Universität Zürich[UZH]

Counting rational points on weighted projective spaces over number fields

Master Thesis
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Introduction

In 2013 R. Harron and A. Snowden gave an expression for the number of elliptic curves over $\mathbb{Q}$ up to height X whose torsion subgroup is isomorphic to one of the possible torsion groups over $\mathbb{Q}$. This result was generalized this year in [1] to all number fields and all level structures G such that the corresponding modular curve is a weighted projective line and the morphism $X_G \to X(1)$ satisfies a certain condition. The main result is stated in [1, Theorem 7.6], which determines a lower and upper bound for the number of elliptic curves over a number field with prescribed level structures and height bounded by X, as $X \to \infty$.

The main motivation of this work is to provide the tools for both generalizing and simplifying the proof of Theorem 7.6. The proof of said theorem makes use of [2, Theorem (A)], which gives an asymptotic formula for the number of rational points on the weighted projective spaces defined over number fields with bounded 'size' (a concrete height function), generalizing the main result obtained in [7]. Namely, one has

$$\#\{x \in \mathbb{P}(w)(\mathbb{K}) \mid S_w(x) \leqslant T\} \sim CT^{|w|}, \tag{1}$$

where C is a constant only depending on the choice of $\mathbb{K}$ and w. Nevertheless, the main result in [1] requires an expression for $S_u(\phi(x))$ (for ϕ a morphism between weighted projective spaces $\mathbb{P}(w)(\mathbb{K})$ and $\mathbb{P}(u)(\mathbb{K})$), which is given in [1, Corollary 6.5]. This approach could be further simplified by the use of a generalized version of Equation 1, in which we look at $S_u(\phi(x)) \leqslant T$ instead of $S_w(x) \leqslant T$. The way in which this method will generalize [1, Theorem 7.6] is by providing a limit to the number of elliptic curves over a number field with prescribed level structures and height bounded by X (as $X \to \infty$) mentioned above, together with not restricting to the conditions $e(G) = 1$ or $w = (1, 1)$ that are found in the statement. Including the 'improved' version of the theorem is beyond the scope of this thesis, but it is a fairly straightforward consequence of our result.

The aim of this thesis is to generalize the results obtained in [2], having as a main goal to determine an asymptotic formula for the above mentioned quantity $\#\{x \in \mathbb{P}(w)(\mathbb{K}) \mid S_u(\phi(x)) \leqslant T\}$ as $T \to \infty$. Despite all the data that the morphism contains, we prove in our main result (Theorem 3.19)

that this quantity can be expressed in a similar way to Equation 1 as

$$\#\{x \in \mathbb{P}(w)(\mathbb{K}) \mid S_u(\phi(x)) \leqslant T\} \sim CT^{|w|/e},$$

where now the constant also depends on the many parameters involved in the choice of the morphism. The contents of this paper are primarily based on [2], as we essentially follow an analogous structure to obtain our result.

The thesis is organized as follows. We have included four chapters. In Chapter 1 we present some elementary tools that are necessary to develop this theory. That includes the definition of weighted projective spaces (and morphisms between them) and the size function, together with some known results on v-adic completions. We also fix the notation we will use throughout the thesis. Chapter 2 introduces new definitions and notations and some of their properties, all with the aim to simplify the proof of the main result. In Chapter 3 we study our main counting problem by following a similar method to that used in [2] and making use of the theory developed in the previous chapter. We state the main result in Theorem 3.19. Chapter 4 focuses on applying our counting result to a specific example of this setting. Furthermore, we see how this theory is related to counting elliptic curves with bounded size. Finally, we include a 'future work' section in which we discuss possible improvements of this thesis.

We further note that this approach may be applied to other areas unrelated to elliptic curves, although the application to elliptic curves will be our main interest.

Chapter 1

Introductory material

In this introductory chapter, we gather some concepts on the theory of weighted projective spaces and we introduce the size function, with the aim of making the reader familiar with the background of this field. On top of that, we define the *v-adic completion* of a ring and we prove some of its most important properties, which we will make use of throughout the thesis.

1.1 Weighted projective spaces

In this section we define weighted projective spaces and morphisms between them. We first define our main working tool.

Definition 1.1. A *number field* is an algebraic field extension of the field of rational numbers $\mathbb{Q}$.

Definition 1.2. Let $\mathbb{K}$ be a number field and $w = (w_1, \ldots, w_m)$ an m-tuple of positive integers, with $m \geqslant 1$. Then the *weighted projective space with weight w over* $\mathbb{K}$ is

$$\mathbb{P}(w)(\mathbb{K}) := \mathbb{K}^{\times} \backslash (\mathbb{K}^m - \{0\}),$$

where $\mathbb{K}^{\times} := \mathbb{K} - \{0\}$ acts on $\mathbb{K}^m - \{0\}$ by

$$(\lambda, (x_1, ..., x_m)) \mapsto \lambda_*(x_1, ..., x_m) := (\lambda^{w_1} x_1, ..., \lambda^{w_m} x_m).$$

We say that λ *acts with the weight w action.*

Lemma 1.3. *Let $\mathbb{K}$ as above and $w = (w_1, \ldots, w_m)$ and $u = (u_1, \ldots, u_m)$ two m-tuples of positive integers. Let $\phi : \mathbb{P}(w)(\mathbb{K}) \to \mathbb{P}(u)(\mathbb{K})$ be a map defined by polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_m]$, where we view $\mathbb{K}[x_1, \ldots, x_m]$ as a graded $\mathbb{K}$-algebra with x_i homogeneous of degree w_i for all i and the polynomials $f_1, \ldots, f_n$ satisfy the following properties*

(i) *There exists $e = e(f_1, \ldots, f_m) \in \mathbb{Z}_{\geqslant 0}$ for which f_i is homogeneous of degree eu_i for all i.*

(ii) *The homogeneous ideal* $\sqrt{(f_1,\dots,f_m)} \subseteq \mathbb{K}[x_1,\dots,x_m]$ *contains the ideal* $(x_1,\dots,x_m)$.

Then, ϕ *defines a morphism between the weighted projective spaces* $\mathbb{P}(w)(\mathbb{K})$ *and* $\mathbb{P}(u)(\mathbb{K})$.

Proof. See [1, Lemma 4.1] for a proof for $m = 2$ (the same proof works for a general m). □

In the rest of the paper we fix the following notations: Let $\mathbb{K}$ denote a number field, $\mathcal{O}_{\mathbb{K}}$ the ring of integers in $\mathbb{K}$, $D_{\mathbb{K}}$ the absolute value of the discriminant, $N := [\mathbb{K} : \mathbb{Q}]$, $Cl(\mathbb{K})$ the group of ideal classes, $\mu(\mathbb{K})$ the group of roots of unity, $\mu_{\mathbb{K}} := \#\mu(\mathbb{K})$, $w = (w_1,\dots,w_m)$ and $u = (u_1,\dots,u_m)$ two tuples of positive integers with $|w| := w_1 + \cdots + w_m$ such that $|w| \geqslant 1$, $\mu_{\mathbb{K}}^w := \frac{\mu_{\mathbb{K}}}{\gcd\{w_1,\dots,w_n,\mu_{\mathbb{K}}\}}$, $\phi : \mathbb{P}(w)(\mathbb{K}) \to \mathbb{P}(u)(\mathbb{K})$ a morphism between weighted projective spaces and e the integer defined in Lemma 1.3. We let $\mathfrak{p}$ vary between integral prime ideals of $\mathbb{K}$, and in the case $\mathbb{K} = \mathbb{Q}$, we write $\mathfrak{p} = p$ for p a prime number.

1.2 The v-adic completion

We now introduce some theory concerning the so-called *v-adic completion* of $\mathbb{K}$. The reader can refer to any book on algebraic number theory for a more extensive explanation of the matter. For reference, we will sometimes utilize [5, Chapter 7]. We start by first introducing the concept of the *order*, followed by classifying the different absolute values (or *places*) in $\mathbb{K}$.

Definition 1.4. Let $x \in \mathbb{Q} - \{0\}$. We denote by $\mathrm{ord}_p(x)$ the *order of* x *at* p, where x factors uniquely as

$$x = \prod_p p^{\mathrm{ord}_p(x)}.$$

This definition can be extended to ideals of a number field in the following way.

Definition 1.5. Let $I \subset \mathbb{K}$ a non-zero ideal. We denote by $\mathrm{ord}_{\mathfrak{p}}(I)$ the *order of* I *at* $\mathfrak{p}$, where I factors uniquely as

$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(I)}.$$

Now, let $x \in \mathbb{K}$.

(i) Infinite places: Let $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$ be an embedding from $\mathbb{K}$ to $\mathbb{C}$. We denote

$$|x|_\sigma := \begin{cases} |\sigma(x)|, \text{ if } \sigma(\mathbb{K}) \subseteq \mathbb{R}; \\ |\sigma(x)|^2, \text{ if } \sigma(\mathbb{K}) \nsubseteq \mathbb{R}, \end{cases}$$

where $|\cdot|$ denotes the usual absolute value. We denote by Ω_∞ the set of infinite places and by N_v the exponent of $|\sigma(x)|$ in $|x|_\sigma$. Note that $N = r_1 + 2r_2$ where r_1 denotes the number of real places ($N_v = 1$) and r_2 the number of complex places ($N_v = 2$) (see [2, Section 3]).

(ii) Finite places: We denote

$$|x|_\mathfrak{p} := N(\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(x\mathcal{O}_\mathbb{K})},$$

where $N(\mathfrak{a}) := \#(\mathcal{O}_\mathbb{K}/\mathfrak{a})$ for $\mathfrak{a}$ an integral ideal of $\mathbb{K}$.

Notation. $\Omega_\mathbb{K}$ is the set of all places in $\mathbb{K}$.

Remark 1.6. It is easy to see that the places defined above are indeed absolute values. It follows that $|u|_v = 1$ for $u \in \mathcal{O}_\mathbb{K}^\times$ and $v \in \Omega_\mathbb{K}$.

Definition 1.7. Let $v \in \Omega_\mathbb{K}$ and let $\sim$ define an equivalence relation between Cauchy sequences with $a_n \sim b_n \iff \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. We define the *$v$-adic completion of* $\mathbb{K}$ to be

$$\mathbb{K}_v := \{\text{Cauchy sequences of } \mathbb{K} \text{ with respect to the absolute value } |\cdot|_v\}/\sim .$$

Throughout the paper our main focus will lie on finite places, for which the following equivalent (algebraic) definition will turn out to be of more help.

Definition 1.8. Let R be a commutative ring and $\mathfrak{q}$ a prime ideal of R. We denote by $R_\mathfrak{q}$ the *$\mathfrak{q}$-adic completion of R*, where

$$\begin{aligned} R_\mathfrak{q} &:= \varprojlim_{n\geqslant 1} R/\mathfrak{q}^n R \\ &:= \{(a_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} R/\mathfrak{q}^n R \,|\, a_n \mod \mathfrak{q}^{n-1} = a_{n-1},\ n = 1, 2, \dots\}. \end{aligned}$$

The topology on $R_\mathfrak{q}$ is defined to be the coarsest topology for which all the maps $r_m : R_\mathfrak{q} \to R/\mathfrak{q}^m R$ sending $(a_n)_{n\in\mathbb{N}}$ to a_m are continuous. In particular, we will be looking at $r_n^{-1}(\{0\}) = \{x \in R_\mathfrak{q} \,|\, |x|_\mathfrak{q} \leqslant N(\mathfrak{q})^{-n}\}$.

Lemma 1.9. *The function* $\operatorname{ord}_\mathfrak{p}$ *extends to a function on* $\mathbb{K}_\mathfrak{p}^\times$ *and is continuous.*

Proof. For the first claim, see [5, Theorem 7.23]. For the second claim, let $x \in \mathbb{K}_\mathfrak{p}^\times$ with $\operatorname{ord}_\mathfrak{p}(x) = k$. Since every locally constant function on a metric space is continuous, it suffices to find an open neighbourhood U of x such that for all $y \in U$, $\operatorname{ord}_\mathfrak{p}(y) = k$. Let $y \in \mathbb{K}_\mathfrak{p}^\times$ with $\operatorname{ord}_\mathfrak{p}(x - y) > k$. These elements form an open neighbourhood of x in $\mathbb{K}_\mathfrak{p}^\times$ because they form the set $r_k^{-1}\{x\}^c \cap \mathbb{K}_\mathfrak{p}^\times$. Moreover, $\operatorname{ord}_\mathfrak{p}$ satisfies the *strong triangle inequality*, i.e. for $a, b \in \mathbb{K}_\mathfrak{p}^\times$ we have

$$\operatorname{ord}_\mathfrak{p}(a + b) \geqslant \min\{\operatorname{ord}_\mathfrak{p}(a), \operatorname{ord}_\mathfrak{p}(b)\}, \text{ with equality if } \operatorname{ord}_\mathfrak{p}(a) \neq \operatorname{ord}_\mathfrak{p}(b),$$

and so $\operatorname{ord}_\mathfrak{p}(y) = k$, where we set $a = x; b = y - x$. □

We will be interested in studying the algebraic and topological properties of the $\mathfrak{p}$-adic completion of the ring of integers $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$.

Lemma 1.10. *Every non-zero fractional ideal of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is a power of $\mathfrak{p}\mathcal{O}_{\mathbb{K},\mathfrak{p}}$.*

Proof. Note that $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is a DVR by definition, since $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is a local Dedekind domain (recall that $\mathcal{O}_{\mathbb{K}}$ is a Dedekind domain by [5, Theorem 3.29]). Consequently, $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ has a unique maximal ideal, namely $\mathfrak{p}\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ (every other prime ideal of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is coprime to $\mathfrak{p}$, and therefore generates the unit ideal of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ by definition), and the result follows. □

Example 1.11. For $\mathbb{K} = \mathbb{Q}$, let $\mathfrak{a}$ be a non-zero fractional ideal of $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}$ and $\mathfrak{a}_p = \mathfrak{a}\mathbb{Z}_p$. Then we can write $\mathfrak{a}_p = (p\mathbb{Z}_p)^{n_p}$ with $n_p = 0$ for almost all p.

