Algebraic Vector Bundles,

From a Homotopic Point of View

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Abstract

This thesis mainly discusses the classification of algebraic vector bundles over spilt quadrics Q_{2n-1}, using a lot of results from $\mathbb{A}^1$-homotopy theory. Based on a modified affine representability theorem, the oriented vector bundle functor can be represented by the classifying space of SL_r in the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$. So the problem can be transformed into the calculation of $\mathbb{A}^1$-homotopy sheaves $\pi_{n-1}^{\mathbb{A}^1}(SL_r)$. An estimate of the $\mathbb{A}^1$-homotopy sheaf in the first unstable case $r = n$ can be obtained by combining various morphisms from algebraic K-theory. Applying this calculation to classifying vector bundles over Q_{2n-1}, we could use unimodular rows over Q_{2n-1} as explicit representatives for vector bundles over Q_{2n-1}. Finally, we use this classification results to answer a question asked by Nori about the completability of unimodular rows. Most of this thesis is based on Asok and Fasel's work [25].

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Chapter 1

Introduction

1.1 Motivation

Let M be a closed smooth real manifold. Vector bundles naturally arise when we want to study the tangent bundle TM. Studying vector bundles leads us to a lot of useful information of M. For instance, the Poincar-Hopf index formula tells us that the zero locus of a suitably generic section of the tangent bundle, namely a vector field, is linked to the Euler characteristic of M, which is a topological feature of M. Using the tools developed by the classical algebraic topology, like homotopies of maps and classifying spaces, the principal GL_r-bundle associated to a vector bundle of rank r over the manifold M can be classified by the homotopy classes of maps from M to the classifying space $BGL(r)$ [18, Theorem 14.2.7 and Theorem 14.4.1]. Rephrasing this result in the language of the category theory, the functor $\mathcal{V}_r(-)$, which assigns a smooth manifold M the isomorphism classes of real vector bundles of rank r, is represented by the classifying space $BGL(r)$ in the homotopy category of topological spaces.

In the 1950s, it is discovered independently by Jean-Pierre Serre and Richard Swan that the category of complex vector bundles over a topological space X is equivalent to the category of finitely generated projective module over the algebra $C(X)$ [16, Chapter 3], where $C(X)$ is the $\mathbb{C}$-algebra of continous complex valued functions on X. This correspondence enables us to understand vector bundles from an algebraic point of view. Quillen [2] and Suslin [3] later proved that every finitely generated projective module over the polynomial ring $k[x_1, x_2, \ldots, x_n]$ is free when k is a field. Put in a geometric way, we have a bijection

$$\mathcal{V}_r(\operatorname{Spec} k) \to \mathcal{V}_r(\mathbb{A}^n_k).$$

This naturally leads to the following conjecture.

Conjecture 1.1.1 (Bass-Quillen)**.** *Let X be a regular Noetherian affine scheme of finite Krull dimension. Then the pullback map $\mathcal{V}_r(X) \to \mathcal{V}_r(X \times \mathbb{A}^n_k)$ is a bijection for all $r \geq 1$ and all $n \geq 1$.*

Here the assumption of affineness is necessary. Denote by Sm_k the category of smooth k-schemes. It is not hard to see that the functor $\mathcal{V}_r(-) : Sm_k^{op} \to Sets$ is not $\mathbb{A}^1$-invariant, even for the projective space $\mathbb{P}^1$.

Example 1.1.2. We want to prove that $\mathcal{V}_2(\mathbb{P}^1)$ is not bijective to $\mathcal{V}_2(\mathbb{P}^1 \times \mathbb{A}^1)$. We are going to construct a vector bundle V of rank 2 on $\mathbb{P}^1 \times \mathbb{A}^1$ such that $V|_{\mathbb{P}^1\times\{0\}} \simeq \mathcal{O}^{\oplus 2}$ and $V|_{\mathbb{P}^1\times\{1\}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Consider the following transition function

$$T(u,t) = \begin{bmatrix} ut & t^2-1 \\ 1 & u^{-1}t \end{bmatrix} \in GL_2(k[u,u^{-1},t]).$$

Here u is the coordinate on $\mathbb{P}^1$ and t is the coordinate on $\mathbb{A}^1$.

When $t = 0$, we have

$$T(u,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which is cohomologous to the identity matrix, thus we get the trivial bundle $\mathcal{O}^{\oplus 2}$. When $t = 1$, we have

$$T(u,1) = \begin{bmatrix} u & 0 \\ 1 & u^{-1} \end{bmatrix},$$

which is cohomologous to $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$. This corresponds to the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

However, when we add some further assumptions to X, including affineness, the Bass-Quillen conjecture can be proved in these special cases. For example, Lindel proved [5] the conjecture when X is essentially of finite type over a field. Additionally, several other constrcutions in cohomology theories for smooth k-schemes are also $\mathbb{A}^1$-invariant. For instance, the pullback maps in Chow rings [6, Theorem 3.3 (a)]

$$CH^*(X) \to CH^*(X \times \mathbb{A}^1),$$

Grothendieck groups [32, Lemma 0FDJ]

$$K_0(X) \to K_0(X \times \mathbb{A}^1),$$

and étale cohomology groups

$$H^*_{\text{ét}}(X, \mu_l) \to H^*_{\text{ét}}(X \times \mathbb{A}^1, \mu_l)$$

are isomorphisms if l is invertible in the field k.

Recall for a smooth manifold M, the vector bundles of rank r over M is bijective to the vector bundles of rank r over $M \times I$, where I is the unit interval $[0, 1]$. All the above results

about $\mathbb{A}^1$-invariance prompts us to establish a homotopy theory for smooth k-schemes, or even more generally, for smooth schemes over a suitable base scheme S. Thanks to the effort of lots of people over the past decades, the (unstable) $\mathbb{A}^1$ (or motivic) homotopy theory is developed and has been successfully applied to study vector bundles on schemes, which we will discuss next.

1.2 Algebraic vector bundles

This part is designed as a survey on several interesting results on algebraic vector bundles obtained by applying $\mathbb{A}^1$-homotopy theory. For the precise definitions, please refer to the later chapters of the thesis. We will mostly focus on vector bundles over smooth affine schemes, with a local trivialization in the Nisnevich topology, in order to use the tools developed in $\mathbb{A}^1$-homotopy theory. Let r be a positive integer and X be a smooth k-scheme. Pulling back from the universal bundle will give a morphism

$$\mathrm{Hom}_{Sm_k}(X, BGL_r) \to \mathcal{V}_r(X)$$

where $\mathcal{V}_r(X)$ is the groupoid of rank r vector bundles over X. In general, this map is neither injective nor surjective. The $\mathbb{A}^1$-invariance of vector bundles coming from various results on Bass-Quillen conjecture inspire us to pass to the $\mathbb{A}^1$-homotopy category, where we could obtain a representability result similar in the classical homotopy category.

Theorem 1.2.1 (Morel, Schlichting, Asok-Hoyois-Wendt [29, Theorem 1]). *Let k be a Noetherian commutative ring which is regular over a Dedekind domain with pefect residue fields. For any smooth affine k-scheme X and any integer $r \geq 0$, there is a bijection*

$$\mathcal{V}_r(X) \cong [X, Gr_r]_{\mathbb{A}^1}$$

where $Gr_r = colim_{N\to\infty} Gr_{r,N}$ is the infinite Grassmannian.

Here Gr_r can be viewed as a model for classifying space BGL_r. As pointed out in [29], this is the best possible $\mathbb{A}^1$-representability theorem for vector bundles for which one could hope. This theorem inspire us to classify vector bundles over spheres, which has a model of affine scheme, by calculating homotopy sheaves of BGL_r. One problem arises, however, that BGL_r(or more precisely $L_{mot}BGL_r$) is not simply connected in the sense of $\mathbb{A}^1$-algebraic topology. This means the unpointed maps $[X, BGL_r]_{\mathbb{A}^1}$ is not bijective with the pointed maps $[(X, x_0), (BGL_r, y_0)]_{\mathbb{A}^1_\bullet}$. To avoid this difficulty, we use $\mathbb{A}^1$-fiber sequence to prove a similar result for oriented vector bundles.

Theorem 1.2.2 (Asok,Fasel [25, Theorem 4.2]). *Let X be an smooth affine scheme and r be*

a positive integer. There is a bijection

$$\mathcal{V}_r^o(X) \cong [X, BSL_r]_{\mathbb{A}^1}$$

where $\mathcal{V}_r^o(X)$ denotes the isomorphism classes of oriented vector bundles over X.

Unlike BGL_r, by explicit computation we have $\pi_1^{\mathbb{A}^1}(BSL_r) = 0$. Later Lemma 5.1.3 points out that since BSL_r is $\mathbb{A}^1$ simply connected, the choice of base point is not essensial if we want to go to the pointed category. Assume X is affine and has the $\mathbb{A}^1$-homotopy type of suspensions of some simple spaces, the oriented vector bundles over X, by the above Theorem, is classified by the morphisms $[X, BSL_r]_{\mathbb{A}^1}$, which is equivalent to certain $\mathbb{A}^1$-homotopy sheaves. The problem of classify oriented vector bundles ultimately boils down to the calculations of some $\mathbb{A}^1$-homotopy sheaves. Furthermore, for an orientable vector bundle $\mathcal{E}$ over X, the orientation is just an explicit choice of isomorphism $\det\mathcal{E} \xrightarrow{\sim} \mathcal{O}_X$. If we could find some explicit representatives for oriented vector bundles over X, using the above strategy, then by comparing these representatives it is possible to classify all the vector bundles of finite rank over X.

1.3 In this thesis

This thesis is largely based on the work of Asok and Fasel [25]. Chapter 2 gives a brief introduction of (unstable) $\mathbb{A}^1$-homotopy theory, including most results we would need to use in the following chapters. Most proofs are referred back to the original work, but some are laid out in details, for example the contraction of $\mathbb{A}^1$-homotopy sheaves (see Theorem 2.7.4) as it is enlightening to see how we work in $\mathbb{A}^1$-homotopy category. Also we include some historical notes in the beginning of this chapter, in hope of shedding some light on the development of some abstract ideas in homotopy category. Chapter 3 introduces Milnor-Witt K-theory, who plays a very important role in $\mathbb{A}^1$-homotopy theory. We are mostly following [23, Chapter 3], where Morel and Hopkins introduce a presentation of Milnor-Witt K-theory using generators and their relationships. A short discussion about the representability of Quillen K-theory and contraction of K-theory sheaves are included. The next two chapters focus on the clacula-tions of $\mathbb{A}^1$-homotopy sheaves for SL_n, and applying these calculations to the classification of vector bundles over the split quadrics Q_{2n-1}. As an application, Nori's question about the completability of unimodular rows can be answered using the classification results. For easy references, an appendix of unimodualr rows is given in the end of the theis, which contains some basics about unimodular rows over a commuatative ring and how unimodular rows are related to the $\mathbb{A}^1$-homotopy theory.

Chapter 2

$\mathbb{A}^1$-homotopy theory

This chapter serves as a brief review of the unstable $\mathbb{A}^1$-homotopy theory. This is a very broad subject and has been revisited and refined over the years, and as a consequence, it is almost impossible to give a thorough introduction within a few pages. We will mostly focus on the important results that are useful for our later discussion of vector bundles on schemes. In section 2.1 of this chapter, a short history and overview of $\mathbb{A}^1$-homotopy theory is included, in order to hopefully convey some intuitions to the readers who are not familiar with this subject. Starting from section 2.2, we will follow [28, Section 3] and define the category of motivic spaces $\mathcal{H}(k)$, which is called the Morel-Voevodsky category. Next, we will recall some very useful classical constructions, for example, the Eilenberg-Maclane spaces and the Postnikov tower in the $\mathbb{A}^1$-homotopy setting. We will also introduce the contraction of homotopy sheaves and show that it characterize applying the $\mathbb{G}_m$-loop space functor. Some details are omitted, and most proofs will be refered back to the original work.

2.1 History and Overview

The main idea of $\mathbb{A}^1$-homotopy theory is to develop a homotopy theory on algebraic schemes using the affine line $\mathbb{A}^1$, obtaining the analogues of classical theorems in the homotopy theory of topological spaces, including the fiber sequence, the Postnikov tower, etc. In this thesis, for simplicity we only consider smooth schemes over a perfect base field k, while in various literatures, $\mathbb{A}^1$-homotopy theory can be defined for smooth schemes over a general base scheme S, if S satisfies certain conditions.

Let Sm_k be category of smooth schemes of finite type over a perfect field k. If we want to define a homotopy theory on Sm_k, the first difficulty we encounter is the nonexistence of colimits. More specifically, you cannot "contract" a scheme with expectation to get a new scheme, unlike what we do in topological spaces by taking the quotient space. A standard way to solove this problem is to embed Sm_k into a larger category using Yoneda embedding, namely considering all the presheaves of simplicial sets on Sm_k, then sheafify it in the Nisnevich

topology. The choice of simplicial sets enable us to constrcut the homotopy category using Quillen's model structure, similar to how we construct the simplicial homotopy theory [20, Chapter I,II]. Morel and Voevodsky [9, Section 3] choose an injective model structure and first define the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ by inverting all the weak equivalences at stalks, plus the projection maps $X \times \mathbb{A}^1 \to X$ for all $X \in Sm_k$, forcing the affine line $\mathbb{A}^1$ to be contractible. Voevodsky also gives an exposition [8] on the topological intuition, following the sequence familiar from the standard textbooks on topological homotopy theory. He points out [8, Theorem 3.6] these two approaches are equivalent.

Dugger later [10, Section 8] presents a universal construction of the same homotopy category $\mathcal{H}(k)$ by formally adjoining homotopy limits to Sm_k and then imposing the homotopy-colimit-type relations coming from the Nisnevich topology and imposing the equivalence

$$X \times \mathbb{A}^1 \xrightarrow{\sim} X \ .$$

The $\mathbb{A}^1$-local space $U(Sm_k)_{\mathbb{A}^1}$ he gets can be summarized in the following expression

$$U(Sm_k)_{\mathbb{A}^1} = L_{\mathbb{A}^1} L_{Nisn} sPreShv(Sm_k)$$

where $L_{\mathbb{A}^1}$ corresponds to the Bousfield localization with respect to all projections $X \times \mathbb{A}^1 \to X$ and L_{Nisn} corresponds to imposing the descent condition on all hypercovers associated to Nisnevich topology. The resulting model structure is Quillen equivalent to the one constructed by Morel and Voevodsky, at the same time being more handy [10, Remark 8.2].

Recently, Robalo in his PhD thesis [26, Section 5.1] has obeserved that, for a given model category, we should focus on the associated $(\infty, 1)$-category, which is supposed to contain all the homotopy information. The construction of $\mathcal{H}(k)$ can be performed directly in the setting of ∞-categories. This construction gives a universal characterization of $\mathcal{H}(k)$ [26, Theorem 5.1.2]. Though it is a more clear way to view $\mathcal{H}(k)$, we will not dive deep into ∞-categories for the reason that they are much more complicated and abstract. In the following sections, we will, however, use this universal characterization to stress that in the definition of $\mathcal{H}(k)$, the choice or even the use of model structures is not essential and proofs of a lot of properties of $\mathbb{A}^1$-homotopy theory are indeed model-independent. As long as working in the same homotopy category $\mathcal{H}(k)$, we are able to utilize a concrete model structures to do calculations without worrying what happens if we choose a different one.

2.2 Motivic spaces

In this section, we recall the basic definitions of unstable $\mathbb{A}^1$-homotopy theory.

Definition 2.2.1 (k-space)**.** Let Sm_k be the category of smooth schemes of finite type over a perfect field k. A presheaf of Kan complexes $\mathcal{X} : Sm_k^{op} \to Kan$ is called space (or k-space).

They form a presentable ∞-category denoted by Spc_k.

Remark.

- The Yoneda embedding gives a fully faithful embedding $Sm_k \hookrightarrow Spc_k$ by sending a smooth k-scheme X to the representable presheaf $\mathrm{Hom}(-, X)$.

- If we instead consider all the presheaves valued in simplicial sets, then Spc_k is just the fibrant objects in the injective model structure [28, Definition 3.48].

Definition 2.2.2 (Nisnevich topology)**.** A family of étale morphism $\{U_i \to X\}_{i\in I}$ is called a Nisnevich cover if for every point $x \in X$, there exists $i_0 \in I$ and a point $x' \in U_{i_0}$ such that the map between residue fields $k(x) \to k(x')$ is an isomorphism.

Nisnevich topology is finer than Zariski topology and coarser than étale topology. It seems to have the good properties of both while avoid the bad ones [9, Section 3]. This is the topology we mostly use in $\mathbb{A}^1$-homotopy theory.

Definition 2.2.3 (Nisnevich descent)**.** Given $\mathcal{X} \in Spc_k$, we say $\mathcal{X}$ has Nisnevich descent if for every hypercover $U_\bullet \to U$ of some $U \in Sm_k$, the map

$$\mathcal{X}(U) \to \mathrm{holim}\mathcal{X}(U_\bullet)$$

is an equivalence. The subcategory spanned by all the k-spaces with Nisnevich descent is denoted by Spc_k^{Nisn}.

Definition 2.2.4 ($\mathbb{A}^1$-invariance)**.** Given $\mathcal{X} \in Spc_k$,we say $\mathcal{X}$ is $\mathbb{A}^1$-invariant or $\mathbb{A}^1$-local if for every $U \in Sm_k$, the projection map $\mathbb{A}^1 \times U \to U$ induces an equivalence $\mathcal{X}(U) \to \mathcal{X}(\mathbb{A}^1 \times U)$.

Remark. There is a more explicit way to check if a k-space $\mathcal{X}$ has Nisnevich descent. $\mathcal{X}$ has Nisnevich descent if and only if the following conditions are satisfied.

(1) $\mathcal{X}(\varnothing) = *$.

(2) Given an elementary distinguished square in Sm_k in the following form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & X \end{array}$$

where p is an étale morphism and i is an open immersion, the induced map $V - W \to X - U$ is an isomorphism, we have the following commutative square.

$$\begin{array}{ccc} \mathcal{X}(W) & \longleftarrow & \mathcal{X}(V) \\ \uparrow & & \uparrow \\ \mathcal{X}(U) & \longleftarrow & \mathcal{X}(X) \end{array}$$

It is homotopy cartesian.

For more details, see [28, Proposition 3.53].

The k-space $\mathcal{X}$ is called a motivic space if $\mathcal{X}$ has Nisnevich descent and it is $\mathbb{A}^1$-invariant. All the motivic spaces form a presentable ∞-subcategory of Spc_k, denoted by $\mathcal{H}(k)$. This is called Morel-Voevodsky category, or $\mathbb{A}^1$-homotopy category. The inclusion functors

$$\mathcal{H}(k) \hookrightarrow Spc_k^{Nisn} \hookrightarrow Spc_k$$

each admits a left adjoint localization functor $L_{\mathbb{A}^1} : Spc_k^{Nisn} \to \mathcal{H}(k)$ and $L_{Nisn} : Spc_k \to Spc_k^{Nisn}$. See [28, Section 3.6]. The localization functor from Spc_k to $\mathcal{H}(k)$ will be denoted by L_{mot}.

Example 2.2.5. We look at some representable presheaves in Spc_k. They all have Nisnevich descent since any representable presheaf is a sheaf in Nisnevich topology. Consider the multiplicative group scheme $\mathbb{G}_m = Spec\ k[T, T^{-1}]$. For any smooth local scheme $X = SpecR$, the morphism $X \times \mathbb{A}^1 \to \mathbb{G}_m$ corresponds to a k-algebra homomorphism $k[T, T^{-1}] \to R[t]$, which must factor through R. This implies $\mathbb{G}_m$ as a presheaf must be $\mathbb{A}^1$-invariant for every local smooth scheme, thus $\mathbb{G}_m$ is $\mathbb{A}^1$-invariant as a k-space. Now we can view $\mathbb{G}_m$ as an object in $\mathcal{H}(k)$. In fact, it is the sheaf associated to the naive presheaf $U \mapsto \mathcal{O}^\times(U)$.

Sometimes we need to use pointed motivic spaces. Denote by $Spc_{k,\bullet}$ the preshaves of pointed Kan complex. We can follow the similar construction as above to get a pointed homotopy category $\mathcal{H}_\bullet(k)$. Moreover, we have the following Quillen adjunction

$$\mathcal{H}_\bullet(k) \rightleftarrows \mathcal{H}(k)$$

given by forgetting the base point and adding a disjoint base point $U \to U_+ = U \sqcup *$. If the context is clear, sometimes we will omit the base point notation.

2.3 $\mathbb{A}^1$-homotopy sheaves

We will write $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1} := Hom_{\mathcal{H}(k)}(\mathcal{X},\mathcal{Y})$(respectively $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1_\bullet}$ for the pointed version), sometimes $\mathbb{A}^1$ is omitted if no confusion arises.

Definition 2.3.1 (homotopy sheaves)**.** Let $\mathcal{X}$ be a motivic space. We write $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ as the Nisnevich sheafification of the presheaf $U \to [U,\mathcal{X}]_{\mathbb{A}^1}$ where $U \in Sm_k$. Now assume $\mathcal{X}$ to be a pointed motivic space. For $i \geq 1$, we write $\pi_i^{\mathbb{A}^1}(\mathcal{X})$ as the Nisnevich sheafification of the presheaf $U \to \pi_i(\mathcal{X}(U))$ where $U \in Sm_k$. More specifically, it is the Nisnevich sheafification of the presheaf $U \to [S^i \wedge U_+,\mathcal{X}]_{\mathbb{A}^1_\bullet}$ where S^i is the constant sheaf associated to the simplicial i-sphere.

By the homotopy theory on simplicial sets, for a motivic space $\mathcal{X}$, we know that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of sets, $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of groups and $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of Abelian groups for $n \geq 2$. We also have the notion of connectiveness and truncatedness as we do in classical algebraic topology.

