Universität Zürich[UZH]

Tautological Rings of Moduli Spaces $\overline{M}_{g,n}$

Master's Thesis

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Abstract

This thesis aims to give a comprehensive introduction to calculation of intersection numbers of tautological classes on $\overline{M}_{g,n}$. On a moduli space of stable curves of genus g and n marked points, we can define the most natural classes that form a subring of the cohomology ring $H^*(\overline{M}_{g,n})$, the tautological ring. This subring captures some interesting classes in $\overline{M}_{g,n}$, hence studying the intersection number of these classes is of great importance.

In 1993, Faber proposed three remarkable conjectures about the structure of the tautological ring. The one of interest here is the Gorenstein part, i.e. the tautological ring $R^*(\overline{M}_{g,n})$ admits a perfect pairing. Faber himself developed a Maple package *KaLa5* that calculates the intersection numbers, however the efficiency is limited when the dimension of $\overline{M}_{g,n}$ exceeds 20. Many people have developed their own methods to calculate intersection numbers. In 2010, Yang introduced a Macaulay2 package *HodgeIntegral* that improved the efficiency significantly, based on new relations discovered by Liu, Xu. This thesis aims to explain Yang's method and develop a more efficient Python program to be incorperated into the SageMath package *admcycles*.

Chapter 1 introduces basic concepts in moduli theory and the motivation of studying it. A good example is the Grassmannian. A bad example is the parameter space of all elliptic curves.

Chapter 2 defines the moduli space of smooth curves of genus g and its compactification $\overline{M}_{g,n}$. We will switch from studying curves of genus g to stable curves. The compactification adds boundary strata to the original moduli space. They are stratified by dual graphs, which encodes the topological properties of stable curves.

Chapter 3 gives definitions of tautological classes and studies their relations. These classes arise from the most natural bundles over $\overline{M}_{g,n}$, the cotangent bundles and Hodge bundle. They form the tautological ring that

is closed under forgetful and gluing morphisms. The intersection of tautological classes of top degree gives the so-called Hodge integral. At the end of this chapter, we will also see Faber's conjectures on structures of the tautological rings.

Chapter 4 defines the notion of a Hodge integral, which is essentially the top intersection number of tautological classes. Due to the beautiful combinatorial structure of the tautological ring, the calculation can be "simplified" by recursive formulae. In particular, I will explain in section 3 how Mumford obtained the expression of Chern characters in terms of boundary classes. The same calculation will be applied to the case of stable curves.

Chapter 1

Basics of Moduli Theory

Roughly speaking, a moduli space of a certain type of geometric objects is a collection of isomorphism classes of them. It is a way to classify objects. To specify, we need three pieces of information:

i) What is the object we want to classify?

ii) What conditions do they satisfy?

iii) What does it mean for two objects to be isomorphic?

In this chapter, we will see some examples of moduli spaces, as well as how to formalize this idea. In practice, we require a stronger version, the fine moduli space, which in general doesn't exist. The reasons why there's usually just a coarse moduli space will be explained.

1.1 A prototypical example: the Grassmannian

In general, Grassmannian can be defined for vector spaces of any dimension.

Definition 1.1. *Given a vector space V, define the Grassmannian $Gr(k, V)$ of V to be the collection of all subspaces of V of dimension k, where $k \leqslant \dim V$.*

When $\dim V = n$, the Grassmannian is denoted by $Gr(k, n)$. As a set, it contains points each of which represents a k-dimensional subspace of V.

The Grassmannian $Gr(k,n)$ naturally carries a vector bundle structure, the tautological bundle $\pi : E \to Gr(k,n)$ which assigns to each point $[W] \in Gr(k,n)$ the k-dimensional vector space W. We can also say that $Gr(k,n)$ parametrizes all k-dimensional subspaces of V.

The Grassmannian also possesses a nice structure. When we only want to study a small part of k-dimensional subspaces, it is reasonable to consider these subspaces parametrized by just a variety B. For every morphism $f : B \to Gr(k,n)$, we can pull the tautological family back to a family $\mathcal{C}$ over B. This operation is in fact a bijection.

$$\begin{array}{ccc} \mathcal{C}_B & \longrightarrow & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{f} & Gr(k,n) \end{array}$$

From this example, we can now summarize what nice properties a moduli space M should have:

i) Each closed point of M represents an isomorphism class.

ii) There is a universal family $E \to M$ satisfying that, for any morphism $f : B \to M$, the family over B is precisely the pullback family f^*E.

Example. *A silly example is to consider a family over a single point $[x] \in Gr(k,n)$. Then a morphism $f : \{[x]\} \to Gr(k,n)$ gives $f^*E = \pi^{-1}(f(x))$.*

1.2 Fine and coarse moduli space

In categorical languages, we have a contravariant functor h from $\mathbf{Sch}_{\mathbb{C}}$ to **Sets**. For every $B \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$, $h(B)$ is the universal family parametrized by B. For every morphism $f : B \to B'$, there is a pullback morphism $f^* : h(B') \to h(B)$ of universal families. This leads to the formal definition of a moduli functor.

Definition 1.2. *Let h be a functor from $\mathrm{Sch}_{\mathbb{C}}^{op}$ to* Set*. It assigns to each scheme B a set $h(B)$ consisting of families parametrized by B. Then h is a moduli functor.*

The moduli functor for Grassmannians is an example of a representable functor.

Definition 1.3. *A functor h is said to be representable if it's of the form $h \simeq h^M$ for some scheme $M \in Ob(\mathrm{Sch}_{\mathbb{C}})$ such that $h^M(B) = \mathrm{Mor}(B, M)$. The scheme M is called a fine moduli space for h.*

Reflecting with our previous example, the set $\mathrm{Mor}(B, M)$ is in one-to-one correspondence with families parametrized by B. Then M is indeed a fine moduli space.

However, life is not always that easy. moduli functors fail to have a moduli space even in good cases. One obstacle is that families over B are not in general in bijection with $\mathrm{Mor}(B, M)$ for some scheme M.

Example. *Elliptic curves are classified by the j-invariant. Any elliptic curve can be transformed into the one that branches at $\{0, 1, \infty, \lambda\}$ for some λ distinct from the other three. Two elliptic curves $(0, 1, \infty, \lambda_1)$ and $(0, 1, \infty, \lambda_2)$ are isomorphic if and only if they have the same j-invariant, i.e. $j(\lambda_1) = j(\lambda_2)$.*

The map $j : \mathbb{P}^1 \backslash \{0, 1, \infty\} \to \mathbb{C}$ is surjective. So it's a reasonable assumption that $\mathbb{C}$ is a fine moduli space for elliptic curves. But sadly it's not.

Every elliptic curve E admits an involution, an automorphism $i : E \to E, (x, y) \mapsto (x, -y)$. Now we construct a family over a scheme B such that each fibre is isomorphic to an elliptic curve E_0, but not trivially. Thus, the morphism from B to $\mathbb{C}$ must be constant. However, there's another family over B that also corresponds to the constant morphism, the trivial family $E_0 \times B \to B$.

Alternatively, we may seek for a moduli space that is weaker, the coarse moduli space.

Definition 1.4. *Let h be a moduli functor. A scheme M is said to be the coarse moduli space for h if given any family $X \to B$, there exists a unique morphism $f : B \to M$, and if there's another M' also satisfies this property, the unique morphism $g : B \to M'$ factors through M.*

Remark. *For a coarse moduli space, a universal family over M may not exist.*

Chapter 2

Moduli Space of Stable Curves

The moduli space M_g of smooth curves of genus g is an example of a coarse moduli space. There are various ways to construct it. We will see the sketch of the approach using geometric invariant theory in section 1.

It is often useful to have a compact moduli space. Deligne and Mumford [1] introduced the compactification $\overline{M}_{g,n}$. The extra curves in $\overline{M}_{g,n}$ correspond to stable nodal curves. We shall see how the compactification remedies the problem of nontrivial automorphisms, how the space $\overline{M}_{g,n}$ is stratified by topological structures of nodal curves.

In section 3, we briefly review results on the structures of $\overline{M}_{g,n}$, as well as intersections of cohomology classes on $\overline{M}_{g,n}$, which will be useful in chapter 3.

2.1 Moduli space M_g of smooth curves

Throughout the entire thesis, a curve refers to a 1-dimensional connected and reduced scheme of finite type over $\mathbb{C}$. In particular, this section deals with smooth curves.

Smooth curves of genus 0 are the easiest to study, because they are all isomorphic to $\mathbb{P}^1$. For any $n \geqslant 1$, there is a canonical embedding of $\mathbb{P}^1$ into $\mathbb{P}^n$

$$\begin{aligned} &\varphi : \mathbb{P}^1 \to \mathbb{P}^n \\ &[x : y] \mapsto [x^n : x^{n-1}y : \cdots : xy^{n-1} : y^n] \end{aligned}$$

For curves of higher genera, such embeddings are also possible. For example, hyperelliptic curves of genera $g \geqslant 2$ can be explicitly described using the linear series $|\mathcal{K}_C|$.

In our case, we can choose any $n \geqslant 3$. Then a curve C of genus g can be embedded into $\mathbb{P}^N$ where $N = (2n-1)(g-1) - 1$ as a curve of degree $2n(g-1)$ and genus g using complete linear series $|n\mathcal{K}_C|$. Therefore, the collection of smooth curves of genus g is a closed subset $\mathcal{K}$ in $\mathbb{P}^N$ and can be parametrized by the Hilbert scheme $\mathcal{H}_{2(g-1)n,(2n-1)(g-1)-1}$. Next, we quotient $\mathcal{K}$ by isomorphisms between these curves, which are given by $\mathrm{PGL}(N+1, \mathbb{C})$. Then M_g should be the quotient space.

The construction of the Hilbert scheme mentioned above is complicated. The readers may refer to [6].

Again, M_g is not a fine moduli space, because a curve of genus g could have nontrivial automorphisms(e.g. elliptic curves). It's also not very useful to consider solely the locus M_g^0 of curves that only has trivial automorphism. Although it is indeed a fine moduli space, the codimension is at least $g-2$. One way to remedy this issue is to view M_g as a stack instead of a scheme. But we won't be able to explain this approach here. A good reference is [2].

2.2 Nodal curves and normalization

The moduli space M_g constructed above is not compact. The well-known Deligne-Mumford compactification includes nodal curves into the inventory. The isomorphism classes of these curves form the boundary of M_g.

Definition 2.1 (nodal curves)**.** *A node of a curve is a point that has a neighbourhood in which the curve is analytically isomorphic to $xy = 0$ at the origin. A curve that admits a closed nodal point is called a nodal curve.*

The nodes considered here are all double points that are not a cusp. The reason to only include them is that they can be easier to understand via normalization than multiple points.

Fact Given any nodal curve C of genus g and with nodes $\delta_1, \ldots, \delta_r$. Then there exists a smooth curve $\tilde{C}$ with a morphism $\nu : \tilde{C} \to C$ such that $\nu^{-1}(p)$ contains a single point for p smooth and $\nu^{-1}(\delta_i) = \{q_i, q_i'\}$ for any node δ_i. In other words, ν is an isomorphism on the smooth points of C.

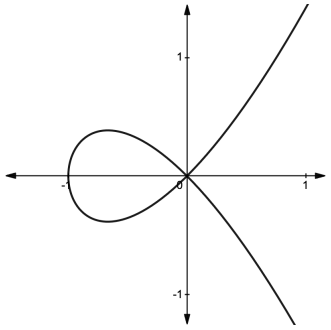

Figure 2.1: Nodal curve

Same as before, a nodal curve could also have nontrivial automorphisms because each component is a smooth curve of smaller genera, where those nontrivial automorphisms come from. So instead of studying the curve itself, we study n-tuples $(C; p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are smooth points on C. Then we slighty modify the definition of automorphism, i.e. $\phi : C \to C$ is an automorphism of the n-tuple if it is an automorphism of the curve C that preserves all marked points $p_1, \ldots, p_n$. This leads to the notion of a stable curve:

Definition 2.2 (stable curves)**.** *An algebraic curve with n marked points is a tuple $(C; p_1, \ldots, p_n)$ such that all p_i's are distinct and the automorphism group*

$$Aut(C; p_1, \ldots, p_n) = \{\sigma : C \to C : \sigma(p_i) = p_i\}$$

is finite.

From the definition, it is immediate that elliptic curves can only be stable with at least one marked point, and a curve of genus 0 must have three marked points to be stable.