We would now like to go a step further and compute $\mathfrak{a} := \bigcap_p(\mathfrak{a}_p \cap \mathbb{Q})$, where we take the intersection inside $\mathbb{Q}_p$. To do so, let π_p be the canonical map $\mathbb{Q} \xhookrightarrow{\pi_p} \mathbb{Q}_p$ embedding $\mathbb{Q}$ into its p-adic completion. Then,

$$\mathfrak{a} = \bigcap_p \{x \in \mathbb{Q} \mid \pi_p(x) \in \mathfrak{a}_p\} = \{x \in \mathbb{Q} \mid \text{ for all } p : \operatorname{ord}_p(x) \geqslant n_p\} = \prod_p p^{n_p}\mathbb{Z},$$

which is well defined by the condition that $n_p = 0$ for all but finitely many p.

Lemma 1.12. *$\mathcal{O}_{\mathbb{K},\mathfrak{p}}^m$ is compact.*

Proof. $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is closed in $\prod_n \mathcal{O}_{\mathbb{K}}/\mathfrak{p}^n\mathcal{O}_{\mathbb{K}}$, since every condition in the expression of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ defines a closed subspace of $\prod_n \mathcal{O}_{\mathbb{K}}/\mathfrak{p}^n\mathcal{O}_{\mathbb{K}}$. Further, $\prod_n \mathcal{O}_{\mathbb{K}}/\mathfrak{p}^n\mathcal{O}_{\mathbb{K}}$ endowed with the product topology is compact by Tychonoff´s Theorem (note that $\mathcal{O}_{\mathbb{K}}/\mathfrak{p}^n\mathcal{O}_{\mathbb{K}}$ endowed with the discrete topology is compact since it is finite). Thus, $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is compact and the result follows. □

Lemma 1.13. *The set*

$$\{\mathcal{U}_{\mathfrak{p}}^{a,b} := \prod_{i=1}^{m}(a_i + \mathfrak{p}^{b_i}\mathcal{O}_{\mathbb{K},\mathfrak{p}}) \mid a = (a_1, \ldots, a_m) \in \mathcal{O}_{\mathbb{K}}^m, b = (b_1, \ldots, b_m) \in (\mathbb{Z}_{\geqslant 0})^m\}$$

forms a basis for $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^m$.

Proof. The set $\{a' + \mathfrak{p}^{b'}\mathcal{O}_{\mathbb{K},\mathfrak{p}} \mid a' \in \mathcal{O}_{\mathbb{K}}, b' \in \mathbb{Z}_{\geqslant 0}\}$ forms a basis for $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$. Since $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^m$ is a finite product, we get a basis for $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^m$ by taking the product of the basis of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$. □

Definition 1.14. We define the *completion of* $\mathcal{O}_{\mathbb{K}}$ to be

$$\hat{\mathcal{O}}_{\mathbb{K}} := \prod_{\mathfrak{p}} \mathcal{O}_{\mathbb{K},\mathfrak{p}},$$

endowed with the product topology.

1.3 The size function

We define the scaling ideal and the size function as done in [2, Definition 3.2].

Definition 1.15. Let $x \in \mathbb{K}^m - \{0\}$. We define the *scaling ideal of x with respect to w* to be $\mathfrak{I}_w(x)$, where

$$\mathfrak{I}_w^{-1}(x) := \{a \in \mathbb{K} : a_* x \in \mathcal{O}_\mathbb{K}\}$$

and a acts on x with the weight w action.

Lemma 1.16. *The scaling ideal satisfies the following properties*

(i) $\mathfrak{I}_w(a_* x) = a\mathfrak{I}_w(x)$ *for* $a \in \mathbb{K}^\times$. *Moreover,* $\mathfrak{I}_w(x)$ *is an integral ideal of* $\mathcal{O}_\mathbb{K}$ *for* $x = (x_1, \ldots, x_m) \in \mathcal{O}_\mathbb{K}^m$.

(ii) *We can also express the scaling ideal as*

$$\mathfrak{I}_w(x) = \prod_\mathfrak{p} \mathfrak{p}^{\min_{0 \leqslant i \leqslant m} \left\lfloor \frac{\operatorname{ord}_\mathfrak{p} x_i}{w_i} \right\rfloor}.$$

Proof. (i) See [2, Proposition 3.3].

(ii) $\mathfrak{I}_w(x)$ is by definition the intersection of all fractional ideals $\mathfrak{a} \subset \mathbb{K}$ satisfying $x_i \in \mathfrak{a}^{w_i}$ for all i. We have

$$\begin{aligned} x_i \in \mathfrak{a}^{w_i} &\iff \operatorname{ord}_\mathfrak{p} x_i \geqslant w_i \operatorname{ord}_\mathfrak{p} \mathfrak{a} \text{ for all } \mathfrak{p} \\ &\iff \operatorname{ord}_\mathfrak{p} \mathfrak{a} \leqslant \left\lfloor \frac{\operatorname{ord}_\mathfrak{p} x_i}{w_i} \right\rfloor \text{ for all } \mathfrak{p} \\ &\iff \mathfrak{a} \supseteq \mathfrak{p}^{\left\lfloor \frac{ord_\mathfrak{p} x_i}{w_i} \right\rfloor} \text{ for all } \mathfrak{p}. \end{aligned}$$

Thus, $\mathfrak{I}_w(x)$ is the intersection of all fractional ideals $\mathfrak{a} \subset \mathbb{K}$ satisfying $\mathfrak{a} \supseteq \bigcap_\mathfrak{p} \mathfrak{p}^{\min_{0 \leqslant i \leqslant m} \left\lfloor \frac{ord_\mathfrak{p} x_i}{w_i} \right\rfloor} = \prod_\mathfrak{p} \mathfrak{p}^{\min_{0 \leqslant i \leqslant m} \left\lfloor \frac{ord_\mathfrak{p} x_i}{w_i} \right\rfloor}$. Taking the intersection yields the result.

□

Definition 1.17. Let $x \in \mathbb{K}^m - \{0\}$. We define the *size of x with respect to u* to be

$$S_u(x) := \frac{1}{N(\mathfrak{I}_u(x))} H_{u,\infty}(x),$$

where we set

$$H_{u,\infty}(x) := \prod_{v \in \Omega_\infty} \max_{i=1,\ldots,m} |x_i|_v^{\frac{1}{u_i}}.$$

The size function is a special case of height functions. For some background on this field, it is worth mentioning other works that have also had an influence in the theory of heights on different spaces. For general machinery of heights on projective spaces, see [8, Section 8.5]. On a more abstract level, [3, Section 2.2] defines heights on a very general type of stacks.

Chapter 2

Useful Tools

In this chapter we will introduce various new concepts in order to give the proof of our main result. We will focus on defining new notions and studying some of their properties, which will become relevant in the next chapter.

2.1 First definitions and notations

Our first step will be to name several sets that we will often make use of throughout the paper.

Notation. $Q_w := \mathcal{O}_{\mathbb{K}}^{\times} \backslash (\mathbb{K}^m - \{0\})$, where $\mathcal{O}_{\mathbb{K}}^{\times}$ acts with the weight w action.

We set $R_{\mathbb{K}}$ to be a set of representatives of $Cl(\mathbb{K})$ and, without loss of generality, we assume every element of $R_{\mathbb{K}}$ is an integral ideal. Since the class group of $\mathbb{K}$ is finite (see [5, Chapter 4]), the set $R_{\mathbb{K}}$ is finite. For the remainder of this paper, let us fix $c \in Cl(\mathbb{K})$ and $\mathfrak{a} \in R_{\mathbb{K}}$ so that $[\mathfrak{a}] = c$.

We can then define the following set.

Definition 2.1. We define $Q_w^{\mathfrak{a}} := \{x \in Q_w \,|\, \mathfrak{I}_w(x) = \mathfrak{a}\}$ to be the *set of points with scaling ideal* $\mathfrak{a}$. Note that:

(i) $Q_w^{\mathfrak{a}} \cong \{x \in \mathbb{P}(w)(\mathbb{K}) \,|\, [\mathfrak{I}_w(x)] = c\}$ by Lemma 1.16 (i).

(ii) $Q_w^{\mathfrak{a}} = \{x \in \mathcal{O}_{\mathbb{K}}^{\times} \backslash ((\mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m}) - \{0\}) \,|\, \mathfrak{I}_w(x) = \mathfrak{a}\}$. This follows from $\mathfrak{I}_w(x) = \mathfrak{a} \implies \mathfrak{I}_w(x) \subseteq \mathfrak{a} \implies x \in \mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m}$, by definition of the scaling ideal.

Remark 2.2. Note that $Q_w^{\mathfrak{a}}$ is well defined by Lemma 1.16 (i) and the fact that multiplication of a fractional ideal of a ring R by a unit of R leaves the ideal unchanged.

Definition 2.3. We denote by $\mathcal{D}_\phi$ the *set of discrepancy ideals of the map* ϕ, i.e. $\mathcal{D}_\phi := \{\mathfrak{I}_u(\phi(x)) \cdot \mathfrak{I}_w(x)^{-e} \,|\, x \in \mathbb{P}(w)(\mathbb{K})\}$.

Remark 2.4. The quotient $\mathfrak{I}_u(\phi(x)) \cdot \mathfrak{I}_w(x)^{-e}$ (and therefore the set $\mathcal{D}_\phi$) is well defined for $x \in \mathbb{P}(w)(\mathbb{K})$, even if the individual factors are not. To see this, note that for $\alpha \in \mathcal{O}_\mathbb{K}^\times$ we have $\phi(\alpha x) = \alpha^e \phi(x)$, where α acts on x with the weight w action and α^e acts on $\phi(x)$ with the weight u action. Then, by Lemma 1.16 (i), $\mathfrak{I}_u(\phi(\alpha x)) \cdot \mathfrak{I}_w(\alpha x)^{-e} = \alpha^e \mathfrak{I}_u(\phi(x)) \cdot \alpha^{-e} \mathfrak{I}_w(x)$, and the result follows.

Lemma 2.5. *$\mathcal{D}_\phi$ is finite.*

Proof. The group of non-zero fractional ideals of $\mathbb{K}$ is isomorphic to $\bigoplus_\mathfrak{p} \mathbb{Z}$ (see [5, Theorem 3.20]). We then have the following diagram:

$$\begin{array}{ccccccc}
\mathbb{P}(w)(\mathcal{O}_\mathbb{K}) & \xrightarrow{=} & \mathbb{P}(w)(\mathbb{K}) & & & & \\
\downarrow & & \downarrow & \searrow^{\delta_\phi} & & & \\
\mathbb{P}(w)(\hat{\mathcal{O}}_\mathbb{K}) & & & & & & \\
\downarrow = & & & & & & \\
\prod_\mathfrak{p} \mathbb{P}(w)(\mathcal{O}_{\mathbb{K},\mathfrak{p}}) & \xrightarrow{=} & \prod_\mathfrak{p} \mathbb{P}(w)(\mathbb{K}_\mathfrak{p}) & \xrightarrow{(\alpha_{\phi,\mathfrak{p}})_\mathfrak{p}} & \bigoplus_\mathfrak{p} \mathbb{Z} & \xrightarrow{\sim} & \{\text{frac. id. of } \mathbb{K}\}
\end{array}$$

Here the maps δ_ϕ and $(\alpha_{\phi,\mathfrak{p}})_\mathfrak{p}$ are defined by $\delta_\phi(x) := \mathfrak{I}_u(\phi(x)) \cdot \mathfrak{I}_w(x)^{-e}$ the discrepancy ideal of x at ϕ and

$$\begin{aligned}
\alpha_{\phi,\mathfrak{p}} : \mathbb{P}(w)(\mathbb{K}_\mathfrak{p}) &\to \mathbb{Z} \\
x &\mapsto \min_{1 \leqslant i \leqslant m} \left\lfloor \frac{\operatorname{ord}_\mathfrak{p}(\phi(x)_i)}{u_i} \right\rfloor - e \min_{1 \leqslant i \leqslant m} \left\lfloor \frac{\operatorname{ord}_\mathfrak{p}(x_i)}{w_i} \right\rfloor.
\end{aligned}$$

Claim: $\alpha_{\phi,\mathfrak{p}}$ is well defined and continuous for the quotient topology on $\mathbb{P}(w)(\mathbb{K}_\mathfrak{p})$ and the discrete topology on $\mathbb{Z}$.
Proof of claim: Let $\lambda \in \mathbb{K}_\mathfrak{p}^\times$. We have

$$\begin{aligned}
\alpha_{\phi,\mathfrak{p}}(\lambda_* x) &= \alpha_{\phi,\mathfrak{p}}(\lambda^{w_1} x_1, \ldots, \lambda^{w_m} x_m) \\
&= \min_{1 \leqslant i \leqslant m} \left\lfloor \frac{\operatorname{ord}_\mathfrak{p}(\lambda^{eu_i} \phi(x)_i)}{u_i} \right\rfloor - e \min_{1 \leqslant i \leqslant m} \left\lfloor \frac{\operatorname{ord}_\mathfrak{p}(\lambda^{w_i} x_i)}{w_i} \right\rfloor \\
&= e \operatorname{ord}_\mathfrak{p}(\lambda) - e \operatorname{ord}_\mathfrak{p}(\lambda) + \alpha_{\phi,\mathfrak{p}}(x) = \alpha_{\phi,\mathfrak{p}}(x).
\end{aligned}$$

Thus, the map $\alpha_{\phi,\mathfrak{p}}$ is well defined. To prove its continuity, it is sufficient to prove that $\beta(x) := \min_{1 \leqslant i \leqslant m} \left\lfloor \frac{ord_\mathfrak{p}(x_i)}{w_i} \right\rfloor$ is continuous for $x \in \mathbb{K}_\mathfrak{p}^m - \{0\}$. We will do so by proving it is locally constant, as we have previously done in the proof of Lemma 1.9.

Fix i such that for all j : $\frac{\operatorname{ord}_\mathfrak{p}(x_j)}{w_j} \geqslant \frac{\operatorname{ord}_\mathfrak{p}(x_i)}{w_i}$ and denote $\gamma := \frac{\operatorname{ord}_\mathfrak{p}(x_i)}{w_i}$. Then the set $B := \left\{ x' \in \mathbb{K}_\mathfrak{p}^m - \{0\} \mid \frac{\operatorname{ord}_\mathfrak{p}(x'_i)}{w_i} = \gamma \text{ and for all } j : \frac{\operatorname{ord}_\mathfrak{p}(x'_j)}{w_j} \geqslant \gamma \right\}$

contains x and $\beta(x') = \gamma$ for any $x' \in B$. Furthermore, B is open in $\mathbb{K}_\mathfrak{p}^m - \{0\}$ since

$$\begin{aligned} B &= \{x' \in \mathbb{K}_\mathfrak{p}^m - \{0\} \mid |x'_i|_\mathfrak{p} = N(\mathfrak{p})^{-w_i\gamma} \text{ and for all } j : |x'_j|_\mathfrak{p} \leqslant N(\mathfrak{p})^{-w_j\gamma}\} \\ &= \left((r_{w_i\gamma}^{-1}(\{0\}) \cap r_{w_i\gamma+1}^{-1}(\{0\})^c) \times \prod_{j\neq i} r_{w_j\gamma}^{-1}(\{0\}) \right) \cap \mathbb{K}_\mathfrak{p}^m - \{0\}, \end{aligned}$$

where all of the sets we multiply are open subsets of $\mathbb{K}_\mathfrak{p}$. So we have found a neighbourhood B of x for every $x \in \mathbb{K}_\mathfrak{p}^m - \{0\}$ such that β is constant on B, implying β is locally constant. This finishes the proof of the claim.

