. Let f be a measurable function and $a, b > 0$. We introduce the rescaled function $\widetilde{f}(x,v) = f(ax, bv)$. Since $x, v \in \mathbb{R}^3$, a change of variables yields

$$\mathcal{C}(\widetilde{f}) = \iint Q(f(ax,bv))\, dvdx = (ab)^{-3}\mathcal{C}(f). \tag{2.12}$$

Hence, $f \in \mathcal{F}_M$ if and only if $\widetilde{f} \in \mathcal{F}_{\widetilde{M}}$, where $\widetilde{M} := (ab)^{-3}M$. Similarly, the kinetic energy of $\widetilde{f}$ is given by

$$E_{kin}(\widetilde{f}) = \frac{1}{2}\iint |v|^2 f(ax,bv)\, dvdx = a^{-3}b^{-5}E_{kin}(f), \tag{2.13}$$

and the potential energy of $\widetilde{f}$ is given by

$$\begin{aligned} E_{pot}(\widetilde{f}) &= -\frac{1}{8\pi}\iint \frac{\rho_{\widetilde{f}}(x)\rho_{\widetilde{f}}(y)}{|x-y|}\, dxdy \\ &= -\frac{1}{8\pi}\iiiint \frac{f(ax,bv)f(ay,bw)}{|x-y|}\, dvdwdxdy \\ &= a^{-5}b^{-6}E_{pot}(f), \end{aligned} \tag{2.14}$$

where we inserted the definition of $\rho_{\widetilde{f}}$ into Lemma 2.4.2 *(i)*. Now let $f \in \mathcal{F}_M$ and $a = b^{-1}$. Then $\widetilde{f} \in \mathcal{F}_M$. Furthermore,

$$\begin{aligned} \mathcal{H}(\widetilde{f}) &= E_{kin}(\widetilde{f}) + E_{pot}(\widetilde{f}) \\ &= b^{-2}E_{kin}(f) + b^{-1}E_{pot}(f). \end{aligned}$$

Choosing b large enough, the potential term in the Hamiltonian is dominating and hence, since $E_{pot} < 0$, $\mathcal{H}(\widetilde{f}) < 0$, what proves *(i)*.

To demonstrate *(ii)*, choose $b = a^{-2}$, so that $a^{-3}b^{-5} = a^{-5}b^{-6}$. Consequently,

$$\begin{aligned} \mathcal{H}(\widetilde{f}) &= a^7 E_{kin}(f) + a^7 E_{pot}(f) \\ &= a^7 \mathcal{H}(f) \\ &= \left(\frac{\widetilde{M}}{M}\right)^{\frac{7}{3}} \mathcal{H}(f), \end{aligned} \tag{2.15}$$

since $\widetilde{M} = a^3 M$. As the rescaling is a bijective map from $\mathcal{F}_M$ to $\mathcal{F}_{\widetilde{M}}$, it holds in particular for h_M. □

Remark 2.7.5. The assertions in Corollary 2.7.3 and Lemma 2.7.4 remain valid in the limiting case where $k = \frac{7}{2}$ and $n = 5$.

Remark 2.7.6 ([LMR08])**.** In the case of the minimization problem

$$h_{M_1,M_j} := \inf_{f \in \mathcal{F}_{M_1,M_j}} \mathcal{H}(f)$$

one can show similar properties as in Corollary 2.7.3 and Lemma 2.7.4. In particular, we have for some $0 < \alpha < 1$

$$-\infty < h_{M_1,M_j} < 0,$$
$$h_{M_1,M_j} \leq h_{M_1,\alpha M_j},$$
$$h_{\alpha M_1,\alpha M_j} = \alpha^{\frac{7}{3}} h_{M_1,M_j}.$$

Moreover, if $\frac{9}{7} < p < \infty$, we have that for all $\alpha, \beta > 0$,

$$h_{\alpha M_1,\beta M_j} = \alpha^{\frac{7p-9}{3(p-1)}} \beta^{\frac{2}{3(p-1)}} h_{M_1,M_j}. \tag{2.16}$$

Remark 2.7.7 ([LMR08])**.** Similarly to Remark 2.7.6, the minimization problem

$$h_{M_1,M_\infty} := \inf_{\|f\|_1 = M_1,\ \|f\|_\infty = M_\infty} \mathcal{H}(f)$$

also has scaling properties: If $p = \infty$, we have that for all $0 \leq \alpha, \beta \leq 1$,

$$h_{\alpha M_1,\beta M_\infty} = \alpha^{\frac{7}{3}} \beta^{\frac{2}{3}} h_{M_1,M_\infty} \geq \alpha^{\frac{7}{3}} h_{M_1,M_\infty}. \tag{2.17}$$

Another preliminary result consists of criteria under which the so-called nondichotomy condition is valid. The significance of this will become clear in Chapter 5. The associated lemma is proven with rescaling arguments. A full proof can be found in Chapter 4 of [LMR08].

Lemma 2.7.8. *Assume one of the following:*

(i) j is a polytrope, i.e. for some $\frac{9}{7} < p < \infty$, $j(f) = f^p$.

(ii) There exists $\frac{3}{2} < p_1 \leq p_2 < \infty$ such that for all $b \geq 1$ and for all $f > 0$

$$b^{p_1} j(f) \leq j(bf) \leq b^{p_2} j(f).$$

Then for every $0 < \alpha < 1$ and for every $0 \leq \beta \leq 1$,

$$h_{M_1,M_j} < h_{\alpha M_1,\beta M_j} + h_{(1-\alpha)M_1,(1-\beta)M_j}.$$

Existence of Minimizers

In the previous chapter, we introduced the minimization problem h_M in Corollary 2.7.3 and characterized how the Hamiltonian $\mathcal{H}$ is bounded. Our goal now is to prove the existence of the minimizers, meaning that there exists a distribution function f such that the total energy is minimal and hence there exists an equilibrium of the kinetic and potential energies. Let $M > 0$. In this context, if $f_0 \in \mathcal{F}_M$, then f_0 is a minimizer if

$$\mathcal{H}(f_0) = h_M = \inf_{f \in \mathcal{F}_M} \mathcal{H}(f).$$

If such minimizers exist, we are able to investigate the dynamical stability of states in the Vlasov-Poisson system. The statements and proofs in this chapter are based on Chapter 3, 4 and 6 of [GR01].

3.1 Existence of Minimizers for $k < \frac{7}{2}$

First, we will focus on studying minimizers for $0 < k < \frac{7}{2}$, excluding the limiting case. For such a k, we let $n = k + \frac{3}{2}$ as in Lemma 2.7.1. To establish the existence of minimizers, we will rely on two results given in the following two lemmas. The first one demonstrates that the mass does not run off to spatial infinity along a minimizing sequence.

Lemma 3.1.1. *Let $(f_i) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}$ and denote by $B_R = \{x \in \mathbb{R}^3 \,:\, |x| \leq R\}$ the ball of radius R in the position space. Then there exist $\varepsilon_0 > 0$, $R_0 > 0$ and a sequence $(a_i) \subset \mathbb{R}^3$ such that*

$$\int_{a_i + B_{R_0}} \int Q(f_i)\, dvdx > \varepsilon_0$$

for $i \in \mathbb{N}$ sufficiently large.

Proof. To establish the boundedness of the mass, we first want to find a lower bound for the potential energy. In this way, we find a lower bound for the Hamiltonian depending on the potential energy

only. Let $R > 0$ and for $j \in \{1, 2, 3\}$ define the functions $F_j : \mathbb{R}^3 \to \mathbb{R}$ by

$$F_1(x) := \begin{cases} R, & |x| < 1/R, \\ \frac{1}{|x|}, & 1/R \leq |x| \leq R, \\ 0, & |x| > R, \end{cases}$$
$$F_2(x) := \frac{1}{|x|} \mathbb{1}_{\{|x|>R\}}(x)$$

and

$$F_3(x) := \left(\frac{1}{|x|} - R \right) \mathbb{1}_{\{|x|<1/R\}}(x)$$

where $\mathbb{1}_A$ denotes the characteristic function of a set A. Using these functions allows us to split the kernel:

$$\frac{1}{|x|} = F_1 + F_2 + F_3.$$

According to this splitting, we are able to separate the potential energy into three terms as

$$\begin{aligned} -8\pi E_{pot}(f_j) &= \iint \frac{\rho_j(x)\rho_j(y)}{|x-y|}\, dxdy \\ &= \iint \sum_{i=1}^{3} F_i(x-y)\ \rho_j(x)\rho_j(y)\, dxdy \\ &=: I_1 + I_2 + I_3, \end{aligned} \tag{3.1}$$

where we used the notation $\rho_j = \rho_{f_j}$. To estimate these three integrals, we observe that ρ_j is in $L^{1+\frac{1}{n}}(\mathbb{R}^3)$ and $\rho_j^{1+\frac{1}{n}}$ is bounded by the kinetic energy concerning Lemma 2.7.1 *(ii)* for every j. Moreover, as $f_j \in \mathcal{F}_M$, Lemma 2.7.1 *(i)* implies that ρ_j is in L^1. Thus

$$\begin{aligned} |I_1| &\leq R \iint_{|x-y|} \rho_j(x)\rho_j(y)\, dxdy \\ &\leq C\, R \sup_{y\in\mathbb{R}^3} \int_{B_R+y} \rho_j(x)\, dx \\ &\leq C\, R \sup_{y\in\mathbb{R}^3} \left(\int_{B_R+y} \rho_j^{1+\frac{1}{n}}\, dx \right)^{\frac{n}{n+1}} \|\mathbb{1}_{\{B_R+y\}}\|_{n+1} \\ &= C\, R \sup_{y\in\mathbb{R}^3} \left(\int_{B_R+y} \rho_j^{1+\frac{1}{n}}\, dx \right)^{\frac{n}{n+1}} R^{\frac{3}{n+1}} \\ &= C\, R^{\frac{n+4}{n+1}} \sup_{y\in\mathbb{R}^3} \left(\int_{B_R+y} \rho_j^{1+\frac{1}{n}}\, dx \right)^{\frac{n}{n+1}}, \end{aligned}$$

where we applied Theorem 1.2.2 with Hölder coefficients $\frac{n+1}{n}$ and $n+1$, and hence

$$|I_1| \leq C\, R^{\frac{n+4}{n+1}} \sup_{y\in\mathbb{R}^3} \left(\int_{B_R+y} \int f_j^{1+\frac{1}{k}}\, dvdx \right)^{\frac{k}{n+1}}.$$

Furthermore,

$$|I_2| \leq \frac{1}{R} \iint \rho_j(x)\rho_j(y)\, dxdy \leq CR^{-1}.$$

For the last integral we apply Theorem 1.2.2 with the Hölder exponents $\frac{n+1}{n}$ and $n+1$ as before, as well as Theorem 1.2.5, which leads to

$$|I_3| \leq \|\rho_j\|_{1+\frac{1}{n}} \|\rho_j * F_3\|_{n+1} \leq C \|F_3\|_{\frac{n+1}{2}}.$$

Compute with the use of spherical coordinates

$$\begin{aligned}
\|F_3\|_{\frac{n+1}{2}}^{\frac{n+1}{2}} &= C \int_0^{\frac{1}{R}} \left(\frac{1}{r} - R\right)^{\frac{n+1}{2}} r^2\, dr \\
&\leq C \int_0^{\frac{1}{R}} \left(\frac{1}{r}\right)^{\frac{n+1}{2}} r^2\, dr \\
&= C \int_0^{\frac{1}{R}} \left(\frac{1}{r}\right)^{\frac{3-n}{2}} dr \\
&= C\, R^{-\frac{5-n}{2}}
\end{aligned}$$

and therefore

$$|I_3| \leq C\, R^{-\frac{5-n}{n+1}}.$$

By assumption, (f_j) is a minimizing sequence and hence for any $R > 1$ and for j sufficiently large,

$$\frac{1}{2} h_M > \mathcal{H}(f_j) \geq -\frac{1}{8\pi} \sum_{i=1}^{3} |I_i|, \tag{3.2}$$

which gives a lower bound for the Hamiltonian. To conclude the statement of the lemma, we consider again the first condition on Q and find approximations using the estimate in (3.2). We find

$$\begin{aligned}
\liminf_{j\to\infty} \left(\sup_{y\in\mathbb{R}^3} \int_{y+B_R} \int Q(f_j)\, dvdx \right)^{\frac{k}{n+1}} &\geq \liminf_{j\to\infty} \left(\sup_{y\in\mathbb{R}^3} \int_{y+B_R} \int f_j^{1+\frac{1}{k}}\, dvdx \right)^{\frac{k}{n+1}} \qquad (3.3) \\
&\geq C \liminf_{j\to\infty} R^{-\frac{n+4}{n+1}} |I_1| \\
&\geq C\, R^{-\frac{n+4}{n+1}} \left(-\frac{1}{2} h_M - R^{-1} - R^{-\frac{5-n}{n+1}} \right), \qquad (3.4)
\end{aligned}$$

where one can replace y by a sequence that approaches y in the supremum with respect to y. As h_M is finite and negative by Lemma 2.7.4 *(i)*, we have that for R large enough, (3.4) is positive and therefore the claim is proven. □

In the next Lemma we will show the strong convergence of the force field $G = \nabla U$ when the potential is limited on a ball with finite radius.

Lemma 3.1.2. *Let $(\rho_i) \subset L^1 \cap L^{1+\frac{1}{n}}(\mathbb{R}^3)$ be a bounded sequence and let $\rho_0 \in L^1 \cap L^{1+\frac{1}{n}}(\mathbb{R}^3)$ such that*

$$\rho_i \rightharpoonup \rho_0 \text{ weakly in } L^{1+\frac{1}{n}}(\mathbb{R}^3).$$

Then for every $R > 0$,

$$\nabla U_{\mathbb{1}_{B_R}\rho_i} \to \nabla U_{\mathbb{1}_{B_R}\rho_0} \text{ strongly in } L^2(\mathbb{R}^3).$$

Proof. Fix $R > 0$ and let $\tilde{R} > R$. By the relation $n = k + \frac{3}{2}$ and since $0 < k < \frac{7}{2}$, we have that $1 + \frac{1}{n} \in \left(\frac{6}{5}, \frac{5}{3}\right)$. Observe that by the Poisson equation (2.1),

$$\| - (-\Delta)^{-\frac{1}{2}}\rho\|_2 = \| - (-\Delta)^{-\frac{1}{2}}(-\Delta)U\|_2 = \|(-\Delta)^{\frac{1}{2}}U\|_2 = \| - \nabla U\|_2$$

and thus it follows that the map

$$\begin{aligned} L^{1+\frac{1}{n}}(B_{\tilde{R}}) &\longrightarrow L^2(B_{\tilde{R}}), \\ \rho_i &\longmapsto \nabla U_{\mathbb{1}_{B_{\tilde{R}}}\rho_i} \end{aligned}$$

is compact. Indeed, we can show that this is a compact embedding of $W^{1,1+\frac{1}{n}}(B_{\tilde{R}})$ into $L^2(B_{\tilde{R}})$. Apply *Sobolev Embedding* (Theorem 1.3.4 *(i)*) with $\kappa = 1$, $p = 1 + \frac{1}{n}$ and $N = 3$. Then $\kappa p = 1 + \frac{1}{n} \in \left(\frac{6}{5}, \frac{5}{3}\right)$ and thus $\kappa p < 3 = N$. Hence $W^{1,1+\frac{1}{n}}(B_{\tilde{R}}) \hookrightarrow L^q(B_{\tilde{R}})$ for every $1 \leq q \leq q_0$, where

$$\frac{1}{q_0} = \frac{1}{p} - \frac{\kappa}{N} = \frac{n}{n+1} - \frac{1}{3} \in \left(\frac{4}{15}, \frac{1}{2}\right)$$

as $\frac{n}{n+1} \in \left(\frac{3}{5}, \frac{5}{6}\right)$ and thus $q_0 \in \left(2, \frac{15}{4}\right)$. In particular, for $q = 2 < q_0$, this embedding is compact. Therefore, by Definition 1.1.9, for a subsequence $(\rho_i) \in L^1 \cap L^{1+\frac{1}{n}}(B_{\tilde{R}})$, $\nabla U_{\mathbb{1}_{B_{\tilde{R}}}\rho_i}$ converges strongly in $L^2(B_{\tilde{R}})$. Using Lemma 2.4.2 *(i)*, we deduce that

$$\int_{|x|\geq\tilde{R}} \left|\nabla U_{\mathbb{1}_{B_R}\rho_i}\right|^2 dx \leq C\,\frac{1}{\tilde{R} - R}\|\rho_i\|_1^2 \leq C\,\frac{1}{\tilde{R} - R} \tag{3.5}$$

for $i \in \mathbb{N}_{\geq 0}$. If $\tilde{R}$ is large, the last expression can become arbitrarily small and hence the strong convergence also holds on $\mathbb{R}^3$. □

With the results of these lemmas, we are now prepared to prove the existence of minimizers of the Hamiltonian $\mathcal{H}$ over the set $\mathcal{F}_M$ for a given $M > 0$.

Theorem 3.1.3. *Let $(f_i) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}$. Then there exists a minimizer $f_0 \in \mathcal{F}_M$, a subsequence, which is still denoted by (f_i), and a sequence of translations*

$$T_i f_i(x, v) = f(x + a_i, v),$$

where $(a_i) \subset \mathbb{R}^3$ is a sequence, such that

$$\mathcal{H}(f_0) = \inf_{\mathcal{F}_M} \mathcal{H} = h_M$$

and moreover,

$$\begin{aligned} T_i f_i &\rightharpoonup f_0 \text{ weakly in } L^{1+\frac{1}{k}}(\mathbb{R}^6), \\ \nabla U_{T_i f_i} &\to \nabla U_0 \text{ strongly in } L^2(\mathbb{R}^3). \end{aligned}$$

Proof. Let (f_i) be a sequence that minimizes $\mathcal{H}$ over $\mathcal{F}_M$. Choose a sequence $(a_i) \subset \mathbb{R}^3$ for which the statement in Lemma 3.1.1 is true. As a consequence of the translation invariance given in Proposition 2.2.1, since the kinetic and potential energy are depending on f, the Hamiltonian $\mathcal{H}(f)$ is also translation invariant. Hence, a sequence of translations $(T_i f_i)$ is again minimizing. From Lemma 2.7.1 *(i)* follows that $T_i f_i \in L^{1+\frac{1}{k}}(\mathbb{R}^6)$ for all i, which implies that there exists a weakly convergent subsequence according to *Banach-Alaoglu Theorem* (Theorem 1.3.3), which we again denote by $(T_i f_i)$, i.e.

$$T_i f_i \rightharpoonup f_0 \text{ weakly in } L^{1+\frac{1}{k}}(\mathbb{R}^6),$$

where $f_0 \geq 0$ because $T_i f_i \in \mathcal{F}_M$ for all i. The kinetic energy of this sequence, given by $E_{kin}((T_i f_i))$, is bounded in terms of the total energy as shown in Corollary 2.7.3. Therefore, by Lemma 2.7.1 *(ii)*, the sequence of densities $(\rho_i) = (\rho_{T_i f_i})$ is bounded in $L^{1+\frac{1}{n}}(\mathbb{R}^3)$. Moreover, by Definition 2.5.1 *(i)*, (ρ_i) is also bounded in $L^1(\mathbb{R}^3)$. Applying *Banach-Alaoglu Theorem* (Theorem 1.3.3) again allows us to extract another subsequence, which is still denoted by (ρ_i), and we have

$$\rho_i \rightharpoonup \rho_0 \text{ weakly in } L^{1+\frac{1}{n}}(\mathbb{R}^3),$$

where $\rho_0 = \rho_{f_0}$ by the definition of weak convergence (Definition 1.1.6). As $T_i f_i$ is weakly convergent, weakly lower semicontinuity is given by Theorem 1.2.3, i.e.

$$\|f_0\|_{1+\frac{1}{k}} \leq \liminf_{i\to\infty} \|T_i f_i\|_{1+\frac{1}{k}}.$$

By the weak convergence of $T_i f_i$, we have that $L(T_i f_i)$ converges for every linear functional L in the dual space of $L^{1+\frac{1}{k}}$, so in particular for the kinetic energy E_{kin}. Thus

$$E_{kin}(f_0) \leq \liminf_{i\to\infty} E_{kin}(T_i f_i) < \infty.$$

By *Mazur's Lemma* (Lemma C.5), there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of sets $\{\alpha(l)_i\}_{i=l}^{N(l)}$, where $\alpha(l)_i \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=l}^{N(l)} \alpha(l)_i = 1$, such that

$$g_l := \sum_{i=l}^{N(l)} \alpha(l)_i T_i f_i \to f_0 \text{ strongly.}$$

Hence,

$$\mathcal{C}(f_0) = \mathcal{C}\left(\liminf_{l\to\infty} g_l\right) \leq \liminf_{l\to\infty} \mathcal{C}\left(g_l\right),$$

where we also applied *Fatou's Lemma* (Lemma 1.2.1). Moreover, since the functional $\mathcal{C}$ is convex,

$$\mathcal{C}(f_0) \leq \sum_{i=l}^{N(l)} \alpha(l)_i \limsup_{i\to\infty} \mathcal{C}\left(T_i f_i\right) = M$$

and in particular, we find that $\rho_0 \in L^1(\mathbb{R}^3)$.

