Master Thesis

# The Hardy-Littlewood Circle Method and Applications

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Goldbach's Conjectures from 1742 are among the most important questions in additive number theory. The binary Goldbach Conjecture states that "Every even integer greater than 2 can be represented as the sum of two prime numbers". The ternary Goldbach Conjecture states that: "Every odd integer greater than 5 can be represented as the sum of three prime numbers". To be able to prove Goldbach's ternary Conjecture we will need the circle method developed by Ramanujan, Hardy, Littlewood and Vinogradov. This paper aims to show how to apply the Hardy-Littlewood circle method to Goldbach's ternary Conjecture and why it is not strong enough to prove the binary Goldbach Conjecture. Using the circle method will enable us to convert the ternary Goldbach Conjecture into a problem in complex analysis. This will require the construction of Major and Minor arcs to partition $[0, 1]$. Finally, we will investigate the use of the circle method when applied to counting rational points on quartic hypersurfaces.

# 1 Introduction

In this thesis we will take a closer look at the Hardy-Littlewood circle method. It is a number theoretic technique that goes back to the work of Ramanujan and Hardy on the partition function $P(n)$ for $n \in \mathbb{N}$. The circle method was later on developed by Hardy and Littlewood and is applicable to a variety of additive problems.

In the main part of this thesis Chapter 4 will first look at the application of the circle method within the proof of Vinogradov's Theorem, which is a weaker version of Goldbach's ternary Conjecture.

In 1742 Goldbach was writing letters to Euler. His statements made in these letters have since been rephrased, due to the fact that 1 is not considered a prime number in modern mathematics. The statements known today as Goldbach's binary (strong) and ternary (weak) Conjecture can be stated as follows:

**Conjecture 1.1.** *(Ternary Goldbach Conjecture - TGC)*

*Every odd integer greater than 5 can be represented as the sum of three prime numbers. [10, p. 1]*

**Conjecture 1.2.** *(Binary Goldbach Conjecture - BGC)*

*Every even integer greater than 2 can be represented as the sum of two prime numbers. [10, p. 1]*

**Theorem 1.3.** *(Vinogradov's Theorem)*

*There exists a natural number $N$, such that every odd positive integer $n$, with $n \geq N$, can be represented as the sum of three prime numbers. [10, p. 5]*

In 1923 Hardy and Littlewood proved that there exists a constant $C$ such that every odd integer $n \geq C$ is the sum of three primes. Their proof was based on the assumption that the General Riemann Hypothesis (GRH) holds true. Vinogradov proved his Theorem in 1937; unlike Hardy and Littlewood he managed to prove it without assuming that the GRH holds true. In 1939 Borodzkin, who was Vinogradov's student, proved that we can set $N = 3^{3^{15}}$. In 1989 Wang and Chen showed that $N$ can be taken to be approximately $N = 10^{43000}$. In 2002 Liu and Wang proved that one can set $N = 2 \cdot 10^{1346}$. Even though the constant $N$ was refined several times over the years it always remained much too high for the problem to be manageable by existing computation power. In 2012 Oliveira e Silva announced that he was able to verify the BGC for every $n \leq 4 \cdot 10^{18}$. In 2013 Helfgott and Platt combined this fact with a prime staircase and managed to show that the TGC holds true for $5 < n \leq 8.875 \cdot 10^{30}$. But the gap was still much too big, as $8.875 \cdot 10^{30}$ is much smaller than $2 \cdot 10^{1346}$. That is why the math had to improve where the computability failed to apply. The goal was to reduce $N$ from $2 \cdot 10^{1346}$ to $\sim 10^{30}$. In 2013 Helfgott finally managed to bring the constant down to $10^{27}$, which proved the TGC unconditionally. The information in this paragraph is taken from [6, pp. 3-6].

Secondly Chapter 5 of this thesis will show why the circle method is not strong enough to prove Goldbach's binary Conjecture. The BGC has so far remained unproven. Lastly Chapter 6 concludes the main part dealing with a treatment of the interface of algebraic geometry and analytic number theory. For thousands of years, mathematicians have been studying Diophantine equations. The investigations of the Diophantine equations led to many points of contact between different mathematical fields, algebraic geometry and analytic number theory being one of them. The goal of Chapter 6 is to show the role of analytic number theory in the study of Diophantine equations. Analytic number theory is concerned with counting objects, which is why it is suitable for counting rational points on algebraic varieties. We specifically will look at the application of the Hardy-Littlewood circle method to counting rational points on quartic hypersurfaces.