Now define

$$\begin{aligned} R_\mathfrak{p} &:= \left\{ x \in \mathcal{O}_{\mathbb{K},\mathfrak{p}}^m \mid \min_i \left\lfloor \frac{\operatorname{ord}_\mathfrak{p}(x_i)}{w_i} \right\rfloor = 0 \right\} \\ &= \mathcal{O}_{\mathbb{K},\mathfrak{p}}^m - (\mathfrak{p}^{w_1}\mathcal{O}_{\mathbb{K},\mathfrak{p}} \times \cdots \times \mathfrak{p}^{w_m}\mathcal{O}_{\mathbb{K},\mathfrak{p}}). \end{aligned}$$

Then $R_\mathfrak{p}$ is compact as it is the complement of an open subset in a compact space (see Lemma 1.12 and Lemma 1.13). The quotient map from $R_\mathfrak{p}$ to $\mathbb{P}(w)(\mathbb{K}_\mathfrak{p})$ is surjective (let $y = (y_1, \ldots, y_m) \in \mathbb{P}(w)(\mathbb{K}_\mathfrak{p})$ and set $T = \min_i \left\lfloor \frac{\operatorname{ord}_\mathfrak{p}(y_i)}{w_i} \right\rfloor$, then $\mathfrak{p}_*^{-T} y \in R_\mathfrak{p}$), and we conclude that $\mathbb{P}(w)(\mathbb{K}_\mathfrak{p})$ is also compact, since the continuous image of a compact space is again compact.

We proved that the image of $\alpha_{\phi,\mathfrak{p}}$ is both compact and discrete, thus finite. Applying Lemma 1.16 (ii), it is then clear by the diagram that $\mathcal{D}_\phi$ is also finite. □

We fix an ideal $\mathfrak{d} \in \mathcal{D}_\phi$.

Definition 2.6. We define the *set of points with discrepancy ideal* $\mathfrak{d}$ to be

$$\begin{aligned} Q_w^{\mathfrak{a},\mathfrak{d}} &:= \{x \in Q_w^\mathfrak{a} \mid \mathfrak{I}_u(\phi(x)) \cdot \mathfrak{I}_w(x)^{-e} = \mathfrak{d}\} \\ &= \{x \in Q_w^\mathfrak{a} \mid \mathfrak{I}_u(\phi(x)) = \mathfrak{d}\mathfrak{a}^e\}. \end{aligned}$$

Note that $Q_w^{\mathfrak{a},\mathfrak{d}} = \delta_\phi^{-1}\{\mathfrak{d}\}$, for the map δ_ϕ defined in the previous proof.

We now define the $\mathfrak{p}$-adic version of Q_w.

Notation. $Q_{w,\mathfrak{p}} := \mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times \backslash (\mathbb{K}_\mathfrak{p}^m - \{0\})$, where again $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times$ acts with the weight w action.

Let us decompose $\mathfrak{d} = \prod_\mathfrak{p} \mathfrak{p}^{j_\mathfrak{p}}$ and $\mathfrak{a} = \prod_\mathfrak{p} \mathfrak{p}^{k_\mathfrak{p}}$. It then follows that $Q_w^\mathfrak{a} = \{x \in Q_w \mid \text{ for all } \mathfrak{p} : \mathfrak{I}_w(x)_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}\}$, where $\mathfrak{I}_w(x)_\mathfrak{p} = \mathfrak{I}_w(x)\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ and $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}\mathcal{O}_{\mathbb{K},\mathfrak{p}} = (\mathfrak{p}\mathcal{O}_{\mathbb{K},\mathfrak{p}})^{k_\mathfrak{p}}$.

Definition 2.7. We denote by $R_\mathfrak{p}^\mathfrak{a} := \left\{ x \in \mathbb{K}_\mathfrak{p}^m \mid \min_i \left\lfloor \frac{\operatorname{ord}_\mathfrak{p} x_i}{w_i} \right\rfloor = k_\mathfrak{p} \right\}$ the *set of* $\mathfrak{p}$*-primitive points.*

Lemma 2.8. *The following equality holds*

$$R_{\mathfrak{p}}^{\mathfrak{a}} = (\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}) - ((\mathfrak{p}\mathfrak{a}_{\mathfrak{p}})^{w_1} \times \cdots \times (\mathfrak{p}\mathfrak{a}_{\mathfrak{p}})^{w_m}).$$

Proof. This is a consequence of the prime decomposition of $\mathfrak{a}$. For $x \in \mathbb{K}_{\mathfrak{p}}$ and $n \in \mathbb{N}$, we have that $x \in \mathfrak{a}_{\mathfrak{p}}^n - (\mathfrak{p}\mathfrak{a}_{\mathfrak{p}})^n \iff \left\lfloor \frac{ord_{\mathfrak{p}} x}{n} \right\rfloor = k_{\mathfrak{p}}$. □

Remark 2.9. $R_{\mathfrak{p}}^{\mathfrak{a}}$ is compact. To see this, note that there is an isomorphism of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$-modules $\mathcal{O}_{\mathbb{K},\mathfrak{p}} \cong \mathfrak{a}_{\mathfrak{p}}^{w_i}$ sending r to $r\mathfrak{a}^{w_i}$ where $\mathcal{O}_{\mathbb{K},\mathfrak{p}}$ is compact, and use Lemma 2.8 to conclude that $R_{\mathfrak{p}}^{\mathfrak{a}}$ is the complement of an open subset in a compact space, thus compact.

We then consider the following diagram, where $\pi_{\mathfrak{p}}$ maps a set to its local version.

$$\begin{array}{ccc} Q_w & \xrightarrow{\pi_{\mathfrak{p}}} & Q_{w,\mathfrak{p}} \\ \uparrow & & \uparrow q \\ Q_w^{\mathfrak{a}} & & \mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times} \backslash R_{\mathfrak{p}}^{\mathfrak{a}} \end{array}$$

By Lemma 2.8, $R_{\mathfrak{p}}^{\mathfrak{a}}$ is simply the local subset of $\mathbb{K}_{\mathfrak{p}}^m$ defined by the equality $\mathfrak{I}_w(x)_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}$; the local component of the requirement that $\mathfrak{I}_w(x) = \mathfrak{a}$. It is then clear that $Q_w^{\mathfrak{a}} = \bigcap_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(\mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times} \backslash R_{\mathfrak{p}}^{\mathfrak{a}})$.

Notation. We denote $Q_{\mathfrak{p}}^{\mathfrak{a}} := \mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times} \backslash R_{\mathfrak{p}}^{\mathfrak{a}}$.

We may now try to get a similar expression for $Q_w^{\mathfrak{a},\mathfrak{d}}$. To do so, we have the following diagram, in which $[\cdot]$ sends a point to its image in the projective space and $\delta_{\phi,\mathfrak{p}}(x) := \mathrm{ord}_{\mathfrak{p}}(\mathfrak{I}_u(\phi(x)) \cdot \mathfrak{I}_w(x)^{-e})$ is the $\mathfrak{p}$-adic valuation of $\delta_{\phi}(x)$.

$$\begin{array}{ccc} R_{\mathfrak{p}}^{\mathfrak{a}} & \hookrightarrow & \mathbb{K}_{\mathfrak{p}}^m - \{0\} \\ \downarrow & & \downarrow \\ Q_{\mathfrak{p}}^{\mathfrak{a}} & \xrightarrow{[\cdot]} & \mathbb{P}(w)(\mathbb{K}_{\mathfrak{p}}) \\ & & \downarrow \delta_{\phi,\mathfrak{p}} \\ & & \mathbb{Z} \end{array}$$

Definition 2.10. We define by $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} := \{x \in R_{\mathfrak{p}}^{\mathfrak{a}} \mid \delta_{\phi,\mathfrak{p}}([x]) = j_{\mathfrak{p}}\}$ the *set of* $\mathfrak{p}$*-primitive points with local discrepancy ideal* $\mathfrak{d}$.

We can conclude that $Q_w^{\mathfrak{a},\mathfrak{d}} = \bigcap_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(\mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times} \backslash R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}})$ by following the same argument as above, as we combine the requirements that the scaling ideal should be $\mathfrak{a}$ and the discrepancy ideal should be $\mathfrak{d}$.

Notation. We denote $Q_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} := \mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times} \backslash R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$.

2.2 Moduli and periodic sets

Our next step will be to introduce some moduli together with the remaining few relevant definitions, which we will utilise to ultimately study our main counting issue.

By construction, working with $Q_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ (resp. $Q_{\mathfrak{p}}^{\mathfrak{a}}$) is equivalent to working with elements in $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ (resp. $R_{\mathfrak{p}}^{\mathfrak{a}}$) that are stable under multiplication by $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times}$. Therefore, for simplicity we will only refer to $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ (resp. $R_{\mathfrak{p}}^{\mathfrak{a}}$), but note that the results also apply for the corresponding Q´s.

Remark 2.11. It follows from the continuity of δ_{ϕ} (Lemma 2.5) and of $\mathrm{ord}_{\mathfrak{p}}$ (Lemma 1.9) that $\delta_{\phi,\mathfrak{p}}$ is continuous. We then have that

(i) $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is defined by conditions determined by open subsets of $R_{\mathfrak{p}}^{\mathfrak{a}}$ and is therefore open in $R_{\mathfrak{p}}^{\mathfrak{a}}$, and

(ii) $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is closed in $R_{\mathfrak{p}}^{\mathfrak{a}}$, thus compact by Remark 2.9.

As a consequence, we get the following lemma.

Lemma 2.12. *For every prime $\mathfrak{p}$ there exists an open subset of the form $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} = \mathfrak{m}_{\mathfrak{p},1}^{\mathfrak{a},\mathfrak{d}} \times \cdots \times \mathfrak{m}_{\mathfrak{p},m}^{\mathfrak{a},\mathfrak{d}} \subseteq \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$, such that $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is determined by congruence conditions modulo $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} := \bigcap_{\mathfrak{d}} \mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ in $R_{\mathfrak{p}}^{\mathfrak{a}}$. Moreover, for every $\mathfrak{p}$ with $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{d} = 0$ for all $\mathfrak{d} \in \mathcal{D}_{\phi}$, we can choose $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} = \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$.*

Proof. We view $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ as a subset of $\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$. Following a similar argument as in Lemma 1.13, the products

$$\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p}},b^{\mathfrak{p}}} := \prod_{i=1}^{m} (a_i^{\mathfrak{p}} + \mathfrak{p}^{b_i^{\mathfrak{p}}} \mathfrak{a}_{\mathfrak{p}}^{w_i})$$

for varying $a^{\mathfrak{p}} = (a_1^{\mathfrak{p}}, \ldots, a_m^{\mathfrak{p}}) \in \mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m}$ and $b^{\mathfrak{p}} = (b_1^{\mathfrak{p}}, \ldots, b_m^{\mathfrak{p}}) \in \mathbb{Z}_{\geqslant 0}^{m}$ will define a basis of open subsets for $\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$. For all but finitely many $\mathfrak{p}$, let us choose $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p}},b^{\mathfrak{p}}}$ to be the whole $\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$. That way, $\prod_{\mathfrak{p}} \mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p}},b^{\mathfrak{p}}}$ is also a basis for $\prod_{\mathfrak{p}} (\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m})$. Without loss of generality, we may choose $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p}},b^{\mathfrak{p}}} = \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$ for every $\mathfrak{p}$ such that $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{d} = 0$ for all $\mathfrak{d} \in \mathcal{D}_{\phi}$.

By Remark 2.11, we can find a basis for and cover $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ by some finite collection of the above mentioned basic open subsets intersected with $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$. That is, we have that

$$R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} = \bigcup_{j \in \mathcal{J}} (\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}} \cap R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}})$$

where we can take $\mathcal{J}$ to be finite. Because of the finiteness of the index set $\mathcal{J}$, there is a maximum value of $b_i^{\mathfrak{p},j}$ which occurs as j varies and i is fixed.

We denote this value by $s_i := \max_{j\in\mathcal{J}}(b_i^{\mathfrak{p},j})$, and $s := (s_1, \ldots, s_m)$. Note that for $\mathfrak{p}$ with $\operatorname{ord}_{\mathfrak{p}} \mathfrak{d} = 0$ for all $\mathfrak{d} \in \mathcal{D}_\phi$ we have $b_1^{\mathfrak{p},j} = \cdots = b_m^{\mathfrak{p},j} = 0$ and thus $s_i = 0$ for all i.

By abuse of notation, let us denote $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}} := \mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}} \cap R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$. We have that each $a_i^{\mathfrak{p},j} + \mathfrak{p}^{b_i^{\mathfrak{p},j}} \mathfrak{a}_{\mathfrak{p}}^{w_i}$ is stable under translation by $\mathfrak{p}^{s_i}\mathfrak{a}_{\mathfrak{p}}^{w_i}$ and we conclude that $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}}$ is $(\mathfrak{p}^{s_1}\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{p}^{s_m}\mathfrak{a}_{\mathfrak{p}}^{w_m})$-periodic. We can therefore consider the following diagram:

$$\begin{array}{ccc} \mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}} & \twoheadrightarrow & \mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}}/(\mathfrak{p}^{s_1}\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{p}^{s_m}\mathfrak{a}_{\mathfrak{p}}^{w_m}) \\ & & \downarrow \\ \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m} & \xrightarrow{q} & (\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m})/(\mathfrak{p}^{s_1}\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{p}^{s_m}\mathfrak{a}_{\mathfrak{p}}^{w_m}) \end{array}$$

We can easily deduce from the diagram that $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}}$ is the inverse image of $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}}/(\mathfrak{p}^{s_1}\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{p}^{s_m}\mathfrak{a}_{\mathfrak{p}}^{w_m})$ under the map q, implying that $\mathcal{U}_{\mathfrak{p}}^{a^{\mathfrak{p},j},b^{\mathfrak{p},j}}$ is determined by congruence conditions modulo $\mathfrak{p}^{s_1}\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{p}^{s_m}\mathfrak{a}_{\mathfrak{p}}^{w_m}$. The result follows. □

Definition 2.13. We call $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}$ the *local modulus at* $\mathfrak{p}$.

