Quadratic Forms and Homogeneous Dynamics

Mikael Jonas Konrad

Supervisors:
Prof. Dr. Alexander Gorodnik
Dr. Tsviqa Lakrec

Master Thesis

Institute of Mathematics
University of Zürich
Switzerland
February to September 2023

Acknowledgements

First and foremost I would like to thank my Supervisors Prof. Dr. Alexander Gorodnik and Dr. Tsviqa Lakrec for providing me with the opportuinity to do a master thesis and also for their support throughout the whole process. Furthermore I would like express my gratitude to Prof. Dr. Corinna Ulcigrai, Prof. Dr. Joachim Rosenthal and to Dr. Tsviqa Lakrec for reinvigorating my love for mathematics.

Also I would like to thank my family, who not only blessed me with their unconditional support but also made me the man I am today.

Asante sana Alex, rafiki yangu, for grinding the σ-grind with me before it was cool and for showing me da wae to rock climbing. Muchas gracias mi cuate Oswaldo for all the great moments and for sharing a sense of humor without any standards or respect. I would also like to thank all the other wonderful people I met lurking in the deep dark dungeons of the math institute: Kuvuna, Jenny, Mattia, Sevi, Marinja, Peng-Cheng, Irati, Simon, Joel, Fede, Jonathan and many more. Thanks to my friends Alexander, Ansel, Samu P., Samu M., Marc, Tom, Gustaf and Etienne for keeping me in touch with reality[1]. Thanks to la Uva, Tobirri, Nils, Jaime, Sigi and Luki. EHC Allez!

Thanks to the ASVZ, Frau Eich and all of the above for keeping me sufficiently sane to complete my thesis. Thanks to Coca-Cola for fueling my efforts[2]. Thanks to my orange Freitag bag, may it rest peace. Thanks to the words “hence”, “thus”, “therefore”, “moreover” and “furthermore” for always keeping my thesis going.

[1] Reality is a concept from physics, with no basis in mathematics.

[2] This thesis was not sponsored by the Coca-Cola Company. If the Coca-Cola Company is interested in future sponsorship agreements please contact me via my email mikael.k.8@gmail.com.

Contents

1 Introduction **4**
1.1 Quadratic Forms . . . 4
1.2 Orbit Closures . . . 5
1.3 History of Oppenheim's conjecture . . . 5
1.4 Notations . . . 6

2 Prerequisites **7**
2.1 Group-Actions and Homogeneous Spaces . . . 7
2.1.1 Convergence in Homogeneous Spaces . . . 9
2.1.2 Invariant Measures on Homogeneous Spaces . . . 10
2.2 Basic Notions of Dynamical Systems . . . 11
2.2.1 Orbits . . . 13
2.2.2 The Tripling Map . . . 15
2.2.3 Dynamical Systems as Group actions . . . 17
2.2.4 The Space of Orbits . . . 18
2.2.5 Ergodicity . . . 18
2.3 Orbit Closures of Subgroups of a Topological Group . . . 19

3 Hedlund's Theorem in the Cocompact Case **22**
3.1 Introduction . . . 22
3.1.1 Outline of the Chapter . . . 22
3.2 P Acts Minimally on Compact Subgroups . . . 23
3.3 No N-Periodic Orbits on G/Γ . . . 26
3.4 Existence of Minimal Sets . . . 27
3.5 Proof of Hedlund's Theorem . . . 28

4 $\mathrm{SL}(3,\mathbb{R})$ **30**
4.1 Important Subsets of $\mathrm{SL}(3,\mathbb{R})$ . . . 30
4.2 Quadratic Forms . . . 33
4.2.1 Action of $\mathrm{SO}(Q_0)$ on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ . . . 35
4.3 Properties of $H = SO(Q_0)$ . . . 36
4.4 V_2^+-Orbits . . . 39

5 Non-Divergence **41**
5.1 The Space of Lattices . . . 41
5.2 Grayson Polygons of Lattices . . . 42
5.3 Mahler's Compactness Criterion . . . 44
5.4 (C,α)-Good Functions . . . 50
5.5 Non-Divergence . . . 52

6 Margulis' Proof of Oppenheim's Conjecture **57**
6.1 Introduction . . . 57
6.1.1 Outline of the Chapter . . . 57
6.2 Counterexample for $n = 2$ . . . 57
6.3 Reduction to $n = 3$ . . . 58
6.4 H-Orbits of Irrational Quadratic Forms are not Closed . . . 60
6.5 Existence of Minimal Sets . . . 62
6.5.1 Existence of DV_1-Minimal Sets . . . 62
6.5.2 Existence of V_1-Minimal Sets . . . 64
6.6 Orbits of One-Parameter Groups of Unipotent Linear Transformations . . . 68
6.7 Proof of Oppenheim's Conjecture . . . 70

References **76**

1 Introduction

One of the great joys of mathematics lies in discovering connections between fields, which at first glance seem completely unrelated. Oftentimes the discovery of such connections greatly increases the understanding of all fields involved. One such connection is that between number theory and dynamical systems.

This connection was first used by Margulis to prove Oppenheim's conjecture. A number theoretical conjecture, which has remained unproven for 58 years and which to this day has not been proven without using dynamical systems.

Theorem 1.1 (Oppenheim's Conjecture)**.** *Let Q be a non-degenerate, indefinite, irrational quadratic form Q in $n \geq 3$ variables. Then the set $Q(\mathbb{Z}^n) \subseteq \mathbb{R}$ is dense.*

This thesis will give an introduction to the methods used by Margulis in this proof. Furthermore it will contain a complete proof of Oppenheim's conjecture

1.1 Quadratic Forms

Definition 1.2 (Quadratic Form)**.** *A quadratic form in n variables is a homogeneous polynomial $Q(x) \in \mathbb{R}[x_1, ..., x_n]$ of degree two. Homogeneous of degree n means that for any $\lambda \in \mathbb{R}$ we have $p(\lambda x_1, ..., \lambda x_n) = \lambda^2 p(x_1, ..., x_n)$.*

It is easy to check that Oppenheim's conjecture fails, when $n = 1$. Furthermore we will show in Section 6.2 that it fails for $n = 2$.

Oppenheim's Conjecture is adresses quadratic forms that are non-degenerate, indefinite and irrational. This is because in cases where a quadratic form is degenerate, definite or not irrational, it can be easily proven that $Q(\mathbb{Z}^3)$ is not dense in $\mathbb{R}$. In this section, we define these notions and prove they prevent denseness.

In Section 6.2 we will prove that for the quadratic form

$$Q(x, y) = y^2 - (4 + 2\sqrt{3})x^2,$$

the set $Q(\mathbb{Z}^2)$ is not dense in $\mathbb{R}$. Note that we can view the same quadratic form as a quadratic form in 3 variables, one where all the coefficients of z are 0. Then the set $Q(\mathbb{Z}^3)$ is not dense in $\mathbb{R}$. However, since Q does not depend on z this quadratic form can be consider as flawed, we will call such quadratic forms degenerate. Therefore not all quadratic form in $n \geq 3$ variables have a dense image. Nevertheless, we can exclude this kind of obstruction with the following definition.

Definition 1.3 (Degenerate Quadratic Form)**.** *A quadratic form Q in n variables is called degenerate if there exists a linear subspace $V \subseteq \mathbb{R}^n$ such that for all $v \in V$ $Q(v) = 0$.*

Let $Q(x, y, z)$ be a quadratic form with integer coefficients. Then for any $x \in \mathbb{Z}^3$, $Q(x)$ is an integer, and $Q(\mathbb{Z}^3)$ is not dense in $\mathbb{R}$.

Now let $Q(x, y, z)$ be a quadratic form whose coefficients are all rationally dependent. This means that there exists $c \in \mathbb{R}$ such that all the coefficients of Q are contained in $c\,\mathbb{Q}$. In this case the quadratic form $\frac{1}{c}Q(x, y, z)$ has rational coefficients. Since $\frac{1}{c}Q$ has a finite number of rational coefficients, at most 6, the least common multiple $q \in \mathbb{Z}$ of the denominators of these coefficients is well defined. Consequentially all the coefficients of the quadratic form $\frac{q}{c}Q(x, y, z)$ are integers. Thus $\sqrt{\frac{q}{c}}Q(\mathbb{Z}^3) \subseteq \mathbb{Z}$ and hence $Q(\mathbb{Z}^3) \subseteq \frac{c}{q}\mathbb{Z}$ thus clearly not dense in $\mathbb{R}$.

Definition 1.4 (Irrational Quadratic Form)**.** *A quadratic form Q in n variables is called irrational if there does not exist a quadratic form Q' with integer coefficients and a $c \in \mathbb{R}$ such that $Q = cQ'$.*

Finally, consider the quadratic form $Q(x, y, z) = ax^2 + by^2 + cz^2$, where $a, b, c \geq 0$. In this case we have that $Q(\mathbb{Z}^3)$ contains no negative values and is therefore clearly not dense in $\mathbb{R}$.

Definition 1.5 (Indefinite Quadratic Form)**.** *A quadratic form Q in n variables is called indefinite if it takes both values smaller and larger than 0.*

1.2 Orbit Closures

Definition 1.6 (Orbit)**.** *Let X be some set and $f : X \to X$.*
We will call the set

$$\mathcal{O}_f^{\mathbb{N}}(x) := \{f^i(x) | i \in \mathbb{N}\},$$

the x-Orbit of $\mathbb{N}$ with regards to f, where $f^0(x) = x$.

If X is a topological space then $\mathcal{O}_f^{\mathbb{N}}(x)$ has a closure. Depending on the map f and the point $x \in X$ orbit closures can look wildly different. To illustrate this diversity of orbits we will give three examples throughout this thesis all of which are maps on S^1.

- Rational rotation R_α (Example 2.9)
- Irrational rotation R_β (Example 2.10)
- Tripling map T (Section 2.2.2)

We will see that the orbits of R_α all have the same finite cardinality, we call orbits of finite cardinality periodic. On the other hand we will see that orbits of R_β are not periodic and that all of them are dense in S^1. Note that both orbits of R_α and R_β all look the same, but this does not hold in general. For instance T is an example of where is not the case. We will see that some orbits of T are periodic, while others are dense in S^1, and others dense in subsets of S^1. In fact we will see that the Cantor set $\mathcal{C}$ is an orbit closure of T.

Since in some cases maps have very regular orbits and other cases orbits can be very to study it makes sense to investigate, which maps give nice orbits and which do not.

Definition 1.7 (Unipotent Map)**.** *We call a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ unipotent if and only if all the eigenvalues of its corresponding matrix are equal to 1.*

In fact Theorems 1.12 and 5.1, will show us that unipotent maps have very predictable orbits. Sometimes we will use a more algebraic definition of unipotency.

Definition 1.8. *Let R be a ring. An element $u \in R$ is called unipotent if there exists $n \in \mathbb{N}$ such that $(u - e)^n = 0$*[3]*.*

1.3 History of Oppenheim's conjecture

In the late 19 century Arnold Meyer proved the following theorem.

Theorem 1.9 (Meyer, [11])**.** [4] *Let Q be an indefinite, rational, non-degenerate quadratic form Q in $n \geq 5$ variables. Then there exists $x \in \mathbb{R}^n - \{0\}$ such that $Q(x) = 0$ if and only if there also exists $z \in \mathbb{Z}^n - \{0\}$ such that $Q(z) = 0$.*

Does Meyer's Theorem still holds if Q is not rational?

The answer is no. Observe that if a quadradic form Q in 5 variables is irrational there cannot exist a non-trivial $z \in \mathbb{Z}^5$ such that $Q(z) = 0$, this is because 0 and z are both rational. Thus Meyer's theorem cannot hold if we drop the requirement of irrationality. However a slightly different question is not as easy to answer. Is 0 an accumulation point of $Q(\mathbb{Z})$?

In a paper published in 1929 ([12]), Oppenheim stated that he was very sure answer is yes, however he did not know how to prove it. Since then this question has been referred to as Oppenheim's conjecture.

Conjecture 1.10 (Oppenheim's original Conjecture)**.** *Let Q be an indefinite, irrational, non-degenerate quadratic form Q in $n \geq 5$ variables. Then for any $\varepsilon > 0$ there exists a $z \in \mathbb{Z}^n - \{0\}$ such that $|Q(z)| < 0$.*

[3] Here e and 0 denote the neutral elements of multiplication and addition respectively.

[4] Note that [11] does not contain a proof of the above theorem. However it states that theorem is correct and that Meyer had proven it in his "Habilitationsschrift", which the author of this thesis was unable to find. This reference was chosen since it was also written by Meyer and because it is used as reference for Meyers theorem in literature for example in [2].

It would remain unproven for 58 years. The reason for this was that traditional methods were not sufficient to prove this conjecture.

The first attempts to prove Oppenheim's conjecture all placed strong restrictions on which quadratic forms were examined. For example, in 1958 Ridout showed that Oppenheims Conjecture holds, when it is assumed that $n \geq 21$ and a stronger form of indefiniteness holds for the quadratic form [14].

However, no advancements were made in disproving Oppenheim's conjecture. In 1946 Davenport stated that Oppenheim's Conjecture might even hold for $n = 3$ [2].

The first step in the homogeneous dynamics direction was done by Cassels and Swinnerton-Dyer in 1955. They showed that proving Oppenheim's Conjecture was equivalent to showing that for a non-degenerate, irrational, indefinite quadratic form Q we have that $\overline{SO(Q)\mathrm{SL}(3,\mathbb{Z})} \subseteq \mathrm{SL}(3,\mathbb{R})$ contains a V_2^+ or V_2^- orbit [1][5]. This equivalence will be explained in Chapter 4.

The establishment of this equivalence allowed for a whole field of mathematics to join the party, namely the field of dynamical systems. In particular now mathematicians started studying the behaviour of $SO(Q)$ orbits in $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$.

Recall that in Examples 2.9 and 2.10, we will see that orbits of rational rotations are periodic and those of irrational rotations are dense in S^1. In the 1930's Hedlund had expanded this concept onto the Torus by proving the following theorem

Theorem 1.11 (Hedlund's Theorem)**.** *Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R})$. Then a unipotent flow on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ is either dense or periodic.*

A proof of a slightly weaker version of this theorem will be given in 3.

One can easily prove that $\mathrm{SO}(Q)$ consists of orbits of unipotent one-parameter subgroups. Thus it was thought that the orbit of any unipotent group on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is either dense or closed. Then by showing that $\mathrm{SO}(Q)$ orbits are closed if and only if Q was the multiple of some rational quadratic form, would prove Oppenheim's conjecture. In 1987 Margulis published [8], in which he finally manged to prove Oppenheim's conjecture, by extending the methods used in the proof of Hedlund's Theorem.

However this was not the end of story, since a much stronger conjecture was believed to hold, an it was proven by Ratner in 1991 (see [13]).

Theorem 1.12 (Ratner's Orbit Closure Theorem)**.** *Let G be a Lie group and let $\Gamma \subseteq G$ be lattice subgroup. Let $H \subseteq G$ be a subgroup generated by one-parameter, unipotent subgroups. Then for any $x \in \Gamma\backslash G$ there exists a closed connected subgroup $U \subseteq S \subseteq G$, such that $\overline{xH} = xS$.*

Consider that this result is quite astonishing, given that, as Section 2.2.2 will point out, we cannot say much about orbit closures in general.

There still remain many open questions regarding orbit closures of subgroups that are not unipotent. In fact in [9], published in the year 2000, Margulis conjectured the following.

Conjecture 1.13 (Conjecture 9 in [9])**.** *Let $n \geq 3$. Let $D \subseteq \mathrm{SL}(n,\mathbb{R})$ be the set of diagonal matrices. Then for any $x \in \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ such that $\overline{Dx}$ is compact, Dx is closed.*

This conjecture is interesting to note, since it being true would imply that Littlewood's conjecture holds true[6].

Conjecture 1.14 (Littlewood)**.** *For any $x \in \mathbb{R}$ let $\|x\| = \min_{z\in\mathbb{Z}} |x - z|$. Then for any $a, b \in \mathbb{R}$ we have that*

$$\liminf_{n\to\infty} n\|na\|\|nb\| = 0.$$

1.4 Notations

We will use the following notation

$$A - B := \{x \in A \mid x \notin B\}.$$

This notation will be used, instead of $A\backslash B$, to avoid confusion with quotient spaces.

Furthermore we will be using e and id to denote the neutral elements of groups.

[5] One should note that they used a somewhat different language and thus in said publication this equivalence is only mentioned implicitly.

[6] This implication is shown in [1].

2 Prerequisites

2.1 Group-Actions and Homogeneous Spaces

Definition 2.1 (Left Group-Action)**.** *Let $(G, *)$ be a group and X some space. Then a left group-action is a map*

$$\begin{aligned} \text{"}\cdot\text{"} \ : \ & G \times X \to X \\ & (g, x) \mapsto g \cdot x, \end{aligned}$$

having the properties

$$\begin{aligned} \textit{Identity: } & e \cdot x = x \\ \textit{Compatibility: } & h \cdot (g \cdot x) = (h * g) \cdot x. \end{aligned}$$

Definition 2.2 (Right Group-Action)**.** *Let $(G, *)$ be a group and X some space. Then a right group-action is a map*

$$\begin{aligned} \text{"}\cdot\text{"} \ : \ & X \times G \to X \\ & (x, g) \mapsto x \cdot g, \end{aligned}$$

having the properties

$$\begin{aligned} \textit{Identity: } & x \cdot e = x \\ \textit{Compatibility: } & (x \cdot g) \cdot h = x \cdot (g * h). \end{aligned}$$

Example 2.1. $(\mathbb{Z}, +)$ acts on $\mathbb{R}$ be the addition on $\mathbb{R}$. With notation used in the definitions above

$$\begin{aligned} \text{"}+\text{"} \ : \ & \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \\ & (g, x) \mapsto g + x, \end{aligned}$$

as a left action, and

$$\begin{aligned} \text{"}+\text{"} \ : \ & \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \\ & (x, g) \mapsto x + g, \end{aligned}$$

as a right action. Note that since addition on $\mathbb{Z}$ is commutative, left and right action coincide.

In general we have that for an action of an abelian group left and right group actions coincide.

Example 2.2. $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^2 - \{0\}$ by matrix multiplication. We then get the following left (resp. right) actions

$$\begin{aligned} L \ : \ & \mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2 \\ & (A, x) \mapsto Ax, \end{aligned}$$

as a left action, and

$$\begin{aligned} R \ : \ & \mathbb{R}^2 \times \mathrm{SL}(2, \mathbb{R}) \to \mathbb{R}^2 \\ & (x, A) \mapsto x^T A, \end{aligned}$$

Note that L and R do not coincide. Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

Then we have that

$$L(B, L(A, x)) = L(BA, x) \quad \text{and} \quad R(R(x, A), B) = R(x, AB)$$

but

$$BA = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus left and right action do not coincide.

Proposition 2.3. *Let G be a group acting on a set X from the left (or right), for any $g \in G$ and $x \in X$ denoted by $g \cdot x$ (resp. $x \cdot g$), then there exists a right (resp. left) action given by $x \cdot g := g^{-1} \cdot x$ (resp. $g \cdot x := x \cdot g^{-1}$).*

Proof. Here we will only show that for any left group there exists a right-action, because the other way uses the same methods. We will denote the left-action by $\cdot_L$ and define, for any $g \in G$ and any $x \in X$, $x \cdot_R g := g \cdot_L x$.

Note that the identity property is for $\cdot_R$ is trivial because $id^{-1} = id$.

For the compatibility property of $\cdot_R$ let $g_1, g_2 \in G$ and $x \in X$ then we have

$$(x \cdot_R g_1) \cdot_R g_2 = g_2^{-1} \cdot_L (g_1^{-1} \cdot_L x).$$

Since we have that by assumption the compatibility property holds for $\cdot_L$

$$g_2^{-1} \cdot_L (g_1^{-1} \cdot_L x) = (g_2^{-1} g_1^{-1}) \cdot_L x = (g_1 g_2)^{-1} \cdot_L x = x \cdot_R g_1 g_2.$$

We have thus shown that $\cdot_R$ is a right-action. □

Remark 2.1. In most literature left-actions are consider more standard than right-actions, therefore often times "left" is dropped, when talking about left-actions.

Note that the mentioning of the handedness of a group-action may also be dropped if the other handedness makes no sense and that this will be done regardless of weather it is a right- or left-action.

Definition 2.4 (Fundamental Domain)**.** *Let G be a group acting on a set X. Then a subset $F \subseteq X$ is called a fundamental domain of the action of G on X if for any $x \in X$ there exists exactly one $u \in F$ and exactly one $g \in G$ such that $gu = x$.*

Definition 2.5 (Fundamental Region)**.** *Let G be a group acting on a topological space X. Then an open subset $U \subseteq X$ is called a fundamental region if for any $g_1, g_2 \in G$ we have that $g_1 U \cap g_2 U = \varnothing$ and $\bigcup_{g \in G} g\overline{U} = X$.*

Example 2.3. An example for the fundamental domain of the action of $(\mathbb{Z}, +)$ on $\mathbb{R}^2 - \{0\}$ as defined in *Example* 2.1 is the interval $[0, 1) \subseteq \mathbb{R}$. An example for a fundamental region of a group action would be the open interval $(0, 1)$.

The two extreme cases of fundamental domains are when $F = X$ and when for all $x \in X$ we have that $\{x\}$ is a fundamental domain. The case where $F = X$ is rather, because this is just the action of the trivial group i.e. $G = \{e\}$. The case when $\{x\}$ is a fundamental domain is called a transitive group action and turns out to be much more interesting.

Definition 2.6 (Transitive Group-Action)**.** *Let G be a group acting on some space X. Then the action of G on X is called transitive if $\forall x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$.*

Example 2.4. The action of $(\mathbb{Z}, +)$ on $\mathbb{R}^2 - \{0\}$ as defined in *Example* 2.1 is not transitive since there exists no $x \in \mathbb{Z}$ such that $1 + x = \pi$.

Example 2.5. The action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}$ as defined in *Example* 2.2 is transitive. To see this consider that for any two vectors in $x \in \mathbb{R}^2 - \{0\}$ there exists a matrix $A_x \in \mathrm{SL}(2, \mathbb{R})$ such that the first column of A_x is x. Thus $A_x e_1 = x$ and hence $A_x^{-1} x = e_1$. Therefore $A_y A_x^{-1} x = y$, thus showing that the action is transitive.

Definition 2.7 (Homogeneous Space)**.** *Let X be some space. Let G be a group acting on X. Then X is called a homogeneous G-space if the action of G on X is transitive.*

Example 2.6. $\mathbb{R}/\mathbb{Z}$ is a homogeneous space. Since $\mathbb{R}$ acts transitively[7] on $\mathbb{R}/\mathbb{Z}$

Remark 2.2. Any group G is a homogeneous space, because it acts transitively on itself transitively.

The above are very general definitions and will not be of much interest to us in this general form. However if we add structure to G and X we get more interesting group actions.

The first structures we will add to X and G are topologies and hence we want the map given by the group action to be continuous. More formally

[7] Both addition and multiplication give rise to a transitive action of $\mathbb{R}$ on $\mathbb{R}/\mathbb{Z}$

Definition 2.8 (Continuous Group-Action)**.** *Let G be a topological group* [8] *acting on topological space X. Then we call the action of G on X continuous if the map $(g,x) \mapsto g \cdot x$ is continuous, with regards to product topology on $G \times X$.*

If we have an other topological group H and a continuous group-homomorphism $f : H \to G$, then we can define a continuous action of H on X, where $h \cdot_H x := f(h) \cdot_G x$ defines the action of H on X [9].

Lemma 2.9. *Let a group G, which acts transitively and continuously on some space X. Let H be the stabilizer of some $x_0 \in X$. Then G/H is homeomorphic to X*

Proof. Let us first define $\varphi : G \to X;\ g \mapsto gx_0$ and let $\bar{\varphi} : G/H \to X;\ gH \mapsto gHx_0$. Note that φ is continuous because the group action is continuous. Also observe that $\bar{\varphi}$ is well defined, since H is the stabilizer of x_0. More over we have that $\bar{\varphi}$ is surjective because G acts transitively on X. Finally if for some $g_1 = g_2 \in G$ we have that $\varphi(g_1) = \varphi(g_2)$ then we have that $g_1x_0 = g_2x_0$ and thus $g_2^{-1}g_1x_0 = x_0$, which implies that $g_2^{-1}g_1 \in H$ thus $g_2^{-1}g_1H = H$, which gives us $g_2H = g_1H$ i.e. $\bar{\phi}$ is injective. Therefore $\bar{\phi}$ is bijective and continuous thus X and G/H are homeomorphic. □

Now we will now add the structure of a differentiable manifold to our requirements for G and X thus making them a Lie-group and differentiable manifold respectively.

Definition 2.10 (Lie-Group Action)**.** *Let G be a Lie-group acting on a smooth manifold M by the action $\sigma : (g,x) \mapsto g \cdot x$. Then we call the action σ smooth (or a Lie-group action) if the map σ is differentiable.*

Note that the action of a Lie-Group G on itself, by the group operation of the underlying topological group, is a Lie-Group action. Now consider $\Gamma \subseteq G$ a discrete subgroup of G and the action of G on G/Γ. Note that this action is transitive thus G acting on G/Γ gives us a homogenous space. It will be spaces of this form, which will our main interest throughout this thesis. In particular we will be interested for which subgroups of $H \subseteq G$ and which $x \in G/\Gamma$ the sets Hx are dense in G/Γ.

2.1.1 Convergence in Homogeneous Spaces

Let again G be a Lie-Group and Γ a discrete subgroup of G. In this section we will derive the notions of distance and convergence in the space $X := \Gamma\backslash G$ from those on G.

As topology on $\Gamma\backslash G$ we will simply be using the quotient topology i.e. the coarsest topology such that the quotient map

$$\begin{aligned} q : G &\to \Gamma\backslash G \\ g &\mapsto \Gamma g \end{aligned}$$

is open. Then $\{g_n\}_{n\in\mathbb{N}} \subseteq \Gamma\backslash G$ converges to some $g \in \Gamma\backslash G$ if for any open neighbourhood U of g there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $g_n \in U$.

Remark 2.3. The action of G on $\Gamma\backslash G$ is continuous with regards to this topology.

In order to prove density it would be nice to rely on a metric, instead of relying solely on topological arguments.

Note that since G is a Lie-Group it has the structure of a smooth manifold and thus there exist multiple Riemannian metrics on G.

Proposition 2.11. *On any Lie-Group G there exists a G-invariant Riemannian metric d_G* [10].

[8] A topological group is a group equipped with a Hausdorff topology. Recall that a topological space (X,τ) is Hausdorff if for all $x,y \in X$ there exists $U \in \tau$ such that $x \in U$ and $y \notin U$.

[9] Here "$\cdot_G$" denotes the the action of G on X.

[10] G-invariant with regards to a metric means that $\forall g,x,y \in G$ we have that $d_G(x,y) = d_G(gx,gy)$.

Proof. Let $\langle \bullet, \bullet \rangle$ be an inner product on T_eG, where e is the identity on G. Now consider for some $a \in G$ the function

$$\begin{aligned} L_a : G &\to G \\ g &\mapsto a \cdot g, \end{aligned}$$

where "$\cdot$" is the left action of G on itself. Note that since G is a Lie-Group L_a is differentiable. We can now define an inner product $\langle \bullet, \bullet \rangle_a$ on T_aG for every $a \in G$ as follows

$$\langle dL_a u, dL_a v \rangle_a := \langle u, v \rangle,$$

for all $u, v \in T_eG$. Or equivalently

$$\langle u, v \rangle_a := \langle dL_{a^{-1}} u, dL_{a^{-1}} v \rangle,$$

for all $u, v \in T_aG$. This then gives a Riemannian metric d_G invariant by left-action of G. □

We will now proceed to constructing a G-invariant metric d_X on $X = \Gamma\backslash G$ from d_G. Let

$$d_X(\Gamma a, \Gamma b) = \inf_{\alpha,\beta \in \Gamma} d_G(\alpha a, \beta b),$$

for any $\Gamma a, \Gamma b \in X$.

A metric enables us to define open balls. These open balls then form a basis of a topology on X.

We can now study convergence on X consider the sequence $\{\Gamma g_n\}_{n\in\mathbb{N}} \subseteq X$ this sequence is convergent if $\lim_{n\to\infty} d_X(\Gamma g_n, \Gamma g) = 0$ for some $g\Gamma \in X$, hence there exists sequences $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \subseteq \Gamma$ such that

$$0 = \lim_{n\to\infty} d_G(\alpha_n g_n, \beta_n g) = \lim_{n\to\infty} d_G(\beta_n^{-1}\alpha_n g_n, g) := \lim_{n\to\infty} d_G(\gamma_n g_n, g).$$

2.1.2 Invariant Measures on Homogeneous Spaces

We will now construct a G-invariant measure on $\Gamma\backslash G$. Recall that on any Lie group G there must exist a left Haar measure i.e. a measure invariant under left action of G on its self. The following proposition will use the existence of a Haar measure on G to define a G-invariant measure on $\Gamma\backslash G$.

Proposition 2.12. *Let G be a Lie-Group and let μ_G be the Haar-meausure on G. Moreover let Γ be a discrete subgroup of G. Then there exists a G-invariant probability measure μ_X on $X := \Gamma\backslash G$ if and only if there exists F a fundamental domain of Γ in G such that $\mu_G(F) < \infty$*

Remark 2.4. Proof and statement of Proposition 2.12 are based on Proposition 4.6. in [3] Proposition 4.6.

Proof. First we show that if μ_X exist then then $\mu_G(F)$ must be finite. Let π be the projection of G to $\Gamma\backslash G$. Then for any measurable set $U \subset G$ we define

$$\mu(U) := \int_X \#(\pi^{-1}(x) \cap U)\, d\mu_X(x).$$

Note that μ is G invariant, because for any $g \in G$

$$\begin{aligned} \mu(Ug) &= \int_X \#(\pi^{-1}(x) \cap Ug)\, d\mu_X(x) \\ &= \int_X \#(\pi^{-1}(xg) \cap Ug)\, d\mu_X(x) \\ &= \int_X \#(\pi^{-1}(x) \cap U)g\, d\mu_X(x) \\ &= \int_X \#(\pi^{-1}(x) \cap U)\, d\mu_X(x) = \mu(U). \end{aligned}$$

We will show that μ is finite on any compact set $K \subset G$. For this notice that because K is compact and Γ is discrete for any $g \in G$ we have that $\#(K \cap \Gamma g) < \infty$. Thus we have

$$\begin{aligned}\mu(K) &= \int_X \#(\pi^{-1}(x) \cap K)\, d\mu_X(x) \\ &= \int_X \left(\sum_{\gamma \in \Gamma} \mathbb{1}_K(\gamma(\pi^{-1}(x) \cap F)) \right) d\mu_X(x) \\ &\leq \int_X \max_{g \in G} \#(K \cap \Gamma g)\, d\mu_X(x) \\ &= \max_{g \in G} \#(K \cap \Gamma g)\mu_X(X) = \max_{g \in G} \#(K \cap \Gamma g) < \infty.\end{aligned}$$

Thus by uniqueness of the Haar-mesure we have that $\mu = \mu_G$. Now note that

$$\mu_G(F) = \int_X \#(\pi^{-1}(x) \cap F) d\mu_X(x) = \mu(X) = 1 < \infty.$$

We will now show the other direction. Therefore let us assume $\mu_G(F) = 1$, this is true up to multiplication by a constant. Furthermore we will define a new μ this time μ will be a function from X to $\mathbb{R}_{\geq 0}$ given by

$$\mu(U) = \mu_G(\pi^{-1}(U) \cap F).$$

Now note

$$\mu(X) = \mu_G(G \cap F) = \mu_G(F) = 1$$

hence μ is a probability measure on X. Moreover for any $g \in G$

$$\mu(Ug) = \mu_G(\pi^{-1}(U)g \cap F) = \mu_G(\pi^{-1}(U)g \cap Fg) = \mu_G(\pi^{-1}(U) \cap F) = \mu(U).$$

Therefore μ is also G-invariant. Thus μ has the properties of μ_X. □

Proposition 2.12 shows us that any discrete subgroup $\Gamma \subseteq G$, whose fundamental domains have finite measure, there exists a finite G-invariant measure on $\Gamma \backslash G$. We will call such subgroups Γ lattices .

Definition 2.13 (Lattice)**.** *A discrete subgroup $\Gamma \subseteq G$ is called a lattice if there exists a fundamental domain F of Γ in G such that $\mu_G(F) < \infty$.*

Example 2.7. $\mathbb{Z}$ is a Lattice in $\mathbb{R}$. Because the fundamental domain $[0,1)$ has finite measure i.e. $\lambda([0,1)) = 1$

Example 2.8. The subgroup $\Lambda := \{(z,0) \mid z \in \mathbb{Z}\}$ is not a lattice in $\mathbb{R}^2$. To see this note that connected fundamental domains of Λ are of the form $F := [a, a+1) \times \mathbb{R}$ and that $\lambda(F) = \infty$. See Figure 1.

Definition 2.14 (Uniform Lattice)**.** *A lattice $\Gamma \subseteq G$ is called uniform or cocompact if the closure of its Fundamental domains are compact.*

Note that the lattice of *Example* 2.7 is uniform. In general any lattice in $\mathbb{R}^n$ must be uniform. This is due to the fact that any compact set in $\mathbb{R}^n$ must be bounded, by Heine-Borel theorem, and there fore it has finite lebesgue measure.

For an example of a non-uniform Lattice see *Example* 3.1.

Remark 2.5. Lattices will be discussed further in Section 5.1.

2.2 Basic Notions of Dynamical Systems

Let X be a set and let $f : X \to X$ be a map from X to itself. Now consider some $x \in X$ and let $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$ and so on, as seen in Figure 2.

Notation 2.15. *We will sometimes write $f : X \circlearrowleft$, instead of $f : X \to X$.*

Notation 2.16. *When studying the behavior of such f^n for a function $f : X \circlearrowleft$, we will call the tuple (X, f) a dynamical system.*

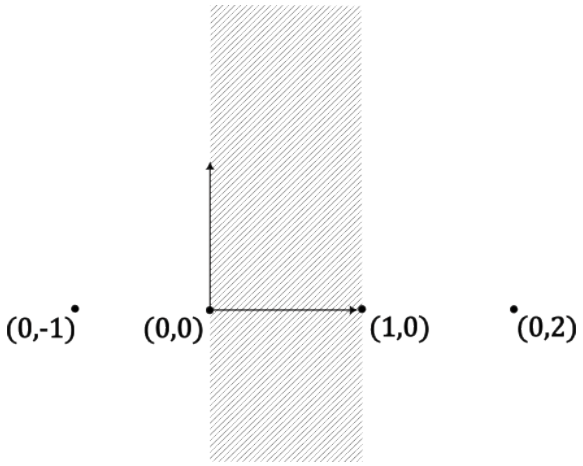


Figure 1: This figure gives a visualisation of the discrete subgroup $\Lambda := \{(z, 0) \mid z \in \mathbb{Z}\}$ in $\mathbb{R}^2$. Note that the marked area is the set F a fundamental domain of Λ.

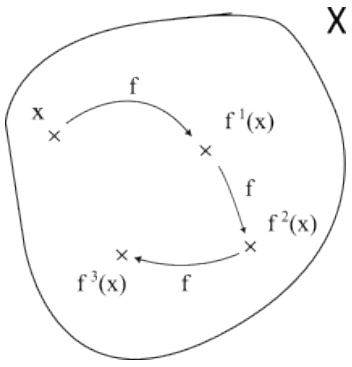


Figure 2: Illustration of an Orbit.