Now we want to prove that $\nabla U_{T_i f_i}$ converges strongly to ∇U_0 in $L^2(\mathbb{R}^3)$. Let R_0, R be positive numbers in $\mathbb{R}$ such that $R_0 < R$. Consider the hollow sphere given by

$$B_{R_0,R} := \{x \in \mathbb{R}^3 \ : \ R_0 \leq |x| \leq R\}.$$

Then we split the sequence $T_i f_i$ according to

$$\begin{aligned} T_i f_i &= T_i f_i \mathbb{1}_{B_{R_0}\times\mathbb{R}^3} + T_i f_i \mathbb{1}_{B_{R_0,R}\times\mathbb{R}^3} + T_i f_i \mathbb{1}_{B_{R,\infty}\times\mathbb{R}^3} \\ &= f_i^1 + f_i^2 + f_i^3. \end{aligned} \tag{3.6}$$

For any fixed $R > 0$, Lemma 3.1.2 already implies that $\nabla U_{f_i^1+f_i^2}$ converges strongly in $L^2(\mathbb{R}^3)$. Thus it is enough to show that

$$\liminf_{i\to\infty} \int \left|\nabla U_{f_i^3}\right|^2 dx < \varepsilon \tag{3.7}$$

for any $\varepsilon > 0$ if R is sufficiently large. By Lemma 2.7.1, $\|\nabla U_{f_i^3}\|_2$ is bounded by the integral of $f_i^3 + (f_i^3)^{1+\frac{1}{k}}$, which in turn is bounded by $Q(f_i^3)$ by Definition 2.5.1. Thus it is already sufficient to show

$$\liminf_{i\to\infty} \iint Q(f_i^3)\, dvdx < \varepsilon \tag{3.8}$$

for any $\varepsilon > 0$ if R is sufficiently large. In order to do so, we split the Hamiltonian based on (3.6) and find by the symmetry of convolutions

$$\begin{aligned} \mathcal{H}(T_i f_i) &= \mathcal{H}(f_i^1 + f_i^2 + f_i^3) \qquad (3.9)\\ &= \sum_{k=1}^3 E_{kin}(f_i^k) + E_{pot}\left(f_i^1 + f_i^2 + f_i^3\right) \\ &= \sum_{k=1}^3 E_{kin}(f_i^k) - \frac{1}{8\pi}\iint \frac{\rho_{f_i^1+f_i^2+f_i^3}(x)\rho_{f_i^1+f_i^2+f_i^3}(y)}{|x-y|}\, dxdy \\ &= \sum_{k=1}^3 E_{kin}(f_i^k) - \sum_{k=1}^3 \frac{1}{8\pi}\iint \frac{\rho_i^k(x)\rho_i^k(y)}{|x-y|}\, dxdy \\ &\quad - \frac{1}{8\pi}\iint \frac{2\rho_i^1(x)\rho_i^2(y) + 2\rho_i^1(x)\rho_i^3(y) + 2\rho_i^2(x)\rho_i^3(y)}{|x-y|}\, dxdy \\ &= \sum_{k=1}^3 \mathcal{H}(f_i^k) - \frac{1}{4\pi}\left(\iint \frac{\rho_i^2(x)(\rho_i^1+\rho_i^3)(y)}{|x-y|}\, dxdy + \iint \frac{\rho_i^1(x)\rho_i^3(y)}{|x-y|}\, dxdy\right) \\ &=: \sum_{k=1}^3 \mathcal{H}(f_i^k) - \frac{1}{4\pi}\left(I_1 + I_2\right), \qquad (3.10) \end{aligned}$$

where ρ_i^k is the corresponding density function to f_i^k for each $k = 1, 2, 3$ and every i. Since $(T_i f_i)$ is a minimizing sequence, we know in particular that $\|\nabla U_{\rho_i^1+\rho_i^3}\|_2$ is bounded and thus, for some constant $C > 0$, we have

$$I_1 \le C\|\nabla U_{\rho_i^2}\|_2.$$

Further, the weak convergence of ρ_i deduces that $\rho_i^2 \rightharpoonup \rho_0^2$ weakly in $L^{1+\frac{1}{n}}(\mathbb{R}^3)$, where we denote $\rho_0^2 := \rho_0 \mathbb{1}_{B_{R_0,R}\times\mathbb{R}^3}$. Therefore

$$\|\nabla U_{\rho_i^2} - \nabla U_{\rho_0^2}\|_2 \to 0 \tag{3.11}$$

as $i \to \infty$ as a consequence of Lemma 3.1.2. To estimate I_2 choose $R > 2R_0$ and observe that for any $y \in B_{R,\infty}$ we have $|y| \leq 2|x-y|$ for every $x \in B_{R_0}$. Hence

$$\begin{aligned} I_2 &= \int_{B_{R,\infty}} \int_{B_{R_0}} \frac{\rho_i(x)\rho_i(y)}{|x-y|}\, dxdy \\ &\leq 2 \int_{B_{R_0}} \rho_i(x)\, dx \int_{B_{R,\infty}} \frac{\rho_i}{|y|}\, dy \end{aligned} \tag{3.12}$$

Applying *Hölder's Inequality* (Theorem 1.2.2) with Hölder coefficients $\frac{6}{5}$ and 6 on the first integral in (3.12) gives the estimate

$$\int_{B_{R_0}} \rho_i(x)\, dx \leq \|\rho_i\|_{\frac{6}{5}} \|\mathbb{1}_{B_{R_0}}\|_6 = C \|\rho_i\|_{\frac{6}{5}} R_0^{\frac{1}{2}}.$$

Similarly, for the second integral in (3.12) we find

$$\int_{B_{R,\infty}} \rho_i \frac{1}{|y|}\, dx \leq \|\rho_i\|_{\frac{6}{5}} \|\mathbb{1}_{B_{R,\infty}} |y|^{-1}\|_6 = C \|\rho_i\|_{\frac{6}{5}} R^{-\frac{1}{2}},$$

where we used spherical coordinates to determine that

$$\|\mathbb{1}_{B_{R,\infty}} |y|^{-1}\|_6^6 = \int_{B_{R,\infty}} \left(\frac{1}{|y|}\right)^6 dy = C \int_R^\infty \frac{1}{r^4}\, dr = C\, R^{-3}.$$

Combining these results gives us

$$I_2 \leq C \|\rho_i\|_{\frac{6}{5}}^2 \left(\frac{R_0}{R}\right)^{\frac{1}{2}}.$$

We claim that for every $\tau \in [0,1]$

$$\tau^{\frac{7}{3}} + (1-\tau)^{\frac{7}{3}} \leq 1 - \frac{7}{3}\tau(1-\tau). \tag{3.13}$$

To show (3.13), define the continuous function ϕ, defined on $[0,1]$, as

$$\phi(\tau) = \tau^{\frac{7}{3}} + (1-\tau)^{\frac{7}{3}} + \frac{7}{3}\tau(1-\tau).$$

It follows directly that $\phi(0) = \phi(1) = 1$ and one can find that $0, 1$ are maxima. Moreover, ϕ has it's minimum at $\frac{1}{2}$. Hence, by continuity, $\phi(\tau) \leq 0$ and the claim is proven. Define M_i^1, M_i^2, M_i^3 according to (3.6) and notice that $M = M_i^1 + M_i^2 + M_i^3$. Then by Lemma 2.7.4 and since h_M is negative,

$$\begin{aligned} \mathcal{H}(f_i^1) + \mathcal{H}(f_i^2) + \mathcal{H}(f_i^3) &\geq h_{M_i^1} + h_{M_i^2} + h_{M_i^3} \\ &= \left[\left(\frac{M_i^1}{M}\right)^{\frac{7}{3}} + \left(\frac{M_i^2}{M}\right)^{\frac{7}{3}} + \left(\frac{M_i^3}{M}\right)^{\frac{7}{3}}\right] h_M \\ &\geq \left[\left(\frac{M_i^1 + M_i^2}{M}\right)^{\frac{7}{3}} + \left(\frac{M_i^3}{M}\right)^{\frac{7}{3}}\right] h_M \\ &\geq \left[1 - \frac{7}{3}\frac{M_i^1 + M_i^2}{M}\frac{M_i^3}{M}\right] h_M, \end{aligned} \tag{3.14}$$

where we also used (3.13). Combining (3.10) with (3.14), setting $C_1 = \frac{7}{3M^2}$ and choosing again $R > 2R_0$ yields

$$\begin{aligned} h_M - \mathcal{H}(T_i f_i) - C_1 h_M (M_i^1 + M_i^2) M_i^3 &\leq \frac{1}{4\pi}(I_1 + I_2) \\ &\leq C_2 \left\| \nabla U_{\rho_i^2} \right\|_2 + C_3 \left\| \rho_i \right\|_{\frac{6}{5}}^2 \left(\frac{R_0}{R} \right)^{\frac{1}{2}} \\ &\leq C_3 \left[\left\| \nabla U_{\rho_i^2} - \nabla U_{\rho_0^2} + \nabla U_{\rho_0^2} \right\|_2 + \left(\frac{R_0}{R} \right)^{\frac{1}{2}} \right] \\ &\leq C_3 \left[\left\| \nabla U_{\rho_i^2} - \nabla U_{\rho_0^2} \right\|_2 + \left\| \nabla U_{f_0^2} \right\|_2 + \left(\frac{R_0}{R} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Notice that the constants C_1, C_2 and C_3 are independent of R, R_0. To show (3.8), suppose it does not hold, meaning that there exists $\varepsilon_1 > 0$ such that for every $R > 0$

$$\liminf_{i \to \infty} \iint Q(f_i^3)\, dvdx \geq \varepsilon_1$$

for i sufficiently large. Let $\varepsilon_0 > 0$ such that it fulfills the conditions in Lemma 3.1.1 and define

$$\varepsilon_2 := -C_1 h_M \varepsilon_0 \varepsilon_1.$$

As h_M is negative, $\varepsilon_2 > 0$. Now we choose R_0 in 3.1.1 such that

$$C_3 \left\| \nabla U_{f_0^2} \right\|_2 \leq \frac{\varepsilon_2}{4}.$$

Further we select $R > 2R_0$ such that

$$C_3 \left(\frac{R_0}{R} \right)^{\frac{1}{2}} \leq \frac{\varepsilon_2}{4}.$$

Combining these estimates implies

$$\begin{aligned} h_M - \mathcal{H}(T_i f_i) + \varepsilon_2 &\leq h_M - \mathcal{H}(T_i f_i) - C_1 h_M (M_i^1 + M_i^2) M_i^3 \\ &\leq C_3 \left\| \nabla U_{\rho_i^2} - \nabla U_{\rho_0^2} \right\|_2 + \frac{\varepsilon_2}{2} \end{aligned}$$

for i large enough. But as $\left\| \nabla U_{\rho_i^2} - \nabla U_{\rho_0^2} \right\|_2 \to 0$ by (3.11),

$$h_M + \frac{\varepsilon_2}{2} \leq \mathcal{H}(T_i f_i),$$

which means that $(T_i f_i)$ does not minimize the energy. This contradicts the assumptions and hence (3.8) holds and therefore (3.7) is true. As $\varepsilon > 0$ is arbitrary, it follows that $\nabla U_{T_i f_i}$ converges strongly to ∇U_0 in $L^2(\mathbb{R}^3)$. To conclude that f_0 is a minimizer of $\mathcal{H}$ over $\mathcal{F}_M$, we first notice that

$$\mathcal{H}(f_0) \leq \lim_{i \to \infty} \mathcal{H}(T_i f_i),$$

because of the weakly lower semicontinuity. Secondly, it remains to be proven that $\mathcal{C}(f_0) = M$ in order to show that $f_0 \in \mathcal{F}_M$. Suppose by contradiction that $M_0 := \mathcal{C}(f_0) < M$. If $M_0 = 0$, $f_0 = 0$, which would contradict $\mathcal{H}(f_0) < 0$. Thus $M_0 > 0$. Define

$$b := \left(\frac{M_0}{M} \right)^{\frac{2}{3}}, \quad a := b^{-\frac{1}{2}}.$$

Then by (2.12),

$$(ab)^{-3}\mathcal{C}(f_0) = \frac{M}{M_0}\mathcal{C}(f_0) = \mathcal{C}(\widetilde{f}_0)$$

and thus $\widetilde{f}_0 \in \mathcal{F}_M$. Then from (2.15) and as $b < 1$ follows

$$\mathcal{H}(\widetilde{f}_0) = a^7\mathcal{H}(f_0) = b^{\frac{7}{2}}\mathcal{H}(f_0) \leq b^{-\frac{7}{2}}h_M < h_M,$$

which contradicts the definition of h_M as the infimum of $\mathcal{H}$ over the set $\mathcal{F}_M$. Therefore f_0 is a minimizer of $\mathcal{H}$ in $\mathcal{F}_M$. □

3.2 Properties of Minimizers

In the previous section, we have proven the existence of minimizers. Now we want to examine various properties of these minimizers. First, we will show that the minimizers are steady states of the Vlasov-Poisson system.

Theorem 3.2.1. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}$. Then for a.e. $(x,v) \in \mathbb{R}^6$*

$$f_0(x,v) = \begin{cases} \phi(E), & E < E_0, \\ 0, & E \geq E_0, \end{cases}$$

where

$$E := \frac{1}{2}|v|^2 + U_0(x),$$
$$E_0 := \lambda_0 Q'(0),\ \lambda_0 := \frac{\iint E\, f_0\, dvdx}{\iint Q'(f_0)\ f_0\, dvdx} < 0,$$

with the potential U_0 induced by f_0 and

$$\phi(E) := \inf\{f \geq 0\ :\ \lambda_0 Q'(f) = E\}, \quad E \leq E_0.$$

In particular, f_0 is a steady state of the Vlasov-Poisson system.

Proof. Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}$ and denote by U_0 its induced potential. To show that the Euler-Lagrange relation is given by (2.9), we first argue on a set K_ε defined by

$$K_\varepsilon := \left\{(x,v) \in \mathbb{R}^6\ :\ \varepsilon \leq f_0(x,v) \leq \frac{1}{\varepsilon}\right\}$$

where $\varepsilon > 0$ is small. By definition, K_ε is of positive finite measure. Let $w \in L^\infty(\mathbb{R}^6)$ have compact support with $w \geq 0$ in K_ε^c, where we denote by A^c the complement of a set A. Further we define for σ, τ close to zero, $\tau \geq 0$,

$$G(\sigma,\tau) := \iint Q(f_0 + \sigma\mathbf{1}_{K_\varepsilon} + \tau w)\, dvdx.$$

Notice that $G(0,0) = \mathcal{C}(f_0) = M$. As f_0 is positive and by definitions of w, τ, the function $f_0 + \sigma \mathbf{1}_{K_\varepsilon} + \tau w$ is bounded on K_ε and, by positivity of f_0 and by definitions of w, τ, non-negative on K_ε^c. Hence, G is continuously differentiable. We have

$$\partial_\sigma G(0,0) = \iint_{K_\varepsilon} Q'(f_0)\, dvdx \neq 0$$

and hence, by the *Implicit Function Theorem* (Theorem A.1), there exists a continuously differentiable function $\tau \mapsto \sigma(\tau)$ satisfying the initial condition $\sigma(0) = 0$ such that $G(\sigma(\tau), \tau) = M$ holds. Therefore $f_0 + \sigma(\tau)\mathbf{1}_{K_\varepsilon} + \tau w \in \mathcal{F}_M$. Moreover, again by the *Implicit Function Theorem* (Theorem A.1),

$$\partial_\tau(\sigma_0) = -(\partial_\sigma G(\sigma_0, \tau_0))^{-1} \cdot \partial_\tau G(\sigma_0, \tau_0).$$

Here, $\sigma_0 = \sigma(\tau_0)$ with $\tau_0 = 0$ and thus

$$\sigma'(0) = -\frac{\partial_\tau G(0,0)}{\partial_\sigma G(0,0)} = -\frac{\iint Q'(f_0) w\, dvdx}{\iint_{K_\varepsilon} Q'(f_0)\, dvdx}. \tag{3.15}$$

Since $\mathcal{H}$ is bounded from below by the kinetic energy by Corollary 2.7.3, which in turn increases with an increasing support, the minimum of $\mathcal{H}(f_0 + \sigma(\tau)\mathbf{1}_{K_\varepsilon} + \tau w)$ is attained at $\tau = 0$. The Taylor expansion of the Hamiltonian around this point is given by

$$\mathcal{H}(f_0 + \sigma(\tau)\mathbf{1}_{K_\varepsilon} + \tau w) = \mathcal{H}(f_0) + \tau \left.\frac{d}{d\tau}\mathcal{H}(f_0 + \sigma(\tau)\mathbf{1}_{K_\varepsilon} + \tau w)\right|_{\tau=0} + \mathcal{O}(\tau^2).$$

Substituting $\tilde{f}(\tau) := f_0 + \sigma(\tau)\mathbf{1}_{K_\varepsilon} + \tau w$ we have by the chain rule and Lemma 2.4.2 *(ii)*

$$\begin{aligned}
\left.\frac{d}{d\tau}\mathcal{H}(\tilde{f})\right|_{\tau=0} &= \left.\frac{d}{d\tau}\left(\iint \frac{1}{2}|v|^2 \tilde{f}(\tau)\, dvdx - \frac{1}{2}\int |\nabla U_{\tilde{f}}(x)|^2\, dx\right)\right|_{\tau=0} \\
&= \left[\iint \frac{1}{2}|v|^2\, \partial_\tau \tilde{f}(\tau)\, dvdx - \int \nabla U_{\tilde{f}}(x)\, \partial_\tau \left(\nabla U_{\tilde{f}}(x)\right)\, dx\right]_{\tau=0} \\
&= \iint \frac{1}{2}|v|^2 [\sigma'(0)\mathbf{1}_{K_\varepsilon} + w]\, dvdx + \left[\iint U_{\tilde{f}}(x)\, \partial_\tau \tilde{f}(\tau)\, dvdx\right]_{\tau=0} \\
&= \iint \frac{1}{2}|v|^2 [\sigma'(0)\mathbf{1}_{K_\varepsilon} + w]\, dvdx + \iint U_0(x)[\sigma'(0)\mathbf{1}_{K_\varepsilon} + w]\, dvdx \\
&= \iint E\, [\sigma'(0)\mathbf{1}_{K_\varepsilon} + w]\, dvdx.
\end{aligned}$$

It follows that

$$0 \leq \mathcal{H}(f_0 + \sigma(\tau)\mathbf{1}_{K_\varepsilon} + \tau w) - \mathcal{H}(f_0) = \tau \iint E\, [\sigma'(0)\mathbf{1}_{K_\varepsilon} + w]\, dvdx + O(\tau^2)$$

with $\tau \geq 0$ small. Hence, using (3.15),

$$\begin{aligned}
0 &\leq \iint \mathbf{1}_{K_\varepsilon} E \frac{-\iint Q'(f_0) w\, dvdx}{\iint_{K_\varepsilon} Q'(f_0)\, dvdx} + Ew\, dvdx \\
&= \iint (-\lambda_\varepsilon Q'(f_0) + E)\, w\, dvdx,
\end{aligned}$$

where we introduced

$$\lambda_\varepsilon := \frac{\iint_{K_\varepsilon} E\, dvdx}{\iint_{K_\varepsilon} Q'(f_0)\, dvdx}.$$

Since w is non-negative in K_ε^c, we then must have

$$E = \lambda_\varepsilon Q'(f_0) \text{ a.e. on } K_\varepsilon,$$
$$E \geq \lambda_\varepsilon Q'(f_0) \text{ else.}$$

As this holds for every $\varepsilon > 0$ small, we can let $\varepsilon \to 0$ and find

$$E = \lambda_0 Q'(f_0) \text{ a.e. on } f_0^{-1}((0,\infty)), \tag{3.16}$$
$$E \geq \lambda_0 Q'(0) =: E_0 \text{ a.e. on } f_0^{-1}(0). \tag{3.17}$$

From (3.16) we deduce

$$\lambda_0 = \frac{\iint E\, f_0 \, dvdx}{\iint Q'\,(f_0)\ f_0 \, dvdx},$$

which corresponds to λ_0 in the claim. By definition, $f_0 \geq 0$ and moreover, $Q(f_0)$ is positive as f_0 is a minimizer. Thus the denominator is positive and since

$$\begin{aligned}\iint E\, f_0 \, dvdx &= \iint \left(\frac{1}{2}|v|^2 + U_0(x)\right) f_0 \, dvdx \\ &= E_{kin}(f_0) + 2E_{pot}(f_0) \\ &< E_{kin}(f_0) + E_{pot}(f_0) \\ &= \mathcal{H}(f_0) < 0,\end{aligned}$$

we conclude that $\lambda_0 < 0$. To express the minimizer f_0 in terms of the energy, we need to invert (3.16). This is possible whenever Q' is bijective. If this is the case, $(Q')^{-1}$ exists and we can examine the function $(Q')^{-1}$ in order to express

$$(Q')^{-1}\left(\frac{E}{\lambda_0}\right).$$

According to the assumptions on Q, given in Definition 2.5.1, the function $Q' : [0,\infty) \to [Q'(0),\infty)$ is continuous, increasing and surjective. Therefore, for every $\eta \geq Q'(0)$, the set $(Q')^{-1}(\eta)$ is a closed and bounded interval. Further, there exists an at most countable set, such that $(Q')^{-1}(\eta)$ consists of one point for η that is not in this set. We define this set as V_{crit} and notice that V_{crit} is the set where Q' fails to be bijective. Since V_{crit} is at most countable, it has measure zero and $(Q')^{-1}$ exists a.e. Let $a \in \mathbb{R}$ be a constant. Then the set

$$E^{-1}(a) = \{(x,v) \in \mathbb{R}^6 \ : \ E(x,v) = a\}$$

is of measure zero. Indeed, for instance fix $x \in \mathbb{R}^3$. Then the potential term in E is fixed and

$$|v|^2 = 2a - 2U_0(x),$$

meaning that the set $E^{-1}(a)$ is a sphere in the v-space and therefore has measure zero. Therefore the set

$$S_{crit} := \left\{(x,v) \in \mathbb{R}^6 \ : \ \frac{E(x,v)}{\lambda_0} \in V_{crit}\right\}$$

is again a set of measure zero. Hence on $f_0^{-1}((0,\infty))\backslash S_{crit}$ we can invert (3.16) and since f_0 is a minimizer, we obtain

$$f_0(x,v) = \phi(E) = \inf\{f \geq 0 \ : \ \lambda_0 Q'(f) = E\}$$

for $E \leq E_0$. If we take into account the energy at zero given in (3.17), the minimizer f_0 is a.e. a function which depends on the particle energy E and therefore a steady state of the Vlasov-Poisson equation as described in Section 2.6. □

Remark 3.2.2. Notice that (3.16) is the so-called *Euler-Lagrange Equation*. This is a fundamental equation in variational calculus and it is used to describe extremal conditions of a physical system, in our case the energy E of the minimizer f_0 using the Lagrange multiplier λ_0. An introduction to Lagrangian mechanics is provided by physics textbooks about Classical Mechanics as for example [GPS02].

Remark 3.2.3. Recall from Example 2.5.4 that for

$$Q(f) = f + f^{1+\frac{1}{k}}$$

with $0 < k < \frac{7}{2}$, a minimizer $f_0 \in \mathcal{F}_M$ of $\mathcal{H}$ is of the form of a polytrope by Theorem 3.2.1. In general, polytropes satisfy the properties of minimizers that we established in Theorem 3.2.1.

Another property to consider involves the symmetries of minimizers. The following result demonstrates that a minimizer remains unchanged when exposed to a spherical rearrangement.

Theorem 3.2.4. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}$. Then $f_0(x,v)$ is spherically symmetric with respect to some $x \in \mathbb{R}^3$.*

Proof. Let f_0^* be the spherically symmetric rearrangement of f_0 with respect to some $x \in \mathbb{R}^3$. Then $\rho_{f_0^*}$ is the spherically symmetric rearrangement of ρ_{f_0} by definition. By substitution we have

$$E_{kin}(f_0^*) = \frac{1}{2}\iint |v|^2 \, f_0^*(x,v)\, dvdx = \frac{1}{2}\iint |v|^2 \, f_0(\tilde{x},v)\, dvd\tilde{x} = E_{kin}(f_0)$$

and as f_0 is a minimizer,

$$\mathcal{H}(f_0) = E_{kin}(f_0) + E_{pot}(f_0) \leq E_{kin}(f_0^*) + E_{pot}(f_0^*) = \mathcal{H}(f_0^*)$$

Hence $E_{pot}(f_0) \leq E_{pot}(f_0^*)$. Taking into account the minus-sign in the definition of the potential energy, we have by *Riesz's Rearrangement Inequality* (Theorem C.2), using the expression given in Lemma 2.4.2 *(i)*, the opposite inequality and therefore $E_{pot}(f_0) = E_{pot}(f_0^*)$. As the terms are equal, we can apply the *Strict Rearrangement Inequality* (Theorem C.3) and find that ρ_0 is spherically symmetric with respect to some point in $\mathbb{R}^3$. Again by Lemma 2.4.2 *(i)* it is implied that U_0 is also spherically symmetric. Since f_0 is a function depending on the energy as seen in Theorem 3.2.1, f_0 is spherically symmetric with respect to $x \in \mathbb{R}^3$ as well. □

In the next theorem our goal is to elaborate the regularity of minimizers. we will prove the continuity properties of of the potential U_0 and the mass density ρ_0.

Theorem 3.2.5. *Let $f_0 \in \mathcal{F}_M$ and assume that if $k \geq \frac{1}{2}$,*

$$\phi(E) \leq C(-E)^k, \quad E \to -\infty,$$

with ϕ defined as in Theorem 3.2.1. Then $U_0 \in C_b^2(\mathbb{R}^3)$ with $\lim_{|x|\to\infty} U_0(x) = 0$ and $\rho_0 \in C_c^1(\mathbb{R}^3)$.