Preliminaries to this main part are covered in chapter 2 where the reader can find Notations, Definitions, Lemmas and Theorems that will be used within this paper. Chapter 2 is meant for reference and can be omitted by the experienced reader. Also as a preliminary Chapter 3 explains the concept of the Hardy-Littlewood circle method.

# 2 Preliminaries

## 2.1 Notations

We shall use the Landau notation $f(x) = O(g(x))$ equivalently with $f \ll g$ and $g \gg f$, which means that there exists a positive constant $C$ such that $|f(x)| \leq Cg(x)$ for sufficiently large $x$.

Furthermore we will take $f(x) = o(g(x))$ to mean

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

And finally the relation $f(x) \sim g(x)$ will mean

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

and is used interchangeably with $f(x) = g(x) + o(g(x))$.

The little-O estimate $o(g(x))$ is strictly stronger than the big-O-estimate $O(g(x))$.

We will write $(a, b)$ for the greatest common divisor (gcd) of a and b.

For a real number $x$, let $e(x) := e^{2\pi\imath x} = \cos(2\pi x) + \imath \sin(2\pi x)$. Therefore $e(x)$ is a periodic function of period 1.

## 2.2 Definitions and Theorems

**Definition 2.1.** Arithmetic Function

An arithmetic function is a function $f : \mathbb{N} \to \mathbb{C}$ with domain of definition the set of natural numbers $\mathbb{N}$ and range a subset of the set of complex numbers $\mathbb{C}$. [9, p. 15]

**Definition 2.2.** Euler Totient Function

The Euler totient function $\varphi(n)$ is defined as the number of positive integers which are less than or equal to n and at the same time relatively prime to n. Equivalently, the Euler function $\varphi(n)$ can be defined by the formula $\varphi(n) = \sum_{m=1}^{n} \left\lfloor \frac{1}{(n,m)} \right\rfloor$. [9, p. 20]

**Theorem 2.3.** *Let $\varepsilon > 0$. Then*

$$n^{1-\varepsilon} < \varphi(n) < n$$

*for all sufficiently large $n$.* [8, p. 315]

**Definition 2.4.** Möbius Function

Let $n \in \mathbb{N}$. The Möbius function $\mu(n)$ is defined as follows

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 \cdots p_k \text{ where } p_1, p_2, \ldots, p_k \text{ are } k \text{ distinct primes} \\ 0, & \text{in every other case} \end{cases}$$

[10, p. 10]

**Theorem 2.5.**

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \textit{if } n = 1 \\ 0, & \textit{if } n \in \mathbb{Z}_{>1} \end{cases}$$

*where the sum extends over all positive divisors of the positive integer $n$.*

[10, p. 10]

**Theorem 2.6.** *(The Möbius Inversion Formula)*

*Let $n \in \mathbb{N}$. If*

$$g(n) = \sum_{d|n} f(d)$$

*then*

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d)$$

*The converse also holds.* [10, p. 11]

**Definition 2.7.**
The prime counting function $\pi(x)$ is defined to be the number of primes which do not exceed a given real number $x$. [9, p. 79]

**Theorem 2.8.** *(Prime Number Theorem)*

$$\pi(x) \approx \frac{x}{\log x}, \textit{as} \quad x \longrightarrow \infty$$

[9, p. 291]

**Definition 2.9.** Chebychev's Theta Function

By the definition of Chebyshev's function $\vartheta(x)$, one has

$$\vartheta(x) = \sum_{p \leq x} \log p.$$

[8, p. 153]

**Definition 2.10.** Chebychev's Psi Function

$$\psi(x) = \sum_{p^k \leq x} \log p$$

[8, p. 153]

**Theorem 2.11.**
*There exist positive constants $c_1$ and $c_2$ such that*

$$c_1 x \leq \vartheta(x) \leq \psi(x) \leq \pi(x) \log x \leq c_2 x$$

*for all $x \geq 2$. Moreover,*

$$\liminf_{x\to\infty} \frac{\vartheta(x)}{x} = \liminf_{x\to\infty} \frac{\psi(x)}{x} = \liminf_{x\to\infty} \frac{\pi(x)\log x}{x} \geq \log 2$$