A famous formula for curves is the Hurwitz bound, which states that $|\mathrm{Aut}(C)| \leqslant 84(g-1)$ for a curve C of genus $g \geqslant 2$. It is immediate that curves of genera $g \geqslant 2$ are stable. For n large enough, a stable curve can only have trivial automorphism. The above discussion is summarized as:

Proposition 2.3. *The automorphism group $\mathrm{Aut}(C; p_1, \ldots, p_n)$ of a smooth curve is finite if and only if $2g - 2 + n > 0$. For nodal curves, the number of marked points on each component is those from $p_1, \ldots, p_n$ plus preimages of nodes. Then $\mathrm{Aut}(C; p_1, \ldots, p_n)$ is finite if and only if the automorphism group of every component is finite.*

Dual graph

A much more concise way of encoding the topological information of a curve is to use dual graph.

Definition 2.4. *Let $\nu : \tilde{C} \to C$ be the normalization of a stable nodal curve $(C; p_1, \ldots, p_n)$. Denote by V the set of all smooth components of $\tilde{C}$, H the set of all special points on $\tilde{C}$, namely all marked points and preimages of nodes, and E the set of edges consisting of pairs of special points that connect two smooth components. The dual graph of $(C; p_1, \ldots, p_n)$ is the tuple together with a map $h : H \to V$ which sends a special point to the vertex it belongs to, and a map $g : V \to \mathbb{N}$ that labels every vertex with its geometric genus.*

Example. *Below is a figure of the dual graph of a curve in $\overline{M}_{3,3}$, where $V = \{v_1, v_2\}$, $H = \{p_1, p_2, p_3, \delta_1, \delta_2, \delta_3\}$, $E = \{(\delta_1, \delta_2), (\delta_3, \delta_3)\}$, and the maps are given by*

$$\begin{aligned} &h(p_1) = v_1, h(\delta_1) = v_1, \\ &h(p_2) = v_1, h(\delta_2) = v_2, \\ &h(p_3) = v_2, h(\delta_3) = v_2, \\ &g(v_1) = 1, g(v_2) = 1 \end{aligned}$$

Note that loops and multiple edges are allowed in a dual graph.

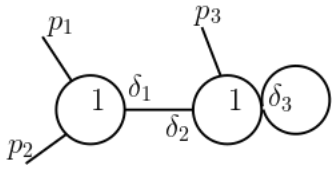

Figure 2.2: Dual graph of a curve in $\overline{M}_{3,3}$

2.3 Boundary strata

The Deligne-Mumford compactification adds boundary to the space $M_{g,n}$. The boundary $\partial\overline{M}_{g,n}$ is in general divisor of codimension 1.

Contraction and stablization

The definition of the forgetful morphism is not simply getting rid of a marked point, because the resulting graph is not necessarily stable. For example, a component of genus 0 has precisely 3 special points, including at least one marked points and half-edges corresponding to nodes on this component. If a marked point is removed, this component is no longer stable.

The method to solve this issue is contraction, namely, to eliminate the genus 0 component and connect the other parts. There are three cases.

- The component C_0 contains 2 marked points p_i, p_{n+1} and is connected to another component by a node q. The contraction map merges p_i and the node q.

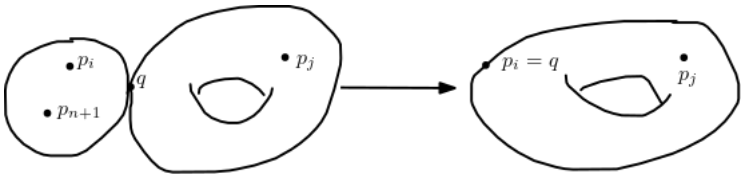

Figure 2.3: Genus 0 component contracts to a point

- The component C_0 contains 1 marked point p_{n+1} and is connected to two other components by two nodes q, q'. The contraction map merges the two nodes.

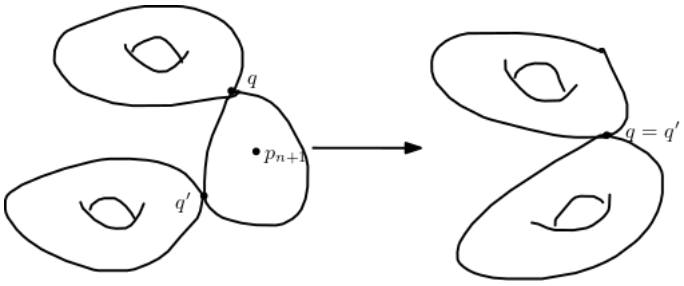

Figure 2.4: Genus 0 component connects two others

- The component C_0 contains 1 marked point p_{n+1} and is connected to another component by two separate nodes q, q'. The contraction map merges the two nodes.

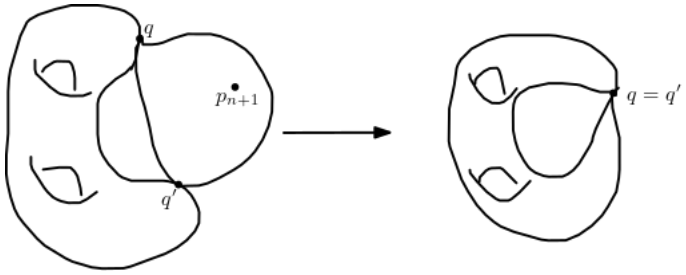

Figure 2.5: Genus 0 component contracting gives another node

The above operation gives a morphism from $\overline{M}_{g,n+1}$ to $\overline{M}_{g,n}$, called the forgetful morphism.

Definition 2.5. *Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the forgetful morphism that forgets the $n+1$-th marked points of a curve. More explicitly, for any $(C; p_1, \ldots, p_{n+1}) \in \overline{M}_{g,n+1}$, the image is either $(C; p_1, \ldots, p_n)$, or its contraction $(C'; \varphi(p_1), \ldots, \varphi(p_n))$.*

The following theorem shows that the forgetful morphism can be used to express universal curves over an open subset of $\overline{M}_{g,n}$.

Theorem 2.6. *Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the forgetful morphism defined above. Then π restricted on $\overline{M}^0_{g,n}$, the locus of curves with trivial automorphism group, is isomorphic to the universal curve. Moreover, it admits n universal sections $p_1, \ldots, p_n$, each corresponding to a marked point. The image of a point $(C; p_1, \ldots, p_n)$ under p_i is a point $(C'; p_1, \ldots, p_{n+1})$ obtained by distributing the marked points p_i and p_{n+1} on one component of genus 0, and all other marked points on the other component of genus g.*

The graph below is an illustration of how the sections are defined.

$$p_i : \overline{M}_{g,n} \to \overline{M}_{g,n+1}$$

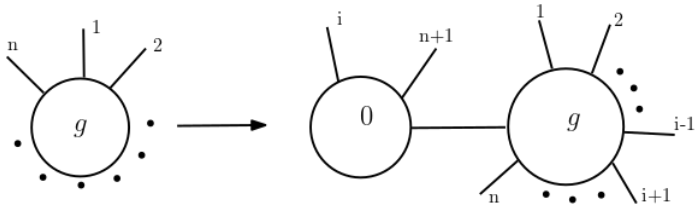

Figure 2.6: The image of a curve in $\overline{M}_{g,n+1}$

However, the forgetful morphism doesm't give the universal curve over the entire moduli space $\overline{M}_{g,n}$. Take any curve $(C; p_1, \ldots, p_n)$ with a non-trivial automorphism group for example. Consider the group action defined by

$$\begin{aligned} Aut(C; p_1, \ldots, p_n) \times C &\to C \\ (\sigma, x) &\mapsto \sigma(x) \end{aligned}$$

Let q be any smooth point on C. Then the two points $(C; p_1, \ldots, p_n, q)$ and $(C; p_1, \ldots, p_n, \sigma(q))$ in $\overline{M}_{g,n+1}$ are in the same orbit under this action. Hence, points in $\pi^{-1}(C; p_1, \ldots, p_n)$ are in one-to-one correspondence with $Aut(C; p_1, \ldots, p_n)$-orbits of C. Then the fibre over $(C; p_1, \ldots, p_n)$ is actually C quotient by its automorphism group $Aut(C; p_1, \ldots, p_n)$.

Gluing morphism

We have already seen an example of gluing in the theorem above. The universal sections are essentially gluing $(C; p_1, \ldots, p_n)$ and a smooth curve of genus 0 and 3 marked points. To explain how this process works for more general curves, we need the following definition.

Definition 2.7. *Let Γ be a stable graph with vertex set $V(\Gamma)$, $g(v)$ and $n(v)$ be the number of genus and half-edges of a component $v \in V(\Gamma)$. Then there exists a "gluing" morphism*

$$\xi_\Gamma : \overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)} \to \overline{M}_{g,n}$$

such that pairs of half-edges $\{h, h'\}$ are sent to edges of Γ, and the rest of half-edges are sent to marked points of Γ.

Figure 12 is an illustration of how gluing works. Here, Γ has two vertices v_1, v_2, and $g(v_1) = 2, g(v_2) = 1$. The gluing morphism ξ_Γ maps $\overline{M}_{2,2} \times \overline{M}_{1,1}$ into $\overline{M}_{3,1}$.

However, the image of the gluing morphism not only consists of curves with this particular dual graph. The reason is that there are tons of stable graphs available in $\overline{M}_{g(v),n(v)}$ for $v \in \Gamma$. For example, figure 13 gives another example of gluing curves in $\overline{M}_{2,2} \times \overline{M}_{1,1}$.

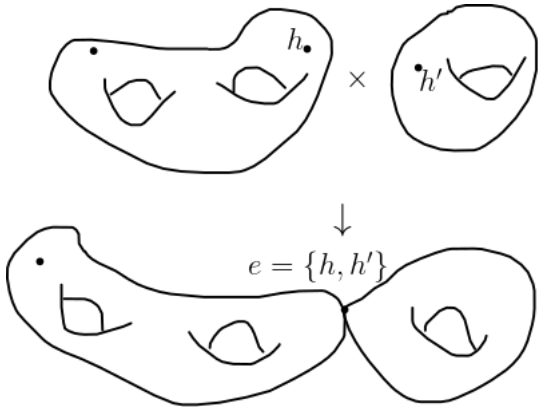

Figure 2.7: Two marked points create a node

Here is a figure that describes how a nodal curve with two components of genus 1 and two marked points and a torus with one marked point are glued into a nodal curve of genus 3 with one marked point.

Chapter 3

Tautological Rings and their Structures

As in algebraic topology, cohomology is a key to understand spaces. The study of moduli spaces also leads to studying the cohomology of them. However, in our situation, the cohomology ring of $\overline{M}_{g,n}$ is enormous, and almost impossible to be expressed explicitly. Instead, we study a subring of it, namely the tautological ring, which is generated by the most natural cohomology classes.

Section 1 reviews dualizing and relative dualizing sheaf, and shows how it forms bundles over moduli spaces, and gives a "vague" definition of the tautological rings. Section 2, 3, and 4 study three natural classes arising from these bundles in the previous section, which turn out to be the fundamental blocks of tautological rings. Section 4 contains useful calculations of expressing λ classes in terms of Chern characters.

The last section introduces Faber's conjecture, for both smooth case and compactified case.

3.1 Tautological rings of moduli spaces

By tautological, we mean the most natural classes, also the easiest to think of. As a subring of the cohomology ring, the tautological ring should be closed under all operations among cohomology classes in it, namely, addition, multiplication, pushing down and gluing.

A natural way of obtaining such classes is to take characteristic classes

of vector bundles over a moduli space. In this section, we consider two such bundles, cotangent bundle and the Hodge bundle.

Relative dualizing sheaf

If a curve is smooth, it's easy to define the cotangent bundle on it. However, we have included nodal curves, which makes the sheaf of differentials fail to be locally free of rank 1. We instead consider the dualizing sheaf ω_C of a nodal curve C.

Let $\nu : \tilde{C} \to C$ be the normalization of C with nodes $\{\delta_1, \ldots, \delta_r\}$. Let $\nu^{-1}(\delta_i) = \{q_i, q_i'\}, 1 \leqslant i \leqslant r$ be preimages of nodes. Then $\tilde{C}$ admits a sheaf of differentials $\Omega_{\tilde{C}}$. We define the dualizing sheaf ω_C to be the subsheaf of $\nu_*(\Omega_{\tilde{C}})$ that assigns to each open $U \in C$ that contains a node δ_i the space of 1-forms $\eta \in \Gamma(\Omega_{\tilde{C}}, \nu^{-1}(U))$ which are allowed to have at most simple poles at q_i and q_i' and satisfy that

$$\mathrm{Res}_{q_i}(\eta) + \mathrm{Res}_{q_i'}(\eta) = 0.$$

Dualizing sheaf can easily fit together for a family of nodal curves. Let $\pi : C \to B$ be a family of curves parametrized by B. We then define the relative dualizing sheaf $\omega_{C/B}$ on C to be the cokernel of $\mathrm{d}\pi : \pi^*\Omega_B \to \Omega_C$ such that the rational 1-forms on each fibre all satisfy the residue condition.