Proof. First let $k < \frac{1}{2}$. Then $n = k + \frac{3}{2} < 2$ and $\rho_0 \in L^1 \cap L^p$ by Lemma 2.7.1, where $p = 1 + \frac{1}{n} > \frac{3}{2}$. By the results of Lemma 2.7.1, it is given that $U_0 \in W^{2,p}(\mathbb{R}^3)$. Set $\kappa = 2$ and $N = 3$. Then $N = 2 \cdot \frac{3}{2} < 2p$ and, moreover, $0 < 2 - \frac{3}{p} = \kappa - \frac{N}{p} \notin \mathbb{N}$. Hence, *Sobolev Embedding* (Theorem 1.3.4 *(iii)*) implies that $W^{2,p}(\mathbb{R}^3) \hookrightarrow C^{l,\alpha}(\mathbb{R}^3)$ with $l = \max\{j \in \mathbb{N} : j \leq 2 - \frac{3}{p}\} \in \mathbb{N}_0$ and $0 \leq \alpha \leq 2 - \frac{3}{p} - l$ and in particular, we have that $U_0 \in C(\mathbb{R}^3)$.

Next we split the potential U_0 for an arbitrary $R > 0$ and $x \in \mathbb{R}^3$ and find

$$\begin{aligned}
-4\pi U_0 &= \int_{\mathbb{R}^3} \frac{\rho_0(y)}{|x-y|}\, dy \\
&= \int_{|x-y|<\frac{1}{R}} \frac{\rho_0(y)}{|x-y|}\, dy + \int_{\frac{1}{R}\leq|x-y|<R} \frac{\rho_0(y)}{|x-y|}\, dy + \int_{|x-y|\geq R} \frac{\rho_0(y)}{|x-y|}\, dy \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Let $q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Since $p > \frac{3}{2}$, $q < 3$. Applying *Hölder's Inequality* (Theorem 1.2.2) and using spherical coordinates gives the estimate

$$\begin{aligned}
I_1 &\leq \|\rho_0\|_p \left(\int_{|x-y|<\frac{1}{R}} \frac{1}{|x-y|^q} dy \right)^{\frac{1}{q}} \\
&= C \|\rho_0\|_p \left(\int_0^{\frac{1}{R}} r^{2-q} dr \right)^{\frac{1}{q}} \\
&= C \|\rho_0\|_p \frac{1}{3-q} \left(\frac{1}{R} \right)^{3-q}.
\end{aligned}$$

For the approximation of I_2 with R large,

$$\begin{aligned}
I_2 &\leq R \int_{\frac{1}{R}\leq|x-y|<R} \rho_0(y)\, dy \\
&\leq R \int_{|x-y|\leq R} \rho_0(y)\, dy \\
&\leq R \int_{|y|\geq|x|-R} \rho_0(y)\, dy
\end{aligned}$$

Finally, I_3 can be bounded by the mass M similar to Lemma 2.7.1 *(i)*:

$$\begin{aligned}
I_3 &\leq \frac{1}{R} \int_{|x-y|\geq R} \rho_0(y)\, dy \\
&\leq \frac{M}{R}.
\end{aligned}$$

Therefore U_0 is bounded, i.e. $U_0 \in C_b(\mathbb{R}^3)$. Take $R > 0$ large and let $|x| \to \infty$. Then I_2 vanishes and I_1, I_3 become arbitrarily small. Hence

$$\lim_{|x|\to\infty} U_0(x) = 0. \tag{3.18}$$

Hence, by Theorem 3.2.1 $E > E_0$ for $|x|$ sufficiently large, where $E_0 < 0$, and moreover, f_0 and ρ_0 are compactly supported. Theorem 3.2.1 also implies that f_0 is a function given by $\phi(E)$ which depends only on the energy of the particles E. Define for $u \in \mathbb{R}$

$$h_\phi(u) := 4\pi\sqrt{2}\int_u^\infty \phi(E)\sqrt{E-u}\,dE.$$

Let $u \geq E_0$. Then the integral boundaries are from E_0 to ∞ and by Theorem 3.2.1, $\phi(E) = 0$ in this region. Thus $h_\phi(u) = 0$ for $u \geq E_0$. We have that f_0 is the infimum of $f \geq 0$ such that $\lambda_0 Q'(f) = E$ for any $E \leq E_0$. Hence, using spherical coordinates with $r = |v| = \sqrt{2(E-U_0)}$,

$$\begin{aligned}
\rho_0(x) &= \int f_0(x,v)\,dv \\
&= 4\pi\int_0^{\sqrt{2(E_0-U_0)}} f_0(x,r)r^2\,dr \\
&= 4\pi\int_{U_0}^{E_0} \phi(E)\sqrt{2(E-U_0)}\,dE \\
&= h_\phi(U_0(x))
\end{aligned} \tag{3.19}$$

From the fact that $Q \in C^1([0,\infty))$, it is implied that $h_\phi \in C^1(\mathbb{R}^3)$ as $\phi(E) = f_0$ for $E < E_0$ and $\phi(E) = 0$ else. By the regularity of U_0 and by (3.18) we therefore find that $\rho_0 \in C_c(\mathbb{R}^3)$. Recall the Definition given in (2.3). U_0 is defined as an integral depending on ρ_0 and therefore $U_0 \in C_b^1(\mathbb{R}^3)$. Consequently, repeating these arguments, $\rho_0 \in C_c^1(\mathbb{R}^3)$ and thus $U_0 \in C_b^2(\mathbb{R}^3)$.

The next step is to consider the second case, i.e. for $k \geq \frac{1}{2}$. Here, $n = k + \frac{3}{2} \geq 2$ and $p = 1 + \frac{1}{n} \leq \frac{3}{2}$. The argument with the splitted potential is not applicable in this case because now the conjugate exponent to p is $q \geq 3$. Taking this into account, the determined upper bound for I_1 could increase for large R. So the idea to prove the claim in this case is to show that ρ_0 is also in some L^p with $p > \frac{3}{2}$. If this is true, we are able to use the same arguments as before and we are finished. To verify this, we use a so-called bootstrap argument, which is a method where we leverage the information gained in (3.19) to conclude that it also holds for the second case. Consider our assumption that $\phi(E) \leq C(-E)^k$ as $E \to -\infty$ for some constant C. This assumption is necessary to control the growth of h_ϕ. In fact, we have

$$\begin{aligned}
h_\phi(u) &\leq C\int_u^{E_0} E^k\sqrt{E-u}\,dE \\
&\leq C\sqrt{E_0-u}\int_u^{E_0} E^k\,dE \\
&= \frac{C}{k+1}\sqrt{E_0-u}\left(E_0^{k+1} - u^{k+1}\right) \\
&\leq C\left(1 + (E_0-u)^{k+\frac{3}{2}}\right)
\end{aligned} \tag{3.20}$$

with $u \leq E_0$. Consider the set of points $x \in \mathbb{R}^3$ where ρ_0 is large. Since ρ_0 is compactly supported, this set is of finite measure and ρ_0 is integrable on the complement of this set. Using the estimate given in (3.20) for $u = U_0(x)$ leads to

$$\int \rho_0(x)^p \, dx \leq C + \int (-U_0(x))^{np} \, dx. \tag{3.21}$$

Observe that for $n = 5$, i.e. in the limiting case where $\rho \in L^{\frac{6}{5}}$, we have that $U_0 \in L^6$ as seen in Lemma 2.7.1 *(iii)*. Bootstrapping this with (3.21) returns $\rho \in L^{\frac{6}{5}}$. In this case, the bootstrap argument is a short closed loop. To acquire additional information, we will use a similar technique while choosing $n < 5$ in the beginning. For that, we want to introduce a faster method to gain details about the integrability of U_f. We claim that if $q > 1$ with $\frac{1}{q} = \frac{1}{3} + \frac{1}{p_0} - 1$, then $U_0 \in L^q$. The proof of this claim is adapted from Chapter 2 in [Kaz83]. Set the parameters $p = p_0 = 1 + \frac{1}{n}$, $\lambda = 1$, $r = q$, $N = 3$ and let $\frac{1}{q} + \frac{1}{q'} = 1$. Then

$$\frac{1}{p_0} + \frac{1}{3} + \frac{1}{q'} = \frac{1}{p_0} + \frac{1}{3} + 1 - \frac{1}{q} = 2$$

and thus we can apply *Hardy-Littlewood-Sobolev Inequality* (Theorem 1.2.4) for the functions $g(x) \in L^{q'}$ and $h(x) = \rho_0(x)$. We obtain

$$\left| \int_{\mathbb{R}^3} g(y) U_0(y) \, dy \right| \leq C \|\rho\|_{p_0} \|g\|_{q'}$$

and therefore, $U_0 \in (L^{q'})^* \cong L^q$ with $q = \left(\frac{1}{p_0} - \frac{2}{3}\right)^{-1}$, which proves the claim. Now we can substitute this result into (3.21) gives back that $\rho_0 \in L^{p_1}$ with $p_1 = \frac{q}{n}$. The goal now is to iterate this process until $\rho_0 \in L^p$ for $p > \frac{3}{2}$ to finish the proof. To this end, we differ three cases and notice that $p_0 < \frac{3}{2}$. First, if $p_1 > \frac{3}{2}$, we are finished. In the second case we consider $p_1 = \frac{3}{2}$. But since n is not fixed, we can decrease p_1 by $\varepsilon > 0$ small. Then $U_0 \in L^q$ with $q = \left(\frac{1}{p_1 - \varepsilon} - \frac{2}{3}\right)^{-1}$ and thus, since this q is small, p_2 can be chosen arbitrarily large after another bootstrap. Lastly we consider $p_1 < \frac{3}{2}$ and claim that

$$p_k = \frac{3(1 + \frac{1}{n})(n-1)}{n^k(n-5) + 2n + 2}. \tag{3.22}$$

This can be shown using induction. Plugging in $k = 0$ yields

$$\begin{aligned} p_0 &= \frac{3(1 + \frac{1}{n})(n-1)}{3n - 3} \\ &= 1 + \frac{1}{n} \end{aligned}$$

which proves the base case. Hence we can assume that (3.22) holds for every k and for $k+1$ we obtain

$$\begin{aligned} p_{k+1} &= \frac{1}{n} \left(\frac{1}{p_k} - \frac{2}{3} \right)^{-1} \\ &= \frac{3p_k}{n\,(3 - 2p_k)} \\ &= \frac{3(n^2 - 1)}{n\,(n^{k+2} - 5n^{k+1} + 2n + 2)} \\ &= \frac{3(1 + \frac{1}{n})(n-1)}{n^{k+1}(n-5) + 2n + 2} \end{aligned}$$

and the claim is established. Observe that if $p_{k-1} < \frac{3}{2}$, we have $p_k > 1$. Due to the fact that $2 \leq n < 5$, it is possible for the denominator in (3.22) to become negative. Hence, the process will stop at after finitely many steps and we find that $\rho_0 \in L^p$ with $p \geq \frac{3}{2}$. □

Remark 3.2.6. If $Q(f) = f + f^{1+\frac{1}{k}}$ for $f \geq 0$ and $k \in (0, \frac{7}{2})$, then there exist at most two minimizers of $\mathcal{H}$ in $\mathcal{F}_M$ up to a shift in x-space. This fact is proven in Theorem 3 of [GR01] where the details follow from Theorem 3 of [Guo00].

3.3 The Plummer Sphere - $k = \frac{7}{2}$

The case for $k = \frac{7}{2}$ describes a so-called Plummer sphere, c.f. Example 2.5.5. This is an idealized assumption for the distribution of stars in a spherical galaxy or globular cluster. It involves a spherical distribution of particles, in our case stars, which is highest at the at the center of the galaxy. As the distance from the center increases, the particle density decreases. However, the actual distribution can vary, as stars are usually not perfectly spherically distributed.

3.3.1 Scaling Transformation

In this limit case we have that $p = 1 + \frac{1}{k} = \frac{9}{7}$ and $n = k + \frac{3}{2} = 5$. Thus the minimization of the Hamiltonian $\mathcal{H}$ will be done on the constraint set

$$\mathcal{F}_M := \{f \in L^{\frac{9}{7}}\left(\mathbb{R}^6\right) \,:\, f \geq 0,\ \iint f^{\frac{9}{7}}\, dvdx = M,\ E_{kin}(f) < \infty\},$$

where we use $Q(f) = f^{\frac{9}{7}}$. For this case, we introduce the scaling transformation

$$(S_\lambda f)(x, v) := \lambda^{-7} f(\lambda^{-4} x, \lambda v), \tag{3.23}$$

where $\lambda \in \mathbb{R}$. This is well chosen, as it leaves both $\mathcal{H}$ and $\mathcal{C}$ invariant.

Lemma 3.3.1. *Let $S_\lambda f$ be the scaling transformation as defined in* (3.23) *and let $Q(f) = f^{\frac{9}{7}}$. Then*

(i) $\mathcal{H}$ is invariant, i.e. $\mathcal{H}(S_\lambda f) = \mathcal{H}(f)$,

(ii) and $\mathcal{C}$ is invariant, i.e. $\mathcal{C}(S_\lambda f) = \mathcal{C}(f)$.

Proof. In both cases we will use the scaling properties that were computed in the proof of Lemma 2.7.4. *(i)* For the kinetic energy, we use (2.13) to get

$$\begin{aligned} E_{kin}(S_\lambda f) &= \lambda^{-7} \left(\lambda^{-4}\right)^{-3} \lambda^{-5} E_{kin}(f) \\ &= \lambda^{-7+12-5} E_{kin}(f) \\ &= E_{kin}(f) \end{aligned}$$

and similarly, using (2.14) yields

$$\begin{aligned} E_{kin}(S_\lambda f) &= \left(\lambda^{-7}\right)^2 \left(\lambda^{-4}\right)^{-5} \lambda^{-6} E_{pot}(f) \\ &= \lambda^{-14+20-6} E_{pot}(f) \\ &= E_{pot}(f). \end{aligned}$$

Conclude

$$\mathcal{H}(S_\lambda f) = E_{kin}(f) + E_{pot}(f) = \mathcal{H}(f).$$

To establish *(ii)*, we will apply the formula in (2.12) to find

$$\begin{aligned} \mathcal{C}(S_\lambda f) &= \iint \left(\lambda^{-7} f(\lambda^{-4}x, \lambda v)\right)^{\frac{9}{7}} \, dvdx \\ &= \lambda^{-9} \left(\lambda^{-4}\right)^{-3} \lambda^{-3} \iint \left(f(\tilde{x}, \tilde{v})^{\frac{9}{7}} \, d\tilde{v}d\tilde{x}\right. \\ &= \lambda^{-9+12-3} \mathcal{C}(f) \\ &= \mathcal{C}(f). \end{aligned}$$

□

3.3.2 Existence of Minimizers for $k = \frac{7}{2}$

The idea to show the existence of minimizers in the limit case is to show that the steps in Section 3.3.2 also hold under the scaling transformation (3.23). As a basis for this, Corollary 2.7.3 and Lemma 2.7.4 are utilized again, which, according to Remark 2.7.5, remain valid in the limiting case. Thus there exists a minimizing sequence (f_i) of $\mathcal{H}$ in $\mathcal{F}_M$. Denote by (f_i^*) the sequence of spherically symmetric rearrangements of (f_i) with respect to x. Similar to Theorem 3.2.4, (f_i^*) is again a minimizing sequence in $\mathcal{F}_M$. Following Theorem 2 in [Guo00], there exists a symmetric minimizer g such that

$$S_i f_i^* \rightharpoonup g \text{ weakly in } L^{\frac{9}{7}}(\mathbb{R}^6),$$

and furthermore,

$$\|\nabla U_{S_i f_i^*} - \nabla U_g\|_2 \to 0,$$

with $S_i := S_{\lambda_i}$ for some $\lambda_i > 0$. From here, we can even find that the corresponding spatial mass densities converge strongly in $L^{\frac{6}{5}}$.

Lemma 3.3.2. *Let (f_i) be a minimizing sequence and denote by g the symmetric minimizer such that $S_i f_i^* \rightharpoonup g$ weakly in $L^{\frac{9}{7}}(\mathbb{R}^6)$. Then*

$$\lim_{i\to\infty} \|\rho_{S_i f_i^*} - \rho_g\|_{\frac{6}{5}} = 0$$

up to a subsequence.

Proof. For simplicity, denote $\rho_i^* = \rho_{S_i f_i^*}$. By Lemma 2.7.1 and Remark 2.7.2, both ρ_i^* and ρ_g are in $L^{\frac{6}{5}}(\mathbb{R}^3)$. Then, according to *Banach-Alaoglu Theorem* (Theorem 1.3.3) and by the Definition 1.1.6, since $S_i f_i^* \rightharpoonup g$ weakly in $L^{\frac{9}{7}}(\mathbb{R}^6)$,

$$\rho_i^* \rightharpoonup \rho_g \text{ weakly in } L^{\frac{6}{5}}(\mathbb{R}^3), \tag{3.24}$$

as $i \to \infty$. Next, we notice that because of the spherical rearrangement, it is enough to only consider the radius as an argument for the spatial mass density. We want to show that for every $r \in (0, \infty)$, such that ρ_g is continuous in r, we have

$$\rho_i^*(r) \to \rho_g(r) \tag{3.25}$$

as $i \to \infty$. Indeed, suppose by contradiction that $r_0 \in (0, \infty)$ such that ρ_g is continuous at r_0 and (3.25) does not hold. Notice that since ρ_g is monotone, the set of discontinuities is of measure zero by Theorem B.2 and therefore not of interest. Then there exists $\varepsilon_0 > 0$ and a subsequence, which we still denote by ρ_i^*, such that

$$|\rho_i^*(r_0) - \rho_g(r_0)| \geq \varepsilon_0$$

for any $i \in \mathbb{N}$. Suppose that

$$\rho_i^*(r_0) - \rho_g(r_0) \geq \varepsilon_0, \tag{3.26}$$

otherwise the following computations can be done similarly with change in signs. Then there exists $\delta > 0$ such that

$$|\rho_g(r) - \rho_g(r_0)| < \frac{\varepsilon_0}{2} \tag{3.27}$$

for $r \in [r_0 - \delta, r_0]$. Then, using (3.26) and (3.27),

$$\begin{aligned}
\int_{r_0-\delta}^{r_0} \rho_g \, dr &\leq \int_{r_0-\delta}^{r_0} \rho_g(r_0) + \frac{\varepsilon_0}{2} \, dr \\
&\leq \int_{r_0-\delta}^{r_0} \rho_i^*(r_0) - \frac{\varepsilon_0}{2} \, dr \\
&\leq \int_{r_0-\delta}^{r_0} \rho_i^*(r) - \frac{\varepsilon_0}{2} \, dr,
\end{aligned}$$

where the last step is due to the fact that ρ_i^* is monotonically decreasing. However, as $\varepsilon_0 > 0$, this contradicts (3.24) and therefore (3.25) holds true.

In order to find strong convergence, let (r_j) and (R_j) be sequences with $r_j \to 0$ and $R_j \to \infty$, respectively, at which ρ_g is continuous. Again by the monotonicity of ρ_i^* as well as (3.25), for each $j \in \mathbb{N}$ the sequence (ρ_i^*) is uniformly bounded in $r_j \leq r \leq R_j$. Hence we can apply *Dominated Convergence Theorem* (Theorem 1.3.1) to find

$$\lim_{i \to \infty} \|\rho_i^* - \rho_g\|_{L^{\frac{6}{5}}(r_j \leq r \leq R_j)} = 0.$$

Left to check are the cases at the origin $r = 0$ or at $r = \infty$. Either way, there is no concentration by (0.47) of [Guo00], which means that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\limsup_{i\to\infty} \iint_{\{r\leq \frac{1}{R}\}\cup\{r\geq R\}} (S_i f_i^*)^{\frac{9}{7}} < \varepsilon.$$

Hence from Lemma 2.7.1 *(ii)* follows that up to a subsequence and for i large enough,

$$\|\rho_i^*\|_{L^{\frac{6}{5}}(\{r\leq r_j\}\cup\{r\geq R_j\})} < C\varepsilon \tag{3.28}$$

for a constant $C > 0$. Since $\varepsilon > 0$ is choosen arbitrarily, we conclude that up to a subsequence,

$$\|\rho_i^* - \rho_g\|_{\frac{6}{5}} \to 0$$

as $i \to \infty$. □

Let ϕ be a non-negative function and let $N > 0$ be a cut-off parameter. Then we introduce the notation

$$\phi_{|N} := \begin{cases} 0, & \phi(x) \in [\frac{1}{N}, N], \\ \phi(x), & \text{else.} \end{cases}$$

The next lemma will show that $S_i f_i$ are equiintegrable (c.f. Definition B.3), i.e. we can always find a uniform bound for every i. If so, there is no concentration, meaning that there is no interval where the density accumulates. This could lead to the function converging to a distribution, a case we aim to exclude.

Lemma 3.3.3. *Let $\varepsilon > 0$. Then there exists $N > 0$ such that up to a subsequence*

$$\|[\rho_{S_i f_i}]_{|N}\|_{\frac{6}{5}} < \varepsilon$$

for every $i \in \mathbb{N}$.

Proof. For simplicity, denote $\rho_i^* = \rho_{S_i f_i^*}$ and $\rho_i = \rho_{S_i f_i}$. The first step is to prove the statement for ρ_i^*. In order to do so, define the sets

$$A_i := \{r \ : \ \rho_i^* < \frac{1}{N}\} \cup \{r \ : \ \rho_i^* > N\},$$
$$A_g := \{r \ : \ \rho_g \leq \frac{1}{N}\} \cup \{r \ : \ \rho_g \geq N\}.$$

Similar to (3.28), for any $\varepsilon > 0$ we can choose $N > 0$ large enough such that

$$\|\rho_g\|_{L^{\frac{6}{5}}(A_g)} < \frac{\varepsilon}{2}. \tag{3.29}$$

Then the triangle inequality and Lemma 3.3.2 imply

$$\begin{aligned} \|(\rho_i^*)_{|N}\|_{\frac{6}{5}} &= \|\rho_i^*\|_{L^{\frac{6}{5}}(A_i)} \\ &\leq \|\rho_i^* - \rho_g\|_{L^{\frac{6}{5}}(A_i)} + \|\rho_g\|_{L^{\frac{6}{5}}(A_i)} \\ &\leq \frac{\varepsilon}{2} + \|\rho_g\|_{L^{\frac{6}{5}}(A_i)} \end{aligned}$$

for i large enough. By the convergence in $L^{\frac{6}{5}}$, it is also given that $\rho_i \to \rho_g$ pointwise a.e. It follows that

$$\limsup_{i\to\infty} \mathbb{1}_{A_i} \leq \mathbb{1}_{A_g} \quad \text{a.e.}$$

This relation allows us to apply Fatou's Lemma (Lemma 1.2.1), which gives us

$$\begin{aligned}
\limsup_{i\to\infty} \|\rho_g\|_{L^{\frac{6}{5}}(A_i)} &= \limsup_{i\to\infty} \left(\int \mathbb{1}_{A_i} |\rho_g|^{\frac{6}{5}} \right)^{\frac{5}{6}} \\
&\leq \left(\int \limsup_{i\to\infty} \mathbb{1}_{A_i} |\rho_g|^{\frac{6}{5}} \right)^{\frac{5}{6}} \\
&\leq \left(\int \mathbb{1}_{A_g} |\rho_g|^{\frac{6}{5}} \right)^{\frac{5}{6}} \\
&= \|\rho_g\|_{L^{\frac{6}{5}}(A_g)}
\end{aligned}$$

and therefore by (3.29)

$$\limsup_{i\to\infty} \|\rho_g\|_{L^{\frac{6}{5}}(A_i)} < \frac{\varepsilon}{2}.$$

Thus for $i \in \mathbb{N}$ and up to a subsequence,

$$\|(\rho_i^*)_{|N}\|_{\frac{6}{5}} < \varepsilon. \tag{3.30}$$

The next step is to check that (3.31) also holds for ρ_i. To this end, we will use the fact that rearrangements are equimeasurable (c.f. Definition B.4), meaning that their level sets have the same measure. To use this, let $\phi \geq 0$ be a function and let $p \geq 1$. Then

$$\begin{aligned}
\int (\phi_{|N})^p &= p \int_0^\infty s^{p-1} \mu\{\phi \mathbb{1}_{\{\phi < \frac{1}{N}\} \cup \{\phi > N\}} > s\}\, ds \\
&= p \int_0^\infty s^{p-1} \left(\mu\{\phi \mathbb{1}_{\{\phi < \frac{1}{N}\}} > s\} + \mu\{\phi \mathbb{1}_{\{\phi > N\}} > s\} \right) ds \\
&= p \int_0^\infty s^{p-1} \left(\mu\{s < \phi < \frac{1}{N}\} + \mu\{\phi > \max[N, s]\} \right) ds \\
&= p \int_0^\infty s^{p-1} \left(\mu\{s < \phi^* < \frac{1}{N}\} + \mu\{\phi^* > \max[N, s]\} \right) ds \\
&= \int (\phi^*_{|N})^p.
\end{aligned}$$

For $\phi = \rho_i$ we therefore find that

$$\|(\rho_i)_{|N}\|_{\frac{6}{5}} < \varepsilon, \tag{3.31}$$

which finishes the proof. □

With this preliminary work, we are now able to prove, analogously to Theorem 3.1.3, the existence of minimizers in the case of a plummer sphere.