Definition 2.3.2 (connective and truncated)**.** Let $\mathcal{X}$ be a motivic space. We say that $\mathcal{X}$ is n-connective if $\pi_0^{\mathbb{A}^1}(\mathcal{X}) = *$ and for any choice of base point $x \in \mathcal{X}(k)$ and any integer $i \leq n-1$, we have $\pi_i^{\mathbb{A}^1}(\mathcal{X},x) = 0$. We say that $\mathcal{X}$ is n-truncated if for all base point $x \in \mathcal{X}(k)$ and any integer $i \geq n+1$, we have $\pi_i^{\mathbb{A}^1}(\mathcal{X},x) = 0$. Sometimes we will call a 1-connective space $\mathcal{X}$ $\mathbb{A}^1$-connected, following the tradition.

These sheaves are very useful in $\mathbb{A}^1$-homotopy theory because they detect $\mathbb{A}^1$-equivalence. It is the $\mathbb{A}^1$-version of Whitehead theorem proved by Morel and Voevodsky.

Theorem 2.3.3 ($\mathbb{A}^1$-Whitehead theorem [9, Chapter 3, Proposition2.14])**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism between pointed, $\mathbb{A}^1$-connected motivic spaces. The following conditions are euqivalent.*

- *f is an $\mathbb{A}^1$-equivalence between $\mathcal{X}$ and $\mathcal{Y}$.*
- *For every $i > 0$, the induced morphism between $\mathbb{A}^1$-homotopy sheaves of groups $f_* : \pi_i^{\mathbb{A}^1}(\mathcal{X}) \to \pi_i^{\mathbb{A}^1}(\mathcal{Y})$ is an isomorphism.*

An important result of Morel tells us that L_{mot} preserves connectivity.

Theorem 2.3.4 (Unstable $\mathbb{A}^1$-connectivity theorem [23, Theorem 6.38])**.** *Let $n > 0$ be an integer and $\mathcal{X}$ be a pointed, n-connective k-space. Then $L_{mot}\mathcal{X}$ is still n-connective in $\mathcal{H}_\bullet(k)$.*

Among the $\mathbb{A}^1$-invariant Nisnevich sheaves on Sm_k, we are more interested in a "stronger" version of $\mathbb{A}^1$-invariance. To be more specific, we want the cohomology functor associated to the sheaves should also have this invariance property. We have the following definition.

Definition 2.3.5 (Strictly $\mathbb{A}^1$-invariance)**.** Let $H^i_{Nisn}(-;\mathcal{G})$ be the Nisnevich sheafification of the presheaf $U \to H^i(-;\mathcal{G})$ where $U \in Sm_k$ and $H^i(-;\mathcal{G})$ is the cohomology functor valued in a sheaf of groups $\mathcal{G}$.

(1) A Nisnevich sheaf of groups $\mathcal{G}$ on Sm_k is said to be strongly $\mathbb{A}^1$-invariant if for any $X \in Sm_k$, the projection $\mathbb{A}^1 \times X \to X$ induces a map

$$H^i_{Nisn}(X;\mathcal{G}) \to H^i_{Nisn}(\mathbb{A}^1 \times X;\mathcal{G})$$

which is a bijection for $i = 0, 1$.

(2) A Nisnevich sheaf of abelian groups $\mathcal{M}$ on Sm_k is said to be strictly $\mathbb{A}^1$-invariant if for any $X \in Sm_k$, the projection $\mathbb{A}^1 \times X \to X$ induces a map

$$H^i_{Nisn}(X;\mathcal{G}) \to H^i_{Nisn}(\mathbb{A}^1 \times X;\mathcal{G})$$

which is a bijection for $i \in \mathbb{N}$.

Let $\mathcal{A}\rfloor_k^{\mathbb{A}^1}$ denote the category of strictly $\mathbb{A}^1$-invariant sheaves of groups. In fact, $\mathcal{A}\rfloor_k^{\mathbb{A}^1}$ is an abelian category, but we do not need this fact here. An important result of Morel points out that homotopy sheaves are indeed satisfy these conditions.

Theorem 2.3.6 ([23, Corollary 6.2])**.** *Let $\mathcal{X} \in \mathcal{H}_\bullet(k)$ be a pointed motivic space. Then the sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and for $n \geq 2$, the sheaf of abelian groups $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant.*

Remark. The proof is highly non-trivial and relies heavily on the fact that k is a field.

Example 2.3.7. We calculate the homotopy sheaves of $\mathbb{G}_m$ as in Example 2.2.5. Consider the presheaf $U \mapsto [U, \mathbb{G}_m]_{\mathbb{A}^1}$. Since $\mathbb{G}_m$ is already a motivic space, we know that $[U, \mathbb{G}_m]_{\mathbb{A}^1} = \pi_0(\mathrm{Hom}_{Spc_k}(U, \mathbb{G}_m))$. Here we view $\mathbb{G}_m$ and U as discrete Kan complexes. Therefore, it is just $\mathrm{Hom}_{Sm_k}(U, \mathbb{G}_m)$. We conclude that $\pi_0^{\mathbb{A}^1}(\mathbb{G}_m) = \mathbb{G}_m$.

Next, fix $n \geq 1$ and assume $\mathbb{G}_m = \mathbb{A}^1 - 0$ is pointed at 1. Consider the presheaf $U \mapsto [S^n \wedge U_+, \mathbb{G}_m]_{\mathbb{A}^1}$. Becuase $\mathbb{G}_m$ is $\mathbb{A}^1$-invariant as a sheaf, so

$$[S^n \wedge U_+, \mathbb{G}_m]_{\mathbb{A}^1_\bullet} = [S^n \wedge U_+, \mathbb{G}_m]_{Spc_{k,\bullet}} = [S^n, \mathrm{Hom}_\bullet(U_+, \mathbb{G}_m)]_{Spc_{k,\bullet}}.$$

This is trivial because the target $\mathrm{Hom}_\bullet(U_+, \mathbb{G}_m)$ is a discrete Kan complex. We conclude that $\pi_n^{\mathbb{A}^1}(\mathbb{G}_m) = 0$ for $n \geq 1$. We can say that $\mathbb{G}_m$ is discrete from the standpoint of $\mathbb{A}^1$-homotopy theory.

2.4 Eilenberg-Maclane spaces

In this section, we discuss the Eilenberg-Maclane objects in Spc_k. They will play a crucial role in the Postnikov tower in the next section. Let $\mathcal{F}_\bullet$ be a complex of presheaves concentrated in the positive degrees, namely $\mathcal{F}_i = 0$ for $i < 0$. We have the following proposition.

Proposition 2.4.1 (Dold-Kan equivalence [9, Proposition 1.24])**.** *The category of complexes of presheaves of abelian groups $\mathcal{F}_\bullet$ with $\mathcal{F}_i = 0$ for $i < 0$ is equivalent to the category of presheaves of simplicial abelian groups.*

By the above correspondence we can associate $\mathcal{F}_\bullet$ to a simplicial set $K(\mathcal{F}_\bullet)$, which is a Kan complex.

Definition 2.4.2 (Eilenberg-Maclane spaces)**.** Let $\mathcal{M}$ be a Nisnevich sheaf of abelian groups on Sm_k. Fix an integer $n \geq 0$. We define the associated Eilenberg-Maclane space $K(\mathcal{M}, n)$ by viewing $\mathcal{M}$ as a complex with only nontrivial entry in degree n and then apply the Dold-Kan correspondence.

Note that by definition, for $U \in Sm_k$ we have

$$\pi_i K(\mathcal{M}, n)(U) = \begin{cases} 0, & \text{if } i \geq n+1 \\ H^{n-i}_{Nisn}(U, \mathcal{M}), & \text{if } 0 \leq i \leq n \end{cases}$$

Remark. Let $\mathcal{G}$ be a sheaf of abelian groups.

(1) Similar to what we have in classical topology, $\pi_i^{\mathbb{A}^1} K(\mathcal{G}, n) = \mathcal{G}$ if $i = n$ and it equals to 0 otherwise. And as we are workin in Nisnevich topology, the cohomology functor is representable by the Eilenberg-Maclane space. Namely, for any pointed motivic space $\mathcal{X}$, we have a bijection

$$[\mathcal{X}, K(\mathcal{G}, n)]_{\mathbb{A}^1} \xrightarrow{\sim} H^n_{Nisn}(\mathcal{X}; \mathcal{G}).$$

(2) $K(\mathcal{G}, 1)$ still makes sense even if $\mathcal{G}$ is not abelian. It is usually called the classifying space of $\mathcal{G}$. We can check by a straightforward computation that $\mathcal{G}$ is $\mathbb{A}^1$-invariant if and only if $K(\mathcal{G}, 1)$ is an $\mathbb{A}^1$-local space.

(3) Similarly, if $\mathcal{G}$ is strictly $\mathbb{A}^1$-invariant if and only if for any $n \in \mathbb{N}$ the Eilenberg-Maclane space $K(\mathcal{G}, n)$ is $\mathbb{A}^1$-local.

2.5 $\mathbb{A}^1$-fiber sequences

In classicial topology, for a topological group G, the principal G-bundles (or sometimes they are called G-torsors), together with the associated fiber sequences, are very useful for

various computations as they induce long exact sequences in homotopy groups. We have simialr constructions in $\mathbb{A}^1$-homotopy theory. First we define the (co)fiber sequences in $\mathcal{H}_\bullet(k)$.

Definition 2.5.1 ((co)fiber sequences)**.** Let $\mathcal{F}, \mathcal{X}, \mathcal{Y} \in \mathcal{H}_\bullet(k)$ be pointed motivic spaces. We say

$$\mathcal{F} \to \mathcal{X} \to \mathcal{Y}$$

is a (co)fiber sequence (or (co)fibration) if the following communitative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathcal{Y} \end{array}$$

is homotopy (co)cartesian. A sequence of pointed k-spaces is called $\mathbb{A}^1$-(co) fiber sequence if it is a (co)fiber sequence in $\mathcal{H}_\bullet(k)$ after applying the localization L_{mot}.

Remark. Fiber sequences are very useful because they induce a long exact sequence in homotopy sheaves, as in the classical topology. More precisely, given a $\mathbb{A}^1$-fiber sequence $\mathcal{F} \to \mathcal{X} \to \mathcal{Y}$ of pointed motivic space, we have a long exact sequence of $\mathbb{A}^1$-homotopy sheaves.

$$\cdots \to \pi_n^{\mathbb{A}^1}(\mathcal{F}) \to \pi_n^{\mathbb{A}^1}(\mathcal{X}) \to \pi_n^{\mathbb{A}^1}(\mathcal{Y}) \to \pi_{n-1}^{\mathbb{A}^1}(\mathcal{F}) \to \cdots \to \pi_0^{\mathbb{A}^1}(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{Y}).$$

We are mostly interested in the case where the fiber $\mathcal{F}$ is a sheaf of groups, usually denoted by $\mathcal{G}$. This is an analogue of G-principal bundles (or sometimes called G-torsors) in classical topology.

Definition 2.5.2 ($\mathcal{G}$-principal fibration)**.** Let $\mathcal{G} \to \mathcal{X} \to \mathcal{Y}$ be an $\mathbb{A}^1$-fiber sequence. We say it is a ($\mathbb{A}^1$-)$\mathcal{G}$-principal fibration if $\mathcal{G}$ is a Nisnevich sheaf of groups and $\mathcal{X}$ has a structure of Nisnevich locally trivial $\mathcal{G}$-torsor over $\mathcal{Y}$.

Given a $\mathcal{G}$-fibration of simplicial sheaves, how can we get an $\mathbb{A}^1$-fibration? The localization functor L_{mot} is left adjoint to the inclusion functor. By the adjoint functor theorem, it preserves all cofiber sequences but not necessarily fiber sequences in pointed k-spaces. Morel showed that we have a criterion to see when it is an $\mathbb{A}^1$-fiber sequence.

Theorem 2.5.3 ([23, Theorem 6.50])**.** *Let $\mathcal{G} \to \mathcal{X} \to \mathcal{Y}$ be a $\mathcal{G}$-principal fibration of k-spaces. If $\pi_0^{\mathbb{A}^1}(G)$ is strongly $\mathbb{A}^1$-invariant, then it is an $\mathbb{A}^1$-principal fibration.*

Given a vector bundle of rank n, we can associate it with a GL_n-torsor. In this case we care about principal fibrations with the structure group GL_n or its subgroup. A result of Balwe, Hogadi and Sawant tells us they are all $\mathbb{A}^1$-principal fibrations.

Theorem 2.5.4 ([31, Theorem 5.2])**.** *If G is a reductive algebraic group over a perfect field k, then $\pi_0^{\mathbb{A}^1}(G)$ is strongly $\mathbb{A}^1$-invariant.*

Remark. It is pointed out in [31, Remark 1.2] that the conlusion of above theorem holds for any algebraic group over a perfect field.

Example 2.5.5 ([9, Example 4.1.11])**.** Let G be a simplicial sheaf of groups. Write EG as a sheaf of simplicial groups with n-th term $(EG)_n = G^{n+1}$. The face maps are induced by rpojections and the degeneracy maps are induced by diagonals. For simplicity, we still denote by EG after applying L_{mot}. It is equivalent to $*$ in $\mathcal{H}_\bullet(k)$ since localization functor preserves weak equivalences in the simplicial category [28, Proposition 5.2]. EG naturally admits a left and right G-action. The orbit space $EG/G \cong BG$ is the classifying space of G. Apply L_{mot} to BG and now we can view BG as a space in $\mathcal{H}_\bullet(k)$. By theorem 2.5.4, for an algebraic group G, we have an $\mathbb{A}^1$-principal fibration

$$G \to EG \to BG.$$

Later we will use such $\mathbb{A}^1$-fiber sequences for $G = GL_n$ or $G = SL_n$.

Proposition 2.5.6. *If G is a Nisnevich sheaf of groups, then $\pi_0^{\mathbb{A}^1}(BG)$ is trivial. In other words, the classifying space BG is $\mathbb{A}^1$-connected.*

Proof. By the unstable $\mathbb{A}^1$-connectivity theorem 2.3.4, we need to show that the sheafification of the presheaf $\pi_0(L_{Nisn}BG)$ is trivial. By definition this is the sheafification of the presheaf $H^1_{Nisn}(-;G)$ on Sm_k, which classifies all G-torsors with Nisnevich-local trivialization. All the G-principal fibrations give us locally Nisnevich trivial G-torsors, so we are done. □

Corollary 2.5.7. *Let G be a Nisnevich sheaf of groups, we have $\pi_i^{\mathbb{A}^1}(BG) \cong \pi_{i-1}^{\mathbb{A}^1}(G)$ for $i \geq 1$.*

Proof. Consider the $\mathbb{A}^1$-fiber sequence $G \to EG \to BG$ and passing to the long exact sequence in homotopy sheaves, we have

$$\cdots \to \pi_n^{\mathbb{A}^1}(G) \to \pi_n^{\mathbb{A}^1}(EG) \to \pi_n^{\mathbb{A}^1}(BG) \to \pi_{n-1}^{\mathbb{A}^1}(G) \to \cdots \to \pi_0^{\mathbb{A}^1}(EG) \to \pi_0^{\mathbb{A}^1}(BG).$$

Since EG is $\mathbb{A}^1$-contractible, we have $\pi_i^{\mathbb{A}^1}(BG) \cong \pi_{i-1}^{\mathbb{A}^1}(G)$ for $i \geq 1$. □

Other than the fiber sequences coming from torsors, we also need fiber sequences from inclusion of group schemes. Morel gives a criterion when it is also an $\mathbb{A}^1$-fiber sequence.

Theorem 2.5.8 ([23, Theorem 6.53])**.** *Let $\mathcal{F} \to \mathcal{X} \to \mathcal{Y}$ be a fiber sequence of pointed k-spaces. If $\mathcal{Y}$ is $\mathbb{A}^1$-connected and the sheaf of groups $\pi_0^{\mathbb{A}^1}(\Omega^1\mathcal{Y}) = \pi_n(L_{mot}(\Omega^1\mathcal{Y}))$ is strongly $\mathbb{A}^1$-invariant, then it is also an $\mathbb{A}^1$-fiber sequence.*

Corollary 2.5.9 ([23, Proposition 8.11]). *Let $H \subset G$ be a monomorphism of sheaves of groups and assume that $\pi_0^{\mathbb{A}^1}(G)$ is strongly $\mathbb{A}^1$-invariant. Then the fiber sequences of pointed k-spaces*

$$G/H \to BH \to BG$$

is an $\mathbb{A}^1$-fiber sequence.

2.6 Postnikov tower

Recall that $\mathcal{H}(k)$ is a presentable ∞-category. Let $\mathcal{H}(k)_{\leq n}$ be the subcategory consisting of n-truncated objects. The inclusion functor admits a left adjoint truncation functor $\tau_{\leq n}$ [19, Proposition 5.5.6.18].

Definition 2.6.1 (Postnikov tower). Given a motivic spaces $\mathcal{X}$, we have a sequence from repeatly applying truncations.

$$\ldots \xrightarrow{\tau_{\leq n}} \tau_{\leq n}\mathcal{X} \xrightarrow{\tau_{\leq n-1}} \tau_{\leq n-1}\mathcal{X} \longrightarrow \ldots \xrightarrow{\tau_{\leq 2}} \tau_{\leq 2}\mathcal{X} \xrightarrow{\tau_{\leq 1}} \tau_{\leq 1}\mathcal{X} \xrightarrow{\tau_{\leq 0}} \tau_{\leq 0}\mathcal{X}$$

together with morphisms $\mathcal{X} \to \tau_{\leq n}\mathcal{X}$.

Remark.

(1) $\pi_i^{\mathbb{A}^1}(\tau_{\leq n}\mathcal{X}) = \pi_i^{\mathbb{A}^1}(\mathcal{X})$ if $0 \leq i \leq n$ and $\pi_i(\tau_{\leq n}\mathcal{X}) = 0$ if $i \geq n+1$.

(2) The Postnikov tower converges to $\mathcal{X}$. Namely, the canonical map $\mathcal{X} \to \lim_n \tau_{\leq n}\mathcal{X}$ is an equivalence.

In an arbitary presentable ∞-category, the above construction does not lead to a convergent Postnikov tower. It converges in the case of $\mathcal{H}(k)$ in which every space is Nisnevich-local. Now we want to analyze the homotopy fiber of $\tau_{\leq n+1}\mathcal{X} \to \tau_{\leq n}\mathcal{X}$. Recall that [20, Chapter VI.2] for a Kan complex X, the Postnikov tower of X consisting of fiber sequences

$$K(\pi_n X, n) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$$

where the homotopy fiber $K(\pi_n X, n)$ is the Eilenberg-Maclane space. If $\mathcal{X}$ is a sheaf of Kan complex, by theorem 2.3.6, $K(\pi_n^{\mathbb{A}^1}\mathcal{X}, n)$ is thus $\mathbb{A}^1$-local for $n \geq 1$. We have $\mathbb{A}^1$-fiber sequences

$$K(\pi_n\mathcal{X}, n) \to \tau_{\leq n}\mathcal{X} \to \tau_{\leq n-1}\mathcal{X}.$$

We use the Postnikov tower to prove one way to understand Nisnevich cohomology for motivic spaces.

Proposition 2.6.2. *Fix an integer $n \geq 2$. Let $\mathcal{X} \in \mathcal{H}_\bullet(k)$ be a pointed n-connective motivic space and $\mathcal{A}$ be a strictly $\mathbb{A}^1$-invariant sheaf of groups. Then we have a bijection*

$$H^n_{Nisn}(\mathcal{X}; \mathcal{A}) \xrightarrow{\sim} Hom_{\mathcal{A}|_k^{\mathbb{A}^1}}(\pi_n^{\mathbb{A}^1}(\mathcal{X}), \mathcal{A}).$$

Proof. Because $n \geq 2$ and $\mathcal{A}$ is strictly $\mathbb{A}^1$-invariant, the Eilenberg-Maclane space $K(\mathcal{A}, n)$ is motivic and 2-connective, so we can identify $H^n_{Nisn}(\mathcal{X}; \mathcal{A})$ with $[\mathcal{X}, K(\mathcal{A}, n)]_{\mathbb{A}^1_\bullet}$.

Given a map between pointed motivic spaces $\mathcal{X} \to K(\mathcal{A}, n)$, since $K(\mathcal{A}, n)$ is n-truncated, we have a map $\tau_{\leq n}\mathcal{X} \to K(\mathcal{A}, n)$. Consider the Postnikov tower at nth stage, we have a fiber sequence

$$K(\pi_n^{\mathbb{A}^1}(\mathcal{X}), n) \to \tau_{\leq n}\mathcal{X} \to \tau_{\leq n-1}\mathcal{X}.$$

Note that $\mathcal{X}$ is n-connective, we get an $\mathbb{A}^1$-equivalence $K(\pi_n^{\mathbb{A}^1}(\mathcal{X}), n) \xrightarrow{\sim} \tau_{\leq n}\mathcal{X}$. Now we have a map between pointed motivic spaces $K(\pi_n^{\mathbb{A}^1}(\mathcal{X}), n) \to K(\mathcal{A}, n)$, which induces a map between strictly $\mathbb{A}^1$-invariant sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{A}$.

Conversely, if we have a map between strictly $\mathbb{A}^1$-invariant sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{A}$, we get a map between motivic spaces $K(\pi_n^{\mathbb{A}^1}(\mathcal{X}), n) \to K(\mathcal{A}, n)$. Identify $\tau_{\leq n}\mathcal{X}$ with $K(\pi_n^{\mathbb{A}^1}(\mathcal{X}), n)$ using Postnikov tower and compose the map with the canonical map $\mathcal{X} \to \tau_{\leq n}\mathcal{X}$, we have the result. □

2.7 Contracted homotopy sheaves

In $\mathbb{A}^1$-homotopy theory, there are two spaces which play the role of spheres in classical topology. One is the constant sheaf S^i on Sm_k associated to the simplicial i-sphere. The other is the Tate sphere $\mathbb{G}_m := \mathbb{A}^1 - 0$ pointed at 1. For a pointed motivic space $\mathcal{X}$, the morphisms from simplicial spheres gives us the homotopy sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ and for the $\mathbb{G}_m$-loop spaces, we need to introduce the contraction of sheaves. Let $\mathcal{X}$ be a pointed motivic space. We denote by $\mathcal{X}^{\mathbb{G}_m}$ the Nisnevich sheaf mapping a smooth k-scheme U to $\mathcal{X}(U \times \mathbb{G}_m)$.