Example. *Let C be the vanishing locus of $xy = t^k$ for some integer $k \geqslant 0$. Consider the projection*

$$f : C \to \mathbb{A}^1$$
$$(x, y, t) \mapsto t$$

Then we can say that $\mathbb{A}^1$ parametrizes the family C of curves $xy = t^k$.

The relative dualizing sheaf gives rise to almost all tautological classes we are about to introduce in the following sections. Let's end this section with a general definition of the tautological ring:

Definition 3.1. *The tautological ring $R^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$ is the smallest among $\mathbb{Q}$-subalgebras that contain the unit element $[\overline{M}_{g,n}]$ and classes closed under all pushforwards $\xi_{\Gamma,*}$ and forgetful morphisms.*

Remark. *The tautological ring of an open subset of $\overline{M}_{g,n}$ is defined by restriction. So $R^*(M_g)$ can be defined by restriction of $R^*(\overline{M}_{g,n})$.*

Now let's see some examples(in fact all) of tautological classes.

3.2 ψ-classes of cotangent bundles

Although $\overline{M}_{g,n}$ is not a fine moduli space, we still view the forgetful morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ as a universal family(it is indeed if $\overline{M}_{g,n}$ is considered as a Deligne-Mumford stack, but we won't be able to elaborate here). We denote $\overline{M}_{g,n}$ by $\overline{C}_{g,n}$ indicating that it is the universal curve. The preimage of a point $[(C; p_1, \ldots, p_n)] \in \overline{M}_{g,n}$ is precisely the curve $(C; p_1, \ldots, p_n)$. This gives rise to a relative dualizing sheaf ω_π on $\overline{C}_{g,n}$. The fibre of ω_π at a smooth point is just the cotangent space.

There are n sections $p_1, \ldots, p_n$ with respect to the forgetful morphism π. By abuse of notations, the i-th section p_i sends a point $[(C; p_1, \ldots, p_n)]$ to the i-th marked point $p_i \in C$. Since the marked points are all smooth, the pullback of ω_π via p_i is a line bundle $\mathbb{L}_i$ on $\overline{M}_{g,n}$, whose fiber at $[(C : p_1, \ldots, p_n)]$ is the cotangent space $T^*_{p_i}C$.

Definition 3.2. *We define the i-th ψ class to be the first Chern class of $\mathbb{L}_i$*

$$\psi_i = c_1(\mathbb{L}_i)$$

Example. *Consider a simple case.*

$$\begin{array}{c} \overline{M}_{0,5} \\ {\scriptstyle p_i}\uparrow \big\downarrow{\scriptstyle \pi} \\ \overline{M}_{0,4} \end{array}$$

The moduli space $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$. The curve $(C; p_1, p_2, p_3, p_4)$ over a point $\lambda \in \mathbb{P}^1$ has the cross ratio

$$\lambda = \frac{(p_4 - p_1)(p_2 - p_3)}{(p_4 - p_3)(p_2 - p_1)}.$$

There are only three exceptions, $\lambda = 0, 1, \infty$, where p_4 coincides one of the first three distinct points. We've already seen in chapter 2 that one way to resolve this is to introduce nodal curves formed by sprouting another branch of genus 0 containing p_4, p_i when p_4 and p_i coincide.

Figure 14 illustrates how fibres of the line bundle $\mathbb{L}_1$ as λ approaches 0. At 0, the fibre is sort of "twisted". So $c_1(\mathbb{L}_1)$ reveals where the nodal curve appears.

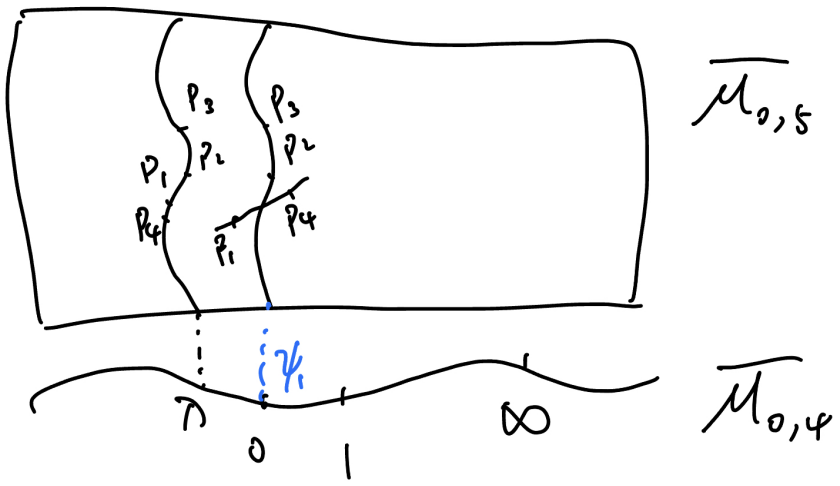

Figure 3.1: Cotangent bundle over $\overline{M}_{0,4}$

The following comparison lemma provides a way to calculate ψ-classes based on those on moduli spaces with fewer marked points and boundary divisors.

Lemma 3.3. *Let $D_{i,n+1} \subset \overline{M}_{g,n+1}$ be the boundary divisor corresponding to curves with two components, one has genus 0 and two marked points p_i, p_{n+1}, the other one has genus g and all the other marked points. We have the relation*

$$\psi_i = \pi^* \psi_i + D_{i,n+1},$$

where the ψ_i on the left is in $H^2(\overline{M}_{g,n+1})$ and ψ_i on the right is in $H^2(\overline{M}_{g,n})$.

Proof. We prove by comparing line bundles on both sides.

The fibre of $\mathbb{L}_i$ at the point $(C; p_1, \ldots, p_{n+1})$ is the same as that at the point $(C; p_1, \ldots, p_n)$ if forgetting the extra mark doesn't change the structure of the curve(i.e. the curve is stable without it). Thus, the only curves that are unstable after forgetting p_{n+1} is those having only p_i and p_{n+1} on a component of genus 0.

Therefore, $c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n+1})$ is essentially the pullback of $c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n})$ via π plus a multiple of $D_{i,n+1}$:

$$\psi_i = \pi^* \psi_i + k D_{i,n+1}.$$

In terms of line bundles, $\mathbb{L}_i$ over $\overline{M}_{g,n+1}$ can be obtained by

$$\mathbb{L}_i \simeq \pi^*\mathbb{L}_i \otimes \mathscr{O}(kD_{i,n+1}).$$

Then we pullback both sides of the above identity along p_i:

$$\begin{aligned}\mathscr{O} \simeq p_i^*(\mathbb{L}_i) &= p_i^*\pi^*\mathbb{L}_i \otimes p_i^*\mathscr{O}(D_{i,n+1})^{\otimes k} \\ &= \mathbb{L}_i \otimes (\mathbb{L}_i^\vee)^{\otimes k}\end{aligned}$$

The first isomorphism is due to the fact that the fibre of $p_i^*(\mathbb{L}_i)$ at any point $(C; p_1, \ldots, p_n)$ is $T^*_{p_i}(C)$, which is isomorphic to $T^*_0(\mathbb{P}^1)$ since p_i is on the component of genus 0 and with two marked points p_i and p_{n+1}. The identity $p_i^*\pi^*(\mathbb{L}_i) = \mathbb{L}_i$ is trivial. The last step $p_i^*(\mathscr{O}(D_{i,n+1}))$ uses the excess intersection formula, and the normal bundle N_{p_i} given by the dual of $\mathbb{L}$.

Now the left hand side is the trivial bundle and the right hand side is $\mathbb{L}_i \otimes (\mathbb{L}^\vee)^{\otimes k}$, which is only true when $k = 1$. We conclude that $\psi_i = \pi^*\psi_i + D_{i,n+1}$.

□

Example. *Consider the forgetful morphism $\pi : \overline{M}_{0,4} \to \overline{M}_{0,3}$. Since $\overline{M}_{0,3}$ is just a point, it only has a trivial line bundle. Then $\psi_i = 0 + D_{i,4} = c_1(\mathscr{O}(D)) = c_1(\mathscr{O}(1))$.*

Intersection of ψ-classes with boundary divisors

The comparison lemma gives a way to understand tautological classes of moduli spaces of higher dimensions via those of lower dimensions. It turns out that cotangent bundles have a close relation with each other.

The gluing morphism allows us to obtain stable curves of higher genera and more marked points by gluing curves of lower genera and fewer marked points. Consider a partition $D = (S_1, g_1|S_2, g_2)$ of g and $S = \{p_1, \ldots, p_n\}$ such that $g_1 + g_2$ and $S_1 \amalg S_2 = S$. Then there exists a gluing morphism

$$\rho_D : \overline{M}_{g_1, S_1\cup\{p\}} \times \overline{M}_{g_2, S_2\cup\{p'\}} \to \overline{M}_{g,n}.$$

Intuitively, line bundles on $\overline{M}_{g_1, S_1\cup\{p\}} \times \overline{M}_{g_2, S_2\cup\{p'\}}$ depends on line bundles on $\overline{M}_{g_1, S_1\cup\{p\}}$ and $\overline{M}_{g_2, S_2\cup\{p'\}}$. Indeed, we have the following lemma [8]:

Lemma 3.4. *Let π_1, π_2 be two projections from $\overline{M}_{g_1, S_1\cup\{p\}} \times \overline{M}_{g_2, S_2\cup\{p'\}}$ to its two components respectively. Then we have the following isomorphisms:*

$$\rho_D^*\mathbb{L}_i = \begin{cases} \pi_1^*\mathbb{L}_i & p_i \in S_1 \\ \pi_2^*\mathbb{L}_i & p_i \in S_2 \end{cases} \tag{3.1}$$

For a detailed proof, the reader may refer to [8].

A direct consequence of this lemma is that we can take D to be the partition that has $g_1 = 0, g_2 = g$ and $S_1 = \{p_i, p_{n+1}\}, S_2 = \{p_1, \ldots, \hat{p_i}, \ldots, p_n\}$. Now the gluing morphism is just the section p_i. Hence, $p_i^*(\mathbb{L}_i) = \rho_D^*(\mathbb{L}_i) = \pi_1^*(\mathbb{L}_i)$ where the last $\mathbb{L}_i$ is on $\overline{M}_{0,3}$, which is trivial. In terms of Chern classes, we have $p_i^*(\psi_i) = 0$.

This insight leads to the following relations:

Corollary 3.5. *i)* $D_{i,n+1} \cdot D_{j,n+1} = 0$ *for* $i \neq j$.

ii) $\psi_i \cdot D_{i,n+1} = 0, \forall 1 \leqslant i \leqslant n$.

iii) $\psi_{n+1} \cdot D_{i,n+1} = 0, \forall 1 \leqslant i \leqslant n$.

iv) $\psi_i^a = \pi^* \psi_i^a + (-1)^{a-1} D_{i,n+1}^a$

Proof. i) Since $D_{i,n+1}$ and $D_{j,n+1}$ are disjoint, their intersection is empty.

ii) The excess intersection formula shows that, intersecting a class $\alpha \in H^*(\overline{M}_{g,n+1})$ with the boundary divisor $D_{i,n+1}$ then pushing down is equivalent to pulling it back along p_i. Then we have

$$\pi_*(\psi_i \cdot D_{i,n+1}) = p_i^*(\psi_i) = 0.$$

Since π is an isomorphism when restricted on $D_{i,n+1}$, we conclude that $\psi_i \cdot D_{i,n+1} = 0$.

iii) Using the same argument as above. The pullback bundle $p_i^*(\mathbb{L}_{n+1})$ is trivial. Hence $p_i^*(\psi_{n+1}) = \pi_*(\psi_{n+1} \cdot D_{i,n+1}) = 0$. We conclude that $\psi_{n+1} \cdot D_{i,n+1} = 0$ for all i.

iv) We calculate the power ψ_i^a inductively. First consider the case $a = 2$.

$$\begin{aligned}
\psi_i^2 &= \psi_i \cdot (\pi^*(\psi_i) + D_{i,n+1}) \\
&= \pi^*(\psi_i)^2 + 2\pi^*(\psi_i) \cdot D_{i,n+1} + D_{i,n+1}^2 \\
&= \pi^*(\psi_i^2) + 2(\psi_i - D_{i,n+1}) D_{i,n+1} + D_{i,n+1}^2 \\
&= \pi^*(\psi_i^2) - D_{i,n+1}^2
\end{aligned}$$

The last equality holds because of the comparison lemma and ii). Then we apply the above argument inductively to get

$$\psi_i^a = \pi^*(\psi_i^a) + (-1)^{a-1} D_{i,n+1}^a.$$

□

3.3 κ-classes

Another natural class is the κ class, obtained by pushing down powers of ψ classes. So the tautological ring has to include it since this subring is closed. Mumford originally defined κ classes as the pushforward of $c_1(\omega_\pi)^{a+1}$ with respect to a family of curves $\pi : \mathfrak{X} \to B$, where ω_π is the relative dualizing sheaf. This works well in the case $\overline{M}_g$.