2.2.1 Orbits

Recall the definition of an orbit (Definition 1.6). Let X be some set and $f : X \circlearrowleft$. Now note that if $|\mathcal{O}_f^{\mathbb{N}}(x)| < \infty$, then we have that the sequence $\{f^i(x)\}_{i\in\mathbb{N}}$ is pre-periodic or periodic [11].

Definition 2.17 (Periodic-Orbits)**.** *If the sequence $\{f^i(x)\}_{i\in\mathbb{N}}$ is periodic (resp. pre-periodic) we call the orbit $\mathcal{O}_f^{\mathbb{N}}(x)$ periodic (resp. pre-periodic).*[12]

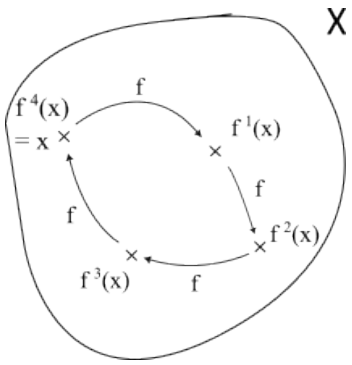


Figure 3: Illustration of an Orbit of period 4.

Example 2.9 (Rational Rotation). Consider the map

$$R_\alpha : S^1 \to S^1$$
$$x \mapsto x + \alpha \mod 1,$$

where $\alpha \in (0, 1)$. We will call this map a rotation by α (illustrated by Figure: 4). Now Consider

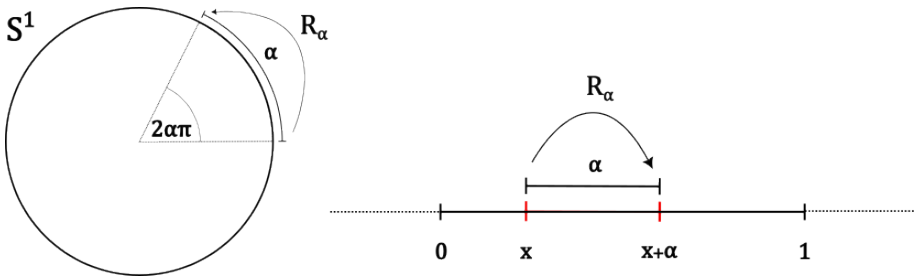


Figure 4: Rotation by α, where on the left-hand side S^1 is displayed as a circle and on the right-hand side as an interval.

the rotation by $\frac{1}{4}$. Note that $\forall x \in S^1$, we have that $R_{\frac{1}{4}}^4(x) = x$. Thus every Orbit of $R_{\frac{\pi}{2}}$ has period 4 (Illustrated by Figure 2.9). Thus the orbit closure of any point in $x \in S^1$ is $\mathcal{O}_{R_\alpha}^{\mathbb{N}}(x)$ its self, which is a set consisting of 4 points.

Note we can generalize this to any $\alpha := \frac{a}{b}$, where are $a, b \in \mathbb{N}$ such that $gcd(a, b) = 1$. Then every orbit has period b.

If $|X| \geq \infty$ then periodicity of an orbit can viewed as an extraordinary behaviour. In some sense periodic orbits are the smallest orbits possible. The other extreme one might consider, i.e. the largest possible orbits, is the case, when $|X| = |\mathcal{O}_f^{\mathbb{N}}(x)|$. Note that because $|\mathcal{O}_f^{\mathbb{N}}(x)|$ is a set indexed by $\mathbb{N}$ it has to be countable. Thus if $|X|$ is uncountable we have that $|X| > |\mathcal{O}_f^{\mathbb{N}}(x)|$, in this case it is in general hard to find a largest possible orbit. If we have that X is a topological space, then we can view orbits, which are dense in X with regards to that topology, as the "largest" possible orbits.

[11] Pre-periodic means that there exist $n, p \in \mathbb{R}$ such that for all $m \geq n$, we have that $f^m(x) = f^{m+p}(x)$, where p the smallest such p is called the period of $\{f^i(x)\}_{i\in\mathbb{N}}$. If $n = 0$ then we call the sequence $\{f^i(x)\}_{i\in\mathbb{N}}$ periodic.

[12] An illustration of a periodic orbit is provided in Figure 3.

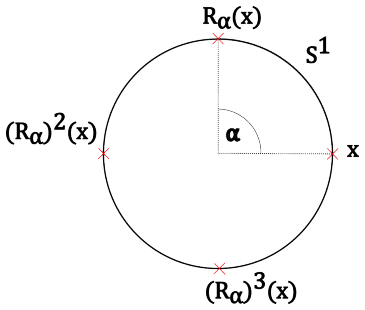


Figure 5: Periodic R_α orbit of some $x \in S^1$, with $\alpha = \frac{1}{4}$.

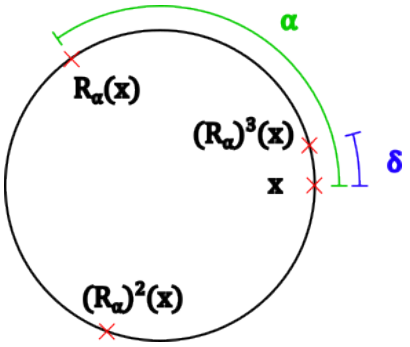


Figure 6: In this example $R_\alpha^3 = R_\delta$. Now note that the four points marked above divide S^1 into four segments, thus $\delta \leq \frac{1}{4}$.

Definition 2.18 (Dense Orbit)**.** *Let X be topological space and f be a map from X to itself. If there exists $x \in X$ such that $\mathcal{O}_f^{\mathbb{N}}(x)$ is dense in X, then we say f is topologically transitive.*

Example 2.10 (Irrational Rotation). Consider the rotation R_α, where $\alpha \in (\mathbb{R} - \mathbb{Q})$.

Claim 2.19. *For any $x \in S^1$ the set $\mathcal{O}_{R_\alpha}(x)$ is dense in S^1.*

Proof. Fix some $x \in S^1$ and let $\mathcal{O}_{R_\alpha}^n(x) := \{R_\alpha^j(x) \mid \forall j \in \{0, 1, ..., n\}\}$, for any $n \in \mathbb{N}$. We now want to show that for any open interval $I \subset S^1$ there exists $n \in \mathbb{N}$ such that $I \cap \mathcal{O}_{R_\alpha}^n(x) \neq \varnothing$.

Let $\varepsilon = |I| > 0$ be the length of the interval. Now pick some $N_1 > \frac{1}{\varepsilon}$. Then the set $\mathcal{O}_{R_\alpha}^{N_1}(x)$ contains $N_1 + 1$ distinct points, hence by the pigeonhole-principal there must exist two points $R_\alpha^{j_1}(x), R_\alpha^{j_2}(x) \in \mathcal{O}_{R_\alpha}^{N_1}(x)$ such that $|R_\alpha^{j_1}(x) - R_\alpha^{j_2}(x)| \leq \frac{1}{N_1} = \varepsilon$. Thus $R_\alpha^{|j_1 - j_2|}$ is a rotation by $\delta \leq \varepsilon$ (This is illustrated by Figure 6). Now let $N_2 > \frac{1}{\delta}$, then $\mathcal{O}_{R_\delta}^{N_2}(x)$ is a set of equally spaced points of distance δ of each other. Hence for any $y \in S^1$ we have that $d(y\mathcal{O}_{R_\delta}^{N_2}(x)) \leq \frac{\delta}{2} \leq \frac{\varepsilon}{2}$. Now consider that this also holds for m the centre-point of I, i.e. m such that $B(\frac{\varepsilon}{2}, m) = I$.

Thus there must exist a point in $\mathcal{O}_{R_\delta}^{N_2}(x)$ that is also in I. Now note that $\mathcal{O}_{R_\delta}^{N_2}(x) \subseteq \mathcal{O}_{R_\alpha}^{N_1 N_2}(x) \subseteq \mathcal{O}_{R_\alpha}(x)$. Hence $\varnothing \neq I \cap \mathcal{O}_{R_\delta}^{N_2}(x) \subseteq I \cap \mathcal{O}_{R_\alpha}(x)$, which was what we had to show. □

Thus the orbit closure of any point in $x \in S^1$ is S^1 itself.

After having defined these nice properties of $\mathcal{O}_f^{\mathbb{N}}(x)$ for a single $x \in X$ a natural thing to study is the orbit of multiple points. Therefore let $A \subseteq X$ then we define

$$\mathcal{O}_f^{\mathbb{N}}(A) := \bigcup_{x \in A} \mathcal{O}_f^{\mathbb{N}}(x) = \bigcup_{i \in \mathbb{N}} f^i(A)$$

as the orbit of a set. Similar to, what we defined for orbits of points, we may also define periodic and dense orbits of a set A:

Definition 2.20. *If there exists an $i \in \mathbb{N}$ such that $f^i(A) = A$ then we call $\mathcal{O}_f^{\mathbb{N}}(A)$ periodic. Moreover we call the smallest such i the period of $\mathcal{O}_f^{\mathbb{N}}(A)$.*

Moreover if the period of $\mathcal{O}_f^{\mathbb{N}}(A)$ is one, i.e. $f(A) = A$, we say that f is A-invariant.

Now assume $\mathcal{O}_f^{\mathbb{N}}(A)$ is a periodic orbit of some $A \subseteq X$, which is not dense in X. Then there cannot exist any $x \in \mathcal{O}_f^{\mathbb{N}}(A)$, that is dense in X. However x might be dense in $\mathcal{O}_f^{\mathbb{N}}(A)$. A particularly nice behaviour is when for every $x \in \mathcal{O}_f^{\mathbb{N}}(A)$ we have that $\mathcal{O}_f^{\mathbb{N}}(x)$ is dense in $\mathcal{O}_f^{\mathbb{N}}(A)$.

Definition 2.21 (Minimal Set)**.** *Let $f : X \circlearrowleft$. Then we call $A \subseteq X$ f-minimal set if for any $x \in A$ we have that $\overline{\mathcal{O}_f^{\mathbb{N}}(x)} = A$.*

It is not a coincidence that these sets are called minimal, in fact we have that the following proposition holds

Proposition 2.22. *Let $B \subseteq X$ be closed and let f be a continuous map from X to itself. Then B is a closed minimal f-invariant [13] subset of X if and only if for any $b \in B$ we have that $\overline{\mathcal{O}_f^{\mathbb{N}}(b)} = B$.*

Proof. First we assume that $B \subseteq X$ and $\forall b \in B$ we have that $\overline{\mathcal{O}_f^{\mathbb{N}}(b)} = B$. Now consider a set $A \subseteq B \subseteq X$ such that $f(A) = A$ then $\forall x \in A$ $\mathcal{O}_f^{\mathbb{N}}(x) \subseteq A$ that since $x \in B$ this implies that A is dense in B. Now since A is closed we must have that $A = B$.

Now we assume that for some closed $B \subseteq X$, we have that $A = B$, for all closed $A \subseteq B \subseteq X$, such that $f(A) = A$. We will now show by contradiction that $\forall b \in B$ we have that $\overline{\mathcal{O}_f^{\mathbb{N}}(b)} = B$. We will therefore assume $\exists b \in B$ such that $\overline{\mathcal{O}_f^{\mathbb{N}}(b)} \neq B$. Note that $\mathcal{O}_f^{\mathbb{N}}(b)$ is f-invariant and hence continuity of f gives us that $\overline{\mathcal{O}_f^{\mathbb{N}}(b)}$ is f-invariant as well. Which gives us the desired contradiction. □

2.2.2 The Tripling Map

The purpose of this section is to given an example of just how complicated orbits can get.

Consider the map $T : [0,1) \to [0,1)$; $x \mapsto 3x \bmod 1$, called the tripling map. At first glance this the orbits of T might be quite hard to study. However when we study T on the 3-adic numbers this becomes a lot simpler.

Note that we can write any $x \in [0,1)$ as a sum of powers of $\frac{1}{3}$ i.e. there exist $x_i \in 0,1,2$ such that $x = \sum_{n\in\mathbb{N}} \frac{x_n}{3^n}$. Consider the map $\varphi : [0,1) \to \{0,1,2\}^{\mathbb{N}}$; $x = \sum_{n\in\mathbb{N}} \frac{x_n}{3^n} \mapsto (x_1, x_2, x_3, ...)$.

Definition 2.23. *For any $x \in [0,1)$ we call $\phi(x)$ the 3-adic number corresponding to x.*

Now let us study T on the 3-adic numbers. Let σ be the T on the 3-adic numbers i.e. let the following diagramm commute.

$$\begin{array}{ccc} [0,1) & \xrightarrow{\quad T \quad} & [0,1) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \{0,1,2\}^{\mathbb{N}} & \xrightarrow{\quad \sigma \quad} & \{0,1,2\}^{\mathbb{N}} \end{array} .$$

Note that for $x = \sum_{n\in\mathbb{N}} \frac{x_n}{3^n} \in [0,1)$, we have that

$$T(x) = \sum_{n\in\mathbb{N}} \frac{x_n}{3^{n-1}} \bmod 1 = x_1 \sum_{n\in\mathbb{N}} \frac{x_{n+1}}{3^n} \bmod 1 = \sum_{n\in\mathbb{N}^+} \frac{x_{n+1}}{3^n}$$

Thus we can conclude that $\sigma(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...)$. Now consider that the set $\mathcal{C} = \{0,2\}^{\mathbb{N}} \subseteq \{0,1,2\}^{\mathbb{N}}$ is preserved by σ i.e. for any $x \in \{0,2\}^{\mathbb{N}}$ we have that $\sigma(x) \in \{0,2\}^{\mathbb{N}}$ as well.

Observe that the inverse of φ is given by $\varphi' : \{0,1,2\}^{\mathbb{N}} \to [0,1); (x_1, x_2, x_3, ...) \mapsto \sum_{n\in\mathbb{N}} \frac{x_n}{3^n}$. Also note that if we define the distance

$$d((x_1, x_2, x_3, ...), (y_1, y_2, y_3, ...)) := d_e(\varphi'(x_1, x_2, x_3, ...), \varphi'(y_1, y_2, y_3, ...)) = \left| \sum_{n\in\mathbb{N}} \frac{x_n - y_n}{3^n} \right|,$$

Figure 7: This image demonstrates the idea of the Cantor set. Each line resembles an interval and the collection of intervals on the same horizontal correspond to one set $\mathcal{C}_n$.

where d_e is the euclidean distance on $[0,1)$. Then φ and φ' are isometries w.r.t. the distances d and d_e.

We will proceed to studying how $\varphi'(\mathcal{C})$ looks. For this let us first study where φ' maps a generic $(x_1, x_2, x_3, ...) \in \{0,1,2\}^{\mathbb{N}}$ to. We first focus on the first element x_1. Note that since $x = \sum_{n\in\mathbb{N}} \frac{x_n}{3^n}$ and $\sum_{2\leq n} \frac{x_n}{3^n} \leq \frac{1}{3}$ we have that

$$\varphi'(x_1, x_2, x_3, ...) \in [\frac{x_1}{3}, \frac{x_1+1}{3}).$$

When considering what the other elements of the sequence mean it gets a bit more complicated. For instance considering the second element x_2 we have that

$$\varphi(T(\varphi'(x_1, x_2, x_3, ...))) = \sigma(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...)$$

and that

$$\varphi'(x_2, x_3, x_4, ...) \in [\frac{x_2}{3}, \frac{x_2+1}{3}),$$

and thus

$$\varphi'(x_1, x_2, x_3, ...) \in [\frac{x_1}{3}, \frac{x_1+1}{3}) \cap T^{-1}([\frac{x_2}{3}, \frac{x_2+1}{3})).$$

Note that

$$T^{-1}([\frac{x_2}{3}, \frac{x_2+1}{3})) = [\frac{x_2}{3^2}, \frac{x_2+1}{3^2}) \cup [\frac{x_2+3}{3^2}, \frac{x_2+4}{3^2}) \cup [\frac{x_2+6}{3^2}, \frac{x_2+7}{3^2}).$$

We can now generalize this to any $n \in \mathbb{N}$ giving us

$$T^{-n+1}(x_n, x_{n+1}, ...) \in \mathcal{S}^n(x_n) := \bigcup_{0\leq j\leq 3^{n-1}} [\frac{x_n+3j}{3^n}, \frac{x_n+3j+1}{3^n}).$$

Therefore $\varphi'(\mathcal{C}) = \bigcap_{n\in\mathbb{N}} \mathcal{S}^n(0) \cup \mathcal{S}^n(2)$. This set is known as the Cantor set. It is well known that the Cantor set is uncountable, closed and nowhere dense in $[0,1)$. Moreover we have just shown that the Cantor set is a T-invariant set. The question which we will now investigate is, whether $\mathcal{C}$ is an Orbit closure or not.

Consider that the set $\{(0,2,0,2,...),(2,0,2,0,...)\}$ is the σ-orbit of the point $(0,2,0,2,...) \in \{0,2\}^{\mathbb{N}}$. But this orbit is clearly not dense in $\mathcal{C}$. Thus certainly not every point in $\mathcal{C}$ has an orbit that is dense in $\mathcal{C}$. In general terms orbits of periodic sequences will not be dense in $\mathcal{C}$.

But are orbits of non-periodic sequences dense in $\mathcal{C}$? The answer is not necessarily consider for instance the sequence $(0,2,0,0,2,0,0,0,2,...)$ the orbit of this sequence will never be closer than $\frac{1}{9}$ to a sequence starting with $(2,2,...)$, yet this sequence is not periodic.

However despite this there are still sequences in $\mathcal{C}$, whose orbit is dense in $\mathcal{C}$. In the following paragraphs such a sequence will be constructed.

We will start by investigating $\{0,2\}$ and $\mathcal{C}$ further. Notice that that $\mathcal{S}^1(0) \cup \mathcal{S}^1(2)$ consists of 2 intervals, and $(\mathcal{S}^1(0) \cup \mathcal{S}^1(2)) \cap (\mathcal{S}^2(0) \cup \mathcal{S}^2(2))$ has 4. In general we have that $S^{n+1}(x)$ contains one interval in each interval of $S^n(y)$. In our case we have that $\mathcal{S}^{n+1}(0) \cup \mathcal{S}^{n+1}(2)$ contains exactly two intervals in every interval of

$$\mathcal{C}_n := \bigcap_{1\leq i\leq n} \mathcal{S}^i(0) \cup \mathcal{S}^i(2).$$

[13] Minimal here means that $\forall A \subseteq B \subseteq X$ closed such that $f(A) = A$ we have that $A = B$.

Thus $\mathcal{C}_{n+1}$ consists of twice as many intervals as $\mathcal{C}_n$. We can thus conclude that $\mathcal{C}_n$ consists of 2^n intervals, each of which has length $\frac{1}{3^n}$. More over note in such an interval of $I \subseteq \mathcal{C}_{n+1}$ we have that any two points $y, x \in I$ begin with the same $n+1$ entries.

Let $(x_1, ..., x_n) \in \{0,2\}^n$ and $(x_1, ..., x_k) \in \{0,2\}^k$ then we define concatenation as follows

$$(x_1, ..., x_n) \circ (y_1, ..., y_k) := (x_1, ..., x_n, y_1, ..., y_k) \in \{0,2\}^{n+k}.$$

Let $C_n(x) := \{(x_1, x_2, ...) \in \{0,1,2\}^{\mathbb{N}} \mid x_n = x\}$ and let for any $K \subseteq \mathbb{N}$ and any $(x_k)_{k\in K} \in \{0,1,2\}^{|K|}$ we will define

$$C_K((x_k)_{k\in K}) := \bigcap_{k\in K} C_k(x_k).$$

These sets are called cylinders and are frequently used when working with p-adic numbers.

Consider that for any $(x_1, x_2, ...) \in C_{\{1,2\}}(0,2)$ the T-orbit of any $\varphi'(x_1, x_2, ...)$, intersects both intervals in $\mathcal{C}_1$. In general if we set $\{x_i\}_{1\leq i\leq 2^n} = \{0,2\}^n$ we can say that the σ-orbit of a sequence starting with $y_n := x_1 \circ x_2 \circ ...x_{2^n}$ intersects every interval of $\mathcal{C}_n$. We will now recursively define a sequence of sequences as

$$z_1 = y_1$$
$$z_n = z_{n-1} \circ y_n.$$

Note that the limit of z_n as $n \to \infty$ must have a σ-orbit dense on $\mathcal{C}$.

This shows, contrary to what Examples 2.9 and 2.10 might have indicated, that different orbits of the same map can look radically different.

Note that using the same method on $\{0,1,2\}^{\mathbb{N}}$, instead of just $\{0,2\}^{\mathbb{N}}$, yields sequences $(x_1, x_2, ...) \in \{0,1,2\}^{\mathbb{N}}$, whose φ-orbits are dense in the whole of S^1.

2.2.3 Dynamical Systems as Group actions

The goal of this section will be to understand how a dynamical system given by an invertible function f on a space X can be viewed as a group action of $(\mathbb{Z}, +)$ on X. Furthermore we will then use this to generalize the definition of a dynamical system to an action of some group G on some space X.

Consider the map

$$\sigma_f : \mathbb{Z} \times X \longrightarrow X$$
$$(n, x) \longmapsto f^n(x).$$

Note that this is indeed an action of $(\mathbb{Z}, +)$ on X. This is because $\sigma_f(0, x) = f^0(x) = x$ and $\sigma_f(n, \sigma_f(m, x)) = \sigma_f(n, f^m(x)) = f^{n+m}(x)$.

We now want to generalize $(\mathbb{Z}, +)$ to a general group G. First we realize that $(\mathbb{Z}, +)$ is the group we use for counting and thus just counting the number of timmes we concatenate f is a very natural way of defining a group action. However for a general G we do not have such a simple intuition.

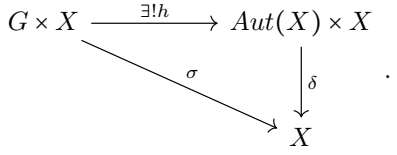
To this end we will introduce $Aut(X)$ the group of automorphisms on X. $Aut(X)$ is a set of functions on X where: "Auto-" means that the functions got from X to itself and "-morphism" means that the maps respect the structure on X, which structure should be respected is deliberately left open here, since it depends on what structures X has and on what properties we want our resulting group action have. For instance in the most general case where X is just some set $Aut(X)$ are all the functions from X to itself, if X is a topological space and G topological group and we might be interested in finding a continuous group-action on X, we will want $Aut(X)$ to be the continuous maps from X to itself.

Now consider that the $Aut(X)$ acts on X by the map

$$\delta : Aut(X) \times X \to X$$
$$(f, x) \to f(x).$$

δ indeed gives us a group-action, because $\delta(id_X, x) = id_X(x) = x$ and $\delta(g, \delta(f, x)) = \delta(g, f(x)) = g \circ f(x)$. More over note that all group-actions can described by elements of $Aut(X)$

Now recall that in Section 2.1 we had that, since $Aut(X)$ acts on X, if there exists a group-homomorphism $h : G \to Aut(X)$, then there exist an action of G on X. We can view this an universal property for any action σ of some G on X we have that there exists a unique homomorphism h such that the following diagram commutes:



For σ_f, as defined above we indeed have such a homomorphism[14]

$$h_f : \mathbb{Z} \to Aut(X)$$
$$n \mapsto f^n.$$

We can now generalize our definition of an orbit

Definition 2.24. *Let G be a group acting on some space X by the action "$\cdot$". Then the orbit of the action $x \in X$ is the set*

$$\mathcal{O}^G(x) := \{g \cdot x \mid \forall g \in G\}.$$

Sometimes we will also use the notation $G \cdot x$ for $\mathcal{O}^G(x)$.

Moreover we can also extend the definition of a dynamical system to the definition.

Definition 2.25 (Dynamical System)**.** *Let G be a group acting on some space X. Then we call the pair (G, X) a dynamical system.*

2.2.4 The Space of Orbits

Assume we have a group G which acts non-transitively on a space some X by the action "$\cdot$", i.e. there exist $x, y \in X$ such that there does not exist any $g \in G$ having the property that $g \cdot x = y$. Note that we have $\forall y \in \mathcal{O}^G(x)$; $\mathcal{O}^G(x) = \mathcal{O}^G(y)$. Now the questions arise how many different G orbits do there exist in X, how do these different orbits behave and so on. To study these questions we will now introduce the space of orbits.

Now consider the equivalence relation "$\sim$" on X, where $x \sim y$ if and only if $y \in \mathcal{O}^G(x)$. Now note that then every element $X/\sim$ is a G-orbit in X. We will call this space the space of orbits. Now in general studying $X/\sim$, instead of just studying the orbits of G on X directly, might not be that insightful, however in some cases doing this may greatly improve our understanding of an action. For instance let H be some group and $L \subseteq H$ some subgroup of H. Now consider the action of L on H given by the Group operation on G. In this case the space of orbits is the quotient space $L\backslash H$.

2.2.5 Ergodicity

Definition 2.26 (Invariant Measure)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let G be a group acting on X in such a way that for any $g \in G$ the function $x \mapsto g \cdot x$ is $\mathcal{A}$-measurable.*

Then G is said to be μ-invariant[15] *when for all $A \in \mathcal{A}$ we have that $\mu(g \cdot A) = \mu(A)$ for any $g \in G$.*

Remark 2.6. Invariant measures need not exist! However recall that in Section 2.1.2 we showed that if G is a Lie group and $\Gamma \subseteq G$ is a lattice then there exists a G invariant probability measure on $\Gamma\backslash G$.

Note that the sum of two G-invariant measures μ and ν is again G-invariant and also note that $Supp(\mu) \cup Supp(\nu) = Supp(\mu + \nu)$. Thus the question arises if invariant measures with non-trivial, minimal support exist[16].

[14] It is a homomorphism because $h_f(n+m) = f^{n+m} = f^n \circ f^m = h_f(n) \circ h_f(m)$.

[15] We can equivalently say that μ is G-invariant.

[16] Here μ having non-trivial support, means that $\mu(X) > 0$. And minimal support means that $\forall$ G-invariant measures ν s.t. $Supp(\nu) \subseteq Supp(\mu)$ we have that $\mu = \nu$.

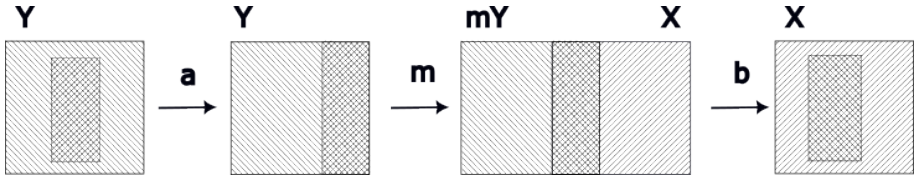


Figure 8: Example of a map in AMB, where $a \in A$, $m \in M$ and $b \in B$.

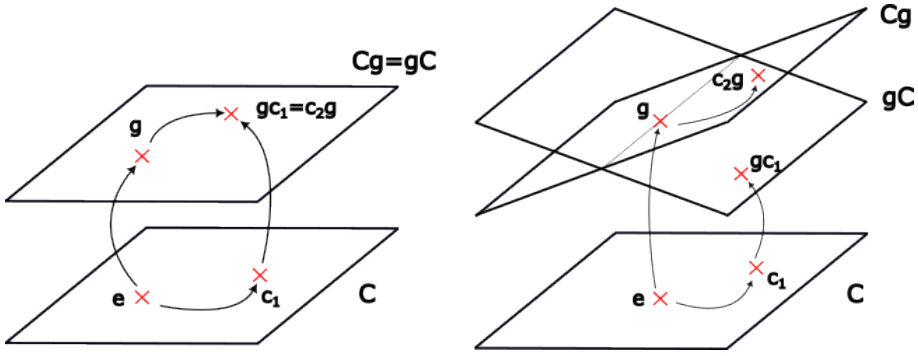


Figure 9: On the left-hand side $g \in N_G(C)$ on the right-hand side $g \notin N_G(C)$

Definition 2.27 (Ergodic Measure)**.** *A finite*[17]*, G-invariant measure μ on $(X, \mathcal{A})$ is called ergodic if for all $A \in \mathcal{A}$, that satisfy $g \cdot A = A$ for every $g \in G$, it must hold that $\mu(A) \in \{0, \mu(X)\}$.*

2.3 Orbit Closures of Subgroups of a Topological Group

The two Lemmas of this section will be used in the proofs of Hedlund's Theorem and Oppenheim's conjecture.

As mentioned in the introduction we will be studying orbit closures in this proof. Will also have to find sets that contain a certain orbit. The following lemma will help us to do that.

Lemma 2.28 ([10] Lemma IV.2.2.)**.** *Let G be a locally compact group acting continuously on a topological space Ω. Let A, B and C be closed subgroups of G, where C is contained in $A \cap B$. Let X and Y be closed subsets of Ω, such that X is invariant under A, and Y is invariant under B, and Y is compact and minimal under the action of C. Furthermore let M be a subset of G with*

$$mY \cap X \neq \varnothing \qquad \forall m \in M.$$

Then

$$gY \subseteq X, \qquad \forall g \in N_G(C) \cap \overline{AMB},$$

where $N_G(C)$ denotes the normalizer of C in G. In particular if $A = B$ and $X = Y$, then

$$gY = Y \qquad \forall g \in N_G(C) \cap \overline{AMA}.$$

Remark 2.7. Lemma 2.28 gives us a description of all the $x \in X$, whose orbit $\overline{Cx}$ is contained in X. The set of such x being $\{gy \mid g \in N_G(C) \cap \overline{AMB} \text{ and } y \in Y\}$. Note that thus the crucial part of finding C-orbits in X has been relegated to finding C minimal sets, which is therefore part of both the proofs of Hedlund's Theorem and Oppenheim's Conjecture Sections 3.4 and 6.5 respectively.

The basic idea is that on the one hand $g \in \overline{AMB}$ grants us existence of a point in $y \in Y$ that is mapped to point in X by g (see Figure 8). On the other hand, $g \in N_G(C)$ grants us that that the C-orbit of gy coincides with the g-image of the C-orbit of y (see Figure 9). And then finally Y being C-minimal implies that the C-orbit of gy spreads on the whole gY. Since X is C-invariant, we get that $gY \subseteq X$.

More formally:

[17] Finite here meaning that $\mu(X) < \infty$.

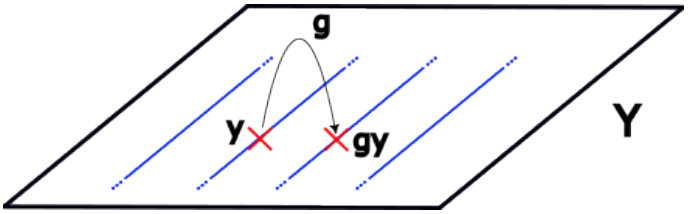


Figure 10: Here the blue lines indicate the C-Orbit of y and $g \in M$. In some sense g can be thought of as shortcut.

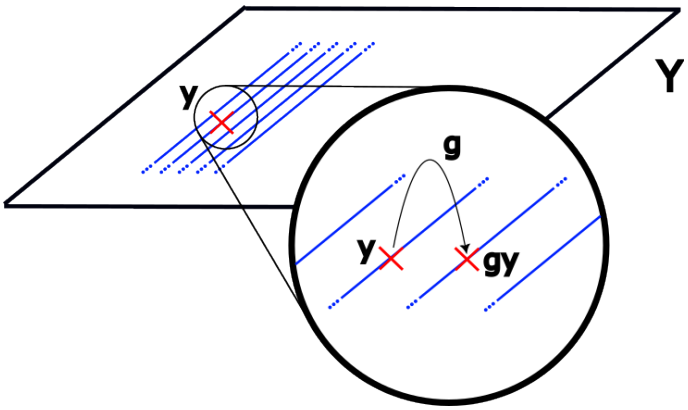


Figure 11: The blue lines indicate the C-Orbit of y and $g \in M$. Now since Cy is dense in Y the C-orbit of y will hit points arbitrarily close to y and by homogeneity there must exist a shortcut from y to this point as well.

Proof. Let $b \in B$, $a \in A$ and $m \in M$ and let $u = amb \in AMB$. Since $m \in M$ its action on Ω maps some subset U_Y of Y to a subset U_X of X. Since X and Y are invariant under action of A and B respectively, u maps $b^{-1}U_Y \subseteq Y$ to $aU_X \subseteq X$ i.e. $ub^{-1}U_Y = aU_X$.

In addition to the above let us now assume that $u \in N_G(C)$ then we have $uC = Cu$, thus also $uCb^{-1}U_Y = CaU_X$, since Y is minimal under the action of C we have $\overline{CaU_X} = \overline{uCb^{-1}U_Y} = uY$. Furthermore since $C \subseteq A$ we have that $CaU_X \subseteq X$ and since X is closed also $\overline{CaU_X} \subseteq X$. Thus we can conclude that $uY \subseteq X$.

If we pick $A = B$ and $X = Y$ then by minimality $\overline{CaU_X} = Y$, thus proving the second claim. □

We will now take a closer look at what we can say about orbit closures. First of all note that if an orbit is already close its self then it is identical to its closure. The more interesting cases are orbits, that do not close. One can think of such orbits as curling up densely in a certain region, where as closing orbits form a closed loop or diverge. The following lemma will give us a useful fact to work with.

Lemma 2.29 ([10] Lemma IV.2.3)**.** *Let G be a locally compact group and let Ω be a homogenous space of G. Let C be a closed subgroup of G. Let $Y \subseteq \Omega$ be closed set that is invariant and minimal under the action of C.*

If there exists $y \in Y$ such that $Y \neq Cy$, then for $M := \{g \notin C \mid gy \in Cy\}$, we have $e \in \overline{M}$.

Proof. We will prove that if $e \notin \overline{M}$ then $Cy = Y$.
Note that in any case due to minimality $\overline{Cy} = Y$, thus $Cy \subseteq Y$, and we are left to show $Y \subseteq Cy$. Note also that by minimality for any open neighbourhood U of e in G, we have

$$CUy \cap Y = Y. \tag{1}$$

Assuming following equality, which will be shown to hold later,

$$Uy \cap Y = Uy \cap Cy, \tag{2}$$

we get in combination with (1) that

$$Y = CUy \cap Y = C(Uy \cap Y) = C(Uy \cap Cy) = CUy \cap Cy \subseteq Cy.$$

□

Proof of Eq. (2)**.** We will prove this by showing inclusion in both directions:

1. Since $Cy \subseteq Y$ we have that $Uy \cap Cy \subseteq Uy \cap Y$.
2. Since, by our assumption, $e \notin \overline{M}$ there exists $V \subseteq G$ an open neighbourhood of e such that $\forall v \in V\colon\ vy \in Cy \iff v \in C$. Let now U be a neighbourhood of e such that

   $$U^2 := \{u_1 u_2 \mid u_1, u_2 \in U\} \subseteq V.$$

   Let $uy \in Uy \cap Y$. Since $Y = \overline{Cy}$, there exists a sequence $(c_i)_{i\in\mathbb{N}} \subseteq C$ such that $\lim_{i\to\infty} c_i y = uy$. In other words, for large enough i, $c_i y$ is in a neighborhood $U_i \subseteq U$ of uy, or equivalently there is some $u_i \in U$ such that $u_i uy = c_i y \in Cy$, and since $u_i u \in U^2 = V$ this implies that $u_i u \in C$. We may take the neighborhood U_i to be smaller and smaller as i grows, giving us also that $\lim_{i\to\infty} u_i = e$. In summary, there are elements $u_i u \in C$ arbitrarily close to u, and since C is closed then $u \in C$ and finally $uy \in Cy$.