Proposition 2.7.1. *Let $\mathcal{X}$ be a pointed motivic space and it is $\mathbb{A}^1$-connected. We have $\pi_k^{\mathbb{A}^1}(\mathcal{X})^{\mathbb{G}_m} = \pi_k^{\mathbb{A}^1}(\mathcal{X}^{\mathbb{G}_m})$ for positive integers $k \geq 1$.*

Proof. We first prove this in the case of $\mathcal{X}$ is an Eilenberg-Maclane space, and for a general motivic space, we use its Postnikov tower.

Assume $\mathcal{G}$ is a strictly $\mathbb{A}^1$-invariant sheaf of groups and $\mathcal{X} = K(\mathcal{G}, n)$ is $\mathbb{A}^1$-local as a space. Let $U \in Sm_k$ be a smooth k-scheme. On the left hand side,

$$(\pi_k^{\mathbb{A}^1}(\mathcal{X})^{\mathbb{G}_m})(U) = (\pi_k^{\mathbb{A}^1} K(\mathcal{G}, n))(U \times \mathbb{G}_m).$$

It equals to $\mathcal{G}(U \times \mathbb{G}_m)$ if $n = k$ and 0 otherwise. On the right hand side,

$$\pi_k^{\mathbb{A}^1}(\mathcal{X}^{\mathbb{G}_m})(U) = \pi_k^{\mathbb{A}^1}(K(\mathcal{G}, n)^{\mathbb{G}_m})(U).$$

This is the Nisnevich sheafification of the presheaf $U \mapsto \pi_k(K(\mathcal{G}, n)(U \times \mathbb{G}_m)) = H^{n-k}(U \times \mathbb{G}_m; \mathcal{G})$. Now we only need to prove that $H^i(U \times \mathbb{G}_m; \mathcal{G})$ vanishes for $i \geq 1$ and U is a spectrum of a local henselian ring. This follows from Lemma 6.12 and Lemma 6.14 in [23].

Now let $\mathcal{X}$ be a general $\mathbb{A}^1$-connected, pointed motivic space. Consider the Postnikov tower of $\mathcal{X}$ and we have the following $\mathbb{A}^1$-fiber sequences.

$$K(\pi_n^{\mathbb{A}^1}\mathcal{X}, n)^{\mathbb{G}_m} \to (\tau_{\leq n}\mathcal{X})^{\mathbb{G}_m} \to (\tau_{\leq n-1}\mathcal{X})^{\mathbb{G}_m}. \tag{2.1}$$

Now we will prove for all $n \geq 0$, $(\pi_k^{\mathbb{A}^1}(\tau_{\leq n}\mathcal{X}))^{\mathbb{G}_m} = \pi_k^{\mathbb{A}^1}((\tau_{\leq n}\mathcal{X})^{\mathbb{G}_m})$. Use induction on n. For $n = 0$, $\tau_{\leq 0}\mathcal{X}$ is $\mathbb{A}^1$-contractible, so this is trivial. Assume this is true for $n-1$ and consider the long exact sequence in homotopy sheaves associated to the fiber sequence 2.1, use the fact that we have proved for Eilenberg-Maclane space and we are done. Finally, recall that $\pi_k^{\mathbb{A}^1}(\tau_{\leq n}\mathcal{X}) = \pi_k^{\mathbb{A}^1}(\mathcal{X})$ for $0 \leq k \leq n-1$, we conclude that $\pi_k^{\mathbb{A}^1}(\mathcal{X})^{\mathbb{G}_m} = \pi_k^{\mathbb{A}^1}(\mathcal{X}^{\mathbb{G}_m})$ for positive integers $k \geq 1$. □

Definition 2.7.2 (contraction)**.** Let $\mathcal{G}$ be a sheaf of abelian groups on Sm_k. We define

$$\mathcal{G}_{-1} := \ker(\mathcal{G}^{\mathbb{G}_m} \xrightarrow{ev_1} \mathcal{G}).$$

The n-fold contraction $\mathcal{G}_{-n} := (\mathcal{G}_{-(n-1)})_{-1}$ is defined by induction.

Example 2.7.3. We can calculate $(\mathbb{G}_m)_{-1}$ from the definition. Let $U = Spec\ R$ be a local scheme. By Example 2.2.5, we know that

$$\begin{aligned}(\mathbb{G}_m)_{-1}(U) &= \ker(\mathcal{O}^\times(Spec\ k[T, T^{-1}] \times Spec\ R) \xrightarrow{ev_1} \mathcal{O}^\times(Spec\ R)) \\ &= \mathcal{O}^\times(Spec\ k[T, T^{-1}]) \\ &= \bigoplus_{n \in \mathbb{Z}} kT^n\end{aligned}$$

So at every stalk, $(\mathbb{G}_m)_{-1}$ is isomorphic to the constant sheaf $\underline{\mathbb{Z}}$. Thus, $(\mathbb{G}_m)_{-1} \cong \underline{\mathbb{Z}}$.

Now we have a nice way to describe $\mathbb{G}_m$-loop space.

Theorem 2.7.4. *Let $\mathcal{X}$ be a pointed motivic space and it is $\mathbb{A}^1$-connected. Write the homotopy sheaves of mapping space $\pi_{i,j}^{\mathbb{A}^1}(\mathcal{X}) := \pi_i^{\mathbb{A}^1}(Hom_\bullet(\mathbb{G}_m^{\wedge j}, \mathcal{X}))$. There is an isomorphism for all $i \geq 1$ and $j \geq 0$.*

$$\pi_{i,j}^{\mathbb{A}^1}(\mathcal{X}) \cong \pi_i^{\mathbb{A}^1}(\mathcal{X})_{-j}.$$

Proof. We only need to prove $j = 1$ and the rest follows from the induction on j. By Proposition 2.7.1, $\pi_i^{\mathbb{A}^1}(\mathcal{X})$ is isomorphic to the kernel of $\pi_i^{\mathbb{A}^1}(\mathcal{X}^{\mathbb{G}_m}) \xrightarrow{ev_1} \pi_i^{\mathbb{A}^1}(\mathcal{X})$. And according to the definition of contraction, this is just $\pi_i^{\mathbb{A}^1}(\mathrm{Hom}_\bullet(\mathbb{G}_m, \mathcal{X}))$. □

Corollary 2.7.5. *Let $\mathcal{G}$ be a sheaf of abelian groups on Sm_k and it is strictly $\mathbb{A}^1$-invariant. Then $\mathcal{G}_{-1}$ is also strictly $\mathbb{A}^1$-invariant.*

Proof. We need to show that $K(\mathcal{G}_{-1}, n)$ is a motivic space for all $n \in \mathbb{N}$. Fix a positive integer n, because $\mathcal{G}$ is strictly $\mathbb{A}^1$-invariant, $K(\mathcal{G}, n)$ is $\mathbb{A}^1$-local. We can conclude by Proposition 2.7.1 and the definition of contraction. □

Another nice property of contraction is that for a strongly $\mathbb{A}^1$-invariant sheaf of groups $\mathcal{G}$, $\mathcal{G} \mapsto \mathcal{G}_{-1}$ is an exact functor from strongly $\mathbb{A}^1$-invariant sheaves to strongly $\mathbb{A}^1$-invariant sheaves.

Proposition 2.7.6 ([23, Lemma 7.33])**.** *Suppose we have an short exact sequence*

$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 1 \tag{2.2}$$

where $\mathcal{K}$,$\mathcal{G}$,$\mathcal{H}$ are strongly $\mathbb{A}^1$-invariant sheaves of groups. Then the diagram

$$1 \to \mathcal{K}_{-1} \to \mathcal{G}_{-1} \to \mathcal{H}_{-1} \to 1$$

is still a short exact sequence of strongly $\mathbb{A}^1$-invariant sheaves of groups.

Proof. Strongly $\mathbb{A}^1$-invariance is preserved by contraction due to Corollary 2.7.5. Now we will prove the exactness. According to the definition of contraction of sheaves, we only need to check that $\mathcal{G} \to \mathcal{H}$ is an epimorphism. More specifically, we need to show that the morphism between sheaves $\mathcal{G}^{\mathbb{G}_m} \to \mathcal{H}^{\mathbb{G}_m}$ is an epimorphism.

We check it on stalks. Let $X \in Sm_k$ be a smooth k-scheme and $x \in X$ is a closed point. Denote by $X_x := \mathrm{Spec}(\mathcal{O}_{X,x})$ the localization of X at x. We want to show that $\mathcal{G}(\mathbb{G}_m \times X_x) \to \mathcal{H}(\mathbb{G}_m \times X_x)$ is an epimorphism between groups. Let α be an element in $\mathcal{H}(\mathbb{G}_m \times X_x)$. It must come from an element $\overline{\alpha} \in \mathcal{H}(\mathbb{G}_m \times U)$ where U is an open neighborhood of x in X. Evaluate the short exact sequence 2.2 at $\mathbb{G}_m \times U$, and we can see that $\overline{\alpha}$ defines a $\mathcal{K}$-torsor over $\mathbb{G}_m \times U$. Every $\mathcal{K}$-torsor over $\mathbb{G}_m \times X_x$ is classified by $H^1(\mathbb{G}_m \times X_x; \mathcal{K})$. We know that $H^1(\mathbb{G}_m \times X_x; \mathcal{K}) = 0$ because $\mathcal{K}$ is strongly $\mathbb{A}^1$-invariant [23, Lemma 6.14]. (cf. the proof of Proposition 2.7.1) By shrinking U, the $\mathcal{K}$-torsor defined by $\overline{\alpha}$ is the trivial torsor, which admits a global section. This means that there exists an element $\beta \in \mathcal{G}(\mathbb{G}_m \times U)$ which is the preimage of $\overline{\alpha}$. □

Example 2.7.7. Let $n \geq 2$ be an integer. Consider the $\mathbb{A}^1$-fiber sequence

$$GL_n \to EGL_n \to BGL_n.$$

For $j \geq 0$, the Hom functor $\mathrm{Hom}_\bullet(\mathbb{G}_m^{\wedge j}, -)$ preserves fiber sequence and we look at the long exact sequence in homotopy sheaves. Recall that EGL_n is $\mathbb{A}^1$-contractible, so we have isomorphisms $\pi_{i,j}^{\mathbb{A}^1}(GL_n) \cong \pi_{i+1,j}^{\mathbb{A}^1}(BGL_n)$ for $i \geq 1$. Since BGL_n is $\mathbb{A}^1$-connected, by Theorem 2.7.4, we have

$$\pi_{i,j}^{\mathbb{A}^1}(GL_n) \cong \pi_{i+1,j}^{\mathbb{A}^1}(BGL_n) \cong \pi_{i+1}^{\mathbb{A}^1}(BGL_n)_{-j} \cong \pi_i^{\mathbb{A}^1}(GL_n)_{-j}.$$

Note that here we cannot directly apply the theorem to GL_n because it is not $\mathbb{A}^1$-connected.

Example 2.7.8. We look more closely at $GL_1 = \mathbb{G}_m$. Consider the $\mathbb{A}^1$-fiber sequence

$$\mathbb{G}_m \to E\mathbb{G}_m \to B\mathbb{G}_m.$$

Passing to the long exact sequence in homotopy sheaves and note that $E\mathbb{G}_m$ is $\mathbb{A}^1$-contractible, we have $\pi_i^{\mathbb{A}^1}(\mathbb{G}_m) \cong \pi_{i+1}^{\mathbb{A}^1}(B\mathbb{G}_m)$ for any $i \geq 0$. Since $B\mathbb{G}_m$ is $\mathbb{A}^1$-connected, we can use Example 2.3.7 and Theorem 2.7.4 to compute the following

$$\pi_{i,j}^{\mathbb{A}^1}(B\mathbb{G}_m) = \begin{cases} \mathbb{G}_m & \text{if } i = 1 \text{ and } j = 0, \\ \mathbb{Z} & \text{if } i = 1 \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lastly we prove a technical result about cohomology for the punctured affine space $\mathbb{A}^n \backslash 0$, which will be useful later in our classification of vector bundles.

Proposition 2.7.9. *Let A be a strictly $\mathbb{A}^1$-invariant sheaf and $n \geq 2$ be an integer. We have*

$$H^i(\mathbb{A}^n \backslash 0, A) = \begin{cases} A(k) & \text{if } i = 0, \\ A_{-n}(k) & \text{if } i = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $i = 0$, $H^0(\mathbb{A}^n \backslash 0, A) = H^0(\mathbb{A}^n, A) = A(k)$ since $0 \in \mathbb{A}^n$ is at least of codimension 2 and A is strictly $\mathbb{A}^1$-invariant. For $i \geq 1$, because A is strictly $\mathbb{A}^1$-invariant, the Eilenberg-Maclane space $K(A, i)$ is a motivic space for $i \geq 0$. Use the adjunction in $\mathcal{H}_\bullet(k)$ and Theorem 2.7.4, we have

$$\begin{aligned} H^i(\mathbb{A}^n \backslash 0, A) &= [\mathbb{A}^n \backslash 0, K(A, i)]_{\mathbb{A}^1_\bullet} \\ &= [\Sigma^{n-1} \mathbb{G}_m^{\wedge n}, K(A, i)]_{\mathbb{A}^1_\bullet} \\ &= [S^{n-1}, \mathrm{Hom}_\bullet(\mathbb{G}_m^{\wedge n}, K(A, i))]_{\mathbb{A}^1_\bullet} \\ &= \pi_{n-1,n}^{\mathbb{A}}(K(A, i))(k) \\ &= \pi_{n-1}^{\mathbb{A}^1}(K(A, i))_{-n}(k). \end{aligned}$$

Note that $K(A, i)$ only has nontrivial $\mathbb{A}^1$-homotopy sheaves in degree i. □

Chapter 3

Algebraic K-Theory

From now on we assume our base field k has a characteristic different from 2. The reason for this is that various calculations of homotopy sheaves will be given by K-theory sheaves, which may have a different presentation over a base filed of characteristic 2. We will give some example of motivic spaces in $\mathcal{H}(k)$ associated to various K-theory, of which the unramified Milnor-Witt K-theory sheaves play an impotant role in $\mathbb{A}^1$-homotopy theory. As shown by Morel [23, Theorem 6.40], for $n \geq 2$, the homotopy sheaves of $\mathbb{A}^n - 0$ can be calculated as

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - 0) \cong K_n^{MW}.$$

This will become very useful in the calculation of next chapter. In section 3.2, we will see one of the motivating reasons for $\mathcal{H}(k)$ is the presheaf $K(-)$ associated to Quillen K-theory is both Nisnevich local and $\mathbb{A}^1$-invariant, and indeed it is representable in $\mathcal{H}(K)$. And finally we show how to calculate the contraction of these K-theory sheaves.

3.1 Unramified Milnor-Witt K-theory

First we give a simple presentation of Milnor-Witt K-theory over the field k, following Morel in [23, Definition 3.1].

Definition 3.1.1. The Milnor-Witt K-theory of the field k is the graded associative ring $K_*^{MW}(k)$ generated by the symbols $[u]$ of degree $+1$, for each unit $u \in k^\times$, and one symbol η of degree -1. These symbols follow four relations.

(1) (Steinberg relation) $[a][1-a] = 0$ for each $a \in k^\times$ and $a \neq 1$.

(2) $[ab] = [a] + [b] + \eta[a][b]$ for $a, b \in k^\times$.

(3) $[u]\eta = \eta[u]$ for $u \in k^\times$.

(4) $\eta(1 + \langle -1 \rangle) = 0$ where $\langle a \rangle = \eta[a] + 1$ for $a \in k^\times$.

This ring $K_*^{MW}(k)$ is closely related to the Grothendieck-Witt ring $GW(k)$. Recall that the Grothendieck-Witt ring $GW(k)$ is the group completion of the commutative mnonoid of isomorphism classes of non-degenerate, symmetric bilinear forms for the orthogonal direct sum. Let $a \in k^\times$, we denote by $\langle a \rangle \in GW(k)$ the associated bilinear form $(x, y) \mapsto axy$. We have the following presentation of $GW(k)$.

Lemma 3.1.2 ([12, Chapter II, section 4])**.** *Assume k is a perfect field with chark $\neq 2$. The Grothendieck-Witt ring $GW(k)$ is generated by the elements $< a >$, $a \in k^\times$, and the following relations give a presentation of $GW(k)$ as an abelian group:*

(1) $\langle a(b)^2 \rangle = \langle a \rangle$*;*

(2) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ *if* $a + b \neq 0$*;*

(3) $\langle a \rangle + \langle -a \rangle = 1 + \langle -1 \rangle$.

Denote by $h := 1 + \langle -1 \rangle \in GW(k)$ the hyperbolic space. They generate an ideal $\mathbb{Z}h$ in $GW(k)$ and $W(k) := GW(k)/\mathbb{Z}h$ is called the Witt ring of the field k. We have a homomorphism $W(k) \to \mathbb{Z}/2$ induced from the rank homomorphism $GW(k) \to \mathbb{Z}$. The kernel $\ker(W(k) \to \mathbb{Z}/2)$ will be denoted by $I(k)$. This is called the fundamental ideal of $W(k)$. $I(k)$ is generated by the Pfister form $\langle\langle a \rangle\rangle := (x, y) \mapsto x^2 - ay^2$. Write $I^m(k)$ for the additive group generated by the m-fold Pfister form [12, Proposition 1.2, Chapter X] if $m > 0$ and $I^m(k) = W(k)$ if $m \leq 0$. This construction gives us a $\mathbb{Z}$-graded ring $I^*(k)$ called the powers of the fundamental ideal $I(k)$ in the Witt ring $W(k)$. Sending a symbol $a \in k^\times$ to the Pfister form defines a gorup homomorphism $K_1^{MW}(k) \to I(k)$, which extends to a ring homomorphism $K_*^{MW}(k) \to I^*(k)$. On the other hand, comparing the relations defining Milnor-Witt K-theory with those defining Milnor K-theory [12, Chapter X, section 6], it is not hard to see that $K_*^{MW}(k)/\eta \cong K_*^M(k)$.

Now consider the following homomorphism

$$\begin{aligned} K_*^M(k) &\to I^*(k)/I^{*+1}(k) \\ ([a_1], [a_2], \ldots, [a_m]) &\mapsto \langle\langle a_1, a_2, \ldots, a_m \rangle\rangle \end{aligned}$$

where $a_1, a_2, \ldots, a_m \in k^\times$ and $\langle\langle a_1, a_2, \ldots, a_m \rangle\rangle$ are corresponding m-fold Pfister forms. Compare with the ring homomorphism from Milnor-Witt K-theory to the powers of fundamental ideal and we get the following commutative square.

$$\begin{array}{ccc} K_*^{MW}(k) & \longrightarrow & K_*^M(k) \\ \downarrow & & \downarrow \\ I^*(k) & \longrightarrow & I^*(k)/I^{*+1}(k) \end{array}$$

Using the Milnor conjecture on quadratic forms $K_*^M(k)/2K_*^M(k) \xrightarrow{\sim} I^*(k)/I^{*+1}(k)$, Morel [11, Théorème 5.3] proved that we have a Cartesian square.

$$\begin{array}{ccc} K_*^{MW}(k) & \longrightarrow & K_*^M(k) \\ \downarrow & & \downarrow \\ I^*(k) & \longrightarrow & K_*^M(k)/2K_*^M(k) \end{array}$$

Moreover, all the above constructions can be used to define sheaves on Sm_k and they are strictly $\mathbb{A}^1$-invariant. For Milnor-Witt K-theory, see [23, Chapter 2]. For Milnor K-theory and other related graded rings, see [13]. Let $n \in \mathbb{Z}$, K_n^{MW} will also denote a Nisnevich sheaf on Sm_k, called unramified Milnor Witt k-theory in weight n. Similar for other graded rings. Note that the residue morphisms from this sheafification process is compatible with the Cartesian square above, thus giving us a Cartesian square of stictly $\mathbb{A}^1$-invariant sheaves of graded rings.

$$\begin{array}{ccc} K_*^{MW} & \longrightarrow & K_*^M \\ \downarrow & & \downarrow \\ I^* & \longrightarrow & K_*^M(k)/2K_*^M \end{array} \tag{3.1}$$

Corollary 3.1.3. *For every $n \in \mathbb{Z}$, we have the following short exact sequences of sheaves.*

$$0 \to I^{n+1} \to K_n^{MW} \to K_n^M \to 0.$$

$$0 \to 2K_n^M \to K_n^{MW} \to I^n \to 0.$$

Moreover, the morphism $K_n^{MW} \to K_{n-1}^{MW}$ induced by the degree -1 element η factors through I^n.

Proof. The two short exact sequences can be directly derived from the Cartesian square 3.1. To check the last statement, for $a_1, a_2, \ldots, a_n \in k^\times$, note that the morphism $K_n^{MW}(k) \to I^n(k)$ is defined by sending $[a_1][a_2]\cdots[a_n]$ to the n-fold Pfister form $\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle$ and the morphism $I^n(k) \to K_{n-1}^{MW}(k)$ is defined by sending $\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle$ to $\eta[a_1][a_2]\cdots[a_n]$. These morphisms are compatible with the sheafification process. □

As mentioned in the beginnning of this chapter, the importance of Milnor-Witt K-theoryis that it is a nontrivial example of calculation of $\mathbb{A}^1$-homotopy sheaves.

Theorem 3.1.4 ([23, Theorem 6.40])**.** *For $n \geq 2$, the pointed k-space $\mathbb{A}^n - 0$ is $\mathbb{A}^1$-$n-1$-connective and is $\mathbb{A}^1$-equivalent to $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$. Moreover, we have*

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - 0) \cong K_n^{MW}.$$

Remark. The proof relies on the detailed construction of Gersten complex related to K_n^{MW} and the universal property of K_n^{MW}, it also gives the identification that $\mathbb{A}^n\backslash 0$ is $\mathbb{A}^1$-equivalent to $\Sigma^{n-1}\mathbb{G}_m^{\wedge n}$. For the details, see [23, Chapter 2].