Definition 3.6. *Consider the universal family $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ that forgets p_{n+1}. Then the a-th κ classes is defined as follows:*

$$\kappa_a = \pi_*(\psi_{n+1}^{a+1}) \in R^{2a}(\overline{M}_{g,n})$$

It is well-defined since $\psi_{n+1} \in H^2(\overline{M}_{g,n+1})$ and $\psi_{n+1}^{a+1} \in H^{2(a+1)}(\overline{M}_{g,n+1})$. Hence, the pushforward sends ψ_{n+1}^{a+1} to the κ class in $H^{2a}(\overline{M}_{g,n})$.

Remark. *Similarly in the case $\pi : \overline{C}_g \to \overline{M}_g$, the a-th κ class is the pushforward of $K = c_1(\omega_\pi) \in R^{a+1}(\overline{C}_g)$.*

The κ class is originally defined to be $\kappa_a = \pi_*(c_1(\omega_\pi)^{a+1})$ for $\pi : \overline{C}_g \to \overline{M}_g$. In the case of $\overline{M}_{g,n}$, we have to consider the sheaf $\omega_\pi(\sum_{i=1}^n D_{i,n+1})$.

Example. *As a consequence, $\psi_{n+1} = K_\pi + \sum_{i=1}^n D_{i,n+1}$. Then $\kappa_0 = \pi_*(K_\pi + \sum_{i=1}^n D_{i,n+1})$. Recall that $\deg K_\pi = 2g-2$. Therefore we have $\kappa_0 = (2g-2+n)[\overline{M}_{g,n}]$.*

We shall see another relation between κ and ψ classes which will be used in section 4.2. For simplicity of notations, we write M for $\overline{M}_{g,n}$, M_x for $\overline{M}_{g,n+1}$(the subscript just means we add an extra mark x), and $M_{x,y}$ for $\overline{M}_{g,n+2}$. Then the forgetful morphisms $\pi_x : M_x \to M$ and $\pi_y : M_y \to M$ give rise to a fibre product

$$\begin{array}{ccc} M_x \times_M M_y & \xrightarrow{\tau_y} & M_x \\ \downarrow{\scriptstyle \tau_x} & & \downarrow{\scriptstyle \pi_x} \\ M_y & \xrightarrow[\pi_y]{} & M \end{array}$$

where τ_x, τ_y are projection maps that forget x, y respectively.

Lemma 3.7. *Following the above notations, we have the following identity on M_x:*

$$\kappa_a = \pi_x^*(\kappa_a) + \psi_x^a,$$

where the κ_a on the right is on M.

Proof. The moduli space $M_{x,y}$ is constructed by blowing up $M_x \times_M M_y$ along the diagonal. Denote by ε the blow-up morphism. The we have a commutative diagram:

$$\begin{array}{ccc} M_{x,y} & \xrightarrow{\tilde{\pi}_y = \tau_y \circ \varepsilon} & M_x \\ \Big\downarrow{\scriptstyle \tilde{\pi}_x = \tau_x \circ \varepsilon} & & \Big\downarrow{\scriptstyle \pi_x} \\ M_y & \xrightarrow[\pi_y]{} & M \end{array}$$

Recall the pullback formula from corollary 3.5 (iv) and apply it to $\tilde{\pi}_y : M_{x,y} \to M_x$:

$$\psi_y^{a+1} = \tilde{\pi}_x^*(\psi_y^{a+1}) + (-1)^a D_{x,y}^{a+1}$$

We push both sides down via $\tilde{\pi}_y$:

$$\begin{aligned} \kappa_a &= \tilde{\pi}_y(\psi_y^{a+1}) \\ &= \tilde{\pi}_y(\tilde{\pi}_x^*(\psi_y^{a+1})) + (-1)^a \tilde{\pi}_y(D_{x,y} \cdot D_{x,y}^a) \end{aligned} \tag{3.2}$$

Since intersecting with $D_{x,y}$ then pushing down via $\tilde{\pi}_y$ is equivalent to pulling back along σ_y, the second term becomes $(-1)^a \sigma_y^*(D_{x,y}^a) = (-1)^a (-1)^a \psi_x^a = \psi_x^a$. Then the pullback formula becomes $\kappa_a = \tilde{\pi}_y(\tilde{\pi}_x^*(\psi_y^{a+1})) + \psi_x^a$. According to the above commutative diagram, $\tilde{\pi}_y(\tilde{\pi}_x^*(\psi_y^{a+1})) = \pi_x^*(\pi_{y,*}(\psi_y^{a+1}))$. Eventually we have

$$\kappa_a = \pi_x^*(\kappa_a) + \psi_x^a.$$

□

Tautological ring is finitely generated

It turns out that ψ and κ classes generate the entire tautological ring. We won't be able to prove this result here, but here is the basic idea. Just like boundary strata in small moduli spaces glue up to be strata in larger moduli spaces, ψ and κ classes also can be glued to be classes in $H^*(\overline{M}_{g,n})$. They are called decoration strata classes, denoted by $[\Gamma, \alpha]$ where α is a class in the image of the gluing morphism ξ_Γ.

Theorem 3.8. *The set of decoration strata classes* $[\Gamma, \alpha]$ *is finite and generates the tautological ring* $RH^*(\overline{M}_{g,n})$ *as a* $\mathbb{Q}$*-vector space.*

3.4 Hodge bundle and the λ classes

Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be a universal family. Each curve C has genus g, i.e. the cohomology $H^0(C, \Omega_C)$ is a vector space of dimension g. These spaces fit together to form a vector bundle of rank g over $\overline{M}_{g,n}$, denoted by $\mathbb{E}$.

Definition 3.9. *The* λ*-classes are the Chern classes of* $\mathbb{E}$*:*

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{M}_{g,n})$$

Remark. *It is immediate that* $\lambda_i(\mathbb{E}) = 0, \ \forall i > g$.

It turns out that these λ classes also lie in the tautological ring, i.e. they can be obtained by finitely many operations among ψ and κ classes. Mumford [12] firstly expressed λ classes in terms of Chern characters, which are further calculated by boundary classes.

The splitting principle assures that we can treat any bundle as if they are a direct sum of line bundles when calculating Chern classes. The total Chern class of the Hodge bundle is

$$c(\mathbb{E}) = 1 + c_1(\mathbb{E}) + \cdots + c_g(E).$$

It's easier to do the following calculation using generating functions, i.e. the Chern polynomial

$$c(t) = 1 + c_1 t + \cdots + c_g t^g,$$

with coefficients in $H^*(\overline{M}_{g,n})$.

Applying the splitting principle on the logarithm of $c(t)$, we have

$$\log c(t) = \log \prod_{i=1}^{g} (1 + \alpha_i t) = \sum_{i=1}^{g} \log(1 + \alpha_i t).$$

Since the expansion of the product $\prod_{i=1}^{g}(1+\alpha_i t)$ equals $c(t)$, these Chern classes as coefficients are essentially symmetric polynomials in $\alpha_1, \ldots, \alpha_g$. Below are an example when $g = 3$.

$$\begin{aligned}1 + c_1t + c_2t^2 + c_3t^3 &= (1+\alpha_1t)(1+\alpha_2t)(1+\alpha_3t)\\ &= 1 + (\alpha_1+\alpha_2+\alpha_3)t + (\alpha_1\alpha_2\\ &+ \alpha_1\alpha_3 + \alpha_2\alpha_3)t^2 + \alpha_1\alpha_2\alpha_3t^3\end{aligned}$$

The coefficients are the three fundamental symmetric polynomials in three Chern roots.

In the mean time, we can expand logarithm as power series.

$$\log c(t) = \sum_{j=1}^{\infty}\sum_{i=1}^{g}(-1)^{j-1}\frac{(\alpha_i t)^j}{j} = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{(c_1t+\cdots+c_gt^g)^j}{j},$$

where the right hand side is a direct expansion of $\log(1+c_1t+\cdots+c_gt^g)$. By comparing coefficients on both sides, we are able to express the logarithm of the total Chern class in terms of polynomials of Chern classes themselves. The first three identity is calculated below:

$$\begin{aligned}\sum_{i=1}^{g}\alpha_i &= c_1\\ \sum_{i=1}^{g}\alpha_i^2 &= c_1^2 - 2c_2\\ \sum_{i=1}^{g}\alpha_i^3 &= c_1^3 - 3c_1c_2 + 3c_3\end{aligned}$$

Furthermore, recall that the Chern character is defined to be

$$ch(\mathbb{E}) = \sum_{i=1}^{g}\exp(\alpha_i) = \sum_{j=1}^{\infty}\sum_{i=1}^{g}\frac{\alpha_i^j}{j!}.$$

Denote the degree k part of $ch(\mathbb{E})$ by $ch_k(\mathbb{E})$. Then the Chern classes and Chern characters can be related by the following identity:

$$\sum_{i=0}^{\infty}c_i(\mathbb{E})t^i = \exp\left(\sum_{i=1}^{\infty}(-1)^{i+1}(i-1)!\cdot ch_i(\mathbb{E})t^i\right). \tag{3.3}$$

We will see in chatper 4 how $ch_k(\mathbb{E})$ can be further expressed as ψ and κ classes.

3.5 Faber's conjectures

While studying the intersection numbers of tautological classes, Faber proposed a series of conjectures regarding the structure of tautological rings, including their generators, dimensions, and pairings. Here we shall briefly introduce these conjectures in two cases.

Mumford [12] already revealed that tautological classes λ_i, κ_i restricted to $A(\overline{M}_g)$ are all polynomials in $\kappa_1, \ldots, \kappa_{g-2}$. Faber's first conjecture is a lot stronger than this.

Vanishing conjecture

The i-th component of $R^*(M_g)$ vanishes for $i > g-2$. Moreover, the highest non-zero component $R^{g-2}(M_g)$ is isomorphic to $\mathbb{Q}$. This result was proved by Looijenga [10] in 1996.

Gorenstein conjecture

Here, the tautological ring $R^*(M_g)$ being Gorenstein means it admits a perfect pairing

$$R^i(M_g) \times R^{g-2-i}(M_g) \to R^{g-2}(M_g) \simeq \mathbb{Q},$$

which resembles Poincaré duality.

This conjecture hasn't been proved. A lot of simple cases are already verified. We will explain in chapter 4 why it fails for $M_3^{\lambda_2}$.

The last part is not very related to the theme of the thesis. Readers who are interested can find more details in [4].

We have similar conjectures for $R^{3g-3+n}(\overline{M}_{g,n})$ and the tautological rings of other types of compactifications. However, these conjectures have not yet been proved. They are proved to be true in many cases. But there are also some exceptions.

Chapter 4

Hodge Integral and Recursive Formulae

This chapter aims at explaining the formulae and algorithms in [15]. The idea is to recursively eliminate tautological classes, i.e. move the calculation over $\overline{M}_{g,n}$ to moduli spaces of smaller genera and fewer marked points. Yang followed Mumford's method to develop a Macaulay2 package *HodgeIntegral* and used it to disprove Faber's conjecture for $M_{3,0}^{\lambda_2}$.

Integrals involving ψ, κ classes and Chern characters are explained in section 4.1, 4.2, and 4.3 respectively.

4.1 Integral involving ψ classes

Intersection of cohomology classes increases codimension, hence gives another class in higher cohomology groups.