Theorem 3.3.4. *Let $(f_i) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}$. Then there exists a minimizer $f_0 \in \mathcal{F}_M$, a subsequence, which is still denoted by (f_i), and a sequence of translations*

$$T_i S_i f_i(x, v) = \lambda_i^7 f_i(\lambda_i^{-4} x + a_i, \lambda_i v),$$

where $(a_i) \subset \mathbb{R}^3$ *and* λ_i *such that*

$$\mathcal{H}(f_0) = \inf_{\mathcal{F}_M} \mathcal{H} = h_M$$

and moreover,

$$T_i S_i f_i \rightharpoonup f_0 \text{ weakly in } L^{\frac{9}{7}}(\mathbb{R}^6),$$
$$\nabla U_{T_i S_i f_i} \to \nabla U_0 \text{ strongly in } L^2(\mathbb{R}^3).$$

Proof. It is enough to show that Lemma 3.1.1 and Lemma 3.1.2 also hold in this case, meaning if we set $\rho_i = \rho_{S_i f_i}$. Fix a cut-off parameter $N > 0$ to split

$$\rho_i =: \rho_i^b + (\rho_i)_{|N},$$

where ρ_i^b denotes a part that is bounded in any L^p for $1 \leq p \leq \infty$ and for every i. Hence, Lemma 3.1.1 as well as Lemma 3.1.2 are valid for ρ_i^b. We know want to check their validity for (ρ_i).
In order to verify Lemma 3.1.1 in this case, notice that by the triangle inequality

$$\|\nabla U_{\rho_i}\|_2 \leq \|\nabla U_{\rho_i^b}\|_2 + \|\nabla U_{(\rho_i)_{|N}}\|_2.$$

Thus, due to the negative sign in the definition, the potential energy with respect to ρ_i is bounded from below by the sum of the potential energies of ρ_i^b and $(\rho_i)_{|N}$. Choose $\varepsilon < -\frac{h_M}{2}$ in Lemma 3.3.3 and use Lemma 2.7.1 *(iii)* to bound the potential energy with respect to $(\rho_i)_{|N}$. Lemma 3.1.1 holds for ρ_i^b and hence, similar as in (3.2), splitting the potential energy according to (3.1) leads to

$$\frac{1}{2} h_M > \mathcal{H}(f_i) \geq -\sum_{k=1}^{3} |I_k^b| - \varepsilon.$$

Here, I_k^b is induced by ρ_i^b for $k = 1, 2, 3$. Notice that

$$\sup_{y \in \mathbb{R}^3} \int_{y+B_R} \rho_i^b \, dx \leq \sup_{y \in \mathbb{R}^3} \int_{y+B_R} \rho_i^b \, dx.$$

Inserting this into (3.3) lets us finish the proof of Lemma 3.1.1 for ρ_i.
To establish the assertion of Lemma 3.1.2 for (ρ_i), we need to examine $\|\nabla U_{(\rho_i)_{|N}}\|_2$. From Lemma 2.7.1 *(iii)* follows that

$$\|\nabla U_{(\rho_i)_{|N}}\|_2 \leq C \|(\rho_i)_{|N}\|_{\frac{6}{5}}$$

for a constant $C > 0$. Consequently, by utilizing Lemma 3.3.3, one can make $\|\nabla U_{(\rho_i)_{|N}}\|_2$ arbitrarily small for N large and uniformly for all i. Hence the estimate in (3.5) is still valid for (ρ_i). □

3.3.3 Properties of Minimizers for $k = \frac{7}{2}$

For the sake of completeness, we will study several properties of the minimizers of $\mathcal{H}$ in the case where $k = \frac{7}{2}$ as in Section 3.2. First we start off with the analogous version of Theorem 3.2.1.

Theorem 3.3.5. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}$. Then for a.e. $(x,v) \in \mathbb{R}^6$*

$$f_0(x,v) = \begin{cases} \left(\frac{E}{\lambda_0}\right)^{\frac{7}{2}}, & E < 0, \\ 0, & E \geq 0, \end{cases}$$

where $\lambda_0 < 0$ is a constant. In particular, f_0 is a steady state of the Vlasov-Poisson system.

Proof. Regarding the proof of 3.2.1, we can simply replace the general Q that satisfies Definition 2.5.1 with the function $f^{\frac{9}{7}}$. From that follows that we will replace Q' with $\frac{9}{7}f^{\frac{2}{7}}$. The steps now follow as in the proof of the case for $k < \frac{7}{2}$. □

The spherical symmetry of a minimizer can be proven in the exact same way as in Theorem 3.2.4. This allows us to restate the assertion for the case of a Plummer sphere.

Theorem 3.3.6. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}$. Then $f_0(x,v)$ is spherically symmetric with respect to some $x \in \mathbb{R}^3$.*

Dynamical Stability

In this chapter, we will investigate the dynamical stability of our system. Prior to this, we will explain the concept of stability theory and establish a distance in our system. The content of this chapter is based on Chapter 5 of [GR01].

The examination of dynamical stability is a fundamental concept in analysis with practical applications as for example in physics. Its primary objective is to comprehend and predict the behavior of dynamical systems. This involves considering a differential equation with initial conditions, where one expects that a small change or perturbation in the initial value should produce only a minor change in the solution.

In the context of the gravitational Vlasov-Poisson system, stability theory assumes particular significance as it aids in our comprehension of the long-term evolution of stellar systems such as galaxies or star clusters. The total energy of a state is described using the Hamiltonian $\mathcal{H}$. We aim to utilize this quantity to investigate stability. We will show that a specific quantity, which characterizes the energetic distance between a minimizer and any solution of the Vlasov-Poisson system, remains arbitrarily small over time.

4.1 Notion of the Distance

Let $f_0 \in \mathcal{F}_M$, with $\mathcal{F}_M$ defined as in Section 2.5. In order to study the dynamical stability, it is necessary to define a distance between the minimizer of $\mathcal{H}$ and any $f \in \mathcal{F}_M$ to compare the change in the energy over time.

Definition 4.1.1. *The distance $d : \mathcal{F}_M \to \mathbb{R}$ is defined as*

$$d(f, f_0) := \iint \left(\frac{1}{2}|v|^2 + U_0\right)(f - f_0)\, dvdx.$$

For any $f \in \mathcal{F}_M$ the difference of it's Hamiltonian to the minimizer's Hamiltonian can be computed as

$$\begin{aligned}
\mathcal{H}(f) - \mathcal{H}(f_0) &= \iint \frac{1}{2}|v|^2 \,(f - f_0)\; dvdx - \frac{1}{2}\int |\nabla U_f|^2 - |\nabla U_0|^2 \, dx \\
&= d(f, f_0) - \frac{1}{2}\int |\nabla U_f|^2 + |\nabla U_0|^2 \, dx - \iint U_0 f \, dvdx \qquad (4.1) \\
&= d(f, f_0) - \frac{1}{2}\int |\nabla U_f|^2 + |\nabla U_0|^2 - 2|\nabla U_0 \nabla U_f| \, dx \\
&= d(f, f_0) - \frac{1}{2}\|\nabla U_f - \nabla U_0\|_2^2, \qquad (4.2)
\end{aligned}$$

where we used the methods in Lemma 2.4.2.

Remark 4.1.2. In general, $f_0, f \in \mathcal{F}_M$ does not imply $d(f_0, f) = d(f, f_0)$. Hence, since symmetry of d is not given, d is not a metric on $\mathcal{F}_M$.

Lemma 4.1.3. *If $f \in \mathcal{F}_M$, then the distance $d(f, f_0)$ is non-negative, i.e. $d(f, f_0) \geq 0$.*

Proof. We have

$$\begin{aligned}
d(f, f_0) &= \iint \left(\frac{1}{2}|v|^2 + U_0\right)(f - f_0)\, dvdx \\
&= \iint \left(\frac{1}{2}|v|^2 + U_0\right)(f - f_0)\, dvdx - \lambda_0(\mathcal{C}(f) - \mathcal{C}(f_0)) \\
&= \iint E(f - f_0) - \lambda_0(Q(f) - Q(f_0))\, dvdx,
\end{aligned}$$

because $\mathcal{C}(f) = \mathcal{C}(f_0)$ and with λ_0 as well as E defined as in Theorem 3.2.1. Since Q is convex, for any $t \in [0, 1]$

$$\begin{aligned}
Q(f_0 + t(f - f_0)) &= Q(tf + (1 - t)f_0) \\
&\leq tQ(f) + (1 - t)Q(f_0)
\end{aligned}$$

and thus

$$\frac{1}{t}\left(Q(f_0 + t(f - f_0)) - Q(f_0)\right) \leq Q(f) - Q(f_0).$$

Taking the limit for $t \to 0$ we obtain

$$Q'(f_0)(f - f_0) \leq Q(f) - Q(f_0)$$

and therefore, as $\lambda_0 < 0$,

$$d(f, f_0) \geq \iint \left(E - \lambda_0 Q'(f_0)\right)(f - f_0)\, dvdx. \qquad (4.3)$$

As seen in Theorem 3.2.1 in (3.16), the integrand of (4.3) is zero on the support of f_0, i.e. on $f_0^{-1}((0, \infty))$. On $f_0^{-1}(0)$ we have

$$\left(E - \lambda_0 Q'(0)\right) f = (E - E_0) \leq 0$$

since $E \leq E_0$. We conclude that

$$d(f, f_0) \geq 0$$

for every $f \in \mathcal{F}_M$. □

4.2 Stability Result

With the notion of the distance d it is possible to state the proof of the dynamical stability of solutions of the Vlasov-Poisson system. In Section 2.2 we already showed that a solution of the system remains a solution after a space translation. Indeed, as we minimize over $\mathcal{H}$ and as E_{kin} and E_{pot} are invariant under such change of variables, one finds that a minimizer, which is shifted in space, remains a minimizer. Since the uniqueness of minimizers up to spatial shifts is not known, the result is stated in two versions.

Theorem 4.2.1. *Denote by $\mathcal{M}_M \subset \mathcal{F}_M$ the set of all minimizers of $\mathcal{H}$ in $\mathcal{F}_M$ and let $f_0 \in \mathcal{M}_M$.*

(i) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for any solution $t \to f(t)$ of the Vlasov-Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$,

$$\inf_{f_0 \in \mathcal{M}_M} \left[d(f(0), f_0) + \frac{1}{2} \|\nabla U_{f(0)} - \nabla U_0\|_2^2 \right] < \delta$$

implies that for every $t \geq 0$,

$$\inf_{f_0 \in \mathcal{M}_M} \left[d(f(t), f_0) + \frac{1}{2} \|\nabla U_{f(t)} - \nabla U_0\|_2^2 \right] < \varepsilon.$$

(ii) Suppose that the minimizer $f_0 \in \mathcal{M}_M$ is isolated, i.e.

$$\inf \left\{ \|\nabla U_{\tilde{f}_0} - \nabla U_{f_0}\|_2 \ : \ \tilde{f}_0 \in \mathcal{M}_M \setminus \{T^a f_0 \ : \ a \in \mathbb{R}^3\} \right\} > 0,$$

where $T^a f(x,v) := f(x+a, v)$ denotes a translation in space by a shift vector $a \in \mathbb{R}^3$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $t \to f(t)$ of the Vlasov-Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$,

$$d(f(0), f_0) + \frac{1}{2} \|\nabla U_{f(0)} - \nabla U_0\|_2^2 < \delta$$

implies that for every $t \geq 0$, there is a shift vector $a \in \mathbb{R}^3$ such that

$$d(f(t), T^a f_0) + \frac{1}{2} \|\nabla U_{f(t)} - \nabla U_{T^a f_0}\|_2^2 < \varepsilon.$$

Proof. To establish *(i)*, by contradiction suppose the statement does not hold. Then there exists $\varepsilon > 0$, $t_n > 0$ and $f_n(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ such that for every $n \in \mathbb{N}$,

$$\inf_{f_0 \in \mathcal{M}_M} \left[d(f_n(0), f_0) + \frac{1}{2} \|\nabla U_{f_n(0)} - \nabla U_0\|_2^2 \right] < \frac{1}{n} \tag{4.4}$$

but

$$\inf_{f_0 \in \mathcal{M}_M} \left[d(f_n(t_n), f_0) + \frac{1}{2} \|\nabla U_{f_n(t_n)} - \nabla U_0\|_2^2 \right] \geq \varepsilon. \tag{4.5}$$

Then, by (4.2) and (4.4),

$$\begin{aligned}\inf_{f_0\in\mathcal{M}_M}\mathcal{H}(f_n(0))-\mathcal{H}(f_0) &= \inf_{f_0\in\mathcal{M}_M}\left[d(f_n(0),f_0)-\frac{1}{2}\|\nabla U_{f_n(0)}-\nabla U_0\|_2^2\right]\\ &\le \inf_{f_0\in\mathcal{M}_M}\left[d(f_n(0),f_0)+\frac{1}{2}\|\nabla U_{f_n(0)}-\nabla U_0\|_2^2\right]\\ &< \frac{1}{n}\to 0\end{aligned}$$

as $n\to\infty$ and hence, since f_0 is a minimizer and therefore the limit exists,

$$\lim_{n\to\infty}\mathcal{H}(f_n(0))=h_M.$$

By Proposition 2.4.3 and Proposition 2.5.2, we know that $\mathcal{H}$ and $\mathcal{C}$ are conserved in time and thus

$$\lim_{n\to\infty}\mathcal{H}(f_n(t_n))=h_M.$$

Therefore $(f_n(t_n))$ is a minimizing sequence of $\mathcal{H}$ in $\mathcal{F}_M$. By Theorem 3.1.3, up to a subsequence, there exists a minimizer $f_0\in\mathcal{F}_M$, a sequence of translations $T^{a_n}f_n(x,v,t_n)=f_i(x+a_n,v,t_n)$ with $(a_n)\in\mathbb{R}^3$ such that

$$\|\nabla U_{T^{a_n}f(t_n)}-\nabla U_{f_0}\|_2^2\to 0,\quad n\to\infty.$$

Applying a change of variables yields

$$\|\nabla U_{f_n(t_n)}-\nabla U_{T^{b_n}f_0}\|_2^2\to 0,\quad n\to\infty, \tag{4.6}$$

where the sequence $(b_n)\in\mathbb{R}$ is defined such that $b_n=-a_n$ for every n. Notice that we have $d(T^{a_n}f_n(t_n),f_0)=d(f_n(t_n),T^{b_n}f_0)$ and because T^{b_n} only causes a space shift, $T^{b_n}f_0$ remains a minimizer, i.e. $T^{b_n}f_0\in\mathcal{M}_M$, and we have

$$\lim_{n\to\infty}\mathcal{H}(f_n(t_n))=h_M=\mathcal{H}(T^{b_n}f_0).$$

By (4.2) and (4.6),

$$d(f_n(t_n),T^{b_n}f_0)=\mathcal{H}(f_n(t_n))-\mathcal{H}(T^{b_n}f_0)+\frac{1}{2}\|\nabla U_{f_n(t_n)}-\nabla U_{T^{b_n}f_0}\|_2^2\to 0$$

as $n\to\infty$, which is a contradiction to (4.5) and *(i)* is proven.

For the proof of *(ii)*, suppose f_0 is an isolated minimizer in $\mathcal{M}_M$ and define

$$\delta_0:=\frac{1}{2}\inf\left\{\|\nabla U_{\tilde{f}_0}-\nabla U_{f_0}\|_2\ :\ \tilde{f}_0\in\mathcal{M}_M\setminus\{T^a f_0\ :\ a\in\mathbb{R}^3\}\right\}.$$

By the definition of isolated, $\delta_0>0$. Now let any $\varepsilon>0$. Our goal is to find the corresponding $\delta>0$ such that the claim holds. Without loss of generality, we can assume $\varepsilon<\frac{\delta_0}{4}$. Again without loss of generality, choose $\delta>0$ such that it holds in part *(i)* for some $\varepsilon>\delta$, and let $f(0)\in C_c^1(\mathbb{R}^6)\cap\mathcal{F}_M$ such that

$$d(f(0),f_0)+\frac{1}{2}\|\nabla U_{f(0)}-\nabla U_0\|_2^2<\delta. \tag{4.7}$$

For further considerations we define the function $h : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ by

$$h(t,a) := d(f(t), T^a f_0) + \frac{1}{2}\|\nabla U_{f(t)} - \nabla U_{T^a f_0}\|_2^2. \tag{4.8}$$

Using similar computations as in (4.1),

$$\begin{aligned} h(t,a) &= d(f(t), T^a f_0) + \frac{1}{2}\int |\nabla U_{f(t)}|^2 + |\nabla U_{T^a f_0}|^2 \, dx + \iint U_{T^a f_0} f(t) \, dvdx \\ &= \iint \left(\frac{1}{2}|v|^2 + U_{T^a f_0}\right)(f(t) - T^a f_0) \; dvdx + \frac{1}{2}\left(\|\nabla U_{f(t)}\|_2^2 + \|\nabla U_{f_0}\|_2^2\right) \\ &\quad + \int U_{T^a f_0} \rho_{f(t)} \, dvdx \end{aligned}$$

Notice that we used the fact that $\|\nabla U_{T^a f_0}\|_2 = \|\nabla U_{f_0}\|_2$. Since

$$\begin{aligned} &\iint \left(\frac{1}{2}|v|^2 + U_{T^a f_0}\right)(f(t) - T^a f_0) \; dvdx \\ &= \iint \frac{1}{2}|v|^2 (f(t) - T^a f_0) \; dvdx + \iint U_{T^a f_0} (f(t) - T^a f_0) \; dvdx \\ &= \iint \frac{1}{2}|v|^2 (f(t) - f_0) \; dvdx + \int U_{T^a f_0} \rho_{f(t)} \, dvdx + \|\nabla U_{T^a f_0}\|_2^2, \\ &= \iint \frac{1}{2}|v|^2 (f(t) - f_0) \; dvdx + \int U_{T^a f_0} \rho_{f(t)} \, dvdx + \|\nabla U_{f_0}\|_2^2, \end{aligned}$$

we have that

$$\begin{aligned} h(t,a) = &\iint \left(\frac{1}{2}|v|^2\right)(f(t) - f_0) \; dvdx + \frac{1}{2}\left(\|\nabla U_{f(t)}\|_2^2 + 3\|\nabla U_{f_0}\|_2^2\right) \\ &+ 2\int U_{T^a f_0} \rho_{f(t)} \, dvdx. \end{aligned}$$

As h is a composition of continuous functions, it is continuous itself. The last term, which is the interaction term, vanishes as $|a| \to \infty$ uniformly on compact time intervals because of $\lim_{|x|\to\infty} U(x,t) = 0$ in (2.1). Thus $\inf_{a\in\mathbb{R}^3} h(t,a)$ is continuous as well. Suppose by contradiction that there exists $t > 0$ such that

$$\inf_{a\in\mathbb{R}^3} h(t,a) \geq \varepsilon. \tag{4.9}$$

Because of (4.7), at $t = 0$ the left hand side of (4.9) must be smaller than ε and therefore, continuity implies that there exists $t^* > 0$ such that

$$\inf_{a\in\mathbb{R}^3} h(t^*,a) = \varepsilon. \tag{4.10}$$

Hence, there is $a_0 \in \mathbb{R}^3$ such that

$$\varepsilon \leq h(t^*, a_0) = d(f(t^*), T^{a_0}) + \frac{1}{2}\|\nabla U_{T^{-a_0} f(t^*)} - \nabla U_{f_0}\|_2^2 \leq \left(1 + \frac{1}{m}\right)\varepsilon \tag{4.11}$$

for every $m \in \mathbb{N}$. For such an a_0, from *(i)* it follows for some $f_0^* \in \mathcal{M}_M$ we have

$$\begin{aligned} &d(f(t^*), f_0^*) + \frac{1}{2}\|\nabla U_{T^{-a_0} f(t^*)} - \nabla U_{T^{-a_0} f_0^*}\|_2^2 \\ &= d(f(t^*), f_0^*) + \frac{1}{2}\|\nabla U_{f(t^*)} - \nabla U_{f_0^*}\|_2^2 < \varepsilon \leq \frac{\delta_0}{4}. \end{aligned} \tag{4.12}$$

Since a minimizer remains a minimizer after spacial translation, $T^{-a_0} f_0^* \in \mathcal{M}_M$. By the triangle inequality and since d is positive by Lemma 4.1.3, we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla U_{f_0} - \nabla U_{T^{-a_0} f_0^*}\|_2^2 &\leq \frac{1}{2}\|\nabla U_{f_0} - \nabla U_{T^{-a_0} f(t^*)}\|_2^2 + \frac{1}{2}\|\nabla U_{T^{-a_0} f(t^*)} - \nabla U_{T^{-a_0} f_0^*}\|_2^2 \\ &\leq h(t^*, a_0) + d(f(t^*), f_0^*) + \frac{1}{2}\|\nabla U_{f(t^*)} - \nabla U_{f_0^*}\|_2^2 \\ &< \left(2 + \frac{1}{m}\right)\varepsilon,\end{aligned}$$

where we also made use of (4.11) and (4.12). Hence, for m large enough,

$$\frac{1}{2}\|\nabla U_{f_0} - \nabla U_{T^{-a_0} f_0^*}\|_2^2 \leq \frac{\delta_0}{2}.$$

From the definition of δ_0 it follows that there exists some $a^* \in \mathbb{R}^3$ such that $f_0^* = T^{a^*} f_0$. But then a contradiction arises if we insert (4.10) into (4.12) and therefore

$$\inf_{a \in \mathbb{R}^3} h(t, a) < \varepsilon$$

for every $t \in \mathbb{R}_+$, which finishes the proof of *(ii)*. □

Remark 4.2.2. (i) For the case of polytropes, we know by Remark 3.2.6 that up to a shift there exist at most two minimizers. Hence the assumptions in Theorem 4.2 *(ii)* are valid for polytropes.