Corollary 3.1.5 (Brouwer degree [23, Corollary 24, Introduction])**.** *For $n \geq 2$, the canonical morphism*

$$[\mathbb{A}^n\backslash 0, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet} \to K_0^{MW}(k) = GW(k)$$

is an isomorphism.

Proof. Use the previous Theorem 3.1.4 and the identification $\mathbb{A}^n\backslash 0 \simeq \Sigma^{n-1}\mathbb{G}_m^{\wedge n}$, we have

$$\begin{aligned}
[\mathbb{A}^n\backslash 0, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet} &= [\Sigma^{n-1}\mathbb{G}_m^{\wedge n}, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet} \\
&= [S^{n-1}, \mathrm{Hom}_\bullet(\mathbb{G}_m^{\wedge n}, \mathbb{A}^n\backslash 0)]_{\mathbb{A}^1_\bullet} \\
&= \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0)_{-n}(k) \\
&= (K_n^{MW})_{-n}(k) \\
&= K_0^{MW}(k) \\
&= GW(k).
\end{aligned}$$

□

3.2 Representability of Quillen K-theory

One important result of $\mathbb{A}^1$-homotopy theory is that algebraic K-theory, or Quillen K-theory, is representable in $\mathcal{H}(k)$. Let $U \in Sm_k$ be a smooth k-scheme. Write $K(U)$ as the algebraic k-theory space of perfect complexes of quasi-coherent sheaves on U. The Nisnevich sheafification of the presheaf $U \mapsto \pi_n(K(U))$ is denoted by K_n^Q. Take the inlcusion $GL_N \to GL_{N+1}$ as adding 1 to the bottom right corner and write $GL_\infty := colim_N GL_N$ as a group scheme and we take its classifying space BGL_∞. We will sketch a proof that the K-theory sheaf $K(-)$ is represented by $L_{mot}(\mathbb{Z} \times BGL_\infty)$. First we show that $K(-)$ is indeed an object in $\mathcal{H}(k)$.

Proposition 3.2.1. *$K(-)$ is motivic, namely it is an object in $\mathcal{H}(k)$.*

Proof. We need to show that $K(-)$ as a presheaf on Sm_k has Nisnevich descent and is $\mathbb{A}^1$-invariant. The Nisnevich local property is proved in [17, Theorem 10.8], and the $\mathbb{A}^1$-invariance follows from [17, Proposition 6.8]. □

For any associative ring R, the K-thoery space $K(R)$ can be defined as the group completion of $\amalg_n BGL_n(R)$. Quillen has proved the following.

Theorem 3.2.2. *Let R be an associative ring with unit, then the group completion of $\amalg_n BGL_n(R)$ is weakly equivalent to $\mathbb{Z} \times BGL_\infty(R)^+$. Here $BGL_\infty(R)^+$ refers to Quillen's +-constrcution applying to $BGL_\infty(R)$.*

See [24, Chapter IV] for a proof. The next thing we want to show is that the localization functor $L_{\mathbb{A}^1}$ imposing $\mathbb{A}^1$-invariance provides an alternative for the +-construction in $\mathcal{H}(k)$. We need a model for this localization functor.

Consider the cosimplicial scheme $\Delta^\bullet$ defined as follows.

$$\Delta^n := \mathrm{Spec} k[x_0, x_1, \ldots, x_n]/(x_0 + x_1 + \cdots + x_n = 1)$$

with face and degeneracy map similar from the standard topological simplex. It is a closed subscheme in $\mathbb{A}^{n+1}$.

Definition 3.2.3. Let $\mathcal{X}$ be a k-space (a presheaf on Sm_k). We define a presheaf $\mathrm{Sing}^{\mathbb{A}^1}\mathcal{X} := (U \mapsto |\mathcal{X}(U \times \Delta^\bullet)|)$, where $|-|$ is the geometric realization of the Kan complex.

Theorem 3.2.4 ([28, Theorem 4.25])**.** *Let $\mathcal{X}$ be a k-space. Then $Sing^{\mathbb{A}^1}\mathcal{X}$ is $\mathbb{A}^1$-invariant and for any $\mathbb{A}^1$-invariant k-space $\mathcal{Y}$, the natural map $\mathcal{X} \to Sing_{\mathbb{A}^1}\mathcal{X}$ induces an equivalence $Map_{Spc_k}(Sing^{\mathbb{A}^1}\mathcal{X}, \mathcal{Y}) \to Map_{Spc_k}(\mathcal{X}, \mathcal{Y})$.*

Remark. The motivic localization functor L_{mot} is not simply the composition $\mathrm{Sing}^{\mathbb{A}^1} \circ L_{Nisn}$. Note that the geometric realization functor in general does not preserve homotopy limits, so for a Nisnevich local k-space $\mathcal{X}$, $\mathrm{Sing}^{\mathbb{A}^1}\mathcal{X}$ is not neccessarily still Nisnevich local.

Proposition 3.2.5. *Let R be an associative ring with unit. Then the natural map $Sing^{\mathbb{A}^1}BGL_\infty(R) \to Sing^{\mathbb{A}^1}BGL_\infty(R)^+$ is a homotopy equivalence.*

Proof. Recall that Quillen's plus construction does not change the homology and since the fundamental group of both sides are nilpotent [24, Exercise IV.11.9], this is indeed a homotopy equivalence. □

Corollary 3.2.6. *Let $K(-)$ be the motivic space associated to Quillen K-theory in $\mathcal{H}(k)$. Then the map $\mathbb{Z} \times BGL_\infty \to K$ is a $\mathbb{A}^1$-equivalence.*

Proof. We need to show that $L_{mot}(\mathbb{Z} \times BGL_\infty) \to K$ is an $\mathbb{A}^1$-equivalence. Both sides are Nisnevich local so we only need to check on stalks. Combine Theorem 3.2.2 and Proposition 3.2.5, and we are done. □

3.3 Contractions on K-theory sheaves

We will later see that some calculations of homotopy sheaves lead us to K-theory sheaves defined above. In this section we show how to calculate the contraction of K-theory sheaves.

Proposition 3.3.1. *Fix $n \in \mathbb{Z}$. Let I^n be the strictly $\mathbb{A}^1$-invariant sheaf related to nth power of the fundamental ideal. For any integers j, we have $(I^n)_{-j} \cong I^{n-j}$.*

Proof. We prove this for $j = 1$ and on the stalks, then use induction to get the result. Let $X = \mathrm{Spec}A$ be the spectrum of a local ring A. By the definition of contraction, we have a short exact sequence.

$$0 \to I^n(X) \to I^n(X \times \mathbb{G}_m) \to (I^n)_{-1}(X) \to 0.$$

In order to prove $(I^n)_{-1} \cong I^{n-1}$, it suffices to show that I^{n-1} can be fitted into a short exact sequence.

$$0 \to I^n(X) \to I^n(X \times \mathbb{G}_m) \to I^{n-1}(X) \to 0.$$

Use the closed immersion $X \times \{0\} \hookrightarrow X \times \mathbb{A}^1$ and identify X with $X \times \{0\}$, we have a long exact sequence in local cohomology.

$$0 \longrightarrow H^0_X(X \times \mathbb{A}^1; I^n) \longrightarrow H^0(X \times \mathbb{A}^1; I^n) \longrightarrow H^0(X \times \mathbb{G}_m; I^n)$$

$$\longrightarrow H^1_X(X \times \mathbb{A}^1; I^n) \longrightarrow H^1(X \times \mathbb{A}^1; I^n) \longrightarrow \cdots$$

Since X is local and I^n is strictly $\mathbb{A}^1$-invariant, $H^1(X \times \mathbb{A}^1; I^n) = 0$. Apply the lemma 3.3.2 we will prove next and again by strictly $\mathbb{A}^1$-invariance of I^n, we have a short exact sequence.

$$0 \to I^n(X) \to I^n(X \times \mathbb{G}_m) \to I^{n-1}(X) \to 0.$$

□

Lemma 3.3.2. *Let X be a smooth k-scheme and I^n be the unramified sheaf associated to the nth power of fundamental ideal. Then the local cohomology can be computed as follows.*

$$H^*_X(X \times \mathbb{A}^1; I^n) \cong H^{*-1}(X; I^{n-1}).$$

In particular, $H^0_X(X \times \mathbb{A}^1; I^n) = 0$ and $H^1_X(X \times \mathbb{A}^1; I^n) = I^{n-1}(X)$.

Proof. We use the closed immersion $X \times \{0\} \hookrightarrow X \times \mathbb{A}^1$ and identify $X \times \{0\} \cong X$. Choose the obvious trivialization of the normal sheaf of $X \times \{0\}$ in $X \times \mathbb{A}^1$. Recall that I^n is strictly $\mathbb{A}^1$-invariant, we can use the Gersten complex to calculate the cohomology, and the local cohomology in X by definition is

$$H^*_X(X \times \mathbb{A}^1; I^n) = H^*(\mathrm{Fiber} : R\Gamma(X \times \mathbb{A}^1; I^n) \to R\Gamma(X \times \mathbb{G}_m; I^n)).$$

Using the fact that X is smooth, we have the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & I^{n-1}(k(X)) & \longrightarrow & \oplus_{x\in X^{(1)}} I^{n-1}(k(x)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I^n(k(X\times\mathbb{A}^1)) & \longrightarrow & \oplus_{x\in(X\times\mathbb{A}^1)^{(1)}} I^{n-1}(k(x)) & \longrightarrow & \oplus_{x\in(X\times\mathbb{A}^1)^{(2)}} I^{n-2}(k(x)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I^n(k(X\times\mathbb{G}_m)) & \longrightarrow & \oplus_{x\in(X\times\mathbb{G}_m)^{(1)}} I^{n-1}(k(x)) & \longrightarrow & \oplus_{x\in(X\times\mathbb{G}_m)^{(2)}} I^{n-2}(k(x)) \longrightarrow \cdots
\end{array}$$

The fiber is just the Gersten complex for I^{n-1} with a shift in degree. □

Remark. In the proof of Proposition 3.3.1, we only use the fact that I^n is strictly $\mathbb{A}^1$-invariant and admits a flasque resolution, the Gersten complex. This is also true for Milnor K-theory sheaves K_n^M and Quillen K-theory sheaves K_n^Q, see [7] and [33]. So we also have

$$(K_n^M)_{-j} \cong K_{n-j}^M \quad \text{and} \quad (K_n^Q)_{-j} \cong K_{n-j}^Q.$$

Corollary 3.3.3. *For any interger n and j, there is also a canonical isomorphism of sheaves $(K_n^{MW})_{-j} \cong K_{n-j}^{MW}$.*

Proof. We look at the long exact sequence from Corollary 3.1.3.

$$0 \to I^{n+1} \to K_n^{MW} \to K_n^M \to 0.$$

Contracting it j times and by Proposition 3.3.1 and the exactness of contraction, we get the result. □

Using the exactness of contraction again, we have the following.

Corollary 3.3.4. *For any integers m, n and j, we have canonical isomorphisms of sheaves.*

$$(K_n^M/mK_n^M)_{-j} \cong K_{n-j}^M/mK_{n-j}^M \quad \textit{and} \quad (K_n^Q/mK_n^Q)_{-j} \cong K_{n-j}^Q/mK_{n-j}^Q.$$

Chapter 4

Computations of homotopy sheaves of SL_n

In this chapter, we will compute the homotopy sheaves of SL_n. It is closely related to the homotopy sheaves of BSL_n.

4.1 In the stable range

We review the computation of the homotopy sheaves of SL_n and GL_n in the stable range. The calculations were originally done by Morel [23, Chapter 8, Section 2], using K-theory sheaves to describe homotopy sheaves of GL_n and SL_n.

Lemma 4.1.1. *The k-space SL_n is $\mathbb{A}^1$-connected and the classifying space BSL_n is $\mathbb{A}^1$-2-connective (or $\mathbb{A}^1$-simply connected). Namely, $\pi_1^{\mathbb{A}^1}(BSL_n) \cong \pi_0^{\mathbb{A}^1}(SL_n) = *$ for $n \geq 1$.*

Proof. Using Corollary 2.5.7, we only need to show that $\pi_0^{\mathbb{A}^1}(SL_n)$ is trivial by showing that the stalks of this sheaf is trivial. By the unstable $\mathbb{A}^1$ connectivity theorem 2.3.4, it suffices to prove that the Nisnevich sheafification of the presheaf $\pi_0(\mathrm{Sing}^{\mathbb{A}^1} SL_n)$ is trivial at every stalk. Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. We need to prove that

$$\mathrm{Sing}^{\mathbb{A}^1} SL_n(SpecR) = *.$$

In other words, for every matrix M in $SL_n(R)$, M can be connected to the identity matrix I_n via a chain of naive $\mathbb{A}^1$-homotopy. Let $k = R/\mathfrak{m}$ be the residue field. Denote the image of M in $SL_n(k)$ by $\overline{M}$. Recall that an elementary (or transvection) matrix in $SL_n(k)$ is taking the identity matrix and replacing one of the zero elements with a nonzero entry. It is a well known algebraic fact that the elementary matrices generate $SL_n(k)$ for a field k, so we can write $\overline{M}$ as a product of elementary matrices in $SL_n(k)$. Now we lift each of these to $SL_n(R)$ and write $M = E(I_n + P)$ where E is a product of elementary matrices in $SL_n(R)$ and

$P = (p_{ij}) \in M_n(\mathfrak{m})$ is matrix with entries from the maximal ideal $\mathfrak{m}$. Take the determinant of M and expand, we have

$$1 = \det(I_n + P) = (1 + p_{11})|M_{11}| - p_{12}|M_{12}| + \cdots + (-1)^{n+1}p_{1n}|M_{1n}|$$

where M_{ij} is the ijth minor of $I_n + P$. Since p_{1k} is an element of the maximal ideal $\mathfrak{m}$ for $2 \leq k \leq n$, we can write $1 - a = (1 + p_{11})|M_{11}|$ where $a \in \mathfrak{m}$. Because R is a local ring and both a and p_{11} are in $\mathfrak{m}$, we know that $1 - a$ and $1 + p_{11}$ are invertible, thus $|M_{11}|$ is also invertible. Now we can write

$$p_{11} = \frac{1 - a}{|M_{11}|} - 1.$$

Define a new matrix $Q = (q_{ij}) \in M_n(\mathfrak{m}[t])$ where the entries $q_{ij} = tp_{ij}$ except for $i = j = 1$. The p_{11} can be solved in $\mathfrak{m}$ indicates that we can choose q_{11} in such a way that $\det(I_n+Q) = 1$. When $t = 0$, $Q = I_n$ and when $t = 1$, $Q = P$. Therefore, we have found a naive $\mathbb{A}^1$-homotopy from I_n to $I_n + P$. And since every elementary matrix is $\mathbb{A}^1$-homotopic to I_n, we have proved that M can be connected to I_n via a chain of naive $\mathbb{A}^1$-homotopy. □

Proposition 4.1.2. *The natural map $BSL_n \to BGL_n$ induces isomorphisms $\pi^{\mathbb{A}^1}_{i+1}(BSL_n) \cong \pi^{\mathbb{A}^1}_{i+1}(BGL_n)$ for $i \geq 1$ and $n \geq 1$.*

Proof. The canonical inclusion $SL_n \to GL_n$ induces a fiber sequence

$$\mathbb{G}_m \to BSL_n \to BGL_n.$$

Here we identify the fiber GL_n/SL_n with $\mathbb{G}_m$ using the determinant map. Looping and note that BSL_n is simplicially simply connected, so we have a simplicial fiber sequence

$$\Omega BGL_n \cong GL_n \to \mathbb{G}_m \to BSL_n.$$

By Lemma 4.1.1, BSL_n is $\mathbb{A}^1$-simply connected, and we have a $\mathbb{A}^1$-fiber sequence. Passing to the long exact sequence in $\mathbb{A}^1$-homotopy sheaves and using Example 2.3.7, we have $\pi^{\mathbb{A}^1}_i(GL_n) \cong \pi^{\mathbb{A}^1}_{i+1}(BSL_n)$ for $i \geq 1$ and $n \geq 1$. Combine with Corollary 2.5.7 and we have the result. □

Theorem 4.1.3. *Suppose $i \geq 1$ and $n \geq 2$ are integers. The morphisms*

$$\pi^{\mathbb{A}^1}_i(SL_{n-1}) \to \pi^{\mathbb{A}^1}_i(SL_n)$$

are epimorphisms for $i \leq n-2$ and isomorphism for $i \leq n-3$. Moreover, we have $\pi^{\mathbb{A}^1}_i(SL_n) \simeq K^Q_{i+1}$ for $1 \leq i \leq n - 2$.

Proof. Consider the following pointed fiber sequence

$$SL_{n-1} \to SL_n \to SL_n/SL_{n-1}.$$

We have proved in Lemma 4.1.1 that $\pi_0^{\mathbb{A}^1}(SL_{n-1})$ is trivial for $n \geq 2$. So this is an $\mathbb{A}^1$-fiber sequence. Note that the inclusion $SL_{n-1} \to SL_n$ is just adding 1 to the right bottom corner of the matrix. Now consider the morphism $SL_n \to \mathbb{A}^n\backslash 0$ given by the projection to the last column. So the morphism $SL_n \to \mathbb{A}^n\backslash 0$ factor through SL_{n-1} and we get a morphism $SL_n/SL_{n-1} \to \mathbb{A}^n\backslash 0$, whose fiber is $\mathbb{A}^{n-1}$. Thus, we have an $\mathbb{A}^1$-equivalence $SL_n/SL_{n-1} \xrightarrow{\sim} \mathbb{A}^n\backslash 0$ in $\mathcal{H}_\bullet(k)$. Recall that by Theorem 3.1.4, $\mathbb{A}^n\backslash 0$ is $\mathbb{A}^1$-$n-1$-connective, which means $\pi_i^{\mathbb{A}^1}(SL_n/SL_{n-1}) = 0$ for $1 \leq i \leq n-2$. Now passing to the long exact sequence in homotopy sheaves, the first assertion is proved.

Furthermore, by representability of Quillen K-theory, we have $\pi_i^{\mathbb{A}^1}(\mathbb{Z} \times BGL_\infty) \cong K_i^Q$ for $i \geq 0$. For $i \geq 1$, the homotopy sheaves are pointed sheaves, so we only need to consider the $\mathbb{A}^1$-connected components. Therefore, $K_i^Q \cong \pi_i^{\mathbb{A}^1}(\mathbb{Z} \times BGL_\infty) \cong \pi_i^{\mathbb{A}^1}(BGL_\infty)$ for $i \geq 1$. By Corollary 2.5.7 and Proposition 4.1.2, we have $\pi_i^{\mathbb{A}^1}(SL_n) \simeq K_{i+1}^Q$ for $1 \leq i \leq n-2$. □

Example 4.1.4. Again consider the fiber suquence

$$\mathbb{G}_m \to BSL_n \to BGL_n.$$

This is an $\mathbb{A}^1$-fiber sequence since $\pi_0^{\mathbb{A}^1}(GL_n)$ is strongly $\mathbb{A}^1$-invariant by Theorem 2.5.4. Passing to the long exact sequence and note that $\pi_1^{\mathbb{A}^1}(BSL_n) = *$ by Lemma 4.1.1, we have that

$$\pi_1^{\mathbb{A}^1}(BGL_n) \cong \pi_0^{\mathbb{A}^1}(\mathbb{G}_m) \cong \mathbb{G}_m.$$

Using Theorem 2.7.4, we find that

$$\pi_{0,j}^{\mathbb{A}^1}(GL_n) \cong \pi_{1,j}^{\mathbb{A}^1}(BGL_n) = \begin{cases} \mathbb{G}_m & \text{if } j = 0, \\ \mathbb{Z} & \text{if } j = 1, \\ 0 & \text{if } j \geq 2. \end{cases}$$

4.2 The first unstable $\mathbb{A}^1$-homotopy sheaves for SL_n

Fix $n \geq 3$. In the previous section we have used K_{i+1}^Q to describe the homotopy sheaves of $\pi_i^{\mathbb{A}^1}(SL_n)$ for $1 \leq i \leq n-2$. Now we want to discuss $\pi_{n-1}^{\mathbb{A}^1}(SL_n) \cong \pi_n^{\mathbb{A}^1}(BSL_n)$ in detail. This is the first non-stable $\mathbb{A}^1$-homotopy sheaf of the space BSL_n. Consider the following fiber sequence

$$SL_n/SL_{n-1} \to BSL_{n-1} \to BSL_n.$$

This is an $\mathbb{A}^1$-fiber sequence by Corollary 2.5.9. Passing to the long exact sequence and we have

$$\pi_n^{\mathbb{A}^1}(BSL_n) \to \pi_{n-1}^{\mathbb{A}^1}(SL_n/SL_{n-1}) \to \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1}) \to \pi_{n-1}^{\mathbb{A}^1}(BSL_n) \to \pi_{n-2}^{\mathbb{A}^1}(SL_n/SL_{n-1}) \to \dots.$$

Note that in the proof of Theorem 4.1.3, we have identified SL_n/SL_{n-1} with $\mathbb{A}^n\backslash 0$ in $\mathcal{H}_\bullet(k)$. Therefore, $\pi^{\mathbb{A}^1}_{n-1}(SL_n/SL_{n-1}) \cong K^{MW}_n$ and $\pi^{\mathbb{A}^1}_{n-2}(SL_n/SL_{n-1}) = 0$. Use Theorem 4.1.3, we have the following exact sequence

$$\pi^{\mathbb{A}^1}_{n-1}(SL_n) \xrightarrow{q_{n-1}} K^{MW}_n \xrightarrow{\delta_{n-1}} \pi^{\mathbb{A}^1}_{n-2}(SL_{n-1}) \to K^Q_{n-1} \to 0. \tag{4.1}$$

The ultimate goal is to better understand the morphism $q_{n-1} : \pi^{\mathbb{A}^1}_{n-1}(SL_n) \to K^{MW}_n$. Consider another $\mathbb{A}^1$-fiber sequence

$$SL_{n+1}/SL_n \to BSL_n \to BSL_{n+1}.$$

Take the morphism $SL_{n+1}/SL_n \to BSL_n$ and apply the looping space Ω^1, then compose with the quotient map. Now we have a composition of morphisms

$$\Omega^1(SL_{n+1}/SL_n) \to \Omega^1 BSL_n \cong SL_n \to SL_n/SL_{n-1}.$$

Applying $\pi^{\mathbb{A}^1}_{n-1}$, we obtain a composition of morphisms

$$K^{MW}_{n+1} \xrightarrow{\delta_n} \pi^{\mathbb{A}^1}_{n-1}(SL_n) \xrightarrow{q_{n-1}} K^{MW}_n.$$

Here the morphism q_{n-1} coincides with the morphisms in 4.1 because the long exact sequence in homotopy sheaves comes from the looping of the $\mathbb{A}^1$-fiber sequence. We will first try to understand this morphism $q_{n-1} \circ \delta_n$. Oberserve that this is a morphism between strictly $\mathbb{A}^1$-invariant sheaves.