Definition 4.1. *The Hodge integral over a moduli space $\overline{M}_{g,n}$ is the top intersection number of tautological classes, i.e. degree of the top intersection in $H_0(\overline{M}_{g,n})$, due to Poincaré duality and the isomorphism between $\overline{M}_{g,n}$ and $\mathbb{Q}$.*

This section deals with intersections involving only ψ classes:

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n},$$

where $a_1, \ldots, a_n$ are non-negative integers satisfying $\sum_{i=1}^n a_i = 3g-3+n$. Otherwise, either the intersection number is 0, or the intersection doesn't yield a cohomology class in $H^{2(3g-3+n)}(\overline{M}_{g,n})$.

The comparison lemma helps us reduce the calculation over $\overline{M}_{g,n+1}$ to that over moduli spaces with strictly fewer marked points.

$$\begin{aligned}
\int_{\overline{M}_{g,n+1}} \psi_1^{a_1} \cdots \psi_n^{a_n} &= \int_{\overline{M}_{g,n+1}} \prod_{i=1}^n (\pi^* \psi_i^{a_i} + \pi^* \psi_i^{a_i - 1} D_{i,n+1}) \\
&= \int_{\overline{M}_{g,n+1}} \prod_{i=1}^n \pi^* \psi_i^{a_i} + \sum_{j=1}^n \pi^* \psi_j^{a_j - 1} \pi^* (\prod_{i \neq j} \psi_i^{a_i}) D_{j,n+1} \\
&= \int_{\overline{M}_{g,n+1}} \pi^* (\psi_1^{a_1} \cdots \psi_n^{a_n}) + \sum_{j=1}^n \pi^* (\psi_j^{a_j - 1} \prod_{i \neq j} \psi_i^{a_i}) D_{j,n+1} \\
&= \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} + \sum_{j=1}^n \psi_j^{a_j - 1} \prod_{i \neq j} \psi_i^{a_i} \\
&= \sum_{j=1}^n \int_{\overline{M}_{g,n}} \psi_j^{a_j - 1} \prod_{i \neq j} \psi_i^{a_i}
\end{aligned}$$

In the above calculation, we applied the projection formula to reduce the integral over $\overline{M}_{g,n+1}$ to integral over $\overline{M}_{g,n}$. Since $D_{i,n+1} \cdot D_{j,n+1} = 0$ for $i \neq j$, there are only $n+1$ terms left in the sum. The first term has 0 integral because $\sum_{i=1}^n a_i = 3g-2+n > 3g-3+n$. This relation is also called the **string equation**.

Remark. *Notice that here we assume $a_i \geqslant 1$, $\forall i$, but in fact it's not necessary. If some a_i's are 0, we can just remove them from the final summation.*

Genus 0

When $g = 0$, $\overline{M}_{0,n}$ has dimension $n-3$, whereas we have n ψ-classes. So at least two of a_i's are 0, which means this induction is always possible. We just need to choose different marked points to forget in each step. The trivial case is easy to compute,

$$\int [\overline{M}_{0,3}] = 1$$

Lemma 4.2. *Given a sum of non-negative integers $\sum a_i = n-3$. We have the following identity:*

$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \frac{(n-3)!}{a_1! \cdots a_n!}$$

This identity can be easily proved using induction and the multinomial version of Pascal's identity:

$$\frac{n!}{p_1! \cdots p_k!} = \sum_{j=1}^{k} \frac{(n-1)!}{(p_j-1)! \prod_{i \neq j} p_i!},$$

where $p_1 + \cdots + p_k = n$.

We can see the string equation only reduces marked points, but doesn't help much with the initial condition and reducing genera. Liu and Xu [9] proved a formula for computing all top intersection numbers only involving ψ classes:

Theorem 4.3. *Given a sum of non-negative integers $\sum a_i = 3g-3+n$. Then we have a recursive formula*

$$\begin{aligned}
&(2g+n-1)(2g+n-2) \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \\
&= \frac{2a_1+3}{12} \int_{\overline{M}_{g-1,n+4}} \psi_1^{a_1+1} \psi_2^{a_2} \cdots \psi_n^{a_n} \\
&- \frac{2g+n-1}{6} \int_{\overline{M}_{g-1,n+3}} \psi_1^{a_1} \cdots \psi_n^{a_n} \\
&+ \sum_{I \sqcup J = \{2,\ldots,n\}} (2a_1+3) \int_{\overline{M}_{g',|I|+3}} \psi_1^{a_1+1} \prod_{i \in I} \psi_i^{a_i} \int_{\overline{M}_{g-g',|J|+2}} \prod_{j \in I} \psi_j^{a_j} \\
&+ \sum_{I \sqcup J = \{2,\ldots,n\}} (2g-n-1) \int_{\overline{M}_{g',|I|+2}} \psi_1^{a_1} \prod_{i \in I} \psi_i^{a_i} \int_{\overline{M}_{g-g',|J|+2}} \prod_{j \in J} \psi_j^{a_j}
\end{aligned}$$

Genus 1

The identity [9] effectively reduces calculation to smaller genera. We only need to determine the initial condition for $g = 1$.

We now sketch why the base case $\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}$ is true.

When genus $g = 1$, $\overline{M}_{1,1}$ has dimension 1. Therefore, ψ_1 represents where a nodal curve appears. Since every curve has arithmetic genus 1 and only 1 nodal curve, it only has one component of genus 0 and with one loop.

It turns out that there are 12 of them and $\psi_1 = \frac{1}{12}D$, where D is the locus of all irreducible uni-nodal curves. Indeed, we can verify this by examining a family of genus 1 curves over $\overline{M}_{1,1}$. I'm cheating a little bit because it is well-known that $\overline{M}_{1,1}$ can be obtained from $\mathbb{P}^1$. So let's construct a family over $\mathbb{P}^1$.

In general, elliptic curves can be parametrized by the family $\{y^2 = x(x-1)(x-t), t \in \mathbb{C} \cup \{\infty\}\}$. Since we are working in projective setting, this family can be described as all linear combinations of two general homogeneous cubic polynomials $f(x, y, z), g(x, y, z)$ modulo scalar multiplication. Singularities occur when $f = g = 0$, which gives 9 points in $\mathbb{P}^2$(Bezout's theorem). Let $\mathcal{C}$ be the blow up of $\mathbb{P}^2$ at these 9 points. Then $\mathcal{C} \to \mathbb{P}^1$ is a universal family of elliptic curves over $\mathbb{P}^1$. We can easily calculate the Euler characteristic of $\mathcal{C}$, which is $\chi(\mathbb{P}^2 \backslash \{p_1, \ldots, p_9\}) + 9\chi(\mathbb{P}^1) = 12$ because blowing up at $p_1, \ldots, p_9$ gives rise to 9 exceptional fibres. Any smooth elliptic curve is isomorphic to a torus, hence has Euler characteristic 0. Any singular elliptic curve has a node, hence has Euler characteristic 1. Therefore, $\chi(\mathcal{C})$ is precisely the number of singular curves in this family.

Now that every elliptic curve E naturally admits a non-trivial automorphism. By the extended definition of the degree map, the degree of ψ_1 is supposed to be $\dfrac{1}{12|\mathrm{Aut}(E)|} = \dfrac{1}{12|\mathbb{Z}/2\mathbb{Z}|} = \dfrac{1}{24}$.

Example. *Now we are ready to actually calculate a Hodge integral only involving ψ classes with theorem 4.3 and the base cases. Let's consider the Hodge integral of $\psi_1^2\psi_2^3$ on $\overline{M}_{2,2}$.*

$$\int_{\overline{M}_{2,2}} \psi_1^2\psi_2^3 = \frac{1}{20}\left(\frac{7}{12}\int_{\overline{M}_{1,6}} \psi_1^3\psi_2^3 - \frac{5}{6}\int_{\overline{M}_{1,5}} \psi_1^2\psi_2^3 + 14\int_{\overline{M}_{1,3}} \psi_2^3\right).$$

Each of the four terms can be further reduced to the base case $\int_{\overline{M}_{1,1}} \psi_1 = \dfrac{1}{24}$.

4.2 Integral involving κ classes

Now we add κ classes to our integral:

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_i} \kappa_{b_1} \cdots \kappa_{b_m},$$

where the powers satisfy the condition $\sum_{i=1}^n a_i + \sum_{i=1}^m b_i = 3g-3+n$. First, by the definition of the κ class,

$$\int_{\overline{M}_{g,n}} \kappa_{b_1} \prod_{i=1}^n \psi_i^{a_i} \prod_{j=2}^m \kappa_{b_j} = \int_{\overline{M}_{g,n}} (\pi_{n+1*}\psi_{n+1}^{b_1+1}) \prod_{i=1}^n \psi_i^{a_i} \prod_{j=2}^m \kappa_{b_j}$$

The integral can be calculated in $\overline{M}_{g,n+1}$ by pulling these classes back via π_{n+1}. Then we apply the relations $\pi^*\psi_i = \psi_i - D_{i,n+1}, \pi^*\kappa_a = \kappa_a - \psi_{n+1}^a$.

$$\begin{aligned}
&\int_{\overline{M}_{g,n+1}} \psi_{n+1}^{b_1+1} \pi_{n+1}^* \left(\prod_{i=1}^n \psi_i^{a_i} \prod_{j=2}^m \kappa_{b_j} \right) \\
&= \int_{\overline{M}_{g,n+1}} \psi_{n+1}^{b_1+1} \prod_{i=1}^n (\psi_i^{a_i} - D_{i,n+1}) \prod_{j=2}^m (\kappa_{b_j} - \psi_{n+1}^{b_j}) \\
&= \int_{\overline{M}_{g,n+1}} \psi_{n+1}^{b_1+1} \prod_{i=1}^n \psi_i^{a_i} \prod_{j=2}^m (\kappa_{b_j} - \psi_{n+1}^{b_j}) \\
&= \sum_{S \subset \{b_2,\ldots,b_m\}} (-1)^{m-1-|S|} \int_{\overline{M}_{g,n+1}} \prod_{b \in S} \kappa_b \prod_{i=1}^n \psi_i^{a_i} \psi_{n+1}^{b_1+1+(\sum_{b \in \{b_2,\ldots,b_m\} \setminus S} b)}
\end{aligned}$$

By expanding the second line, we can see any term involving $D_{i,n+1}$ vanishes since $\psi_{n+1} \cdot D_{i,n+1} = 0$. So last step has only $\prod_{i=1}^n \psi_i^{a_i}$ left. Then we expand the rest to get Hodge integrals over $\overline{M}_{g,n+1}$ with strictly fewer κ classes. Applying the above operation recursively to each term, eventually we have integrals only involving ψ classes.

Last but not least, we should do dimension check. The moduli space $\overline{M}_{g,n}$ has dimension $3g-3+n$, so the original integral has dimension $\sum_{i=1}^n a_i + \sum_{j=1}^m b_j = 3g-3+n$. After eliminating one κ class, we have now a $3g-2+n$ dimensional space $\overline{M}_{g,n+1}$, and each term of the last summation has dimension

$$\sum_{i=1}^n a_i + \sum_{b \in S} b + b_1 + \sum_{b \in \{b_2,\ldots,b_m\} \setminus S} b + 1 = \sum_{i=1}^n a_i + \sum_{j=1}^m b_j + 1 = 3g-2+n.$$

Finally, let's calculate an integral over $\overline{M}_{3,2}$.

$$\begin{aligned}
&\int_{\overline{M}_{3,2}} \psi_1^2\psi_2^3\kappa_1\kappa_2 \\
=&\int_{\overline{M}_{3,2}} \pi_{3,*}(\psi_3^2)\psi_1^2\psi_2^3\kappa_2 \\
=&\int_{\overline{M}_{3,3}} \psi_1^2\psi_2^3\psi_3^2(\kappa_2-\psi_3^2) \\
=&\int_{\overline{M}_{3,3}} \psi_1^2\psi_2^3\psi_3^2\kappa_2 - \int_{\overline{M}_{3,3}} \psi_1^2\psi_2^3\psi_3^4 \\
=&\int_{\overline{M}_{3,4}} \psi_1^2\psi_2^3\psi_3^2\psi_4^3 - \int_{\overline{M}_{3,3}} \psi_1^2\psi_2^3\psi_3^4
\end{aligned}$$

We have now eliminated all κ classes. Using LiuXu's formula, we can evaluate the integral.

4.3 Integral involving Chern characters

We have seen that λ classes can be expressed in terms of Chern characters. In his paper, Mumford remarkably obtained an expression of Chern characters in terms of boundary strata and other tautological classes. We shall review his approach for $\overline{M}_g$ first.