(ii) Physically it makes sense to restrict on the cases where $f(0) \in \mathcal{F}_M$ for the perturbed initial data. If a perturbation of a given galaxy would result in a perturbed state, it is just a rearrangement of the original state, meaning that $\mathcal{C}(f)$ remains the same. Such a perturbation can be caused for example by the gravitational pull of some object outside of the galaxy.

4.3 Stability Result for $k = \frac{7}{2}$

In conclusion, we present the stability theorem for the limiting condition. For this purpose, define $\mathcal{F}_M$ as in Section 3.3.

Theorem 4.3.1. *Let $f_0 \in \mathcal{F}_M$ be a minimizers of $\mathcal{H}$. Then For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any solution $t \to f(t)$ of the Vlasov-Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$,*

$$d(f(0), f_0) + \frac{1}{2}\|\nabla U_{f(0)} - \nabla U_0\|_2^2 < \delta$$

implies that there is a shift vector $a \in \mathbb{R}^3$ and a scaling parameter $\lambda > 0$ such that

$$d(f(t), T^a S_\lambda f_0) + \frac{1}{2}\|\nabla U_{f(t)} - \nabla U_{T^a S_\lambda f_0}\|_2^2 < \varepsilon$$

for every $t \geq 0$.

Proof. In the proof of Theorem 4.2, we had to consider the case where f_0 is translated in space. Now, in addition, we also have to take into account the scaling transformation given in (3.23). Comparing the definition of the function $h(t, a)$ in (4.8) with (4.2), one finds that h uses the same terms as the Hamiltonian $\mathcal{H}$. Hence, the scaling transformation is done similarly to Lemma 3.3.1 *(i)* and the computations follow. □

Remark 4.3.2. It is possible to relax the condition $f_0 \in \mathcal{F}_M$ as shown in Theorem 4 of [Guo00]. This implies that the claim holds for a broader set of f_0, specifically for those that undergo scaling transformations of f_0.

Dynamical Stability through Concentration Compactness

After deriving our first result for the dynamical stability of minimizers of the Vlasov-Poisson system in Chapter 4, we aim to achieve a stronger result in the energy space of a larger family of ground states. To establish this, we will make use of a concentration compactness argument, and the details for this will be elaborated in Section 5.2. The basis for this chapter is given in Chapter 2, 3 and 4 of [LMR08]. For $p \in [p_{crit}, \infty]$ with $p_{crit} = \frac{9}{7}$ as seen in Example 2.5.5, define the energy space

$$\mathcal{E}_p := \{f \in L^1(\mathbb{R}^6) \,:\, f \geq 0, \mathcal{C}(f) < \infty, E_{kin}(f) < \infty\}.$$

Notice that

$$\mathcal{E}_p \backslash \{0\} \subset \bigcup_{M>0} \mathcal{F}_M$$

and hence the results on the existence of minimizers remain valid for $\mathcal{E}_p$. Further, define the norm

$$\|f\|_{\mathcal{E}_p} := \|f\|_1 + \|f\|_p + \||v|^2 f\|_1.$$

This is indeed a norm as the sum of norms satisfies the norm conditions. Moreover, by Definition 2.4.1, we can rewrite

$$\mathcal{E}_p = \{f \geq 0 \,:\, \|f\|_{\mathcal{E}_p} < \infty\}.$$

We will show that in the homogeneous case, i.e. for $j(f) = f^p$ with j defined in Section 2.5, the spherically symmetric polytropes may be attained as minimizers for a suitable inequality. Before we prove this, we will fix some notations. For $p \in [p_{crit}, \infty]$, denote by U_p the radial solution to

$$-\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d}{dr} U_p) + \gamma_p(-1 - U_p)_+^{\frac{1}{p-1}+\frac{3}{2}} = 0 \tag{5.1}$$

with $U_p(r) \to 0$ as $r \to \infty$ and

$$\gamma_p = 4\pi \int_0^1 (2t)^{\frac{1}{2}} (1-t)^{\frac{1}{p-1}} \, dt.$$

The computation will be done in the proof of Theorem 5.1.4. Notice that U_p is a unique solution since by computations similar to (2.5), (5.1) is invariant under scaling. The equation (5.1) corresponds to

the Poisson equation in radial coordinates, where the density is given by the second term, c.f. Lemma 2.6.3. By the same lemma, the corresponding the spherically symmetric polytrope is given by

$$\phi_p(x,v) = \begin{cases} (-1-E)^{\frac{1}{p-1}}, & E < -1, \\ 0, & E \geq -1, \end{cases} \tag{5.2}$$

where we used as the subscript letter p instead of k, as in Definition 2.6.1, with the relation $p = 1 + \frac{1}{k}$ and moreover, $E = \frac{1}{2}|v|^2 + U_p$. Remark that in this case the cut-off energy is $E_0 = -1$.

5.1 Variational Characterization of the Polytropes

Once again, we aim to utilize minimization properties, this time involving polytropes. To achieve this, we need to formulate a variational problem and examine the functions, which turn out to be polytropes, on which it is minimized. In order to do so, we want to define a functional, which will let us describe the new minimization problem.

Definition 5.1.1. *Let $p \in (p_{crit}, \infty]$. Then we define*

$$J_p(f) = \frac{\||v|^2 f\|_1^{\frac{1}{2}} \, \|f\|_1^{\frac{7p-9}{6(p-1)}} \, \|f\|_p^{\frac{p}{3(p-1)}}}{\|\nabla U_f\|_2^2}.$$

Remark 5.1.2. In the functional of J_p one has in the numerator, up to powers and constants, the kinetic energy and some type of mass represented by the L^1- and L^p-norm of f. These quantities are divided by the potential energy up to sign. Further, we will see in Section 5.2.2 that the potential term has an upper bound. This is necessary, as otherwise J_p can become infinitely small. In the Hamiltonian $\mathcal{H}$, c.f. Definition 2.4.1, the absolute value of the potential energy was subtracted from the kinetic energy, while the information on the mass were just stored inside of f instead of giving an direct contribution to the minimization problem.

For a bounded sequence (f_n), we have already established the strong convergence of ∇U_{f_n} in Lemma 3.1.2 and Theorem 3.1.3 on $\mathcal{F}_M$. A similar result that holds on the energy space $\mathcal{E}_p$ is given by the following lemma and the proof can be found in the appendix of [LMR08].

Lemma 5.1.3. *Let $p \in (p_{crit}, \infty]$ and let (f_n) be a bounded sequence in $\mathcal{E}_p$ such that f_n is spherically symmetric in x for all n, i.e. f_n only depends on $|x|$ and v. Then there exists a function $f \in \mathcal{E}_p$ such that, up to a subsequence,*

$$\begin{aligned} f_n &\rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^6) \text{ for } p < +\infty, \\ f_n &\rightharpoonup f \text{ weak}\star \text{ in } L^\infty(\mathbb{R}^6) \text{ for } p = \infty, \\ \nabla U_{f_n} &\to \nabla U_f \text{ strongly in } L^2(\mathbb{R}^3). \end{aligned}$$

Theorem 5.1.4. *Let $p \in (p_{crit}, \infty]$. Then the minimum value of the minimization problem*

$$\mathcal{J}_p = \inf_{f\in\mathcal{E}_p,\, f\neq 0} J_p(f) = \inf_{f\in\mathcal{E}_p,\, f\neq 0} \frac{\||v|^2 f\|_1^{\frac{1}{2}}\, \|f\|_1^{\frac{7p-9}{6(p-1)}}\, \|f\|_p^{\frac{p}{3(p-1)}}}{\|\nabla U_f\|_2^2}$$

is attained exactly on the family

$$\gamma\phi_p\left(\frac{x-x_0}{\lambda}, \mu v\right),$$

where $\gamma, \lambda, \mu \in \mathbb{R}_{\geq 0}$ and $x_0 \in \mathbb{R}^3$.

Proof. The proof is divided into three steps, where we first proof the existence of the minimizers for our minimization problem and afterwards we will characterize the minimizers in the cases $p < +\infty$ and $p = +\infty$.

Step 1 - Existence of minimizers: Let $(f_n) \subset \mathcal{E}_p$ be a minimizing sequence of J_p. Further, let $\gamma, \lambda, \mu \in \mathbb{R}_{\geq 0}$ and define for $f \in \mathcal{E}_p\backslash\{0\}$ the rescaling

$$\tilde{f}(x,v) = \gamma f\left(\frac{x}{\lambda}, \mu v\right). \tag{5.3}$$

Under this rescaling, J_p will remain unchanged, i.e.

$$J_p(\tilde{f}) = J_p(f). \tag{5.4}$$

Indeed, clearly we have

and from (2.13), (2.12) and (2.14) follows that

$$\||v|^2\tilde{f}\|_1 = \frac{\gamma\lambda^3}{\mu^5}\||v|^2 f\|_1, \quad \|\tilde{f}\|_p = \gamma\left(\frac{\lambda}{\mu}\right)^{\frac{3}{p}}\|f\|_p \tag{5.5}$$

and

$$\|\nabla U_{\tilde{f}}\|_2^2 = \frac{\gamma^2\lambda^5}{\mu^6}\|\nabla U_f\|_2^2, \tag{5.6}$$

respectively. Further, by a simple computation we get

$$\begin{aligned}
\left(\frac{\gamma\lambda^3}{\mu^5}\right)^{\frac{1}{2}} &= \gamma^{\frac{1}{2}}\lambda^{\frac{3}{2}}\mu^{-\frac{5}{2}},\\
\left(\gamma\left(\frac{\lambda}{\mu}\right)^3\right)^{\frac{7p-9}{6(p-1)}} &= \gamma^{\frac{7p-9}{6(p-1)}}\lambda^{\frac{7p-9}{2(p-1)}}\mu^{-\frac{7p-9}{2(p-1)}},\\
\left(\gamma\left(\frac{\lambda}{\mu}\right)^{\frac{3}{p}}\right)^{\frac{p}{3(p-1)}} &= \gamma^{\frac{p}{3(p-1)}}\lambda^{\frac{1}{p-1}}\mu^{-\frac{1}{p-1}}
\end{aligned}$$

and moreover

$$\begin{aligned}
&\gamma^{\frac{1}{2}+\frac{7p-9}{6(p-1)}+\frac{p}{3(p-1)}-2} = \gamma^{\frac{1}{2}+\frac{3}{2}-2} = 1, \quad \lambda^{\frac{3}{2}+\frac{7p-9}{2(p-1)}+\frac{1}{p-1}-5} = \lambda^{\frac{3}{2}+\frac{7}{2}-5} = 1,\\
&\mu^{-\frac{5}{2}-\frac{7p-9}{2(p-1)}-\frac{1}{p-1}+5} = \mu^{-\frac{5}{2}-\frac{7}{2}+6} = 1.
\end{aligned}$$

Therefore, (5.4) holds. Choose

$$\gamma = \|f\|_1^{-\frac{1}{p-1}} \|f\|_p^{-\frac{p}{p-1}}, \quad \lambda = \|f\|_1^{-\frac{5p-3}{6(p-1)}} \|f\|_p^{\frac{p}{3(p-1)}} \||v|^2 f\|_1^{\frac{1}{2}},$$
$$\mu = \|f\|_1^{-\frac{1}{2}} \||v|^2 f\|_1^{\frac{1}{2}}.$$

in (5.5) to find that

$$\||v|^2 \tilde{f}\|_1 = \|f\|_1^{-\frac{1}{p-1}-\frac{5p-3}{2(p-1)}+\frac{5}{2}} \|f\|_p^{-\frac{p}{p-1}+\frac{p}{p-1}} \||v|^2 f\|_1^{\frac{3}{2}-\frac{5}{2}+1} = 1,$$
$$\|\tilde{f}\|_p = \|f\|_1^{-\frac{1}{p-1}-\frac{5p-3}{2(p-1)}+\frac{3}{2}+1} \|f\|_p^{-\frac{p}{p-1}+\frac{p}{p-1}+1} \||v|^2 f\|_1^{\frac{3}{2p}-\frac{3}{2p}} = 1.$$

Consider J_p, given in Definition 5.1.1, and notice that the first term in the numerator decreases under a non-increasing symmetric rearrangement in x by Lemma C.4 and moreover, L^q norms remain the same for $1 \leq q \leq \infty$ under such rearrangement. Therefore, for every $f \in \mathcal{E}_p$,

$$J_p(f^*) \leq J_p(f), \tag{5.7}$$

where f^* denotes the symmetric rearrangement in x. This means that J_p decreases under this rearrangement. From this follows that we can choose, without loss of generality, that the minimizing sequence (f_n) has the following properties:

$$\|f_n\|_1 = \||v|^2 f_n\|_1 = \|f_n\|_p = 1,$$
$$f_n = f_n(|x|, v).$$

By Lemma 5.1.3 we find that, up to a subsequence, there exists $f \in \mathcal{E}_p$ such that

$$f_n \rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^6) \text{ for } p < +\infty,$$
$$f_n \rightharpoonup f \text{ weak}\star \text{ in } L^\infty(\mathbb{R}^6) \text{ for } p = \infty,$$
$$\nabla U_{f_n} \to \nabla U_f \text{ strongly in } L^2(\mathbb{R}^3).$$

Notice that $f \neq 0$. Indeed, by the weak convergence as well as by the choice of f_n we have

$$\|\nabla U_f\|_2 = \lim_{n\to\infty} \|\nabla U_{f_n}\|_2 = \lim_{n\to\infty} (J_p(f_n))^{-\frac{1}{2}} = (\mathcal{J}_p)^{-\frac{1}{2}} \neq 0.$$

Furthermore, the weak convergence in combination with Fatou's Lemma (Lemma 1.2.1) implies

$$\|f_n\|_1 \leq 1, \quad \||v|^2 f_n\|_1 \leq 1, \quad \|f_n\|_p \leq 1$$

and therefore

$$J_p(f) \leq \frac{1}{\|\nabla U_f\|_2^2} = \mathcal{J}_p.$$

Since $\mathcal{J}_p$ is defined as the infimum, $J_p(f) = \mathcal{J}_p$ and therefore the minimizer is attained, i.e.

$$J_p(f) = \min_{g \in \mathcal{E}_p \setminus \{0\}} J_p(g)$$

with

$$\|f\|_1 = \||v|^2 f\|_1 = \|f\|_p = 1. \tag{5.8}$$

From (5.8) together with the weak convergence we can even conclude that for $p < \infty$, the sequence (f_n) converges strongly in L^p. This is done by the fact that $\|f\|_p = \lim_{n\to\infty} \|f_n\|_p$. Moreover, from $\||v|^\alpha f\|_1 = \lim_{n\to\infty} \||v|^\alpha f_n\|_1$ for $\alpha \in \{0, 2\}$, one can also conclude that f_n and $|v|^2 f_n$ converges strongly in L^1 to f and $|v|^2 f$, respectively.

Step 2 - Characterization of minimizers for $p < +\infty$: Applying Lemma C.4 with the same arguments as for (5.7), we find that if ϕ is a minimizer of J_p, then there exists $x_0 \in \mathbb{R}^3$ such that $\rho_\phi(x - x_0)$ is spherically symmetric. Further we recall the invariance under the rescaling given in (5.3) and set again

$$\tilde{f}(x, v) = \gamma f\left(\frac{x}{\lambda}, \mu v\right).$$

Now fix $a_1, a_2, a_3 \in \mathbb{R}_{\geq 0}$. From (5.5) and (5.6), for every $f \in \mathcal{E}_p \backslash \{0\}$ one can find a unique set $(\gamma, \lambda, \mu) \in \mathbb{R}^3_{\geq 0}$ that satisfies

$$\||v|^2 \tilde{f}\|_1 = a_1, \quad \|\tilde{f}\|_p = a_2, \quad \|\tilde{f}\|_p = a_3.$$

The set of all minimizers of J_p is described by the minimizers $\phi \in \mathcal{E}_p$ such that

$$\||v|^2 \phi\|_1 = a_1, \quad \|\phi\|_p = a_2, \quad \|\phi\|_p = a_3, \quad \rho_\phi = \rho_\phi(|x|). \tag{5.9}$$

We now aim to develop an *Euler-Lagrange* relation as introduced in Theorem 3.2.1. Since $\mathcal{E}_p$ is not an open set, we can not differentiate J_p in an arbitrary direction. Pick $\varepsilon > 0$ and define

$$K_\varepsilon := \{(x, v) \in \mathbb{R}^6 \ : \ \phi(x, v) \geq \varepsilon\}.$$

Let $g \in L^\infty(\mathbb{R}^6)$ be a compactly supported function with $g \geq 0$ a.e. on $\mathbb{R}^6 \backslash K_\varepsilon$. Observe that

$$\phi + sg \in \mathcal{E}_p \backslash \{0\} \quad \forall s \in \left(0, \|g\|_\infty^{-1} \varepsilon\right)$$

and therefore

$$J_p(\phi + sg) - J_p(\phi) \geq 0.$$

For simplification, denote

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{7p - 9}{6(p - 1)}, \quad \alpha_3 = \frac{p}{3(p - 1)}. \tag{5.10}$$

Then we can compute the half *Gâteaux derivative* (c.f. Definition C.6) of $\log J_p$ at ϕ in direction g. In this context, the term "half" is intended to signify that we are considering only the one-sided

derivative, i.e. the limit from above. We have

$$\begin{aligned}
d(\log J_p)(\phi, g) &= \lim_{s\to 0^+} \frac{\log J_p(\phi + sg) - \log J_p(\phi)}{s} \\
&= \frac{d}{ds} \log J_p(\phi + sg)\bigg|_{s=0} \\
&= \Bigg[\frac{\frac{d}{ds}\||v|^2\phi + sg\|_1^{\alpha_1}}{\||v|^2\phi + sg\|_1^{\alpha_1}} + \frac{\frac{d}{ds}\|\phi + sg\|_1^{\alpha_2}}{\|\phi + sg\|_1^{\alpha_2}} + \frac{\frac{d}{ds}\|\phi + sg\|_p^{\alpha_3}}{\|\phi + sg\|_p^{\alpha_3}} \\
&\quad - \frac{\frac{d}{ds}\|\nabla U_{\phi+sg}\|_2^2}{\|\nabla U_{\phi+sg}\|_2^2} \Bigg]_{s=0} \\
&= \alpha_1 \frac{\int |v|^2 g \, dxdv}{\int |v|^2 \phi \, dxdv} + \alpha_2 \frac{\int g \, dxdv}{\int \phi \, dxdv} + \alpha_3 \frac{\int g\phi^{p-1} \, dxdv}{\int \phi^p \, dxdv} + 2\frac{\int U_\phi g \, dxdv}{\|\nabla U_\phi\|_2^2},
\end{aligned}$$

where we used the relation given in Lemma 2.4.2 *(ii)* for the last term. By our choice of g,

$$d(\log J_p)(\phi, g) \geq 0$$

and therefore, with the use of (5.9) and Definition 5.1.1,

$$\begin{aligned}
&\int \left(\frac{\alpha_1}{a_1}|v|^2 + \frac{\alpha_2}{a_2} + \frac{\alpha_3}{a_3}\phi^{p-1} + \frac{2}{\|\nabla U_\phi\|_2^2} U_\phi \right) g \, dxdv \\
&= \int \left(\frac{\alpha_1}{a_1}|v|^2 + \frac{\alpha_2}{a_2} + \frac{\alpha_3}{a_3}\phi^{p-1} + \frac{2J_p}{a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}} U_\phi \right) g \, dxdv \geq 0.
\end{aligned}$$

We have discussed that the choice of $(a_1, a_2, a_3) \in \mathbb{R}^3_{\geq 0}$ is arbitrary, so in particular it is possible to fix a_1, a_2, a_3 such that

$$2\frac{\alpha_1}{a_1} = \frac{\alpha_2}{a_2} = \frac{\alpha_3}{a_3} = \frac{2J_p}{a_1^{\alpha_1} a_2^{\alpha_2} a_1^{\alpha_2}}.$$

Therefore we obtain

$$\int \left(\frac{1}{2}|v|^2 + 1 + \phi^{p-1} + U_\phi \right) g \, dxdv \geq 0 \tag{5.11}$$

and since $g \geq 0$ a.e. only on $\mathbb{R}^6 \backslash K_\varepsilon$, the function g on K_ε can be positive or negative. Thus, in order to make (5.11) valid in both cases, we have

$$\frac{1}{2}|v|^2 + 1 + \phi^{p-1}(x, v) + U_\phi(x) \equiv 0 \quad \forall (x, v) \in K_\varepsilon.$$

As this is true for every $\varepsilon > 0$, it holds on the support of ϕ. Outside of it's support we have

$$\frac{1}{2}|v|^2 + 1 + \phi^{p-1}(x, v) + U_\phi(x) \geq 0, \quad (x, v) \in \mathbb{R}^6 \backslash \operatorname{supp}(\phi),$$

because g has to be non-negative, since the remaining terms in (5.11) are positive. It follows that

$$\phi(x, v) = \begin{cases} (-1 - \frac{1}{2}|v|^2 - U_\phi(x))^{\frac{1}{p-1}}, & \frac{1}{2}|v|^2 + U_\phi(x) < -1, \\ 0, & \text{else}, \end{cases}$$

which corresponds to the form of polytropes given in (5.2) with $E = \frac{1}{2}|v|^2 + U_\phi(x)$. To show the existence of a unique spherically symmetric function of this form that satisfies the Poisson equation, we apply Lemma 2.6.3 using the relation $k = \frac{1}{p-1}$ and $E_0 = -1$. We obtain

$$\begin{aligned}
\rho_\phi(x) &= \left(4\pi \int_0^1 (2t)^{\frac{1}{2}} (1-t)^k \, dt \right) (-1 - U_\phi(x))_+^{k+\frac{3}{2}} \\
&=: \xi_p \, (-1 - U_\phi(x))_+^{\frac{1}{p-1}+\frac{3}{2}}.
\end{aligned} \tag{5.12}$$

By assumption, ρ_ϕ is spherically symmetric and thus the Poisson equation in (2.1) is satisfied if U_ϕ is a solution to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}U_\phi\right) = \xi_p\left(-1-U_\phi\right)_+^{\frac{1}{p-1}+\frac{3}{2}} \tag{5.13}$$

with $U_\phi(r)\to 0$ as $r\to+\infty$. Notice that the left hand side of (5.13) is the Laplacian of U_ϕ in spherical coordinates, independent of the angles. It is a ordinary differential equation which is invariant under scaling and hence the solutions are unique.