Theorem 4.2.1. *Let $\mathcal{X}$ be a pointed, n-connective motivic space, $n \geq 2$ and A be a strictly $\mathbb{A}^1$-invariant sheaf. We have a bijection*

$$H^n_{Nisn}(\mathcal{X}, A) \xrightarrow{\sim} Hom_{\mathcal{A}\lfloor^{\mathbb{A}^1}_k}(\pi^{\mathbb{A}^1}_n(\mathcal{X}), A).$$

Proof. On the left hand side, since $\mathcal{X}$ is a pointed motivic space and A is strictly $\mathbb{A}^1$-invariant, we can rewrite $H^n_{Nisn}(\mathcal{X}, A) \cong [\mathcal{X}, K(A,n)]_{\mathbb{A}^1_\bullet}$. Note that the Eilenberg-Maclane space $K(A,n)$ is n-truncated and $\mathcal{X}$ is n-connective, we obtain that

$$\begin{aligned} H^n_{Nisn}(\mathcal{X}, A) &= [\mathcal{X}, K(A,n)]_{\mathbb{A}^1_\bullet} \\ &= [\tau_{\leq n}\mathcal{X}, K(A,n)]_{\mathbb{A}^1_\bullet} \\ &= [K(\pi^{\mathbb{A}^1}_n(\mathcal{X}), n), K(A,n)]_{\mathbb{A}^1_\bullet} \end{aligned}$$

In the last equation both A and $\pi^{\mathbb{A}^1}_n(\mathcal{X})$ are strictly $\mathbb{A}^1$-invariant for $n \geq 2$, □

Applying the above theorem in the case that $\mathcal{X} = SL_{n+1}/SL_n \cong \mathbb{A}^{n+1}\backslash 0$ and $A = K^{MW}_n$,

the morphism $q_{n-1} \circ \delta_n$ corresponds to an element in $H^n_{Nisn}(SL_{n+1}/SL_n, K^{MW}_n)$. Now consider the SL_n-torsor related to the morphism $SL_{n+1} \to SL_{n+1}/SL_n$ and the Euler class e in its associated oriented vector bundle.

Proposition 4.2.2. *Keep the assumptions and notations as above. The morphism $q_{n-1} \circ \delta_n$ corresponds to the Euler class $e \in H^n_{Nisn}(SL_{n+1}/SL_n, K^{MW}_n)$ related to the SL_n-torsor $SL_{n+1} \to SL_{n+1}/SL_n$.*

Proof. Given the SL_n-torsor $SL_{n+1} \to SL_{n+1}/SL_n$, recall [23, Corollary B.10] that the Euler class related to this oriented vector bundle is the composition of morphism $SL_{n+1}/SL_n \to BSL_n \to K(K^{MW}_n, n)$. Here we do not need to twist the Eilenberg-Maclane space because BSL_n is $\mathbb{A}^1$-simply connected. Now take the $\mathbb{A}^1$-homotopy sheaves and note that $SL_{n+1}/SL_n \simeq \mathbb{A}^{n+1}\backslash 0$ is n-connective and the Eilenberg-Maclane Space $K(K^{MW}_n, n)$ is n-truncated, we have the result. □

Next we want to compute the cohomology group $H^n_{Nisn}(\mathbb{A}^{n+1}\backslash 0, K^{MW}_n)$. Recall that $\mathbb{A}^{n+1}\backslash 0$ is $\mathbb{A}^1$-equivalent to $\Sigma^n \mathbb{G}_m^{\wedge n+1}$, using the adjointness of loop space Ω and the suspension Σ in $\mathcal{H}_\bullet(k)$, we have that

$$\begin{aligned} H^n(\mathbb{A}^{n+1}\backslash 0, K^{MW}_n) &\cong H^n(\Sigma^n \mathbb{G}_m^{\wedge n+1}, K^{MW}_n) \\ &\cong [\Sigma^n \mathbb{G}_m^{\wedge n+1}, K(K^{MW}_n, n)]_{\mathbb{A}^1_\bullet} \\ &\cong [\mathbb{G}_m^{\wedge n+1}, \Omega^n K(K^{MW}_n, n)]_{\mathbb{A}^1_\bullet} \\ &\cong [\mathbb{G}_m^{\wedge n+1}, K^{MW}_n]_{\mathbb{A}^1_\bullet} \\ &\cong \pi_0^{\mathbb{A}^1}(\mathrm{Hom}_\bullet(\mathbb{G}_m^{\wedge n+1}, K^{MW}_n)) \\ &\cong (K^{MW}_n)_{-(n+1)} \\ &\cong K^{MW}_{-1}. \end{aligned}$$

Here we are using Theorem 2.7.4 and Corollary 3.3.4. Recall that by Corollary 3.1.3 that $K^{MW}_{-1}(k)$ is isomorphic to the Witt ring $W(k)$. We will describe the morphism $q_{n-1} \circ \delta_n$ using the notations from the Milnor-Witt K-theory. But first we need the following lemma. Write $Q_{2n+1} \subset \mathbb{A}^{2n+2}$ as the quadric hypersurface defined by

$$k[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]/(\sum_{i=1}^{n+1} x_i y_i - 1).$$

Denote by A_{2n+1} the ring of global sections on Q_{2n+1}. Given a matrix in SL_{n+1}/SL_n, projecting to its first row and the first column of its inverse will give an isomorphism $SL_{n+1}/SL_n \simeq Q_{2n+1}$.

Lemma 4.2.3. *The vector bundle on Q_{2n+1} corresponding to the torsor $SL_{n+1} \to SL_{n+1}/SL_n \simeq Q_{2n+1}$ is the total space of the stably free module of rank n on Q_{2n+1} defined by the split eaxct*

sequence

$$0 \to P_n \to (A_{2n+1})^{n+1} \xrightarrow{(x_1,\ldots,x_{n+1})} A_{2n+1} \to 0.$$

Proof. Let V be the total space of P_n over Q_{2n+1}. Using the split exact sequence, we can explicitly write

$$V = \operatorname{Spec} k[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}, z_1, \ldots, z_{n+1}]/(\sum_{i=1}^{n+1} x_i y_i - 1, \sum_{i=1}^{n+1} x_i z_i).$$

Define a morphism

$$\begin{aligned} f : \mathbb{A}^n \times SL_{n+1} &\to V \\ (v, M) &\mapsto (M^t e_1,\ M^{-1} e_1, M^{-1} v') \end{aligned}$$

where $e_1 = (1, 0, \ldots, 0)$ and $v' = (0, v)$. Assume that we have an inclusion $SL_n \to SL_{n+1}$ by adding 1 to the upper left corner and consider SL_n acting on $\mathbb{A}^n \times SL_{n+1}$ by $P \cdot (v, M) = (Pv, PM)$. If V is endowed with the trivial group action by SL_n, then f is SL_n-equivariant, and we have a morphism from the orbit space to V (still denoted by f)

$$f : \mathbb{A}^n \times^{SL_n} SL_{n+1} \to V.$$

Note that the projection map $\mathbb{A}^n \times^{SL_n} SL_{n+1} \to SL_{n+1}/SL_n$ is the vector bundle assocaiated to the SL_n-torsor $SL_{n+1} \to SL_{n+1}/SL_n$ and we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}^n \times^{SL_n} SL_{n+1} & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ SL_{n+1}/SL_n & \xrightarrow{g} & Q_{2n+1} \end{array}$$

Here g is the isomorphism induced by mapping $M \in SL_{n+1}$ to $(M^t e_1,\ M^{-1} e_1)$. Thus, by definition locally f is an isomorphism and we are done. □

Proposition 4.2.4. *Use the notations from above. We have*

$$q_{n-1} \circ \delta_n = \begin{cases} \eta & \text{if } n = 2m, \\ 0 & \text{if } n = 2m+1. \end{cases}$$

where η is the symbol in Milnor-Witt K-theory of degree -1.

Proof. From the previous discussion, we know that $q_{n-1} \circ \delta_n$ corresponds to the Euler class $e \in H^n_{Nisn}(SL_{n+1}/SL_n, K_n^{MW})$ related to the SL_n-torsor $SL_{n+1} \to SL_{n+1}/SL_n$. Use the affine scheme $Q_{2n+1} \simeq SL_{n+1}/SL_n$ and by Lemma 4.2.3, the vector bundle assocaited to this Euler

class is stably free. If $n = 2m + 1$ is an odd number, consider the short exact sequence for Lemma 4.2.3

$$0 \to P_n \to (A_{2n+1})^{2m+2} \xrightarrow{(x_1,\ldots,x_{2m+2})} A_{2n+1} \to 0.$$

We know that P_n is given by a unimodular row $(x_1, x_2, \ldots, x_{2m+2}) \in (A_{2n+1})^{2m+2}$. Now we look at the 1-dimensional subspace generated by $(-x_2, x_1, \ldots, -x_{2m+1}, x_{2m+2})$. This is a summand of P_n. So this vector bundle splits off a trivial line bundle. Morel proved [23, Theorem 8.14] that the Euler class is the obstruction class for the vector bundle spiltting off a trivial line bundle, thus we conclude in this case $q_{n-1} \circ \delta_n = 0$. If $n = 2m$ is an even number, note that the Euler class from the obstruction theory coincides with the Euler class defined as a characteristic class valued in Chow-Witt theory [27], and it is calculated in [22, Proposition 3.2] using the latter definition. □

4.3 The torsion sheaf S_n

Recall that for all integers $n \in \mathbb{Z}$, we have defined a surjective morphism of sheaves $K_n^{MW} \to K_n^M$ sending the symbol η of degree -1 to 0. Now for $n \geq 3$, consider the following diagram from $\mathbb{A}^1$-fiber sequence.

$$\begin{array}{ccccc} & & & & SL_{n+1}/SL_n \\ & & & & \downarrow \\ SL_n/SL_{n-1} & \longrightarrow & BSL_{n-1} & \longrightarrow & BSL_n \\ & & & & \downarrow \\ & & & & BSL_{n+1} \end{array}$$

Passing to the long exact sequences in homotopy sheaves and we have the following.

$$\begin{array}{ccccccccc} & & K_{n+1}^{MW} & & & & & & \\ & & \downarrow{\scriptstyle \delta_n} & & & & & & \\ \pi_{n-1}^{\mathbb{A}^1}(SL_n) & \xrightarrow{q_{n-1}} & K_n^{MW} & \xrightarrow{\delta_{n-1}} & \pi_{n-2}^{\mathbb{A}^1}(SL_{n-1}) & \longrightarrow & K_{n-1}^Q & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & & & \\ K_n^Q & \dashrightarrow & K_n^M & & & & & & \\ \downarrow & & & & & & & & \\ 0 & & & & & & & & \end{array} \tag{4.2}$$

We have an induced morphism from $K_n^Q \to K_n^M$ by Proposition 4.2.4 and the exactness of the vertical line.

Definition 4.3.1. Denote by $\psi_n : K_n^Q \to K_n^M$ the induced morphism from the above diagram. Set S_n to be the cokernel of ψ_n.

In fact, S_n is a sheaf of torsion abelian groups. To see this, we need to reinterpret the morphism ψ_n. Next we will introduce the Suslin matrices and use them to defines morphisms between K-theory sheaves.

Definition 4.3.2 (Suslin matrices)**.** Let R be a commutative ring with unity and $m \geq 1$ be an integer. For any row $v = (v_1, v_2, \ldots, v_m) \in R^m$, we write $v' = (v_2, \ldots, v_m) \in R^{m-1}$. For any positive integer $n \geq 1$ and $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, we inductively define a matrix $S_n(a, b) \in M_{2^{n-1}}(R)$.

$$\begin{aligned} S_1(a, b) &= a_1, \\ S_n(a, b) &= \begin{pmatrix} a_1 \mathrm{Id}_{2^{n-2}} & S_{n-1}(a', b') \\ -S_{n-1}(b', a')^t & b_1 \mathrm{Id}_{2^{n-2}} \end{pmatrix}. \end{aligned}$$

Lemma 4.3.3. *R is a ring and r is a positive integer. Let $A_1, A_2, B_1, B_2 \in M_r(R)$ satisfying $A_1A_2 = A_2A_1$. Then*

$$det \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = det(B_2A_1 - B_1A_2).$$

Proof. Note that

$$\begin{aligned} \det \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \cdot \det A_1 &= \det \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix} = \det \begin{pmatrix} A_1 & A_2A_1 \\ B_1 & B_2A_1 \end{pmatrix} \\ &= \det[\begin{pmatrix} A_1 & A_2A_1 \\ B_1 & B_2A_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -A_2 \\ 0 & 1 \end{pmatrix}] = \det \begin{pmatrix} A_1 & 0 \\ B_1 & B_2A_1 - B_1A_2 \end{pmatrix} \\ &= \det(B_2A_1 - B_1A_2) \cdot \det A_1 \end{aligned}$$

If $\det A_1$ is not a zero divisor in R, then we are done. Now we consider the matrix $\begin{pmatrix} A_1 + X & A_2 \\ B_1 & B_2 \end{pmatrix}$ with entries in the polynomial ring $R[X]$. Here $\det(A_1 + X)$ is not a zero divisor in the ring $R[X]$, and we can apply the previous result and get

$$\det \begin{pmatrix} A_1 + X & A_2 \\ B_1 & B_2 \end{pmatrix} = \det[B_2(A_1 + X) - B_1A_2].$$

Setting $X = 0$ and we prove the lemma. □

Now assume that $\sum_{i=1}^n a_ib_i = 1$ and we inductively apply Lemma 4.3.3 to the Suslin matrices $S_n(a, b)$. In this case $S_n(a, b) \in GL_{2^{n-1}}(R)$. This construction is functorial in R and

we have a morphism

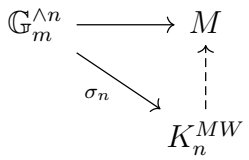
$$\phi_n : Q_{2n-1} \to GL_{2^{n-1}}$$

where Q_{2n-1} is the hypersurface cut out by $\sum_{i=1}^{n} a_i b_i - 1$. Compose ϕ_n with the standard map into the stable general linear group GL_∞, and this is a pointed map if we choose $(1, 0, \ldots, 0, 1, 0, \ldots, 0)$ as the base point of Q_{2n-1}. Recall that Q_{2n-1} is $\mathbb{A}^1$-equivalent to $\mathbb{A}^n \backslash 0$ by the projection $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n)$. Apply $\pi_{n-1}^{\mathbb{A}^1}$, and by the representability of algebraic K-theory, we have a morphism

$$\Phi_n : K_n^{MW} \to K_n^Q.$$

There exists another way to define a morphism $K_n^{MW} \to K_n^Q$. Morel has proved the following universal property of Milnor-Witt K-theory sheaf.

Theorem 4.3.4 ([23, Theorem 3.37])**.** *$n \geq 1$ is an integer. Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, together with a morphism of sheaves $\mathbb{G}_m^{\wedge n} \to M$. Then there exists a unique morphism $K_n^{MW} \to M$ such that the following diagram commutes.*



Here $\sigma_n : \mathbb{G}_m^{\wedge n} \to K_n^{MW}$ is the canonical symbol map.

Thus, for $n \geq 3$, if we want to get a morphism $K_n^{MW} \to K_n^Q$, we first need to define a morphism of sheaves $\mathbb{G}_m^{\wedge n} \to K_n^Q$. Let L/k be a finitely generated field extension. We send a section $a_1, \ldots, a_n$ of $\mathbb{G}_m^{\wedge n}(L)$ to the cup product $(a_1) \smile \ldots \smile (a_n)$ in $K_n^Q(L)$. This will give us a morphism of sheaves, and by the above Theorem, gives us a morphism $K_n^{MW} \to K_n^Q$. Recall that the symbol map σ_n sends a section $(a_1, \ldots, a_n)$ in $\mathbb{G}_m^{\wedge n}(L)$ to the corresponding symbol $[a_1] \cdots [a_n]$ in $K_n^{MW}(L)$. Using the relationship $[ab] = [a] + [b] + \eta[a][b]$ in $K_n^{MW}(L)$ and the relationship $(ab) = (a) \smile (b)$ in K_n^Q, this morphism factors through Milnor K-theory K_n^M and we have a composition of morphism.

$$K_n^{MW} \to K_n^M \xrightarrow{\mu_n} K_n^Q$$

where the first map sends the symbol η to 0. We have the following comparison lemma with the previous Φ_n.

Lemma 4.3.5. *Notations as above. The following diagram*

$$\begin{array}{ccc} K_n^{MW} & \xrightarrow{\Phi_n} & K_n^Q \\ \downarrow & & \downarrow id \\ K_n^M & \xrightarrow[\mu_n]{} & K_n^Q \end{array}$$

commutes up to a sign.

Proof. Let $n \geq 1$ be a positive integer. Consider the morphism

$$\begin{aligned} \mathbb{G}_m &\to GL_n \\ \alpha &\mapsto diag(\alpha, 1, \ldots, 1) \end{aligned}$$

Stablizing and we get a morphism $\nu_1 : \mathbb{G}_m \to GL_\infty = colim_n GL_n$. Use the $\mathbb{A}^1$-equivalence $GL_\infty \to \Omega^1 BGL_\infty$ and we have

$$\begin{aligned} &[\mathbb{G}_m, GL_\infty]_{\mathbb{A}^1_\bullet} \\ =&[\mathbb{G}_m, \Omega^1 BGL_\infty]_{\mathbb{A}^1_\bullet} \\ =&[\Sigma^1 \mathbb{G}_m, BGL_\infty]_{\mathbb{A}^1_\bullet} \\ =&[\mathbb{P}^1, BGL_\infty]_{\mathbb{A}^1_\bullet} \end{aligned}$$

By Corollary 3.2.6, Quillen K-theory space is represented by $\mathbb{Z} \times BGL_\infty$ in $\mathcal{H}(k)$. In the pointed category $\mathcal{H}_\bullet(k)$, we only need to consider the connected components. Therefore, the morphism of sheaves $\nu_1 : \mathbb{G}_m \to GL_\infty$ defined above corresponds to an element in $K_0^Q(\mathbb{P}^1)$. Note that here $\mathbb{P}^1$ is pointed at ∞ (will be denoted by $\mathbb{P}^{1,\infty}$ to emphasize it is pointed at ∞), thus $K_0^Q(\mathbb{P}^1)$ is the connected component of the Grothendieck group of $\mathbb{P}^1$. Recall that for the unpointed projective space, we have an isomorphism

$$\begin{aligned} K_0(\mathbb{P}^1) &\to \mathrm{Pic}(\mathbb{P}^1) \times \mathbb{Z} \\ [V] &\mapsto (\det V, \mathrm{rank}\, V). \end{aligned}$$

By definition, for the pointed space $\mathbb{P}^{1,\infty}$, the Grothendieck Group

$$K_0(\mathbb{P}^{1,\infty}) := ker(K_0(\mathbb{P}^1) \to K_0(\{\infty\}))$$

is given by the kernel of the rank map. Thus, ν_1 corresponds to a rank 0 vector bundle on $\mathbb{P}^1$, and by definition it is the generator $\mathcal{O}(1) - \mathcal{O}$.

Now consider the smash product $(\mathbb{P}^1)^{\wedge n}$ and the multiplicative structure in algebraic K-

theory spectrum, we have a morphism

$$\nu_n : (\mathbb{P}^1)^{\wedge n} \to BGL_\infty$$

where $\nu_n := \nu_1 \wedge \cdots \wedge \nu_1$. Use the $\mathbb{A}^1$-equivalence $(\mathbb{P}^1)^{\wedge n} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$, viewed as a k-space $\mathbb{P}^n$ pointed at $\mathbb{P}^{n-1}$, and deduce that ν_n corresponds to an element in $K_0(\mathbb{P}^n/\mathbb{P}^{n-1}) := ker(K_0(\mathbb{P}^n) \to K_0(\mathbb{P}^{n-1}))$. Apply the projective bundle formula and we see that ν_n corresponds to the generator $(\mathcal{O}(1) - \mathcal{O})^{\otimes(n-1)}$. Applying the functor $\pi_n^{\mathbb{A}^1}(-)$ to ν_n gives us a morphism between K-theory sheaves

$$K_n^{MW} \to K_n^Q.$$

And since it is the generator, this morphism coincides with the previous morphism we defined using the universal property of K_n^{MW}.

On the other hand, Φ_n is induced from the map $Q_{2n-1} \to GL_\infty$. Suslin has proved that it corresponds to the generator of $[Q_{2n-1}, GL_\infty]_{\mathbb{A}^1_\bullet} = \tilde{K}_1(Q_{2n-1}) = \mathbb{Z}$ [15, Theorem 2.3]. Compose with the $\mathbb{A}^1$-equivalence $GL_\infty \to \Omega^1 BGL_\infty$ generates the group

$$[Q_{2n-1}, \Omega^1 BGL_\infty]_{\mathbb{A}^1_\bullet} = [\Sigma^1(\mathbb{A}^n \backslash 0), BGL_\infty]_{\mathbb{A}^1_\bullet} = [(\mathbb{P}^1)^{\wedge n}, BGL_\infty]_{\mathbb{A}^1_\bullet} = \mathbb{Z}.$$

Therefore, the generator coincides with ν_n up to a sign and we have the following diagram

$$\begin{array}{ccc} K_n^{MW} & \xrightarrow{\Phi_n} & K_n^Q \\ \downarrow & & \downarrow id \\ K_n^M & \xrightarrow[\mu_n]{} & K_n^Q \end{array}$$

commutes up to a sign. □

Corollary 4.3.6. *The morphism $(\Phi_n)_{-1} : K_{n-1}^{MW} \to K_{n-1}^Q$ coincides with Φ_{n-1}.*

Proof. Use the diagram from Lemma 4.3.5 and the results from Section 3.3. Note that the morphism $K_n^{MW} \to K_n^Q$ defined via the universal property of K_n^{MW} is valid for each n and is compatible with contraction. □

Lemma 4.3.7. *For $n \geq 2$, we have the following commutative triangle.*

$$\begin{array}{ccc} K_n^{MW} & \xrightarrow{\mu_n} & K_n^Q \\ & \searrow^{(n-1)!} & \downarrow \psi_n \\ & & K_n^M \end{array}$$

Recall that here ψ_n is from Definition 4.3.1.