Remark 2.14. Let $\mathfrak{p}$ be such that $k_{\mathfrak{p}} = 0$ and $\operatorname{ord}_{\mathfrak{p}} \mathfrak{d} = 0$ for all $\mathfrak{d} \in \mathcal{D}_\phi$. Then $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} = \mathcal{O}_{\mathbb{K},\mathfrak{p}}^m$, since for all the primes $\mathfrak{p}$ that do not divide $\mathfrak{a}$ we have that $\mathfrak{a}_{\mathfrak{p}} = (\mathfrak{p}\mathcal{O}_{\mathbb{K},\mathfrak{p}})^{k_{\mathfrak{p}}}$ is trivial.

Remark 2.15. The first statement of the lemma can be translated as follows: there exists some subset $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ in $(\mathfrak{a}_{\mathfrak{p}}^{w_1}/\mathfrak{m}_{\mathfrak{p},1}^{\mathfrak{a}}) \times \cdots \times (\mathfrak{a}_{\mathfrak{p}}^{w_m}/\mathfrak{m}_{\mathfrak{p},m}^{\mathfrak{a}})$ such that $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is the inverse image of $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ under the reduction map shown in the diagram below. To see this, note that taking the inverse image under the reduction map is essentially the same as taking the primitive points satisfying some congruence conditions modulo $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}$.

$$\begin{array}{ccccc} R_{\mathfrak{p}}^{\mathfrak{a}} & \hookrightarrow & \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m} & \twoheadrightarrow & (\mathfrak{a}_{\mathfrak{p}}^{w_1}/\mathfrak{m}_{\mathfrak{p},1}^{\mathfrak{a}}) \times \cdots \times (\mathfrak{a}_{\mathfrak{p}}^{w_m}/\mathfrak{m}_{\mathfrak{p},m}^{\mathfrak{a}}) \\ \uparrow & & & & \uparrow \\ R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} & & \longrightarrow & & S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} \end{array}$$

All the components in the first row of the above diagram are stable under the (weight w) action of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times$ (this is easy to see, and it comes down to the fact that multiplication of a fractional ideal of a ring R by a unit of R leaves the ideal unchanged). In fact, all components are, since $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is by definition also $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times$-stable and without loss of generality we can choose $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ to be $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times$-stable as well. If not, we may replace $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ by $\bigcup_{\kappa\in\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times} \kappa S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$, making it $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times$-stable while maintaining $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ as its inverse image. For the remaining of the thesis we fix such a choice of $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^\times$-stable sets $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$.

Definition 2.16. We define the *global modulus* to be

$$\mathfrak{m}^{\mathfrak{a}} := \bigcap_{\mathfrak{p}} (\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} \cap \mathbb{K}^m) = \mathfrak{m}_1^{\mathfrak{a}} \times \cdots \times \mathfrak{m}_m^{\mathfrak{a}},$$

where $\mathfrak{m}_i^{\mathfrak{a}} := \bigcap_{\mathfrak{p}} (\mathfrak{m}_{\mathfrak{p},i}^{\mathfrak{a}} \cap \mathbb{K})$ and the intersection inside the brackets takes place in $\mathbb{K}_{\mathfrak{p}}$. Equivalently, $\mathfrak{m}^{\mathfrak{a}} = \{x \in \mathbb{K}^m \mid \text{for all } \mathfrak{p} : x \in \mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}\}$. Note that $\mathfrak{m}^{\mathfrak{a}}$ is a lattice in $\mathbb{K}^m$.

Remark 2.17. It follows from the second result of Lemma 2.12 that this set is well defined. Else, we would have non-trivial conditions at every $\mathfrak{p}$.

Definition 2.18. We define the *set of points with local discrepancy ideal* $\mathfrak{d}$ to be the inverse image of $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ in $\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m} \subseteq \mathbb{K}_{\mathfrak{p}}^m$. We denote this set by $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$.

Remark 2.19. Note that $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} = V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} \cap R_{\mathfrak{p}}^{\mathfrak{a}}$.

Remark 2.20. Let $\mathfrak{p}$ such that $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} = \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$ is trivial. Then, $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is also trivial. As a consequence, we get that $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} = \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$ and $R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} = R_{\mathfrak{p}}^{\mathfrak{a}}$.

We will next introduce one of our most important concepts, which we will make use of in the next chapter for counting points within a suitable weighted expanding compact set, leading us to our main result.

Definition 2.21. We define the *global set* $V^{\mathfrak{a},\mathfrak{d}}$ to be

$$V^{\mathfrak{a},\mathfrak{d}} = \bigcap_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}).$$

In other words, $V^{\mathfrak{a},\mathfrak{d}}$ is the set of points in $\mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m}$ that, localized, satisfy all the congruence conditions modulo $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}$ at every $\mathfrak{p}$.

Example 2.22. Let ϕ be the identity map. Then $e = 1$ and there is only one choice for $\mathfrak{d} \in \mathcal{D}_\phi$, namely $\mathfrak{d} = (1)$, and thus by Lemma 2.12 we can take $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}$ to be $\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$ for all $\mathfrak{p}$, which by Remark 2.20 implies that for all $\mathfrak{p}$ we have $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} = \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}$ and therefore $V^{\mathfrak{a},\mathfrak{d}} = \mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m}$.

Lemma 2.23. *$V^{\mathfrak{a},\mathfrak{d}}$ satisfies the following properties:*

(i) *$V^{\mathfrak{a},\mathfrak{d}}$ is $\mathfrak{m}^{\mathfrak{a}}$-periodic. As a result, $V^{\mathfrak{a},\mathfrak{d}}$ is a union of cosets of $\mathfrak{m}^{\mathfrak{a}}$ and we could either view $V^{\mathfrak{a},\mathfrak{d}}$ as the whole set or as $V^{\mathfrak{a},\mathfrak{d}} = \bigsqcup_{v \in X \text{ finite}} (v + \mathfrak{m}^{\mathfrak{a}})$.*

(ii) *$V^{\mathfrak{a},\mathfrak{d}}$ is stable under the (weight w) action of $\mathcal{O}_{\mathbb{K}}^\times$ on $\mathbb{K}^m$.*

Proof. Both results follow from analogous properties of $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$:

(i) Let $x \in V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ and $y \in \mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}$. Then it is clear by definition that $x + y \in V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ and so $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}}$-periodic for each $\mathfrak{p}$. Thus, the result.

(ii) By definition of $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ and the fact that $S_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times}$-stable together with the compatibility of the reduction map, we ensure that $V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}}$ is $\mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times}$-stable as well. Again, the construction of $V^{\mathfrak{a},\mathfrak{d}}$ makes the result straightforward, as $\mathcal{O}_{\mathbb{K}}^{\times} \subseteq \mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times}$ for any $\mathfrak{p}$.

□

Remark 2.24. Note that $Q_w^{\mathfrak{a},\mathfrak{d}} = \{x \in \mathcal{O}_{\mathbb{K}}^{\times} \backslash (V^{\mathfrak{a},\mathfrak{d}} - \{0\}) \,|\, \mathfrak{I}_w(x) = \mathfrak{a}\}$. This follows from the expression $Q_w^{\mathfrak{a},\mathfrak{d}} = \bigcap_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(\mathcal{O}_{\mathbb{K},\mathfrak{p}}^{\times} \backslash R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}})$ and the fact that $\{x \in V^{\mathfrak{a},\mathfrak{d}} - \{0\} \,|\, \mathfrak{I}_w(x) = \mathfrak{a}\} = \bigcap_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(V_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}} \cap R_{\mathfrak{p}}^{\mathfrak{a}}) = \bigcap_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(R_{\mathfrak{p}}^{\mathfrak{a},\mathfrak{d}})$.

Chapter 3

The asymptotic formula on $\mathbb{P}(w)(\mathbb{K})$

In the following sections, we will compute the $\mathbb{K}$-rational points on the weighted projective space $\mathbb{P}(w)(\mathbb{K})$ satisfying certain property with respect to the size function defined in Definition 1.17. Our final aim and main counting problem will be to determine the asymptotic behaviour of the quantity

$$N_\phi(T) := \#\{x \in \mathbb{P}(w)(\mathbb{K}) \,|\, S_u(\phi(x)) \leqslant T\} \text{ as } T \to \infty.$$

We do so by following an analogous method to that of [2].

3.1 Introduction to the counting problem

We will first introduce some similar values to $N_\phi(T)$, through which we intend to ease the computation of $N_\phi(T)$.

Definition 3.1. We define the *counting function of* ϕ to be

$$N_\phi(c, T) := \#\{x \in \mathbb{P}(w)(\mathbb{K}) \,|\, [\mathfrak{I}_w(x)] = c,\, S_u(\phi(x)) \leqslant T\}.$$

Thus, by summing $N_\phi(c, T)$ over all the ideal classes c of $\mathbb{K}$ we will achieve our counting goal.

Remark 3.2. Note that

$$\begin{aligned} N_\phi(c, T) &= \#\{x \in Q_w^{\mathfrak{a}} \,|\, S_u(\phi(x)) = \frac{H_{u,\infty}(\phi(x))}{N(\mathfrak{I}_u(\phi(x)))} \leqslant T\} \\ &= \sum_{\mathfrak{d} \in \mathcal{D}_\phi} \#\{x \in Q_w^{\mathfrak{a}} \,|\, \mathfrak{I}_u(\phi(x)) = \mathfrak{d}\mathfrak{a}^e, H_{u,\infty}(\phi(x)) \leqslant N(\mathfrak{d}\mathfrak{a}^e)T\} \\ &= \sum_{\mathfrak{d} \in \mathcal{D}_\phi} \#\{x \in Q_w^{\mathfrak{a},\mathfrak{d}} \,|\, H_{u,\infty}(\phi(x)) \leqslant N(\mathfrak{d}\mathfrak{a}^e)T\}. \end{aligned}$$

Let $\mathfrak{b} \subseteq \mathfrak{a}$ be an ideal. We define the following two quantities.

$$\bar{\mathcal{N}}_\phi(\mathfrak{b}, \mathfrak{a}, \mathfrak{d}, T) := \#\{x \in \mathcal{O}_\mathbb{K}^\times \backslash (V^{\mathfrak{a},\mathfrak{d}} - \{0\}) \,|\, \mathfrak{I}_w(x) = \mathfrak{b}, H_{u,\infty}(\phi(x)) \leqslant T\}.$$

$$\mathcal{N}_\phi(\mathfrak{b}, \mathfrak{a}, \mathfrak{d}, T) := \#\{x \in \mathcal{O}_\mathbb{K}^\times \backslash (V^{\mathfrak{a},\mathfrak{d}} - \{0\}) \,|\, \mathfrak{I}_w(x) \subseteq \mathfrak{b},\ H_{u,\infty}(\phi(x)) \leqslant T\}.$$

By Lemma 2.23 (ii), these sets are well defined, and it is clear that the following equality holds;

$$\mathcal{N}_\phi(\mathfrak{b}, \mathfrak{a}, \mathfrak{d}, T) = \sum_{\mathfrak{e} \subseteq \mathcal{O}_\mathbb{K}} \bar{\mathcal{N}}_\phi(\mathfrak{be}, \mathfrak{a}, \mathfrak{d}, T).$$

Also note there is an equivalent way of rewriting $\mathcal{N}_\phi(\mathfrak{b}, \mathfrak{a}, \mathfrak{d}, T)$, namely as $\#\{x \in \mathcal{O}_\mathbb{K}^\times \backslash \big(V^{\mathfrak{a},\mathfrak{d}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) - \{0\}\big) \,|\, H_{u,\infty}(\phi(x)) \leqslant T\}$. On top of that, it is not hard to see, following Remark 3.2 and Remark 2.24, that the counting function of ϕ can be expressed as

$$N_\phi(c, T) = \sum_{\mathfrak{d} \in \mathcal{D}_\phi} \bar{\mathcal{N}}_\phi(\mathfrak{a}, \mathfrak{a}, \mathfrak{d}, TN(\mathfrak{d})N(\mathfrak{a})^e).$$

We define the Möbius function of an ideal as done in [10, Section 1].

Definition 3.3. Let $I \subseteq \mathcal{O}_\mathbb{K}$ be an integral ideal of $\mathbb{K}$. We define the *Möbius function of I over $\mathbb{K}$* to be

$$\mu_\mathbb{K}(I) := \begin{cases} 1, \text{ if } I = \mathcal{O}_\mathbb{K}; \\ 0, \text{ if } I \subset \mathfrak{p}^2 \text{ for some } \mathfrak{p}; \\ (-1)^k, \text{ if } I = \mathfrak{p}_1 \cdots \mathfrak{p}_k \text{ for } \mathfrak{p}_1, \ldots, \mathfrak{p}_k \text{ distinct prime ideals of } \mathbb{K}. \end{cases}$$

The following two lemmas follow by definition.

Lemma 3.4. *Let I, J be coprime integral ideals of $\mathbb{K}$. Then, we have that $\mu_\mathbb{K}(IJ) = \mu_\mathbb{K}(I)\mu_\mathbb{K}(J)$.*

Lemma 3.5. *Let I, J be integral ideals of $\mathbb{K}$. Then,*

$$\sum_{I \subseteq J} \mu_\mathbb{K}(J) = \begin{cases} 1, \text{ if } I = \mathcal{O}_\mathbb{K}; \\ 0, \text{ if } I \neq \mathcal{O}_\mathbb{K}. \end{cases}$$

We can now apply Möbius inversion, from which we deduce that

$$\begin{aligned} \sum_{\mathfrak{c} \subseteq \mathcal{O}_\mathbb{K}} \mu_K(\mathfrak{c}) \mathcal{N}_\phi(\mathfrak{bc}, \mathfrak{a}, \mathfrak{d}, T) &= \sum_{\mathfrak{c}, \mathfrak{e} \subseteq \mathcal{O}_\mathbb{K}} \mu_K(\mathfrak{c}) \bar{\mathcal{N}}_\phi(\mathfrak{bce}, \mathfrak{a}, \mathfrak{d}, T) \\ &= \sum_{\mathfrak{n} \subseteq \mathcal{O}_\mathbb{K}} \Big(\sum_{\mathfrak{n} \subseteq \mathfrak{c}} \mu_K(\mathfrak{c}) \Big) \bar{\mathcal{N}}_\phi(\mathfrak{bn}, \mathfrak{a}, \mathfrak{d}, T) \\ &= \bar{\mathcal{N}}_\phi(\mathfrak{b}, \mathfrak{a}, \mathfrak{d}, T), \end{aligned}$$

This way, we can express the term $\bar{\mathcal{N}}_\phi$ as a linear combination of $\mathcal{N}_\phi$, reducing the counting of points with fixed scaling ideal (which is what we ultimately need to do) to simply counting points with scaling ideal a subset of a fixed ideal.