□

Lemma 2.29 gives us that if Cy is not closed then there exists $\{g_n\}_{n\in\mathbb{N}} \subseteq G - C$ such that $Cg_n y \to Cy$ as n goes to ∞. If now in addition $\{g_n\}_{n\in\mathbb{N}} \subseteq N_G(C) \cap \overline{AMB}$ we can apply Lemma 2.28 to get that for all $n \in \mathbb{N}$, $g_n Cy = Cg_n y \subseteq X$. Also note that since $\{g_n\}_{n\in\mathbb{N}} \subseteq G - C$, we cannot have that $g_n Cy \subseteq Y$. Thus $\{\overline{g_n Cy} = g_n Y\}_{n\in\mathbb{N}}$ is a sequence of closed sets in X which accumulates in Y.

Let us now assume there exists P a subgroup of G such that $P - \{e\} \subseteq (N_G(C) \cap \overline{AMB}) - C$ and that $e \in \overline{P - \{e\}}$. Then we have that $PY \subset X$, which implies that X must contain a P-orbit.

Remark 2.8. This is exactely what we will be doing in the proof of Hedlund's theorem except there we will have that $A = B = C$, $X = Y$ and P will minimal on G/Γ. Thus proving that C is minimal on G/Γ.

3 Hedlund's Theorem in the Cocompact Case

3.1 Introduction

In Chapter 6, we'll prove Oppenheim's Conjecture (Theorem 1.1). In this chapter we give a proof of Hedlund's Theorem (Theorem 3.3). The proof of Hedlund's given here, uses arguments similar to the proof we will give for Oppenheim's Conjecture. However in the proof of Hedlund's Theorem we will be working with 2×2-matrices, instead of 3×3 matrices as in Oppenheim's Conjecture. This reduces the overall complexity of the proof and we can focus more on comprehending the core ideas outlined in Section 2.3. In some sense this chapter can be thought of as giving an introductory example, to get acquainted with ideas used in the proof Oppenheim's Conjecture.

In order to state Hedlund's theorem we will first need to introduce horocyclic subgroups and the horocycle flow in $\mathrm{SL}(2,\mathbb{Z})$

Definition 3.1 (Horocyclic Subgroups)**.** *The subgroups of* $\mathrm{SL}(2,\mathbb{R})$ *given by*

$$N := \left\{ u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\},$$

and

$$N^- := \left\{ h(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\},$$

will be referred to as the horocyclic subgroups.

Definition 3.2 (Horocycle Flow)**.** *Let* $\mathrm{SL}(2,\mathbb{R})$ *act on some space* X*. For any* $x \in X$ *the we will be calling the* N*-orbit of* x *the horocycle flow of* x[18]*.*

Having introduced these two definitions, we can now state Hedlund's Theorem.

Theorem 3.3 (Hedlund's Theorem, cocompact case)**.** *Let* Γ *be a cocompact lattice in* $\mathrm{PSL}(2,\mathbb{R})$*. Then the horocycle flow on* $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ *is minimal.*

In the Proof of Hedlund's theorem we will also be using the following subgroups.

Definition 3.4 (Geodesic Subgroup)**.** *We will call the subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *given by*

$$A := \left\{ a(\tau) := \begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix} \middle| \tau \in \mathbb{R} \right\}$$

the geodesic subgroup of $\mathrm{SL}(2,\mathbb{R})$*.*

Definition 3.5 (Parabolic Subgroup)**.** *Let*

$$P := AN = \left\{ \begin{pmatrix} e^{\tau} & x \\ 0 & e^{-\tau} \end{pmatrix} \middle| \tau, x \in \mathbb{R} \right\}$$

be called the parabolic subgroup of $\mathrm{SL}(2,\mathbb{R})$*.*

Definition 3.6 (Geodesic and Parabolic Flows)**.** *If we let again* $\mathrm{SL}(2,\mathbb{R})$ *act on some space* X*, and let again* $x \in X$*. Then, as with the horoycle flow, we will be calling the* A*-orbit (resp.* P*-orbit) of* x *the geodesic flow (resp. parabolic flow) of* x*.*

3.1.1 Outline of the Chapter

We will start off with section 3.2, in which we will be showing that P acts minimally on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$, i.e. for all $x \in \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ we have that $\overline{Px} = \mathrm{PSL}(2,\mathbb{R})$. Note that if we can in addition show that any N-orbit closure contains some P-orbit on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$, then for all $x \in \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ we have that

$$\overline{Nx} \supseteq \overline{Py} = \mathrm{PSL}(2,\mathbb{R}),$$

[18] Often times N^--orbits are also called horocycle flows. However here we will only study N- orbits.

for some $y \in \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ and thus giving us $\overline{Nx} = \mathrm{PSL}(2,\mathbb{R})$, which means that N acts minimally on $\mathrm{PSL}(2,\mathbb{R})$.

Showing that any N-orbit contains some P-orbit on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ is where we will use the ideas outlined in section 2.3. Now recall that to use these ideas we will have to show that N does not have any periodic orbits and that $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ contains an N-minimal subset. This will be done in sections 3.3 and 3.4 respectively.

Finally, in section 3.5 we will conclude this chapter by giving a rigorous proof of Hedlund's Theorem.

3.2 P Acts Minimally on Compact Subgroups

The goal of this section will be to show that the parabolic subgroup P acts minimally on compact subgroups. We will then use this fact to apply Lemma 2.28 in the proof of Hedlund's Theorem (3.3).

Lemma 3.7 ([10] Proposition IV.1.5 i))**.** *Let G be a locally compact group acting continuously on a second countable topological space X. Assume that X is equipped with a borel measure μ such that $\mu(U) > 0$ for all open sets $U \subseteq X$ and that μ is invariant and ergodic under the action of G. Let S be a subsemigroup of G such that $G = S \cup S^{-1}$. Then for almost any $x \in X$, the orbit Sx is dense in X.*

In order to prove the above statement we will first prove an auxiliary lemma.

Lemma 3.8. *For all $U \subseteq X$ open, $\mu(X - \bigcup_{s\in S} s^{-1}(U)) = 0$.*

Proof. Let $B := \bigcup_{s\in S} s^{-1}(U)$. We have that

$$s^{-1}B = s^{-1}\bigcup_{\sigma\in S}\sigma^{-1}(U) = \bigcup_{\tau\in s^{-1}S^{-1}}\tau(U).$$

Note that $s^{-1}S^{-1} = (Ss)^{-1} \subset S^{-1}$, because S is a semigroup, and thus

$$s^{-1}B \subset B. \tag{3}$$

Multiplying this from the right with s gives us

$$B = ss^{-1}B \subseteq sB. \tag{4}$$

Since U is open and $s \in G$ acts continuously on X we have that all the $s^{-1}U$ are open as well. By the assumption that the measure of any open set in X is non-zero. We therefore have that $\mu(B) > 0$, because $\mu(B) \geq \mu(s^{-1}U) > 0$.

Since μ is G-invariant $\mu(s^{-1}B) = \mu(B) = \mu(sB)$. Observe that this together with (3) and (4) implies that

$$\mu(s^{-1}B \triangle B) = \mu(B \triangle sB) = 0.$$

Now because we chose S such that $S \cup S^{-1} = G$, this implies that the characteristic function χ_B is μ a.e. G-invariant. Since we assumed that G acts ergodically on X with regards to μ, χ_B must be μ a.e. constant. Now since $\mu(B) > 0$ and $\chi_B = 1$ on B, χ_B must not be almost everywhere zero, and thus is is almost everywhere equal to 1. Therefore

$$\mu(X - B) = \mu(\{x \in X \mid \chi_B(x) = 0\}) = 0,$$

proving the lemma. □

Now we can move on to the proof of the proposition.

Proof of Lemma 3.7 . Let $\{U_n\}_{n\in\mathbb{N}}$ be a basis of the topology on X. By the auxiliary Lemma 3.8 we have that for any $n \in \mathbb{N}$

$$\mu(X - \bigcup_{s\in S} s^{-1}(U_n)) = 0.$$

Thus also

$$\mu\left[\bigcup_{n\in\mathbb{N}}\left(X-\bigcup_{s\in S}s^{-1}(U_n)\right)\right]=\mu\left[X-\bigcap_{n\in\mathbb{N}}\bigcup_{s\in S}s^{-1}(U_n)\right]=0.$$

Note that this means that μ a.e. $x\in X$ is contained in $\bigcap_{n\in\mathbb{N}}\bigcup_{s\in S}s^{-1}(U_n)$, and thus for all $n\in\mathbb{N}$ we have that $Sx\cap U_n\neq\varnothing$. Therefore $Sx\cap U\neq\varnothing$ for any non-empty open subset $U\subseteq X$, or in other words, Sx is dense in X.

□

Theorem 3.9. *Let Γ be a lattice in $PSL(2,\mathbb{R})$ and let μ be the Haar measure on $\Gamma\backslash G$. Then A acts ergodically on $\Gamma\backslash G$.*

To show this we will again first prove an auxiliary lemma.

Lemma 3.10. *For $u\in N\cup N^-$ we have that for any $g\in A-\{e\}$, either $g^nug^{-n}\to e$ as n goes to ∞, or $g^nug^{-n}\to e$ as n goes to $-\infty$.*

Proof. Let

$$g:=\begin{pmatrix}e^{\tau}&0\\0&e^{-\tau}\end{pmatrix},\quad \tau\neq 0.$$

If $u\in N$ then

$$g^nug^{-n}=\begin{pmatrix}e^{\tau n}&0\\0&e^{-\tau n}\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}e^{-\tau n}&0\\0&e^{\tau n}\end{pmatrix}=\begin{pmatrix}1&xe^{2\tau n}\\0&1\end{pmatrix}.$$

Now note that $|2\tau n|\to\infty$ as $|n|\to\infty$. Thus for either $n\to-\infty$ or $n\to\infty$ we have that $xe^{2\tau n}\to 0$ and consequently $g^nug^{-n}\to e$. For $u\in N^-$ we can the same argumentation. □

Lemma 3.11. *For any sequence $\{v_n\}_{n\in\mathbb{N}}\subseteq\Gamma\backslash G$ converging to e and any $f\in L_2(\Gamma\backslash G)$ we have that*

$$\|f\circ v_n-f\|_{L^2}^2\to 0\quad \text{as } n\to\infty.$$

Proof . Because $f\in L_2(\Gamma\backslash G)$, there exists $\varphi\in C_\infty$ such that for any $\varepsilon>0$

$$\|f-\varphi\|_{L_2}^2<\frac{\varepsilon}{3}.$$

Since φ is continuous we have that there exists $n_0\in\mathbb{N}$ such that for any $n>n_0$

$$\|\varphi\circ v_n-\varphi\|_{L_2}^2<\frac{\varepsilon}{3}.$$

Then

$$\begin{aligned}\|f\circ v_n-f\|_{L_2}^2&\le\|f\circ v_n-\varphi\circ v_n+\varphi\circ v_n-\varphi+\varphi-f\|_{L_2}^2\\&\le\|f\circ v_n-\varphi\circ v_n\|_{L_2}^2+\|\varphi\circ v_n-\varphi\|_{L_2}^2+\|\varphi-f\|_{L_2}^2\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

And thus we have proven the lemma. □

Proof of Theorem 3.9. Let $f\in L^2(\Gamma\backslash G)$ be A-invariant and fix arbitrary $n\in\mathbb{Z}$, $g\in A$ and $u\in N\cup N^-$. Note we will use $f\circ u$ to denote the function f precomposed with the action of u on $\Gamma\backslash G$ i.e. $f\circ u(\Gamma g)=f(\Gamma gu)$. Since f is A-invariant we have that

$$\begin{aligned}\|f\circ u-f\|_{L^2}^2&=\int_{\Gamma\backslash G}|f(ux)-f(x)|^2d\mu(x)\\&=\int_{\Gamma\backslash G}|f(xug^{-n})-f(xg^{-n})|^2d\mu(x).\end{aligned}$$

We can now change variables $y:=xg^{-n}$ giving us

$$\int_{\Gamma\backslash G}|f(xug^{-n})-f(xg^{-n})|^2d\mu(x)=\int_{\Gamma\backslash G}|f(yg^nug^{-n})-f(y)|^2d\mu(y).$$

Putting the above two equations together gives us

$$\|f \circ u - f\|_{L^2}^2 = \|f \circ g^n u g^{-n} - f\|_{L^2}^2.$$

Observe that by Lemma 3.11

$$\lim_{n\to\infty} \|f \circ g^n u g^{-n} - f\|_{L^2}^2 = \|f \circ \left(\lim_{n\to\infty} g^n u g^{-n}\right) - f\|_{L^2}^2. \tag{5}$$

Thus Lemma 3.10 shows us that $\|f \circ u - f\|_{L^2}^2 = 0$, since $u \in N \cup N^-$ was chosen arbitrarily, we have shown that any A-invariant function is also $N \cup N^-$-invariant. Now note that N, N^- and A generate G. Thus we can conclude that any A-invariant is G-invariant, because G acts transitively on $\Gamma\backslash G$ it means that f is a constant. Now since any A-invariant function on $\Gamma\backslash G$ is constant A acts ergodically on $\Gamma\backslash G$, which concludes the proof. □

Lemma 3.12 ([10], Lemma IV.2.4)**.** *Let Γ be a lattice in $G = SL(2,\mathbb{R})$. The left-action of Γ on $B = G/P$ is minimal. Equivalently, the right-action of P on $\Gamma\backslash G$ is minimal.*

Proof. We will prove this by showing that for any $x \in G$ the double coset $\Gamma x P$ is dense in G. Because $\Gamma x P = x(x^{-1}\Gamma x)P$ and $x^{-1}\Gamma x$ is a lattice, it is sufficient to show that for any lattice Γ we have that $\overline{\Gamma P} = G$. Also let $A^+ := \{a(t) \mid t > 0\}$.

Since the action of A on $\Gamma\backslash G$ is ergodic (Theorem 3.9), we have that by Proposition 3.7

$$\Gamma u (A^+)^{-1}$$

is dense in G for almost all $u \in G$, and therefore the set

$$U = \{u \in G \mid \Gamma u (A^+)^{-1} \text{ is dense in } G\}$$

is itself dense in G.
Let us now fix some $g \in G$. Let $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Gamma$, $\{a_n\}_{n\in\mathbb{N}} \subseteq A^+$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq N^- P$ having the following properties:

$$u_n \to e \quad \text{and} \quad \gamma_n u_n^{-1} a_n^{-1} \to g \quad \text{as} \quad n \to \infty.$$

Now consider the sequence

$$w_n := \gamma_n u_n^{-1} a_n^{-1} g^{-1}.$$

Note that $\{w_n\}_{n\in\mathbb{N}}$ has the properties:

$$\gamma_n = w_n g a_n u_n \quad \text{and} \quad w_n \to e \quad \text{as} \quad n \to \infty.$$

Also note that since $\{u_n\}_{n\in\mathbb{N}} \subseteq N^- P$ there exists $\{v_n\}_{n\in\mathbb{N}} \subseteq N^-$ and $\{p_n\}_{n\in\mathbb{N}} \subset P$ such that $u_n = v_n p_n$ for any $n \in \mathbb{N}$.

Putting the above together we get that

$$\gamma_n P = w_n g a_n v_n P$$

as $p_n P = P$, then

$$w_n g a_n v_n P = w_n g (a_n v_n a_n^{-1}) a_n P.$$

We have that $w_n \to e$ as $n \to \infty$ and $\{a_n\}_{n\in\mathbb{N}} \subset A^+ \subseteq P$, hence $a_n P = P$.

If now

$$a_n v_n a_n^{-1} \to e \quad as \quad n \to \infty \tag{6}$$

holds then we have that

$$\gamma_n P = w_n g (a_n v_n a_n^{-1}) a_n P \to gP \quad as \quad n \to \infty.$$

Since we have chosen g to be an arbitrary element of G, we have shown that $\overline{\Gamma P} = G$. This implies that both the left-action of Γ on G/P and the right-action of P on $\Gamma\backslash G$ are minimal.

It remains to showing that (6) holds. Let

$$a_n := \begin{pmatrix} e^{\tau_n} & 0 \\ 0 & e^{-\tau_n} \end{pmatrix}, \quad v_n := \begin{pmatrix} 1 & 0 \\ x_n & 1 \end{pmatrix}, \quad p_n := \begin{pmatrix} e^{\alpha_n} & y_n \\ 0 & e^{-\alpha_n} \end{pmatrix}.$$

Note that by definition $u_n = v_n p_n \to e$ as $n \to \infty$ hence we have:

$$v_n p_n = \begin{pmatrix} e^{\alpha_n} & y_n \\ x_n e^{\alpha_n} & e^{-\alpha_n} + x_n y_n \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad as \quad n \to \infty,$$

we thus have that $e^{\alpha_n} \to 1$ and therefore also $x_n, y_n, \alpha_n \to 0$ as $n \to \infty$. We also have that

$$a_n v_n a_n^{-1} = \begin{pmatrix} 1 & 0 \\ x_n e^{-2\tau_n} & 1 \end{pmatrix}.$$

Because $a_n \in A^+$, we have $\tau_n > 0$, and thus $e^{-2\tau_n} < 1$. Thus $x_n e^{-2\tau_n} \to 0$, and (6) holds. □

3.3 No N-Periodic Orbits on G/Γ

As mentioned in section 3.1.1 we need to show that horocycle flows have no periodic orbits. To see this more easily we will introduce an action of $\mathrm{PSL}(2,\mathbb{R})$ on the upper half plane $\mathbb{H}$[19]. We will then see that, the horocycle flow cannot have periodic orbits.

Consider the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

for any $z \in \mathbb{H}$ and any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example 3.1 (Modular Group). $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ then for any $z, y \in \mathbb{H}$ $z \sim y$ if and only if there exists $\frac{az+b}{cz+d} = y$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Recall that $\mathrm{PSL}(2,\mathbb{Z})$ is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with corresponding Möbius transformations $\varphi_T : z \mapsto z+1$ and $\varphi_S : z \mapsto \frac{-1}{z}$. Now since $S^2 = Id$ and $(ST)^3 = Id$. Note that a fundamental region of the action of the group generated by T is $\{z \in \mathbb{C} \mid \frac{1}{2} < Re(z) < \frac{1}{2}\}$.

Note that a fundamental domain of the action of the group generated by S is $\{z \in \mathbb{C} \mid |z| > 1\}$

Thus a fundamental domain of the action $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}$ is

$$F = \{z \in \mathbb{C} \mid \frac{-1}{2} < Re(z) < \frac{1}{2} \text{ and } |z| > 1\}$$

Proposition 3.13. *The measure* $\frac{1}{\mathrm{Im}(z)^2} dz$ *is preserved under the action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathbb{H}$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$. Let $z := x_1 + iy_1$ $A(z) = \frac{az+b}{cz+d} = x_2 + iy_2$. Note that $y_2 = \frac{y_1}{|cz+d|^2}$ Then for any $U \subseteq \mathbb{H}$ we have that

$$\int_{A(U)} \frac{1}{\mathrm{Im}(z)^2} dz = \int_{A(U)} \frac{dy_2 dx_2}{y_2^2} = \int_U |\frac{dA}{dz}|^2 |(cz+d)^4| \frac{dy_1 dx_1}{y_1^2}$$

Now observe that

$$\frac{dA}{dz} = \frac{1}{dz} \frac{az+b}{cz+d} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-cb}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

And thus we have

$$\int_U |\frac{dA}{dz}|^2 |(cz+d)^4| \frac{dy_1 dx_1}{y_1^2} = \int_U \frac{dy_1 dx_1}{y_1^2}.$$

□

[19] Recall that the upper half plane is the set of all elements of $\mathbb{C}$ with positive imaginary part i.e. $\mathbb{H} = \{a + bi \in \mathbb{C} \mid b > 0\}$.

Now note that

$$\int_F \frac{dxdy}{y^2} \leq \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{1}{y^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} dxdy = \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{1}{y^2} dy \leq \int_{\frac{\sqrt{3}}{2}}^{1} \frac{1}{y^2} dy + \sum_{y \in \mathbb{N}} \frac{1}{y^2} \leq \frac{4}{3} \frac{\pi^2}{6} < \infty.$$

Thus $\mathrm{PSL}(2,\mathbb{Z})$ is a lattice in $\mathrm{PSL}(2,\mathbb{R})$. However observe that F is not bounded and therefore not compact. Thus $\mathrm{PSL}(2,\mathbb{Z})$ is an example of a non-uniform lattice.

Note that in *Example* 3.1 we have chosen a very nice fundamental domain. That is one which is connected. This must not be case for a fundamental domain. We will therefore introduce the notion of a Dirichlet domain.

Definition 3.14. *For any subgroup $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ and any point $p \in \mathbb{H}$ we call the set*

$$D(p) := \{z \in \mathbb{H} \mid d(z,p) < d(z,\gamma p) \quad \forall \gamma \in \Gamma - \{e\}\},$$

a Dirichlet domain.

Lemma 3.15 ([10] Lemma II.2.6.)**.** *Let $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ be a discrete subgroup*[20]*. Then for any $p \in \mathbb{H}$ the Dirichlet domain $D(p)$ for Γ is a fundamental domain of Γ.*

Remark 3.1. The above Lemma will not be proven here. A proof can be found in [10] on page 45.

Lemma 3.16 ([10] Corollary IV.1.8.)**.** *The horocycle flow on $\Gamma\backslash SL(2,\mathbb{R})$ has no periodic orbits if Γ is cocompact.*

Proof. For any $p \in \mathbb{H}$ consider the *Dirichlet region* of Γ on $\mathbb{H}$ in p

$$D(p) := \{z \in \mathbb{H} \mid d(z,p) < d(z,\gamma p) \quad \forall \gamma \in \Gamma - \{e\}\}.$$

Note that if Γ is cocompact and p is an interior point of $\mathbb{H}$, then $D(p)$ does not contain a point at ∞ i.e. $\Gamma\backslash PSL(2,\mathbb{R})$ doesn't have a cusp.

Assume $\Gamma g \in \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ is a periodic point of the horocyle flow. Therefore there exists $t \in \mathbb{R}_{>0}$ such that $\Gamma g u(t) = \Gamma g$. Thus there exists $\gamma \in \Gamma$ such that $gu(t) = \gamma g$. Now consider that $\gamma = gu(t)g^{-1}$ and that $u(t)$ has the unique fixed point ∞ in $\partial\mathbb{H}$. Thus γ has a unique fixed point $x = g \cdot \infty$ on the boundary of $\mathbb{H}$.

Now note that for any $p \in \mathbb{H}$ we have that $d(x,p) = d(\gamma x, \gamma p) = d(x,\gamma p) = \infty$. Therefore for all $p \in \mathbb{H}$, we have that x is contained in $D(p)$.

Thus for some $p \in \mathbb{H}$ we have that $x \in D(p)$, thus $D(p)$ contains a point at ∞. By Lemma 3.15, we have that $D(p)$ is a fundamental domain of Γ, which is not bounded, hence it cannot be compact. But this contradicts the assumption that Γ is cocompact. □

3.4 Existence of Minimal Sets

To justify existence of minimal sets in different contexts we will keep this lemma quite general.

Lemma 3.17 ([10] Lemma IV.2.1.)**.** *Let G be a locally compact group acting continuously on a compact space Ω. Then there exists a nonempty closed G-invariant $Y \subseteq \Omega$ which is minimal with respect to these properties.*

Observe the strong prerequisite in the above lemma is that the space Ω is compact. This is because using properties of compact sets will make the proof much simpler. More precisely we will use the following characterization of compactness.

Definition 3.18. *A topological space X is compact if for every family $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ of closed subsets of X, having the finite intersection property, we have that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$.*

Proof. Let $\mathcal{O}$ be the family of all non-empty, closed G-invariant sets in Ω. Note that $\mathcal{O}$ is not empty since $\Omega \in \mathcal{O}$, it is closed to intersection, and that $(\mathcal{O}, \subset)$ is a partially ordered set.

Let $\mathcal{F}$ be a chain in $\mathcal{O}$. Then $\mathcal{F}$ must have the finite intersection property, and since Ω is compact, by Definition 3.18 $\bigcap_{A \in \mathcal{F}} A$ is a non-empty element of $\mathcal{O}$. Moreover, it is a lower bound of $\mathcal{F}$ in $\mathcal{O}$. Thus by Zorn's lemma $\mathcal{O}$ contains minimal G-invariant sets. By Proposition 2.22 this is equivalent to the existence of G-minimal sets. □

[20] Discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ are often called Fuchsian groups

Remark 3.2. One of the difficulties in proving Oppenheim's Conjecture (Theorem 1.1) compared to Theorem 3.3, is the fact that we act on a non-compact space there. We will solve this problem in Chapter 5 by proving a non-divergence property, that gives us recurrence of the flow to a compact subset, and proceed with arguments similar to the proof of Lemma 3.17. The same strategy can also be applied in the extension of Hedlund's Theorem to non-uniform lattices.

3.5 Proof of Hedlund's Theorem

Proof of Theorem 3.3. Let $G := SL(2,\mathbb{R})$, and let Γ be any cocompact lattice in G. In the following part we will be using N as defined in Remark 3.1.

By Lemma 3.17 there must exist a minimal closed N-invariant non-empty $X \subseteq \Gamma\backslash G$. We will now proceed by showing that $X = \Gamma\backslash G$.

Recall that there are no periodic N-orbits (Lemma 3.16). Consider that $n([t_1,t_2])$ is closed but $n(\mathbb{R})$ is not, hence Nx is closed if and only if there exist $-\infty < t_1 \leq t_2 < \infty$ such that $n([t_1,t_2])x = Nx$ i.e. when the N-orbit of x is periodic. Thus we have that Nx is not closed but X is closed, which gives us that $X - Nx \neq \emptyset$. Therefore we can apply Lemma 2.29 to every $x \in X$, which gives us that

$$M := \{g \notin N | Xg \cap X \neq \emptyset\} \supseteq \{g \notin N | gx \in Nx\} =: M_x,$$
$$e \in \overline{M_x} \subseteq \overline{M}.$$

Since $P \subseteq N_G(N)$ we have that

$$X = Xg \quad \forall g \in P \cap \overline{NMN},$$

by Lemma 2.28 with $N = A = B = C$ and $X = Y$.

Claim 3.19. $P \subseteq \overline{NMN}$

By this claim, $X = Xg$ for any $g \in P$. By Lemma 3.12, P acts minimally on $\Gamma\backslash G$. Combined, $\Gamma\backslash G = \overline{xP} = XP = X$, and we are done. Thus all that is left to prove is Claim 3.19. □

In order to prove Claim 3.19 we will first prove some properties of the sets M and N:

Claim 3.20. $M = NMN$

Proof. Since $e \in N$, clearly $NMN \supseteq M$, so we need just to show $M \supseteq NMN$. Let $g^* \in NMN$ then there exists $n_1, n_2 \in N$ and $g \in M$ such that $g^* = n_1 g n_2$. Since $g \in M$ we have that $Xg \cap X \neq \emptyset$. Also because of N-invariance of X we have that $Xn_i = X$ for $i \in \{1,2\}$. Thus

$$\emptyset \neq Xg \cap X = Xn_1 g \cap X$$

and so

$$\emptyset \neq (Xn_1 g \cap X)n_2 = Xn_1 g n_2 \cap Xn_2 = Xg^* \cap X.$$

Therefore $g^* \in M$. □

Claim 3.21. *The set $N \cup M = \{g \in G | Xg \cap X \neq \emptyset\}$ is a closed subset of G.*

Proof. Denote by $\pi_G : X \times G \to G$ the projection to G and by $m : X \times G \to G$ the multiplication $(x,g) \mapsto xg$.

Note that X is compact and thus closed, and G acts on X continuously. Therefore $m^{-1}(X)$ is closed. If we can also show that π_G is a closed map[21], then

$$\pi_G(m^{-1}(X)) = \pi_G(\{(x,g) \in X \times G | xg \in X\}) = \{g \in G | Xg \cap X \neq \emptyset\},$$

thereby concluding the proof of Claim 3.21.

To show $\pi_G(A)$ is closed, assume some closed set $A \subseteq X \times G$ exists such that $\pi_G(A)$ is not closed. Let $\{g_n\}_{n\in\mathbb{N}} \subseteq \pi_G(A)$ be such that $g_n \to g^*$ as $n \to \infty$, but $g^* \notin \pi_G(A)$. Now consider a sequence $\{(x_n, g_n)\}_{n\in\mathbb{N}} \subseteq A$. Since X is compact there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $x_{n_k} \to x^* \in X$ as $k \to \infty$, and therefore $(x_{n_k}, g_{n_k}) \to (x^*, g^*)$ as $k \to \infty$. Since A is closed, $(x^*, g^*) \in A$. But this implies $g^* \in \pi_G(A)$, which gives us the desired contraction. □

[21] Recall we call $\pi_G(A)$ closed if we have that for any closed set A, $\pi_G(A)$ is closed as well

Proof of Claim 3.19. Now by lemma 3.21:

$$\overline{M} \subseteq \overline{N \cup M} = N \cup M \tag{7}$$

Note that since $e \in \overline{M}$ and from lemma 3.20 we have that $N \subseteq \overline{NMN} = \overline{M}$ and thus $\overline{M}$ contains N and M. This gives us

$$N \cup M \subseteq \overline{M} = \overline{NMN},$$

and together with 7 then gives us

$$N \cup M = \overline{M} = \overline{NMN}.$$

Finally, we will use the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2 - \{0\}$. Consider the N-orbit of $\binom{a}{b} \in \mathbb{R}^2 - \{0\}$,

$$O_N\begin{pmatrix} a \\ b \end{pmatrix} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+bx \\ b \end{pmatrix} \middle| x \in \mathbb{R} \right\}.$$

Note that $O_N\binom{a}{b}$ is a line parallel to the x-axis if $b \neq 0$ and a point on the x-axis if $b = 0$.

Let $e_1 := (1,0)^T$Let $\Phi : SL(2,\mathbb{R}) \to \mathbb{R}^2 - \{0\}$ be such that $\Phi(g) = ge_1$. Since $\mathrm{Stab}_G(e_1)$[22] $= N$, the induced map $\mathrm{SL}(2,\mathbb{R})/N \to \mathbb{R}^2 - \{0\}$ is a bijection. Denote $\overline{M}e_1 = \Phi(\overline{M})$. note that $\overline{M}e_1$ has the following properties.

- Since Φ is an open map $\overline{M}e_1$ has to be a closed subset of $\mathbb{R}^2 - \{0\}$.
- Since $\overline{M} = \overline{NMN} = N\overline{M}$, $\overline{M}e_1$ is a union of N orbits in $\mathbb{R}^2 - \{0\}$, i.e N-invariant.
- Recall that, by definition of M, $N \cap M = \varnothing$ and thus $\overline{M} = M \cup N$ is a disjoint union. Therefore $\Phi(N) = e_1$ gives us $\Phi(M) = \overline{M}e_1 - \{e_1\}$.
- We have shown that $e \in \overline{M}$ and thus $e_1 \in \overline{\overline{M}e_1 - \{e_1\}}$.

These four properties of $\overline{M}e_1$ imply there are two possibilities for how $\overline{M}e_1$ can look like:

1. $\overline{M}e_1$ contains a sequence of horizontal lines converging to the x-axis. Thus since $\overline{M}e_1$ is closed it must also contain the the x-axis itself[23].
2. $\overline{M}e_1$ contains a sequence of points on the x-axis, not containing e_1, converging to e_1.

In either case, there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq \overline{M}e_1$ such that $x_i \to e_1$ as $i \to \infty$ and all x_i-s lie on the x-axis and are not equal to e_1.

Since P is exactly the Φ pre-image of the x-axis, we can find a sequence $\{m_i\}_{i\in\mathbb{N}} \subseteq P \cap \overline{M}$ such that $m_i e_1 = x_i, m_i \notin N$ for any i, and $m_i \to m \in N$. Recall that

$$P \cap \overline{M} = \{g \in P | X = Xg\}$$

the stabilizer subgroup of X in G, and thus $P \cap \overline{M}$ is a closed subgroup of P. Therefore $(P \cap \overline{M})/N$ is a closed subgroup of P/N. On the other hand, it is not discrete since $m_i N \to eN$ as $i \to \infty$, but $m_i N \neq eN$. Since $P/N \cong (\mathbb{R}, +)$ as topological groups, and the only closed non-discrete subgroup of $\mathbb{R}$ is $\mathbb{R}$ itself, we have that

$$(P \cap \overline{M})/N = P/N.$$

This implies that $P \subseteq \overline{M} = \overline{NMN}$. □

[22] Recall that for any $x \in X$ $\mathrm{Stab}_G(x) := \{g \in G \mid gx = x\}$.

[23] In this case we have $\Phi(P) \subseteq \overline{\overline{M}e_1 - \{e_1\}}$, thus $P \subseteq \overline{NMN}$, which was what we had to show. However the Proof of 2. also works for 1.

4 $\mathrm{SL}(3,\mathbb{R})$

Oppenheim's conjecture is a statement about quadratic forms, yet the blueprint given in Section 2.3 is in the setting of Lie groups. The connection between the two field is as follows. Consider some quadratic form Q in n variables. There must exist an $n \times n$-matrix A such that

$$Q(x) = x^T A x \quad \forall x \in \mathbb{R}.$$

It turns that $\mathrm{GL}(n,\mathbb{R})$, which is a subset of the set of all $n \times n$-matrices and a well known Lie group, corresponds to the set of all non-degenerate quadratic forms in n variables. This correspondence between $\mathrm{GL}(n,\mathbb{R})$ and non-degenerate quadratic forms will give the possibility to apply homogeneous dynamics to questions regarding quadratic forms.

Throughout this chapter we will restrict our efforts to the case $n = 3$. The reason for this will become apparent in Section 6.3, where we will show that proving Oppenheim's conjecture for quadratic forms in three variables implies it for any larger number of variables.

The aim of this chapter will therefore be to explain the link between quadratic forms in three variables and the Lie group $\mathrm{SL}(3,\mathbb{R})$. We will also discuss properties of $\mathrm{SL}(3,\mathbb{R})$ that are used in the proof of Oppenheim's conjecture.

4.1 Important Subsets of $\mathrm{SL}(3,\mathbb{R})$

In this section we will take a closer look at some subgroups of $\mathrm{SL}(3,\mathbb{R})$, which will be heavily used in the proof of Oppenheim's Theorem.

We have that

$$\mathfrak{gl}(3,\mathbb{R}) := \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid \mathrm{tr}(A) = 0\},$$

is the Lie algebra of the Lie group $\mathrm{GL}(3,\mathbb{R})$. We can now use this Lie algebra to define flows on $\mathrm{GL}(3,\mathbb{R})$.

Consider the subalgebra

$$\mathfrak{D} = \left\{ \mathfrak{d}(t) := \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \subseteq \mathfrak{gl}(3,\mathbb{R}).$$

Using the exponential function we get

$$d(t) := \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} = exp(\mathfrak{d}(t)).$$

The set $D = \{d(t) \mid t \in \mathbb{R}\}$ is a one parameter subgroup of $\mathrm{GL}(3,\mathbb{R})$.