Proof. Recall that at the beginning of this section we define the morphism $\phi_n : Q_{2n-1} \to GL_{2^{n-1}}$ using the Suslin matrices

$$S_n(a,b) = \begin{pmatrix} a_1\mathrm{Id}_{2^{n-2}} & S_{n-1}(a',b') \\ -S_{n-1}(b',a')^t & b_1\mathrm{Id}_{2^{n-2}} \end{pmatrix}.$$

There are at most n nonzero elements in each row and in each column of $S_n(a,b)$, thus the order of this matrix can be lowered to n by performing elementary transformations on the rows and columns. In other words, given a point $(a,b) \in Q_{2n-1}$, there exists a matrix $\beta_n(a,b) \in GL_n$ such that its image in $GL_{2^{n-1}}$ is precisely the Suslin matrix $S_n(a,b)$ up to multiplication by a product of elementary matrices. Recall that in the proof of Lemma 4.1.1, we have shown that a product of elementary matrices is $\mathbb{A}^1$-homotopic to the identity. As a result, $\phi_n : Q_{2n-1} \to GL_{2^{n-1}}$ can be lifted to a morphism $\phi'_n : Q_{2n-1} \to GL_n$, and ϕ_n and ϕ'_n are $\mathbb{A}^1$-homotopic.

Apply $\pi^{\mathbb{A}^1}_{n-1}(-)$ to ϕ'_n, and we have a morphism $\Phi'_n : K^{MW}_n \to \pi^{\mathbb{A}^1}_{n-1}(GL_n)$, along with a cummutative diagram.

$$\begin{array}{ccc} K^{MW}_n & \xrightarrow{\Phi'_n} & \pi^{\mathbb{A}^1}_{n-1}(GL_n) \\ & \searrow^{\Phi_n} & \downarrow \\ & & K^Q_n \end{array}$$

Now compose $\phi'_n : Q_{2n-1} \to GL_n$ with the first row projection map $GL_n \to \mathbb{A}^n \backslash 0$. By Suslin [4, §5], this maps $(a,b) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ to $(a_1, a_2, a_3^2, \ldots, a_n^{n-1})$. Apply $\pi^{\mathbb{A}^1}_{n-1}(-)$, and we have a composition of morphisms of K-theory sheaves

$$K^{MW}_n \xrightarrow{\Phi'_n} \pi^{\mathbb{A}^1}_{n-1}(GL_n) \to K^{MW}_n \xrightarrow{\eta \mapsto 0} K^M_n.$$

Suppose L/k is a finitely generated field extension. On sections over the field L, the above morphism sends the symbol $[a_1][a_2]\cdots[a_n] \in K^{MW}_n$ to the symbol $[a_1][a_2][a_3^2]\cdots[a_n^{n-1}] = (n-1)![a_1][a_2]\cdots[a_n] \in K^M_n$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} K^{MW}_n(L) & \xrightarrow{\Phi_n} & K^Q_n(L) \\ {\scriptstyle \eta=0}\downarrow & & \downarrow{\scriptstyle \psi_n} \\ K^M_n(L) & \xrightarrow[(n-1)!]{} & K^M_n(L) \end{array}$$

for every finitely generated field extension L/k. Note that the K-theory sheaves above are strictly $\mathbb{A}^1$-invariant, which are uniquely determined by their values on sections over finitely generated extensions of the base field k. Combine the above diagram with Lemma 4.3.5 and we are done. □

With all the results we obtained so far, we can finally give a torsion bound to the sheaf S_n

from Definition 4.3.1.

Corollary 4.3.8. *The morphism of sheaves $K_n^M \to S_n$ induces an epimorphism $K_n^M/(n-1)! \to S_n$. For any $m \geq n-2$, this epimorphism becomes an isomorphism after m-fold contraction.*

Proof. Recall that S_n is defined as the cokernel of $\psi_n : K_n^Q \to K_n^M$, and the first assertion is the result of Lemma 4.3.7. Now apply m-fold contraction to the triangle in Lemma 4.3.7, and we have

$$\begin{array}{ccc} K_{n-m}^M & \xrightarrow{(\mu_n)_{-m}} & K_{n-m}^Q \\ & \searrow^{(n-1)!} & \downarrow{\scriptstyle (\psi_n)_{-m}} \\ & & K_{n-m}^M \end{array}.$$

In the proof of Corollary 4.3.6, we have shown that $(\mu_n)_{-m} = \mu_{n-m}$. If $m = n-2$, by Matsumoto's theorem for K-theory, for any field F, the second K-group

$$K_2^Q(F) = F^\times \otimes_{\mathbb{Z}} F^\times/(a \otimes (1-a) | a \neq 0, 1).$$

By definition, $\mu_2 : K_2^M \to K_2^Q$ is an isomorphism and compare the relations for generators in higher K-groups, we have $\mu_{n-m} : K_{n-m}^M \to K_{n-m}^Q$ is an isomorphism for $m \geq n-2$. □

Remark. In general, it is not true that $(\psi_n)_{-m} = \psi_{n-m}$. For example, in the case $n = 3$ and $m = 1$, we have a commutative triangle

$$\begin{array}{ccc} K_2^M & \xrightarrow{\mu_2} & K_2^Q \\ & \searrow^{2} & \downarrow{\scriptstyle (\psi_3)_{-1}} \\ & & K_2^M \end{array}.$$

Here μ_2 is an isomorphism by Matsumoto's theorem. Let $L = \mathbb{C}$ and we know that $K_2^M(\mathbb{C})/2K_2^M(\mathbb{C})$ is not trivial, then $(\psi_3)_{-1}$ is not surjective but ψ_2 is an isomorphism over the field $\mathbb{C}$.

4.4 Computation of $\pi^{\mathbb{A}^1}_{n-2}(SL_{n-1})$

Now we can return to the computation of $\pi^{\mathbb{A}^1}_{n-2}(SL_{n-1})$. Recall we have the folloing diagram 4.2.

$$\begin{array}{ccccccccc}
 & & K^{MW}_{n+1} & & & & & & \\
 & & \downarrow{\scriptstyle \delta_n} & & & & & & \\
\pi^{\mathbb{A}^1}_{n-1}(SL_n) & \xrightarrow{q_{n-1}} & K^{MW}_n & \xrightarrow{\delta_{n-1}} & \pi^{\mathbb{A}^1}_{n-2}(SL_{n-1}) & \longrightarrow & K^Q_{n-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
K^Q_n & \xrightarrow{\psi_n} & K^M_n & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

First assume $n = 2m$ is an even number. Then by Proposition 4.2.4, we have $q_{2m-1} \circ \delta_{2m} = \eta$. For $m \geq 1$, the above diagram yields a short exact sequence

$$0 \to S_{2m} \to \pi^{\mathbb{A}^1}_{2m-2}(SL_{2m-1}) \to K^Q_{2m-1} \to 0.$$

If $n = 2m + 1$ is an odd number, the situation is more complicated. Now By Proposition 4.2.4, $q_{2m-1} \circ \delta_{2m} = 0$. This means there exists a morphism $\psi'_{2m+1} : K^Q_{2m+1} \to K^{MW}_{2m+1}$ such that the following diagram commutes.

$$\begin{array}{ccccccccc}
 & & K^{MW}_{2m+2} & & & & & & \\
 & & \downarrow{\scriptstyle \delta_{2m+1}} & & & & & & \\
\pi^{\mathbb{A}^1}_{2m}(SL_{2m+1}) & \xrightarrow{q_{2m}} & K^{MW}_{2m+1} & \xrightarrow{\delta_{2m}} & \pi^{\mathbb{A}^1}_{2m-1}(SL_{2m}) & \longrightarrow & K^Q_{2m} & \longrightarrow & 0 \\
\downarrow & \overset{\psi'_{2m+1}}{\nearrow} & \downarrow{\scriptstyle \eta=0} & & & & & & \\
K^Q_{2m+1} & \xrightarrow[\psi_{2m+1}]{} & K^M_{2m+1} & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

and we have the following exact sequence:

$$K^Q_{2m+1} \xrightarrow{\psi'_{2m+1}} K^{MW}_{2m+1} \xrightarrow{\delta_{2m}} \pi^{\mathbb{A}^1}_{2m-1}(SL_{2m}) \to K^Q_{2m} \to 0.$$

Now use the Cartesian square 3.1:

$$\begin{array}{ccc} K^{MW}_{2m+1} & \longrightarrow & K^{M}_{2m+1} \\ \downarrow & & \downarrow \\ I^{2m+1} & \longrightarrow & K^{M}_{2m+1}(k)/2K^{M}_{2m+1} \end{array} .$$

We need to understand the composition of morphisms $K^{Q}_{2m+1} \xrightarrow{\psi'_{2m+1}} K^{MW}_{2m+1} \to I^{2m+1}$.

Lemma 4.4.1. *$K^{Q}_{2m+1} \xrightarrow{\psi'_{2m+1}} K^{MW}_{2m+1} \to I^{2m+1}$ is trivial.*

Proof. Combine the short exact sequence from Corollary 3.1.3 with the diagram 4.2, we have a large commutative diagram.

$$\begin{array}{ccccccccc} & & & & \pi^{\mathbb{A}^1}_{2m}(SL_{2m+1}) & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 2K^{M}_{2m+1} & \longrightarrow & K^{MW}_{2m+1} & \longrightarrow & I^{2m+1} & \longrightarrow & 0 \\ & & & & \downarrow & \searrow^{\eta} & \downarrow & & \\ & & & & \pi^{\mathbb{A}^1}_{2m-1}(SL_{2m}) & \longrightarrow & K^{MW}_{2m} & \longrightarrow & \pi^{\mathbb{A}^1}_{2m-2}(SL_{2m-1}) \\ & & & & \downarrow & & \downarrow & & \\ & & & & K^{Q}_{2m} & \xrightarrow[\psi_{2m}]{} & K^{M}_{2m} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

We still need to explain why the following part of the diagram commutes.

$$\begin{array}{ccc} K^{MW}_{2m+1} & \longrightarrow & I^{2m+1} \\ \downarrow & \searrow^{\eta} & \downarrow \\ \pi^{\mathbb{A}^1}_{2m-1}(SL_{2m}) & \longrightarrow & K^{MW}_{2m} \end{array}$$

The upper triangle commutes is a result of $K^{MW}_{2m+1} \xrightarrow{\eta} K^{MW}_{2m}$ factoring through I^{2m+1} from Corollary 3.1.3. The lower triangle commutes is a result of Proposition 4.2.4 and $n = 2m$ is even.

Let α be an element in $\pi^{\mathbb{A}^1}_{2m}(SL_{2m+1})$. It goes to $0 \in \pi^{\mathbb{A}^1}_{2m-1}(SL_{2m})$ by exactness of the vertical column. By commutativity, the composition

$$\pi^{\mathbb{A}^1}_{2m}(SL_{2m+1}) \to K^{MW}_{2m+1} \xrightarrow{\eta} K^{MW}_{2m}$$

is trivial. Again by commutativity, the composition

$$\pi^{\mathbb{A}^1}_{2m}(SL_{2m+1}) \to K^{MW}_{2m+1} \to I^{2m+1}$$

is trivial. Recall that $\pi^{\mathbb{A}^1}_{2m}(SL_{2m+1}) \to K^{MW}_{2m+1}$ factors through K^Q_{2m+1} by definition and we are done. □

Now again use the second short exact sequence from Corollary 3.1.3

$$0 \to 2K^M_{2m+1} \to K^{MW}_{2m+1} \to I^{2m+1} \to 0.$$

By the above lemma, ψ'_{2m+1} must factor through $2K^M_{2m+1}$:

$$K^Q_{2m+1} \xrightarrow{\psi_{2m+1}} 2K^M_{2m+1} \to K^{MW}_{2m+1}.$$

By definition of coker, we have a morphism of sheaves $S_{2m+1} \to K^M_{2m+1}/2$.

Definition 4.4.2. Notations as above. For any integer $m \in \mathbb{Z}$, we define the sheaves T_{2m+1} as the fibre product

$$\begin{array}{ccc} T_{2m+1} & \longrightarrow & I^{2m+1} \\ \downarrow & & \downarrow \\ S_{2m+1} & \longrightarrow & K^M_{2m+1}/2K^M_{2m+1} \end{array}$$

Note that T_{2m+1} is also strictly $\mathbb{A}^1$-invariant. Remember that we are trying to find the cokernel of $\psi'_{2m+1} : K^Q_{2m+1} \to K^{MW}_{2m+1}$. Combine all the discussion so far, we have the following theorem.

Theorem 4.4.3. *Let k be a perfect field with characteristic not equal to 2. For any integer $m \in \mathbb{Z}$, we have short exact sequences of strictly $\mathbb{A}^1$-invariant sheaves*

$$0 \to T_{2m+1} \to \pi^{\mathbb{A}^1}_{2m}(BSL_{2m}) \to K^Q_{2m} \to 0,$$

$$0 \to S_{2m+2} \to \pi^{\mathbb{A}^1}_{2m+1}(BSL_{2m+1}) \to K^Q_{2m+1} \to 0.$$

Note that SL_n is the $\mathbb{A}^1$-connected component of the identity I_n in GL_n, and $\pi^{\mathbb{A}^1}_i(GL_n) \simeq \pi^{\mathbb{A}^1}_i(SL_n)$ for $i \geq 2$. Thus, we have also obtained a computation for the homotopy sheaves of GL_n, which would be useful later when we want to discuss isomorphism classes of vector bundles. Combine Corollary 3.3.4 with the above theorem, we have the following results.

Proposition 4.4.4. *Let $m \geq 1$ and $j \geq 0$ be integers. We have the following short exact sequence*

$$0 \to (T_{2m+1})_{-j} \to \pi^{\mathbb{A}^1}_{2m-1,j}(GL_{2m}) \to K^Q_{2m-j} \to 0,$$

$$0 \to (S_{2m+2})_{-j} \to \pi^{\mathbb{A}^1}_{2m,j}(GL_{2m+1}) \to K^Q_{2m+1-j} \to 0.$$

Here S_{2m+2} and T_{2m+1} are the same sheaves from Theorem 4.4.3 and second index of the homotopy sheaves indicates the Contractions.

Chapter 5

Vector bundles on split quadrics

In the previous chapters we have encoutered the split quadric

$$Q_{2n-1} = \operatorname{Spec} k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum_{i=1}^{n} x_i y_i - 1),$$

which is affine and has the $\mathbb{A}^1$-homotopy type of $\mathbb{A}^n \backslash 0$. As mentioned in the Introduction, Morel has observed that vector bundles of rank r on the smooth affine scheme Q_{2n-1} can be classified via the morphisms $Q_{2n-1} \to BGL_r$ in the $\mathbb{A}^1$-homotopy category. Using the $\mathbb{A}^1$-equivalence $Q_{2n-1} \simeq \Sigma^{n-1}\mathbb{G}_m^{\wedge n}$, it boils down to the calculation of $\mathbb{A}^1$-homotopy sheaves $\pi_{n-1}^{\mathbb{A}^1}(BGL_r)$. In the next sections, we will first introduce oriented vector bundles because the k-space BSL_r is $\mathbb{A}^1$-simply connected, whose $\mathbb{A}^1$-homotopy sheaves are easier to calculate. Then we could use what have achieved in the previous chapter about the $\mathbb{A}^1$-homotopy sheaves $\pi_{n-1}^{\mathbb{A}^1}(BSL_r)$ to classify the (oriented) vector bundles over the split quadrics Q_{2n-1}. Finally, as an application, in the last section we use the classification results to answer a question of M. V. Nori about the completability of some unimodular rows.

5.1 Oriented vector bundles

Let X be a smooth affine scheme over a perfect filed k. Let $\mathcal{E}$ be an algebraic vector bundle of rank r over X with a local trivialization in the Nisnevich topology.

Definition 5.1.1. We say $\mathcal{E}$ is orientable if the assocaiated determinant bundle $\det\mathcal{E}$ is a trivial line bundle over X. And with a fixed trivialization $\det\mathcal{E} \simeq \mathcal{O}_X$, we say $\mathcal{E}$ is a oriented vector bundle.

Theorem 5.1.2 ([25, Theorem 4.2])**.** *Let X be a smooth affine k-scheme and $r \geq 1$ be an integer. We denote by $\mathcal{V}_r^o(X)$ the orientable vector bundle over X. We have a bijection*

$$[X, BSL_r]_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{V}_r^o(X).$$

Proof. Given a morphism $X \to BSL_r$ in the category Sm_k, by pulling back from the universal bundle we obtain a vector bundle $\mathcal{E}$. $\mathcal{E}$ is orientable since the determinant bundle $\det\mathcal{E}$ is given by the composition of morphisms $X \to BSL_r \xrightarrow{det} B\mathbb{G}_m$, where the second morphism is induced by taking the determinant. And by definition of SL_r, this corresponds to the trivial line bundle. Thus, we have a morphism

$$\mathrm{Hom}_{Sm_k}(X, BSL_r) \to \mathcal{V}_r^o(X).$$

This morphism factors through $\mathrm{Hom}_{\mathbb{A}^1}(X, BSL_r)$ since by Bass-Quillen conjecture 1.1.1, vector bundles over smooth affine k-schemes are $\mathbb{A}^1$-invariant. Now we need to show the morphism

$$\mathrm{Hom}_{\mathbb{A}^1}(X, BSL_r) \to \mathcal{V}_r^o(X)$$

is a bijection.

Take an oriented vector bundle $\mathcal{E} \in \mathcal{V}_r^o(X)$. By affine representability theorem [29, Theorem 5.2.3], there exists a unique morphism $X \to BGL_r$ in $\mathcal{H}(k)$ corresponding to $\mathcal{E}$. $\mathcal{E}$ is orientable means the composition $X \to BGL_r \xrightarrow{det} B\mathbb{G}_m$ is trivial and the fiber of the determinant map $GL_n \xrightarrow{det} \mathbb{G}_m$ is just SL_n. So $X \to BGL_r$ must factor through BSL_r and since we have already chosen an orientation, $X \to BSL_r$ is also unqiue. □

The main reason BSL_r is easier to work with compared to BGL_r is that BSL_r is $\mathbb{A}^1$-simply connected by Lemma 4.1.1. When the affine scheme X has a nice $\mathbb{A}^1$-model, the calculation of morphisms $X \to BSL_r$ can be transformed into calculations of $\mathbb{A}^1$-homotopy sheaves of BSL_r, where we have to choose a base point. The following lemma points out that the choice of the base point does not matter too much. But this is not the case for BGL_r.

Lemma 5.1.3. *Let $\mathcal{X}, x_0$ and $\mathcal{Y}, y_0$ be pointed motivic spaces and assume $\mathcal{Y}$ is $\mathbb{A}^1$-simply connected. Then forgetting the base point induces a morphism*

$$f : [(\mathcal{X}, x_0), (\mathcal{Y}, y_0)]_{\mathbb{A}^1_\bullet} \to [\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}.$$

Here f is a bijection.

Proof. This can be proved using the simplicial model structure in Spc_k. Without loss of generality, we may assume $\mathcal{X}$ is cofibrant and $\mathcal{Y}$ is both fibrant and cofibrant. Since Spc_k and $Spc_{k,\bullet}$ is an equivalence of categories, we only need to show that

$$[(\mathcal{X}, x_0), (\mathcal{Y}, y_0)]_{Spc_{k,\bullet}} \to [\mathcal{X}, \mathcal{Y}]_{Spc_k}$$

is a bijection. Abuse some notations and we also denote this morphism by f.

We first prove that f is surjective. Take a morphism $h : \mathcal{X} \to \mathcal{Y}$. $\mathcal{Y}$ is $\mathbb{A}^1$-simply connected implies that it is also simplicially 1-connected. This means we could find a simplicial homotopy

between $h(x_0)$ and y_0. More precisely, there exists a morphism $H : \Delta^1_s \to \mathcal{Y}$ such that $\Delta^1_s(0) = h(x_0)$ and $\Delta^1_s(1) = y_0$. Here Δ^1_s is the usual 1-simplex. We have a commutative diagram

$$\begin{array}{ccc} * & \xrightarrow{H} & \underline{\mathrm{Hom}}(\Delta^1_s, \mathcal{Y}) \\ {\scriptstyle x_0}\downarrow & \overset{F}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ \mathcal{X} & \xrightarrow[h]{} & \mathcal{Y}. \end{array}$$

Here ev_0 is the evaluation map at $\Delta^1_s(0)$, which is a weak equivalence. Since $\mathcal{Y}$ is fibrant, ev_0 is also a fibration. By homotopy lifting property, there exists a map $F : \mathcal{X} \to \underline{\mathrm{Hom}}(\Delta^1_s, \mathcal{Y})$ such that the above square commutes. Compose F with the evaluation map ev_1 at 1, we obtain a morphism $ev_1 \circ F : \mathcal{X} \to \mathcal{Y}$ which maps x_0 to y_0.