Step 3 - Characterization of minimizers for $p=+\infty$: Our first aim in this step is to prove that

$$\phi(x,v)=1, \quad (x,v)\in\operatorname{supp}(\phi).$$

Pick $\varepsilon>0$ and define

$$K_\varepsilon := \{(x,v)\in\mathbb{R}^6 \ : \ \varepsilon\le\phi(x,v)\le\|\phi\|_\infty-\varepsilon\}.$$

Similar as in step 2, we let $g\in L^\infty(\mathbb{R}^6)$ be a compactly supported function with $g\ge 0$ a.e. on $\mathbb{R}^6\backslash K_\varepsilon$ and in addition, we require that

$$g=0 \quad \text{a.e. on } \{(x,v)\in\mathbb{R}^6 \ : \phi(x,v)\ge\|\phi\|_\infty-\varepsilon\}.$$

Let $s\in\left(0,\frac{\varepsilon}{\|g\|_\infty}\right)$ and observe that

$$0\le(\phi+sg)(x,v)\le\|\phi\|_\infty \tag{5.14}$$

for a.e. $(x,v)\in\mathbb{R}^6$. On the energy space $\mathcal{E}_\infty\backslash\{0\}$, we introduce the functional

$$\tilde{J}_\infty(f) := \frac{\||v|^2f\|_1^{\alpha_1}\,\|f\|_1^{\alpha_2}}{\|\nabla U_f\|_2^2}, \tag{5.15}$$

where we did not include the term with the $L^\infty-$norm compared to Definition 5.1.1. If we choose $s>0$ small enough,

$$\begin{aligned}\|\phi+sg\|_{\mathcal{E}_p} &= \|\phi+sg\|_1+\|\phi+sg\|_\infty+\||v|^2(\phi+sg)\|_1\\ &\le \|\phi\|_1+s\|g\|_1+\|\phi\|_\infty+\||v|^2\phi\|_1+s\||v|^2g\|_1\\ &<\infty\end{aligned}$$

by (5.14) and therefore $\phi+sg\in\mathcal{E}_\infty\backslash\{0\}$. Moreover,

$$\begin{aligned}\tilde{J}_\infty(\phi) &= \frac{1}{\|\phi\|_\infty^{\alpha_3}}J_\infty(\phi)\\ &\le \frac{1}{\|\phi+sg\|_\infty^{\alpha_3}}J_\infty(\phi+sg)\\ &= \tilde{J}_\infty(\phi+sg)\end{aligned}$$

where we used the notation from (5.10). Again we compute the half *Gâteaux derivative* (c.f. Definition

C.6) with the use of Lemma 2.4.2 *(ii)* as

$$\begin{aligned} d(\log \tilde{J}_\infty)(\phi, g) &= \lim_{s\to 0^+} \frac{\log \tilde{J}_\infty(\phi + sg) - \log \tilde{J}_\infty(\phi)}{s} \\ &= \frac{d}{ds} \log \tilde{J}_\infty(\phi + sg)\Big|_{s=0} \\ &= \left[\frac{\frac{d}{ds}\||v|^2\phi + sg\|_1^{\alpha_1}}{\||v|^2\phi + sg\|_1^{\alpha_1}} + \frac{\frac{d}{ds}\|\phi + sg\|_1^{\alpha_2}}{\|\phi + sg\|_1^{\alpha_2}} - \frac{\frac{d}{ds}\|\nabla U_{\phi+sg}\|_2^2}{\|\nabla U_{\phi+sg}\|_2^2} \right]_{s=0} \\ &= \alpha_1 \frac{\int |v|^2 g\, dxdv}{\int |v|^2 \phi\, dxdv} + \alpha_2 \frac{\int g\, dxdv}{\int \phi\, dxdv} + 2\frac{\int U_\phi g\, dxdv}{\|\nabla U_\phi\|_2^2}. \end{aligned}$$

Therefore,

$$d(\log \tilde{J}_\infty)(\phi, g) \geq 0$$

and by (5.9),

$$\begin{aligned} &\int \left(\frac{\alpha_1}{a_1}|v|^2 + \frac{\alpha_2}{a_2} + \frac{2}{\|\nabla U_\phi\|_2^2} U_\phi \right) g\, dxdv \\ &= \int \left(\frac{\alpha_1}{a_1}|v|^2 + \frac{\alpha_2}{a_2} + \frac{2J_p}{a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}} U_\phi \right) g\, dxdv \geq 0. \end{aligned}$$

Set a_1, a_2, a_3 such that

$$2\frac{\alpha_1}{a_1} = \frac{\alpha_2}{a_2} = \frac{2J_p}{a_1^{\alpha_1} a_2^{\alpha_2} a_1^{\alpha_2}}, \quad a_3 = \|\phi\|_\infty = 1$$

to obtain

$$\int \left(1 + \frac{1}{2}|v|^2 + U_\phi \right) g\, dxdv \geq 0. \tag{5.16}$$

The function g on K_ε can be positive or negative. For the validity of (5.16), we have

$$1 + \frac{1}{2}|v|^2 + U_\phi(x) = 0 \quad \text{for a.e. } (x, v) \in K_\varepsilon$$

for every $\varepsilon > 0$. Hence, by definition of K_ε, the measure λ is

$$\lambda(\{(x, v) \in \mathbb{R}^6 \ : \ 0 < \phi(x, v) < 1\}) = 0, \tag{5.17}$$

meaning that $\phi(x, v) = 1$ a.e. Denote by $\phi^{*,v}$ the non-increasing symmetric rearrangement of ϕ with respect to v, i.e. $\phi^{*,v}(x, \cdot) = (f(x, \cdot))^*$ for some parameter x. Since ϕ is a minimizer of J_∞, one can argue with Lemma C.4 as for (5.7) to conclude that $\phi^{*,v} = \phi$, i.e. $\phi(x, v) = \phi(x, |v|)$ is spherically symmetric and decreasing with respect to v and ϕ. As a consequence of (5.17), we have that either $\phi = 0$ or $\phi = 1$. Further, we can consider a sphere, whose volume is ρ_ϕ. Using spherical coordinates, we can integrate ϕ with respect to v to find

$$\phi(x, v) = \begin{cases} 1, & |v| \leq \left(\frac{3}{4\pi}\rho_\phi(x)\right)^{\frac{1}{3}}, \\ 0, & \text{else.} \end{cases} \tag{5.18}$$

With this dependence on the density ρ_ϕ, we want to rewrite the minimization problem such that we can minimize over ρ_ϕ instead of f. To this end, define $f[\rho]$ by

$$f[\rho](x, v) = \begin{cases} 1, & |v| \leq \left(\frac{3}{4\pi}\rho(x)\right)^{\frac{1}{3}}, \\ 0, & \text{else} \end{cases}$$

for any $\rho \in L^1 \cap L^{\frac{5}{3}}(\mathbb{R}^3)$. Using spherical coordinates in the variable v, one can calculate

$$\begin{aligned}\|f[\rho]\|_1 &= 4\pi \int \int_0^{\left(\frac{3}{4\pi}\rho(x)\right)^{\frac{1}{3}}} r_v^2 \, dr_v dx \\ &= \frac{4\pi}{3} \int \frac{3}{4\pi}\rho(x) \, dx \\ &= \|\rho\|_1 \end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\||v|^2 f[\rho]\|_1 &= 4\pi \int \int_0^{\left(\frac{3}{4\pi}\rho(x)\right)^{\frac{1}{3}}} r_v^4 \, dr_v dx \\ &= \frac{4\pi}{5} \int \left(\frac{3}{4\pi}\rho(x)\right)^{\frac{5}{3}} \, dx \\ &= \frac{3}{5}\left(\frac{4\pi}{3}\right)^{1-\frac{5}{3}} \int (\rho(x))^{\frac{5}{3}} \, dx \\ &= \frac{3}{5}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \|\rho\|_{\frac{5}{3}}^{\frac{5}{3}}. \end{aligned} \tag{5.20}$$

Then, for $\rho \neq 0$, define the functional

$$I(\rho) := C_0 \frac{\|\rho\|_{\frac{5}{3}}^{\frac{5}{3}\alpha_1} \|\rho\|_1^{\alpha_2}}{\|\nabla U_{f[\rho]}\|_2^2}$$

with

$$C_0 = \left(\frac{3}{5}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}\right)^{\alpha_1}.$$

Inserting (5.19) and (5.20) into (5.15), it follows that

$$I(\rho) = \tilde{J}_\infty(f[\rho]) = J_\infty(f[\rho]) \geq \mathcal{J}_\infty,$$

where we also used the fact that $\|f[\rho]\|_\infty = 1$. By the existence of minimizers and since $\phi = f[\rho_\phi]$ by (5.18),

$$I(\rho_\phi) = \mathcal{J}_\infty = \min_{\rho \in L^1 \cap L^{\frac{5}{3}}(\mathbb{R}^3)\setminus\{0\}} I(\rho).$$

Therefore, ρ_ϕ is a minimizer of I that is spherically symmetric in x. To find an expression for ρ_ϕ for $p = \infty$ analogous to (5.12), observe that

$$\begin{aligned}\frac{1}{4\pi}\xi_\infty &= \int_0^1 (2t)^{\frac{1}{2}} \, (1-t)^0 \, dt \\ &= \int_0^1 (2t)^{\frac{1}{2}} \, dt \\ &= \frac{2^{\frac{3}{2}}}{3}. \end{aligned}$$

Thus

$$\rho_\phi(x) = \frac{2^{\frac{7}{2}}\pi}{3}(-1 - U_\phi(x))_+^{\frac{3}{2}}.$$

Again, due to the spherically symmetry of ρ_ϕ, the Poisson equation in (2.1) is satisfied if U_ϕ is a solution to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}U_\phi\right) = \frac{2^{\frac{7}{2}}\pi}{3}\left(-1-U_\phi\right)_+^{\frac{3}{2}} \tag{5.21}$$

with $U_\phi(r) \to 0$ as $r \to +\infty$. Because of the scaling invariance, the solutions to this ordinary differential equation are unique. If we insert (5.21) into the condition for $\phi(x,v) = 1$ in (5.18) we have

$$|v| \leq \left(2^{\frac{3}{2}}(-1-U_\phi)_+^{\frac{3}{2}}\right)^{\frac{1}{3}}$$

and thus

$$\frac{1}{2}|v|^2 \leq (-1-U_\phi)_+.$$

Using $E = \frac{1}{2}|v|^2 + U_\phi$, this coincides with the definition of polytropes given in (5.2) for $p = \infty$. □

5.2 Concentration Compactness Argument

In this section, we will first state and explain the standard concentration compactness Lemma. It is one of the methods in the calculus of variations. An insight in different variational methods is provided by [Str08]. Afterwards we will utilize it to analyze the characteristics of the potential energy within each regime.

Often we will only consider positive L^1 functions and therefore we define the set $L^1_+(\Omega)$ for some $\Omega \subseteq \mathbb{R}^N$ by

$$L^1_+(\Omega) := \{f \in L^1(\Omega) \,:\, f \geq 0\}.$$

5.2.1 Concentration Compactness Lemma

The following lemma is adapted from Section I.3. in [Lio84], and its proof is given in Appendix D. It is the standard concentration compactness lemma, which states that in a bounded sequence of functions, in our case a sequence of spatial mass densities, there exists a subsequence that has either compactness properties, vanishes or is splitted into two states.

Lemma 5.2.1. *Let $M > 0$ and let $(\rho_n) \subset L^1(\mathbb{R}^3)$ be a sequence of non-negative functions such that*

$$\int_{\mathbb{R}^3} \rho_n(x)\,dx = M.$$

Then there exists a subsequence (ρ_{n_k}) such that one of the following possibilities occurs:

(i) Compactness: There are $y_k \in \mathbb{R}^3$ such that for every $\varepsilon > 0$, there exists $R < +\infty$ such that for all $k \geq 1$,

$$\int_{y_k + B_R} \rho_{n_k}(x)\,dx \geq M - \varepsilon.$$

(ii) Vanishing: For every $R < +\infty$,

$$\lim_{k\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{y+B_R} \rho_{n_k}(x)\,dx = 0.$$

(iii) Dichotomy: There is $m \in (0, M)$ such that for every $\varepsilon > 0$, there exist subsequences $(\rho_k^1), (\rho_k^2) \subset L^1_+(\mathbb{R}^3)$ and $k_0 \geq 1$ such that for all $k \geq k_0$,

$$\begin{cases} \rho_{n_k} = \rho_k^1 + \rho_k^2 + w_k, \quad \rho_k^1, \rho_k^2, w_k \in [0, \rho_{n_k}], \\ \rho_k^1\rho_k^2 = \rho_k^1 w_k = \rho_k^2 w_k = 0 \quad a.e., \\ \operatorname{dist}(\operatorname{supp}(\rho_k^1), \operatorname{supp}(\rho_k^2)) \to +\infty \quad as\ k \to +\infty, \\ \|\rho_{n_k} - \rho_k^1 - \rho_k^2\|_1 \leq \varepsilon, \\ \left|\int \rho_k^1(x)\,dx - m\right| + \left|\int \rho_k^2(x) - (M-m)\right| < \varepsilon. \end{cases}$$

Remark 5.2.2. Let (ρ_n) be a sequence as in Lemma 5.2.1. To gain a better understanding, we will briefly explain the meaning of the three possibilities.

(i) Compactness implies that there is a subsequence (ρ_{n_k}) that contains almost all of the mass somewhere in x-space.

(ii) In the vanishing case, there is a subsequence (ρ_{n_k}) that converges to zero, meaning that the mass density disappears for any ball in space, i.e. for any $B_R(y)$ with $y \in \mathbb{R}^3$ and $R < +\infty$.

(iii) If we have dichotomy, there are two distinct states, represented by the subsequences (ρ_k^1) and (ρ_k^2), that contain almost all of the spatial mass. The parameter m indicates the distribution of the mass density between these states.

Remark 5.2.3. It is possible to rewrite the dichotomy property in terms of the distribution function instead of the spatial mass density. Indeed, let $(f_n) \subset L^1_+(\mathbb{R}^6)$ be a sequence of distribution functions such that

$$\rho_n(x) = \int f_n(x, v)\,dv$$

with (ρ_n) as in Lemma 5.2.1. If *(iii)* holds for (ρ_n), define

$$f_k^i := f_{n_k} \mathbb{1}_{\operatorname{supp}(\rho_k^i)}$$

for $i = 1, 2$, where (ρ_k^i) is defined in Lemma 5.2.1 *(iii)* for $i = 1, 2$. Then for all $k \geq k_0$ we have

$$\begin{cases} f_{n_k} = f_k^1 + f_k^2 + u_k, \quad f_k^1, f_k^2, u_k \in [0, f_{n_k}], \\ f_k^1 f_k^2 = f_k^1 u_k = f_k^2 u_k = 0 \quad \text{a.e.}, \\ \operatorname{dist}(\operatorname{supp}(f_k^1), \operatorname{supp}(f_k^2)) \to +\infty \quad \text{as } k \to +\infty, \\ \|f_{n_k} - f_k^1 - f_k^2\|_1 \leq \varepsilon, \\ \left|\|f_k^1(x)\|_1 - m\right| + \left|\|f_k^2(x)\|_1 - (M-m)\right| < \varepsilon. \end{cases}$$

5.2.2 Control of the Potential Energy

In order to describe the control of the potential energy, we introduce an expression for the mass fraction m as seen in Lemma 5.2.1.

Definition 5.2.4. *Let $(\rho_n) \subset L^1_+(\mathbb{R}^3)$ be a sequence as in Lemma 5.2.1. Moreover, let $R \geq 0$. Then the concentration function of measure is defined by*

$$\Gamma_n(R) = \sup_{y \in \mathbb{R}^3} \int_{|x-y|<R} \rho_n(x)\, dx.$$

Let $(\rho_n) \subset L^1_+(\mathbb{R}^3)$ be a sequence as in Lemma 5.2.1. For each $R \in \mathbb{Q}_+$, we find a subsequence $(\Gamma_{n_k}(R))$ that converges by *Banach-Alaoglu* (Theorem 1.3.3). Then by Cantor's diagonal argument, there exists a subsequence (Γ_{n_k}) and a non-negative function Γ such that $\Gamma_{n_k}(R) \to \Gamma(R)$ as $n \to +\infty$ for every $R \in \mathbb{Q}_+$ and by continuity, this is valid for every $R \in \mathbb{R}_+$. Furthermore, this implies that there is a subsequence (ρ_{n_k}) such that $\Gamma_{n_k}(R) \to \Gamma(R)$ as $n \to +\infty$ for every $R \geq 0$. Notice that Γ is non-decreasing since for a larger R, the integration domain becomes larger. For such a function Γ we therefore define the mass fraction m as follows:

Definition 5.2.5. *Let $R > 0$ Then the mass fraction m is defined by*

$$m := \lim_{R \to +\infty} \Gamma(R).$$

Remark 5.2.6. Notice that $m \in [0, M]$ and moreover, the possibilities *(i), (ii)* and *(iii)* in Lemma 5.2.1 correspond to the cases $m = M$, $m = 0$ and $m \in (0, M)$, respectively.

Recall by (2.4) that the Poisson force field for a time-independent distribution function f and the corresponding density function ρ is given by

$$G_f = \frac{1}{4\pi} \frac{x}{|x|^3} * \rho = \frac{1}{4\pi} \int \frac{(x-y)\,\rho\,(y)}{|x-y|^3} dy$$

with the relation $\nabla U_f = G_f$. We are now ready to state and proof the result about the control of the potential energy.

Lemma 5.2.7. *Let $M > 0$ and let $(f_n) \subset L^1_+(\mathbb{R}^6)$ be a sequence. Denote by (ρ_n) the corresponding sequence of spatial mass densities. Suppose that*

$$\int_{\mathbb{R}^3} \rho_n(x)\, dx = M$$

and

$$\sup_{n \in \mathbb{N}} \iint |v| f_n(x)\, dxdv < \infty.$$

Moreover, assume that there exists $p \in (p_{crit}, \infty]$ such that

$$\limsup_{n \to +\infty} \|f_n\|_p < \infty.$$

Let ρ_{n_k} be a subsequence of ρ_n that satisfies one of the three possibilities in Lemma 5.2.1 and denote by

$$T_k f_{n_k}(x,v) = f_{n_k}(x + y_k, v)$$

a sequence of translations, where $(y_k) \subset \mathbb{R}^3$ is a sequence. Then it follows one of the following:

(i) In the case of compactness, up to a subsequence

$$T_k f_{n_k} \rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^6) \text{ for } p < +\infty,$$
$$T_k f_{n_k} \rightharpoonup f \text{ weak-}* \text{ in } L^\infty(\mathbb{R}^6) \text{ for } p = \infty.$$

For $p < \infty$ as well as in the case for $p = \infty$ we have that $f \in \mathcal{E}_p$ and moreover,

$$\int_{\mathbb{R}^6} f = M,$$
$$\nabla U_{T_k f_{n_k}} \to \nabla U_f \text{ strongly in } L^2(\mathbb{R}^3).$$

(ii) In the case of vanishing,

$$\|\nabla U_{f_{n_k}}\|_2 \to 0.$$

(iii) In the case of dichotomy,

$$\limsup_{k\to+\infty} \left| \|\nabla U_{f_{n_k}}\|_2^2 - \|\nabla U_{f_k^1}\|_2^2 - \|\nabla U_{f_k^2}\|_2^2 \right| < C\varepsilon^{\frac{2(p-p_{crit})}{3(p-1)}}$$

with $C > 0$.

Proof. Pick $p \in (p_{crit}, \infty]$ and let

$$q = \frac{5p-3}{3p-1}. \tag{5.22}$$

Then, since $p_{crit} = \frac{9}{7}$,

$$q > \frac{5\frac{9}{7} - 3}{3\frac{9}{7} - 1} = \frac{6}{5}.$$

As a consequence of interpolation (c.f. Proposition B.1), we find that

$$\|\rho\|_q \leq \|f\|_p^{\frac{2p}{5p-3}} \|f\|_1^{\frac{3p-3}{5p-3}} \leq C \|f\|_p^{\frac{2p}{5p-3}} \||v|^2 f\|_1^{\frac{3p-3}{5p-3}}.$$

By assumption,

$$\int_{\mathbb{R}^3} \rho_n(x)\, dx = M$$

and since

$$\limsup_{n\to+\infty} \|f_n\|_p < \infty,$$

we conclude that for every $r \in [1, q]$,

$$\limsup_{n \to +\infty} \|\rho_n\|_r < \infty. \tag{5.23}$$

Similar to Lemma 2.7.1 *(iii)*, it is possible to control the potential energy with terms used in Definition 5.1.1 by

$$\|\nabla U\|_2^2 \leq C \, \||v|^2 f\|_1^{\frac{1}{2}} \, \|f\|_1^{\frac{7p-9}{6(p-1)}} \, \|f\|_p^{\frac{p}{3(p-1)}}.$$

We will now first show *(ii)* and *(iii)* and then utilize intermediate results to establish (i).

(ii) Vanishing: Suppose that vanishing occurs, i.e. Lemma 5.2.1 *(ii)* is valid. Here, we will make use of a splitting method similar to the one we used in the proof of Theorem 3.2.5 for the potential. Let $\varepsilon > 0$. Then, for $\eta, R > 0$ such that $\eta < R$, we have

$$\begin{aligned} 4\pi \|\nabla U_{f_{n_k}}\|_2^2 &= \iint_{|x-y|<\eta} \frac{\rho_{n_k}(x)\rho_{n_k}(y)}{|x-y|} \, dxdy + \iint_{\eta \leq |x-y| < R} \frac{\rho_{n_k}(x)\rho_{n_k}(y)}{|x-y|} \, dxdy \\ &\quad + \iint_{|x-y| \geq R} \frac{\rho_{n_k}(x)\rho_{n_k}(y)}{|x-y|} \, dxdy \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{5.24}$$

For the first term we define the kernel function $K_\eta(x) := \frac{1}{|x|} \mathbb{1}_{\{|x| \leq \eta\}}$ in order to rewrite I_1 as a convolution (c.f. Definition 1.1.4). Let $r = \frac{q}{2q-2}$ with q defined as in (5.22). Then $\frac{2}{q} + \frac{1}{r} = 2$ and we can apply *Young's Inequality* (Theorem 1.2.5) to find

$$\begin{aligned} I_1 &= \left| \int_{\mathbb{R}^3} \rho_{n_k}(x) \left(K_\eta * \rho_{n_k}\right)(x) \, dx \right| \\ &\leq C \|\rho_{n_k}\|_q^2 \|K_\eta\|_r. \end{aligned}$$

The term $\|\rho_{n_k}\|_q$ is bounded and $\|K_\eta\|_r$ can be calculated using spherical coordinates by

$$\begin{aligned} \|K_\eta\|_r^r &= \int_{|x| \leq \eta} \frac{1}{|x|^r} \, dx \\ &= C \int_0^\eta s^{2-r} \, ds \\ &= C\eta^{3-r} \end{aligned}$$

since $r < 3$. Hence,

$$I_1 \leq C\eta^{\frac{3-r}{r}} < \varepsilon \tag{5.25}$$

for η small enough. The second term can be estimated as

$$\begin{aligned} I_2 &\leq \frac{1}{\eta} \int \rho_{n_k}(y) \left(\int_{|x-y| \leq R} \rho_{n_k}(x) \, dx \right) dy \\ &\leq \frac{M}{\eta} \sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_{n_k}(x) \, dx \end{aligned}$$

and thus, using *(ii)* in Lemma 5.2.1, we have $I_2 \to 0$ as $k \to \infty$. From the assumption it follows directly that

$$I_3 \leq \frac{M^2}{R} < \varepsilon, \tag{5.26}$$

if we choose R large enough. Therefore,

$$\|\nabla U_{f_{n_k}}\|_2^2 \leq C\varepsilon \longrightarrow 0$$

for $\varepsilon \to 0$, which finishes the proof in the vanishing case.