In a similar fashion we get $V_1 = \{v_1(t) \mid t \in \mathbb{R}\}$ and $V_2 = \{v_2(t) \mid t \in \mathbb{R}\}$, where v_1 and v_2 are defined by

$$v_1(t) := \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = exp(\mathfrak{v}_1(t)),$$

$$v_2(t) := \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = exp(\mathfrak{v}_2(t)).$$

and subalgebras

$$\mathfrak{V}_1 = \left\{ \mathfrak{v}_1(t) := \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \subseteq \mathfrak{gl}(3,\mathbb{R}),$$

$$\mathfrak{V}_2 = \left\{ \mathfrak{v}_2(t) := \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \subseteq \mathfrak{gl}(3,\mathbb{R}).$$

We also define $V_i^+ := \{v_i(t) \mid t > 0\}$ and $V_i^- := \{v_i(t) \mid t < 0\}$ for $i \in \{1,2\}$.

Lemma 4.1. *V_2 is contained in centralizer of V_1 and vice versa i.e. for any $s,t \in \mathbb{R}$ we have that $v_2(t)v_1(s) = v_2(s)v_1(t)$.*

Proof. Let $s,t \in \mathbb{R}$, then

$$v_1(t)v_2(s) = \begin{pmatrix} 1 & t & \frac{t^2}{2}+s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = v_2(s)v_1(t).$$

□

Remark 4.1. This Lemma 4.1 also implies that V_2 is contained in the normalizer $N_{Sl(3,\mathbb{R})}(V_1)$.

Remark 4.2. Note that since

$$v_1(t)v_2(s) = \begin{pmatrix} 1 & t & \frac{t^2}{2}+s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

we may define the group

$$V := V_1V_2 = \left\{ \left. \begin{pmatrix} 1 & t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right| s,t \in \mathbb{R} \right\}.$$

Lemma 4.2. *D is contained in the normalizer of V_1,V_2 and V.*

Proof. Let let $s,t,r \in \mathbb{R}$ then we have that

$$d(r)v_1(t)d(r)^{-1} = v_1(e^r t),$$
$$d(r)v_2(s)d(r)^{-1} = v_2(e^{2r}s).$$

This implies that D is contained in the normalizers of V_1 and V_2 Then we can use the equations above to show that

$$d(r)v_1(t)v_2(s)d(r)^{-1} = d(r)v_1(t)d(r)^{-1}d(r)v_2(s)d(r)^{-1} = v_1(e^r t)v_2(e^{2r}s).$$

This implies that D is also contained in the normalizer of V. □

Lemma 4.3. *Let Δ be a discrete subgroup of $DV = DV_1V_2$. Then*

1. *either $\Delta \subseteq V$ or Δ is a cyclic group generated by an element vdv^{-1} for some $d \in D - \{e\}$ and some $v \in V$. In the latter case, if $\Delta \subseteq DV_1$, then $v \in V_1$;*
2. *if Δ is generated by an element vdv^{-1}, $d \in D - \{e\}$, $v \in V$, the projection of Δ on the V_2-component does not have the identity as an accumulation point.*

Proof. 1. Assume Δ is not contained in V. Now let

$$w := \begin{pmatrix} 1 & t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad d := \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix}$$

be such that $dw \in \Delta - V$. Then

$$dw = \begin{pmatrix} \delta & \delta t & \delta s \\ 0 & 1 & t \\ 0 & 0 & \delta^{-1} \end{pmatrix}$$

has eigenvectors

$$v^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad v^2 = \begin{pmatrix} \frac{\delta t}{1-\delta} \\ 1 \\ 0 \end{pmatrix}; \quad v^3 = \begin{pmatrix} \frac{\delta^3 t^2 + s\delta^2(1-\delta)}{(1-\delta)(1-\delta^2)} \\ \frac{\delta t}{1-\delta} \\ 1 \end{pmatrix},$$

with eigenvalues δ, 1 and δ^{-1} respectively. Now let us define

$$v := \begin{pmatrix} | & | & | \\ v^1 & v^2 & v^3 \\ | & | & | \end{pmatrix} \in V.$$

Then we have that $v^{-1}dwv = d$ and thus $d \in \Delta'$, where $\Delta' := v^{-1}\Delta v$. Let $a \in D$ and $u \in V$ be such that $au \in \Delta'$, then we have that

$$d^j aud^{-j} = ad^j ud^{-j} \in \Delta' \quad \forall j \in \mathbb{Z}.$$

Claim 4.4. *$d^j ud^{-j} \to e$ for either $j \to \infty$ or $j \to -\infty$.*

Then we have that

$$d^j aud^{-j} \to a \in \Delta'.$$

Now since Δ' is discrete, there must exist $j \in \mathbb{Z}$ such that $d^j aud^{-j} = a$, and hence we have that $u = e$. Therefore Δ' is a discrete subgroup of D. By definition of D is a one-parameter subgroup and thus isomorphic to $\mathbb{R}$, all of whose discrete subgroups are cyclic. Hence Δ' is cyclic and generated by some $d \in D - \{e\}$, and therefore also Δ is cyclic and generated by vdv^{-1}.

We still have to show that, if $\Delta \subseteq DV_1$ then Δ is generated by vdv^{-1}, where $v \in V_1$. For this consider some $d \in D$ and $v_1 \in V_1$ then

$$dv_1 = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta & \delta t & \delta\frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & \delta^{-1} \end{pmatrix},$$

is a generic element of DV_1 now all we have to show is that if all elements of Δ have the form of dv_1, then for its generator $vd'v^{-1}$ we have that $v \in V_1$.

Let again

$$v := \begin{pmatrix} 1 & t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad d := \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix}.$$

Then we have that

$$vdv^{-1} = \begin{pmatrix} \delta & (1-\delta)t & (\delta-1)(t^2 - \delta^{-1}(1+\delta)s) \\ 0 & 1 & \delta^{-1}t(1-\delta) \\ 0 & 0 & \delta^{-1} \end{pmatrix} = \begin{pmatrix} \delta^* & \delta^* t^* & \delta^* \frac{(t^*)^2}{2} \\ 0 & 1 & t^* \\ 0 & 0 & (\delta^*)^{-1} \end{pmatrix}$$

for some t^*, δ^*. We thus get the system of equations

$$\delta = \delta^* \qquad\qquad (1-\delta)t = \delta^* t^*$$

$$(\delta-1)(t^2 - \delta^{-1}(1+\delta)s) = \delta^* \frac{(t^*)^2}{2} \qquad\qquad \delta^{-1}t(1-\delta) = t^*.$$

This system gives us

$$\delta = \delta^* \qquad\qquad t^* = \delta^{-1}t(1-\delta)$$

and thus

$$(\delta-1)(t^2 - \delta^{-1}(1+\delta)s) = \frac{\delta^{-1}t^2(1-\delta)^2}{2},$$

which gives us that

$$s = \frac{t^2}{2}.$$

Which gives us that

$$v = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

i.e. $v \in V_1$, which was what we wanted to show.

2. For this let $s,t,r \in \mathbb{R}$ and $v = v_1(s)v_2(t)$. Now let us assume that Δ is generated by vdv^{-1}. Then all elements of Δ are of the form

$$\begin{aligned} vd(r)^j v^{-1} &= d(r)^j (d(r)^{-j} v_1(s) d(r)^j)(d(r)^{-j} v_2(t) d(r)^j) v_1(-s) v_2(-t) \\ &= d^j v_1(e^{-rj}s - s) v_2(e^{-2rj}t - t). \end{aligned}$$

Thus the projection on to the V_2 component of Δ is the set

$$\{v_2(e^{-2rj}t - t) \mid j \in \mathbb{Z}\}.$$

Now note that the set

$$B := \{e^{-2rj}t - t \mid j \in \mathbb{Z}\},$$

is not dense around 0. [24]

Thus the identity is not an accumulation point of the projection on to the V_2 component of Δ. Thus proving the statement.

□

Proof of Claim 4.4. Note that

$$d(r)^j u d(r)^{-j} = \begin{pmatrix} e^{jr} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-jr} \end{pmatrix} \begin{pmatrix} 1 & t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-jr} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{jr} \end{pmatrix} = \begin{pmatrix} 1 & te^{jr} & s\delta^{2jr} \\ 0 & 1 & t\delta^{jr} \\ 0 & 0 & 1 \end{pmatrix}.$$

Now $e^{jr} \to 0$ either as $j \to -\infty$ or as $j \to \infty$, proving the claim. □

4.2 Quadratic Forms

In the proof Oppenheim's Conjecture we will use properties of $\mathrm{SL}(3,\mathbb{R})$ and its subgroups to show facts about quadratic forms. More specifically we will study density properties of the set $Q(\mathbb{Z}^3)$ in $\mathbb{R}$. It is therefore crucial to have a good understanding of the relation between quadratic forms and $\mathrm{SL}(3,\mathbb{R})$.

Consider that if a quadratic form Q is non-degenerate there exist a matrix $A \in \mathrm{GL}(n,\mathbb{R})$ such that for any $x \in \mathbb{R}^3$ we have

$$Q(x) = x^T A x.$$

However note that A is not unique. Consider for instance the quadratic form

$$Q_0(x) = Q_0(x_1, x_2, x_3) = 2x_1x_2 - x_2^2$$

but

$$x^t \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x = x^T \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x = Q(x).$$

This considered, a natural question might be: how does the set

$$U(Q) := \{A \in \mathrm{GL}(3,\mathbb{R}) \mid Q(x) = x^T A x\}$$

look like?

Note that

$$x^T A x = Q(x) = Q(x)^T = x^T A^T x,$$

and therefore for any $B \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ such that $B^t = -B$,

$$x^T B x = 0,$$

[24] The interval $(0, |(\delta^{-2} - 1)t|)$ does not contain any elements of B.

and thus $Q(x) = x^T(A+B)x$ and $A+B \in U(Q)$. On the other hand, if some C satisfies that for any x we have $x^TAx = x^TCx$, then $x^T(A-C)x = 0$. Thus for any x, y

$$\begin{aligned} 0 &= (x+y)^T(A-C)(x+y) - x^T(A-C)x - y^T(A-C)y \\ &= x^T(A-C)y + y^T(A-C)x \\ &= x^T(A-C)y + x^T(A-C)^Ty \\ &= x^T((A-C)+(A-C)^T)y. \end{aligned}$$

Thus $(A-C)+(A-C)^T = 0$, i.e. $C = A+B$ for $B = C-A$ antisymmetric. So we get that $C \in U(Q)$ if and only if $C+A$ is antisymmetric. Additionally, for any matrix A, $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$ where $\frac{A+A^T}{2}$ is symmetric and $\frac{A-A^T}{2}$ is antisymmetric.

This means for each Q there is a unique symmetric matrix in $U(Q)$, which we will from now on denote by σ_Q, and since we will be working with Q_0 a lot we will let $\sigma_{Q_0} := \sigma_0$.

Consider an action of $\mathrm{GL}(3,\mathbb{R})$ on the space of Quadratic forms. This action is given for $g \in \mathrm{GL}(3,\mathbb{R}), Q$ a quadratic form by

$$g \cdot Q(x) = Q(g^{-1}x).$$

Consider also the action of $\mathrm{GL}(3,\mathbb{R})$ on the symmetric matrices given for $g \in \mathrm{GL}(3,\mathbb{R})$ and $\sigma \in \mathrm{Mat}_{3\times3}(\mathbb{R})$ symmetric by

$$g \cdot \sigma = g^{-T}\sigma g^{-1}. \tag{8}$$

Note that $g^{-T}\sigma g^{-1}$ is also symmetric. Then the two actions are conjugate in the following way.

Lemma 4.5. *For any $g \in \mathrm{GL}(3,\mathbb{R})$ and Q quadratic form, we get*

$$g \cdot \sigma_Q = \sigma_{g\cdot Q}.$$

Proof. Simply, for any $x \in \mathbb{R}^3$ we get

$$x^t(g\cdot\sigma_Q)x = x^t g^{-T}\sigma g^{-1}x = Q(g^{-1}x) = g\cdot Q(x) = x^T(\sigma_{g\cdot Q})x.$$

□

Remark 4.3. By Proposition 2.3 there exists a right action of $\mathrm{GL}(3,\mathbb{R})$ on both the space of symmetric 3×3 matrices and on the space of quadratic forms, which is given by the left action of inverses.

Denote the vector space of symmetric $d\times d$ matrices by Sym_d. Now consider a symmetric matrix $\sigma \in \mathrm{Sym}_3$. How does its $\mathrm{GL}(3,\mathbb{R})$ orbit under the action described in (8) look like? To answer this, we will recall Sylvester's law of innertia.

Theorem 4.6 (Sylvester's law of Inertia)**.** *Let $A, B \in \mathrm{Sym}_n$. Then A and B have the same signature if and only if there exists $S \in \mathrm{GL}(n,\mathbb{R})$ with the property that $B = S^TAS$.*

This shows us that the $\mathrm{GL}(3,\mathbb{R})$ orbit of σ consists of all the symmetric matrices with the same signature as σ.

When studying quadratic forms, this gives us that for any non-degenerate and indefinite quadratic form Q the $\mathrm{GL}(3,\mathbb{R})$-orbit of Q consists of non-degenerate and indefinite quadratic forms, whose corresponding symmetric matrix has the same signature as the symmetric matrix corresponding to Q.

Moreover in the three dimensional case due to the scarcity of signatures we have that any symmetric, non-degenerate, indefinite matrix is contained either in the orbit of σ_0 or $-\sigma_0$. More specifically the signature of an indefinite, non-degenerate symmetric 3×3 matrix can only ever be either $(0,2,1)$ or $(0,1,2)$ and the signature of σ_0 is $(0,1,2)$ and that of $-\sigma_0$ is $(0,2,1)$[25].

[25] An $n\times n$ matrix has signature (n_0, n_+, n_-) where n_0 is the multiplicity of the eigenvalue 0, and n_+, n_- are the multiplicities of positive and negative eigenvalues respectively.

Remark 4.4. The action of $\mathrm{GL}(3,\mathbb{R})$ on Sym_3 is not free. Consider for example

$$\sigma_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad v_1(t) := \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Then for any $t \neq 0$

$$v_1(t)^T \sigma_0 v_1(t) = \sigma_0.$$

This implies that both $v_1(t)^T \sigma_0 v_1(t)$ and Id are mapped to σ_0.

Let us now define

$$\begin{aligned} O(Q_0) &= \{g \in \mathrm{GL}(3,\mathbb{R}) \mid g \cdot Q_0 = Q_0\} \\ &= \{g \in \mathrm{GL}(3,\mathbb{R}) \mid g \cdot \sigma_0 = \sigma_0\} \\ &= \{g \in \mathrm{GL}(3,\mathbb{R}) \mid g^{-T} \sigma_0 g^{-1} = \sigma_0\} \\ &= \{g \in \mathrm{GL}(3,\mathbb{R}) \mid g^T \sigma_0 g = \sigma_0\}, \end{aligned}$$

where the last equality is by the following lemma.

Lemma 4.7. *$O(Q_0)$ is a subgroup of $\mathrm{GL}(3,\mathbb{R})$.*

Proof. Clearly $O(Q_0)$ contains the identity element. Also, if $o \in O(Q_0)$ then $o^{-1} \in O(Q_0)$:

$$\sigma_0 = (o^{-1}o)^{-T} \sigma_0 o^{-1} o = o^T o^{-T} \sigma_0 o^{-1} o = o^T \sigma_0 o = o^{-1} \cdot \sigma_0.$$

Finally, for $o_1, o_2 \in O(Q_0)$ we have $o_1 o_2 \in O(Q_0)$:

$$(o_1 o_2)^{-T} \sigma_0 (o_1 o_2)^{-1} = o_1^{-T} o_2^{-T} \sigma_0 o_2^{-1} o_1^{-1} = o_1^{-T} \sigma_0 o_1^{-1} = \sigma_0.$$

□

Lemma 4.8. *Let Q be a non-degenerate, indefinite quadratic form, $g \in \mathrm{GL}(3,\mathbb{R})$ such that $Q = g \cdot Q_0$. Then any $o' \in gO(Q_0)$ satisfies $o' \cdot Q_0 = Q$.*

Proof. Let $o \in O(Q)$. Since

$$(go)^{-T} \sigma_0 (go)^{-1} = g^{-T} o^{-T} \sigma_0 o^{-1} g^{-1} = g^{-T} \sigma_0 g^{-1} = \sigma_Q,$$

we have $go \cdot Q_0 = Q$. □

Lemma 4.8 and Lemma 4.7 show that there exists a 1-1 correspondence between the set of non-degenerate, indefinite quadratic forms and the quotient space $O(Q_0)\backslash \mathrm{GL}(3,\mathbb{R})$.

4.2.1 Action of $\mathrm{SO}(Q_0)$ on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$

Recall that in Oppenheim's conjecture we are interested in density of sets $Q(\mathbb{Z}^3)$ in $\mathbb{R}$, where Q is a non-degenerate, indefinite, and irrational quadratic form in 3-variables.

For every Q non-degenerate, indefinite we have that there exists some $g \in \mathrm{GL}(3,\mathbb{R})$ such that

$$\sigma_Q = g^T \sigma_0 g = \det(g)^2 \left(\frac{1}{\det(g)} g\right)^T \sigma_0 \frac{1}{\det(g)} g.$$

Since $\frac{1}{det(g)} g \in \mathrm{SL}(3,\mathbb{R})$ and $\frac{1}{\det(g)^2} Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$ if and only if $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$, it suffices to consider $\mathrm{SL}(3,\mathbb{R})$ instead of $\mathrm{GL}(3,\mathbb{R})$. Consequently, we will be working on the space $SO(Q_0)\backslash \mathrm{SL}(3,\mathbb{R})$, where $SO(Q_0)$ is the subgroup

$$SO(Q_0) = \{g \in O(Q_0) \mid det(g) = 1\}.$$

Since all elements $\mathrm{SL}(3,\mathbb{Z})$ are automorphisms of $\mathbb{Z}^3$ we have that for any quadratic form Q and any $A \in \mathrm{SL}(3,\mathbb{Z})$

$$Q(A\mathbb{Z}^3) = Q(\mathbb{Z}^3).$$

Now note that for any $x \in \mathbb{Z}^3$ we have that

$$Q(Ax) = (Ax)^T \sigma_Q Ax = x^T A^T \sigma_Q Ax = Q(x) \cdot A.$$

Thus the action of $\mathrm{SL}(3,\mathbb{Z})$ from the right on the space of quadratic forms preserves the image set of $\mathbb{Z}^3$.

In summary, we get the following.

Proposition 4.9. *A double coset $SO(Q_0)g\mathrm{SL}(3,\mathbb{Z}) \in SO(Q_0)\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is a set of matrices whose corresponding quadratic forms all have the same image set.*

For any $z \in \mathbb{Z}^3$ consider the map $f_z : A \mapsto z^T Az$ note that

$$z^T Az = a_{11}z_1^2 + a_{22}z_2^2 + a_{33}z_3^2 + (a_{12}+a_{21})z_1z_2 + (a_{13}+a_{31})z_1z_3 + (a_{23}+a_{32})z_2z_3,$$

which is clearly continuous. This then implies that for any $U \subseteq \mathrm{GL}(3,\mathbb{R})$ we have that $f_z(\overline{U}) = \overline{f_z(U)}$.

Thus by showing that for any $x \in \mathbb{R}$ there exists $z \in \mathbb{Z}^3$ and $A \in \overline{SO(Q_0)g\mathrm{SL}(3,\mathbb{Z})}$ such that $z^T Az = x$, we will prove that $\overline{Q_g(\mathbb{Z}^3)} = \mathbb{R}$.

In the case of Oppenheim's conjecture we will show that if Q_g is irrational and indefinite then some orbit of either V_2^+ or V_2^- is contained in $\overline{SO(Q_0)g\mathrm{SL}(3,\mathbb{Z})}$, and in Section 4.4 we will show that this implies $\overline{Q_g(\mathbb{Z}^3)} = \mathbb{R}$.

4.3 Properties of $H = SO(Q_0)$

Let $H := SO(Q_0)$, for $Q_0(x_1,x_2,x_3) = 2x_1x_2 - x_3^2$ as before. Let D, V_1 and V_2 be the ones defined in Section 4.1.

Lemma 4.10. $H = V_1^T D V_1$

Proof. Note that there exist L and U subgroups of H, where L consists of lower triangular matrices and U consists of upper triangular matrices[26].

We can further decompose L and U respectively into a diagonal group and a group of triangular matrices, whose diagonal entries are all equal to 1. Let $L = L^*D_L$ and $U = D_U U^*$ be those decompositions. Thus we have that $H = L^*D'DU^* := L^*D^*U^*$.

Since $H^T\sigma_0 H^T = \sigma_0$, we have that $(L^*)^T\sigma_0 L^* = \sigma_0$, $(R^*)^T\sigma_0 R^* = \sigma_0$ and $(D^*)^T\sigma_0 D^* = \sigma_0$.

Let now

$$l = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \in L^*$$

be a generic element. Then we have that

$$l^T\sigma_0 l = \begin{pmatrix} 2b-a^2 & c-a & 1 \\ c-a & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \sigma_0,$$

which implies that $b = \frac{a^2}{2}$ and also $c = a$. Therefore we have that $L^* = V_1^T$, similarly we can show that $U^* = V_1$. Let now

$$d = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in D^*$$

be a generic. Then

$$l^T\sigma_0 l = \begin{pmatrix} 0 & 0 & d_3d_1 \\ 0 & -d_2^2 & 0 \\ d_1d_3 & 0 & 0 \end{pmatrix} = \sigma_0,$$

which gives us that $d_2 = 1$ and $d_3 = d_1^{-1}$ and thus $D^* = D$. We can thus conclude that $V_1 D V_1^T = H$. □

[26] This is well known from linear algebra called the LU-decomposition, sometimes it is also referred to as LR-decomposition.

Lemma 4.11. $\mathrm{SO}(Q_0)$ *is generated by unipotent one parameter subgroups of* G.

Proof. From Lemma 4.10 we have that $V_1 D V_1^T = H$. Observe that V_1, V_1^T and D are all one-parameter subgroups but D is not unipotent. However consider that the Lie algebras of V_1 and V_1^T are generated by

$$\tau := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau^T := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

respectively, and that

$$[\tau, \tau^T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $[\bullet, \bullet]$, denotes the commutator, which is the Lie bracket on G. Now note that the one parameter subgroup generated by $[\tau, \tau^T]$ is D. Hence H is generated by V_1 and V_1^T, which are both unipotent one parameter subgroups of G. □

Lemma 4.12. $C_G(H_0)$ *then the centralizer of* H_0*, the connected component of the identity* id *in* G *is equal to* $\{\mathrm{id}\}$.

Proof. From Lemma 4.10 we have that $V_1 D V_1^T = H$.

Since any $h \in C_G(H_0)$ as to commute with every $d \in D$ h has to be a diagonal matrix. Let now $\delta \in \mathrm{SL}(3, \mathbb{R}) - \{\mathrm{id}\}$ be a diagonal matrix, then we have that $v_(t)\delta \neq \delta v_1(t)$ and hence $\delta \notin C_G(H_0)$. Thus we can conclude that $C_G(H_0) = \{\mathrm{id}\}$. □

Lemma 4.13. DV_1 *is a cocompact subgroup of* H.

Proof. Consider the action of H on $\mathbb{R}^3$ as a subgroup of $\mathrm{GL}(3, \mathbb{R})$. Let "$\sim$" be the equivalence relation on $\mathbb{R}^3$ where $a \sim b$ if and only if there exists $\lambda \in \mathbb{R} - \{0\}$ such that $a = \lambda b$. The real projective plane $\mathbb{RP}^2$ is $(\mathbb{R}^3 - \{0\})/\sim$. Denote the equivalence class of $p \in \mathbb{R}^3 - \{0\}$ by $\overline{p} \in \mathbb{RP}^2$.

Since for all $a, b \in \mathbb{R}^3 - \{0\}$ and $h \in H$, we have that $ha \sim hb$ if and only if $a \sim b$, we get a well defined action of H on $\mathbb{RP}^2$.

For any $p \in \mathbb{R}^3 - \{0\}$ such that $Q_0(p) = 0$ we also get $Q_0(\lambda p) = 0$ for any $\lambda \in \mathbb{R} - \{0\}$, so the set

$$C := \{\overline{p} \in \mathbb{RP}^2 \mid Q_0(p) = 0\}$$

is well defined.

Moreover, C is invariant under the action of H. Now consider that

$$\tilde{C} = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \mid Q_0(p_1, p_2, p_3) = 2p_1p_3 - p_2^2 = 0\}$$

is a closed algebraic curve. Also C is the projection of $\tilde{C}$ onto $\mathbb{RP}^2$. Thus C is closed subset of $\mathbb{RP}^2$. Now because $\mathbb{RP}^2$ is a compact space C has to be compact.

Observe that $\overline{p} \in C$ is either of the form $\overline{p} = \overline{(1, t, \frac{t^2}{2})} = v_1(t)^T \overline{e_1}$ or of the form $\overline{p} = \overline{e_3} = \sigma_0 \overline{e_1}$. Since $v_1(t), \sigma_0 \in H$, the action of H on C is transitive.

Now consider the stabilizer subgroup $R := \{h \in H \mid h\overline{e_1} = \overline{e_1}\}$. Then H/R is homeomorphic to C, this is a consequence of Lemma 2.9.

By the following Claim 4.14, $R = DV_1$ and then we have that $H/(DV_1) = H/R \cong C$, which is compact, and DV_1 is indeed cocompact in H. □

Claim 4.14. $R = DV_1$.

Proof. In order to see why the claim holds we first take a look at how am matrix in R looks in general.

Let

$$g := \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

be an element of R. Thus we have that $ge_1 = \lambda e_1$, which implies that $g_{21} = g_{31} = 0$ and $g_{11} = \lambda$.

Since $g \in R \subseteq H$ we have that

$$g^T \sigma_0 g = \sigma_0.$$

We have that

$$g^T \sigma_0 g = \begin{pmatrix} 0 & g_{32}\lambda & g_{33}\lambda \\ g_{32}\lambda & g_{32}g_{12} - g_{22}^2 + g_{12}g_{32} & g_{33}g_{12} - g_{23}g_{22} + g_{13}g_{32} \\ g_{33}\lambda & g_{32}g_{13} - g_{22}g_{23} + g_{12}g_{33} & 2g_{33}g_{13} - g_{23}^2 \end{pmatrix}$$

Thus we get

$$g_{32}\lambda = 0$$
$$g_{33}\lambda = 1.$$

The first equation gives us that $g_{32} = 0$ and thus g is an upper triangular matrix. The second equation implies that $g_{33} = \lambda^{-1}$. Thus we get the equations

$$g_{32}g_{12} - g_{22}^2 + g_{12}g_{32} = -g_{22}^2 = -1$$
$$g_{32}g_{13} - g_{22}g_{23} + g_{12}g_{33} = -g_{22}g_{23} + g_{12}\lambda^{-1} = 0$$
$$2\lambda^{-1}g_{13} - g_{23}^2 = 0$$

The first of these equations gives us $g_{22} = \pm 1$. Because g is an upper triangular matrix and $g_{11} = g_{22} = 1$ we have that $\det(g) = g_{11}g_{22}g_{33} = g_{22}$. Furthermore because $g \in \mathrm{SL}(3,\mathbb{R})$ we have that $\det(g) = 1$, thus we must have that $g_{22} = 1$.

Then the second equation gives us $g_{12}\lambda^{-1} = g_{23}$. Finally the third equation gives us $g_{12}^2\lambda^{-2} = 2\lambda^{-1}g_{13}$, which implies that $\frac{1}{2}\lambda g_{23}^2 = g_{13}$.

Thus we have that

$$R = \left\{ d(\log\lambda)v_1(g_{23}) = \begin{pmatrix} \lambda & \lambda g_{23} & \frac{1}{2}\lambda g_{23}^2 \\ 0 & 1 & g_{23} \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \middle| \ \forall \lambda \in \mathbb{R} - \{0\} \text{ and } \forall g_{23} \in \mathbb{R} \right\} = DV_1.$$

□

Lemma 4.15. *Let $h \in H = SO(Q_0)$ and let $v = v_2(t) \in V_2 - \{e\}$ be such that vh lies in HV_2. Then h is an upper triangular matrix. Moreover, if all the diagonal entries of h are positive, then $h \in DV_1$.*

Proof. Let again $Q_0(x_1, x_2, x_3) := 2x_1x_3 - x_2^2$ and let $Q_1(x_1, x_2, x_3) := x_3^2$. Consider that

$$v_2(t)p = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 + tp_3 \\ p_2 \\ p_3 \end{pmatrix},$$

for any $p \in \mathbb{R}^3$. And thus we have that

$$\begin{aligned} Q_0(v_2(t)p) &= 2(p_1 + tp_3)p_3 - p_2^2 \\ &= 2p_1p_3 + 2tp_3^2 - p_2^2 \\ &= Q_0(p) + 2tQ_1(p). \end{aligned}$$

Let $h' \in H$ and $s \in \mathbb{R}$ be such that $v_2(t)h = h'v_2(s)$. Then we have that,

$$Q_0(v_2(t)hp) = Q_0(hp) + 2tQ_1(hp) = Q_0(p) + 2tQ_1(hp) \tag{9}$$

and

$$Q_0(h'v_2(s)p) = Q_0(v_2(s)p) = Q_0(p) + 2sQ_1(p) \tag{10}$$

for all $p \in \mathbb{R}^3$. Putting (9), (10) and the homogeneity of Q_1, together we get that for all p,

$$Q_1(hp) = \frac{s}{t}Q_1(p).$$

Therefore, for any $p \in \mathbb{R}e_1 \oplus \mathbb{R}e_2$ we have $Q_1(hp) = \frac{s}{t}Q_1(p) = 0$, and so we have that $hp \in \mathbb{R}e_1 \oplus \mathbb{R}e_2$. In other words, h must be of the form

$$h = \begin{pmatrix} r_1 & q_1 & h_1 \\ r_2 & q_2 & h_2 \\ 0 & 0 & h_3 \end{pmatrix}.$$

In addition, $h \in H$ satisfies $Q_0(hp) = Q_0(p)$ for any $p \in \mathbb{R}^3$, and thus

$$-r_2^2 = Q_0\begin{pmatrix} r_1 \\ r_2 \\ 0 \end{pmatrix} = Q_0(he_1) = Q_0(e_1) = 0.$$

Thus h must be of the form

$$h = \begin{pmatrix} r_1 & q_1 & h_1 \\ 0 & q_2 & h_2 \\ 0 & 0 & h_3 \end{pmatrix},$$

i.e. h is upper triangular, as we wanted to show.

Now assume that $r_1, q_2, h_3 > 0$. We want to show $h \in DV_1$. Since $h \in H \subseteq \mathrm{SL}(3,\mathbb{R})$ we have that $\det(h) = r_1 q_2 h_3 = 1$ and $h^T \sigma_0 h = \sigma_0$. Specifically,

$$h^T\sigma_0 h = \begin{pmatrix} 0 & 0 & h_3 r_1 \\ 0 & -q_2^2 & h_3 q_1 - h_2 q_2 \\ r_1 h_3 & q_1 h_3 - q_2 h_2 & 2h_3 h_1 - h_2^2 \end{pmatrix} = \sigma_0.$$

From this we get that $h_3 r_1 = 1$, implying that $h_3 = r_1^{-1}$. We also get that $-q_2^2 = -1$ thus $q_2 = \pm 1$, but since by assumption $q_2 > 0$ we have that $q_2 = 1$.

Moreover we have that $h_3 q_1 - h_2 q_2 = 0$ giving us $p_1^{-1} q_1 = h_2$. And thus $2p_1^{-1}h_1 - h_2^2 = 2p_1^{-1}h_1 - p_1^{-2}q_1^2 = 0$, which implies that $h_1 = \frac{1}{2}p_1^{-1}q_1^2$. In summary,

$$h = \begin{pmatrix} r_1 & q_1 & \frac{1}{2}r_1^{-1}q_1^2 \\ 0 & 1 & r_1^{-1}q_1 \\ 0 & 0 & r_1^{-1} \end{pmatrix} = d(r_1)v_1(r_1^{-1}q_1),$$

and thus $h \in DV_1$, concluding the proof. □

4.4 V_2^+-Orbits

The goal of this section will be to prove the following Lemma.

Lemma 4.16. *We have that the sets*

$$Q_0(V_2^+ g\mathbb{Z}^3) = \left\{z(t)^T\sigma_0 z(t) \mid t \in [0,\infty),\ z \in \mathbb{Z}^3,\ z(t) = v_2(t)gz\right\}$$

and

$$Q_0(V_2^- g\mathbb{Z}^3) = \left\{z(t)^T\sigma_0 z(t) \mid t \in (-\infty,0],\ z \in \mathbb{Z}^3,\ z(t) = v_2(t)gz\right\}$$

are equal to $\mathbb{R}$.

In the proof of Oppenheim's conjecture we will then show that the closure of $SO(Q)gSL(3,\mathbb{Z})$ contains an orbit of either V_2^+ or V_2^-. Together with lemma 4.16 this gives us that

$$\overline{Q(\mathbb{Z}^3)} = \overline{Q_0(SO(Q)g\mathrm{SL}(3,\mathbb{Z})\mathbb{Z}^3)} \supseteq Q_0(V_2^{\pm}h\mathbb{Z}^3) = \mathbb{R}$$

for some $h \in \mathrm{SL}(3,\mathbb{R})$. This then implies that $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

Proof. Fix some $z := (z_1, z_2, z_3)^T \in \mathbb{Z}^3$ and

$$g := \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \in SL(3,\mathbb{R}).$$

Then if we define

$$U := \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

$$v_2(t)gz = (Id + U)gz.$$

Thus we have

$$\begin{aligned}(v_2(t)gz)^T \sigma_0 v_2(t)gz &= ((Id+U)gz)^T \sigma_0 (Id+U)gz \\ &= g^T(\sigma_0 + U^T\sigma_0 + \sigma_0 U + U^T\sigma_0 U)g.\end{aligned}$$

Note that

$$U' := U^T\sigma_0 = \sigma_0 U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{pmatrix}$$

and

$$U^T\sigma_0 U = 0.$$

Thus we have

$$\begin{aligned}(v_2(t)gz)^T \sigma_0 v_2(t)gz &= z^T g^T \sigma_0 gz + 2z^T g^T U'(t)gz \\ &= Q_0(gz) + 2(g_{31}z_1 + g_{32}z_2 + g_{33}z_3)^2 t \\ &:= Q_0(gz) + c(z)t,\end{aligned}$$

where $c(z) \geq 0$ for any $z \in \mathbb{Z}^3$. Now choose

$$t := \frac{s - Q_0(gz)}{c(z)}$$

then $(v_1(t)gz)^T \sigma_0 v_2(t)gz = s$. So if we show is that for any $s \in \mathbb{R}$ there exists $z, z^* \in \mathbb{Z}^3$ such that

$$\frac{s - Q_0(gz^*)}{c(z^*)} < 0 < \frac{s - Q_0(gz)}{c(z)}, \tag{11}$$

we are finished. Consider the set

$$\mathcal{C} := \{p \in \mathbb{R}^3 | Q_0(gp) < min\{s, 0\}\}.$$

We now want to show that $\mathcal{C} \cap \mathbb{Z}^3 \neq \varnothing$, i.e. some z satisfies (11) .
Since Q is indefinite, its corresponding matrix $A := g^{-1}\sigma_0 g$ has an eigenvector e_v with negative eigenvalue λ, and thus

$$Q(e_v) = e_v^T A e_v = \lambda \|e_v\|^2 < 0.$$

If $s \geq 0$ then $\mathcal{C} \neq \varnothing$. If $s < 0$, then for $\alpha > \frac{\sqrt{-s}}{\sqrt{-\lambda}\|e_v\|} > 0$ we get

$$s = \frac{-s}{-\lambda\|e_v\|^2}\lambda\|e_v\|^2 > \alpha^2 Q(e_v) = Q(\alpha e_v),$$

and $\alpha e_v \in \mathcal{C}$. Hence again $\mathcal{C} \neq \varnothing$.