Hence, we gather that

$$\begin{aligned} N_\phi(T) &= \sum_{c \in Cl(\mathbb{K})} N_\phi(c,T) \\ &= \sum_{\mathfrak{a} \in R_\mathbb{K}} \sum_{\mathfrak{d} \in \mathcal{D}_\phi} \bar{\mathcal{N}}_\phi(\mathfrak{a},\mathfrak{a},\mathfrak{d}, TN(\mathfrak{d})N(\mathfrak{a})^e) \\ &= \sum_{\mathfrak{a} \in R_\mathbb{K}} \sum_{\mathfrak{d} \in \mathcal{D}_\phi} \sum_{\mathfrak{c} \subseteq \mathcal{O}_\mathbb{K}} \mu_\mathbb{K}(\mathfrak{c}) \mathcal{N}_\phi(\mathfrak{ac},\mathfrak{a},\mathfrak{d}, TN(\mathfrak{d})N(\mathfrak{a})^e). \end{aligned} \tag{3.1}$$

This reduces the original problem to determining the asymptotic behaviour of

$$\mathcal{N}_\phi(\mathfrak{ac},\mathfrak{a},\mathfrak{d},T) \text{ as } T \to \infty$$

for

(i) finitely many fixed representatives $\mathfrak{a}$ of $Cl\mathbb{K}$.

(ii) finitely many $\mathfrak{d} \in \mathcal{D}_\phi$.

(iii) all integral ideals $\mathfrak{c} \subseteq \mathcal{O}_\mathbb{K}$.

3.2 The asymptotic formula for $\mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T)$

Mirroring the method used in [2, Section 4], we may simplify the problem even further by making it into a lattice point counting issue.

The first step to our objective is to generalize [2, Proposition 4.4], which will allow us to count lattice points. We first recall Dirichlet´s unit theorem.

Theorem 3.6. *(Dirichlet) Let*

$$\begin{aligned} \ell : \mathcal{O}_\mathbb{K}^\times &\to \mathbb{R}^{r_1+r_2} \\ u &\mapsto (\log |u|_v)_{v \in \Omega_\infty}. \end{aligned}$$

Then, $\Gamma := Im(\ell)$ *is a lattice of rank* $r := r_1 + r_2 - 1$ *in the hyperplane* $\mathcal{H}$ *defined by* $\sum_{v \in \Omega_\infty} y_v = 0$. *Moreover,* $\ker(\ell) = \mu(\mathbb{K})$.

Let Δ be as in [2, Section 4], i.e. $\Delta := (pr \circ \eta)^{-1} F$, where we define the maps $pr : \mathbb{R}^{r_1+r_2} \to \mathcal{H}$ and $\eta = (\eta_v)_{v \in \Omega_\infty} : \prod_{v \in \Omega_\infty} (\mathbb{K}_v^m - \{0\}) \to \mathbb{R}^{r_1+r_2}$ as $pr(y)_v = y_v - \left(\frac{\sum_{v' \in \Omega_\infty} y_{v'}}{N} \right) N_v$ and

$$\begin{aligned} \eta_v : \mathbb{K}_v^m - \{0\} &\to \mathbb{R} \\ Z_v &\mapsto \log \max_{i=1,\dots,m} |Z_{v,i}|_v^{\frac{1}{w_i}} \end{aligned}$$

for $Z = (Z_v)_{v\in\Omega_\infty}$, and F is the standard fundamental domain for $\mathcal{H}/\Gamma$. As stated in [2, Proposition 4.1, Lemma 4.3], the following properties hold.

Lemma 3.7. (i) *Δ is a fundamental domain for $\prod_{v\in\Omega_\infty}(\mathbb{K}^m - \{0\})/\mathcal{O}_\mathbb{K}^\times$.*

(ii) *Δ is $\mu(\mathbb{K})$-stable.*

(iii) *$\mathcal{O}_{\mathbb{K}*}^\times(\Delta) = \mathcal{H}$.*

(iv) *$u_*(\Delta)\cap\Delta = \varnothing$ for $u \in \mathcal{O}_\mathbb{K}^\times - \mu(\mathbb{K})$.*

(v) *$t_*\Delta = \Delta$ for $t \in \mathbb{R}^\times$.*

We now introduce the analogues of the sets $\mathfrak{D}(T)$ and $\mathfrak{B}(T)$ in [2, Section 4].

Notation. We denote

$$\mathfrak{D}_\phi(T) := \{z \in \prod_{v\in\Omega_\infty}(\mathbb{K}_v^m - \{0\}) \mid \prod_{v\in\Omega_\infty}\max_i |\phi(z)_i|_v^{1/u_i} \leqslant T\}$$

and $\mathfrak{B}_\phi(T) := \mathfrak{D}_\phi(T)\cap\Delta$.

Remark 3.8. We note that $\mathfrak{D}_\phi(T)$ is $\mathcal{O}_\mathbb{K}^\times$-stable. This can be done by

$$\begin{aligned}\max_i |\phi(\lambda z)_i|_v^{1/u_i} &= \max_i |\lambda^{eu_i}\phi(z)_i|_v^{1/u_i} = \max_i(|\lambda^e|_v|\phi(z)_i|_v^{1/u_i})\\ &= \max_i |\phi(z)_i|_v^{1/u_i},\end{aligned}$$

for all $\lambda \in \mathcal{O}_\mathbb{K}^\times$.

By a similar argument to the one used in [2, Lemma 4.3], we get the following result.

Lemma 3.9. *We have $\mathfrak{B}_\phi(T) = T_*^{1/eN}\mathfrak{B}_\phi(1)$ with the weight w action.*

Assumption. We assume in the paper that $\mathfrak{B}_\phi(1)$ has C^1-parametrizable boundary (see [2, Section 4]). We will not prove this claim, as it would require a lengthy and not very illustrative computation using theory of semi-algebraic subsets of $\mathbb{R}^n$. Nevertheless, it is to be expected that this holds for any morphism ϕ (for a specific example, see Chapter 4).

Let $\mathfrak{P}_w(T)$ be the *weighted expanding set* defined in [2, Section 4], i.e.

$$\mathfrak{P}_w(T) := T_*\mathfrak{P} = \{(T^{w_1}x_1,\dots,T^{w_k}x_k) : (x_1,\dots,x_k) \in \mathfrak{P}\},$$

for $\mathfrak{P}$ a bounded subset of $\mathbb{R}^k$ with C^1-parametrizable boundary. We now want to generalize [2, Proposition 4.4] to lattice cosets. To do so, let Λ be a lattice in $\mathbb{R}^k$, and $\Lambda_v = \Lambda + v$ its translate by $v \in \mathbb{R}^k$.

We may now state our generalization of [2, Proposition 4.4].

Lemma 3.10. *Notations as above. We have that*

$$\#\Lambda_v \cap \mathfrak{P}_w(T) = T^{|w|}\frac{\mathrm{Vol}_k(\mathfrak{P})}{\det \Lambda} + O(T^{|w|-1}).$$

Proof. Let $T' = T + |v|$ and $T'' = T - |v|$ be translates of T. Then, applying [2, Proposition 4.4], $\#\Lambda \cap \mathfrak{P}_w(T') = (T')^{|w|}\frac{\mathrm{Vol}_k(\mathfrak{P})}{det\Lambda} + O((T')^{|w|-w_{\min}})$ and $\#\Lambda \cap \mathfrak{P}_w(T'') = (T'')^{|w|}\frac{\mathrm{Vol}_k(\mathfrak{P})}{det\Lambda} + O((T'')^{|w|-w_{\min}})$, for $w_{\min} = \min_{i=1,\dots,m} w_i$. Note that $\#\Lambda \cap \mathfrak{P}_w(T') = \#\Lambda \cap \mathfrak{P}_w(T'') = T^{|w|}\frac{\mathrm{Vol}_k(\mathfrak{P})}{det\Lambda} + O(T^{|w|-1})$. Moreover, $\#\Lambda \cap \mathfrak{P}_w(T'') \leqslant \#\Lambda_v \cap \mathfrak{P}_w(T) \leqslant \#\Lambda \cap \mathfrak{P}_w(T')$ by construction (both inequalities follow from the triangle inequality since, for instance, we have $x \in \Lambda \cap \mathfrak{P}_w(T'') \implies x+v \in \Lambda_v \cap \mathfrak{P}_w(T)$ as $|x+v| \leqslant |x|+|v| \leqslant T''+|v| = T$). The result follows. □

Recall that we can decompose $V^{\mathfrak{a},\mathfrak{d}} = \bigsqcup_{v \in X \text{ finite}}(v + \mathfrak{m}^{\mathfrak{a}})$ into a disjoint union of cosets. In order to determine the asymptotic behaviour of $\mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T)$, our next objective will be to find the number of points in $\left(V^{\mathfrak{a},\mathfrak{d}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) - \{0\}\right) \cap \mathfrak{B}_\phi(T)$ (as we will shortly argue in the proof of Lemma 3.16), and to do so we will apply Lemma 3.10 to translates of $\Lambda = \mathfrak{m}^{\mathfrak{a}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) - \{0\}$, a sublattice of $\mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m} - \{0\}$. To accomplish that, write the ideal $\mathfrak{b}$ as $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$, with $\mathfrak{c} \subseteq \mathcal{O}_{\mathbb{K}}$. Without loss of generality, we may take $\mathfrak{c}$ to be square-free. The reason for this assumption is that we ultimately want to make use of Equation 3.1 to find $N_\phi(T)$, where the factor $\mu_{\mathbb{K}}(\mathfrak{c})$ plays a role, which is trivial on non-square free ideals.

Definition 3.11. We define the *set of non-trivial ideals* to be the set

$$\mathcal{S} := \{\mathfrak{p} \subset \mathcal{O}_{\mathbb{K}} \mid \mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} \neq \mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{a}_{\mathfrak{p}}^{w_m}\}.$$

Remark 3.12. The set $\mathcal{S}$ is finite by Lemma 2.12.

Factor $\mathfrak{c} = \mathfrak{c}_0\mathfrak{c}_1$ as a product of prime ideals, where $\mathfrak{c}_0$ is coprime to all $\mathfrak{p} \in \mathcal{S}$ and $\mathfrak{c}_1$ is a (finite) product of primes $\mathfrak{p} \in \mathcal{S}$. It follows from co-primality of $\mathfrak{c}_0$ and $\mathfrak{c}_1$ that $\mathfrak{c} = \mathfrak{c}_0 \cap \mathfrak{c}_1$.

We fix the following notation.

Notation.

$$\begin{aligned}
&M^{\mathfrak{a}} := \mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_m};\\
&M^{\mathfrak{a}}_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1} := \mathfrak{m}^{\mathfrak{a}} \cap ((\mathfrak{a}\mathfrak{c}_1)^{w_1} \times \cdots \times (\mathfrak{a}\mathfrak{c}_1)^{w_m});\\
&M^{\mathfrak{a}}_{\mathfrak{c}_0} := (\mathfrak{a}\mathfrak{c}_0)^{w_1} \times \cdots \times (\mathfrak{a}\mathfrak{c}_0)^{w_m};\\
&V^{\mathfrak{a},\mathfrak{d}}_{\mathfrak{c}_1} := V^{\mathfrak{a},\mathfrak{d}} \cap ((\mathfrak{a}\mathfrak{c}_1)^{w_1} \times \cdots \times (\mathfrak{a}\mathfrak{c}_1)^{w_m}).
\end{aligned}$$

We then have that $V^{\mathfrak{a},\mathfrak{d}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) = V^{\mathfrak{a},\mathfrak{d}}_{\mathfrak{c}_1} \cap M^{\mathfrak{a}}_{\mathfrak{c}_0}$. Note that $V^{\mathfrak{a},\mathfrak{d}}_{\mathfrak{c}_1}$ can only have finitely many values by Lemma 2.5 and the fact that $\mathfrak{c}_1$ can by construction only take $2^{|\mathcal{S}|}$ values. Further, points $x \in V^{\mathfrak{a},\mathfrak{d}}_{\mathfrak{c}_1} \cap M^{\mathfrak{a}}_{\mathfrak{c}_0}$ satisfy the following congruence conditions:

(i) x lies in the finite set $\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}} := V_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}/M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}$ modulo $M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}$ (recall that $V_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}$ is $M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}$-periodic by Lemma 2.23 (i)),

(ii) x is 0 modulo $M_{\mathfrak{c}_0}^{\mathfrak{a}}$.

We conclude that $V_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}} \cap M_{\mathfrak{c}_0}^{\mathfrak{a}}$ is a $(M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})$-periodic subset of $M_{\mathfrak{c}_0}^{\mathfrak{a}}$ and can therefore be written as the union of (finitely many) translates of $M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}$ in $M_{\mathfrak{c}_0}^{\mathfrak{a}} \subset M^{\mathfrak{a}}$.

We show a version of the Chinese remainder theorem applied to modules.