(iii) Dichotomy: Suppose that dichotomy occurs, i.e. Lemma 5.2.1 *(iii)* is valid. Let $\varepsilon > 0$. We split the density function according to $\rho_{n_k} = \rho_k^1 + \rho_k^2 + w_k$ and find, using similar techniques as in (3.9)-(3.10),

$$\begin{aligned}
&4\pi \left(\|\nabla U_{f_{n_k}}\|_2^2 - \|\nabla U_{f_k^1}\|_2^2 - \|\nabla U_{f_k^2}\|_2^2 \right) \\
&= \iint \frac{(\rho_k^1(x) + \rho_k^2(x) + w_k(x))(\rho_k^1(y) + \rho_k^2(y) + w_k(y))}{|x-y|}\, dxdy \\
&\quad - \iint \frac{\rho_k^1(x)\rho_k^1(y)}{|x-y|}\, dxdy - \iint \frac{\rho_k^2(x)\rho_k^2(y)}{|x-y|}\, dxdy \\
&= 2\iint \frac{\rho_k^1(x)\rho_k^2(y)}{|x-y|}\, dxdy + 2\iint \frac{\rho_{n_k}(x)w(y)}{|x-y|}\, dxdy - \iint \frac{w(x)w(y)}{|x-y|}\, dxdy.
\end{aligned} \tag{5.27}$$

As a property of the dichotomy case, the distance between the supports of ρ_k^1 and ρ_k^2 diverges. Hence, the first term in (5.27) vanishes as

$$\int \frac{\rho_k^1(x)\rho_k^2(y)}{|x-y|}\, dxdy \leq \frac{M^2}{\operatorname{dist}(\operatorname{supp}(\rho_k^1), \operatorname{supp}(\rho_k^2))} \longrightarrow 0$$

for $k \to \infty$. The other terms in (5.27) can by estimated via *Hardy-Littlewood-Sobolev Inequality* (Theorem 1.2.4) as seen in Lemma 2.7.1 *(iii)* (c.f. (2.11)). This yields

$$\left| \iint \frac{\rho_{n_k}(x)w(y)}{|x-y|}\, dxdy \right| \leq C\|\rho_{n_k}\|_{\frac{6}{5}} \|w_k\|_{\frac{6}{5}}$$

and

$$\left| \iint \frac{w(x)w(y)}{|x-y|}\, dxdy \right| \leq C\|w_k\|_{\frac{6}{5}}^2.$$

Therefore, using the upper bound in (5.23) and interpolation (c.f. Proposition B.1) with the use of the fact that $q > \frac{6}{5}$, we obtain

$$\begin{aligned}
\limsup_{k\to+\infty} \left| \|\nabla U_{f_{n_k}}\|_2^2 - \|\nabla U_{f_k^1}\|_2^2 - \|\nabla U_{f_k^2}\|_2^2 \right| &\leq \left(\|\rho_{n_k}\|_{\frac{6}{5}} - \|w_k\|_{\frac{6}{5}} \right) \|w_k\|_{\frac{6}{5}} \\
&\leq C\|w_k\|_{\frac{6}{5}} \\
&\leq C\|w_k\|_1^{\frac{2(p-p_{crit})}{3(p-1)}} \\
&< C\varepsilon^{\frac{2(p-p_{crit})}{3(p-1)}},
\end{aligned}$$

where the last step is given by Lemma 5.2.1 *(iii)*.

(i) Compactness: Suppose that compactness occurs, i.e. Lemma 5.2.1 *(i)* is valid. Let $\varepsilon > 0$ and consider the sequence $T_k f_{n_k}(x,v) = f_{n_k}(x + y_k, v)$ with y_k as in Lemma 5.2.1 *(i)*, which moreover implies, using the complement of the set, that there exists $R_1 > 0$ such that

$$\int_{\mathbb{R}^3} \int_{|x|>R_1} T_k f_{n_k}(x,v)\, dxdy < \frac{\varepsilon}{2}.$$

By assumption, the term that corresponds to the kinetic energy is bounded and therefore for $R_2 > 0$ large enough,

$$\int_{|v|>R_2} \int_{\mathbb{R}^3} T_k f_{n_k}(x,v)\, dxdv \leq \frac{1}{R_2^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 T_k f_{n_k}(x,v)\, dxdv < \frac{\varepsilon}{2}.$$

If we combine these results, we get

$$\int_{|v|>R_2} \int_{|x|>R_1} T_k f_{n_k}(x,v)\, dxdy < \varepsilon.$$

We claim that the sequence is equiintegrable. If it is not, then there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is a measurable set A with $|A| < \delta$ such that

$$\int_A T_k f n_k\, dxdv \geq \varepsilon_0 \tag{5.28}$$

by Definition B.3. By assumption, the L^p norm of f_n is uniformly bounded and thus the L^p norm of $T_k f_{n_k}$ is bounded as well. But then, using *Hölder's Inequality* (Theorem 1.2.2) with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, this implies

$$\int_A T_k f n_k\, dxdv \leq \|T_k f n_k\|_p \|\mathbb{1}_A\|_{p'} < C\delta^{\frac{1}{p'}},$$

which is a contradiction to (5.28) for δ small enough. Hence, the sequence is equiintegrable, which allows us to apply *Dunford-Pettis Theorem* (Theorem C.7). This in turn implies that $(T_k f_{n_k})$ is a relatively compact subset of L^1 in the weak topology, meaning that there is $f \in \mathcal{E}_p$ such that $T_k f_{n_k} \rightharpoonup f$ in L^1 with

$$\iint f(x,v)\, dxdv = M. \tag{5.29}$$

By the uniform bound of the L^p norm of f_n and as a consequence of *Hölder's Inequality* (Theorem 1.2.2), the weak convergence is also valid in L^p for $p < \infty$ and we have weak-$*$ convergence in L^∞ for $p = \infty$. For this f we have

$$\begin{aligned} 4\pi \|\nabla U_{T_k f_{n_k}} - \nabla U_f\|_2^2 &= \iint \frac{(\rho_{T_k f_{n_k}}(x) - \rho_f(x))(\rho_{T_k f_{n_k}}(y) - \rho_f(y))}{|x-y|}\, dxdy \\ &= \iint_{|x-y|<\eta} [...]\, dxdy + \iint_{\eta \leq |x-y| < R} [...]\, dxdy + \iint_{|x-y|\geq R} [...]\, dxdy \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where we split the integral again with the same boundaries as in (5.24). I_1 and I_3 can be estimated analogously to the vanishing case *(ii)* and we get

$$I_1 \leq C\eta^{\frac{3-r}{r}},$$

$$I_3 \leq C\frac{1}{R}$$

from (5.25) and (5.26), respectively. To estimate I_2, define the function

$$g_{\eta,R}(x) := \frac{1}{|x|}\mathbb{1}_{\eta\leq|x-y|<R}$$

and consider

$$u_{n_k}(x) := \int_{\mathbb{R}^3}(\rho_{T_k f_{n_k}}(y) - \rho_f(y))g_{\eta,R}(x-y)\,dy.$$

Clearly, $g_{\eta,R} \in L^\infty$ and therefore by weak convergence (Definition 1.1.6),

$$\int_{\mathbb{R}^3}\rho_{T_k f_{n_k}}(y)g_{\eta,R}(x-y)\,dy \longrightarrow \int_{\mathbb{R}^3}\rho_f(y)g_{\eta,R}(x-y)\,dy$$

as $k\to\infty$ and therefore $u_{n_k}(x)\to 0$ for every $x\in\mathbb{R}^3$. From the assumptions in Lemma 5.2.1 and (5.29), the value of the integral remains the same while increasing k, meaning that there is no loss in mass, and thus

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho_{T_k f_{n_k}}(y)g_{\eta,R}(x-y)\,dxdy &= \left(\int_{\mathbb{R}^3}\rho_{T_k f_{n_k}}(y)\,dy\right)\left(\int_{\mathbb{R}^3}g_{\eta,R}(x)\,dx\right)\\ &= \left(\int_{\mathbb{R}^3}\rho_f(y)\,dy\right)\left(\int_{\mathbb{R}^3}g_{\eta,R}(x)\,dx\right)\\ &= \int_{\mathbb{R}^3}\rho_f(y)g_{\eta,R}(x-y)\,dy.\end{aligned}$$

Hence,

$$\|u_{n_k}\|_p = \left(\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}(\rho_{T_k f_{n_k}}(y)-\rho_f(y))g_{\eta,R}(x-y)\,dy\right|^p dx\right)^{\frac{1}{p}} \longrightarrow 0$$

as $k\to\infty$ and therefore $u_{n_k}\to 0$ in L^p for every $p\in[1,\infty)$. Combining this with (5.23) and *Hölder's Inequality* (Theorem 1.2.2), we get

$$\int_{\mathbb{R}^3}\rho_{T_k f_{n_k}}(x)u_{n_k}(x)\,dx \leq C\|u_{n_k}\|_p \longrightarrow 0,$$

as $k\to\infty$, where p is the conjugate exponent to r in (5.23). Similarly,

$$\int_{\mathbb{R}^3}\rho_f(x)u_{n_k}(x)\,dx \longrightarrow 0$$

as $k\to\infty$ and it follows that I_2 becomes arbitrarily small. Therefore, as $k\to\infty$, we conclude that

$$4\pi\|\nabla U_{T_k f_{n_k}} - \nabla U_f\|_2^2 = I_1 + I_2 + I_3 \longrightarrow 0,$$

which shows the L^2 convergence for ∇U if compactness occurs. □

5.3 Dynamical Stability

The majority of the work in this section will be devoted to proving the compactness of minimizing sequences. In order to do so, we will use the standard conditions for j, as given in Section 2.5 and consider the minimization problem

$$h_{M_1,M_j} = \inf_{f\in\mathcal{F}_{M_1,M_j}}\mathcal{H}(f)$$

as seen in Remark 2.7.6. In the case of $p = \infty$, we will consider the minimization problem

$$h_{M_1,M_\infty} = \inf_{\|f\|_1=M_1,\ \|f\|_\infty=M_\infty} \mathcal{H}(f)$$

as seen in Remark 2.7.7. With the results from Section 5.2, we can construct a sequence that converges to a minimizer within the energy space $\mathcal{E}_p$.

Theorem 5.3.1. *For a sequence $(y_n) \subset \mathbb{R}^3$ and for any $f \in \mathcal{E}_p$, let*

$$T_n f(x,v) := f(x + y_n, v).$$

(i) Let $p_{crit} < p < \infty$, $M_1, M_j > 0$ and assume that j is strictly convex. Suppose that the nondichotomy condition holds, i.e. for every $0 < \alpha < 1$ and for every $0 \leq \beta \leq 1$,

$$h_{M_1,M_j} < h_{\alpha M_1,\beta M_j} + h_{(1-\alpha)M_1,(1-\beta)M_j}.$$

Then for every minimizing sequence (f_n) of h_{M_1,M_j}, there exists $(y_n) \subset \mathbb{R}^3$ such that $T_n f_n$ is relatively compact in $\mathcal{E}_p$. Moreover,

$$T_n f_n \to f \text{ in } \mathcal{E}_p,$$

where f is a minimizer.

(ii) Let $M_1, M_\infty > 0$ and let $(f_n) \subset \mathcal{E}_\infty$ be a minimizing sequence of h_{M_1,M_∞}. Then there exists $(y_n) \subset \mathbb{R}^3$ such that, up to a subsequence,

$$T_n f_n \rightharpoonup f \text{ weakly in } L^1(\mathbb{R}^6),$$
$$T_n f_n \rightharpoonup f \text{ weak-}* \text{ in } L^\infty(\mathbb{R}^6),$$

where f is a minimizer given by Theorem 5.1.4.

Proof. This proof is structured in several steps, where we first exclude the cases of vanishing and dichotomy, in order to establish compactness using Lemma 5.2.1. Subsequently, we can apply Lemma 5.2.7 and prove the statements. In the proof, we distinguish between the cases *(i)* and *(ii)*.
For *(i)*, let (f_n) be a minimizing sequence of

$$h_{M_1,M_j} = \inf_{f \in \mathcal{F}_{M_1,M_j}} \mathcal{H}(f),$$

and similarly for *(ii)*, let (f_n) be a minimizing sequence of

$$h_{M_1,M_\infty} = \inf_{\|f\|_1=M_1,\ \|f\|_\infty=M_\infty} \mathcal{H}(f).$$

From Lemma 2.7.1 *(iii)* it follows that

$$\|\nabla U_{f_n}\|_2^2 \leq C \, \||v|^2 f_n\|_1^{\frac{1}{2}} \, \|f_n\|_1^{\frac{7p-9}{6(p-1)}} \, \|f_n\|_p^{\frac{p}{3(p-1)}}$$

for $p \in [p_{crit}, \infty]$. Hence, the potential energy is bounded and since $\mathcal{H}(f_n)$ is bounded, the kinetic term has to be bounded, too. Thus

$$\limsup_{n\to\infty} \|f_n\|_{\mathcal{E}_p} < \infty.$$

Step 1 - Vanishing cannot occur: In the case of *(i)* we have

$$\begin{aligned} -E_{pot}(f_n) &= -\mathcal{H}(f_n) + E_{kin}(f_n) \\ &\geq -h_{M_1,M_j} + E_{kin}(f_n) \\ &\geq -h_{M_1,M_j} \end{aligned}$$

and thus by definition of the potential energy, given in Definition 2.4.1,

$$\liminf_{n\to\infty} \|\nabla U_{f_n}\|_2^2 \geq -2h_{M_1,M_j} > 0, \tag{5.30}$$

where we used Lemma 2.7.4. If vanishing occurs, by Lemma 5.2.7 *(ii)* there is a subsequence (f_{n_k}) such that

$$\|\nabla U_{f_{n_k}}\|_2 \to 0,$$

which contradicts (5.30). The proof for *(ii)* is done in the same way.

Step 2 - Dichotomy cannot occur: Suppose by contradiction that the case of dichotomy is valid, i.e. Lemma 5.2.1 *(iii)* holds. Let $\varepsilon > 0$. Then there is a subsequence (f_{n_k}) that satisfies (5.2.3). Without loss of generality, choose f_k^1, f_k^2 such that for $1 < \alpha < 1$,

$$\begin{aligned} \lim_{k\to\infty} \|f_k^1\|_1 &= \alpha M_1, \\ \lim_{k\to\infty} \|f_k^2\|_1 &= (1-\alpha) M_1. \end{aligned}$$

(i) As the supports of f_k^1, f_k^2 are distinct, i.e.

$$\operatorname{dist}(\operatorname{supp}(f_k^1), \operatorname{supp}(f_k^2)) \to +\infty \quad \text{as } k \to +\infty,$$

one can choose M_j^1, M_j^2 with

$$M_j^1 + M_j^2 \leq M_j \tag{5.31}$$

such that

$$\begin{aligned} \lim_{k\to\infty} \|j(f_k^1)\|_1 &= M_j^1, \\ \lim_{k\to\infty} \|j(f_k^2)\|_1 &= M_j^2. \end{aligned}$$

Since $f_{n_k} = f_k^1 + f_k^2 + u_k$, we can deduce that

$$\begin{aligned} E_{kin}(f_{n_k}) &= \frac{1}{2} \iint |v|^2 f_{n_k}(x,v)\, dxdv \\ &\geq \frac{1}{2} \left(\iint |v|^2 f_k^1(x,v)\, dxdv + \iint |v|^2 f_k^2(x,v)\, dxdv \right) \\ &= E_{kin}(f_k^1) + E_{kin}(f_k^2). \end{aligned}$$

Combining this with Lemma 5.2.7, we find that

$$\begin{aligned}
h_{M_1,M_j} &= \inf_{f\in\mathcal{F}_{M_1,M_j}} E_{kin}(f)+E_{pot}(f)\\
&\geq \limsup_{k\to\infty} E_{kin}(f_k^1)+E_{pot}(f_k^1)+\limsup_{k\to\infty} E_{kin}(f_k^2)+E_{pot}(f_k^2)\\
&\quad-\limsup_{k\to\infty}\left|E_{pot}(f_{n_k})-E_{pot}(f_k^1)-E_{pot}(f_k^2)\right|\\
&= \limsup_{k\to\infty}\mathcal{H}(f_k^1)+\limsup_{k\to\infty}\mathcal{H}(f_k^2)-\frac{1}{2}\limsup_{k\to\infty}\left|\|\nabla U_{f_{n_k}}\|_2^2-\|\nabla U_{f_k^1}\|_2^2-\|\nabla U_{f_k^2}\|_2^2\right|\\
&\geq \limsup_{k\to\infty}\mathcal{H}(f_k^1)+\limsup_{k\to\infty}\mathcal{H}(f_k^2)-C\varepsilon^\theta
\end{aligned}$$

with $\theta=\frac{2(p-p_{crit})}{3(p-1)}$. For $\varepsilon\to 0$, there is $0<\alpha<1$ and M_j^1, M_j^2 such that (5.31) is satisfied and

$$h_{M_1,M_j}\geq h_{\alpha M_1,M_j^1}+h_{(1-\alpha)M_1,M_j^2}. \tag{5.32}$$

Moreover, for some $0\leq\beta\leq 1$ we have that

$$\begin{aligned}
M_j^1 &= \beta(M_j^1+M_j^2)\leq\beta M_j,\\
M_j^2 &= (1-\beta)(M_j^1+M_j^2)\leq(1-\beta)M_j
\end{aligned}$$

by (5.31). By Lemma 2.7.4 *(ii)*, the negativity of h_{M_1,M_j} gives the relation $h_{\alpha M_1,\beta M_j}\leq h_{\alpha M_1,M_j^1}$ since $\frac{\beta M_j}{M_j^1}\geq 1$. Analogously, we have that $h_{(1-\alpha)M_1,(1-\beta)M_j}\leq h_{(1-\alpha)M_1,M_j^2}$ and thus

$$h_{M_1,M_j}\geq h_{\alpha M_1,\beta M_j}+h_{(1-\alpha)M_1,(1-\beta)M_j}.$$

But this contradicts the nondichotomy condition given in the assumptions and therefore, dichotomy does not occur in this case.

(ii) Starting from (5.32), we can apply (2.17) to find

$$h_{M_1,M_\infty}\geq\left(\alpha^{\frac{7}{3}}+(1-\alpha)^{\frac{7}{3}}\right)h_{M_1,M_\infty} \tag{5.33}$$

for $0<\alpha<1$. As a general result, it is implied from (3.13) that for every $0<\tau<1$,

$$1\geq\tau^{\frac{7}{3}}+(1-\tau)^{\frac{7}{3}}. \tag{5.34}$$

But by the negativity of h_{M_1,M_∞}, (5.33) contradicts (5.34) and thus dichotomy does not occur.

Step 3 - Conclusion: We can exclude vanishing and dichotomy and thus, from Lemma 5.2.1 we have that compactness occurs.

(i) As a consequence of Lemma 5.2.7, there exists $(y_n)\subset\mathbb{R}^3$ such that for a subsequence which we still denote by (f_n),

$$T_n f_n\rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^6) \text{ for } p<+\infty.$$

for

$$T_n f_n(x,v)=f_n(x+y_n,v).$$

Moreover, we have that $f \in \mathcal{E}_p$ and

$$\|f\|_1 = M_1.$$

Applying *Fatou's Lemma* (Lemma 1.2.1) yields

$$\liminf_{n\to\infty} \iint |v|^2 T_n f_n(x,v)\,dxdv \geq \iint |v|^2 f(x,v)\,dxdv. \tag{5.35}$$

Furthermore, lower semicontinuity of norms (Theorem 1.2.3), we find that

$$\begin{aligned}\liminf_{n\to\infty} \|j(f_n)\|_1 &= \liminf_{n\to\infty} \|j(f_n)\|_1 \\ &\geq \|j(f)\|_1\end{aligned}$$

and therefore,

$$M_j \geq \|j(f)\|_1. \tag{5.36}$$

By the strong convergence of $\nabla U_{T_n f_n}$ given in Lemma 5.2.7 and by (5.35), it is given that

$$h_{M_1,M_j} \geq \mathcal{H}(f).$$

Thus, in order to show that f is a minimizer, it is enough to show that there is equality in (5.36). To this end, define $\alpha = \frac{\|j(f)\|_1}{M_j}$. Notice that $\alpha > 1$ is excluded due to (5.36) and if $\alpha = 1$, we are done. Suppose that $\alpha < 1$. Then, by Remark 2.7.6,

$$\begin{aligned}h_{\|f\|_1,\|j(f)\|_1} &= h_{M_1,\alpha M_j} \\ &> h_{M_1,M_j} \\ &\geq \mathcal{H}(f),\end{aligned}$$

which contradicts the definition of $h_{\|f\|_1,\|j(f)\|_1}$ being an infimum. We conclude that

$$M_j = \|j(f)\|_1 \tag{5.37}$$

and therefore f is a minimizer with

$$\|f\|_1 = M_1, \quad \|j(f)\|_1 = M_j, \quad \mathcal{H}(f) = h_{M_1,M_j}$$

and such that

$$\|T_n f_n\|_1 \to \|f\|_1, \quad \|j(T_n f_n)\|_1 \to \|j(f)\|_1, \quad \||v|^2 T_n f_n\|_1 \to \||v|^2 f\|_1. \tag{5.38}$$

To finish the proof, we will use this results to show that

$$T_n f_n \to f \text{ in } \mathcal{E}_p. \tag{5.39}$$

Define the sequence g_n given by

$$g_n := \frac{1}{2}(T_n f_n + f).$$

By the convexity property of j (c.f. Section 2.5),

$$j(g_n) \leq \frac{1}{2}(j(T_n f_n) + j(f)).$$

From Lemma 2.4.2 *(ii)* and from the L^2-convergence of the potential energy in Lemma 5.2.7 *(i)*, we deduce that

$$\nabla U_{g_n} = \frac{1}{2}\left(\nabla U_{T_n f_n} + \nabla U_f\right) \to \nabla U_f \text{ strongly in } L^2.$$

Hence, with the use of (5.40),

$$\|g_n\|_1 = M_1, \quad \limsup_{n\to\infty} \|j(g_n)\|_1 \leq M_j, \quad \mathcal{H}(g_n) \to h_{M_1,M_j}$$

as $n \to \infty$ and repeating the arguments for the proof of (5.37), we have that

$$\|j(g_n)\|_1 \to M_j$$

as $n \to \infty$. Since $\|j(f)\|_1 = M_j$, this implies that for $n \to \infty$,

$$j(T_n f_n) + j(f) - 2j(g_n) \to 0$$

in L^1. Choose $K, \eta > 0$ and define

$$\delta_K(\eta) := \inf\left\{ j(s) + j(t) - 2j\left(\frac{1}{2}(s+t)\right) \ : \ 0 \leq s,t \leq K,\ |s-t| \geq \eta \right\}.$$

Since by assumption, j is strictly convex, we have that $\delta_K(\eta) > 0$ for every η. Further, $\delta_K(\eta)$ is non-decreasing with respect to η as it is defined as an infimum. Define the set

$$\Omega_{K,\eta} := \left\{(x,v) \in \mathbb{R}^6 \ : \ T_n f_n(x,v), f(x,v) \leq K \ |T_n f_n(x,v) - f(x,v)| \geq \eta\right\}.$$

Notice that

$$\begin{aligned}
\lambda(\{(x,v) \in \mathbb{R}^6 \ : \ T_n f_n(x,v) \geq K\}) &\leq \frac{1}{K}\int_{T_n f_n(x,v) \geq K} T_n f_n \, dxdv \\
&\leq \frac{1}{K}\int_{\mathbb{R}^6} T_n f_n \, dxdv \\
&= \frac{M_1}{K},
\end{aligned}$$

where λ denotes the Lebesgue measure. Similarly,

$$\lambda(\{(x,v) \in \mathbb{R}^6 \ : \ f(x,v) \geq K\}) \leq \frac{M_1}{K}$$

and combining these results, we find that

$$\lambda(\{(x,v) \in \mathbb{R}^6 \ : \ T_n f_n(x,v) \geq K \text{ or } f(x,v) \geq K\}) \leq \frac{2M_1}{K}.$$

Using similar techniques, it follows that

$$\begin{aligned}
&\lambda(\{(x,v) \in \mathbb{R}^6 \ : \ |T_n f_n(x,v) - f(x,v)| \geq \eta\}) \\
&\quad \leq \frac{2M_1}{K} + \lambda(\{(x,v) \in \Omega_{K,\eta} \ : \ \delta_K(|T_n f_n(x,v) - f(x,v)|) \geq \delta_K(\eta)\}) \\
&\quad \leq \frac{2M_1}{K} + \frac{1}{\delta_K(\eta)}\int_{\Omega_{K,\eta}} \delta_K(|T_n f_n(x,v) - f(x,v)|) \, dxdv \\
&\quad \leq \frac{2M_1}{K} + \frac{1}{\delta_K(\eta)}\int_{\mathbb{R}^6} j(T_n f_n(x,v)) + j(f(x,v)) - 2j(g_n) \, dxdv
\end{aligned}$$

and therefore $T_n f_n \to f$ in measure (c.f. Definition B.5), i.e. for every $\eta > 0$,

$$\lim_{n\to\infty} \lambda(\{(x,v) \in \mathbb{R}^6 \ : \ |T_n f_n(x,v) - f(x,v)| \geq \eta\}) = 0.$$

Similar to the proof of Lemma 5.2.7 *(i)*, one can show equiintegrability for $(T_n f_n)$ and thus, by *Vitali's Theorem* (Theorem B.6),

$$T_n f_n \to f \text{ in } L^1, \quad j(T_n f_n) \to j(f) \text{ in } L^1, \quad |v|^2 T_n f_n \to |v|^2 f \text{ in } L^1. \tag{5.40}$$

By the assumption on j, we have that for every $\eta > 0$ there exists $R(\eta)$ such that

$$\int_{|(x,v)|\geq R(\eta)} |T_n f_n|^p \leq C \int_{|(x,v)|\geq R(\eta)} j(T_n f_n) < \eta$$

and therefore,

$$T_n f_n \to f \text{ in } L^p.$$

With this, (5.39) is shown and the proof of *(i)* is finished.
(ii) By Lemma 5.2.7 *(i)*, using similar arguments as in the case of *(i)*, there is f such that

$$\|f\|_1 = M_1, \quad M_\infty \geq \|f\|_\infty, \quad h_{M_1,M_\infty} \geq \mathcal{H}(f).$$

Setting

$$\beta = \frac{\|f\|_\infty}{M_\infty} \in [0,1]$$

in (2.17), we obtain that

$$h_{M_1,\|f\|_\infty} \geq h_{M_1,M_\infty}.$$

Therefore, $\|f\|_\infty = M_\infty$ and it follows that f is a minimizer that is given by Theorem 5.1.4. □

We are now ready to prove the dynamical stability of a spherically symmetric polytropic state ϕ_p as given in (5.2). As ϕ_p is a minimizer, it acts as a ground state of the system. To do this, we first consider the following proposition and subsequently deduce stability in a corollary.