Because c and Q_0 are both homogeneous polynomials of degree 2, for any $\lambda \geq 1$ and $p \in \mathcal{C}$ we have that

$$Q_0(\lambda gp) = \lambda^2 Q_0(gp) < min\{s, 0\} \quad and \quad c(\lambda p) = \lambda^2 c(p) > 0,$$

and thus $\lambda p \in \mathcal{C}$.

By continuity of Q_0 and c, $\mathcal{C}$ is open. Because $\mathcal{C}$ is open, there exists a rational vector $q := (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \in \mathcal{C}$, where $q_i, p_i \in \mathbb{Z}$. Then for $\lambda' := \mathrm{lcm}(q_1, q_2, q_3)$, we have that $\lambda' q \in \mathbb{Z}^3 \cap \mathcal{C}$. So $z := \lambda' q$ satisfies the first inequality in (11). We find z^* there similarly.

□

5 Non-Divergence

5.1 The Space of Lattices

This section aims to introduce the reader to Ω the space of lattices in $\mathbb{R}^3$ with covolume one. Moreover Proposition 5.3 will show that there is in fact a one to one correspondence between Ω and $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z}) = G/\Gamma$. Then in the end of this section we will state the main theorem of this chapter (Theorem 5.1).

Definition 5.1 (Covolume in $\mathbb{R}^n$)**.** *The covolume of a discrete subgroup Λ is defined to be the volume of a fundamental domain for the action of Λ on $\mathbb{R}^n$. Denote it by* $\mathrm{Cov}(\Lambda)$.

Observe that the fundamental domain of a subgroup $\Gamma \subseteq \mathbb{R}^n$ is not unique. The volume of any fundamental domain of Γ has to be the same this is because of how we constructed invariant measures on homogeneous spaces in Section 2.1.2.

Definition 5.2 (Lattice on $\mathbb{R}^n$)**.** *A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ with a finite covolume.*

Remark 5.1. The above definition is a special case of Definition 2.13, where we defined a lattice subgroup of a Lie group.

Example 5.1. $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ with covolume one, as seen in Figure 12. An example of a discrete group which is not a lattice in given in *Example* 2.8.

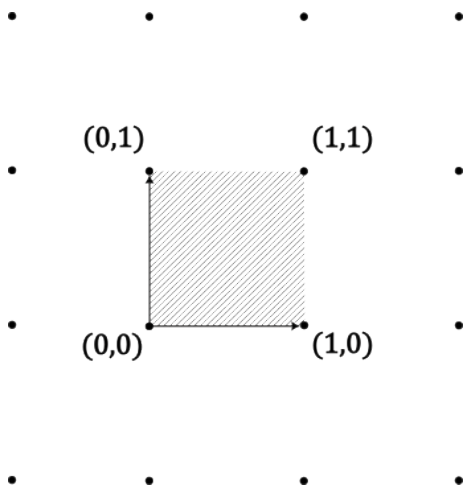


Figure 12: This figure gives a visualisation of the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Note that the marked area is a fundamental domain for $\mathbb{Z}^2$ in $\mathbb{R}^2$ and has area 1.

Let now Ω be the set of all lattices in $\mathbb{R}^3$ of covolume one. Then we have the following proposition.

Proposition 5.3. *There exists a one-to-one correspondence between Ω and G/Γ.*

Proof. Let us define the function $\Phi : G \to \Omega$ by $\Phi(g) = g\mathbb{Z}^3$. First we will show that Φ is well defined and surjective.

1. *Well definition of* Φ. For any $g \in \mathrm{SL}(3,\mathbb{R})$ we have that for any $gx, gy \in g\mathbb{Z}^3$ we have that $gx + gy = g(x+y)$ and $-g(x) = g(-x)$ are in $g\mathbb{Z}^3$. Thus $\Phi(g) = g\mathbb{Z}^3$ is a subgroup of $\mathbb{R}^3$. Moreover, we have that $\mathrm{Cov}(\Phi(g)) = \det(g)\mathrm{Cov}(\mathbb{Z}^3) = 1$, since applying g on a fundamental domain of $\mathbb{Z}^3$ gives a fundamental domain of $g\mathbb{Z}^3$. Thus $\Phi(g) \in \Omega$ for all $g \in G$.

2. *Surjectivity of* Φ. Let $\Lambda \in \Omega$. There exists three linearly independent vectors $\gamma_1, \gamma_2, \gamma_3$ generating Λ. Now consider the matrix

$$g = \begin{pmatrix} - & \gamma_1 & - \\ - & \gamma_2 & - \\ - & \gamma_3 & - \end{pmatrix}.$$

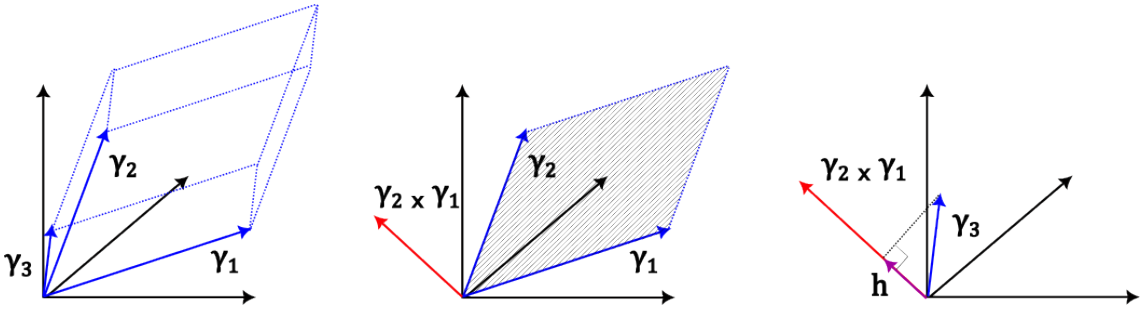


Figure 13: The volume of the parallelepiped is the determinant of g.

Let e_1, e_2, e_3 be the standard basis of $\mathbb{Z}^3$, so that $e_i g = \gamma_i$ for $i \in \{1,2,3\}$. The parallelepiped given by $\gamma_1, \gamma_2, \gamma_3$ is a fundamental domain for the action of Λ on $\mathbb{R}^3$, since the one given by e_1, e_2, e_3 (the unit cube) is a fundamental domain for $\mathbb{Z}^3$. Therefore $\det(g) = \mathrm{Cov}(\Lambda) = 1$ and indeed $g \in G$.

Now note that for any $g \in G$ and $h \in \Gamma$ we have that $ghZ^3 = gZ^3$, and thus Φ maps every element in $g\Gamma$ to the same quadratic form. So Φ induces a well defined and surjective map $\bar{\Phi} : G/\Gamma \to \Omega$. We will now show that it is injective.

Let $g_1, g_2 \in G$ be such that $g_1\mathbb{Z}^3 = g_2\mathbb{Z}^3$. Then $g_2^{-1}g_1\mathbb{Z}^3 = \mathbb{Z}^3$. Let $e_1, e_2, e_3 \in \mathbb{Z}^3$ be the standard basis of $\mathbb{R}^3$. Then $z_i = g_2^{-1}g_1 e_i \in \mathbb{Z}^3$ for $i = 1,2,3$. Consider the matrix h with columns z_1, z_2, z_3. Then $h = g_2^{-1}g_1\mathrm{Id}$ and also $h \in \Gamma$. Thus $g_1 \in g_2\Gamma$, so $\bar{\Phi}$ is also injective, and so a bijection, proving the proposition. □

From now on, identify Ω with G/Γ. This is a homogeneous space and thus by Proposition 2.11 there exists G-invariant metric.

We can now Introduce the main theorem of this chapter.

Theorem 5.4. *Fix a $\sigma > 0$. Then there exists a compact subset K of Ω such that, for any $x = g\Gamma \in G/\Gamma$ and for any unipotent one parameter subgroup $\{u_t\}_{t\in\mathbb{R}}$ of $\mathrm{SL}(3,\mathbb{R})$, exactly one of the following is true.*

- *$\{t \geq 0 \mid u_t x \in K\}$ is unbounded.*
- *There exist $\Delta \subseteq \mathbb{Z}^3$ non trivial subgroup such that $\Delta_{\mathbb{R}}$ is invariant under $\{g^{-1}u_t g\}_{t\in\mathbb{R}}$ with the property that $\mathrm{Cov}(u_t g\Delta) = \mathrm{Cov}(g\Delta) < \sigma$ for any $t \in \mathbb{R}$*

Note that this theorem holds not just for $n = 3$ but for any n and was first proven by Margulis in [7]. The specific wording of Theorem has been chosen similar to that of Theorem V.5.2 in [10], where it is proven differently. The methods used in the proof are mostly taken from a paper by de Saxcé [15], where a more general statement was proven. Furthermore many ideas used in Section 5.4 have been taken from [5] written by Kleinbock.

5.2 Grayson Polygons of Lattices

In this section we will introduce Grayson polygons of three dimensional lattices, which is useful to gain an intuition on how covolumes of lattices behave.

First we to introduce some functions. Let Γ be a lattice in $\mathbb{R}^3$ then define for all $k \in \{1,2,3\}$

$$\lambda_k(\Gamma) := \inf\{\lambda \mid \exists v_1, ..., v_k \in \Gamma, \text{ linearly independent such that } \forall i, |v_i| \leq \lambda\}.$$

Clearly

$$\lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \lambda_3(\Gamma).$$

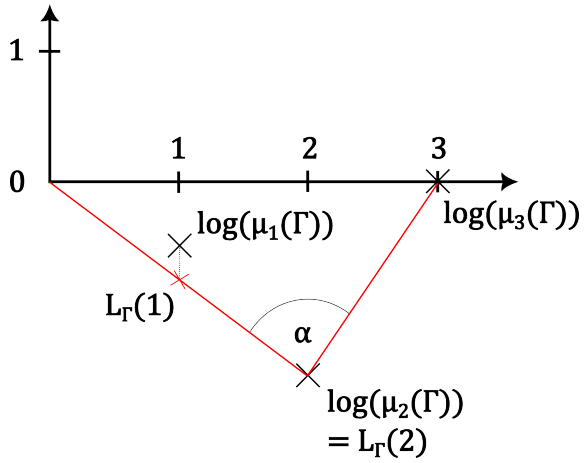


Figure 14: The red line is the graph of a Grayson polygon of a 3-dim lattice with covolume equal to 1 in $\mathbb{R}^3$.

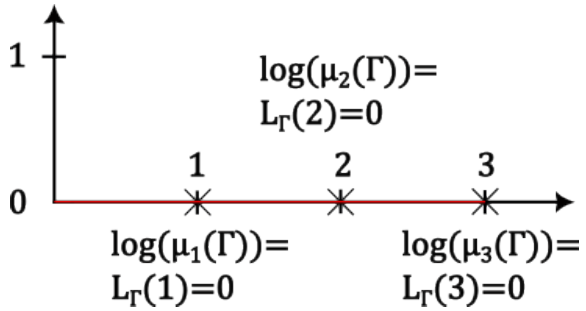


Figure 15: The red line is the graph of a Grayson polygon of the lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$.

Let us also define

$$\mu_1(\Gamma) = inf\{|v|, \text{ where } v \in \Gamma\}^{27},$$
$$\mu_2(\Gamma) = inf\{|v_1 \times v_2|, \text{ where } v_1, v_2 \in \Gamma \text{ linearly independent}\},$$
$$\mu_3(\Gamma) = \mathrm{Cov}(\Gamma).$$

In other words, for $\mathrm{Gr}_{k,3}$ the set of all k-dimensional subspaces of $\mathbb{R}^3$,

$$\mu_k(\Gamma) = \min_{\delta \in \mathrm{Gr}_{k,3}} \mathrm{Cov}_\delta(\Gamma \cap \delta),$$

where $\mathrm{Cov}_\delta(\Gamma \cap \delta)$ is the covolume of $\Gamma \cap \delta$ in δ.

Now we can define the Grayson polygon.

Definition 5.5 (Grayson polygons of three dimensional lattices)**.** *The Grayson polygon of a lattice* Γ *in* $\mathbb{R}^3$ *is the maximal convex function* $L_\Gamma : [0,3] \to \mathbb{R}$ *such that for all* $i \in \{1,2,3\}$ *we have that* $L_\Gamma(i) \le \log(\mu_i(\Gamma))$ *and* $L_\Gamma(0) = 0$

When

$$L_\Gamma(k) - L_\Gamma(k-1) < L_\Gamma(k+1) - L_\Gamma(k),$$

then the graph of L_Γ has an angle at k. Denote

$$\varepsilon_k^\Gamma = \sup\{\varepsilon \ge 0 \mid L_\Gamma(k) - L_\Gamma(k-1) + \varepsilon < L_\Gamma(k+1) - L_\Gamma(k)\}.$$

The larger ε_k^Γ is, the smaller the angle at k is. For example, in Figure 14 $\varepsilon_2 > 0$, hence the angle α at k is smaller than π. On the other hand, in Figure 15 $\varepsilon_k^\Gamma = 0$ for all k and the angle at any k is equal to π.

If $\varepsilon_k > 0$ then $L_\Gamma(k) = \log(\mu_k(\Gamma)) < 0$. This is illustrated in Figure 14, for $k = 2$. In this case there exists some $\delta \in \mathrm{Gr}_{k,3}$ such that $\mathrm{Cov}_\delta(\Gamma \cap \delta) = \mu_k(\Gamma)$. If however $\varepsilon_k^\Gamma = 0$ then $L_\Gamma(k) \leq \log(\mu_k(\Gamma)) \leq 0$. This is illustrated in Figure 14, for $k = 1$.

Now consider the case where $\varepsilon_i > 0$ and $\delta_i \in \mathrm{Gr}_{i,3}$ such that $\mathrm{Cov}_{\delta_i}(\Gamma \cap \delta_i) = \mu_i(\Gamma)$ for $i = 1, 2$. It turns out that this implies that $\Gamma \cap \delta_1 \subset \Gamma \cap \delta_2$. To show this let us first introduce a useful definition.

Definition 5.6 (Flag)**.** *Let V be an n-dimensional vector space. A flag is a sequence $\{V_i\}_{0 \leq i \leq r}$ of subspaces of V such that*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V,$$

where for each i, $V_i \neq V_{i+1}$. In particular, for $d_i := \dim(V_i)$ we have

$$0 = d_0 < d_1 < \cdots < d_r = n.$$

Proposition 5.7. *Let Γ be a lattice in $\mathbb{R}^3$, $\Gamma_0 = \{0\}$, and for each $k > 0$ let Γ_k be the intersection of Γ with a subspace of $\mathbb{R}^3$ achieving the minimum covolume $\mu_k = \mathrm{Cov}(\Gamma_k)$.*

Let $k_1 < \ldots < k_s$ be such that $\varepsilon_{k_i}^\Gamma > 0$ for all $i \in \{1, \ldots, s\}$. Then $\{\Gamma_{k_i}\}_{0 \leq i \leq s}$ form a flag

$$\{0\} \subset \Gamma_{k_1} \subset \ldots \subset \Gamma_{k_s} \subset \Gamma.$$

This flag is called the Harder-Narasimhan filtration.

Proof. Since we are only considering 3 dimensional lattices, s is at most 2. Note that $\{0\} \subset \Gamma_{k_1} \subset \Gamma$ and $\{0\} \subset \Gamma_{k_s} \subset \Gamma$ are trivial, so in the case of $s = 1$ we get the flag

$$\{0\} \subset \Gamma_{k_1} \subset \Gamma.$$

In the case $s = 2$ we have two sublattices of Γ, the 1-dimensional Γ_1 and 2-dimensional Γ_2, both satisfying $\mathrm{Cov}(\Gamma_i) = \mu_i(\Gamma) < 1$[28]. Now assume that Γ_1 is not contained in Γ_2. This means that the shortest vector γ_1 in Γ is linearly independent of the plane containing the smallest parallelogram P spanned by two vectors γ_2, γ_3 in Γ. Now consider that the parallelpiped spanned by $\gamma_1, \gamma_2, \gamma_3$ has base P and a height at most equal to the length of γ_1, and therefore this parallelpiped has volume of at most $|\gamma_1||P|$ (See Figure 17). However then we would get that

$$1 = \mathrm{Cov}(\Gamma) \leq |\gamma_1||P| = \mu_1(\Gamma)\mu_2(\Gamma) < 1.$$

Since this is a contradiction, we get that $\Gamma_1 \subset \Gamma_2$ and hence

$$\{0\} \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma$$

is a flag. □

5.3 Mahler's Compactness Criterion

Recall that we want to study the intersection of orbits with compact subsets in the space of lattices Ω. However in order to do this we need to be able to determine whether a subset of Ω is compact or not. The following theorem, first proven by Mahler in [6], will aid us in doing so.

Theorem 5.8 (Mahler's compactness criterion)**.** *A subset $M \subseteq \Omega$ is relatively compact*[29] *if and only if there exists a neighbourhood $U \subset \mathbb{R}^3$ of the origin such that for any $L \in M$ we have that $L \cap U = \varnothing$.*

Remark 5.2. This chapter is based on Chapter V. of [10], where most things shown in this chapter, are beeing shown to hold for any n. Here however we will restrict ourselves to the 3 dimensional case.

[28] An example of corresponding Grayson Polygon is given in Figure 16

[29] A set U is called relatively compact if its closure is compact.

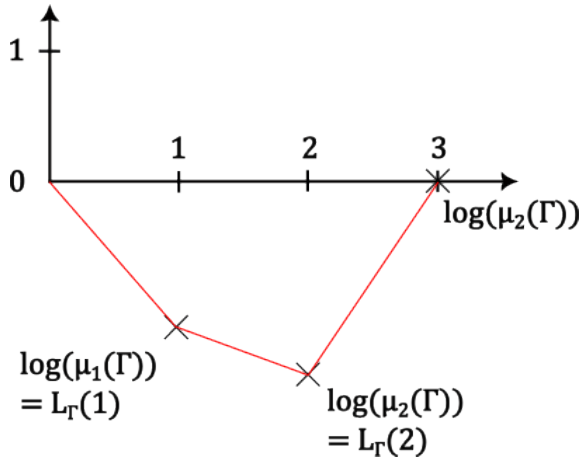


Figure 16: This is a Grayson polygon with two angles.

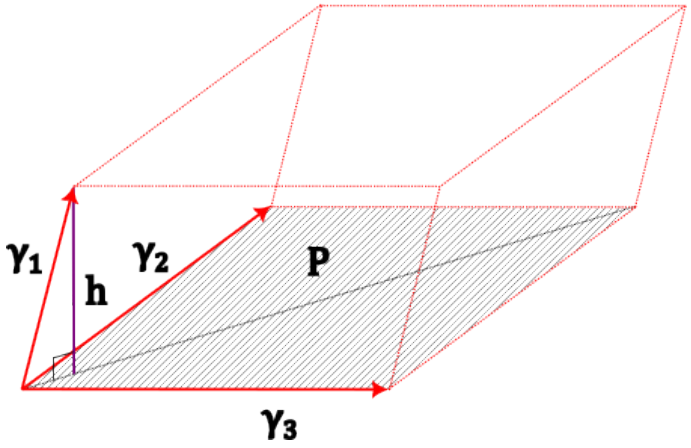


Figure 17: The paralellpiped from the proof of Proposition 5.7.

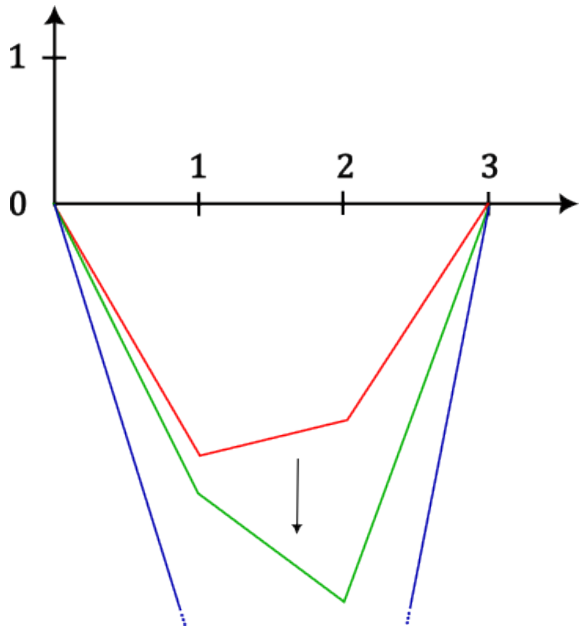


Figure 18: As t gets larger the curve corresponding to $u(t)$ drops further down.

In the proof of Theorem 5.8 we will decompose $G = \mathrm{SL}(3,\mathbb{R})$ into subspaces, which we will then examine separately. The decomposition we will use is called the Iwasawa decomposition and can be defined more general however here we will stick to the definition specific to G.

Theorem 5.9 (Iwasawa decomposition of $\mathrm{SL}(3,\mathbb{R})$)**.** *Let*

$$\begin{aligned}\mathrm{SO}(3,\mathbb{R}) &:= \{a \in G \mid \textit{ The columns of } a \textit{ are pairwise orthogonal}\},\\ D &:= \{d \in G \mid A\ d \textit{ is a diagonal matrix}\},\\ N &:= \{(a_{ij}) \in G \mid a_{ij} = 0, \textit{when } i < j, \textit{ and } a_{ii} = 1\}.\end{aligned}$$

Then we have that $\mathrm{SO}(3,\mathbb{R})DN = G$.

Proof. $\mathrm{SO}(3,\mathbb{R})DN \subseteq G$ is trivial.

To show $G \subseteq \mathrm{SO}(3,\mathbb{R})DN$, let $a \in \mathrm{SL}(3,\mathbb{R})$. We can use the QR-decomposition to get $q \in \mathrm{SO}(3,\mathbb{R})$ and $r \in G$ an upper triangular matrix such that $a = qr$. Observe that we can further decompose the r in to a matrix $d \in D$ and a matrix $n \in N$. This is done as follows.

$$r = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} 1 & \frac{r_{12}}{r_{11}} & \frac{r_{13}}{r_{11}} \\ 0 & 1 & \frac{r_{23}}{r_{22}} \\ 0 & 0 & 1 \end{pmatrix} := dn.$$

Therefore $a \in \mathrm{SO}(3,\mathbb{R})DN$, since a was chosen to be a generic matrix in G we have proven that $G \subseteq \mathrm{SO}(3,\mathbb{R})DN$ and thus concluding the proof. □

How does this help us in determining, whether a subset of $\Omega \cong G/\Gamma = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is dense or not?

To answer this question one needs recall that G is a Manifold and thus locally homeomorphic to $\mathbb{R}^n$. Hence a subset of G is compact if and only if it is closed and bounded. Determining if a set of G is closed can be done using the traditional means[30]. Boundedness of a set $U \subseteq G$ can be proved by verifying that, that the entries of the matrices in U are all bounded.

Let $U = QSM$ be the Iwasawa decomposition of U then if the entries of Q, S and M are bounded then those of U are bounded as well. Moreover it is basic property of topological groups that if two subsets are closed then so is their product. Hence if Q, S and M are closed then so is U.

Note that not any subset $U \subseteq G$ has an Iwasawa decomposition, in the following we will turn to studying Siegel sets a class of sets, which have an Iwasawa decomposition by design.

Definition 5.10 (Siegel set)**.** *For some* $u, t > 0$ *let*

$$N(u) := \{(a_{ij}) \in N \mid a_{ij} \le u \ \ \forall 1 \le i < j \le 3\}$$

$$D(t) := \{(d_{ij}) \in D \mid \frac{d_{ii}}{d_{jj}} \le t, \textit{ when } j = i+1\}.$$

Observe that $N(u)$ *is bounded while* $D(t)$ *is not.*

We define the Siegel set of u, t *to be*

$$S_{t,u} := \mathrm{SO}(n,\mathbb{R})D(t)N(u).$$

One interesting property of Siegel sets is characterized by the following proposition.

Proposition 5.11. *For* $t \ge \frac{2}{\sqrt{3}}$ *and* $u \ge \frac{1}{2}$*, we have that* $G = S_{t,u}\Gamma$

Proof. Let $g \in G$ and let kan be its Iwasawa decomposition. We will start by showing that that $n \in N(u)\Gamma$ And let

$$n = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

[30] I.e. convergence of sequences, finite subcoverings...

Now note that for any two $r, z \in N$ we have that

$$rz = \begin{pmatrix} 1 & r_1 & r_2 \\ 0 & 1 & r_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r_1 + z_1 & r_2 + z_3 r_1 + z_2 \\ 0 & 1 & r_3 + z_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $i \in \{1, 3\}$ let z_i be the closest integer to n_i and let $r_i = n_i - z_i$. Note that $|r_i| \leq \frac{1}{2}$ and $n_i = r_i + z_i$. Let z_2 be the closest integer to $n_2 - z_3 r_1$ and let $r_2 = n_2 - z_3 r_1 - z_1$. Observe that $|r_2| \leq \frac{1}{2}$ and $n_2 = r_2 + z_3 r_1 + z_2$. Hence we can choose $z \in \Gamma \cap N$ and $r \in N(u)$ such that $n = rz$.

Now let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and let

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

Let $v_1 \in \mathbb{Z}_3$ such that

$$\|gv_1\| = \min_{v \in \mathbb{Z}^3} \|gv\|,$$

and $v_2 \in \mathbb{Z}_3$ such that

$$\|gv_2\| = \min_{v \in \mathbb{Z}^3 - \langle v_1 \rangle} \|gv\|.$$

Note that there exists $\gamma \in \Gamma$ such that $\gamma e_1 = v_1$ and $\gamma e_2 = v_2$, thus on G/Γ we may assume that $\|ge_1\| \leq \|ge_2\| \leq \|ge_3\|$

Note that $ne_1 = e_1$ and $ne_2 = e_2 + n_1 e_1$ and $ne_3 = n_2 e_1 + n_3 e_2 + e_3$ Then $\|kane_1\|^2 = a_1^2$ and $\|kane_2\|^2 = n_1^2 a_1^2 + a_2^2$ and $\|kane_3\|^2 = n_2^2 a_1^2 + n_3^2 a_2^2 + a_3^2$.

We have shown above that we may assume that all the $|n_i| \leq \frac{1}{2}$. Thus $n_1^2 a_1^2 + a_2^2 \leq \frac{1}{4} a_1^2 + a_2^2$. Because $\|ge_1\| \leq \|ge_2\|$ we have $a_1^2 \leq \frac{1}{4} a_1^2 + a_2^2$, which implies that

$$\frac{a_1}{a_2} \leq \frac{2}{\sqrt{3}}.$$

Since we also have that $\|ge_2\| \leq \|ge_3\|$, we get that $\frac{1}{4} a_1^2 + a_2^2 \leq \frac{1}{4}(a_1^2 + a_2^2) + a_3^2$ and hence $\frac{3}{4} a_2^2 \leq a_3^2$, which then implies that

$$\frac{a_2}{a_3} \leq \frac{2}{\sqrt{3}}.$$

Thus proving that $an \in D(t)N(u)\Gamma$, which implies that $g = kan \in \mathrm{SO}(n, \mathbb{R})D(t)N(u)\Gamma$. Because $g \in G$ was chosen arbitrarily we have herewith proven this proposition. □

Viewing Lemma 5.11 in the context of Lattices gives us nice results, which will be used in proof of Theorem 5.1.

For some lattice $\Lambda \in \Omega$ we define λ_i, μ_i as in Section 5.1. Let now $g = kan \in S_{\frac{2}{\sqrt{3}}, \frac{1}{2}}$ a representative of λ

Since $\mathrm{SO}(3, \mathbb{R})$ just rotates lattices we will not be interested in k. Thus consider

$$an := \begin{pmatrix} a_1 & n_1 a_1 & a_1 n_2 \\ 0 & a_2 & a_2 n_3 \\ 0 & 0 & a_3 \end{pmatrix} := \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

Note that each column is a lattice point in the corresponding lattice and they correspond to the shortest linearly independent vectors of the lattice.

This implies that

$$\lambda_i(\Lambda) = |v_i|.$$

Lemma 5.12. $\mu_2(\Lambda) = a_1 a_2$

Proof. We have that

$$\mu_2(\Lambda) = \min_{1 \leq i < j \leq 3} |v_i \times v_j|.$$

Moreover we have that

$$|v_1 \times v_2| = a_1 a_2 = \mu_2$$

From the inequalities derived in the proof of Proposition 5.11 we get that

$$a_1 a_3 \leq \sqrt{a_1^2 a_2^2 n_3^2 + a_1^2 a_3^2} = |v_1 \times v_3|,$$

and

$$a_2 a_3 \leq \sqrt{a_2^2 a_3^2 + a_1^2 n_2^2 a_3^2 + (n_1 a_1 a_2 n_3 - a_1 n_2 a_2)^2} = |v_1 \times v_3|.$$

since we assumed that $a_1 \leq a_2 \leq a_3$ we get that

$$\mu_2(\Lambda) = a_1 a_2$$

□

and

$$\mu_2(\Lambda) = \min_{1 \leq i < j \leq 3} |v_i \times v_j|.$$

Lemma 5.13.

$$\frac{\sqrt{3}}{2} \lambda_1(\Lambda)\lambda_2(\Lambda) \leq \mu_2(\Lambda) \leq \lambda_1(\Lambda)\lambda_2(\Lambda)$$

Proof. Using the inequalities from the proof of Proposition 5.11 we get that

$$a_2 \leq \lambda_2(\Lambda) = \sqrt{n_1^2 a_1^2 + a_2^2} \leq a_2 \sqrt{\frac{1}{4}\frac{a_1^2}{a_2^2} + 1} \leq \frac{2}{\sqrt{3}} a_2, \tag{12}$$

which implies that

$$\frac{\sqrt{3}}{2} \lambda_2(\Lambda) \leq a_2 \leq \lambda_2(\Lambda).$$

Recall that $\lambda_1 = \mu_1 = a_1$. This together with Lemma 5.12 proves this lemma □

Remark 5.3. Similarly to (12) we can show that

$$a_3 \leq \lambda_3 = \sqrt{a_3^2 + a_2^2 n_3^2 + a_1^2 n_2^2} \leq \sqrt{a_3^2 + \frac{1}{4}(a_2^2 + a_1^2)} \leq a_3 \sqrt{\frac{4}{3} + \frac{a_1^2}{a_2^2}\frac{a_2^2}{a_3^2}} \leq a_3 \frac{2\sqrt{7}}{3}. \tag{13}$$

Lemma 5.14. *Let $\Delta \subseteq$ Gamma be a 2 dimensional sublattice such that $\mathrm{Cov}(g\Delta) = \mu_2(\Lambda)$. and let Δ' be the two dimensional sublattice not colinear to Δ such that $\mathrm{Cov}(g\Delta')$ is minimal.*

Then we have that $\frac{3}{2\sqrt{7}} \frac{\lambda_3}{\lambda_2} \mu_2 \leq \mathrm{Cov}(\Delta')$

Proof. First notice that using (12) and (13) we get that

$$\frac{3}{2\sqrt{7}} \frac{\lambda_3(\Lambda)}{\lambda_2(\Lambda)} \leq \frac{a_3}{a_2}$$

Note that we have two possibilities either $g\Delta'$ is generated by $\{v_1, v_3\}$ or $\{v_2, v_3\}$. Assuming the first case we get that

$$\frac{3}{2\sqrt{7}} \frac{\lambda_3(\Lambda)}{\lambda_2(\Lambda)} \mu_2(\Lambda) \leq \frac{a_3}{a_2} \mu_2 = a_1 a_3 \leq \mathrm{Cov}(\Delta').$$

Assuming the second case we get

$$\frac{3}{2\sqrt{7}} \frac{\lambda_3(\Lambda)}{\lambda_2(\Lambda)} \mu_2(\Lambda) \leq \frac{a_3}{a_2} \mu_2(\Lambda) \leq \frac{a_3}{a_1} \mu_2(\Lambda) = a_2 a_3 \leq \mathrm{Cov}(\Delta)$$

□

This allows us to prove the following lemma.

Lemma 5.15. *For any $\varepsilon > 0$ and any $\Lambda \in \Omega$. There exist at most finitely many sublattices $\Delta \subseteq \Lambda$ such that $\mathrm{Cov}(\Delta) < \varepsilon$.*

Proof. Note that number of sublattices with covolume less than ε is equal to the sum of the number of two- and one-dimensional sublattices, whose covolume is less than ε. We will therefore first show that there can only exists finitely many one dimensional sublattices and then show the same thing for two dimensional sublattices.

To simplify notation let $\lambda_i(\Lambda) = \lambda_i$.

1. Let v_1, v_2 and v_3 be linearly independent vectors such that $\|v_i\| = \lambda_i$. Let $N_i = \{n \in \mathbb{N} \mid \frac{\varepsilon}{\lambda_i} \geq n\}$ then for all $n \in N_i$ we have that $\|nv_i\| < \varepsilon$. Now note that all vectors $v \in \Lambda$ shorter than ε are linear combinations of vectors of the form $n_i v_i$, where $n_i N_i$. Thus we can conclude that there are finitely many vectors $v \in \Lambda$ that are shorter than ε.

2. Similarly as in the proof of Lemma 5.13 we can show if that for any $u_1, u_2 \in \Lambda$ we have that

$$\frac{2}{\sqrt{3}}\|u_1\|\|u_2\| \geq |u_1 \times u_2|,$$

then there exists $u_1', u_2' \in \Lambda$ such that for at least one $i \in \{1,2\}$ $\|u_i\| > \|u_i'\|$ such that $\langle\{u_1, u_2\}\rangle = \langle\{u_1, u_2\}\rangle$.

Thus if for some sublattice $\Delta \subseteq \Lambda$ such that $\mathrm{Cov}(\Delta) < \varepsilon$ then using Lemma 5.13 there must exist $u_1, u_2 \in V := \{v \in \Lambda \mid \|v\| \leq \frac{2\varepsilon}{\lambda_1\sqrt{3}}\}$ such that $\langle\{u_1, u_2\}\rangle = \Delta$. By the previous case $|V| = n_V < \infty$. Thus the number of sublattices $\Delta \subseteq \Lambda$ such that $\mathrm{Cov}(\Delta) < \varepsilon$ must be less than $\binom{n_V}{2} < \infty$.

□

Proof of Theorem 5.8. 1. We prove that if $M \subseteq \Omega$ is compact then there exists a neighbourhood $U \subset \mathbb{R}^3$ of the origin such that for any $L \in M$ $L \cap U = \{0\}$.

Assume the contrary i.e. that M is compact but such a U does not exist. Then there exists a sequence of lattices $\{L_i\}_{i\in\mathbb{N}} \subseteq M$ and consequentially also a sequence of lattice points $x_i \in L_i - \{0\}$ such that $\lim_{i\to\infty} x_i = 0$. Moreover there must exist a sequence $\{g_i\}_{i\in\mathbb{N}} \subseteq G$ such that $g_i\mathbb{Z}^3 = L_i$ for any $i \in \mathbb{N}$, which implies existence of a sequence $\{y_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}^3$ such that $x_i = g_i y_i$.