Theorem 3.13. *Let M be a module over a ring R, and $M_0, M_1 \subset M$ submodules of M such that $M_0 + M_1 = M$. We then get an isomorphism*

$$\begin{aligned} M/(M_0 \cap M_1) &\cong M/M_0 \times M/M_1 \\ m \mod (M_0 \cap M_1) &\mapsto (m \mod M_0,\, m \mod M_1). \end{aligned}$$

Proof. The map is clearly injective. To prove that it is also surjective, let $a \in M/M_0$ and $b \in M/M_1$, and write $a = a_0 + a_1$, $b = b_0 + b_1$ with $a_0, b_0 \in M_0$ and $a_1, b_1 \in M_1$. Then, the image of $a_1 + b_0 \in M$ is (a_1, b_0), which is equal to (a, b) in the target space. Hence, the map is surjective. □

Note that $M_{\mathfrak{c}_0}^{\mathfrak{a}} + M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}} = M^{\mathfrak{a}}$. Applying Lemma 3.13, we get the following diagram.

$$\begin{array}{ccc} M^{\mathfrak{a}} & & \\ \downarrow & \searrow^{\pi} & \\ M^{\mathfrak{a}}/(M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}) & \xrightarrow{\sim} & M^{\mathfrak{a}}/M_{\mathfrak{c}_0}^{\mathfrak{a}} \times M^{\mathfrak{a}}/M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}} \hookleftarrow \{0\} \times \mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}} \end{array}$$

Thus, $V_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}} \cap M_{\mathfrak{c}_0}^{\mathfrak{a}} = \{m \in M^{\mathfrak{a}} \,|\, \pi(m) \in \{0\} \times \mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}\}$ can be written as the union of $\#\{\{0\} \times \mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}\}\} = \#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}$ translates of $M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}$ inside $M^{\mathfrak{a}}$.

Lemma 3.14. *We have* $\det(M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}) = N(\mathfrak{c}_0)^{|w|}(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})\dfrac{D_{\mathbb{K}}^{m/2}}{2^{mr_2}}$.

Proof. It follows from Lemma 3.13 that

$$\begin{aligned} \det(M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}) &= (M^{\mathfrak{a}} : (M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}}))\det(M^{\mathfrak{a}}) \\ &= (M^{\mathfrak{a}} : M_{\mathfrak{c}_0}^{\mathfrak{a}})(M^{\mathfrak{a}} : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})\det(M^{\mathfrak{a}}). \end{aligned}$$

Moreover, $M^{\mathfrak{a}}/M_{\mathfrak{c}_0}^{\mathfrak{a}}$ has cardinality $N(\mathfrak{c}_0)^{|w|}$, since, as a consequence of $\mathcal{O}_{\mathbb{K}}$ being a Dedekind domain, $(\mathfrak{a}^{w_i} : (\mathfrak{a}\mathfrak{c}_0)^{w_i}) = (\mathcal{O}_{\mathbb{K}} : \mathfrak{c}_0^{w_i}) = N(\mathfrak{c}_0)^{w_i}$ for all i. Further, $(M^{\mathfrak{a}} : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})$ is equal to $(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})/N(\mathfrak{a})^{|w|}$, and by [2, Section 4] we know that $\det(M^{\mathfrak{a}}) = \frac{D_{\mathbb{K}}^{m/2} N(\mathfrak{a})^{|w|}}{2^{mr_2}}$. That concludes the proof of the lemma. □

We can now apply Lemma 3.10 to $\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}$ many translates of the lattice $\Lambda = M_{\mathfrak{c}_0}^{\mathfrak{a}} \cap M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}} - \{0\}$, where we set the two values $\mathfrak{P} = \mathfrak{B}_\phi(1) \subset \mathbb{R}^{Nm}$ and $\mathfrak{P}_w(T) = \mathfrak{B}_\phi(T)$, which is $T_*^{1/eN}\mathfrak{B}_\phi(1)$ by Lemma 3.9.

Lemma 3.15. *The cardinality of* $\left(V^{\mathfrak{a},\mathfrak{d}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) - \{0\}\right) \cap \mathfrak{B}_\phi(T)$ *is*

$$\frac{\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{N(\mathfrak{c}_0)^{|w|}(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})} T^{\frac{|w|}{e}} \frac{2^{mr_2}}{D_{\mathbb{K}}^{m/2}} \operatorname{Vol}_{Nm}(\mathfrak{B}_\phi(1)) + O(T^{\frac{|w|}{e}-\frac{1}{eN}}).$$

We derive a formula for $\mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T)$.

Lemma 3.16. *We have*

$$\mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T) = \frac{\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{N(\mathfrak{c}_0)^{|w|}(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})} \frac{1}{\mu_{\mathbb{K}}^w} T^{\frac{|w|}{e}} \frac{2^{mr_2}}{D_{\mathbb{K}}^{m/2}} \operatorname{Vol}_{Nm}(\mathfrak{B}_\phi(1)) + O(T^{\frac{|w|}{e}-\frac{1}{eN}} N(\mathfrak{c}_0)^{-|w|+\frac{1}{N}}).$$

Proof. The first step is to find a fitting analogue of [2, Proposition 4.2] to our setting, which will let us switch from counting $\mathcal{O}_{\mathbb{K}}^\times$-orbits to counting lattice points instead.
Claim: $\mu_{\mathbb{K}}^w \cdot \mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T)$ is (asymptotically) equivalent to the number of points in $\left(V^{\mathfrak{a},\mathfrak{d}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) - \{0\}\right) \cap \mathfrak{B}_\phi(T)$, with discrepancy at most $O(T^{\frac{|w|}{e}-\frac{1}{eN}})$.
Proof of claim: As argued in [2, Section 4], it suffices to count the number of $\mu(\mathbb{K})$-orbits in $\left(V^{\mathfrak{a},\mathfrak{d}} \cap (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}) - \{0\}\right) \cap \mathfrak{B}_\phi(T)$, the only difference being that we do not assume $\mathbb{P}(w)$ to be *well-formed* (i.e. it does not necessarily hold that each $n-1$ elements of w are coprime). We consequently must replace the factor w in [2, Proposition 4.2] (denoting the number of roots of unity) by $\mu_{\mathbb{K}}^w$ (see [1, Theorem 3.8]). We do not get a strict equality, since only orbits with all coordinates non-zero contain $\mu_{\mathbb{K}}^w$ points. However, points with at least one coordinate equal to zero lie on a lower dimensional sub-variety, and by applying the same method used to obtain Lemma 3.15 to this sub-variety, we deduce that its contribution is at most $O((T^{\frac{1}{eN}})^{|w|N-1})$, that is, $O(T^{\frac{|w|}{e}-\frac{1}{eN}})$.
It follows that

$$\mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T) = \frac{\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{N(\mathfrak{c}_0)^{|w|}(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})} \frac{1}{\mu_{\mathbb{K}}^w} T^{\frac{|w|}{e}} \frac{2^{mr_2}}{D_{\mathbb{K}}^{m/2}} \operatorname{Vol}_{Nm}(\mathfrak{B}_\phi(1)) + O(T^{\frac{|w|}{e}-\frac{1}{eN}}). \quad (3.2)$$

In order to determine the influence of $\mathfrak{c}_0$ in the error term, let us write $\mathfrak{c}_0 = \lambda\mathfrak{r}$ for $\mathfrak{r} \in R_{\mathbb{K}}$ such that $[\mathfrak{c}_0] = [\mathfrak{r}]$ and $\lambda \in \mathbb{K}^\times$ unique up to units.
Next, let $x \in \mathbb{K}^m - \{0\}$. Note that

(i) $\mathfrak{I}_w(x) \subseteq \mathfrak{b} \iff \mathfrak{I}_w(\lambda_* x) \subseteq \lambda\mathfrak{b}$, by Lemma 1.16 (i);

(ii) $H_{u,\infty}(\phi(\lambda_* x)) = N((\lambda))^e H_{u,\infty}(\phi(x))$, since

$$H_{u,\infty}(\phi(\lambda_* x)) = H_{u,\infty}(\lambda_*^e \phi(x)) = \prod_{v\in\Omega_\infty} |\lambda|_v^e H_{u,\infty}(\phi(x))$$

and $\prod_{v\in\Omega_\infty} |\lambda|_v = N((\lambda))$.

Consequently,

$$\begin{aligned}\mathcal{N}_\phi(\mathfrak{b},\mathfrak{a},\mathfrak{d},T) &= \mathcal{N}_\phi(\mathfrak{c}_0\mathfrak{c}_1\mathfrak{a},\mathfrak{a},\mathfrak{d},T) = \mathcal{N}_\phi(\lambda\mathfrak{r}\mathfrak{c}_1\mathfrak{a},\mathfrak{a},\mathfrak{d},T)\\ &= \#\{x \in \mathcal{O}_\mathbb{K}^\times\backslash(V^{\mathfrak{a},\mathfrak{d}} - \{0\}) \,|\, \mathfrak{I}_w(x) \subseteq \lambda\mathfrak{r}\mathfrak{c}_1\mathfrak{a},\ H_{u,\infty}(\phi(x)) \leqslant T\}\\ &= \mathcal{N}_\phi(\mathfrak{r}\mathfrak{c}_1\mathfrak{a},\mathfrak{a},\mathfrak{d},N((\lambda))^{-e}T),\end{aligned}$$

since, as argued before, $\mathfrak{I}_w(x) \subseteq \lambda\mathfrak{r}\mathfrak{c}_1\mathfrak{a} \iff \mathfrak{I}_w(\lambda_*^{-1}x) \subseteq \mathfrak{r}\mathfrak{c}_1\mathfrak{a}$ and, on the other hand, $H_{u,\infty}(\phi(x)) \leqslant T \iff H_{u,\infty}(\phi(\lambda_*^{-1}x)) \leqslant N((\lambda))^{-e}T$.

Applying Equation 3.2 to $\mathcal{N}_\phi(\mathfrak{r}\mathfrak{c}_1\mathfrak{a},\mathfrak{a},\mathfrak{d},N((\lambda))^{-e}T)$ yields the result, since $\mathfrak{c}_0 = \lambda\mathfrak{r}$ by construction. □

Remark 3.17. $\mathrm{Vol}_{Nm}(\mathfrak{B}_\phi(1))$ is the volume of the region defined by some polynomial inequalities determined by the choice of ϕ. We will not give an explicit expression for this volume, as it might not even be possible to do so for an arbitrary ϕ. However, this value can be numerically approximated in any given example of ϕ, as we do in Chapter 4.

3.3 The asymptotic formula for $N_\phi(T)$

We define the Dedekind zeta function as done in [9, Lemma 6.2].

Definition 3.18. Let $t \in \mathbb{C}$ be a complex number with $\Re(t) > 1$. We define the *Dedekind zeta function of t over $\mathbb{K}$* to be

$$\zeta_\mathbb{K}(t) := \sum_{I\neq 0} (N(I))^{-t} = \prod_{\mathfrak{p}} (1 - (N(\mathfrak{p}))^{-t})^{-1},$$

where the sum ranges over all non-zero ideals I of $\mathcal{O}_\mathbb{K}$.

We can now state our main result, which we will prove by combining the information gained from Lemma 3.16 and Equation 3.1.

Theorem 3.19.

$$N_\phi(T) = C_\phi^\mathbb{K} \left(\frac{1}{\zeta_\mathbb{K}(|w|)} \left(\frac{2^{r_2}}{\sqrt{D_\mathbb{K}}} \right)^m \frac{\mathrm{Vol}_{Nm}(\mathfrak{B}_\phi(1))}{\mu_\mathbb{K}^w} \right) T^{\frac{|w|}{e}} + O(T^{\frac{|w|}{e} - \frac{1}{eN}}),$$

where

$$C_\phi^{\mathbb{K}} := \sum_{\mathfrak{a}\in R_{\mathbb{K}}} \sum_{\mathfrak{d}\in\mathcal{D}_\phi} \sum_{\substack{\mathfrak{c}_1\subseteq\mathcal{O}_{\mathbb{K}} \\ \text{product} \\ \text{of } \mathfrak{p}\in\mathcal{S}}} (N(\mathfrak{d})^{1/e}N(\mathfrak{a}))^{|w|} \frac{\mu_{\mathbb{K}}(\mathfrak{c}_1)\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{(\mathcal{O}_{\mathbb{K}}^m : M^{\mathfrak{a}}_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1})} \prod_{\mathfrak{p}\in S}(1-N(\mathfrak{p})^{-|w|})^{-1}.$$

If $\mathbb{P}(w)(\mathbb{K}) = \mathbb{P}(1,1)(\mathbb{Q})$ or $\mathbb{P}(w)(\mathbb{K}) = \mathbb{P}(2)(\mathbb{Q})$, the error term is taken to be $O(T^{1/e}\log T^{1/e})$.

Proof. We first want to determine the formula of the last factor in Equation 3.1, namely of $\sum_{\mathfrak{c}\subseteq\mathcal{O}_{\mathbb{K}}} \mu_{\mathbb{K}}(\mathfrak{c})\mathcal{N}_\phi(\mathfrak{a}\mathfrak{c},\mathfrak{a},\mathfrak{d},TN(\mathfrak{d})N(\mathfrak{a})^e)$. We do so by applying Lemma 3.16 to $\mathcal{N}_\phi(\mathfrak{a}\mathfrak{c},\mathfrak{a},\mathfrak{d},TN(\mathfrak{d})N(\mathfrak{a})^e)$, and decomposing $\mathfrak{c}$ as above. Note that by doing so, by Lemma 3.4 $\mu_{\mathbb{K}}(\mathfrak{c}_0\mathfrak{c}_1) = \mu_{\mathbb{K}}(\mathfrak{c}_0)\mu_{\mathbb{K}}(\mathfrak{c}_1)$, since $\mathfrak{c}_0$ and $\mathfrak{c}_1$ are coprime by construction. We have

$$\begin{aligned}
&\sum_{\mathfrak{c}\subseteq\mathcal{O}_{\mathbb{K}}} \mu_{\mathbb{K}}(\mathfrak{c})\mathcal{N}_\phi(\mathfrak{a}\mathfrak{c},\mathfrak{a},\mathfrak{d},TN(\mathfrak{d})N(\mathfrak{a})^e) \\
&\quad\sim \sum_{\substack{\mathfrak{c}_1\subseteq\mathcal{O}_{\mathbb{K}} \\ \text{product} \\ \text{of } \mathfrak{p}\in\mathcal{S}}} \frac{\mu_{\mathbb{K}}(\mathfrak{c}_1)\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{(\mathcal{O}_{\mathbb{K}}^m : M^{\mathfrak{a}}_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1})\mu_{\mathbb{K}}^w}(TN(\mathfrak{d})N(\mathfrak{a})^e)^{\frac{|w|}{e}} \frac{2^{mr_2}}{D_{\mathbb{K}}^{m/2}} \operatorname{Vol}_{Nm}(\mathfrak{B}_\phi(1)) \\
&\qquad\qquad \times \sum_{\substack{\mathfrak{c}_0\subseteq\mathcal{O}_{\mathbb{K}} \\ \text{product} \\ \text{of } \mathfrak{p}\notin\mathcal{S}}} \frac{\mu_{\mathbb{K}}(\mathfrak{c}_0)}{N(\mathfrak{c}_0)^{|w|}}.
\end{aligned}$$

We rewrite the latter sum in the above equation in the following way (recall that $\mathfrak{c}$ is square-free, and therefore so is $\mathfrak{c}_0$).

$$\begin{aligned}
\sum_{\substack{\mathfrak{c}_0\subseteq\mathcal{O}_{\mathbb{K}} \\ \text{product} \\ \text{of } \mathfrak{p}\notin\mathcal{S}}} \frac{\mu_{\mathbb{K}}(\mathfrak{c}_0)}{N(\mathfrak{c}_0)^{|w|}} &= \prod_{\mathfrak{p}\notin\mathcal{S}} \sum_{k=0}^{1} \frac{\mu_{\mathbb{K}}(\mathfrak{p}^k)}{N(\mathfrak{p})^{k|w|}} = \prod_{\mathfrak{p}\notin\mathcal{S}}(1-N(\mathfrak{p})^{-|w|}) \\
&= \frac{\prod_{\mathfrak{p}\in\mathcal{S}}(1-N(\mathfrak{p})^{-|w|})^{-1}}{\prod_{\mathfrak{p}}(1-N(\mathfrak{p})^{-|w|})^{-1}} = \frac{1}{\zeta_{\mathbb{K}}(|w|)} \prod_{\mathfrak{p}\in\mathcal{S}}(1-N(\mathfrak{p})^{-|w|})^{-1}.
\end{aligned}$$

Substituting this into Equation 3.1 and rearranging some of the factors, we get the desired result.