Consider the universal family $\pi : \overline{C}_g \to \overline{M}_g$. We can apply the G-R-R formula to calculate the Chern character of the Hodge bundle over $\overline{M}_g$. A direct calculation shows that

$$\mathrm{ch}(\pi_!\omega_{\overline{C}_g/\overline{M}_g}) = \pi_*(\mathrm{ch}(\omega_{\overline{C}_g/\overline{M}_g})\mathrm{Td}^\vee(\Omega^1_{\overline{C}_g/\overline{M}_g})),$$

where $\omega_{\overline{C}_g/\overline{M}_g}$ is the relative dualizing sheaf and $\pi_!$ is the alternating sum of direct images: $\pi_!(\omega_{\overline{C}_g/\overline{M}_g}) = \sum_{i=0}^{\infty}(-1)^i R^i_*(\omega_{\overline{C}_g/\overline{M}_g})$.

The 0-th direct image of $\omega_{\overline{C}_g/\overline{M}_g}$ is precisely the Hodge bundle, and $R^1_*(\omega_{\overline{C}_g/\overline{M}_g}) \simeq \mathscr{O}_{\overline{M}_g}$, whose Chern character is trivial. All higher direct images vanish. We denote $c_1(\omega_{\overline{C}_g/\overline{M}_g})$ by K. Together with the expression of Todd class, we have

$$\mathrm{ch}(\mathbb{E}) - 1 = \pi_*(e^K\mathrm{Td}^\vee(\Omega^1_{\overline{C}_g/\overline{M}_g})).$$

Notice that we have the following exact sequence [11]:

$$0 \to \Omega_{\overline{C}_g/\overline{M}_g} \to \omega_{\overline{C}_g/\overline{M}_g} \to \omega_{\overline{C}_g/\overline{M}_g} \otimes \mathscr{O}_{\text{Sing } \overline{C}_g} \to 0,$$

where Sing $\overline{C}_g$ is the variety of singular points of $\overline{C}_g$. Let Sing$'$ $\overline{C}_g$ be the double cover of Sing$\overline{C}_g$. Recall the construction of the relative dualizing sheaf. We notice that the sheaf $\omega_{\overline{C}_g/\overline{M}_g} \otimes \mathscr{O}_{\text{Sing}\overline{C}_g}$ is isomorphic to $\mathscr{O}_{\text{Sing}'\overline{C}_g}$. Taking the dual of Todd classes of the above exact sequence, we get

$$\text{Td}^\vee(\omega_{\overline{C}_g/\overline{M}_g}) = \text{Td}^\vee(\Omega^1_{\overline{C}_g/\overline{M}_g}) \cdot \text{Td}^\vee(\mathscr{O}_{\text{Sing}'\overline{C}_g}).$$

Then the Chern character can be expressed as

$$\begin{aligned}
\text{ch}(\mathbb{E}) &= 1 + \pi_*(e^K \text{Td}^\vee(\omega_{\overline{C}_g/\overline{M}_g}) \cdot \text{Td}^\vee(\mathscr{O}_{\text{Sing}'\overline{C}_g})^{-1}) \\
&= 1 + \pi_*(e^K \cdot \frac{K}{e^K - 1} \cdot (1 + \text{Td}^\vee(\mathscr{O}_{\text{Sing}'\overline{C}_g})^{-1} - 1)) \\
&= 1 + \pi_*(e^K \cdot \frac{K}{e^K - 1} + (1 + f(K))(\text{Td}^\vee(\mathscr{O}_{\text{Sing}'\overline{C}_g})^{-1} - 1)),
\end{aligned}$$

where $f(K)$ is a power series that starts with the linear term in K after expanding $e^K \cdot \dfrac{K}{e^K - 1}$. Similarly, $\text{Td}^\vee(\mathscr{O}_{\text{Sing}\overline{C}_g})^{-1} - 1$ is a power series that starts with the linear term in some cycles on Sing $\overline{C}_g$. Since K intersecting any cycle on Sing $\overline{C}_g$ is 0, we have

$$\text{ch}(\mathbb{E}) = 1 + \pi_*(e^K \cdot \frac{K}{e^K - 1}) + \pi_*((\text{Td}^\vee(\mathscr{O}_{\text{Sing}'\overline{C}_g})^{-1} - 1)).$$

The next step is to express $\text{Td}^\vee(\mathscr{O}_{\text{Sing}'\overline{C}_g})^{-1} - 1$ in terms of boundary strata. This can be done using the following lemma [12]:

Lemma 4.4. *Let Z be a smooth subvariety of X of codimension 2 and i be the inclusion. Then there exists a universal power series P in Chern classes $c_1(N), c_2(N)$ of the normal bundle $N = \mathscr{I}_Z/\mathscr{I}_Z^2$ such that*

$$(\text{Td}^\vee(\mathscr{O}_Z))^{-1} - 1 = i_*[P(c_1(N), c_2(N))].$$

[11] proved that Sing$'(\overline{C}_g)$ is a smooth subvariety of codimension 2. So applying the lemma to Sing$'(\overline{C}_g)$ gives an explicit formula for ch($\mathbb{E}$).

Proof of the lemma. Notice that $(\mathrm{Td}^\vee(\mathscr{O}_Z))^{-1} - 1$ is a polynomial in Chern characters $\mathrm{ch}_k(\mathscr{O}_Z)$. Apply G-R-R to the inclusion $i : Z \to X$.

$$\mathrm{ch}(i_!\mathscr{O}_Z) = i_*(\mathrm{ch}(\mathscr{O}_Z)\mathrm{Td}(\mathscr{T}_Z))/\mathrm{Td}(\mathscr{T}_X).$$

The left hand side is $i_!\mathscr{O}_Z$ on X. The Chern character of $\mathscr{O}_Z$ on Z is trivial. So $\mathrm{ch}(\mathscr{O}_Z)$ on X is $[Z] \cdot i_*(\mathrm{Td}(\mathscr{T}_Z))/\mathrm{Td}(\mathscr{T}_X)$ which can be expressed as the pushforward of a polynomial P in $c_1(N)$ and $c_2(N)$ of the normal bundle N. Therefore the universality of P is true.

To calculate the explicit formula of P, we may consider the case $Z = D_1 \cdot D_2$. Then the Chern classes of N satisfy $i_*(c_1(N)) = D_1 + D_2, i_*(c_2(N)) = D_1 \cdot D_2$. From the definition, we immediately have the following exact sequence

$$0 \to \mathscr{O}_X(-D_1 - D_2) \to \mathscr{O}_X(-D_1) \oplus \mathscr{O}_X(-D_2) \to \mathscr{O}_X \to \mathscr{O}_Z \to 0.$$

We can break it into two short exact sequences:

$$0 \to \mathscr{O}_X(-D_1 - D_2) \to \mathscr{O}_X(-D_1) \oplus \mathscr{O}_X(-D_2) \to \mathscr{O}_X(-Z) \to 0$$
$$0 \to \mathscr{O}_X(-Z) \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$$

Taking Todd classes of them, we have

$$\begin{aligned}\mathrm{Td}^\vee(\mathscr{O}_Z) &= \mathrm{Td}^\vee(\mathscr{O}_X)(\mathrm{Td}^\vee(\mathscr{O}_X(-Z)))^{-1} \\ &= \mathrm{Td}^\vee(\mathscr{O}_X(-D_1 - D_2))\mathrm{Td}^\vee(\mathscr{O}_X(-D_1) \oplus \mathscr{O}_X(-D_2))^{-1} \\ &= \left(\frac{-D_1 - D_2}{e^{-D_1 - D_2} - 1}\right) \cdot \left(\frac{-D_1}{e^{-D_1} - 1}\right)^{-1} \cdot \left(\frac{-D_2}{e^{-D_2} - 1}\right)^{-1},\end{aligned}$$

since the Todd class of $\mathscr{O}_X$ is trivial.

By expanding the right hand side, we have

$$i_*P(c_1(N), c_2(N)) = \frac{D_1 D_2}{D_1 + D_2} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(D_1^{2k-1} + D_2^{2k-1}),$$

where B_k is the k-th Bernoulli number. Notice that $i_*P(c_1(N), c_2(N)) = D_1 D_2 \cdot P(D_1 + D_2, D_1 D_2)$. Then we have the polynomial

$$P(D_1 + D_2, D_1 D_2) = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(D_1^{2k-2} - D_1^{2k-3} D_2 + \cdots + D_2^{2k-2}).$$

□

The polynomial given in the proof is actually not in terms of $c_1(N)$ and $c_2(N)$, but it works better in the following steps.

Now that Sing $\overline{C}_g$ breaks into disjoint pieces, each of which is the locus where a node seperates two components of smaller genera. More precisely,

$$\text{Sing } \overline{C}_g = \coprod_{0 \leqslant h \leqslant \lfloor \frac{g}{2} \rfloor} \Delta_h,$$

where Δ_h can be described in three cases.

• When $0 < h < \lfloor \frac{g}{2} \rfloor$, a node disconnects two components of genera h and $g-h$. So the set of such singularities is equivalent to $\overline{C}_h \times \overline{C}_{g-h}$, which has codimension $3g-2-(3h-2+3(g-h)-2) = 2$.

• When g is even and $h = \frac{g}{2}$, a node disconnects two components of the same genera. So the set of such singularities is equivalent to $(\overline{C}_{g/2} \times \overline{C}_{g/2})/(\mathbb{Z}/2\mathbb{Z})$, which has codimension $3g-2-2(3\frac{g}{2}-2) = 2$.

• When $h = 0$, it doesn't disconnect the dual graph of a curve, but does reduce its arithmetic genus by 1. So the set of such singularities is equivalent to $\overline{M}_{g-1,2}/(\mathbb{Z}/2\mathbb{Z})$, which has codimension $3g-2-(3(g-1)-3+2) = 2$. The reason for the quotient by $\mathbb{Z}/2\mathbb{Z}$ is that we identify these two marked points to create curve of genus g.

In fact, we wouldn't need the quotient in $\text{Sing}'\overline{C}_g$, in which case,

$$\begin{aligned}
\text{Sing}'\overline{C}_g &= \coprod_{0 \leqslant h \leqslant \lfloor \frac{g}{2} \rfloor} \Delta'_h, \\
\Delta'_h &\simeq 2 \text{ copies of } \overline{C}_h \times \overline{C}_{g-h}, \text{if } 0 < h < \frac{g}{2} \\
&\simeq \overline{C}_{\frac{g}{2}} \times \overline{C}_{\frac{g}{2}}, \text{if } g \text{ is even, and } h = \frac{g}{2} \\
&\simeq \overline{M}_{g-1,2}, \text{if } h = 0
\end{aligned}$$

So far, we are able to express the Chern character as

$$\begin{aligned}
\text{ch}(\mathbb{E}) &= 1 + \pi_*(1 + \frac{K}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} K^{2i}) + (\pi \circ i)_*(\sum_{h=0}^{g} P(c_1(N_h), c_2(N_h))) \\
&= g + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} \kappa_{2i-1} + (\pi \circ i)_*(\sum_{h=0}^{g} P(c_1(N_h), c_2(N_h))),
\end{aligned}$$

because $\pi_*(K) = 2g-2$ and $\pi_*(K^{a+1}) = \kappa_a$.

The morphisms $\pi \circ i_0$ and $\pi \circ i_{g/2}$ are restrictions of $\pi \circ i$ to Δ_0 and $\Delta_{g/2}$ respectively. These two pieces admit a non-trivial automorphism, so their degree are both $\frac{1}{2}$. Since every $\Delta_h, 0 < h < \lfloor \frac{g}{2} \rfloor$ is counted twice in the sum, we can rewrite it as

$$\begin{aligned} &\frac{1}{2}\sum_{h=0}^{\lfloor \frac{g}{2} \rfloor} \pi \circ i_h(P(c_1(N_h), c_2(N_h))) \\ =&\frac{1}{2}\sum_{h=0}^{\lfloor \frac{g}{2} \rfloor}\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(K_1^{2k-2} - K_1^{2k-3}K_2 + \cdots + K_2^{2k-3}), \end{aligned} \tag{4.1}$$

where K_1, K_2 are boundary divisors.

Corollary 4.5. *[12] The pieces of* $\mathrm{ch}(\mathbb{E})$ *of even degrees are 0, i.e.*

$$\mathrm{ch}_{2k}(\mathbb{E}) = 0$$

Example. *Here we consider a Hodge integral involving* κ *and* λ *classes.*

$$\int_{\overline{M}_{3,0}} \kappa_1\kappa_2\lambda_1\lambda_2.$$

First, we convert λ *classes to Chern characters.*

$$\begin{aligned} 1 + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 &= exp(\mathrm{ch}_1 t + 2\mathrm{ch}_3 t^3) \\ &= 1 + (\mathrm{ch}_1 t + \mathrm{ch}_3 t^3) + \frac{1}{2!}(\mathrm{ch}_1 t + \mathrm{ch}_3 t^3)^2 + \cdots \end{aligned}$$

So we have $\lambda_1 = \mathrm{ch}_1$ *and* $\lambda_2 = \frac{1}{2}\mathrm{ch}_1^2$.