Assuming compactness of M gives us a sequence $\{\gamma_i\}_{i\in\mathbb{N}} \subseteq \Gamma - \{0\}$, which upon passing to a subsequence converges in G i.e. we may assume that $\lim_{i\to\infty} g_i\gamma_i = g \in \mathrm{SL}(3,\mathbb{R})$. Then we have

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty} g_i\gamma_i\gamma_i^{-1}y_i = 0$$

and thus

$$\lim_{i\to\infty} \gamma_i^{-1}y_i = 0.$$

Now note that $\gamma_i^{-1}y_i \in \mathbb{Z}^3$ for any i and thus $y_i = 0$ for large $i \in \mathbb{N}$. But this contradicts the assumption that $x_i \in L_i - \{0\}$.

2. We will now assume that a neighbourhood U, as described in the statement of the theorem, exists. Let $M' \subseteq \mathrm{SL}(3,\mathbb{R})$ such that $M'\mathbb{Z}^3 = M$. Note that by Proposition 5.11 there exist $0 < t \leq \frac{2}{\sqrt{3}}$ and $0 < u \leq \frac{1}{2}$ such that $M' \subseteq S_{u,t}$.

Since we assumed existence of a U, we have that there exists $c > 0$ such that for any $x \in M - 0$ we have that

$$\|x\| \geq c.$$

Thus for any $g \in M'$ we have that

$$\|ge_1\| = a_1 \geq c,$$

where a_1 is the first diagonal entry of the diagonal component of the Iwasawa decomposition of g. Now because $g \in \Sigma_{u,t}$

$$\frac{c}{t} \leq \frac{a_1}{t} \leq a_2$$

and therefore also $\frac{c}{t^2} \leq a_3$. Hence for $\alpha := \min_{i\in\{1,2,3\}} \frac{c}{t^i}$ we have that $0 < \alpha \leq a_i$ for any $i \in \{1,2,3\}$. Since $g \in G$ we have that $a_1a_2a_3 = \det(g) = 1$, which implies that $a_i \leq \alpha^{-2}$. This then gives us that M' is bounded.

□

5.4 (C,α)-Good Functions

The goal of this section is to introduce the concept of (C,α)-good functions, the notion of which first appeared in [4], by Kleinbock and Margulis.

Moreover in this section we will show that polynomials of bounded degree (C,α)-good. This will be done in Lemma 5.17, which is essentially a reformulation of Proposition 3.2 in [5]. Furthermore the proof given here follows the one given in [5] closely. Finally this section will prove that for any unipotent, one-parameter flow $\{u_t\}_{t\in\mathbb{R}} \subseteq G$ the function $t \mapsto \mathrm{Cov}(u_t g\Delta)$ is (C,α)-good (Corollary 5.20). This fact will then be heavily used in the proof of Theorem 5.1.

Definition 5.16 ((C,α)-Good Functions)**.** *Let $C,\alpha > 0$ and let $B \subseteq \mathbb{R}$. Also let l denote the Lebesgue measure on $\mathbb{R}$.*

A function $f : B \to \mathbb{R}$ is called (C,α)-good on $\mathbb{R}$ if and only if for any open interval $J \subseteq \mathbb{R}$ and any $\varepsilon > 0$ we have that

$$l(\{x \in J \mid |f(x)| < \varepsilon\}) \leq C\left(\frac{\varepsilon}{\sup_{x\in J}|f(x)|}\right)^{\alpha} l(J).$$

Lemma 5.17. *Let $k \in \mathbb{N}$. Then any polynomial of degree $n \leq k$ is $(k(k+1)^{\frac{1}{k}}, \frac{1}{k})$-good on $\mathbb{R}$.*

Proof. Let us arbitrarily chose an open interval $J \subseteq \mathbb{R}$, an $\varepsilon > 0$ and a polynomial p of at most degree k.

Since J is an open interval we have that $0 < l(J)$. Now let $0 < m$ such that

$$m < l(\{x \in J \mid |p(x)| < \varepsilon\}).$$

Note that if such an m does not exist the statement we want to prove is trivially true.

Consider that $I_\varepsilon = \{x \in J \mid |p(x)| < \varepsilon\}$ consists of multiple intervals, the sum of whose lengths exceeds m. Thus we can chose $k+1$ points $x_1, ..., x_{k+1} \in \{x \in J \mid |p(x)| < \varepsilon\}$ for any $1 \leq i < j \leq k+1$ such that

$$\frac{m}{k} = \int_{x_i}^{x_j} \mathbb{1}_{I_\varepsilon}(x)dl(x) \leq \int_{x_i}^{x_j} \mathbb{1}_J(x)dl(x) = |x_i - x_j|. \tag{14}$$

Now note that because p has a degree of at most k the x_i defined above are sufficient to construct p using Lagrange's interpolation formula i.e.

$$p(x) = \sum_{i=1}^{k+1} p(x_i) \prod_{j=1,\, j\neq i}^{k+1} \frac{x - x_i}{x_i - x_j}.$$

Because the $x_i \in \{x \in J \mid |p(x)| < \varepsilon\}$ we have that $p(x_i) < \varepsilon$. Moreover by (14) we have that $\frac{1}{|x_i - x_j|} \leq \frac{k}{m}$. Therefore for any $x \in J$ we get that

$$p(x) < \sum_{i=1}^{k+1} \varepsilon \prod_{j=1,\, j\neq i}^{k+1} \frac{k(x - x_i)}{m} \leq (k+1)\varepsilon \frac{k^k l(J)^k}{m^k}.$$

Thus by continuity of polynomials we also have that

$$\sup_{x\in J} p(x) \leq (k+1)\varepsilon \frac{k^k l(J)^k}{m^k},$$

which implies that

$$m \leq k(k+1)^{\frac{1}{k}} \left(\frac{\varepsilon}{\sup_{x\in J}|p(x)|}\right)^{\frac{1}{k}} l(J).$$

Since the above equation must hold for any $m \leq l(\{x \in J \mid |p(x)| < \varepsilon\})$ it must also hold for $l(\{x \in J \mid |p(x)| < \varepsilon\})$ its self. This proves the lemma. □

To show that for any unipotent, one-parameter flow $\{u_t\}_{t\in\mathbb{R}} \subseteq G$ the function $t \mapsto \mathrm{Cov}(u_t g\Delta)$ is (C,α)-good, we will first prove two lemmas (Lemma 5.18 and Lemma 5.19), which are interesting in their own right.

Lemma 5.18. *Let $\{u_t\}_{t\in\mathbb{R}} \subseteq G$ be a unipotent one parameter subgroup. Then the entries of the matrix u_t are polynomials of at most degree 2.*

Proof. Recall that all of the eigenvalues of a unipotent matrix are equal to one thus the characteristic polynomial of u_t is $\chi_{u_t}(x) = (x-1)^3 = x^3 - 3x^2 + 3x - 1$. Now note that by Cayley-Hamilton we have that

$$u_t^3 - 3u_t^2 + 3u_t - I = 0,$$

which implies that

$$u_t^2 - 3u_t + 3I = u_t^{-1}.$$

Thus for the matrix $\tau \in \mathfrak{gl}(n,3)$ such that $u_t = e^{t\tau}$, we have that

$$e^{2t\tau} - 3e^{t\tau} + 3I = e^{-t\tau} \tag{15}$$

Expanding exponential functions to series we get

$$I - t\tau + \frac{1}{2}(t\tau)^2 + \frac{5}{6}(t\tau)^3 + \ldots$$

for the left hand side of (15) and

$$I - t\tau + \frac{1}{2}(t\tau)^2 + \frac{1}{6}(t\tau)^3 + \ldots$$

for the right hand side of (15). Thus we must have that $t^3\tau^3 = 0$ for any $t \in \mathbb{R}$. Hence $\tau^3 = 0$. We therefore have that

$$u_t = e^{t\tau} = I + t\tau + \frac{1}{2}t^2\tau^2.$$

This shows that the entries of u_t are polynomials in t of degree at most 2.

□

Lemma 5.19. *Let $g \in G$ and let $\{u_t\}_{t\in\mathbb{R}} \subseteq G$ be a unipotent one parameter subgroup. Then, for any sublattice $\Delta \subseteq \Gamma$, the function $t \mapsto \mathrm{Cov}(u_t g\Delta)^2$ is a polynomial in t of at most degree 8.*

Proof. Let

$$u_t g = \begin{pmatrix} | & | & | \\ u_1(t) & u_2(t) & u_3(t) \\ | & | & | \end{pmatrix}$$

Note that by Lemma 5.18 the entries of each $u_i(t)$ are polynomials of at most degree 2. Note that the same thing also holds for any linear combination of the $u_i(t)$. We will now seperate cases by dimension of Δ

1. Let the dimension of Δ be 1. Then $u_t g\Delta$ is generated by some $v(t) \in \mathbb{R}^3$, which is a linear combination of the u_i. We therefore have that $\mathrm{Cov}(u_t g\Delta)^2 = \|v(t)\|^2$ is a polynomial of at most degree 4.

2. Let the dimension of Δ be 2. Then $u_t g\Delta$ is generated by $v_1(t), v_2(t) \in \mathbb{R}^3$, which are both linear combinations of the u_i. Then $\mathrm{Cov}(u_t g\Delta)^2 = \|v_1(t) \times v_2(t)\|^2$. Note that $v_1(t) \times v_2(t)$ is a vector, whose entries are polynomials of at most degree 4. Therefore $\|v_1(t) \times v_2(t)\|^2$ is a polynomial of at most degree 8.

□

Corollary 5.20. *Let $g \in G$ and let $\{u_t\}_{t\in\mathbb{R}} \subseteq G$ be a unipotent one parameter subgroup. Then for any $\Delta \subseteq \Gamma$ the function $t \mapsto \mathrm{Cov}(u_t g\Delta)$ is $(8 \cdot 9^{\frac{1}{8}}, \frac{1}{4})$-good. .*

Proof. Let $J \subseteq \mathbb{R}$ be some open interval. By Lemma 5.19 we have that $\mathrm{Cov}(u_t g\Delta)^2$ is a polynomial of at most degree 8. Thus by Lemma 5.17 we have that $\mathrm{Cov}(u_t g\Delta)^2$ is (C_0, α_0)-good, where $C_0 = 8 \cdot 9^{\frac{1}{8}}$ and $\alpha_0 = \frac{1}{8}$. Thus

$$\begin{aligned} l(\{t \in J \mid |\mathrm{Cov}(u_t g\Delta)| < \varepsilon\}) &= l(\{t \in J \mid |\mathrm{Cov}(u_t g\Delta)^2| < \varepsilon^2\}) \\ &\leq C_0 \left(\frac{\varepsilon^2}{\sup_{t\in J} |\mathrm{Cov}(u_t g\Delta)^2|} \right)^{\alpha_0} l(J) \\ &= C_0 \left(\frac{\varepsilon}{\sup_{t\in J} |\mathrm{Cov}(u_t g\Delta)|} \right)^{2\alpha_0} l(J). \end{aligned}$$

Therefore $\mathrm{Cov}(u_t g\Delta)$ is $(C_0, 2\alpha_0)$-good. □

5.5 Non-Divergence

This main goal of this section is to give a proof of Theorem 5.1. For this we will also prove several auxilliary lemmas (Lemma 5.21, Lemma 5.22, Lemma 5.25 and Lemma 5.26).

We will start by introducing the measure

$$\begin{aligned} \nu : \sigma(G) &\to \mathbb{R} \\ U &\mapsto \int_{\mathbb{R}} \mathbb{1}_U(utg) dl(t), \end{aligned}$$

where $\sigma(G)$ denotes the Borel σ-Algebra on G and l is the one dimensional Lebesgue measure.

We therefore have that

$$\{u_t g \mid \forall t \in \mathbb{R}\} = \mathrm{Supp}(\nu).$$

Remark 5.4. We have that for any ball $B \subseteq G$ centered at some $u_t g$ there exists $t_0 < t_1$ such that $t_0 = \inf\{s \in \mathbb{R} \mid u_s g \in B\}$ and $t_1 = \sup\{s \in \mathbb{R} \mid u_s g \in B\}$ then

$$\begin{aligned} \nu(\{x \in B \mid \mathrm{Cov}(x\Delta) \leq \varepsilon\}) &= \int_{t_0}^{t_1} \mathbb{1}_{[0,\varepsilon]}(\mathrm{Cov}(u_t g\Delta)) dl(x) \\ &= l(\{t \in (t_0, t_1) \mid \mathrm{Cov}(u_t g\Delta) \leq \varepsilon\}) \end{aligned}$$

Thus by Lemma 5.20 we get that

$$\nu(\{x \in B \mid \mathrm{Cov}(x\Delta) \leq \varepsilon\}) \leq C \left(\frac{\varepsilon}{\sup_{x\in B} |\mathrm{Cov}(x\Delta)|} \right)^{\alpha} \nu(B),$$

where $C = 8 \cdot 9^{\frac{1}{8}}$ and $\alpha = \frac{1}{4}$. For the rest of this section fix C and α.

Lemma 5.21. *For any $t, r \in \mathbb{R}$ such that $r > 0$, we have that $\nu(B(u_t g, 2r)) \leq 3\nu(B(u_t g, r))$.*

Proof. Recall that there exists a G-invariant distance d on $\mathrm{SL}(3, \mathbb{R})$. Thus if $h \in B(u_t g, r)$ then $u_\delta h \in B(u_{t+\delta} g, r)$, as a consequence of this we have that

$$\begin{aligned} \nu(B(u_{t+\delta} g, r)) &= \int_{\mathbb{R}} \mathbb{1}_{B(u_{t+\delta} g, r)}(u_s g)\, dl(s) \\ &= \int_{\mathbb{R}} \mathbb{1}_{B(u_t g, r)}(u_{s-\delta} g)\, dl(s) \\ &= \int_{\mathbb{R}} \mathbb{1}_{B(u_t g, r)}(u_s g)\, dl(s) \\ &= \nu(B(u_t g, r)). \end{aligned}$$

Let now $\alpha = \min\{s \in \mathbb{R} \mid \forall s \leq v \leq t\ u_v g \in B(u_t g, r)\}$ and $\beta = \min\{s \in \mathbb{R} \mid \forall t \leq v \leq s\ u_v g \in B(u_t g, r)\}$.

Note that $\mathrm{Supp}(\nu) \cap B(u_t g, 2r) \subseteq B(u_\alpha g, r) \cup B(u_t g, r) \cup B(u_\beta g, r)$. Hence

$$\nu(B(u_t g, 2r)) = \nu(B(u_\alpha g, r) \cup B(u_t g, r) \cup B(u_\beta g, r)) \leq 3\nu(B(u_t g, r)).$$

Thus proving the statement. □

Lemma 5.22. *Let B be a ball in G such that for any $\Delta \subseteq \Gamma$ sublattice there exists $t \in \mathbb{R}^+$ such that $u_t g \in B$ and $\mathrm{Cov}(u_t g\Delta) \geq \sigma$.*

For any $\varepsilon > 0$ small enough and $M > 1$ define

$$\mathcal{U}^k_{\varepsilon,\sigma,M}(B) := \left\{ b \in B \,\middle|\, \varepsilon^M \sigma \leq \mu_k(b\Gamma) \leq \varepsilon\sigma \text{ and } \lambda_k(b\Gamma) < \varepsilon\lambda_{k+1}(b\Gamma) \right\}.$$

Then we have that

$$\nu(\mathcal{U}^k_{\varepsilon,\sigma,M}) \leq 3MC^*\varepsilon^{\alpha_1}\nu(B)$$

where C^* and α_1 are positive constants not depending on the choice of B.

Proof. Then for some $1 > \varepsilon > 0$ define

$$\mathcal{A}^k_{\varepsilon,\sigma} = \left\{ b \in B \,\middle|\, \varepsilon^{\frac{4}{3}}\sigma \leq \mu_k(b\Gamma) \leq \varepsilon\sigma \text{ and } \lambda_k(b\Gamma) < \varepsilon\lambda_{k+1}(b\Gamma) \right\}.$$

Now let Δ be a k-dim sub lattice such that $\mathrm{Cov}(u_t g\Delta) = \mu_k(u_t x)$. Let $R > 0$ be such that for $r \leq R$ we have that if $y\Gamma \in B(u_t g, r) \cap \mathrm{Supp}(\nu)$ then $\mathrm{Cov}(y\Delta) < \varepsilon^{\frac{1}{2}}\sigma$. For simplicity let us define $B_t := B(u_t g, R)$.

Claim 5.23. *If $y\Gamma \in B_t \cap \mathcal{A}^k_{\varepsilon,\sigma}$, then any $\Delta_y \subseteq \Gamma$ k dimensional sub-lattice having $\mathrm{Cov}(y\Delta_y) = \mu_k(y)$, we must have that $\Delta_y = \Delta$.*

Proof. To show this we will assume the converse i.e. Δ is not colinear to Δ_y then we have that

$$\mathrm{Cov}(y\Delta) \geq c_k \frac{\lambda_{k+1}(y\Gamma)}{\lambda_k(y\Gamma)}\mu_k(y\Gamma),$$

where c_k is a constant depending on k, for $k = 1$ this is trivial and we have $c_1 = 1$ for $k = 2$ this follows from Lemma 5.14 and $c_2 = \frac{3}{2\sqrt{7}} \geq \frac{1}{2}$.

Since $y \in \mathcal{A}^k_{\varepsilon,\sigma}$ we have that $\lambda_k(y\Gamma) < \varepsilon\lambda_{k+1}(y\Gamma)$ and thus we have

$$\frac{\lambda_{k+1}(y)}{\lambda_k(y)}\mu_k(y) > \frac{\mu_k(y\Gamma)}{\varepsilon}.$$

Also because $y \in \mathcal{A}^k_{\varepsilon,\sigma}$ we have that $\mu_k(y\Gamma) \geq \varepsilon^{\frac{4}{3}}\sigma$ thus for ε small enough, i.e. $\varepsilon^{\frac{1}{6}} < \frac{3}{2\sqrt{7}}$ we get that

$$\mathrm{Cov}(y\Delta) > c_k\varepsilon^{\frac{1}{3}}\sigma \geq \varepsilon^{\frac{1}{2}}\sigma.$$

Now since $y \in B_t$ we have that $\mathrm{Cov}(y\Delta) \leq \varepsilon^{\frac{1}{2}}\sigma$, which contradicts the equation above. □

Thus for all $y \in \mathcal{A}^k_{\varepsilon,\sigma} \cap B_t$ we have that $\mathrm{Cov}(y\Delta) = \mu_k(y\Gamma) \leq \varepsilon\sigma$, which together with Remark 5.4 implies that

$$\nu(\mathcal{A}^k_{\varepsilon,\sigma} \cap B_t) \leq \nu(\{y \in 2B_t \mid \mathrm{Cov}(y\Delta) \leq \varepsilon\sigma\}) \leq C\varepsilon^{\frac{\alpha}{2}}\nu(2B_t).$$

We can now use Lemma 5.21, which gives us

$$\nu(\mathcal{A}^k_{\varepsilon,\sigma} \cap B_t) \leq 3C\varepsilon^{\frac{\alpha}{2}}\nu(B_t).$$

Theorem 5.24 (Besicovitch's Covering Theorem)**.** *Let M be an n-dimensional manifold. Let $E \subseteq M$ and let F be collection of balls in M, such that for any $x \in E$, F contains a ball centered at x*[31]*.*

Then there exists $U \subseteq F$ such that U covers E and any $x \in E$ is contained in at most C_n balls in U, where C_n is a constant depending only on n.

[31] Such an F is called a Besicovitch covering of E.

Now $E \coloneqq \mathcal{A}^k_{\varepsilon,\sigma} \cap B \cap \mathrm{Supp}(\nu)$ then $F \coloneqq \{B_t \mid t \in (t_0, t_1)\}$ is a Besicovitch covering of E. Thus there exists a subcovering $U \subseteq F$ such that each $x \in E$ is contained in at most C_9 balls in U. And thus we have

$$\begin{aligned}\nu(\mathcal{A}^k_{\varepsilon,\sigma} \cap B) &\leq \sum_{B_t \in U} \nu(\mathcal{A}^k_{\varepsilon,\sigma} \cap B_t) \\ &\leq 3C\varepsilon^{\frac{\alpha_0}{2}} \sum_{B_t \in U} \nu(B_t) \\ &\leq 3C\varepsilon^{\frac{\alpha_0}{2}} C_9 \nu(B) \\ &\coloneqq C^* \varepsilon^{\alpha_1} \nu(B).\end{aligned}$$

Let $M > \frac{4}{3}$ and let M' be the smallest integer larger that $3M$.

Let now $j \in \{3, 4, ..., M'\}$ and $\sigma_j \coloneqq \varepsilon^{\frac{j}{3}-1}\sigma$. By what we have previously shown $\nu(\mathcal{A}^k_{\varepsilon,\sigma_j} \cap B) \leq C^* \varepsilon^{\alpha_1} \nu(B)$ and that

$$\mathcal{U}^k_{\varepsilon,\sigma,M} \subseteq \bigcup_{3 \leq j \leq M'} \mathcal{A}^k_{\varepsilon,\sigma_j}.$$

We can therefore conclude that

$$\nu(\mathcal{U}^k_{\varepsilon,\sigma,M} \cap B) \leq 3MC^* \varepsilon^{\alpha_1} \nu(B).$$

□

Lemma 5.25. *Let $\varepsilon \in (0, \frac{1}{2})$. If for some $x \in G/\Gamma$ there exists $i \in \{1, 2\}$ such that $\mu_i(u_t x) < \varepsilon\sigma$. Then there must exist $k \in \{1, 2\}$ such that both*

$$\mu_k(u_t x) \leq \varepsilon\sigma \tag{16}$$

and

$$\varepsilon\lambda_{k+1}(u_t x) > \lambda_k(u_t x) \tag{17}$$

hold.

Proof. In this proof we will fix $u_t x$ and thus we will write $\lambda_i \coloneqq \lambda_i(u_t x)$ and $\mu_i \coloneqq \mu_i(u_t x)$, for all $i \in \{1, 2, 3\}$.

We will now split this proof into two cases in the first case we will assume that $\mu_1 \leq \sigma\varepsilon$ and in the second case we will assume that $\mu_2 \leq \sigma\varepsilon$.

1. Assume $\mu_1 \leq \sigma\varepsilon$ but $\lambda_1 \geq \varepsilon\lambda_2$, this implies $\sigma \geq \lambda_2$. Hence we get that $\mu_2 \leq \lambda_1\lambda_2 \leq \sigma\lambda_1 \leq \sigma^2\varepsilon$. Note that in this case the Lemma implies that $\varepsilon\lambda_3 > \lambda_2$, therefore we will show that $\lambda_2 \geq \varepsilon\lambda_3$ leads to a contradiction.

   Assume $\lambda_2 \geq \varepsilon\lambda_3$. But then

   $$\mu_3 \leq \mu_2\lambda_3 \leq \sigma^2\varepsilon\lambda_3 \leq \lambda_2\sigma^2 \leq \sigma^2\frac{\lambda_1}{\varepsilon} \leq \sigma^3 < 1,$$

   however this can not be because $\mu_3 = 1$.

2. Assume now that $\mu_2 \leq \sigma\varepsilon$ but $\lambda_2 \geq \varepsilon\lambda_3$. Moreover we will assume that $\mu_1 \geq \sigma\varepsilon$, because if not we would be in the first case, which we have already checked.

   We then get

   $$\frac{\sqrt{3}}{2}\sigma\varepsilon\lambda_2 \leq \frac{\sqrt{3}}{2}\lambda_2\lambda_1 \leq \mu_2 \leq \sigma\varepsilon,$$

   which implies that $\lambda_2 \leq \frac{2}{\sqrt{3}}$. We also have

   $$\mu_3 \leq \lambda_3\mu_2 < \sigma\varepsilon\lambda_3 \leq \sigma\lambda_2,$$

   which gives a contradiction if $\sigma < \frac{\sqrt{3}}{2}$.

For the case where $\sigma \geq \frac{\sqrt{3}}{2}$, define $\varepsilon' = \varepsilon^{\frac{1}{2}}$ and $\sigma' = \varepsilon^{\frac{1}{2}}\sigma$. Note that $\varepsilon\sigma = \varepsilon'\sigma'$ and $\varepsilon'\lambda_3 \geq \varepsilon\lambda_3$. Now since $\varepsilon \in (0, \frac{1}{2})$ we have that $\sigma'\lambda_2 \leq 1$. Thus if we assume that $\lambda_2 \geq \varepsilon'\lambda_3$, we get the same contradiction as above.

If $\varepsilon'\lambda_3 \geq \lambda_2 > \varepsilon\lambda_3$ then

$$\frac{1}{\varepsilon'} \leq \frac{\lambda_2}{\lambda_3} \leq \frac{1}{\varepsilon}$$

and then

$$\frac{1}{\varepsilon'} \leq \frac{\lambda_2}{\lambda_3}\mu_3 \leq \lambda_2\mu_2 \leq \lambda_2\sigma\varepsilon$$

Thus

$$\frac{1}{\sigma'\varepsilon} \leq \lambda_2$$

but then

$$\frac{3}{4\sigma'} \leq \frac{\sqrt{3}}{2\sigma'\varepsilon}\sigma\varepsilon \leq \mu_2 \leq \sigma\varepsilon$$

Which implies that

$$\frac{3}{4} \leq \sigma^2\varepsilon,$$

now note $\sigma \leq 1$ and $\varepsilon \leq \frac{1}{2}$ thus the above is a contradiction.

□

Lemma 5.26. *Let B be a ball in G such that for any $\Delta \subseteq \Gamma$ sublattice there exists $t \in \mathbb{R}^+$ such that $u_t g \in B$ and $\mathrm{Cov}(u_t g\Delta) \geq \sigma$.*

For any $\frac{1}{2} > \varepsilon > 0$ let

$$\mathcal{B}_\varepsilon = \{g \in G \mid \exists k \in \{1,2\} \text{ s.t. } \mu_k(g\Gamma) \leq \sigma\varepsilon\}.$$

then we have that

$$\nu(\mathcal{B}_\varepsilon \cap B) \leq 12C^* \frac{\varepsilon^{\alpha_1}}{1-\varepsilon^{\alpha_1}}\nu(B).$$

Proof. Let

$$\mathcal{B}^0_\varepsilon = \left\{g \in G \mid \exists k \in \{1,2\} \text{ s.t. } \mu_k(g\Gamma) \leq \sigma\varepsilon \text{ and } \forall k \in \{1,2\};\ \mu_k(g\Gamma) \geq \sigma\varepsilon^2\right\}$$

By Lemma 5.25 we have that if $x \in \mathcal{B}^0_\varepsilon$ then $x \in \mathcal{U}^2_{\varepsilon,\sigma,2} \cup \mathcal{U}^1_{\varepsilon,\sigma,2}$, and therefore applying Lemma 5.22 we have that

$$\nu(\mathcal{B}^0_\varepsilon) \leq \nu(\mathcal{U}^1_{\varepsilon,\sigma,2}) + \nu(\mathcal{U}^2_{\varepsilon,\sigma,2}) \leq 12C^*\varepsilon^{\alpha_1}\nu(B).$$

Now note that

$$\mathcal{B}_\varepsilon \subseteq \bigcup_{0\leq n} \mathcal{B}^0_{\varepsilon^{2^n}}$$

and therefore

$$\begin{aligned}
\nu(\mathcal{B}_\varepsilon \cap B) &\leq \sum_{0\leq n} \nu(\mathcal{B}^0_{\varepsilon^{2^n}} \cap B) \\
&\leq 12C^*\nu(B) \sum_{0\leq n} \varepsilon^{2^n\alpha_1} \\
&\leq 12C^*\nu(B) \sum_{1\leq n} (\varepsilon^{\alpha_1})^n \\
&\leq 12C^* \frac{\varepsilon^{\alpha_1}}{1-\varepsilon^{\alpha_1}}\nu(B)
\end{aligned}$$

□

Proof of Theorem 5.1. Recall the Theorem 5.1 stated that for any $0 < \sigma$, there exists a compact set $K \subseteq G/\Gamma$ such that for any unipotent one parameter flow $\{u_t\}_{t\in\mathbb{R}}$ we have that one of the following holds

1. $\{t \geq 0 \mid u_t x \in K\}$ is unbounded.
2. There exist $\Delta \subseteq \mathbb{Z}^3$ non trivial subgroup such that $\Delta_{\mathbb{R}}$ is invariant under $\{g^{-1}u_t g\}_{t\in\mathbb{R}}$ with the property that $\mathrm{Cov}(u_t g\Delta) = \mathrm{Cov}(g\Delta) < \sigma$ for any $t \in \mathbb{R}$

We will split the proof into two parts in the first part we will show that if there does not exist a $\Delta \subseteq \Gamma$ sublattice such that for any $t \in \mathbb{R}$ $\mathrm{Cov}(u_t g\Delta) < \sigma$, then we are in the second case. In the second part we will show that if such a Δ exists we are in the second case.

1. By Lemma 5.15 for any $T \in \mathbb{R}$ the number of sublattices $\Lambda \subseteq \Gamma$ such that that $\mathrm{Cov}(u_T g\Delta) < \sigma$ is finite.

   Since we are assuming that there exists no $\Delta \subseteq \Gamma$ sublattice such that for any $t \in \mathbb{R}$ $\mathrm{Cov}(u_t g\Delta) < \sigma$, then for any $t_0 \in \mathbb{R}$ there must always exist a ball B centered at $u_{t_0} g$ in G such that for any $\Delta \subseteq \Gamma$ sublattice there exists $t \in \mathbb{R}^+$ such that $u_t g \in B$ and $\mathrm{Cov}(u_t g\Delta) \geq \sigma$.

   Such a ball must exist because $\mathrm{Cov}(u_t g\Delta)^2$ is a positive polynomial of finite degree and thus $\mathrm{Cov}(u_t g\Delta)^2 \to \infty$ as t goes to ∞ or $-\infty$. Hence there must also exist a $t_1(\Delta) > t_0$ such that $\mathrm{Cov}(u_{t_1(\Delta)} g\Delta) > \sigma$. If we now let $t_1 = \max\{t_1(\Delta) \mid \Delta \subseteq \Gamma \text{ sublattice s.t. } \mathrm{Cov}(u_{t_0} g\Delta) < \sigma\}$, then any ball centered at t_0, with radius $r = d(u_{t_0} g\Gamma, u_{t_1} g\Gamma)$ has the desired properties.

   We can now use Lemma 5.26 to show that for any ε small enough

   $$\nu(\mathcal{B}_\varepsilon \cap B) \leq 12C^* \frac{\varepsilon^{\alpha_1}}{1-\varepsilon^{\alpha_1}} \nu(B).$$

   Note if we choose ε such that

   $$12C^* \frac{\varepsilon^{\alpha_1}}{1-\varepsilon^{\alpha_1}} < 1.$$

   Then we have that

   $$\nu(\mathcal{B}_\varepsilon^C \cap B) > 0,$$

   since B only contains $u_t g\Gamma$, if $t_0 \leq t$ and because t_0 was chosen arbitrarily, the above equation implies that $u_t g$ is recurrent on $\mathcal{B}_\varepsilon^C$.

   Note that by Mahler's compactness criterion (Theorem 5.8) the set

   $$K := \mathcal{B}_\varepsilon^C = \{x\Gamma \in G/\Gamma \mid \mu_k(x) \geq \varepsilon\sigma \ \forall k \in \{1,2\}\}$$

   is compact, for any $\varepsilon > 0$.

   Hence we are in the first case.

2. Let us now assume that there exists a sublattice $\Delta \subseteq \Gamma$ as described above. Then by Lemma 5.18 we have that $\mathrm{Cov}(u_t g\Delta)^2$ is a positive polynomial of finite degree. Thus because $\mathrm{Cov}(u_t g\Delta)^2 < \sigma^2$ we have that $\mathrm{Cov}(u_t g\Delta)^2$ has to be constant, and therefore $\mathrm{Cov}(u_t g\Delta)$ has to also be constant.

   If Δ is a 1-dimensional lattice then $\mathrm{Cov}(u_t g\Delta)$ being constant implies that $u_t g\Delta$ is a vector, whose entries are constant polynomials, hence we have that $u_t g\Delta_{\mathbb{R}} = \Delta_{\mathbb{R}}$, where $\Delta_{\mathbb{R}} \in \mathrm{Gr}_{1,3}$ such that $\Delta = \Delta_{\mathbb{R}} \cap \Gamma$.

   If Δ is a 2-dimensional lattice then for any two generators $v_1, v_2 \subseteq \Delta$, we have that $|u_t g v_1 \times u_t g v_2|$ has to be constant polynomial and thus $u_t g v_1 \times u_t g v_2$ must be a vector whose entries are constant polynomials as well and thus we must also have that $u_t g\Delta_{\mathbb{R}} = \Delta_{\mathbb{R}}$, where $\Delta_{\mathbb{R}} \in \mathrm{Gr}_{2,3}$ such that $\Delta = \Delta_{\mathbb{R}} \cap \Gamma$.

   We can therefore conclude that we are in case two.

□

6 Margulis' Proof of Oppenheim's Conjecture

6.1 Introduction

Let Q be a quadratic form in n-variables. One might ask how the set $Q(\mathbb{Z}^n) \subseteq \mathbb{R}$ looks like. If all the coefficients of Q lie in $\mathbb{Z}$, then obviously $Q(\mathbb{Z}^n) \subseteq \mathbb{Z}$. If the coefficients all lie in $\mathbb{Q}$, i.e. are of the form $\frac{p_i}{q_i}$, then $Q(\mathbb{Z}^n) \subseteq \frac{1}{\prod q_i}\mathbb{Z}$. In both cases $Q(\mathbb{Z}^n)$ is not dense in $\mathbb{R}$. This raises the question: if the coefficients of Q are rationally independent, is $Q(\mathbb{Z}^n) \subseteq \mathbb{R}$ dense? In [12], published in 1929, Oppenheim conjectured that for any non-degenerate, irrational, indefinite quadratic form Q form in $n \geq 5$ variables $Q(\mathbb{Z}^n) \subseteq \mathbb{R}$ is dense.

However no attempts to prove Oppenheim's conjecture using methods from analytic number theory were fruitful. Furthermore no counter examples were found in the case $n \geq 3$ (see Section 6.2 for a counterexample for $n = 2$). In fact the conjecture would remain open for decades. Finally, in 1986 Margulis settled the conjecture for all $n \geq 3$.

Theorem 6.1 (Margulis, [8])**.** *For any non-degenerate, irrational, indefinite quadratic form Q in $n \geq 3$ variables $Q(\mathbb{Z}^n) \subseteq \mathbb{R}$ is dense.*

In Chapter 4 we have seen that there exists a deep relation between homogeneous dynamics and quadratic forms. It was exactly this relation which Margulis used to prove Oppenheim's Conjecture. The purpose of this chapter will be to give a proof of Theorem 1.1 along those lines.

6.1.1 Outline of the Chapter

Section (6.2) contains a counter example for $n = 2$. In Section 6.3 we will reduce Oppenheim's conjecture to the case when $n = 3$. This reduces it to a question about dynamics on $\mathrm{SL}(3, \mathbb{R})$. This question will be answered using the blueprint given in Section 2.3.

In order to apply this blueprint, we need to show that $\mathrm{SO}(Q_0)$-orbits of irrational quadratic forms are not closed. In fact, we will see in Section 6.4 that this holds only in this case. In Section 6.5 we will find minimal sets of subgroups of $\mathrm{SO}(Q_0)$. Section 6.6 is dedicated to solving some issues that arise when considering the convergence of orbits of one-parameter subgroups. Finally, in Section 6.7 the proof of Theorem 1.1 will be completed.