Next we show that f is injective. Suppose $h_0, h_1 : (\mathcal{X}, x_0) \to (\mathcal{Y}, y_0)$ are two homotopic morphisms in $Spc_{k,\bullet}$. This means that there exists a morphism $H : \mathcal{X} \times \Delta^1_s \to \mathcal{Y}$ such that $H(-, 0) = h_0$ and $H(-, 1) = h_1$. We need to find another base point preserving homotopy between h_0 and h_1. Note that $H(x_0, -) : \Delta^1_s \to \mathcal{Y}$ is a loop based at y_0. Since $\mathcal{Y}$ is simply connected, there exists a homotopy $G : \Delta^1_s \times \Delta^1_s \to \mathcal{Y}$ such that $G(-, 0) = y_0$ and $G(-, 1) = H(x_0, -)$. (Here y_0 is viewed as the constant map.) Now we could define a morphism

$$N : (\mathcal{X} \times \{0\}) \cup (\{x_0\} \times \Delta^1_s) \cup (\mathcal{X} \times \{1\}) \to \underline{\mathrm{Hom}}(\Delta^1_s, \mathcal{Y})$$

where $N|_{\mathcal{X}\times\{0\}} = h_0$, $N|_{\mathcal{X}\times\{1\}} = h_1$ and $N|_{\{x_0\}\times\Delta^1_s} = G$. Consider the following commutative diagram

$$\begin{array}{ccc} (\mathcal{X} \times \{0\}) \cup (\{x_0\} \times \Delta^1_s) \cup (\mathcal{X} \times \{1\}) & \xrightarrow{N} & \underline{\mathrm{Hom}}(\Delta^1_s, \mathcal{Y}) \\ \downarrow & \overset{M}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ \mathcal{X} \times \Delta^1_s & \xrightarrow[H]{} & \mathcal{Y}. \end{array}$$

Here the left vertical map is just the usual inclusion map on each component and it is a cofibration. The right vertical map ev_0 is the evaluation map at 0 and it is a fibration. By homotopy lifting property, there exists a morphism $M : \mathcal{X} \times \Delta^1_s \to \underline{\mathrm{Hom}}(\Delta^1_s, \mathcal{Y})$, making the above diagram commutes. Compose M with the evalutaion map at 1 $ev_1 : \underline{\mathrm{Hom}}(\Delta^1_s, \mathcal{Y}) \to \mathcal{Y}$, we obtain the desired base point preserving homotopy. □

Theorem 5.1.4. *Let (X, x) be a pointed smooth affine k-scheme and $r \geq 1$ be any positive integer. We have a bijection*

$$[(X, x), (BSL_r, *)]_{\mathbb{A}^1_\bullet} \xrightarrow{\sim} [X, BSL_r]_{\mathbb{A}^1}.$$

In particular, if $X = Q_{2n-1}$ is the split quadric, then we have

$$\pi^{\mathbb{A}^1}_{n-1}(BSL_r)_{-n}(k) \xrightarrow{\sim} [Q_{2n-1}, BSL_r]_{\mathbb{A}^1}.$$

Proof. Lemma 4.1.1 tells us that BSL_r is $\mathbb{A}^1$-simply connected and we apply Lemma 5.1.3, the first part is proved. In the case that $X = Q_{2n-1}$, use the $\mathbb{A}^1$-equivalence $Q_{2n-1} \simeq_{\mathbb{A}^1} \mathbb{A}^n\backslash 0 \simeq_{\mathbb{A}^1} \Sigma^{n-1}\mathbb{G}_m^{\wedge n}$ and we have

$$\begin{aligned}[Q_{2n-1}, BSL_r]_{\mathbb{A}^1_\bullet} &= [\Sigma^{n-1}\mathbb{G}_m^{\wedge n}, BSL_r]_{\mathbb{A}^1\bullet} \\ &= [S^{n-1}, \mathrm{Hom}_\bullet(\mathbb{G}_m^{\wedge n}, BSL_r)]_{\mathbb{A}^1_\bullet} \\ &= \pi^{\mathbb{A}^1}_{n-1,n}(BSL_r)(k) \\ &= \pi^{\mathbb{A}^1}_{n-1}(BSL_r)_{-n}(k).\end{aligned}$$

The last step is due to Theorem 2.7.4. □

As an application, we can classify the vector bundles over Q_{2n-1} of large rank since the $\mathbb{A}^1$-homotopy sheaves $\pi^{\mathbb{A}^1}_{n-1}(BSL_r)$ go into the stable ranger when $r \geq n$.

Corollary 5.1.5. *Suppose $r \geq n \geq 1$ are integers. Any vector bundle $\mathcal{E}$ of rank r over Q_{2n-1} is free.*

Proof. When $n = 1$, $Q_1 \simeq \mathbb{G}_m$ has only trivial vector bundles. Now Assume $n \geq 2$. We know that $\mathrm{Pic}(Q_{2n-1}) \simeq \mathrm{Pic}(\mathbb{A}^n\backslash 0) \simeq \mathrm{Pic}(\mathbb{A}^n)$ is trivial, so every vector bundle over Q_{2n-1} has a trivial determinant bundle, thus orientable. By Theorem 5.1.2 and Theorem 5.1.4, vector bundles over Q_{2n-1} are classified by $\pi^{\mathbb{A}^1}_{n-1}(BSL_r)_{-n}(k)$. Because $r \geq n \geq 1$, by Theorem 4.1.3, $\pi^{\mathbb{A}^1}_{n-1}(BSL_r)_{-n}(k) \simeq (K^Q_{n-1})_{-n} = 0$. This shows that Q_{2n-1} has only one vector bundle of rank r, which is the trivial bundle, thus it must free. □

In the next section, we are going to discuss the critical rank $r = n - 1$. We will use the results obtained in Section 4.4. Due to Theorem 4.4.3, the situation will depend on if n is even or odd.

5.2 Vector bundle of critical rank

5.2.1 When n is even

Notations and backgrounds as the previous section. Now we assume n is even.

Theorem 5.2.1. *Let $n \geq 2$ be a positive even integer. There exists a bijection between the set of isomorphism classes of oriented vector bundle of rank $n-1$ over Q_{2n-1} and the group $\mathbb{Z}/(n-1)!$. Moreover, each isomorphism class is represented by the unimodular row $(x_1^m, x_2, \ldots, x_n)$ for $1 \leq m \leq (n-1)!$.*

Proof. Use the bijection

$$\pi^{\mathbb{A}^1}_{n-1}(BSL_{n-1})_{-n}(k) \xrightarrow{\sim} [Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1}$$

from Theorem 5.1.4 and the short exact sequence

$$0 \to (S_n)_{-n}(k) \to \pi^{\mathbb{A}^1}_{n-1}(BSL_{n-1})_{-n}(k) \to (K^Q_{n-1})_{-n}(k) \to 0$$

from Theorem 4.4.3 since $n-1$ is odd and the contraction is an exact functor. Note that $(K^Q_{n-1})_{-n} = K^Q_{-1} = 0$, thus we have a bijection

$$[Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1} \xrightarrow{\sim} (S_n)_{-n}(k).$$

By Corollary 4.3.8, $(S_n)_{-n}(k) \simeq K^M_0(k)/(n-1)! \simeq \mathbb{Z}/(n-1)!$. The first part of the theorem is proved.

Now we want to explicitly describe each isomorphism class. Consider the following $\mathbb{A}^1$-fiber sequence

$$\mathbb{A}^n\backslash 0 \to BSL_{n-1} \to BSL_n.$$

Apply the functor $\mathrm{Hom}_{\mathbb{A}^1_\bullet}(Q_{2n-1}, -)$ in the pointed category $\mathcal{H}_\bullet(k)$ and we obtain an exact sequence of pointed sets

$$[Q_{2n-1}, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet} \to [Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1_\bullet} \to [Q_{2n-1}, BSL_n]_{\mathbb{A}^1_\bullet}.$$

The first map is defined by pulling back the vector bundle associated to the SL_n-torsor $SL_n \to SL_n/SL_{n-1} \simeq \mathbb{A}^n\backslash 0$ along a given morphism $Q_{2n-1} \to \mathbb{A}^n\backslash 0$. Note that $[Q_{2n-1}, BSL_n]_{\mathbb{A}^1_\bullet} = *$ is trivial by Corollary 5.1.5. So the pullback bundle over Q_{2n-1} is stably free, namely it becomes trivial if we add a free rank 1 summand. In other words, any vector bundles of rank $n-1$ over Q_{2n-1} obtained this way corresponds to a unimodular row. By the previous discussion, we have the following identification

$$\begin{aligned}[Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1_\bullet} &\simeq \pi^{\mathbb{A}^1}_{n-1}(BSL_{n-1})_{-n}(k) \simeq (S_n)_{-n}(k) \\ [Q_{2n-1}, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet} &\simeq \pi^{\mathbb{A}^1}_{n-1}(\mathbb{A}^n\backslash 0)_{-n}(k) \simeq K^{MW}_0(k).\end{aligned}$$

By Lemma 4.3.7 and Corollary 4.3.8, the morphism

$$[Q_{2n-1}, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet} \to [Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1_\bullet}$$

corresponds to the projection map

$$K^{MW}_0(k) \xrightarrow{\eta=0} K^M_0(k) \to (S_n)_{-n}(k) = Z/(n-1)!.$$

Now fix a positive integer $m \in \mathbb{N}$ and consider the unimodular row $(x_1^m, x_2, \ldots, x_n)$ over the coordinate ring of the affine scheme Q_{2n-1}. By Theorem A.2.1, this unimodular row can be viewed as an element in $[Q_{2n-1}, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1_\bullet}$. Use the $\mathbb{A}^1$-equivalence $Q_{2n-1} \simeq \mathbb{A}^n\backslash 0 \simeq \Sigma^{n-1}\mathbb{G}_m^{\wedge n}$,

this unimodular row corrresponds to a morphism

$$\begin{aligned} l :&\mathbb{G}_m^{\wedge n} \to \mathbb{G}_m^{\wedge n}, \\ &(x_1, x_2, \ldots, x_n) \mapsto (x_1^m, x_2, \ldots, x_n). \end{aligned}$$

Since $K_n^M(\mathbb{G}_m^{\wedge n}) = (K_n^M)_{-n}(k) = K_0^M(k) = \mathbb{Z}$ is generated by the class of the symbol $\{x_1, \ldots, x_n\}$ and $\{x_1^m, x_2, \ldots, x_n\} = m\{x_1, \ldots, x_n\}$, we know that the unimodular rows $(x_1^m, x_2, \ldots, x_n)$ for $1 \leq m \leq (n-1)!$ give all the isomorphism classes of oriented vector bundles of rank $n-1$ over Q_{2n-1}.

□

Next, we can generalize this result to all vector bundles over Q_{2n-1}, not necessarily oriented.

Corollary 5.2.2. *Let $n \geq 2$ be a positive even integer. The set of isomorphism classes of vector bundles of rank $n-1$ over Q_{2n-1} has $(n-1)!$ elements. Moreover, each isomorphism class admits a representative given by the unimodualr row $(x_1^m, x_2, \ldots, x_n)$ for $1 \leq m \leq (n-1)!$.*

Proof. By Corollary 5.1.5, every vector bundle of rank n over Q_{2n-1} is free. So every vector bundle of rank $n-1$ over Q_{2n-1} is stably free. This means that the set of isomorphism class of vector bundles of rank $n-1$ is given by the unimodular rows $Um_n(Q_{2n-1})/GL_n(Q_{2n-1})$ due to Proposition A.1.2. On the other hand, by Theorem 5.2.1, $Um_n(Q_{2n-1})/SL_n(Q_{2n-1}) = \mathbb{Z}/(n-1)!$ with explicit generators. Now it remains to show that the surjective morphism

$$Um_n(Q_{2n-1})/SL_n(Q_{2n-1}) \to Um_n(Q_{2n-1})/GL_n(Q_{2n-1})$$

is also injective. Write A_{2n-1} as the ring of global sections on Q_{2n-1} and we have $A_{2n-1}^\times = k^\times$. Given $\alpha \in k^\times$, we need to show that two unimodular rows $(x_1^m, x_2, \ldots, x_n)$ and $(\alpha x_1^m, x_2, \ldots, x_n)$ will give the same isomorphism class. According to the proof of Theorem 5.2.1, when viewed as elements in $K_n^M(\mathbb{G}_m^{\wedge n})$, we have

$$\{\alpha x_1^m, x_2, \ldots, x_n\} = \{\alpha, x_2, \ldots, x_n\} + \{x_1^m, x_2, \ldots, x_n\} = \{x_1^m, x_2, \ldots, x_n\}.$$

This is because the unimodular row $(\alpha, x_2, \ldots, x_n)$ is completable as $\alpha \in k^\times$. And we are done. □

5.2.2 When n is odd

Now we assume $n \geq 3$ is an odd integer. We are going to discuss isomorphism classes of vector bundles of rank $n-1$ over Q_{2n-1}. Recall that by Theorem 4.4.3, for $n \geq 3$ odd, we have a short exact sequence

$$0 \to T_n \to \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1}) \to K_{n-1}^Q \to 0.$$

Theorem 5.2.3. *Let $n \geq 3$ be an odd integer. There is a bijection between the set of isomorphism classes of oriented vector bundles of rank $n-1$ over Q_{2n-1} and $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)$, where $W(k)$ is the Witt ring of the field k. Moreover, the representatives are given by the set of unimodular rows $\{(\alpha x_1, x_2, \ldots, x_n), \alpha \in k^\times\}$ together with the rows $(x_1^{2m}, x_2, \ldots, x_n)$ for $1 \leq m \leq (n-1)!/2$.*

Proof. Similar to the proof for the case n is even, we identify the set $[Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1}$ with the $\pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})_{-n}(k)$ and note that the representatives can be given by unimodular rows. Since $n \geq 3$ is odd, by Theorem 4.4.3, we have an isomorphism

$$\pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})_{-n}(k) \simeq (T_n)_{-n}(k).$$

Recall that in Definition 4.4.2, T_n is defined as a fiber product and since the contraction is an exact functor, we obtain a Cartesian square:

$$\begin{array}{ccc} (T_n)_{-n}(k) & \longrightarrow & (I^n)_{-n}(k) \\ \downarrow & & \downarrow \\ \mathbb{Z}/(n-1)! & \longrightarrow & \mathbb{Z}/2. \end{array}$$

By Proposition 3.3.1, $(I^n)_{-n}(k) = I^0(k) = W(k)$ is the Witt ring and we can conclude that the set of isomorphism classes of oriented vector bundles of rank $n-1$ over Q_{2n-1} is bijective with

$$\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k).$$

To find the explicit representatives, similar to the proof in the case n is even, we observe that the morphism

$$[Q_{2n-1}, \mathbb{A}^n \backslash 0]_{\mathbb{A}^1_\bullet} \to [Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1_\bullet}$$

corresponds to the projection map

$$K_0^{MW}(k) \simeq GW(k) \to \mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k).$$

To show that the unimodular rows in the statements indeed represents all the oriented vector bundles, it suffices to calculate their $GW(k)$-valued Brouwer degrees. By the next Lemma 5.2.4, for $\alpha \in k^\times$, the unimodular row $(\alpha x_1, x_2, \ldots, x_n)$ has the degree $(1, \langle \alpha \rangle)$, and for $1 \leq m \leq (n-1)!/2$, the unimodular row $(x_1^{2m}, x_2, \ldots, x_n)$ has the degree $(2m, 0)$. This proves the theorem.

□

To complete the above proof, we need to calculate the Brouwer degree of the unimodular rows. Recall that the Cartesian square 3.1 in degree 0 gives a presentation of $K_0^{MW}(k) =$

$GW(k)$ as follows:

$$\begin{array}{ccc} GW(k) & \longrightarrow & W(k) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/2. \end{array}$$

By Corollary 3.1.5, we can calculate the Brouwer degree of the given unimodular rows to show that they indeed represents oriented vector bundle of rank $n-1$ over Q_{2n-1}.

The presentation of $GW(k) = \mathbb{Z} \times_{\mathbb{Z}/2} W(k)$ tells us that the $GW(k)$-valued Brouwer degree has two parts: $GW(k) \to \mathbb{Z}$ comes from the corresponding rank morphism of quadratic forms, and $W(k)$ is the Witt ring of field k, which is generated by $\langle a \rangle$ for $a \in k^\times$ with relationships in Lemma 3.1.2, plus sending the hyperbolic form $h = 1 + \langle -1 \rangle$ to 0. The next Lemma shows us how to calculate the Brouwer degree for the unimodular row $(\alpha x_1^m, x_2, \ldots, x_n)$.

Lemma 5.2.4. *Notations as above. Let $\alpha \in k^\times$ and $m \neq 1$ be an integer. The Grothendieck-Witt valued Brouwer degree of the unimodualr row $(\alpha x_1^m, x_2, \ldots, x_n)$ is $(m, \sum_{i=1}^m \langle (-1)^{i-1} \alpha \rangle)$.*

Proof. Use the notations and results from [23, Lemma 3.14], we can view $[\alpha x_1^m]$ as an element in $K_1^{MW}(k)$ and we have

$$[\alpha x_1^m] = [\alpha] + [x_1^m] + \eta[\alpha, x_1^m] = [\alpha] + \langle \alpha \rangle [x_1^m] = [\alpha] + \langle \alpha \rangle m_\epsilon [x_1].$$

Here $m_\epsilon = \sum_{i=1}^m \langle (-1)^{i-1} \rangle$. The $\mathbb{Z}$-part of the Brouwer degree for $(\alpha x_1^m, x_2, \ldots, x_n)$ is just $\dim(m_\epsilon) = m$, viewing m_ϵ as a quadratic form. Due to [22, Lemma 2.4, Remark 2.6], the unimodular row $(\alpha x_1, x_2, \ldots, x_n)$ has the degree $(1, \langle \alpha \rangle)$. So the $W(k)$-part of the degree is just $\langle \alpha \rangle m_\epsilon = \sum_{i=1}^m \langle (-1)^{i-1} \alpha \rangle$. □

Corollary 5.2.5. *Let $n \geq 3$ be an odd integer. Then there is a bijection between the set of isomorphism classes of vector bundles of rank $n-1$ over Q_{2n-1} and the quotient $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)/k^\times$.*

Proof. Similar to the proof when n is even, the surjective morphism

$$Um_n(Q_{2n-1})/SL_n(Q_{2n-1}) \to Um_n(Q_{2n-1})/GL_n(Q_{2n-1})$$

can be viewed as a morphism from oriented vector bundles to vector bundles. To calculate what is the kernel of this morphism, consider $\alpha \in A_{2n-1}^\times = k^\times$, by Lemma 5.2.4, we know α acts on a representative $(m, \phi) \in \mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)$ as $\alpha \cdot (m, \phi) = (m, \langle \alpha \rangle \cdot \phi)$. So the resulting orbit is given by $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)/k^\times$. □

Remark. By the representability of vector bundles in $\mathbb{A}^1$-homotopy category, we know that the sets of isomorphism classes of vector bundles over $\mathbb{A}^1$-equivalent spaces are bijective. So the above results also give the classification of vector bundles over other smooth varieties, not

necessarily isomorphic to Q_{2n-1}. These varieties are torsors under vector bundles over $\mathbb{A}^n\backslash 0$. For example, let $f_1, \ldots, f_n \in k[y_1, \ldots, y_n]$ be polynomials such that $V(f_1, \ldots, f_n)$ is a point in $\mathbb{A}^n$. The hypersurface defined by the equation $\sum_{i=1}^n x_i f_i = 1$ is a smooth affine variety whcih is $\mathbb{A}^1$-equivalent to $\mathbb{A}^n\backslash 0$. The set of isomorphism classes of vector bundles over this variety is bijective with the set of isomorphism classes of vector bundles over Q_{2n-1}.

5.3 Application to a question of Nori

Nori has proposed the following question about completability of unimodualr rows. As an application, we could use the results from the previous sections to give some answers. For some technical reason, we assume the field k is infinite.

Question 5.3.1 (Nori)**.** *Let k be a field and $R = k[x_1, \ldots, x_n]$ be the polynomial ring over k. We have a k-algebra homomorphism $\phi : R \to A$ satisfying $\sum_{i=1}^n \phi(x_i)A = A$. Let $f_1, \ldots, f_n$ be the elements in R such that the reduced closed subscheme defined by the ideal $(f_1, \ldots, f_n)$ is $0 \in \mathbb{A}^n$. If the length of $R/(f_1, \ldots, f_n)$ is divisible by $(n-1)!$, then is the unimodular row $(\phi(f_1), \ldots, \phi(f_n))$ completable?*

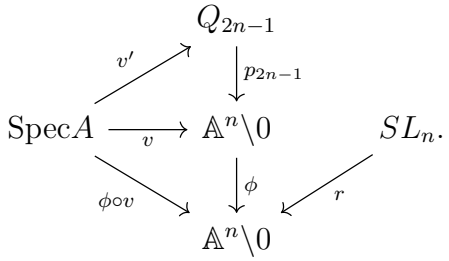
We could reinterpret Nori's question into a lifting problem. The condition $\sum_{i=1}^n \phi(x_i)A = A$ defines a unimodular row $v = (\phi(x_1), \ldots, \phi(x_n))$, and thus a morphism $v : \mathrm{Spec}A \to \mathbb{A}^n$. Now a sequence of polynomials $f_1, \ldots, f_n$ with $rad(f_1, \ldots, f_n) = (x_1, \ldots, x_n)$ defines a morphism $\phi : \mathbb{A}^n\backslash 0 \to \mathbb{A}^n\backslash 0$. If $\mathrm{length}(R/(f_1, \ldots, f_n))$ is divisible by $(n-1)!$, then does the morphism $\phi \circ v : \mathrm{Spec}A \to \mathbb{A}^n\backslash 0$ lift to a morphism $\mathrm{Spec}A \to SL_n$?

Recall that the coordinate k-algebra A_{2n-1} of the smooth affine spilt quadrics Q_{2n-1} is the universal algebra parameterizing unimodular rows of length n, with the choice of a section. More precisely, let A be a k-algebra and v be a unimodular row of length n over A. A choice of $w \in Um_n(A)$ such that $v \cdot w^t = 1$ yields a lift of the morphism $v : \mathrm{Spec}A \to \mathbb{A}^n\backslash 0$ to a morphism $v' : \mathrm{Spec}A \to Q_{2n-1}$, namely we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}A & \overset{v'}{\dashrightarrow} & Q_{2n-1} \\ & \searrow^{v} & \downarrow{\scriptstyle p_{2n-1}} \\ & & \mathbb{A}^n\backslash 0. \end{array}$$

Here $p_{2n-1} : Q_{2n-1} \to \mathbb{A}^n\backslash 0$ is the projection into the first n coordinates. Now let $\phi : \mathbb{A}^n\backslash 0 \to \mathbb{A}^n\backslash 0$ be a morphism and $r : SL_n \to \mathbb{A}^n\backslash 0$ be the projection to the first row, we have the

following commutative diagram:



Whether the morphism $\phi \circ v : \mathrm{Spec}A \to \mathbb{A}^n\backslash 0$ factorize through SL_n can be rephrased as whether the morphism $\phi \circ p_{2n-1} : Q_{2n-1} \to \mathbb{A}^n\backslash 0$ factorize through SL_n.