The error term is estimated using [7], where $B = T^{1/e}$, $\mathfrak{a} = \mathfrak{c}_0$ and $t = |w| - 1/N$ (since there are only finitely many contributions of $\mathfrak{a}$ and $\mathfrak{d}$, we may group these constants together). □

Remark 3.20. The expression for $N_\phi(T)$ is indeed well defined, since the constant $C_\phi^{\mathbb{K}}$ is independent of the choice of the representative $\mathfrak{a}$ in its ideal class. To see this, let $\mathfrak{b}\in\mathcal{O}_{\mathbb{K}}$ be a principal ideal and note that

(i) $N(\mathfrak{ab})^{|w|} = N(\mathfrak{a})^{|w|}N(\mathfrak{b})^{|w|} = N(\mathfrak{a})^{|w|}(\mathcal{O}_{\mathbb{K}}^m : (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m}))$;

(ii) $(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{c}_1}^{\mathfrak{ab}}) = (\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})(\mathcal{O}_{\mathbb{K}}^m : \mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m})$, since by definition of $\mathfrak{m}^{\mathfrak{a}}$ we have $\mathfrak{m}^{\mathfrak{ab}} = \mathfrak{m}^{\mathfrak{a}} \times (\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m})$ (recall by Lemma 2.12 that $\mathfrak{m}_{\mathfrak{p}}^{\mathfrak{a}} = \mathfrak{p}^{s_1}\mathfrak{a}_{\mathfrak{p}}^{w_1} \times \cdots \times \mathfrak{p}^{s_m}\mathfrak{a}_{\mathfrak{p}}^{w_m}$) and $(\mathfrak{abc}_1)^{w_1} \times \cdots \times (\mathfrak{abc}_1)^{w_m}$ is $((\mathfrak{ac}_1)^{w_1} \times \cdots \times (\mathfrak{ac}_1)^{w_m})(\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m})$;

(iii) $\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{ab},\mathfrak{d}} = \#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}$, since $V(\mathfrak{ab},\mathfrak{d}) = V^{\mathfrak{a},\mathfrak{d}}(\mathfrak{b}^{w_1} \times \cdots \times \mathfrak{b}^{w_m})$ by definition,

which implies that $N(\mathfrak{ab})^{|w|}\frac{\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{ab},\mathfrak{d}}}{(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{c}_1}^{\mathfrak{ab}})} = N(\mathfrak{a})^{|w|}\frac{\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})}$.

It is to be expected that we get the same asymptotic formula as in [2, Theorem (A)] except for the factor $C_\phi^{\mathbb{K}}$ built up from all these finite contributions from the different $\mathfrak{a}, \mathfrak{d}, \mathfrak{c}_1$, which stem from the choice of $\mathbb{K}$ and ϕ. Note that we would also have a factor h (denoting the class number of $\mathbb{K}$) in our result, if the summand in $C_\phi^{\mathbb{K}}$ were independent of the choice of $\mathfrak{a} \in R_{\mathbb{K}}$.

Definition 3.21. We define the quantity $C_\phi^{\mathbb{K}}$ to be the *compensation factor for* ϕ.

Chapter 4

Example

In this chapter, we give an explicit example of a morphism between weighted projective lines $\phi : \mathbb{P}(w) \to \mathbb{P}(u)$ and we determine the asymptotic behaviour of the corresponding $N_\phi(T) = \#\{x \in \mathbb{P}(w) \,|\, S_u(\phi(x)) \leqslant T\}$ as $T \to \infty$, by following the method studied in the previous chapter. As it will turn out, in Remark 4.2 we show that this example allows us to count the number of generalized elliptic curves over $\mathbb{Q}$ with a point of order 2 such that $S_{4,6}(c_4, c_6) \leqslant T$.

4.1 The morphism ϕ

We define the following map between weighted projective lines

$$\begin{aligned} \phi : \mathbb{P}(2,4) &\to \mathbb{P}(4,6) \\ (b_2, b_4) &\mapsto (b_2^2 - 24b_4,\ b_2(36b_4 - b_2^2)). \end{aligned} \tag{4.1}$$

Remark 4.1. The map ϕ is a morphism (as defined in Lemma 1.3) with $e = 1$. To prove the second criteria for ϕ to be a morphism, note that we actually have $\sqrt{\phi(b_2, b_4)} = (b_2, b_4)$, as

$$\begin{cases} b_2^2 - 24b_4 = 0 \\ b_2(36b_4 - b_2^2) = 0 \end{cases} \iff b_2 = b_4 = 0;$$

implying $V(\phi(b_2, b_4)) = V(b_2, b_4)$ (which is equivalent to $\sqrt{\phi(b_2, b_4)} = (b_2, b_4)$ because, applying Hilbert's Nullstellensatz, $I(V(\phi(b_2, b_4))) = \sqrt{\phi(b_2, b_4)}$ is equal to $I(V(b_2, b_4)) = (b_2, b_4)$).

It follows that we can make use of the techniques learned in Chapter 3 and apply the results to our example, where now ϕ is the map defined above; with $m = 2$, $w = (2, 4)$, $u = (4, 6)$ and $\mathbb{K} = \mathbb{Q}$.

Remark 4.2. There is a bijection between the set of (isomorphism classes of) elliptic curves over the field $\mathbb{Q}$ with a point of order 2 and the space

$\mathbb{P}(2,4) - \{(b_2,0),(b_2,b_2^2/4)\}$ (for the readers that are familiar with stacks, notice that this set is just the set of $\mathbb{Q}$-points of the modular curve $Y_1(2)$). To see this, let $E(\mathbb{Q}) : y^2 = x^3 + a_2x^2 + a_4x$ be an elliptic curve over $\mathbb{Q}$ with $P = (0,0)$ a point of order 2, where we use a change of variables if necessary. We then note that $E(\mathbb{Q})$ is uniquely parametrized by the invariants a_2 and a_4 up to multiplication by λ^2 and λ^4 for $\lambda \in \mathbb{Q}^\times$, respectively (as a consequence of the curve being defined up to the change of variables $\{x = \lambda^2 x'; y = \lambda^3 y'\}$). We exclude the points of the form $(x,0)$ and $(x,x^2/4)$ to avoid singularities in the curve.

Note that the coordinates of the image of ϕ correspond to the so-called c_4 and c_6-invariants of the elliptic curve $E(\mathbb{Q})$, where

$$\begin{aligned} b_2 &= 4a_2; \\ b_4 &= 2a_4; \\ b_6 &= 0. \end{aligned}$$

It follows that the number of (isomorphism classes of) elliptic curves over $\mathbb{Q}$ with a point of order 2 such that $S_\phi(c_4,c_6) \leqslant T$ is $N_\phi(T)$. This quantity turns out to also count the number of points in $\mathbb{P}(2,4)$ (which is isomorphic to the modular curve $X_1(2)$ of generalized elliptic curves over $\mathbb{Q}$ with a point of order 2) such that $S_\phi(c_4,c_6) \leqslant T$. This follows from the fact that $\mathbb{P}(2,4)$ has no sub-variety of accumulation points with respect to the size function (see [2, Proposition 6.3]), implying $\#\{x \in \mathbb{P}(2,4) \mid S_\phi(c_4,c_6) \leqslant T\}$ is asymptotically equivalent to $N_\phi(T) = \#\{x \in \mathbb{P}(2,4) - \{(b_2,0),(b_2,b_2^2/4)\} \mid S_\phi(c_4,c_6) \leqslant T\}$.

4.2 The compensation factor for ϕ

We will ultimately apply Theorem 3.19 to find an asymptotic formula for $N_\phi(T)$, but, to do so, we first compute the value of $C_\phi^{\mathbb{Q}}$; the compensation factor for ϕ. Recall that

$$C_\phi^{\mathbb{K}} := \sum_{\mathfrak{a} \in R_{\mathbb{K}}} \sum_{\mathfrak{d} \in \mathcal{D}_\phi} \sum_{\substack{\mathfrak{c}_1 \subseteq \mathcal{O}_{\mathbb{K}} \\ \text{product} \\ \text{of } \mathfrak{p} \in \mathcal{S}}} (N(\mathfrak{d})^{1/e} N(\mathfrak{a}))^{|w|} \frac{\mu_{\mathbb{K}}(\mathfrak{c}_1) \# \mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}}{(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})} \prod_{\mathfrak{p} \in \mathcal{S}} (1 - N(\mathfrak{p})^{-|w|})^{-1}.$$

Since $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ is a PID, $Cl(\mathbb{Q})$ is trivial, and thus, we may take $\mathfrak{a} = \mathbb{Z}$ as a representative of the trivial element $c \in Cl(\mathbb{Q})$.

Lemma 4.3. *Let $(b_2,b_4) \in \mathbb{P}(2,4)$. We denote by*

(i) $(*) := \{(b_2,b_4) \in \mathbb{P}(2,4) : 2^3 \mid b_2$ *and* $2 \mid b_4$ *or* $2^2 \mid b_2$ *and* $2^2 \mid b_4\}$;

(ii) $(**) := \{(b_2,b_4) \in \mathbb{P}(2,4) : 3^2 \mid b_2$ *and* $3^3 \mid b_4\}$.

Then,

$$\mathfrak{I}_u(\phi(b_2,b_4))\cdot\mathfrak{I}_w(b_2,b_4)^{-1}=\begin{cases}(2),\ \textit{when } (b_2,b_4)\in(*)\cap(**)^c;\\(3),\ \textit{when } (b_2,b_4)\in(*)^c\cap(**);\\(6),\ \textit{when } (b_2,b_4)\in(*)\cap(**);\\(1),\ \textit{else.}\end{cases}$$

Proof. Without loss of generality we can take $\mathfrak{I}_{(2,4)}(b_2,b_4)$ to be (1) by multiplying the point (b_2,b_4) by p^{-1} (for p a prime number) with the weight $(2,4)$ action if necessary (we say that (b_2,b_4) is *primitive*). We now write $\phi(b_2,b_4)=(\phi(b_2,b_4)_1,\phi(b_2,b_4)_2)$. We want to find primes p such that $p^4\mid\phi(b_2,b_4)_1=b_2^2-24b_4$ and $p^6\mid\phi(b_2,b_4)_2=36b_2b_4-b_2^3$. We check prime by prime.

(i) Let $p=2$. Then, $2^4\mid b_2^2-24b_4\iff 2^2\mid b_2$ and $2\mid b_4$. On the other hand, $2^6\mid 36b_2b_4-b_2^3\iff 2^2\mid b_2$ and $2^4\mid b_2b_4$. Thus, for $x=(b_2,b_4)$ such that either $2^3\mid b_2$ and $2\mid b_4$ or $2^2\mid b_2$ and $2^2\mid b_4$, we have that $\mathfrak{I}_{(4,6)}(\phi(x))\subseteq(2)$.

(ii) Let $p=3$. Then, $3^4\mid b_2^2-24b_4\iff 3^2\mid b_2$ and $3^3\mid b_4$. On the other hand, $3^6\mid 36b_2b_4-b_2^3\iff 3^2\mid b_2$ and $3^4\mid b_2b_4$. Thus, for $x=(b_2,b_4)$ such that $3^2\mid b_2$ and $3^3\mid b_4$, we have that $\mathfrak{I}_{(4,6)}(\phi(x))\subseteq(3)$.

(iii) Let $p\geqslant 5$. Then, $p^4\mid b_2^2-24b_4\iff p^2\mid b_2$ and $p^4\mid b_4$.

Recall that for the points $x\in\mathbb{P}(2,4)$ for which there does not exist a prime p satisfying $p^4\mid\phi(x)_1$ and $p^6\mid\phi(x)_2$, we have that $\mathfrak{I}_{(4,6)}(\phi(x))=(1)$. The result follows by primitivity of the point (b_2,b_4). □

Corollary 4.4. $\mathcal{D}_\phi=\{(1),(2),(3),(6)\}$. *Consequently,* $\mathcal{S}=\{(2),(3)\}$.

Proof. Applying Lemma 4.3, the first statement is obvious. For the second claim, use Lemma 2.12 and the definition of the set $\mathcal{S}$, together with the decomposition of each $\mathfrak{d}\in\mathcal{D}_\phi$ into primes. □

Lemma 4.5. *We have* $\mathfrak{m}^{\mathfrak{a}}=72\mathbb{Z}\times 108\mathbb{Z}$.