Following the above argument, we have

$$\mathrm{ch}_1 = \frac{B_2}{2!}\kappa_1 = \frac{1}{12}\kappa_1$$

Hence, $\lambda_1 = \frac{1}{12}\kappa_1, \lambda_2 = \frac{1}{288}\kappa_1^2$. *The integral then becomes*

$$\frac{1}{3456}\int_{\overline{M}_{3,0}} \kappa_1^4\kappa_2.$$

By the method in sec 4.2, we can eventually eliminate all κ *classes and turn it into an integral involving solely* ψ *classes.*

The story doesn't end here, because we are now dealing with $\overline{M}_{g,n}$. We can either pull the formula back via $\pi : \overline{M}_{g,n} \to \overline{M}_g$, or perform the similar calculation on the morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$. Here I shall do the latter one. Since the process is similar, I will just highlight what extra data need to be added to the formula.

In this case, the Chern class $K \in H^2(\overline{M}_{g,n+1})$ is defined as usual by $K = c_1(\omega_{\pi_{n+1}})$. However, the κ classes on $\overline{M}_{g,n}$ are defined to be $\pi_*(\psi_{n+1}^{a+1}) = \pi_*(c_1(\omega_{\pi_{n+1}}(\sum_{i=1}^n D_{i,n+1})))$. To avoid confusion, the former one will be denoted by $\tilde{\kappa}$. They can be related via the relation $\kappa_a = \tilde{\kappa}_a + \sum_{i=1}^n \psi_i^a$. Therefore, in the final expression of $\mathrm{ch}(\mathbb{E})$, the pushforward of K^a should be replaced with $\kappa_a - \sum_{i=1}^n \psi_i^a$.

Now we should consider the partition of $\mathrm{Sing}'(\overline{C}_{g,n})$. It is the disjoint union of all $\overline{C}_{h,|S|+1} \times \overline{C}_{g-h,|\{1,\ldots,n\}\setminus S|+1}$ when $h \neq 0$, and $\overline{M}_{g-1,n+2}$ when $h = 0$. For example, all possible partitions of $\mathrm{Sing}'(\overline{C}_{2,2})$ are listed below

$$\overline{M}_{1,4}, \overline{C}_{0,2} \times \overline{C}_{2,0}, \overline{C}_{1,1} \times \overline{C}_{1,1}$$

I chose $\mathrm{Sing}'(\overline{C}_{2,2})$ as an example is because it has very few marked points and it's not hard to distribute them to two components. When n is large, the number of possible partitions increases at an exponential rate.

We are able to give the complete formula of $\mathrm{ch}_k(\mathbb{E})$ on $\overline{M}_{g,n}$ now:

$$\mathrm{ch}_k(\mathbb{E}) = \kappa_k - \sum_{i=1}^n \psi_i^k + \sum_{\xi} \sum_{j=0}^{k-1} (-1)^j \xi_*(\psi_\star^{k-1-j} \psi_\bullet^j),$$

where ξ ranges over all possible gluing morphisms

$$\overline{M}_{h,S\cup\{x\}} \times \overline{M}_{g-h,S'\cup\{x'\}} \to \overline{M}_{g,n}$$

with $S \cup S' = \{1, \ldots, n\}$ and

$$\overline{M}_{g-1,n+2} \to \overline{M}_{g,n},$$

and $\psi_\star, \psi_\bullet$ correspond to the boundary divisors on $\overline{M}_{h,S\cup\{x\}}, \overline{M}_{g-h,S'\cup\{x'\}}$ respectively, or in the case of $\overline{M}_{g-1,n+2}$, two extra marked points.

Eliminate Chern characters

After we express $\mathrm{ch}_k(\mathbb{E})$ in terms of κ, ψ, and boundary classes, the next question is to investigate how they interact with the rest of Chern characters pieces. Consider the following integral over $\overline{M}_{g,n}$ and denote the dimension by $d = 3g - 3 + n$. For simplicity, I eliminated λ classes in advance, so the integral only involves ψ, κ classes and Chern characters.

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \kappa_1^{b_1} \cdots \kappa_d^{b_d} \mathrm{ch}_1^{c_1} \mathrm{ch}_3^{c_3} \cdots \mathrm{ch}_{2g-1}^{c_{2g-1}},$$

where $\sum a_i + \sum b_i + \sum c_{2i-1} = d = 3g-3+n$. By replacing one of the Chern characters using the above formula, the list of indices has strictly less Chern characters, but the indices of ψ and κ classes also change.

There are three parts when we expand one ch_k. The first two parts fall into an integral over the same $\overline{M}_{g,n}$ that involves strictly less Chern characters. The third one reduces the calculation down to integrals over two smaller moduli spaces.

$$\begin{aligned}
&\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \kappa_1^{b_1} \cdots \kappa_d^{b_d} (\kappa_1 - \sum_{i=1}^{n} \psi_i) \mathrm{ch}_1^{c_1-1} \mathrm{ch}_3^{c_3} \cdots \mathrm{ch}_{2g-1}^{c_{2g-1}} \\
=&\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \kappa_1^{b_1+1} \cdots \kappa_d^{b_d} \mathrm{ch}_1^{c_1-1} \mathrm{ch}_3^{c_3} \cdots \mathrm{ch}_{2g-1}^{c_{2g-1}} \\
-&\sum_{i=1}^{n} \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_i^{a_i+1} \cdots \psi_n^{a_n} \kappa_1^{b_1} \cdots \kappa_d^{b_d} \mathrm{ch}_1^{c_1-1} \mathrm{ch}_3^{c_3} \cdots \mathrm{ch}_{2g-1}^{c_{2g-1}} \qquad (4.2)
\end{aligned}$$

There's no boundary divisors in the expansion of ch_1. So to understand how recursion works, we expand ch_{2k-1} for some $k > 1$. Then the only term we need to worry about is

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \kappa_1^{b_1} \cdots \kappa_d^{b_d} \mathrm{ch}_1^{c_1} \cdots \mathrm{ch}_{2k-1}^{c_{2k-1}-1} \mathrm{ch}_{2g-1}^{c_{2g-1}} \xi_*(\psi_\star^{2k-2-j} \psi_\bullet^j)$$

where $\psi_\star, \psi_\bullet$ are boundary divisors correspond to $\overline{M}_{h,S\cup\{x\}}, \overline{M}_{g-h,S'\cup\{x'\}}$. By projection formula, the integral becomes

$$\int_{\overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1}} \psi_1^{a_1} \cdots \psi_n^{a_n} \xi^*(\kappa_1^{b_1} \cdots \kappa_d^{b_d} \mathrm{ch}_1^{c_1} \cdots \mathrm{ch}_{2g-1}^{c_{2g-1}}) \psi_\star^{2k-2-j} \psi_\bullet^j.$$

Here the only thing left is to determine what the pullback of κ classes and Chern characters are in $\overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1}$.

Since every time we only split a stratum into two components, it suffices to consider the following diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{\eta} & \overline{C}_{g,n} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{n+1}} \\ \overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1} & \xrightarrow{\xi} & \overline{M}_{g,n} \end{array}$$

where $\Delta = \overline{C}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1} \amalg \overline{C}_{g-h,n_2+1} \times \overline{M}_{h,n_1+1}$. This universal family over $\overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1}$ is obtained by gluing two families along sections p_x and $p_{x'}$. Therefore, the morphism η splits into two parts. We denote the restrictions of η to two components by η_h and η_{g-h}. Similarly we have π_h and π_{g-h}. The projection π_h induces $n_1 + 1$ sections $p_1, \ldots, p_{n_1+1}$. Similarly, π_{g-h} induces $n_2 + 1$ sections.

$$\begin{array}{ccccc} \overline{C}_{g-h,n_2+1} & \longleftarrow & \Delta & \longrightarrow & \overline{C}_{h,n_1+1} \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ \overline{M}_{g-h,n_2+1} & \xleftarrow{\mathrm{pr}_2} & \overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1} & \xrightarrow{\mathrm{pr}_1} & \overline{M}_{h,n_1+1} \end{array}$$

Then the dualizing sheaves on both components give

$$\eta_h^*(c_1(\omega_{\pi_{n+1}}(\sum D_i))) = c_1(\omega_{\pi_h}(\sum \mathrm{Im}(p_i))) \tag{4.3}$$

$$\eta_{g-h}^*(c_1(\omega_{\pi_{n+1}}(\sum D_i))) = c_1(\omega_{\pi_{g-h}}(\sum \mathrm{Im}(p_i))) \tag{4.4}$$

Hence, we are able to pullback κ classes and express them as follows:

$$\xi^*(\kappa_a) = \mathrm{pr}_1^*(\kappa_a) + \mathrm{pr}_2^*(\kappa_a).$$

On the right hand side, the first κ_a is on $\overline{M}_{h,n_1+1}$ and the second one is on $\overline{M}_{g-h,n_2+1}$.

This procedure is similar for Chern characters. In this case, we pullback Hodge bundles.

$$\begin{array}{ccc} \mathbb{E}' & \xrightarrow{\eta} & \mathbb{E} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1} & \xrightarrow{\xi} & \overline{M}_{g,n} \end{array}$$

The fibre at each point in $\overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1}$ splits into two parts since a node connects two components of genera h and $g-h$ respectively, i.e. it's the direct sum of two vector spaces of dimension h and $g-h$. By looking at the following diagram,

$$\begin{array}{ccccc} \mathbb{E}_{g-h} & \xleftarrow{\eta_{g-h}} & \mathbb{E}' & \xrightarrow{\eta_h} & \mathbb{E}_h \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ \overline{M}_{g-h,n_2+1} & \xleftarrow{\mathrm{pr}_2} & \overline{M}_{h,n_1+1} \times \overline{M}_{g-h,n_2+1} & \xrightarrow{\mathrm{pr}_1} & \overline{M}_{h,n_1+1} \end{array}$$

we conclude that $\mathbb{E}' = \eta^*(\mathbb{E}) = \eta_h^*(\mathbb{E}_h) \oplus \eta_{g-h}^*(\mathbb{E}_{g-h})$. Therefore, the Chern characters also splits.

The discrepancy between the cases $\overline{M}_g$ and $\overline{M}_{g,n}$ is that the boundary contains the stratum of genera 0 and g, as long as the genus 0 component has at least three marked points. Nonetheless, the irreducible case still exists, namely $\xi_\Gamma : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ which glues two marked points to form another edge and add genus by 1. The κ classes and Chern characters simply pullback without splitting.

Appendix A

Cohomology Tools

We review some fundamental tools used throughout the thesis.

Appendix A focuses on basic cohomology tools like Poincaré Duality, projection formula, and excess intersection formula. Since the tautological ring is a subring of the cohomology ring $H^*(\overline{M}_{g,n}, \mathbb{Q})$, these tools play an important role in the calculation of Hodge integrals.

Let X be a topological space of dimension n, and $H^*(X;\mathbb{Q})$ be its cohomology ring with coefficients in $\mathbb{Q}$. Then $H^k(X,\mathbb{Q})$ vanishes for $k > 2n$ or $k < 0$. Moreover, the top cohomology group $H^{2n}(X,\mathbb{Q}) \simeq \mathbb{Q}$.

Basic singular homology and cohomology theory is thoroughly developed in [7]. Here we briefly review operations between homology and cohomology groups.

A.1 Cap product and cup product

Usually we have the following identification

$$H^k(X;\mathbb{Q}) \simeq \mathrm{Hom}(H_k(X),\mathbb{Q}).$$

This allows us to grade the cohomology ring $H^*(X,\mathbb{Q})$ by codimensions.

Definition A.1. *Let $\sigma \in H_k(X), \alpha \in H^l(X)$ be two classes with $k \geqslant l$. The cap product gives a homology class $\sigma \frown \alpha \in H_{k-l}(X)$. In particular, we have a perfect pairing for $k = l$:*

$$\begin{aligned} \frown\colon H_k(X) \otimes H^k(X) &\to H_0(X) \simeq \mathbb{Q} \\ \sigma \otimes \alpha &\mapsto \sigma \frown \alpha \end{aligned}$$

The last isomorphism is given by the degree map

$$\deg : H_0(X) \to \mathbb{Q} \tag{A.1}$$

$$\sum_{i=1}^{N} a_i[P_i] \mapsto \sum_{i=1}^{N} a_i, \tag{A.2}$$

where $[P_i]$ is just a class of a point P_i on X. Note that this is a finite sum.