Theorem 5.3.2. *If $n \geq 2$ is an even integer, then Question 5.3.1 has an affirmative answer.*

Proof. The morphism $p_{2n-1} : Q_{2n-1} \to \mathbb{A}^n\backslash 0$ correspond to the unimodular row $(x_1, x_2, \ldots, x_n)$, and by Corollary 5.2.2, it also correponds to the element 1 in $\mathbb{Z}/(n-1)!$.

The unimodular row over A $(f_1, \ldots, f_n)$ corresponds to the following morphism:

$$\begin{aligned} F :&\mathbb{A}^n\backslash 0 \to \mathbb{A}^n\backslash 0, \\ &(x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)). \end{aligned}$$

Consider the following commutative square:

$$\begin{array}{ccc} \mathbb{A}^n\backslash 0 & \longrightarrow & \mathbb{A}^n \\ {\scriptstyle F|_{\mathbb{A}^n\backslash 0}}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{A}^n\backslash 0 & \longrightarrow & \mathbb{A}^n. \end{array}$$

The horizontal morphisms are open immersions. It induces a morphism in long exact sequences and we have

$$\begin{array}{ccc} H^{n-1}(\mathbb{A}^n\backslash 0, K_n^M) & \xrightarrow{\partial} & H^n_{\{0\}}(\mathbb{A}^n, K_n^M) \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H^{n-1}(\mathbb{A}^n\backslash 0, K_n^M) & \xrightarrow[\partial]{} & H^n_{\{0\}}(\mathbb{A}^n, K_n^M). \end{array}$$

Here ∂ is the connecting morphism coming from the open immersion $\mathbb{A}^n\backslash 0 \to \mathbb{A}^n$. Now we use the Gersten complex to find a generator of $H^n_{\{0\}}(\mathbb{A}^n, K_n^M)$. Consider the closed subscheme Y of $\mathbb{A}^n\backslash 0$ defined by the equation $x_1 = x_2 = \cdots = x_{n-1} = 0$. x_n has codimension $n-1$ in Y and can be viewed as an element of $K_1^M(k(Y))$ in the degree $n-1$ part of the Gersten complex. It is easy to see that $\partial(x_n)$ is a generator of $H^n_{\{0\}}(\mathbb{A}^n, K_n^M)$, since the local cohomology can be computed using Gersten complex.

Up to $\mathbb{A}^1$-equivalences, we can assume closed subscheme Z cut out by $f_1 = f_2 = \cdots = f_{n-1} = 0$ is smooth by Swan's version of Bertini's theorem [1, Theorem 1.3]. Then similarly f_n

has codimension $n-1$ in Z and can be viewed as an element in $K_1^M(k(Z))$. We have $F^*(x_n) = f_n$ and by definition of the connecting homomorphism $\partial(f_n) = \text{length}(R/(f_1, f_2, \ldots, f_n))$ in $H^n_{\{0\}}(\mathbb{A}^n, K_n^M) \simeq \mathbb{Z}$. Since the length is divided by $(n-1)!$, the corresponding unimodular row is $(x^{(n-1)!}, x_2, \ldots, x_n)$, which is the free vector bundles, thus it is completable. □

When $n \geq 3$ is odd, Fasel has already constructed a counterexample [22, Theorem 4.7]. He also proposed a refined version of Nori's question. We first introduce the degree of a morphism $\phi : \mathbb{A}^n\backslash 0 \to \mathbb{A}^n\backslash 0$. Proposition 2.7.9 tells us that $H^{n-1}(\mathbb{A}^n\backslash 0, I^n) \simeq (I^n)_{-n}(k) \simeq W(k)$. The morphism ϕ induces a morphism between cohomology groups $\phi^* : H^{n-1}(\mathbb{A}^n\backslash 0, I^n) \to H^{n-1}(\mathbb{A}^n\backslash 0, I^n)$, which can be viewed as a homomorphism between Witt rings $\phi^* : W(k) \to W(k)$. And we call this homomorphism the degree of ϕ and denote it by $\deg(\phi)$. This degree is in fact the Witt ring part of Morel's Grothendieck-Witt valued Brouwer degree [25, Remark 5.3]. The strong Nori's question proposed by Fasel [22, Question 4.8] can be answered affirmatively and we state it as a theorem below.

Theorem 5.3.3. *Notations sames as Question 5.3.1. Assume additionally that $n \geq 3$ is odd and $\deg(\phi) = 0$, then the unimodular row $(\phi(f_1), \ldots, \phi(f_n))$ is completable.*

Proof. Write $l = \text{length}(R/(f_1, \ldots, f_n))$. A similar argument as in Theorem 5.3.2 shows that the unimodular row has Brouwer degree $(l, \deg(\phi))$, and by Corollary 5.2.5, the corresponding vector bundle is free if and only if the length is divided by $(n-1)!$ and $\deg(\phi)$ is the trivial element in $W(k)/k^\times$. This is true since $\deg(\phi) = 0$. □

Appendix A

Unimodular Rows

In this appendix, we give a brief review of unimodular rows in the context of $\mathbb{A}^1$-homotopy theory. Assume R is a commutative ring with unity and $n \geq 3$ is a positive integer.

A.1 Basics

In this section, we briefly review the basic definitions about unimodular rows and some important results. The main reference is [14]. More details can always be found in this book.

Definition A.1.1. A row of elements $\underline{a} = (a_1, a_2, \ldots, a_n) \in R^n$ is called a unimodualr row of length n if there exists another row $(b_1, b_2, \ldots, b_n) \in R^n$ such that $\sum_{i=1}^n a_i b_i = 1$. We denote by $Um_n(R)$ the set of unimodular rows of length n over the ring R.

Remark. Every unimodular row $\underline{a} \in Um_n(R)$ of length n defines a surjective morpism $R^n \xrightarrow{\underline{a}} R$ by multiplication, and we have the following split short exact sequence

$$0 \to P \to R^n \xrightarrow{\underline{a}} R \to 0.$$

Here P is a projective module over R of rank $n-1$ satisfying $P \oplus R \simeq R^n$. We call P the projective module associated with the unimodular row $\underline{a}$ and it is stably free.

Let $GL_n(R)$ be the general linear group valued in the ring R. $GL_n(R)$ has a right action on $Um_n(R)$ by right matrix multiplication. Consider the orbit space and we have the following:

Proposition A.1.2 ([14, Chapter 1, proposition 4.8])**.** *The orbits of $Um_n(R)$ under the $GL_n(R)$-action are in 1-1 correspondence with the isomorphism classes of R-module P for which $P \oplus R \cong R^n$. Under this correspondence, the orbit of $(1, 0, \ldots, 0)$ corresponds to the isomorphism class of free module R^{n-1}.*

Proof. By the previous Remark, we have already seen how we can associate a projective module to a unimodular row. Suppose we have two unimodular rows $\underline{a}, \underline{b} \in Um_n(R)$ and

denote their associated projective module by P_1 and P_2. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \longrightarrow & R^n & \xrightarrow{\underline{a}} & R & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & P_2 & \longrightarrow & R^n & \xrightarrow{\underline{b}} & R & \longrightarrow & 0 \end{array}$$

If P_1 and P_2 is isomorphic as projective R-modules, then we can find an isomorphism in the middle column since these two short exact sequences both splits. Conversely, if there exist a matrix $M \in GL_n(R)$ such that $\underline{a}M = \underline{b}$, then if defines an isomorphism in the middle column $M : R^n \xrightarrow{\sim} R^n$. Thus, the kernel is also isomorphic. □

Corollary A.1.3 ([14, Corollary 4.9, Chapter I]). *A unimodular row $(a_1, \ldots, a_n) \in Um_n(R)$ is completable to a square invertible matrxi if and only if the associated projective module is free, if and only if $(a_1, \ldots, a_n)$ and $(1, 0, \ldots, 0)$ are in the same $GL_n(R)$-orbits.*

A.2 Unimodular rows and $\mathbb{A}^1$-homotopy

In this section we are going to discuss the unimodualr rows from the $\mathbb{A}^1$-homotopy point of view. From now on, we assume k is a perfect field and R is a finitely generated smooth k-algebra. Every unimodular row $\underline{a} \in Um_n(R)$ uniquely determines a surjective morphism $R^n \to R$, which can be viewed as a morphism $\mathrm{Spec}R \to \mathbb{A}^n\backslash 0$. This yields a bijection

$$Um_n(R) \xrightarrow{\sim} \mathrm{Hom}_{Sm_k}(\mathrm{Spec}R, \mathbb{A}^n\backslash 0).$$

Recall that $E_n(R)$ is the set of elementary matrices with input from R and in the proof of Lemma 4.1.1, it is shown that two unimodular rows connected by an elementary matrix is naively $\mathbb{A}^1$-homotopic. We have an induced morphism:

$$Um_n(R)/E_n(R) \to \mathrm{Hom}_{N\mathbb{A}^1_\bullet}(\mathrm{Spec}R, \mathbb{A}^n\backslash 0)$$

where the left-hand side is pointed at the e_1-orbit and $\mathrm{Hom}_{N\mathbb{A}^1_\bullet}$ denote the morphisms modulo naively $\mathbb{A}^1$-homotopy. In fact, this is a bijection when $n \geq 3$.

Be careful here as in general to obtain the Hom-set in $\mathcal{H}_\bullet(k)$ it is not enough to just modulo naive $\mathbb{A}^1$-homotopies, namely $\mathrm{Hom}_{N\mathbb{A}^1_\bullet}(\mathrm{Spec}R, \mathbb{A}^n\backslash 0)$ is not neccessarily isomorphic to $\mathrm{Hom}_{\mathbb{A}^1_\bullet}(\mathrm{Spec}R, \mathbb{A}^n\backslash 0)$. But we do have $\mathrm{Hom}_{N\mathbb{A}^1_\bullet}(\mathrm{Spec}R, \mathbb{A}^n\backslash 0) \simeq \mathrm{Hom}_{\mathbb{A}^1_\bullet}(\mathrm{Spec}R, \mathbb{A}^n\backslash 0)$, due to [23, Theorem 8.9 3)].

Theorem A.2.1 ([21, Theorem 2.1]). *Notation as above. For $n \geq 3$, we have a bijection:*

$$Um_n(R)/E_n(R) \xrightarrow{\sim} Hom_{\mathbb{A}^1_\bullet}(SpecR, \mathbb{A}^n\backslash 0).$$

Proof. According to the above discussion, we only need to prove the reverse direction. Let $\underline{a}, \underline{b} \in Um_n(R)$ and assume they are naively $\mathbb{A}^1$-homotopic. More precisely, there exists an elemnt $T(t) \in Um_n(R[t])$ such that $T(0) = \underline{a}$ and $T(1) = \underline{b}$. Consider the associated short exact sequence

$$0 \to P \to R[t]^n \xrightarrow{T} R[t] \to 0.$$

Note that here P is a projective $R[t]$-module, and by some proof of Bass-Quillen Conjecture (Conjecture 1.1.1), P comes from a projective R-module. Thus, we can write $P = P(0)[t]$, where $P(0)$ is defined by the following short exact sequence

$$0 \to P(0) \to R^n \xrightarrow{\underline{a}} R \to 0.$$

This can be augmented by mapping t to t and abusing some notations

$$0 \to P(0)[t] \to R[t]^n \xrightarrow{\underline{a}} R[t] \to 0.$$

Comparing two short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & R[t]^n & \xrightarrow{T} & R[t] & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & P(0)[t] & \longrightarrow & R[t]^n & \xrightarrow{\underline{a}} & R[t] & \longrightarrow & 0. \end{array}$$

There exists an automorphism $\psi(t) \in SL_n(R[t])$ such that the above diagram commutes. Note that $\psi(0) = Id$. By Suslin's stability theorem [14, Chapter VI, Theorem 4.5], $\psi \in E_n(R[t])$ since $n \geq 3$ and R is a finitely generated k-algebra. Evaluate at $t = 1$, we have $\underline{a}\psi(1) = \underline{b}$. □

Note that $\mathbb{A}^n\backslash 0$ is $\mathbb{A}^1$-(n-1)-connective, and its first nontrivial $\mathbb{A}^1$-homotopy sheaf $\pi^{\mathbb{A}^1}_{n-1}(\mathbb{A}^n\backslash 0) \simeq K^{MW}_n$. We could also obtain a cohomological description of $Um_n(R)/E_n(R)$ as follows when $n \geq 3$. In this case $\mathbb{A}^n\backslash 0$ is $\mathbb{A}^1$-simply connected. Consider the Psotnikov tower of $\mathbb{A}^n\backslash 0$ at stage n

$$\tau_{\leq n}(\mathbb{A}^n\backslash 0) \to \tau_{\leq n-1}(\mathbb{A}^n\backslash 0) \to K(\pi^{\mathbb{A}^1}_n(\mathbb{A}^n\backslash 0), n+1).$$

This is an $\mathbb{A}^1$-fiber sequence and since $\mathbb{A}^n\backslash 0$ is $\mathbb{A}^1$-$n-1$-connective, $\tau_{\leq n-1}(\mathbb{A}^n\backslash 0)$ only has nontrivial $\mathbb{A}^1$-homotopy sheaf K^{MW}_n in degree $n-1$. Therefore, we have the following $\mathbb{A}^1$-fiber sequence

$$\tau_{\leq n}(\mathbb{A}^n\backslash 0) \to K(K^{MW}_n, n-1) \to K(\pi^{\mathbb{A}^1}_n(\mathbb{A}^n\backslash 0), n+1).$$

Now assume X is a pointed smooth scheme and given a morphism $X \to \mathbb{A}^n\backslash 0$ in $\mathcal{H}_\bullet(k)$, we can compose with the canoncial morphism $\mathbb{A}^n\backslash 0 \to \tau_{\leq n}(\mathbb{A}^n\backslash 0)$ from the Postnikov tower of $\mathbb{A}^n\backslash 0$ and $\tau_{\leq n}(\mathbb{A}^n\backslash 0) \to K(K^{MW}_n, n-1)$ as above, we obtain a morphism

$$X \to K(K^{MW}_n, n-1)$$

in $\mathcal{H}_\bullet(k)$, which can be viewed as a Nisnevich cohomology class in $H^{n-1}(X, K_n^{MW})$. Therefore, we establish a morphism

$$[X, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1} \to H^{n-1}(X, K_n^{MW}).$$

Proposition A.2.2. *Suppose X is a smooth scheme and $n \geq 3$ is an integer. X is $\mathbb{A}^1$-equivalent to a smooth scheme of dimension at most n (respectively at most $n-1$), then the morphism*

$$[X, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1} \to H^{n-1}(X, K_n^{MW})$$

is surjective (respectively bijective).

Proof. According to the previous discussion, $H^{n-1}(X, K_n^{MW})$ corresponds to a morphism $X \to K(K_n^{MW}, n-1) \simeq \tau_{\leq n-1}(\mathbb{A}^n\backslash 0)$. $n \geq 3$ implies that $\mathbb{A}^n\backslash 0$ is $\mathbb{A}^1$-simply connected. Let $i \geq 0$ be an integer. Consider the Postnikov tower of $\mathbb{A}^n\backslash 0$ and to define a morphism $X \to \mathbb{A}^n\backslash 0$, we need to lift the morphism $X \to \tau_{\leq n-1}(\mathbb{A}^n\backslash 0)$ to $\tau_{\leq i}(\mathbb{A}^n\backslash 0)$ for $i \geq n$. Assume we have a morphism $X \to \tau_{\leq i}(\mathbb{A}^n\backslash 0)$,

$$\begin{array}{ccccc} & & & & X \\ & & & \swarrow & \downarrow \\ \tau_{\leq i+1}(\mathbb{A}^n\backslash 0) & \longrightarrow & \tau_{\leq i}(\mathbb{A}^n\backslash 0) & \longrightarrow & K(\pi_{i+1}^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0), i+2). \end{array}$$

The morphism $X \to \tau_{\leq i}(\mathbb{A}^n\backslash 0)$ can be lifted to $\tau_{\leq i+1}(\mathbb{A}^n\backslash 0)$ if and only if the compsition $X \to \tau_{\leq i}(\mathbb{A}^n\backslash 0) \to K(\pi_{i+1}^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0), i+2)$ is zero, which corresponds to a cohomology class in $H^{i+2}(X, \pi_{i+1}^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0))$. Here $\pi_{i+1}^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0)$ is strictly $\mathbb{A}^1$-invariant for all $i \geq 0$. If X is $\mathbb{A}^1$-equivalent to a smooth scheme of dimension at most n, then by Nisnevich cohomology vanishing result, we know that $H^{i+2}(X, \pi_{i+1}^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0)) = 0$ for $i \geq n-1$. In conclusion, the obstruction of lifting $X \to \tau_{\leq n-1}(\mathbb{A}^n\backslash 0)$ to $\tau_{\leq i}(\mathbb{A}^n\backslash 0)$ lies in $H^{n+1}(X, \pi_n^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0)) = 0$. So the lifting is always possible, and this proves the surjectivity. If we assume X has the $\mathbb{A}^1$-homotopy type of a smooth scheme of dimension at most $n-1$, note that the lifting is parametrized by $H^n(X, \pi_n^{\mathbb{A}^1}(\mathbb{A}^n\backslash 0)) = 0$. In this case, we get a bijection. □

The above Proposition gives a bijection between unimodular rows $Um_n(R)/E_n(R)$ and cohomology classes in $H^{n-1}(\mathrm{Spec}R, K_n^{MW})$ for $n \geq 3$.

Corollary A.2.3. *When $X = \mathbb{A}^n\backslash 0$, the surjective map*

$$[\mathbb{A}^n\backslash 0, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1} \to H^{n-1}(\mathbb{A}^n\backslash 0, K_n^{MW})$$

is a bijection.

Proof. By Proposition 2.7.9, the cohomology of $\mathbb{A}^n\backslash 0$ in degree n vanishes and then applies the previous Propostion. □

A.3 Completability of unimodular rows

Fix a communitative ring R and a positive integer $n \geq 2$.

Definition A.3.1. We say a unimodular row $\underline{a} \in Um_n(R)$ is completable if the associate projective R-module is free. By the above Proposition, we see that $\underline{a}$ is completable if and only if $\underline{a}$ and $e_1 := (1, 0, \ldots, 0)$ are in the same $GL_n(R)$-orbit. The set $Um_n(R)$ is naturally pointed at e_1.

It is not hard to see that the Completability of a unimodular row can be detected by $SL_n(R)$-action. Indeed, suppose a unimodular row $(a_1, \ldots, a_n) \in Um_n(R)$ is completable. This means that there exists a invertible matrix $M \in GL_n(R)$ such that $(a_1, \ldots, a_n)M = (1, 0, \ldots, 0)$. Multiply each entry of the first column of M with $(\mathrm{det}M)^{-1}$. The resulting matrix is denoted by M' and we have $\mathrm{det}M' = 1$ and $(a_1, \ldots, a_n)M' = ((\mathrm{det}M)^{-1}, 0, \ldots, 0)$. Consider the multiplication $M'diag(\mathrm{det}M, (\mathrm{det}M)^{-1}, 1, \ldots, 1) \in SL_n(R)$ and we have

$$(a_1, \ldots, a_n)M'diag(\mathrm{det}M, (\mathrm{det}M)^{-1}, 1, \ldots, 1) = (1, 0, \ldots, 0).$$

This shows that $(a_1, \ldots, a_n)$ is completable if and only if it is in the same $SL_n(R)$-orbit with $(1, 0, \ldots, 0)$ if $n \geq 2$. Next we will rephrase this in $\mathbb{A}^1$-homotopy.

Consider the morphism between schemes $SL_n \to \mathbb{A}^n\backslash 0$ given by the projection on the first column. Let $X = \mathrm{Spec}R$ be a smooth affine k-scheme. From the discussion in the previous section, a unimodular row over R can be viewed as a morphism $X \to \mathbb{A}^n\backslash 0$. Then this unimodular row is completable if and only if the associated morphism factor through SL_n. More precisely, there exists a morphism $X \to SL_n$ such that the following triangle commutes

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^n\backslash 0 \\ & \searrow & \uparrow \\ & & SL_n. \end{array}$$

We have already seen by Theorem A.2.1 that the morphism $\mathrm{Hom}_{Sm_k}(X, \mathbb{A}^n\backslash 0) \to [X, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1}$ is surjective as they are just taking the $E_n(R)$-orbits. Note that $\mathrm{Sing}^{\mathbb{A}^1} SL_n$ is $\mathbb{A}^1$-invariant and Nisnevich local [30, Theorem 2.3.5], so $\mathrm{Hom}_{Sm_k}(X, SL_n) \to [X, SL_n]_{\mathbb{A}^1}$ is also surjective and we have a commutative square:

$$\begin{array}{ccc} \mathrm{Hom}_{Sm_k}(X, SL_n) & \longrightarrow & \mathrm{Hom}_{Sm_k}(X, \mathbb{A}^n\backslash 0) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ [X, SL_n]_{\mathbb{A}^1} & \longrightarrow & [X, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1}. \end{array}$$

The horizontal maps are induced by the projection to the first column $SL_n \to \mathbb{A}^n\backslash 0$. Thus, if we view a unimodular row over R as a morphism in $[\mathrm{Spec}R, \mathbb{A}^n\backslash 0]_{\mathbb{A}^1}$, it is completable if and

only if the corresponding $\mathbb{A}^1$-homotopy classes can be lifted to an element in $[\mathrm{Spec}R, SL_n]_{\mathbb{A}^1}$.

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