Proof. By definition, $\mathfrak{m}^{\mathfrak{a}}=\bigcap_{\mathfrak{p}}(\mathfrak{m}^{\mathfrak{a}}_{\mathfrak{p}}\cap\mathbb{Q}^2)$, where the intersection takes place in $\mathbb{Q}^2_{\mathfrak{p}}$. It follows from Lemma 2.12 that $\mathfrak{m}^{\mathfrak{a}}_{\mathfrak{p}}=\mathbb{Z}^2_{\mathfrak{p}}$ for $\mathfrak{p}\neq(2),(3)$. In order to determine $\mathfrak{m}^{\mathfrak{a}}_{\mathfrak{p}}$ for a fixed $\mathfrak{p}\in\mathcal{S}$ and $\mathfrak{d}\in\mathcal{D}_\phi$, we need to look at the local congruence conditions that $\mathfrak{d}$ being a given value is determined by. Lemma 4.3 then gives us that

$$\mathfrak{m}^{\mathfrak{a}}_{\mathfrak{p}}=\begin{cases}2^3\mathbb{Z}_{(2)}\times 2^2\mathbb{Z}_{(2)}=8\mathbb{Z}_{(2)}\times 4\mathbb{Z}_{(2)},\ \text{for } \mathfrak{p}=(2);\\3^2\mathbb{Z}_{(3)}\times 3^3\mathbb{Z}_{(3)}=9\mathbb{Z}_{(3)}\times 27\mathbb{Z}_{(3)},\ \text{for } \mathfrak{p}=(3);\\\mathbb{Z}^2_{\mathfrak{p}},\ \text{for } \mathfrak{p}\neq(2),(3).\end{cases}$$

Then we can apply the method learned in Example 1.11 twice, once for determining the factor $(8\mathbb{Z}_{(2)} \cap \mathbb{Q}) \cap (9\mathbb{Z}_{(3)} \cap \mathbb{Q}) \cap \bigcap_{\mathfrak{p}\neq(2),(3)}(\mathbb{Z}_{\mathfrak{p}} \cap \mathbb{Q})$, and another for the factor $(4\mathbb{Z}_{(2)} \cap \mathbb{Q}) \cap (27\mathbb{Z}_{(3)} \cap \mathbb{Q}) \cap \bigcap_{\mathfrak{p}\neq(2),(3)}(\mathbb{Z}_{\mathfrak{p}} \cap \mathbb{Q})$. The result then follows. □

Corollary 4.6. *As a consequence,*

$$M^{\mathfrak{a}}_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1} = \begin{cases} 72\mathbb{Z} \times 108\mathbb{Z}, \text{ for } \mathfrak{c}_1 = (1); \\ 72\mathbb{Z} \times 432\mathbb{Z}, \text{ for } \mathfrak{c}_1 = (2); \\ 72\mathbb{Z} \times 324\mathbb{Z}, \text{ for } \mathfrak{c}_1 = (3); \\ 72\mathbb{Z} \times 1296\mathbb{Z}, \text{ for } \mathfrak{c}_1 = (6). \end{cases}$$

We now describe a method to compute the value of $\#\mathfrak{S}^{\mathfrak{a},\mathfrak{d}}_{\mathfrak{c}_1}$. We do so by studying the case $\mathfrak{c}_1 = (3)$. The rest of the cases can be determined in a similar way.

First, note that $S^{\mathfrak{a},\mathfrak{d}}_{\mathfrak{p}}$ is trivial when $\mathfrak{p} \neq (2), (3)$ by Remark 2.20. For $\mathfrak{p} \in S$, we have the following

(i) $S^{\mathbb{Z},\mathfrak{d}}_{(2)} \subseteq (\mathbb{Z}_{(2)}/8\mathbb{Z}_{(2)}) \times (\mathbb{Z}_{(2)}/4\mathbb{Z}_{(2)})$, where $|S^{\mathbb{Z},(2)}_{(2)}| = 3$; $|S^{\mathbb{Z},(3)}_{(2)}| = 29$;

(ii) $S^{\mathbb{Z},\mathfrak{d}}_{(3)} \subseteq (\mathbb{Z}_{(3)}/9\mathbb{Z}_{(3)}) \times (\mathbb{Z}_{(3)}/27\mathbb{Z}_{(3)})$, where $|S^{\mathbb{Z},(3)}_{(3)}| = 1$; $|S^{\mathbb{Z},(2)}_{(3)}| = 215$.

This is easy to see by, for instance, writing a matrix with the different values of (b_2, b_4) modulo $\mathfrak{m}^{\mathfrak{a}}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{S}$, where in the position corresponding to (b_2, b_4) we get the corresponding value for the local discrepancy $j_{\mathfrak{p}}$, which can either be 0 or 1. For example, for the case $\mathfrak{p} = (2)$ we get the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We then extend the matrices such that we have different values of (b_2, b_4) modulo $M^{\mathfrak{a}}_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1} = 72\mathbb{Z} \times 324\mathbb{Z}$. We first look for points $(b_2, b_4) \in 3^2\mathbb{Z} \times 3^4\mathbb{Z}$ with non-zero local discrepancy. We see that we only get $j_{(2)} = 1$ for the positions (b_2, b_4) in $\{(0,0), (36,0), (0,162)\}$ and $j_{(3)} = 1$ for the positions (b_2, b_4) in $\{0, 9, 18, 27, 36, 45, 54, 63\} \times \{0, 82, 162, 243\}$ (note that for (b_2, b_4) in $\{(0,0), (36,0), (0,162)\}$ we have $j_{(2)} = j_{(3)} = 1$). On the other hand, we notice that $j_{(3)} = 0 \iff (b_2, b_4) \neq (0,0) \mod (3^2, 3^3)$ (see Lemma 4.3), and so there is no point $(b_2, b_4) \in 3^2\mathbb{Z} \times 3^4\mathbb{Z}$ with $j_{(3)} = 0$.

Hence, we get that

$$\#\mathfrak{S}^{\mathfrak{a},\mathfrak{d}}_{(3)} = \begin{cases} 0, \text{ for } \mathfrak{d} = (1); \\ 0, \text{ for } \mathfrak{d} = (2); \\ 29, \text{ for } \mathfrak{d} = (3); \\ 3, \text{ for } \mathfrak{d} = (6). \end{cases}$$

See Table 4.1 for an overview of the elements appearing in the expression for the compensation factor for ϕ.

Factors in $C_\phi^{\mathbb{K}}$	**Value for ϕ**
$R_{\mathbb{K}}$	$\{\mathbb{Z}\}$
$\mathcal{D}_\phi$	$\{(1),(2),(3),(6)\}$
$\mathcal{S}$	$\{(2),(3)\}$
$(N(\mathfrak{d})^{1/e}N(\mathfrak{a}))^{\lvert w\rvert}$	1, for $\mathfrak{d}=(1)$
	64, for $\mathfrak{d}=(2)$
	729, for $\mathfrak{d}=(3)$
	46656, for $\mathfrak{d}=(6)$
$\mu_{\mathbb{K}}(\mathfrak{c}_1)$	1, for $\mathfrak{c}_1=(1),(6)$
	-1, if $\mathfrak{c}_1=(2),(3)$
$\#\mathfrak{S}_{\mathfrak{c}_1}^{\mathfrak{a},\mathfrak{d}}$	7018, for $\mathfrak{d}=(1),\mathfrak{c}_1=(1)$
	0, for $\mathfrak{d}=(1),\mathfrak{c}_1=(2)$
	0, for $\mathfrak{d}=(1),\mathfrak{c}_1=(3)$
	0, for $\mathfrak{d}=(1),\mathfrak{c}_1=(6)$
	726, for $\mathfrak{d}=(2),\mathfrak{c}_1=(1)$
	484, for $\mathfrak{d}=(2),\mathfrak{c}_1=(2)$
	0, for $\mathfrak{d}=(2),\mathfrak{c}_1=(3)$
	0, for $\mathfrak{d}=(2),\mathfrak{c}_1=(6)$
	29, for $\mathfrak{d}=(3),\mathfrak{c}_1=(1)$
	0, for $\mathfrak{d}=(3),\mathfrak{c}_1=(2)$
	29, for $\mathfrak{d}=(3),\mathfrak{c}_1=(3)$
	0, for $\mathfrak{d}=(3),\mathfrak{c}_1=(6)$
	3, for $\mathfrak{d}=(6),\mathfrak{c}_1=(1)$
	2, for $\mathfrak{d}=(6),\mathfrak{c}_1=(2)$
	3, for $\mathfrak{d}=(6),\mathfrak{c}_1=(3)$
	2, for $\mathfrak{d}=(6),\mathfrak{c}_1=(6)$
$(\mathcal{O}_{\mathbb{K}}^m : M_{\mathfrak{m}^{\mathfrak{a}},\mathfrak{c}_1}^{\mathfrak{a}})$	7776, for $\mathfrak{c}_1=(1)$
	31104, for $\mathfrak{c}_1=(2)$
	23328, for $\mathfrak{c}_1=(3)$
	93312, for $\mathfrak{c}_1=(6)$
$(1-N(\mathfrak{p})^{-\lvert w\rvert})^{-1}$	$\frac{1}{1-2^{-6}}$, for $\mathfrak{p}=(2)$
	$\frac{1}{1-3^{-6}}$, for $\mathfrak{p}=(3)$

Table 4.1: *Values of the different factors appearing in the expression of $C_\phi^{\mathbb{K}}$ applied to ϕ in Equation 4.1.*

Using a computer algebra system, we conclude the following.

Lemma 4.7. *Notations as above. The compensation factor for ϕ is*

$$C_\phi^{\mathbb{Q}} = 18.$$

4.3 The asymptotic formula for $N_\phi(T)$

Lemma 4.8. *The set $\mathfrak{B}_\phi(1)$ has C^1-parametrizable boundary. Moreover, we have that*

$$\mathrm{Vol}_2(\mathfrak{B}_\phi(1)) = \frac{12 + 6\ln 2 + 6\alpha}{108} - \frac{\ln \alpha}{18},$$

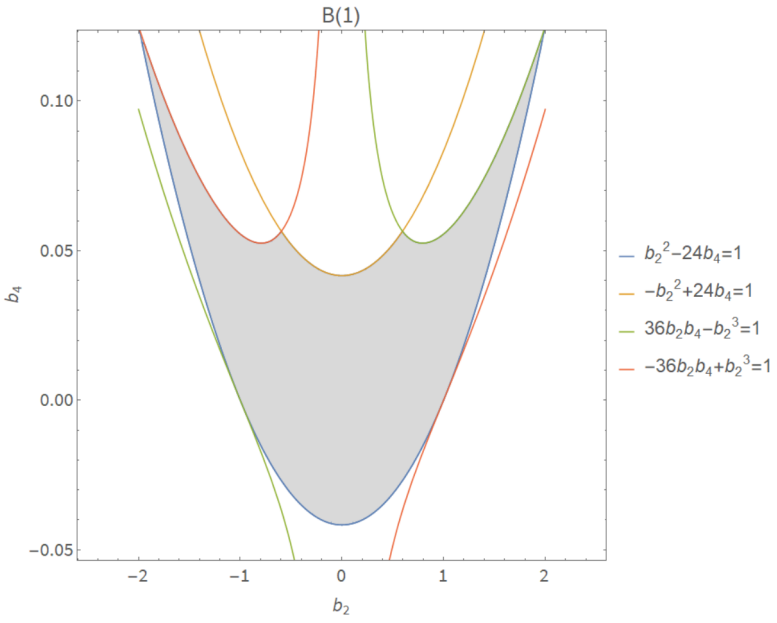
where α is the (unique) real root of $x^3 + 3x - 2 = 0$.

Proof. Recall that $\mathfrak{B}_\phi(1) = \mathfrak{D}_\phi(1) \cap \Delta$, where

$$\mathfrak{D}_\phi(1) = \{(b_2, b_4) \in \mathbb{R}^2 - \{0\} \mid \max\{|b_2^2 - 24b_4|^{1/4}, |b_2(36b_4 - b_2^2)|^{1/6}\} \leqslant 1\}$$

and $\Delta = (pr \circ \eta)^{-1}F$ (see Section 3.2). Note that since, in this setting, $\mathcal{H}$ is the hyperplane defined by a single point $y = 0$ and Γ has rank 0, $F = \mathcal{H}$ is a fundamental domain for $\mathcal{H}/\Gamma$ and Δ is the entirety of $\mathbb{R}^2 - \{0\}$. We then conclude that $\mathfrak{B}_\phi(1) = \mathfrak{D}_\phi(1)$.

The region $\mathfrak{D}_\phi(1)$ has the following shape drawn in grey.



The first claim is then evident. For the second claim, by symmetry we have that

$$\mathrm{Vol}_2(\mathfrak{B}_\phi(1)) = 2\left(\int_0^\alpha \left(\frac{1+x^2}{24} - \frac{x^2-1}{24}\right) dx + \int_\alpha^2 \left(\frac{x^3+1}{36x} - \frac{x^2-1}{24}\right) dx\right)$$

and, integrating the above expression and using $\alpha^3 = -3\alpha + 2$, we get the desired result. □

See Table 4.2 for an overview of the elements appearing in the expression for $N_\phi(T)$.

Factors in $N_\phi(T)$	**Value for ϕ**
$C_\phi^{\mathbb{K}}$	18
$\zeta_{\mathbb{K}}(\lvert w\rvert)$	$\frac{\pi^6}{945}$
r_2	0
$D_{\mathbb{K}}$	1
$\mathrm{Vol}_{Nm}(\mathfrak{B}_\phi(1))$	$\frac{12+6\ln 2+6\alpha}{108} - \frac{\ln\alpha}{18}$
$\mu_{\mathbb{K}}^w$	1

Table 4.2: *Values of the different factors appearing in the expression of $N_\phi(T)$ applied to ϕ in Equation 4.1.*

Applying Theorem 3.19 and using a computer algebra system, we conclude the following.

Theorem 4.9. *Notations as above. We have*

$$N_\phi(T) = C\,T^6 + O(T^5),$$

with

$$C = \frac{945}{\pi^6}(2 + \ln 2 + \alpha - \ln\alpha) \approx 3.74172.$$

Future work

There are still a few directions in which our result could further be generalized. Firstly, it would be interesting to obtain an asymptotic formula for $N_\phi(T)$ for ϕ a morphism between not necessarily weighted projective spaces, but between spaces that are 'close' to being weighted projective spaces. An example for this would be $\mathbb{P}(2) \times \mathbb{P}(1,1)$ (which is isomorphic to the modular curve $X_0(6)$; see [1, Section 9]). However, it is important to note that finding similar results for this kind of spaces quickly becomes complicated. [6] already gives an asymptotic formula for counting elliptic curves over $\mathbb{Q}$ in $X_0(3)$ (which has the form $\mathbb{P}(2) \times \mathbb{P}(1,3)$; see [1, Remark 7.4]) with bounded height by following an entirely different method to ours. It would be of interest to find out if our approach could be extended to their setting, in an attempt to ease the generalization of their result to any number field and to potentially generalize our result to different spaces.

On top of that, a (simpler) generalization would be to find a similar result to Theorem 3.19 for morphisms between products of weighted projective spaces.

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