6.2 Counterexample for $n = 2$

Lemma 6.2. *There exists a non degenerate, indefinite, irrational quadratic form in 2 variables such that $Q(\mathbb{Z}^2) \subseteq \mathbb{R}$ is not dense.*

Proof. Let

$$Q(x, y) = y^2 - (1 + \sqrt{3})^2 x^2 = y^2 - (4 + 2\sqrt{3})x^2.$$

Note that 1 and $-(4 + 2\sqrt{3})$ are rationally independent, thus Q is irrational. We also have that

$$\sigma_Q := \begin{pmatrix} 1 & 0 \\ 0 & -(4 + 2\sqrt{3}) \end{pmatrix}$$

has eigenvalues $1 > 0$ and $-(4 + 2\sqrt{3}) < 0$, so Q is indefinite and non-degenerate.

We will now proceed by showing that $Q(\mathbb{Z}^2)$ is not dense around 0.

Claim 6.3. *There exists a constant $C > 0$ such that*

$$|(1 + \sqrt{3}) - \frac{p}{q}| > \frac{C}{q^2} \qquad \forall (p, q) \in \mathbb{Z}^2,\ q \neq 0.$$

Proof. Let

$$f(x) := x^2 - 2x - 2.$$

Then

$$f((1 + \sqrt{3})) = 4 + 2\sqrt{3} - 2 - 2\sqrt{3} - 2 = 0$$

and

$$f((1-\sqrt{3})) = 4 - 2\sqrt{3} - 2 + 2\sqrt{3} - 2 = 0,$$

and thus by the fundamental theorem of algebra, there cannot exist $a \in \mathbb{Z}$ such that $f(a) = 0$. Now consider $f'(x) = 2x - 2$ the derivative of f. Note that f' is continuous and therefore bounded from above on any bounded interval. Therefore there must exist $\epsilon > 0$ such that $|f'|$ is bounded from above by some constant $m > 0$ on the interval $I_\epsilon := \left[(1+\sqrt{3}) - \epsilon, (1+\sqrt{3}) + \epsilon\right]$.

Moreover, note that f has integer coefficients and thus for any $p \in \mathbb{Z}$

$$f(p) \in \mathbb{Z} - \{0\}.$$

Thus for all $p, q \in \mathbb{Z},\ q \neq 0$ we get

$$\left|f\left(\frac{p}{q}\right)\right| = \frac{1}{q^2}|p^2 - 2pq - q^2| \geq \frac{1}{q^2}. \tag{18}$$

For any $x \in I_\epsilon$, by the fundamental theorem of calculus,

$$|f(x)| = \left|f(x) - f(1+\sqrt{3})\right| = \left|\int_{1+\sqrt{3}}^{x} f'(x)dx\right| \leq m\left|x - (1+\sqrt{3})\right|,$$

and in particular for any $\frac{p}{q} \in I_\epsilon \cap \mathbb{Q}$

$$\left|(1+\sqrt{3}) - \frac{p}{q}\right| \geq \frac{1}{m}\left|f\left(\frac{p}{q}\right)\right|. \tag{19}$$

Now combining (18) and (19) we get

$$\left|(1+\sqrt{3}) - \frac{p}{q}\right| \geq \frac{1}{mq^2}.$$

On the other hand, if $\frac{p}{q} \notin I_\epsilon$, then we have a trivial lower bound on the left hand side of this inequality. In any case, Setting $C = \min(\frac{\epsilon}{2}, \frac{1}{2m}) > 0$ finishes the proof. □

Applying Claim 6.3 to Q we get for any $x, y \in \mathbb{Z}$ such that $\frac{y}{x} > 0, x \neq 0$:

$$\begin{aligned}
|Q(x,y)| &= \left|x^2(\frac{y}{x} - (1+\sqrt{3}))(\frac{y}{x} + (1+\sqrt{3}))\right| \\
&> \left|x^2\frac{C}{x^2}(\frac{y}{x} + (1+\sqrt{3}))\right| \\
&= \left|C(\frac{y}{x} + (1+\sqrt{3}))\right| \\
&= C\left|\frac{y}{x}\right| + C\left|1+\sqrt{3}\right| \\
&\geq C\left|1+\sqrt{3}\right|.
\end{aligned}$$

If $\frac{y}{x} < 0$ we use that $Q(x,y) = Q(x,-y)$, and if $x = 0$, $Q(0,\cdot)$ is clearly discrete. Thus $Q(\mathbb{Z}^2)$ is not dense around 0. □

6.3 Reduction to $n = 3$

The goal of this section will be to show that if Theorem 1.1 holds for $n = 3$ then it holds for all any $n > 3$ as well. We will achieve this by proving the following lemma.

Lemma 6.4 ([10], Lemma VI.2.1)**.** *If Q is a non-degenerate, indefinite, irrational quadratic form in n variables, where $n \geq 3$. Then there exists a rational hyperplane L such that the restriction of Q to L is non-degenerate, indefinite and irrational.*

For Oppenheim's conjecture (Theorem 1.1) the above Lemma shows that if Q is a non-degenerate, indefinite, irrational quadratic form in $n > 3$ variables then there exists an $n-1$ dimensional subspace L in $\mathbb{Q}^n$ such that $Q|_L$ is a non-degenerate, indefinite, irrational quadratic form. Thus inductively there must exist a 3 dimensional subspace L^* of $\mathbb{Q}^n$ such that $Q|_{L^*}$ is a non-degenerate, indefinite, irrational quadratic form. Thus there must exist a non-degenerate, indefinite, irrational quadratic form Q^* in 3 variables such that $Q(\mathbb{Z}^3) \subseteq Q^*(\mathbb{Q}^3) = Q(L^*)$. Thus Lemma 6.4 shows that if Oppenheim's Conjecture holds for $n = 3$ then it also holds for $n \geq 3$.

Proof. Let Q be a quadratic form in n variables and let σ be the corresponding symmetric matrix. Let $B := \{b_1, b_2, ..., b_n\}$ be the set of eigenvectors of σ. Decompose $B = B^+ \cup B^-$ such that for $\sigma b_i = \lambda b_i$, if $\lambda > 0$ then $b_i \in B^+$ and if $\lambda < 0$ then $b_i \in B^-$. Since Q is indefinite, both B^+ and B^- are nonempty. Since $n \geq 3$, either $|B^+| \geq 2$ or $|B^-| \geq 2$. Without loss of generality, assume that $b_n \in B^+$ and $B^+ - \{b_n\} \neq \varnothing$. Now for each $i < n$ let

$$b_i' := b_i - b_n\langle b_i, b_n\rangle$$
$$L := span(b_1', b_2', ..., b_{n-1}').$$

Clearly $L \cap span(B^+) \neq \varnothing$, and since for $b_i \in B^-$ simply $b_i = b_i'$, then also $L \cap span(B^-) \neq \varnothing$. Therefore Q is indefinite when restricted to L. Additionally, the restriction of a non-degenerate quadratic form to a subspace is also non-degenerate by definition.

We will now construct a hyperplane L_t, on which Q is not just indefinite and non-degenerate but also irrational. Specifically, we will construct hyperplanes $\{L_t\}_{t\in\mathbb{Q}}$ such that if t is close to 0 then L_t is close to L. We will show that if L_t is sufficiently close to L, then it is also non-degenerate and indefinite. We will also show that for some sufficiently small t, the form $Q|_{L_t}$ has to be irrational.

Claim 6.5. *There exists $L^* = span(b_1^*, b_2^*, ..., b_{n-1}^*)$ such that Q is indefinite and non-degenerate when restricted to L^* and $b_i^* \in \mathbb{Q}^n$ for all $i \in \{1, 2, ..., n-1\}$.*

Proof. Define the matrix

$$A := (Q(b_i', b_j'))_{1\leq i,j\leq n-1}.$$

Note that A is the matrix of $Q|_L$ in the basis $b_1', b_2', ..., b_{n-1}'$. By continuity of Q, if we chose $b_1^*, b_2^*, ..., b_{n-1}^*$ close to $b_1', b_2', ..., b_{n-1}'$ then the matrix

$$A^* := (Q(b_i^*, b_j^*))_{1\leq i,j\leq n-1}$$

will be close to A and therefore the eigenvalues will be close to each other as well. Hence for $b_1^*, b_2^*, ..., b_{n-1}^*$ close enough to $b_1', b_2', ..., b_{n-1}'$, we have that A being non-degenerate and indefinite is equivalent to A' being non-degenerate and indefinite. Thus by density of $\mathbb{Q}$ in $\mathbb{R}$ we can choose $\{b_1^*, b_2^*, ..., b_{n-1}^*\} \subseteq \mathbb{Q}^n$ s.t. A^* is non-degenerate and indefinite.
Let now $L^* = span(b_1^*, b_2^*, ..., b_{n-1}^*)$. □

Let now $x \in \mathbb{Q}^n - L^*$ and we define

$$L_t := span(b_1^*, b_2^*, ..., b_{n-1}^* + tx) \qquad t \in \mathbb{Q}.$$

Note that for t sufficiently small $Q|_{L_t}$ will still be non-degenerate and indefinite.

Claim 6.6. *We can choose $t \in \mathbb{Q}$ such that Q restricted to L_t is not proportional to a rational quadratic form.*

Proof. We will show this by contradiction. We assume $\forall t \in \mathbb{Q}$ small enough for $Q|_{L_t}$ to still be irrational and indefinite, $Q|_{L_t}$ is proportional to a rational quadratic form.

Let σ be the matrix corresponding to Q. By our assumption for any t there must exist an $\alpha_t \in \mathbb{R}$, such that

$$b_i^*\sigma b_j^* \in \alpha_t\mathbb{Q},$$

for all $i, j \in \{1, 2, ..., n-2\}$. Since $b_i^*\sigma b_j^*$ is independent of the choice of t, α_t is independent of the choice of t i.e. $\alpha_0 = \alpha_t$ for all t. We also have

$$(b_i^*)^T\sigma(b_{n-1}^* + tx) = (b_i^*)^T\sigma b_{n-1}^* + t(b_i^*)^T\sigma x \in \alpha_t\mathbb{Q},$$

for all $i \in \{1, 2, ..., n-2\}$.Note that this also holds for $t = 0$ and thus

$$(b_i^*)^T \sigma b_{n-1}^* \in \alpha_t \mathbb{Q}.$$

Considering the case where $t = 0$ we have that since $t \in \mathbb{Q}$ that

$$t(b_i^*)^T \sigma x \in \alpha_t \mathbb{Q}.$$

Now consider

$$(b_{n-1}^* + tx)^T \sigma (b_{n-1}^* + tx) = (b_{n-1}^*)^T \sigma b_{n-1}^* + 2t b_{n-1}^* \sigma b_{n-1}^* + t^2 x^T \sigma x \in \alpha_t \mathbb{Q}.$$

Again using $t = 0$ gives us

$$(b_{n-1}^*)^T \sigma b_{n-1}^* \in \alpha_t \mathbb{Q}.$$

Then $t \neq 0$ gives us

$$2t(b_{n-1}^*)^T \sigma b_{n-1}^* + t^2 x^T \sigma x \in \alpha_t \mathbb{Q}$$

now since $t \in \mathbb{Q}$ we have that

$$2(b_{n-1}^*)^T \sigma b_{n-1}^* + t x^T \sigma x \in \alpha_t \mathbb{Q},$$

choosing $t = 0$ now gives us

$$2(b_{n-1}^*)^T \sigma b_{n-1}^* \in \alpha_t \mathbb{Q}.$$

We thus also must have that

$$x^T \sigma x \in \alpha_t \mathbb{Q}.$$

Since $\{b_1^*, b_2^*, ..., b_{n-1}^*, x\}$ is a rational basis of the entire space we have just shown that Q is rational, which contradicts our definition of Q. □

Thus we have shown that for t sufficiently small, $Q|_{L_t}$ is an irrational, non-degenerate and indefinite quadratic form in $n-1$ variables. □

6.4 H-Orbits of Irrational Quadratic Forms are not Closed

We have seen in Proposition 4.9 that the image set $Q(\mathbb{Z}^3)$ is connected to some point in the space $H\backslash G/\Gamma$. This space turns out to be quite hard to study [32], but we can instead study the Orbits of H in G/Γ. This will explain using homogeneous dynamics why we need Q to be irrational to get that $Q(\mathbb{Z}^3)$ is dense.

Lemma 6.7 ([10], Proposition VI.2.4)**.** *Let $Q = g_Q^{-1} \cdot Q_0$ be a non-degenerate, indefinite quadratic form with positive determinant, where $g_Q \in G$. Denote $x = g_Q \Gamma \in G/\Gamma$.*

If the H-Orbit of x is closed in G/Γ, then Q is rational.

Remark 6.1. The lemma above is similar to Lemma 3.16. In the proof of Hedlund's Theorem we showed that the orbit of the horocycle flow in $\Gamma\backslash \mathrm{SL}(2, \mathbb{R})$ is not closed, when Γ is a uniform lattice. This was crucial because it allowed us to apply Lemma 2.29.

Proof. Assume Hx is closed in G/Γ. Consider $x_0 = \Gamma$ as an element of G/Γ. Let $\mathrm{SO}(Q) = g_Q^{-1} H g_Q$. Then $\mathrm{SO}(Q)x_0 = g_Q^{-1} Hx$ is closed since we assumed that Hx is closed.

Assume that the following claim holds. It will be proven later.

Claim 6.8. *Assume $\mathrm{SO}(Q)x_0$ is closed, and let $\Delta := \Gamma \cap \mathrm{SO}(Q)$.*

If Q' is a quadratic form where $\Delta \subseteq \mathrm{SO}(Q')$, then Q' is proportional to Q.[33]

[32] If we used Theorem 1.12, we would have got that $Hg\Gamma$ is itself dense in G if and only if g is irrational.

[33] Q is proportional to Q' if and only if the corresponding matrices $\sigma_Q, \sigma_{Q'}$ are proportional.

Let $\sigma = g_Q^T \sigma_0 g_Q$. And consider the set A of all 3×3 matrices S solving the equation

$$\delta^T S \delta = S, \quad \forall \delta \in \Delta := \Gamma \cap \mathrm{SO}(Q).$$

Since $\Delta \subseteq \mathrm{SO}(Q)$ we have that $\delta^T \sigma \delta = \sigma$ for any $\delta \in \Delta$. Thus $\sigma \in A$, and $A \neq \{0\}$. Since $\Delta \subseteq \Gamma$ the entries of all $\delta \in \Delta$ are integers, and A is defined by an homogeneous integral linear equation system, which has a nontrivial solution. Therefore there must exist $S \in A$ such that all coefficients of S are rational.

By Claim 6.8, A is a set of matrices corresponding to quadratic forms proportional to Q. In particular, Q is proportional to the rational, quadratic form given by the matrix $S^T \sigma_0 S$. Hence Q is itself rational. □

In in order to prove Claim 6.8 we will first prove the following claim.

Claim 6.9. *Let $\Delta := \mathrm{SO}(Q) \cap \Gamma$, and assume $\mathrm{SO}(Q)x_0$ is closed.*
If $\Delta \subseteq \mathrm{SO}(Q')$, then all unipotent one-parameter subgroups of $\mathrm{SO}(Q)$ are contained in $\mathrm{SO}(Q')$.

Proof. First consider the function

$$f_p : \mathrm{SO}(Q) \to \mathbb{R}, \quad g \mapsto Q'(g^{-1}p)$$

for any $p \in \mathbb{R}^3$. Note that for $g_1, g_2 \in \mathrm{SO}(Q)$ such that

$$g_1 \in g_2 \mathrm{SO}(Q'),$$

we have that $f_p(g_1) = f_p(g_2)$ for all $p \in \mathbb{R}^3$. Since $\Delta \subseteq \mathrm{SO}(Q')$ and $\Delta \subseteq \mathrm{SO}(Q)$ we have that the function

$$f_p^* : \mathrm{SO}(Q)/\Delta \to \mathbb{R}, \quad g\Delta \mapsto Q'(g^{-1}p)$$

is well defined and continuous.

If now $\{u(t)\}_{t\in\mathbb{R}} \subseteq \mathrm{SO}(Q)$ is a unipotent one parameter subgroup by Theoram 5.1 there exists a compact set $K \subset G/\Gamma$ such that $\{t \in \mathbb{R} \mid u(t)x_0 \in K\}$ is unbounded.

Consider the function

$$\phi : \mathrm{SO}(Q)/\Delta \to \mathrm{SO}(Q)x_0, \quad g\Delta \mapsto gx_0.$$

Since $\Delta \subseteq \Gamma$ and $x_0 = \Gamma$ as element of Ω, we have that

$$hx_0 = gx_0 \qquad \forall h \in g\Delta.$$

By Lemma 2.9 ϕ is a homeomorphism. This gives us that $K' := \phi^{-1}(K)$ is compact. Thus also $f_p^*(K')$ must be compact in $\mathbb{R}$, thus by Heine-Borel bounded and closed.

Now consider that

$$q_p : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f_p(u(t)).$$

Note that since $u(t)$ is unipontent all its entries must be polynomials of finite degree, therefore $q_p(t)$ must be polynomial of finite degree. Therefore $q_p(t)$ is either constant or $q_p(t) \to \pm\infty$ for $t \to \infty$.

By the definition of K and K' we have that

$$\{t \in \mathbb{R} \mid q_p(t) \in f_p^*(K')\}$$

is unbounded but $f_p^*(K')$ is bounded thus $q_p(t)$ does not go to $\pm\infty$ as $t \to \infty$, thus $q_p(t)$ has to be constant. Hence we have that

$$q_p(t) = Q'(u(t)p) = Q'(p) = q_p(0) \quad \forall p \in \mathbb{R}^3.$$

Thus we have that $\{u(t)\}_{t\in\mathbb{R}} \subseteq \mathrm{SO}(Q')$, which concludes the proof of the claim 6.9. □

Proof of Claim 6.8. We have that $\mathrm{SO}_0(Q)$, the connected component of $\mathrm{SO}(Q)$ containing the identity, is generated by unipotent one-parameter subgroups, this is a consequence of Lemmas 4.11 and 4.8.. This together with Claim 6.9 gives us that $\mathrm{SO}_0(Q) \subseteq \mathrm{SO}(Q')$.

Now let σ, σ' be the symmetric matrices corresponding to Q, Q' respectively. Now recall that by definition of SO we have that

$$g^t \sigma g = \sigma \qquad \forall g \in \mathrm{SO}(Q)$$

and

$$g^t \sigma' g = \sigma' \qquad \forall g \in SO(Q').$$

Thus for any $h \in \mathrm{SO}_0(Q)$

$$h\sigma'\sigma^{-1}h^{-1} = h\sigma' h^t (h^{-1})^t \sigma^{-1} h^{-1} = \sigma'\sigma^{-1}$$

Thus $\sigma'\sigma^{-1}$ is contained in the centralizer of $\mathrm{SO}_0(Q)$ in G, which is conjugate to the centralizer of H_0 in G, which consists of multiples of the identity, this is a consequence of Lemma 4.12.. This means there exists a constant $c \in \mathbb{R}$ such that

$$\sigma'\sigma^{-1} = c\,\mathrm{id}.$$

and therefore $\sigma' = c\sigma$, so σ and σ' are indeed proportional. □

Remark 6.2. Note that Lemma 6.7 implies that if Q is a non-degenerate, indefinite, irrational quadratic then its H-Orbit is not closed.

6.5 Existence of Minimal Sets

Recall that in the proof of Hedlund's Theorem we already showed existence of minimal Sets, namely in Lemma 3.17. We used that " $\subseteq$ " is an order relation on families of closed invariant subsets, and then the compactness of Ω gives us that the intersection of sets in such an ordered family is never empty. Then we used Zorn's Lemma to show that there must exist a minimal set.

But in the setting of Oppenheim's Conjecture, we do not have the luxury of Ω being compact. However, to show existence of minimal sets we will nevertheless use a similar argument, but we will use it on a compact subspace of Ω.

6.5.1 Existence of DV_1-Minimal Sets

In this section will prove existence of DV_1 minimal subsets of Ω and discuss some of their properties. We will start by finding a compact set K on which we can use the same argument as in Lemma 3.17.

Lemma 6.10. *For $\lambda \in \mathbb{R}$, let*

$$a(\lambda) := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Then there exists a compact set $K \subseteq \Omega$ with the following property. For every $x = g\Gamma \in \Omega$, there exists $\lambda_0 > 0$ such that the set

$$A_\lambda := \{t \geq 0 \mid v_1(t)a(\lambda)x \in K\}$$

is unbounded for all $\lambda \geq \lambda_0$.

Proof. By Theorem 5.1 with $\sigma = 1$, there exists a compact set $K \subseteq \Omega$ such that, for each $x = g\Gamma \in \Omega$ and for any unipotent one-parameter subgroup $\{u_t\}_{t\in\mathbb{R}}$ and we have two possibilities

1. the set $\{t \geq 0 \mid u_t x \in K\}$ is unbounded.
2. there exists a non-trivial subgroup Δ of $\mathbb{Z}^3$ such that $\Delta_{\mathbb{R}}$ is invariant under $\{g^{-1}u_t g\}_{t\in\mathbb{R}}$ and $\mathrm{Cov}(u_t g\Delta) = \mathrm{Cov}(g\Delta) < 1$ for all $t \in \mathbb{R}$.

In our case we will let $u_t = a(\lambda^{-1})v(t)a(\lambda)$. We will now show that for any $g\Gamma \in \Omega$ there exists λ_0 large enough such that for all $\lambda \geq \lambda_0$ we can exclude the second case of Theorem 5.1.

Fix $g \in G$ and let $x = g\Gamma$. Also let $W_1 := \mathbb{R}e_1$ and $W_2 := \mathbb{R}e_1 \oplus \mathbb{R}e_2$. Now let us pick

$$\lambda_0 := max\{\frac{1}{\mathrm{Cov}(g\mathbb{Z}^3 \cap W_1)}, \frac{1}{\mathrm{Cov}(g\mathbb{Z}^3 \cap W_2)}, 1\},$$

and let $\lambda \geq \lambda_0$. Let $\Delta_{\mathbb{R}}$ be a subgroup of $\mathbb{Z}^3$ such that $\Delta = \Delta_{\mathbb{R}} \cap \Gamma$, for some $\Delta_{\mathbb{R}} \in \mathrm{Gr}_{k,3}$ for $k \in \{1, 2\}$. Moreover let $\Delta_{\mathbb{R}}$ be $g^{-1}u_t g$ invariant.

$$V_1 g\Delta_{\mathbb{R}} = a(\lambda)^{-1}V_1 a(\lambda)g\Delta_{\mathbb{R}} = g\Delta_{\mathbb{R}}.$$

Thus $g\Delta_{\mathbb{R}}$ is a proper V_1-invariant subspace of $\mathbb{R}^3$. Therefore $g\Delta_{\mathbb{R}} \in \{W_1, W_2\}$. Note now that W_1 and W_2 are both invariant under $a(\lambda)$ with

$$\det(a(\lambda)|_{W_1}) = \det(a(\lambda)|_{W_2}) = \lambda.$$

Hence we have that

$$\mathrm{Cov}(a(\lambda)g\Delta) = \lambda \mathrm{Cov}(g\Delta).$$

Furthermore, because $g\Delta$ is either contained in $g\mathbb{Z}^3 \cap W_1$ or in $g\mathbb{Z}^3 \cap W_2$, we have that either

$$\mathrm{Cov}(g\Delta) \geq \mathrm{Cov}(g\mathbb{Z}^3 \cap W_1)$$

or

$$\mathrm{Cov}(g\Delta) \geq \mathrm{Cov}(g\mathbb{Z}^3 \cap W_2).$$

Hence

$$\begin{aligned}\mathrm{Cov}(a(\lambda)g\Delta) &= \lambda \mathrm{Cov}(g\Delta) \\ &\geq \lambda \min\{\mathrm{Cov}(g\mathbb{Z}^3 \cap W_1), \mathrm{Cov}(g\mathbb{Z}^3 \cap W_2), 1\} = \frac{\lambda}{\lambda_0} \\ &\geq 1 = \sigma\end{aligned}$$

We must therefore be in the first case of (Theorem 5.1). Therefore we have A_λ is unbounded for all $\lambda \geq \lambda_0$, which was exactly, what we had to show. □

We will proceed by showing the existence of DV_1-minimal sets.

Proposition 6.11. *Any closed DV_1-invariant subset of Ω contains a minimal non-empty closed DV_1-invariant set.*

Proof. As described before we will use the K from Lemma 6.10 to proceed in a similar way to Lemma 3.17.

For this let Z be closed non-empty DV_1-invariant subset of Ω, let $\{X_\alpha\}_\alpha$ be a totally ordered family of DV_1 invariant subsets of Z. Now consider that since X_α are DV_1 invariant, $X_\alpha \cap K \neq \emptyset$. Now recall that Definition 3.18 gives us that since K is compact, we have that

$$\bigcap_\alpha (X_\alpha \cap K) \neq \emptyset.$$

And then we have

$$\emptyset \neq \bigcap_\alpha (X_\alpha \cap K) = \left(\bigcap_\alpha X_\alpha\right) \cap K \subseteq \bigcap_\alpha X_\alpha.$$

By the finite intersection theorem we have that $\bigcap_\alpha X_\alpha$ is non-empty. By Zorn's Lemma we therefore have that $\{X_\alpha\}_\alpha$ has minimal elements. Thus we can conclude that there exist DV_1-minimal sets. □

Now we will proceed by investigating some properties of DV_1-minimal sets, which we will use later on in the proof of Oppenheim's conjecture (Theorem 1.1).

Let us define

$$M := \{g \notin DV_1 \mid gy \in DV_1 y\},$$

where y is a fixed element of DV_1-minimal set. We will be using Lemma 2.29 to show that $M \neq$. But first we will prove the following proposition.

Proposition 6.12. *There are no closed DV_1-orbits in Ω.*

Proof. We will show this by contradiction. We will therefore assume that there exists an $x \in \Omega$ such that DV_1x is closed. Consider the subgroup

$$\Delta := \{g \in DV_1 \mid gx = x\}$$

Note that Δ is discrete. Since DV_1x is closed, the mapping

$$\begin{aligned}\Phi : DV_1/\Delta &\to DV_1x,\\ g\Delta &\mapsto gx\end{aligned}$$

is a homeomorphism by Lemma 2.9. By Lemma 4.3, Δ is either contained in V_1 or Δ is a cyclic group generated by some element vdv^{-1}, where $d \in D\backslash\{e\}$ and $v \in V_1$. We will now proceed by showing that both possibilities lead to a contradiction. For this choose a compact subset $K \subseteq \Omega$ as defined in Lemma 6.10.

1. Suppose $\Delta \subseteq V_1$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $\lambda_n \to \infty$. Since $\Delta \subseteq V_1$, for no sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ does the sequence $\{a(\lambda_n)v_1(t_n)\Delta\}_{n\in\mathbb{N}}$ have a convergent subsequence in DV_1/Δ . Now since Φ is a homeomorphism, the sequence

   $$\{\Phi(a(\lambda_n)v_1(t_n)\Delta)\}_{n\in\mathbb{N}} = \{a(\lambda_n)v_1(t_n)x\}_{n\in\mathbb{N}}$$

   has no convergent subsequence. But by Lemma 6.10 for all large λ, there exists $t \geq 0$ such that

   $$v_1(t)a(\lambda)x = a(\lambda)v_1(\lambda^{-1}t)x \in K,$$

   this is contradiction.

2. Suppose now that Δ is a cyclic group generated by some element vdv^{-1}, where $d \in D\backslash\{e\}$ and $v \in V_1$. Then for $\lambda > 0$ the set $V_1a(\lambda)\Delta = a(\lambda)V_1\Delta$ is closed and the mapping

   $$\begin{aligned}\Psi : \mathbb{R} &\to V_1a(\lambda)\Delta/\Delta,\\ t &\mapsto v_1(t)a(\lambda)\Delta\end{aligned}$$

   is a homeomorphism. Because Φ is a homeomorphism as well we have that

   $$\begin{aligned}\Phi \circ \Psi : \mathbb{R} &\to V_1a(\lambda)x,\\ t &\mapsto v_1(t)a(\lambda)x\end{aligned}$$

   is a homeomorphism. Note also that $V_1a(\lambda)x$ is closed in Ω. This is because by assumption DV_1x is closed, thus $a(\lambda)V_1x = V_1a(\lambda)x$, which is a closed subset of DV_1x, must be closed as well. Hence

   $$B := \{v_1(t)a(\lambda)x \mid t \geq 0\} \cap K$$

   is compact. Thus its preimage under $\Phi \circ \Psi$

   $$A_\lambda := (\Phi \circ \Psi)^{-1}(B) = \{t \geq 0 \mid v_1(t)a(\lambda)x \in K\}$$

   must be compact too. Note that by Lemma 6.10 A_λ is unbounded for large λ. But by Heine-Borel compact subsets of $\mathbb{R}$ can never be unbounded. Thus we have reached a contradiction.

□

6.5.2 Existence of V_1-Minimal Sets

In the proof of Oppenheim's Conjecture we will also be using the existence of minimal closed V_1-invariant sets, for simplicity we will, throughout this section, refer to such sets as V_1-minimal. We will first study some V_1-invariant sets.

Let $\rho > 0$, $i = 1, 2$, $W_1 = \mathbb{R}e_1$ and $W_2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$. Now consider the sets

$$A(i, \rho) := \{g\Gamma \in \Omega \mid g\mathbb{Z}^3 \cap W_i \text{ spans } W_i, \text{ and } \mathrm{Cov}(g\mathbb{Z}^3 \cap W_i) = \rho\}.$$

In this context we want to think of the elements of $A(i,\rho)$ as lattices in $\mathbb{R}^3$. Recall that in Chapter 5.1, it was explained in detail how G/Γ, can be thought of as the space of lattices in $\mathbb{R}^3$ with covolume 1.

First we can think of the elements of $g\Gamma \in A(1,\rho)$. Then because of the condition that $g\mathbb{Z}^3 \cap W_1$ spans W_1 there exists a $z \in \mathbb{Z}^3$ such that $gz = \lambda e_1$. Moreover we have that $\mathrm{Cov}(g\mathbb{Z}^3 \cap W_i) = \rho$. Thus for all $z \in \mathbb{R}$ we have $\rho \leq |\lambda|$ and we must have that there exists at least one $z^* \in \mathbb{Z}^3$ such that $gz^* = \rho e_1$.

In $A(2,\rho)$ by the first condition we have that there exist $z_1, z_2 \in \mathbb{Z}^3$ such that $gz_1 := b_1$ and $gz_2 := b_2$ form a basis of W_2. The second condition implies that for any two linearly independent $a_1, a_2 \in g\mathbb{Z}^3 \cap W_2$ we have that

$$|a_1 \times a_2| \geq \rho,$$

Note that this bound is met i.e. there exist some $m_1, m_2 \in g\mathbb{Z}^3$ such that

$$|m_1 \times m_2| = \rho.$$

By Theorem 5.1, we have that for every $\sigma > 0$ there exists a compact set $K \subseteq G/\Gamma$, such that either

$$\{t \geq 0 \mid v_1(t)g\Gamma \in K\}$$

is unbounded or there exists some $\rho \leq \sigma$ such that $g\Gamma \in A(i,\rho)$ for some $i \in \{1,2\}$.

Lemma 6.13. *For any $\rho > 0$, $A(1,\rho)$ is closed and V_1-invariant.*

Proof. Let $g\Gamma \in A(1,\rho)$. Choose $m \in \mathbb{Z}^3$ such that gm is a generator of the group $g\mathbb{Z}^3 \cap W_1$. Now let $\gamma \in \Gamma$ such that $\gamma e_1 = m$. Consider that if we define

$$a := \begin{pmatrix} \rho^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then we have that $ag\gamma e_1 = e_1$. Moreover we will define

$$B_1 := G_1 a^{-1} \cap \mathrm{SL}(3,\mathbb{R})$$

where

$$G_1 := \{g \in \mathrm{GL}(3,\mathbb{R}) \mid ge_1 = e_1\}.$$

Note that for any matrix $g \in \mathrm{GL}(3,\mathbb{R})1$ commutes with a and a^{-1} if and only if $g \in G_1$.

Now note that $ag\gamma \in G_1$ and $g\gamma \in G$ we therefore have that $g \in B_1\Gamma$ since $g\Gamma \in A(1,\rho)$ was chosen arbitrarily, we can conclude that

$$A(1,\rho) = B_1\Gamma/\Gamma.$$

We will now show that $A(1,\rho)$ is V_1 invariant. For this consider that if $\forall v_1 \in V_1$, we have that $v_1 e_1 = e_1$. Thus $V_1 \subseteq B_1$. And therefore $V_1 B_1 = B_1$ i.e. $A(1,\rho)$ is V_1-invariant.

We will now show that $A(1,\rho)$ is closed. Consider that $\Gamma\lambda e_1$ is a discrete subgroup of $\mathbb{R}^3$ for any $\lambda \in \mathbb{R}$. Thus for any diagonal matrix $d \in \mathrm{SL}(3,\mathbb{R})$, we have that Γd is a discrete, and therefore also closed, subgroup of $\mathrm{SL}(3,\mathbb{R})$. Now consider the set

$$C_d := \Gamma d G_1 \cap \mathrm{SL}(3,\mathbb{R}) = \{g \in \mathrm{SL}(3,\mathbb{R}) \mid ge_1 \in \Gamma d e_1\},$$

and the continuous function

$$f : \mathrm{SL}(3,\mathbb{R}) \to \mathbb{R}^3; \quad g \mapsto ge_1.$$

Now note that $f(C_d) = \Gamma d e_1$. Thus by continuity of f, C_d must be closed as well. Because d was chosen to be a generic diagonal matrix, $C_{a^{-1}} = B_1\Gamma$ is closed and therefore $B_1\Gamma/\Gamma = A(1,\rho)$ must be closed too. □

Lemma 6.14. *For any $\rho > 0$, $A(2,\rho)$ are closed and V_1-invariant.*

Proof. Let $g\Gamma \in A(2,\rho)$. Choose $m_1, m_2 \in \mathbb{Z}^3$ such that $\{gm_1, gm_2\}$ generates the group $g\mathbb{Z}^3 \cap W_2$. Thus there must exist a $\gamma \in \Gamma$ such that $\gamma e_1 = m_1$, $\gamma e_2 = m_2$.

Observe that since for $i \in \{1,2\}$ $gm_i \subseteq W_2$ and since a is the identity on W_2, we have that $ag\gamma e_i = gm_i$. Now note that because $g\Gamma \in A(2,\rho)$, we have that $|gm_1 \times gm_1| = \rho$. Furthermore since because $\{gm_1, gm_2\}$ generates the group $g\mathbb{Z}^3 \cap W_2$, we must have that

$$agm_1 \times agm_2 = \rho e_3. \tag{20}$$

Thus for $e_3^T ag\gamma e_i = 0$ for any $i \in \{1,2\}$, which implies that for some constant $c \in \mathbb{R}$ we have that $e_3^T g\gamma = ce_3^T$. Hence we have that

$$g\gamma = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & c \end{pmatrix}.$$

To calculate c note that we have that $\det(g\gamma) = 1$ i.e. $\langle g\gamma e_1 \times g\gamma e_2, g\gamma e_3\rangle = 1$. because of (20) we must have that $c = \rho^{-1}$. Thus if we let

$$a := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix},$$

then $e_3^T ag\gamma = e_3^T$.