*and* $\limsup_{x\to\infty} \frac{\vartheta(x)}{x} = \limsup_{x\to\infty} \frac{\psi(x)}{x} = \limsup_{x\to\infty} \frac{\pi(x)\log x}{x} \leq 4 \log 2.$
[8, p. 155]

**Definition 2.12.** Riemann Zeta Function

The zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

for all real values of s with $s > 1$. [9, p. 83]

**Definition 2.13.** von Mangoldt Function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime number } p \text{ and some } k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

[10, p. 14]

**Theorem 2.14.**
*Let $n \in \mathbb{N}$. Then, it holds*

$$\sum_{d|n} \Lambda(d) = \log n$$

*and*

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d$$

[10, p. 14]

**Definition 2.15.**
The exponential sum

$$c_n(m) = \sum_{\substack{1 \le q \le n \\ (q,n)=1}} e\left(\frac{qm}{n}\right)$$

is called the Ramanujan sum $c_n(m)$. [10, p. 15]

**Lemma 2.16.** *For the Ramanujan sum $c_n(m)$, we have*

$$c_n(m) = \sum_{d|(n,m)} \mu\left(\frac{n}{d}\right) d$$

*In particular, if $(n, m) = 1$, then*

$$c_n(m) = \mu(n).$$

[10, p. 15]

**Theorem 2.17.**
*The Ramanujan sum can be expressed in the form*

$$c_n(m) = \frac{\mu(\frac{n}{(n,m)})\varphi(n)}{\varphi(\frac{n}{(n,m)})}$$

[8, p. 322]

**Proposition 2.18.** *Let $x \ge 1$ and $k > 1$. The following estimation holds:*

$$\sum_{n>x} \frac{1}{n^k} = O\left(x^{1-k}\right).$$

[7, p.35]

**Theorem 2.19.** *(Siegel-Walfisz Theorem)*

*Let $B$ be a positive constant. Then there exists a positive constant $C(B)$ such that the following holds: Assume that $x$ is a real number and $a, q$ are integers such that $(a, q) = 1$ with $q \le \log^B x$. Then*

$$\sum_{\substack{p \le x \\ p \equiv a (\bmod q)}} \log(p) = \frac{x}{\varphi(q)} + O\left(\frac{x}{\log^C x}\right)$$

*where $\varphi(n)$ denotes the Euler totient function. The constant above only depends on $C$ and $B$.* [7, p. 312]

**Theorem 2.20.**
*Let $u(n)$ and $f(n)$ be arithmetic functions. Define the sum function*

$$U(t) := \sum_{n \leq t} u(n)$$

*Let $a$ and $b$ be nonnegative integers with $a < b$. Then*

$$\sum_{n=a+1}^{b} u(n)f(n) = U(b)f(b) - U(a)f(a+1) - \sum_{n=a+1}^{b-1} U(n)(f(n+1) - f(n))$$

*Let $x$ and $y$ be real numbers such that $0 \leq y < x$. If $f(t)$ is a function with a continuous derivative on the interval $[y, x]$, then*

$$\sum_{y < n \leq x} u(n)f(n) = U(x)f(x) - U(y)f(y) - \int_y^x U(t)f'(t)dt$$

*In particular, if $f(t)$ has a continuous derivative on $[1, x]$, then*

$$\sum_{n \leq x} u(n)f(n) = U(x)f(x) - \int_1^x U(t)f'(t)dt$$

[8, p. 304]

**Definition 2.21.**
The integer part of a real number $x$ shall be denoted by $[x]$ and the fractional part by $\{x\}$. Then obviously $[x] \in \mathbb{Z}$ and $\{x\} \in [0, 1)$ and $x = [x] + x$.

Let $\|x\|$ denote the distance from the real number $x$ to the nearest integer: $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\} = \min(\{x\}, 1 - \{x\}) \in \left[0, \frac{1}{2}\right]$. [8, p. 103]

Using this definition we get the next Lemma.