Proposition 5.3.2. *Let $p_{crit} < p \leq \infty$ and let ϕ_p be the ground state. Let $(f_n) \subset L^1_+(\mathbb{R}^6)$ be a sequence such that*

$$\limsup_{n\to\infty} \|f_n\|_1 \leq \|\phi_p\|_1, \quad \limsup_{n\to\infty} \|f_n\|_p \leq \|\phi_p\|_p, \quad \limsup_{n\to\infty} \mathcal{H}(f_n) \leq \mathcal{H}(\phi_p).$$

Then, up to a subsequence, there exists a sequence $(y_n) \subset \mathbb{R}^3$ such that

$$T_n f_n(x,v) := f_n(x + y_n, v)$$

satisfies

$$T_n f_n \to \phi_p \text{ in } \mathcal{E}_p$$

in the case $p < \infty$ or

$$T_n f_n \rightharpoonup \phi_\infty \text{ weakly in } L^1(\mathbb{R}^6),$$
$$T_n f_n \rightharpoonup \phi_\infty \text{ weak-} * \text{ in } L^\infty(\mathbb{R}^6)$$

in the case $p = \infty$.

Proof. First consider the case $p < \infty$. We want to make use of Theorem 5.3.1 *(i)* and in order to do so, will prove that (f_n) is a minimizing sequence for

$$h_{M_1,M_j} = \inf_{f \in \mathcal{F}_{M_1,M_j}} \mathcal{H}(f),$$

where $M_1, M_j > 0$. To do this, we claim that

$$\|f_n\|_1 \to \|\phi_p\|_1, \quad \|f_n\|_p \to \|\phi_p\|_p, \quad \mathcal{H}(f_n) \to \mathcal{H}(\phi_p). \tag{5.41}$$

Indeed, set

$$\alpha = \frac{\|f_n\|_1}{\|\phi_p\|_1}, \quad \beta = \frac{\|f_n\|_p^p}{\|\phi_p\|_p^p}$$

in (2.16) to find

$$\mathcal{H}(f_n) \geq h_{\|f_n\|_1, \|f_n\|_p^p} = \left(\frac{\|f_n\|_1}{\|\phi_p\|_1}\right)^{\frac{7p-9}{3(p-1)}} \left(\frac{\|f_n\|_p^p}{\|\phi_p\|_p^p}\right)^{\frac{2}{3(p-1)}} \mathcal{H}(\phi_p).$$

On the one hand, since

$$\limsup_{n\to\infty} \|f_n\|_1 \leq \|\phi_p\|_1, \quad \limsup_{n\to\infty} \|f_n\|_p \leq \|\phi_p\|_p,$$

by assumption, $\alpha, \beta \leq 1$ and as $p > \frac{9}{7}$ aswell as $\mathcal{H}(\phi_p) < 0$, we find that

$$\mathcal{H}(f_n) \geq \mathcal{H}(\phi_p).$$

On the other hand,

$$\limsup_{n\to\infty} \mathcal{H}(f_n) \leq \mathcal{H}(\phi_p).$$

by assumption and therefore (5.41) holds and in particular, (f_n) is a minimizing sequence. Let $j(f) = f^p$ and use Lemma 2.7.8 to show that the nondichotomy condition is fulfilled. Then we can apply Theorem 5.3.1 *(i)* to find that

$$T_n f_n \to \phi_p \text{ in } \mathcal{E}_p.$$

In the case $p = \infty$, we can follow the same steps using (2.17) instead of (2.16) to conclude that (f_n) is a minimizing sequence. After applying Theorem 5.3.1 *(ii)*, it follows that

$$T_n f_n \rightharpoonup \phi_\infty \text{ weakly in } L^1(\mathbb{R}^6),$$
$$T_n f_n \rightharpoonup \phi_\infty \text{ weak-} * \text{ in } L^\infty(\mathbb{R}^6).$$

□

Corollary 5.3.3. *Let $p_{crit} < p < \infty$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any weak solution $f(t) \in L^\infty(\mathbb{R}_+, \mathcal{E}_p)$ of the Vlasov-Poisson system with initial data $f_0 \in \mathcal{E}_p$,*

$$\begin{aligned} &\mathcal{H}(f_0) - \mathcal{H}(\phi_p) \leq \delta, \\ &\|f_0\|_1 \leq \|\phi_p\|_1 + \delta, \quad \|f_0\|_p \leq \|\phi_p\|_p + \delta \end{aligned}$$

implies that there exists a translation shift $x(t) \in \mathbb{R}^3$ such that for all $t \geq 0$,

$$\|f(x + x(t), v, t) - \phi_p\|_{\mathcal{E}_p} < \varepsilon.$$

Proof. Let $f(t) \in L^\infty(\mathbb{R}_+, \mathcal{E}_p)$ be a solution of the Vlasov-Poisson system and suppose by contradiction that the statement does not hold. Then for every $n \in \mathbb{N}$ there exists $\varepsilon > 0$, $t_n > 0$ and initial data $f_0 \in \mathcal{E}_p$ such that

$$\begin{aligned} &\mathcal{H}(f_0) - \mathcal{H}(\phi_p) \leq \frac{1}{n}, \\ &\|f_0\|_1 \leq \|\phi_p\|_1 + \frac{1}{n}, \quad \|f_0\|_p \leq \|\phi_p\|_p + \frac{1}{n} \end{aligned}$$

but for every translation shift $x(t) \in \mathbb{R}^3$,

$$\|f_n - \phi_p\|_{\mathcal{E}_p} \geq \varepsilon, \tag{5.42}$$

where we denoted $f_n = f(x + x(t_n), v, t_n)$ for $n > 0$. Since for $1 \leq q \leq p$ the L^q norm of f is conserved under translation shift, we have

$$\|f_0\|_1 = \|f_n\|_1, \quad \|f_0\|_p = \|f_n\|_p.$$

Furthermore, by Proposition 2.4.3, $\mathcal{H}$ is conserved in time and thus

$$\mathcal{H}(f_n) = \mathcal{H}(f_0).$$

But then the sequence (f_n) satisfies the conditions for Proposition 5.3.2 and therefore, up to a subsequence, there exists a sequence $(y_n) \subset \mathbb{R}^3$ such that

$$T_n f_n(x, v) := f_n(x + y_n, v, t_n) = f(x + x(t_n) + y_n, v, t_n)$$

satisfies

$$T_n f_n \to \phi_p \text{ in } \mathcal{E}_p,$$

which contradicts (5.42). □

Remark 5.3.4. The proof of Corollary 5.3.3 uses the strong convergence of minimizing sequences given in Proposition 5.3.2. However, for the case where $p = \infty$, we only have weak convergence. Unfortunately, this weak convergence cannot be translated into a strong dynamical stability of the ground state ϕ_∞.

Comparison and Conclusion

The prerequisite for proving dynamical stability was the existence of minimizers, which we established in Chapter 3. This is valid under the mass-Casimir constraint given by $\mathcal{F}_M$, both in the case $0 < k < \frac{7}{2}$ and in the critical case $k = \frac{7}{2}$. Our aim was to compare the difference in total energy between a ground state, given by a minimizer, and a solution of the Vlasov-Poisson system. This difference should remain arbitrarily small at any time. In Theorem 4.2.1, we showed that this holds true for both general minimizers and isolated minimizers of the system in the case $0 < k < \frac{7}{2}$. In the critical case with $k = \frac{7}{2}$, the assertion remains valid as seen in Theorem 4.3.1.

After obtaining an initial result concerning dynamical stability, our aim was to achieve a stronger result, namely dynamical stability on the energy space $\mathcal{E}_p$. However, this required more preparation. Initially, we formulated a new minimization problem $\mathcal{J}_p$, which is attained on polytropes as seen in Theorem 5.1.4. Following that, we utilized the concentration compactness lemma, given in Lemma 5.2.1, to prove that minimizing sequences are relatively compact in $\mathcal{E}_p$. This holds true except for translation shifts and was shown in Theorem 5.3.1. Subsequently, we were able to establish strong convergence of minimizing sequences in $\mathcal{E}_p$, providing us with dynamical stability. As $\mathcal{E}_p$ is a larger space, the result is stronger. However, the limiting case is not covered here.

It is noteworthy that this dynamical stability, which holds over all times, is an important result for understanding the long-term behavior of galaxies and globular clusters. What is fascinating is that in our initial approach, we were able to achieve such a result almost solely through intuitive definitions of kinetic and potential energy for the Vlasov-Poisson system. It was only when we aimed to show dynamical stability over a larger set of states that it required more effort. Nonetheless, the key steps outlined above led us to a dynamical stability result in the energy space $\mathcal{E}_p$.

Appendix

This appendix provides additional definitions and results in the fields of Calculus, Analysis and Functional Analysis, referencing primarily [LL01] unless otherwise noted. Additionally, a proof of a lemma will be included.

A Calculus

Theorem A.1 (Implicit Function Theorem [Kön13]). *Let $U \subset \mathbb{R}^M \times \mathbb{R}^N$ open and let $f \in C^1(U; \mathbb{R}^N)$ such that $f(x_0, y_0) = 0$ for $(x_0, y_0) \in U$. If*

$$\det \left(\frac{\partial f_i}{\partial y_j} \right)_{1 \leq i,j \leq N} \neq 0,$$

then there exist open neighborhoods $V \subset \mathbb{R}^M$ of x_0 and $W \subset \mathbb{R}^N$ of y_0 and a continuously differentiable function $g : V \to W$ such that

$$\{(x, y) \in V \times W \ : \ f(x, y) = 0\} = \{(x, g(x)) \times W \ : \ x \in V\}.$$

Moreover,

$$Dg(x_0) = -(D_y f(x_0, y_0))^{-1} \cdot D_x f(x_0, y_0).$$

B Analysis

Proposition B.1 (Interpolation of L^p-norms). *Let p, q be such that $0 < p < q \leq \infty$. Moreover, let r such that $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ with $0 < \theta < 1$. If $f \in L^p(\Omega) \cap L^q(\Omega)$, then*

$$\|f\|_r \leq \|f\|_p^{1-\theta} \|f\|_q^{\theta}.$$

Proof. First split the function $|f| = |f|^{1-\theta}|f|^{\theta}$. Then we obtain

$$\begin{aligned} \|f\|_r &= \left(\int_\Omega |f|^{r(1-\theta)} |f|^{r\theta} \, d\mu \right)^{\frac{1}{r}} \\ &= \left(\int_\Omega |f|^{p \frac{r(1-\theta)}{p}} |f|^{q \frac{r\theta}{q}} \, d\mu \right)^{\frac{1}{r}}. \end{aligned}$$

Since $|f|^{p\frac{r(1-\theta)}{p}} \in L^{\frac{p}{r(1-\theta)}}$ and $|f|^{q\frac{r\theta}{q}} \in L^{\frac{q}{r\theta}}$, this yields

$$\begin{aligned}\|f\|_r &\leq \left(\int_\Omega |f|^{p\frac{r(1-\theta)}{p}\frac{p}{r(1-\theta)}}\,d\mu\right)^{\frac{r(1-\theta)}{pr}} \left(\int_\Omega |f|^{q\frac{r\theta}{q}\frac{q}{r\theta}}\,d\mu\right)^{\frac{r\theta}{qr}} \\ &= \|f\|_p^{1-\theta}\,\|f\|_q^{\theta},\end{aligned}$$

where we applied *Hölder's Inequality* (Theorem 1.2.2) with the conjugate Hölder coefficients $\frac{p}{r(1-\theta)}$ and $\frac{q}{r\theta}$. □

Theorem B.2 ([R$^+$76])**.** *Let f be a monotone function defined on an interval I. Then the set of discontinuities is at most countable.*

Definition B.3 (Equiintegrability [Eva22])**.** *A family $\mathcal{F}$ of measurable functions on Ω is equiintegrable (or uniformly integrable) over Ω, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $A \subset \Omega$ is measurable with $|A| < \delta$, then*

$$\int_A |f| < \varepsilon$$

for every $f \in \mathcal{F}$.

Definition B.4 (Equimeasurability)**.** *Let f, g be two functions on $\mathbb{R}$. They are called equimeasurable, if for every $s > 0$*

$$\mu(\{x \,:\, f(x) > s\}) = \mu(\{x \,:\, g(x) > s\}),$$

where μ denotes the measure.

Definition B.5 (Convergence in Measure [BR07])**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n), f$ be measurable functions. Then $f_n \to f$ in measure if for every $\delta > 0$,*

$$\lim_{n\to\infty} \mu(\{x \in \Omega \,:\, |f_n(x) - f(x)| > \delta\}) = 0.$$

Theorem B.6 (Vitali's Theorem [BR07])**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $(f_n) \subset L^1(\Omega, \mathcal{A}, \mu)$ and let f be measurable with respect to μ. Then, the following are equivalent:*

(i) $f_n \to f$ in measure and (f_n) is equiintegrable;

(ii) $f \in L^1(\Omega, \mathcal{A}, \mu)$ and $f_n \to f$ in $L^1(\Omega, \mathcal{A}, \mu)$.

C Functional Analysis

Definition C.1 (Symmetric Rearrangement)**.** *Let $A \in \mathbb{R}^N$ be a finite Borel set. Then its symmetric rearrangement A^* is defined as*

$$A^* = \{x \,:\, |x| < r\},$$

i.e. the open ball centered at the origin with the same volume as A.

The symmetric-decreasing rearrangement of a characteristic function of A is defined by

$$\mathbb{1}_A^* := \mathbb{1}_{A^*}.$$

Let f be a measurable function. Then its symmetric-decreasing rearrangement f^* is defined by

$$f^*(x) := \int_0^\infty \mathbb{1}^*_{\{|f|>t\}}(x)\, dt,$$

where

$$|f(x)| = \int_0^\infty \mathbb{1}_{\{|f|>t\}}(x)\, dt.$$

Theorem C.2 (Riesz's Rearrangement Inequality)**.** *Let f, g, h be non-negative functions on $\mathbb{R}^N$ and denote by f^*, g^*, h^* their symmetric rearrangements. Define*

$$I(f,g,h) := \iint f(x)g(x-y)h(y)\, dxdy.$$

Then $I(f,g,h) \le I(f^*,g^*,h^*)$ with $I(f^*,g^*,h^*) = \infty$ if $I(f,g,h) = \infty$.

Theorem C.3 (Strict rearrangement inequality)**.** *Let f, g, h be measurable and non-negative functions on $\mathbb{R}^n$ with g strictly symmetric-decreasing. Then there is equality in Theorem C.2 if and only if $f(x) = f^*(x-y)$ and $h(x) = h^*(x-y)$ for some $y \in \mathbb{R}^n$.*

Lemma C.4 ([LMR08])**.** *Let $f \in L^1(\mathbb{R}^3)$ and denote by f^* its spherically symmetric rearrangement. Let Q be a strictly increasing continuous function with $Q(0) = 0$. Then*

$$\int Q(f^*)(x)\, dx = \int Q(f)(x)\, dx$$

and in particular,

$$\|f^*\|_p = \|f\|_p$$

for all $1 \le p \le \infty$.

If g is a non-negative and non-decreasing function on $\mathbb{R}_+$, then

$$\int g(|x|) f*(x)\, dx \le \int g(|x|) f(x)\, dx$$

and moreover, if g is strictly increasing, there is equality unless $f^* \equiv f$.

Lemma C.5 (Mazur's Lemma [RR06])**.** *Let X be a Banach space and let (f_n) be a sequence of functions such that $f_n \rightharpoonup f$ in X. Then there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of sets $\{\alpha(n)_k\}_{k=n}^{N(n)}$, where $\alpha(n)_k \in \mathbb{R}_{\ge 0}$, such that $\sum_{k=n}^{N(n)} \alpha(n)_k = 1$ such that*

$$g_n := \sum_{k=n}^{N(n)} \alpha(n)_k f_k \to f \text{ strongly in } X.$$

Here, g_n is called a convex combination of elements of the set $\{f_k\}_{k=n}^{N(n)}$.

Definition C.6 (Gâteaux Derivative [Gat13], [Gat19]). *Let X, Y be Banach spaces, $U \subseteq X$ open and $F : U \to Y$. Then the Gâteaux differential $dF(u, \psi)$ of F at $u \in U$ in the direction $\psi \in X$ is defined by*

$$dF(u, \psi) = \lim_{s \to 0} \frac{F(u + s\psi) - F(u)}{s} = \frac{d}{ds} F(u + s\psi)\bigg|_{s=0}.$$

F is said to be Gâteaux differentiable at u if this limit exists for every $\psi \in X$.

Theorem C.7 (Dunford-Pettis Theorem [BR07]). *Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n) \subset L^1(X, \mu)$ be a bounded sequence. Then (f_n) is equiintegrable if and only if (f_n) is a relatively compact subset of $L^1(X, \mu)$ in the weak topology.*

D Proof of the Concentration Compactness Lemma

This section is dedicated to the proof of the concentration compactness lemma, which is adapted from Section I.3. in [Lio84].

Proof. (Lemma 5.2.1) Consider the function P_n given by

$$P_n(r) := \sup_{y \in \mathbb{R}^3} \int_{y + B_r} \rho_n(\xi)\, d\xi.$$

This is the so-called concentration function of a measure, which is introduced in [Lev54]. By the assumptions on (ρ_n), the sequence (P_n) of functions on $\mathbb{R}_+$ is non-decreasing, non-negative and uniformly bounded. Furthermore,

$$\lim_{r \to \infty} P_n(r) = M.$$

Thus, by *Arzelà-Ascoli*, there exist a subsequence (P_{n_k}) and a non-decreasing function P such that

$$\lim_{k \to \infty} P_{n_k}(r) = P(r)$$

for every $r \geq 0$. Notice that

$$m := \lim_{r \to \infty} P(r) \in [0, M].$$

(i) Let $m = M$ and let $\mu > \frac{M}{2}$. We can find $R = R(\mu)$ such that for all $k \geq 1$,

$$P_{n_k}(R) = \sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_{n_k}(\xi)\, d\xi > \mu.$$

Hence, there exists $y_k(\mu)$ such that

$$\int_{y_k + B_R} \rho_{n_k}(\xi)\, d\xi > \mu.$$

Since $\frac{M}{2}$ represents half of the total mass, setting $y_k = y_k\left(\frac{M}{2}\right)$ with the same argumentation as before yields

$$|y_k(\mu) - y_k| \leq R(\mu) + R\left(\frac{M}{2}\right)$$

for all $\mu \geq \frac{M}{2}$. Denote

$$\tilde{R} = 2R(\mu) + R\left(\frac{M}{2}\right).$$

Then there exists a sequence $(y_k) \subset \mathbb{R}^3$ such that for all $k \geq 1$ and for all $\mu \geq \frac{M}{2}$,

$$\int_{y_k + B_{\tilde{R}}} \rho_{n_k}(\xi)\, d\xi > \mu. \tag{A.1}$$

In particular, if $\varepsilon > 0$, (A.1) holds for $\mu = M - \varepsilon$ and *(i)* is proven.

(ii) If $m = 0$ it follows directly that *(ii)* holds true.

(iii) Let $m \in (0, M)$ and $\varepsilon > 0$ and pick R such that

$$P(R) > m - \frac{\varepsilon}{4}.$$

Then there exists $k_0 \geq 1$ such that

$$m - \frac{\varepsilon}{4} < P_{n_k}(R) < m + \frac{\varepsilon}{4}$$

for every $k \geq k_0$. Moreover, there exists a sequence (R_k) with $R_k \to \infty$ as $k \to \infty$ such that

$$P_{n_k}(R_k) \leq m + \frac{\varepsilon}{4}.$$

Thus there exists $y_k \in \mathbb{R}^3$ such that

$$\int_{y_k + B_R} \rho_{n_k}(\xi)\, d\xi \in (m - \frac{\varepsilon}{4}, m + \frac{\varepsilon}{4}).$$

Set

$$\rho_k^1 = \rho_{n_k} \mathbb{1}_{\{y_k + B_R\}},$$
$$\rho_k^2 = \rho_{n_k} \mathbb{1}_{\{\mathbb{R}^3 \setminus \{y_k + B_R\}\}}.$$

Then it follows that

$$\begin{aligned}\int_{\mathbb{R}^3} \rho_{n_k}(x) - \rho_k^1(x) - \rho_k^2(x)\, dx &= \int_{R \leq |x - y_k| \leq R_k} \rho_{n_k}(x)\, dx \\ &\leq P_{n_k}(R_k) - P_{n_k}(R) + \frac{\varepsilon}{2} \\ &\leq (m + \frac{\varepsilon}{4}) - (m - \frac{\varepsilon}{4}) + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore, the assertion in *(iii)* is valid. □

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