Definition A.2. *Let $\sigma \in H^k(X), \alpha \in H^l(X)$ be two cohomological classes. Their cup product is a class in $H^{k+l}(X)$:*

$$\begin{aligned} \smile : H^k(X) \otimes H^l(X) &\to H^{k+l}(X) \\ \sigma \otimes \alpha &\mapsto \sigma \smile \alpha \end{aligned}$$

Cup product together with the usual addition among cohomology classes give $H^*(X, \mathbb{Q})$ a ring structure.

A.2 Poincaré Duality

Poincaré Duality gives a connection between homology and cohomology. In other words, we have a perfect pairing

$$\begin{aligned} H^k(X) \otimes H^{2n-k}(X) &\to H^{2n}(X) \simeq \mathbb{Q} \\ \alpha \otimes \beta &\mapsto \alpha \smile \beta \end{aligned}$$

The last isomorphism is given by

$$H^{2n}(X) \to \mathbb{Q} \tag{A.3}$$

$$\alpha \mapsto \int_X \alpha = \deg(\alpha \frown [X]) \tag{A.4}$$

for any class $\alpha \in H^{2n}(X)$.

Together with the identification (A.1), we have a natural isomorphism

$$H^k(X) \to H_{2n-k}(X) \tag{A.5}$$

$$\alpha \mapsto \alpha \frown [X] \tag{A.6}$$

Alternatively, we can formulate the above discussion into the theory of Chow groups. The k-th Chow group $A_k(X)$ is generated by formal sums of subvarieties of dimension k modulo rational equivalence. These are called k-cycles. Similarly, we can grade the Chow groups $A^*(X)$ by codimensions. The product of two cycles are given by the intersection of two corresponding varieties. In this way, $A^*(X)$ is given a ring structure.

Most calculations in this thesis can be carried out in $A^*(X)$ without much modification. Note that some moduli spaces have non-trivial odd cohomology, which makes the Chow ring less adequate for certain calculations.

A.3 Projection formula

The projection formula relates fundamental classes in two cohomology rings via morphisms. Let Z be a irreducible subvariety of dimension k. The fundamental class $[Z]$ can be defined as follows. If Z is not reduced, $[Z]$ is the formal sum of its irreducible components $[Z_i]$ with coefficient $m_i = l_{\mathscr{O}_{Z,Z_i}}\mathscr{O}_{Z,Z_i}$. For example, let $X = \mathbb{C}^2, Z = \mathrm{Spec}(k[x,y]/(y^2 - x^2))$. Then $Z_1 = \mathrm{Spec}(k[x,y]/(y-x)), Z_2 = \mathrm{Spec}(k[x,y]/(y+x))$, and the local rings $\mathscr{O}_{Z,Z_1}$, $\mathscr{O}_{Z,Z_2}$ both have length 1. So the fundamental class $[Z] = [Z_1] + [Z_2]$.

Definition A.3. *Let $f : X \to Y$ be a proper morphism. Given a subvariety $Z \subset X$ and its image $Z' = f(Z) \subset Y$, we define the pushforward of the fundamental class $[Z]$ to be*

$$f_*(Z) = \begin{cases} \deg(Z/Z')[Z'] & if \dim Z = \dim Z' \\ 0 & otherwise, \end{cases}$$

where the degree of Z over Z' can be defined as the degree of field extension $\mathbb{C}(Z') \subset \mathbb{C}(Z)$.

Remark. *This definition indeed makes sense. For example, consider the projection morphism from the parabola $\mathrm{Spec}(k[x,y]/(y-x^2))$ to the affine line $\mathrm{Spec}(k[y])$. The subvariety $Z = \mathrm{Spec}(k[x,y]/(x^2-1))$ consists of two points, whose image is a single point Z'. Hence $\deg(Z/Z') = 2$.*

Definition A.4. *Let $f : X \to Y$ be a flat morphism. Given a closed subvariety $Z \subset Y$ and the closed subscheme $f^{-1}(Z)$. Then the pullback of the fundamental class $[Z]$ is simply $[f^{-1}(Z)]$.*

Theorem A.5 (Projection formula)**.** *Using the same notations, we have the projection formula*

$$f_*(f^*\beta \smile \alpha) = \beta \smile f_*(\alpha), \forall \alpha \in H^*(X), \beta \in H^*(Y).$$

A.4 Excess intersection formula

In general, it's hard to calculate cup product of cohomological classes. We introduce an important technique here to help calculate cup products of certain types of classes.

Theorem A.6. *Let X_1, X_2, Y_1, Y_2 be connected, smooth, proper schemes that satisfy the following fibre diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{g_1} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g_2} & Y_2 \end{array}$$

where g_1, g_2 are local complete intersections of codimensions d_1, d_2 and f_2 is a proper morphism. Then $\forall \alpha \in H^*(X)$, the identity holds

$$g_2^*(f_{2,*}(\alpha)) = f_{1,*}(c_{top}(\mathbb{E}) \smile g_1^*(\alpha)) \in H^*(Y_1),$$

where $\mathbb{E}$ is the vector bundle of rank $d_1 - d_2$ over X_1, the quotient sheaf of $f_1^*\mathcal{N}_{Y_1/Y_2}$ by $\mathcal{N}_{X_1/X_2}$.

Appendix B

Grothendieck-Riemann-Roch theorem

Appendix B provides a quick review of the Grothendieck-Riemann-Roch theorem. It contains the definitions of concepts that appear in the statement, as well as a sketch of proof. Chern characters and Todd classes will be introduced.

I won't present the whole proof here. Readers may refer to [3] and [5] for a complete proof.

B.1 Chern character and Todd class

Let $\mathbb{L}$ be a line bundle over a scheme X. There exists an isomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X)$ that associates to $\mathbb{L}$ a cohomology class $c_1(\mathbb{L})$. Moreover, if $\mathbb{E}$ is a bundle of rank $k > 1$, we can associate to it cohomology classes $c_2(\mathbb{E}) \in H^4(X), \ldots, c_k(\mathbb{E}) \in H^{2k}(X)$. Let 1 be the identity element in $H^0(X)$. Then we can write $c(\mathbb{E}) = 1 + c_1(\mathbb{E}) + \cdots + c_k(\mathbb{E})$ as the total Chern class. It is easy to verify the following two properties

$$c_1(\mathbb{L}_1 \otimes \mathbb{L}_2) = c_1(\mathbb{L}_1) + c_2(\mathbb{L}_2) \tag{B.1}$$

$$c(\mathbb{E}_1 \oplus \mathbb{E}_2) = c(\mathbb{E}_1) \cdot c(\mathbb{E}_2) \tag{B.2}$$

These properties are not satisfying because the Chern class "reverses" operation on bundles. To remedy this, we upgrade it to the Chern character, which is a homomorphism from the K-theory $K(X)$ to the cohomology ring

$H^*(X)$. This means, it must preserve addition and multiplication in each ring.

$$ch(\mathbb{E}_1 \oplus \mathbb{E}_2) = ch(\mathbb{E}_1) + ch(\mathbb{E}_2) \tag{B.3}$$
$$ch(\mathbb{E}_1 \otimes \mathbb{E}_2) = ch(\mathbb{E}_1) \tag{B.4}$$

The Chern character can be easily defined on a line bundle $\mathbb{L}$.

$$ch(\mathbb{L}) = \exp(c_1(\mathbb{L})) = \sum_{i=0}^{\infty} \frac{c_1(\mathbb{L})^i}{i!}$$

Let E be a vector bundle of rank k over X. By splitting principle, equations on $H^*(X)$ works on $H^*(Y)$ for the flag bundle over Y together with a map $p : Y \to X$, which means the Chern character on the direct sum $\mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_k$ can be pulled back to the Chern character on $\mathbb{E}$. More precisely, we have the following

$$ch(\mathbb{E}) = \sum_{i=1}^{k} \exp(c_1(\mathbb{L}_i)).$$

It is often convenient to consider the generating function of Chern classes. Given a vector bundle $\mathbb{E}$ and its total Chern class $c(\mathbb{E}) = 1 + c_1 + \cdots + c_k$, the generating function is a degree k polynomial $c_t(\mathbb{E}) = 1 + c_1 t + \cdots + c_k t^k$ with coefficients in $H^*(X)$. Since any polynomial of degree k must have k solutions(counded with multiplicity), we can write

$$c_t(\mathbb{E}) = (1 + \alpha_1 t) \cdots (1 + \alpha_k t),$$

where $\alpha_i(\mathbb{E}) = c_1(\mathbb{L}_i)$ by splitting principle. These are called the Chern roots. It is well-known that symmetric polynomials in these solutions can be expressed in terms of the coefficients of this generating function. We can expand the above sum to get

$$ch(\mathbb{E}) = rank(\mathbb{E}) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) \cdots .$$

Todd class can be defined in the following way.

Consider the power series expansion of the function

$$Q(X) = \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i}.$$

Now we define the Todd class to be

$$td(\mathbb{E}) = \prod_{i=1}^{k} Q(\alpha_i).$$

The reason for choosing this function is related to Todd class of the tangent bundle over a topological space, and for the space $\mathbb{CP}^n$, we can compute the Todd class via Euler exact sequence for the tangent bundle $T\mathbb{CP}^n$

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{n+1} \to T\mathbb{CP}^n \to 0$$

that $td(T\mathbb{CP}^n) = \left(\frac{\xi}{1-e^{-\xi}}\right)^{n+1}$ where $\xi \in H^2(\mathbb{CP}^2)$ is the hyperplane class. Details are well explained in [13].

B.2 Grothendieck group

The pure categorical definition of a Grothendieck group can be found in any book on homological algebra, e.g. [14]. Here we only treat the case for the coherent sheaves on a scheme.

Definition B.1. *Let X be a noetherian scheme. Then the Grothendieck group $K_0(X)$ is the free abelian group generated by isomorphism classes of coherent sheaves on X modulo exact sequences.*

To be more precise, we can think of the modulo relation in the following way. To any short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathrm{Coh}(X)$, we assign an element $[\mathscr{F}] - [\mathscr{F}'] - [\mathscr{F}'']$ in the set $\mathrm{sCoh}(X)$ of isomorphism classes of coherent sheaves. Then

$$K_0(X) =< \mathrm{sCoh}(X) > / \simeq .$$

Similarly, we can define $K^0(X)$ to be the Grothendieck group of vector bundles on X. When X is smooth, there exists an isomorphism between $K^0(X)$ and $K_0(X)$.

B.3 Proper pushforward

When there is a morphism between schemes, it's natural to study how the Grothendieck groups of them relate via the morphism. However, it's not trivial in this case. Neither pullback nor pushforward preserves exact sequences. We need to modify the definition a little to make things work.

Let $f : X \to Y$ be a proper morphism. It preserves left exactness. In other words, given any exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathrm{Coh}(X)$, we get

$$0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}''$$

in $\mathrm{Coh}(Y)$. Using higher direct images, we can extend this sequence to a long exact sequence

$$\begin{aligned} 0 \to R^0 f_*\mathscr{F}' \to R^0 f_*\mathscr{F} \to R^0 f_*\mathscr{F}'' & \quad \text{(B.5)} \\ \to R^1 f_*\mathscr{F}' \to R^1 f_*\mathscr{F} \to R^1 f_*\mathscr{F}'' & \quad \text{(B.6)} \\ \to \cdots & \quad \text{(B.7)} \end{aligned}$$

Then we can define

$$\begin{aligned} f_! : K_0(X) \to K_0(Y) & \quad \text{(B.8)} \\ [\mathscr{F}] \mapsto \sum_{i=0}^{\infty} (-1)^i [R^i f_*\mathscr{F}] & \quad \text{(B.9)} \end{aligned}$$

Now we are ready to state the Grothendieck-Riemann-Roch theorem:

Theorem B.2 (G-R-R)**.** *Let $f : X \to Y$ be a proper morphism. Then we have the following identity*

$$\mathrm{ch}(f_!\mathscr{F})\mathrm{td}(\mathscr{T}_Y) = f_*(\mathrm{ch}(\mathscr{F})\mathrm{td}(\mathscr{T}_X)),$$

where $[\mathscr{F}] \in K_0(X)$ and $\mathscr{T}_X$ is the tangent bundle on X.

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