We can now proceed similarly as in the proof 6.13. Let

$$G_2 := \{g \in \mathrm{GL}(3,\mathbb{R}) \mid e_3^T g = e_3^T\},$$

and let

$$B_2 := G_2 a^{-1} \cap \mathrm{SL}(3,\mathbb{R}) = a^{-1} G_2 \cap \mathrm{SL}(3,\mathbb{R}).$$

We then have that

$$A(2,\rho) = B_2\Gamma/\Gamma.$$

To show that $A(2,\rho)$ is V_1-invariant, note that $e_3^T v_1 = e_3^T$ for any $v_1 \in V_1$. Thus $V_1 \subseteq B_2$ hence $A(2,\rho)$ must be V_1- invariant.

Using a similar argumentation as in the proof of Lemma 6.13, we can show that $A(2,\rho)$ is closed. □

The following lemma will be used to show that any V_1-minimal set is compact.

Lemma 6.15. *Let X be a locally compact space, and let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a set of homeomorphisms on X such that the map $(x,t) \mapsto \varphi_t(x)$ is continuous. Suppose that there exists a compact subset K of X such that, $\forall x \in X$, the sets*

$$\{t \le 0 \mid \varphi_t x \in K\} \quad \textit{and} \quad \{t \ge 0 \mid \varphi_t x \in K\}$$

are unbounded. Then X is compact.

Proof. First consider the set

$$K' := \{\varphi_s x \mid \forall x \in K \text{ and } -1 \le s \le 1\}.$$

Note that, since $\{\varphi_t\}_{t\in\mathbb{R}}$ is a one parameter subgroup and $\{s \in \mathbb{R} \mid -1 \le s \le 1\}$ is compact by Heine-Borel, K' is compact as well. By the definition of K we have that

$$\{j \in \mathbb{N} \mid \varphi_j x \in K'\} \quad \text{and} \quad \{j \in \mathbb{N} \mid \varphi_{-j} x \in K'\}$$

are both unbounded for any $x \in X$. Let now K^* be a compact neighbourhood of K'. Therefore

$$X = \bigcup_{j=1}^{\infty} \varphi_{-j}(K^*). \tag{21}$$

This is because $\{j \in \mathbb{N} \mid \varphi_j x \in K'\}$ is unbounded for every $x \in X$, thus for some $j \in \mathbb{N}$ $\varphi_j x \in K^*$ and thus $x \in \varphi_{-j}(K^*)$. Similarly

$$X = \bigcup_{j=1}^{\infty} \varphi_j(K^*). \tag{22}$$

Moreover the following claim holds.

Claim 6.16. *There exists $N \in \mathbb{N}$ such that*

$$X = \bigcup_{j=1}^{N} \varphi_{-j}(K^*).$$

Proof. We will prove this claim by contradiction, therefore we will assume that there exists no $N \in \mathbb{N}$ such that

$$X = \bigcup_{j=1}^{N} \varphi_{-j}(K^*).$$

However there then exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that for any $n \in \mathbb{N}$

$$\varphi_j(x_n) \in X - K^* \quad \forall 1 \leq j \leq n.$$

Now note, that from (22) we have that

$$X = \bigcup_{j=1}^{\infty} \varphi_j(K^*).$$

Therefore for every $n \in \mathbb{N}$ there exist some $j \geq n$ such that $\varphi_{-j}(x_n) \in K^*$. We will call the smallest such j j_n. For every n let us also define

$$y_n := \varphi_{-j_n}(x_n).$$

Then we have that

$$\varphi_j(y_n) \in X - K^* \quad \forall j \in \mathbb{N}, n \geq j.$$

Now since K^* is compact and $y_n \in K^*$ for any n, there must exist $y \in K^*$ such that $\lim_{n\to\infty} y_n = y$. Note that therefore for any $j > 0$ we have that

$$\varphi_j(y_n) \notin K^* \quad \forall n \geq j.$$

Thus for any $j > 0$ we have that $\varphi_j(y) \in \overline{X - K^*}$. Note that $\overline{X - K^*} \cap \overline{K'} = \varnothing$. We therefore also get that

$$\varphi_j y \notin K' \quad \forall j \in \mathbb{N}.$$

This contradicts that

$$\{j \in \mathbb{N} \mid \varphi_j y \in K'\}$$

is unbounded. Thus we have proven Claim 6.16. □

By applying Claim 6.16 we get that X is a finite union of compact sets and thus is itself compact, which proves Lemma 6.15. □

The following proposition will guarantee the existence of compact, V_1-minimal subsets of Ω, and should be considered the most import proposition of this section.

Proposition 6.17. *Any non-empty, closed V_1-invariant set $Z \subseteq \Omega$ contains a compact non-empty V_1-invariant set. Hence Z contains a non-empty, compact V_1-minimal set.*

Proof. Note that for any set $K \subseteq X$

$$\bigcup_{i\in\{1,2\}} \bigcup_{\rho\in\mathbb{R}_{>0}} A(i,\rho) \cap K = K.$$

Let now Z be some V_1-invariant subset of Ω. Now by Lemmas 6.13 and 6.14 we have that the $A(i,\rho)$ themselves are closed and V_1-invariant thus there will always exist $i \in \{1,2\}$ and $\rho \in \mathbb{R}_{>0}$ such that $Z' := Z \cap A(i,\rho)$ is a closed, non-empty, V_1-invariant set.

Now fix Z', i and ρ. We will now choose $\sigma \leq 1$ such that $\sigma \leq \rho$ Let now $K \subseteq \Omega$ be a closed set as given by Theorem 5.1 applied to σ.

Claim 6.18. *For all $x \in Z'$ the sets*

$$\{t \leq 0 \mid v_1(t)x \in K\} \quad \textit{and} \quad \{t \geq 0 \mid v_1(t)x \in K\}$$

are unbounded.

Proof. Conversely we will assume that there exist $x = g\Gamma \in Z$ such that

$$\{t \leq 0 \mid v_1(t)x \in K\} \quad \text{or} \quad \{t \geq 0 \mid v_1(t)x \in K\}$$

are bounded. Then Theorem 5.1 gives us that for some $k \in \{1, 2\}$ there exists a non trivial $\Delta_{\mathbb{R}} \in \mathrm{Gr}_{k,3}$ invariant under $g^{-1}V_1 g$ such that for $\Delta := \Delta_{\mathbb{R}} \cap \mathbb{Z}^3$ we have that

$$\mathrm{Cov}(v_1(t)g\Delta) = \mathrm{Cov}(g\Delta) < \sigma$$

Since we have that $\sigma \leq 1$ we can conclude, as done in the proof of Lemma 6.10 that $g\Delta_{\mathbb{R}}$ is either equal to W_1 or W_2, where $W_1 = e_1\mathbb{R}$ and $W_2 = e_1\mathbb{R} \oplus e_2\mathbb{R}$. This then implies that ρ is the determinant of $g\mathbb{Z}^3 \cap W_i$. Thus

$$\rho \leq \mathrm{Cov}(g\Delta) < \sigma,$$

This is a contradiction to our choice of σ such that $\sigma \leq \rho$. Thus we have proven the claim. □

Then Lemma 6.15 gives us that Z' is compact. We can now use the same argumentation as in the proofs of Lemma 3.17 and Proposition 6.11 to get that Z' contains closed V_1- minimal set. Now since this minimal set is a closed subset of a compact set it has to be compact as well. □

6.6 Orbits of One-Parameter Groups of Unipotent Linear Transformations

Recall that in the proof of Hedlund's Theorem (3.3) we showed that the closure of the set $M' \subseteq \mathbb{R}^2$ contained the whole x-axis even though M' itself didn't contain any point on the x-axis. This was of interest to us, because the x-axis is exactly the set of fixed points of the action of N on $\mathbb{R}^2$, and because x-axis is invariant under action of P on $\mathbb{R}^2$. This then allowed us to show that $P \subset \overline{NMN}$, which then allowed us to apply Lemma 2.28.

To prove Oppenheim's conjecture we will use Lemma 2.28 in a similar way, however in this case the set of fixed points resp. the set of invariant points, will not be as simple as in Hedlund's Theorem. To enable us nevertheless to easily work with such arguments we will prove a quite general fact. In order to do this we introduce polynomial mappings.

Definition 6.19 (Polynomial Mapping)**.** *Let E be a finite-dimensional real vector space. We call a mapping $f : \mathbb{R} \to E$ polynomial, if there exists a basis $\{e_1, e_2, ..., e_n\}$ of E such that*

$$f(t) = \sum_{j=1}^{n} p_j(t)e_j,$$

where all the p_j are polynomials on $\mathbb{R}$.

Lemma 6.20. *Let $\{u_t\}_{t\in\mathbb{R}}$ be a unipotent one-parameter group acting on a finite dimensional real vector space E, and let*

$$L := \{x \in E \mid u(t)x = x \ \forall t \in \mathbb{R}\}$$

be the set of all common fixed points of the $u(t)$'s. Let M be a subset $E - L$. Then for every $p_0 \in L \cap \overline{M}$, there exist a non-constant polynomial $\varphi : \mathbb{R} \to L$ and sequences $\{t_i\}_i$, $\{m_i\}_i$ in $\mathbb{R}$ and M respectively such that

$$\varphi(0) = p_0 \quad \textit{and} \quad \lim_{i\to\infty} u(st_i)(m_i) = \varphi(s),$$

for any $s \in \mathbb{R}$.

Note that in the case of Hedlund's Theorem we had $\{u_t\}_{t\in\mathbb{R}} = N$, L was the x-axis and $\varphi : u(t) \mapsto t$.

Proof. Let τ be the an element of the Lie algebra generating $\{u_t\}_{t\in\mathbb{R}}$. Because N is unipotent there exists a basis $\{e_1, ..., e_r\}$ of $\mathbb{R}^r$ in which

$$\tau = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

where

$$J_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

for all $i \in \{1, ..., k\}$[34].

We will now assume without loss of generality that τ consists of only one such Jordan block. Thus in the same basis we have that

$$u(t) = exp(t\tau) = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{r-1}}{(r-1)!} \\ & 1 & t & \cdots & \frac{t^{r-2}}{(r-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & t \\ & & & & 1 \end{pmatrix}.$$

In this case we have that any $x \in E$ having the property $u(t)x = x$ for any $t \in \mathbb{R}$ is of the form re_1 for some $r \in \mathbb{R}$. Thus $L = \mathbb{R}e_1$.

Now consider a generic element $m \in E$, let

$$m = \sum_{j=1}^{r} m(j)e_j,$$

be its representation in the basis $\{e_1, ..., e_r\}$. Then we have that

$$u(t)m = \begin{pmatrix} \sum_{j=1}^{r} \frac{t^{j-1}}{(j-1)!} m(j) \\ \sum_{j=2}^{r} \frac{t^{j-2}}{(j-2)!} m(j) \\ \vdots \\ \sum_{j=r}^{r} \frac{t^{j-r}}{(j-r)!} m(j) \end{pmatrix} = \sum_{i=1}^{r} \sum_{j=i}^{r} \frac{t^{j-1}}{(j-1)!} m(j)e_i.$$

Let now $\theta(m) = \max_{2\le j\le r}\{|m(j)|^{\frac{1}{j-1}}\}$ thus for any $j \ge 2$

$$\left| \frac{m(j)}{\theta(m)^{j-1}} \right| \le 1.$$

Moreover for at least one $j_0 \ge 2$ we must have that

$$\left| \frac{m(j_0)}{\theta(m)^{j_0-1}} \right| = 1.$$

Now let $\{m_i\}_i \subseteq M$ such that $m_i \to p_0$ as $i \to \infty$. We may now assume that

$$\lambda(j) := \lim_{i\to\infty} \frac{m_i(j)}{\theta(m_i)^{j-1}}$$

exists for any $j \ge 2$ and that for at least one $j_0 \ge 2$, $|\lambda(j_0)| = 1$ [35]. We will now proceed by defining a polynomial mapping

$$\varphi(s) := \sum_{j=1}^{r} \frac{\lambda(j)}{(j-1)!} s^{j-1} e_1.$$

[34]This is elementary linear algebra, the above basis is the Jordan basis of τ and the J_i are Jordan blocks. The diagonal entries are all equal to zero because all eigenvalues τ are zero.

[35]Note that this doesn't hold for any such $\{m_i\}_i \subseteq M$, however every such sequence must contain a subsequence for which this holds.

Note that since φ is only summing up multiples of e_1, we have that $\varphi(s) \in \mathbb{R}e_1 = L$ for all $s \in \mathbb{R}$. Moreover since there exists at least one $j_0 \geq 2$ such that $|\lambda(j_0)| = 1$, we have that φ is not constant. Also note that

$$\varphi(0) = \lambda(1)e_1 = \lim_{i\to\infty} m_i(1) = p_0.$$

All we are left to do now is to find a sequence of t_i's such that

$$\lim_{i\to\infty} u(st_i)(m_i) = \varphi(s).$$

For this note that

$$u(st_i)(m_i) = \sum_{k=1}^{r}\sum_{j=k}^{r} \frac{t_i^{j-k}s^{j-k}}{(j-k)!} m(j)e_k.$$

Now note that

$$\lim_{i\to\infty} \frac{m_i(j)}{\theta(m_i)^l} = \begin{cases} 0 & if\ l < j-1 \\ \lambda(j) & if\ l = j-1 \end{cases}$$

Thus if we choose $t_i := \frac{1}{\theta(m_i)}$ we get the desired properties.

□

6.7 Proof of Oppenheim's Conjecture

Proof of Theorem 1.1. Let $Q = g_Q^{-1}Q_0$ be an indefinite, non-degenerate, irrational, quadratic form in three variables. Recall that we want to prove that $\overline{Q(\mathbb{Z}^3)} = \mathbb{R}$.

Now recall that in section 4.2 we showed, that we can identify the space $H\backslash G$ with the space with the space of non-degenerate quadratic forms in three variables and that $H\backslash G/\Gamma$ can be identified with the space of non-degenerate quadratic forms in three variables, where quadratic forms Q_1 and Q_2 are identified iff $\overline{Q_1(\mathbb{Z}^3)} = \overline{Q_2(\mathbb{Z}^3)}$.

Furthermore Lemma 4.16 showed us that the sets

$$Q_0(V_2^+ g\mathbb{Z}^3) = \left\{ z(t)^T\sigma_0 z(t) \;\middle|\; \forall t \in \mathbb{R}_{\geq 0},\ z(t) = v_1(t)gz,\ z \in \mathbb{Z}^3 \right\}$$

and

$$Q_0(V_2^- g\mathbb{Z}^3) = \left\{ z(t)^T\sigma_0 z(t) \;\middle|\; \forall t \in \mathbb{R}_{\leq 0},\ z(t) = v_1(t)gz,\ z \in \mathbb{Z}^3 \right\}$$

are equal to $\mathbb{R}$, for any $g \in G$.

Therefore our goal will be to prove that for any $x := g_Q\Gamma$ there exists a $y \in G$, such that $V_2^+ y$ or $V_2^- y$ is contained in $\overline{Hx}$, which will imply Theorem 1.1.

Proposition 6.11 shows that there exists a closed, non-empty DV_1-minimal subset of $\overline{Hx}$, which we will call X_1. Furthermore Propostition 6.17 gives us existence of a non-empty, compact V_1-minimal subset of X_1, which we will call Y. We will now show that for any $y \in Y$ either $V_2^+ y$ or $V_2^- y$ is contained in $\overline{Hx}$.

For this consider the set

$$R := \{g \in G - H | gy \in \overline{Hx}\}.$$

We will later use the following fact:

Claim 6.21.

$$e \in \overline{R}$$

Proof. We will show this by contradiction. Therefore we will assume that $e \notin \overline{R}$. Then there is a neighbourhood U of e in G such that $Uy \cap \overline{Hx} \subseteq Hy$. Because $y \in \overline{Hx}$, we have that $Uy \cap Hx$ is a non-empty open subset. Thus we have

$$\varnothing \neq Uy \cap Hx = (Uy \cap \overline{Hx}) \cap Hx \subseteq Hy \cap Hx.$$

Since H is a group and $\varnothing \neq Hy \cap Hx$ we have that $Hy = Hx$.

Now since y is both an element of X_1 and Hy, minimality of X_1 implies that $X_1 \subseteq Hy$ (by Lemma 4.13 $H \supset V_1$), and hence also $HX_1 \subseteq Hy$. But since $y \in X_1$ we have that $Hy \subseteq HX_1$. Thus we have that $Hy = HX_1$

Now recall that in Lemma 4.13, we showed that DV_1 is cocompact in H. Therefore there must exist a compact set $K \subseteq H$ such that $H = KDV_1$. Hence

$$KX_1 = K(DV_1)X_1 = HX_1 = Hy,$$

where the first equality holds because X_1 was chosen to be DV_1-invariant. Now note that KX_1 is closed, since both K and X_1 are closed. But this implies that $Hy = Hx$ is closed, which contradicts Remark 6.2. □

Let E be the vector space of all real symmetric 3×3 matrices. Now consider the group action of G on E given by the operation

$$\begin{aligned} G \times E &\to E \\ (g, \sigma) &\mapsto g \cdot \sigma := g^{-T} \sigma g^{-1} \end{aligned}$$

Claim 6.22. *V_1 acts unipotentely on E.*

Proof. Let

$$\sigma = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix} \in E.$$

We now want to show that the matrix τ is nilpotent, where τ has the property

$$v_1(t) \cdot \sigma = exp(-t\tau)\sigma.$$

Consider that

$$\begin{aligned} v_1(t) \cdot \sigma &= v_1(-t)^T \sigma v_1(-t) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ \frac{t^2}{2} & -t & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 - ta_1 & a_3 - ta_2 + \frac{t^2}{2}a_1 \\ a_2 - a_1 t & a_4 - 2ta_2 + t^2 a_1 & a_5 - ta_3 + \frac{3t^2}{2}a_2 - \frac{t^3}{2}a_1 \\ a_3 - a_2 t + \frac{t^2}{2}a_2 & a_5 - t(a_3 + a_4) + \frac{3t^2}{2}a_2 - \frac{t^3}{2}a_1 & a_6 - 2ta_5 + t^2(a_3 + a_4) - t^3 a_2 + \frac{t^4}{4}a_1 \end{pmatrix} \\ &= exp(-t\tau)\sigma = (Id - t\tau + \frac{t^2}{2}\tau^2 - \frac{t^3}{6}\tau^3 + \frac{t^4}{24}\tau^4 + ...)\sigma. \end{aligned}$$

Note that this implies that

$$\tau\left[\begin{pmatrix} b_1 & b_2 & b_3 \\ b_2 & b_4 & b_5 \\ b_3 & b_5 & b_6 \end{pmatrix}\right] = \begin{pmatrix} 0 & b_1 & b_2 \\ b_1 & 2b_2 & b_3 + b_4 \\ b_2 & b_3 + b_4 & 2b_5 \end{pmatrix}.$$

Since $\tau^5 \equiv 0$, we have that τ is nilpotent and thus V_1 acts unipotentely on E. □

Let L be the set of common fixed points of all the $v_1(t)$ i.e.

$$L = \{x \in E \mid \ v_1(t) \cdot x = x, \ \forall t \in \mathbb{R}\}.$$

From the above proof we have that

$$v_1(t) \cdot x = (Id - t\tau + \frac{t^2}{2}\tau^2 - \frac{t^3}{6}\tau^3 + \frac{t^4}{24}\tau^4)x,$$

which implies that $L = ker(\tau)$. It is not hard to see that $ker(\tau) = \sigma_1\mathbb{R} + \sigma_0\mathbb{R}$, where

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 6.3. Note that we have defined Q_0 as the Quadratic form corresponding to σ_0. We will now define Q_1 to be the quadratic form corresponding to σ_1 i.e. $Q_0(x_1, x_2, x_3) = x_3^2$. Also note that $Q_0(v_2(t)p) = Q_0(p) + 2tQ_1(p)$, which in turn implies that

$$V_2\sigma_0 = \sigma_0 + \mathbb{R}\sigma_1. \tag{23}$$

Moreover note that

$$\sigma_0 + \mathbb{R}\sigma_1 = L \cap G \tag{24}$$

We now want to use Lemma 6.20. For this we need a set $M \subseteq E - L$ such that $\overline{M} \cap L \neq \varnothing$. Consider now that we have shown, that $e \in \overline{R}$ thus there exists $\{r_i\}_i \subseteq R$, with $\lim_i r_i = e$. Hence $\{r_i^{-1}\}_i \subseteq G$ with $\lim_i r_i^{-1} = e$. Now let $M := \{r_i^{-1}\sigma_0\}_{i\in\mathbb{N}}$, then $\sigma_0 \in \overline{M}$ thus since $\sigma_0 \in L$, we therefore have that $\overline{M} \cap L \neq \varnothing$. However we do not necessarily have that $M \subseteq E - L$, for this we have to exclude the case, where $r_i^{-1}\sigma_0 \in L$. Since both r_i^{-1} and σ_0 have determinant equal to one we can reduce this to the case $r_i^{-1}\sigma_0 \in L \cap G = V_2\sigma_0$, which is equivalent to $\sigma_0 \in r_i V_2\sigma_0$, which occurs when $r_i \in HV_2$.

Thus a desired M exists and we may therefore apply Lemma 6.20 only if there exists a sequence

$$\{r_i\}_i \subseteq R - HV_2,$$

with the property that

$$\lim_{i\to\infty} r_i = e.$$

We will now separate the Proof into two cases. The first case will be when such an M exists, which we will show using Lemma 6.20. The second Case, when there does not exist such an M i.e. for all sequences of the form $\{r_i\}_i$ that converge to e, we have that $r_j \in HV_2$ for large enough $j \in \mathbb{N}$. In this Case we will use a different approach.

1. In the first case we want to apply Lemma 2.28, with $X = \overline{Hx}$, $Y = Y$, $M = R$, $A = H$ and $B = C = V_1$.

   Claim 6.23. *The properties required by Lemma 2.28, hold for $X = \overline{Hx}$, $Y = Y$, $M = R$, $A = H$ and $B = C = V_1$.*

   Proof. V_1 and H are closed subgroups of G, this is given from their definition.

   $V_1 \subseteq V_1 \cap H$ since $V_1 \subseteq H$.

   $\overline{Hx}$ is closed, since its the closure of Hx. We already showed that Y is closed.

   $\overline{Hx}$ is clearly invariant under H. Y is invariant, minimal, and compact under action of V_1 by definition.

   Since $\forall r \in R$; $ry \in \overline{Hx}$ we have that $\varnothing \neq rY \cap \overline{Hx}$.

   □

   Applying Lemma 2.28, then gives us that

   $$gY \subseteq \overline{Hx}, \quad \forall g \in N_{Sl(3,\mathbb{R})}(V_1) \cap \overline{HRV_1}.$$

   Now note that by Remark 4.1 we have that $V_2 \subseteq N_{Sl(3,\mathbb{R})}(V_1)$. Thus if we can show that either V_2^+ or V_2^- is contained in $\overline{HRV_1}$ we are done with the Proof.

   Since we are in the first case we can now apply Lemma 6.20, which gives us a non-constant polynomial $\varphi : \mathbb{R} \to L$ such that $\varphi(0) = \sigma_0$ and a sequence $\{t_i\}_i \subseteq \mathbb{R}$, with

   $$\lim_{i\to\infty} v_1(st_i)r_i^{-1}\sigma_0 = \varphi(s) \quad \forall s \in \mathbb{R}.$$

   Additionally since all $v_1(st_i)r_i^{-1}\sigma_0$ have determinant one, so does their limit and thus $\varphi(s) \in L \cap G = \sigma_0 + \mathbb{R}\sigma_1$. There therefore must exist a non-constant polynomial $\psi : \mathbb{R} \to \mathbb{R}$ such that

   $$\varphi(s) = \sigma_0 + \psi(s)\sigma_1.$$

Note that since $\varphi(0) = \sigma_0$ we have that $\psi(0) = 0$. Moreover since ψ is a polynomial it must be continuous and therefore its image contains either $\mathbb{R}_+$ or $\mathbb{R}_-$. Hence either

$$V_2^+\sigma_0 \subseteq Im(\varphi) \quad or \quad V_2^-\sigma_0 \subseteq Im(\varphi).$$

Now since

$$\lim_{i\to\infty} v_1(st_i)r_i^{-1}\sigma_0 = \varphi(s) \quad \forall s \in \mathbb{R},$$

we have that

$$Im(\varphi) \subseteq \overline{V_1R^{-1}H}\sigma_0.$$

This is because $r_i^{-1} \in R^{-1}$, $v_1(st_i) \in V_1$ and σ_0 is invariant under H.
Now note that $\overline{V_1R^{-1}H}^{-1} = \overline{HRV_1}$ and that $(V_2^+)^{-1} = V_2^-$ and $(V_2^-)^{-1} = V_2^+$. Thus we can conclude that either

$$V_2^- \subseteq \overline{HRV_1} \quad or \quad V_2^+ \subseteq \overline{HRV_1}.$$

Thus proving Oppenheim's conjecture for the first case.

2. In the second case our goal will be to show that Y is V_2 invariant, which gives us that

$$V_2Y = Y \subseteq \overline{Hx}.$$

Note that since we are in the second case, we have that there exists an open neighbourhood U of e, such that

$$\{g \in G \mid gy \in \overline{Hx}\} \cap U \subseteq HV_2. \tag{25}$$

Recall that by proving Theorem 6.12 we have already showed that DV_1y is not closed. Since X_1 is minimal under DV_1, Lemma 2.29 shows us that there exists a sequence $\{m_j\}_{j\in\mathbb{N}} \subseteq G - DV_1$, with $\lim_{j\to\infty} m_j = e$ and $m_jy \in DV_1y$ for any $j \in \mathbb{N}$. Thus there exists a sequence $\{g_j\}_{j\in\mathbb{N}} \subseteq DV_1$ such that

$$m_jy = g_jy.$$

Without loss of generality, we can also assume that $m_j \in U$ for any $j \in \mathbb{N}$.
Now consider (25) gives us that $m_j \in HV_2$, thus there must exist two sequences $\{h_j\}_{j\in\mathbb{N}} \subseteq H$ and $\{v_j := v_2(t_j)\}_{j\in\mathbb{N}} \subseteq V_2$

$$m_j = h_jv_j,$$

hence we get

$$g_j^{-1}h_jv_jy = y \quad \forall j \in \mathbb{N}.$$

Claim 6.24. *The sequence $\{h_j\}_{j\in\mathbb{N}}$ is contained in DV_1.*

The claim gives us that

$$g_j^{-1}h_jv_j \in DV.$$

We thus have that

$$g_j^{-1}h_jv_j \in \Delta := \{g \in DV | gy = y\}.$$

Moreover the claim shows us that, since

$$h_jv_j = m_j \notin DV_1, \quad g_j \in DV_1,$$

then $g_j^{-1}h_jv_j \notin DV_1$ and $\Delta \nsubseteq DV_1$. Moreover, this shows that $v_j \neq e$. Furthermore since $\lim_{j\to\infty} m_j = \lim_{j\to\infty} h_jv_j = e$, we must also have that $\lim_{j\to\infty} v_j = e$. And thus the projection of Δ to its V_2 component contains an accumulation point at the identity. Now note that it can easily be checked that $\{g \in V_2 \mid gy = y\}$ is discrete. Thus because by Lemmas 4.2 and 4.1 both V_1 and D are contained in the normalizer of V_2, we thus have that Δ must be discrete (For an illustration of the normalizer see Figure 9). We can thus apply Lemma 4.3, to see that

$$\Delta \subseteq V.$$

We thus have that

$$g_j^{-1}h_j \in DV_1 \cap V = V_1,$$

and therefore also

$$h_j^{-1}g_j \in DV_1 \cap V = V_1.$$

We then have that

$$v_2(t_j)y = h_j^{-1}g_j y \in V_1 Y = Y.$$

Now recall that V_2 centralizes V_1 (Lemma 4.1) and that Y is V_1 minimal, hence we have that

$$v_2(t_j)Y = v_2(t_j)\overline{V_1 y} = \overline{v_2(t_j)V_1 y} = Y.$$

Now consider that $v_j \neq e$ and $v_j \to e$ implies that the set $\{t_j\}_{j\geq 0}$ is not discrete in $\mathbb{R}$. Hence $\{t_j\}_{j\geq 0}$ generates a dense subgroup of $\mathbb{R}$. Thus also $\{v_2(t_j)\}_{j\geq 0}$ must generate a dense subgroup of V_2. Now by compactness of Y this implies that

$$V_2 Y = Y \subseteq \overline{Hx}.$$

This concludes the proof of the second case.

□

In order to finish the Proof we still have to prove Claim 6.24. For this we will begin by proving another claim.

Claim 6.25. *If there exists a neighbourhood U of e such that*

$$\{g \in G \mid gy \in \overline{Hx}\} \cap U \subseteq HV_2$$

then there exists a sequence

$$\{r_i\}_i \subseteq R \cap V_2$$

with $\lim_{i\to\infty} r_i = e$.

Proof. By Claim 6.21, we may assume that there exists $\{f_j\}_{j\in\mathbb{N}} \subseteq U \cap R$, with $\lim_{j\to\infty} f_j = e$. By (25) there exist two sequences $\{h_j\}_{j\in\mathbb{N}} \subseteq H$ and $\{r_j := v_2(s_j)\}_{j\in\mathbb{N}} \subseteq V_2$ such that

$$f_j = h_j r_j.$$

We then have that for any $p \in \mathbb{R}^3$,

$$Q_0(f_j p) = Q_0(h_j r_j p) = Q_0(r_j p) = Q_0(p) + s_j Q_1(p)$$

Since $f_j \to e$, this implies that $s_j \to 0$, which in turn implies that $r_j \to e$. Now, since R is H-invariant we have that $\{r_j\}_{j\in\mathbb{N}} \subseteq R$. Thus proving the Claim. □

Proof of Claim 6.24. Recall that we want to show that the sequence $\{h_j\}_{j\in\mathbb{N}}$ is contained in DV_1. For this let $\{m_j\}_{j\in\mathbb{N}}$ and $\{g_j\}_{j\in\mathbb{N}}$ be defined as in the proof of Theorem 1.1 (second case) i.e. $\{m_j\}_{j\in\mathbb{N}} \subseteq HV_2$ such that $\lim_{j\to\infty} m_j = e$ and $\{g_j\}_{j\in\mathbb{N}} \subseteq DV_1$ such that $g_j y = m_j y$. Recall that $\{h_j\}_{j\in\mathbb{N}} \subseteq H$ was defined as the H-component of $\{m_j\}_{j\in\mathbb{N}}$ i.e for any $j \in \mathbb{N}$ $m_j = h_j v_j$, where $v_j \in V_2$. Furthermore let $\{r_i\}_{i\in\mathbb{N}}$ be as in the proof of Claim 6.25.

We therefore have that

$$r_i y \in \overline{Hx},$$

Now we fix a $j \in \mathbb{N}$. Then we get that

$$g_j r_i g_j^{-1} m_j y = g_j r_i y \in g_j \overline{Hx} \subseteq \overline{Hx}.$$

We can observe that

$$g_j r_i g_j^{-1} m_j \to m_j \quad as \quad i \to \infty.$$

By (25) we now have that $g_j r_i g_j^{-1} m_j \in HV_2$. Thus for large i we will have that $g_j r_i g_j^{-1} m_j \in U$. For large i we therefore also have that

$$r_i g_j^{-1} h_j = g_j^{-1}(g_j r_i g_j^{-1} m_j) v_j^{-1} \in H(HV_2)V_2.$$

Because we have that $r_i \in V_2 - \{e\}$ and $g_j, h_j \in H$, we can apply Lemma 4.15, which implies that $g_j^{-1} h_j$ is an upper triangular matrix. Now note that g_j^{-1} is upper triangular as it is an element of DV_1. Thus h_j has to be an upper triangular matrix as well.

Observe that since $h_j v_j = m_j \to e$ as $j \to \infty$ and that h_j and m_j must have the same diagonal entries, which is case since $v_j \in V_2$. Thus for some j large enough the diagonal entries of h_j must all be so close to 1 that they are positive. We may now apply Lemma 4.15 again this time it gives us that $h_j \in DV_1$, which proves the claim. □

References

[1] J. W. S. Cassels and H. P. F. Swinnerton-Dyer. "On the Product of Three Homogeneous Linear Forms and Indefinite Ternary Quadratic Forms". English. In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 248.940 (1955), pp. 73–96. ISSN: 00804614. URL: http://www.jstor.org/stable/91633 (visited on 07/29/2023).

[2] Harold Davenport and H. Heilbronn. "On indefinite quadratic forms in five variables". English. In: *J. Lond. Math. Soc.* 21 (1946), pp. 185–193. ISSN: 0024-6107. DOI: 10.1112/jlms/s1-21.3.185.

[3] D. Y. Kleinbock. *Dynamics on homogeneous spaces and new applications to measure Theory.* 2022. URL: https://people.brandeis.edu/~kleinboc/Nachdiplom/index.html (visited on 09/11/2023).

[4] D. Y. Kleinbock and G. A. Margulis. "Flows on Homogeneous Spaces and Diophantine Approximation on Manifolds". In: *Annals of Mathematics* 148.1 (1998), pp. 339–360. ISSN: 0003486X. URL: http://www.jstor.org/stable/120997 (visited on 09/11/2023).

[5] D.Y. Kleinbock. "Quantitative nondivergence and its Diophantine applications". English. In: *Clay Mathematical Proceedings; Volume 10; Homogeneous Flows, Moduli Spaces, and Arithmetic.* Centro di Recerca Matematica Ennio De Giorgi, Pisa, Italy, 2010, pp. 131–153.

[6] K. Mahler. "On Lattice Points in n-Dimensional Star Bodies. I. Existence Theorems". English. In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 187.1009 (1946), pp. 151–187. ISSN: 00804630. URL: http://www.jstor.org/stable/97965 (visited on 09/11/2023).

[7] G A Margulis. "On The Action of Unipotent Groups in a Lattice Space". In: *Mathematics of the USSR-Sbornik* 15.4 (Apr. 1971), p. 549. DOI: 10.1070/SM1971v015n04ABEH001560. URL: https://dx.doi.org/10.1070/SM1971v015n04ABEH001560.

[8] G. A. Margulis. "Discrete Subgroups and Ergodic Theory". English. In: *Number Theory, Trace Formulas, and Discrete Groups: Symposium in Honor of Atle Selberg, Oslo, Norway, July 14-21, 1987.* Academic Press, 1989. ISBN: 9780120675708.

[9] G. A. Margulis. "Problems and Conjectures in Rigidity Theory". English. In: *Mathematics: Frontiers and Perspectives: Frontiers and Perspectives.* American Mathematical Soc, 2000.

[10] M. Bachir Bekka & Matthias Mayer. *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces.* English. The Pitt Building, Trumpington Street, Cambridge, United Kingdom: The Press Syndicate of the University of Cambridge, 2000. ISBN: 0521-66030-0.

[11] A. Meyer. "Mathematische Mittheilungen". Deutsch. In: *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich* 29 (1884), pp. 241–250.

[12] A. Oppenheim. "The minima of indefinite quaternary quadratic forms." English. In: *Proc. Natl. Acad. Sci. USA* 15 (1929), pp. 724–727. ISSN: 0027-8424. DOI: 10.1073/pnas.15.9.724.

[13] Marina Ratner. "Raghunathan's topological conjecture and distributions of unipotent flows". English. In: *Duke Math. J.* 64.1 (1991), pp. 235–280. URL: http://dml.mathdoc.fr/item/1077295782.

[14] D. Ridout. "Indefinite quadratic forms". English. In: *Mathematika* 5.2 (1958), pp. 122–124. DOI: 10.1112/S0025579300001443.

[15] Nicolas de Saxcé. "Non-divergence in the space of lattices". English. In: *Groups, Geometry, and Dynamics* (Nov. 2022). URL: https://hal.science/hal-03852638.