**Lemma 2.22.**
*For every $x \in \mathbb{R}$ und $N_1, N_2 \in \mathbb{Z}$ with $N_1 < N_2$*

$$\sum_{n=N_1+1}^{N_2} e(xn) \ll \min\left(N_2 - N_1, \|x\|^{-1}\right)$$

[8, p. 104]

**Lemma 2.23.**
*Let $x$ be a real number, and let $q$ and $a$ be integers such that $q \geq 1$ and $(a, q) = 1$. If*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

*then*

$$\sum_{1 \leq r \leq q/2} \frac{1}{\|xr\|} \ll q \log q$$

[8, p. 104]

**Theorem 2.24.** *(Dirichlet's Approximation)*

*Let $x$ and $Q$ be real numbers and $Q \geq 1$. Then there exist integers $a$ and $q$ such that $1 \leq q \leq Q$,$(a, q) = 1$ and $\left| x - \frac{a}{q} \right| < \frac{1}{qQ}$* [8, p. 98]

**Lemma 2.25.**
*Let $x$ be a real number, if*

$$\left| x - \frac{a}{q} \right| < \frac{1}{q^2}$$

*with $q \geq 1$ and $(a, q) = 1$, then for any real number $U \geq 1$ and positive integer $n$*

$$\sum_{1 \leq k \leq U} \min\left( \frac{n}{k}, \frac{1}{\|xk\|} \right) \ll \left( \frac{n}{q} + U + q \right) \log 2qU$$

[8, p. 108]

**Theorem 2.26.**
*An Euler product is an infinite product over the prime numbers.*

*Let $f(n)$ be a multiplicative function that is not identically zero. If the series*

$$\sum_{n=1}^{\infty} f(n)$$

*converges absolutely, then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left( 1 + f(p) + f\left(p^2\right) + \cdots \right) = \prod_{p} \left( 1 + \sum_{k=1}^{\infty} f\left(p^k\right) \right)$$

*If $f(n)$ is completely multiplicative, then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} (1 - f(p))^{-1}$$

[8, p. 325]

# 3 The Hardy-Littlewood Circle Method

The circle method originated in investigations by Hardy and Ramanujan on the partition function $P(n)$. For $n \in \mathbb{N}, P(n)$ is the partition function, the number of ways of writing $n$ as a sum of positive integers where we do not distinguish re-orderings. For example, if $n = 4$ then

$$\begin{aligned} 4 &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

and $P(4) = 5$. [7, p. 301]

Hardy and Littlewood developed the method so that it could be used to investigate problems in additive number theory. It connects exponential sums with general problems of additive number theory.

A problem that motivated the development of the circle method is: "writing $n$ as the sum of $s$ perfect $k$-powers". For $k = 1$ the problem translates to "writing $n$ as the sum of $s$ non-negative integers" and we have the combinatorial solution $\binom{n+s-1}{s-1}$. Due to the fact, that this argument doesn't work for higher $k$ we need another approach. The analytic approach works for $k = 1$ and can be generalized.

For $|x| < 1$, we define the generating function

$$f(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}. \tag{3.1}$$

We claim

$$f(x)^s = \left( \sum_{m_1=0}^{\infty} x^{m_1} \right) \cdots \left( \sum_{m_s=0}^{\infty} x^{m_s} \right) = \sum_{n=0}^{\infty} r_{1,s}(n) x^n. \tag{3.2}$$

With $r_{1,s}(n)$ being defined as the number of solutions to $m_1 + ... + m_s = n$, where each $m_i$ denotes a non-negative integer. [7, p. 301]

*Proof.* We have terms of the form $x^{m_1} \cdots x^{m_s} = x^{m_1 + ... + m_s} = x^n$, for some n. Expanding the product we obtain $x^n$ many times; once for each choice of $m_1, \cdots, m_s$ that adds up to $n$. This means, that the coefficient of $x^n$ in the expansion is $r_{1,s}(n)$.

On the other hand for $s \geq 1$ and $|x| \neq 1$ we have

$$f(x)^s = \left(\frac{1}{1-x}\right)^s = \frac{1}{(s-1)!}\frac{d^{s-1}}{dx^{s-1}}\frac{1}{1-x} \tag{3.3}$$

We can substitute $\frac{1}{1-x}$ by its geometric series expansion and for $s \geq 1$ and $|x| < 1$ we get:

$$f(x)^s = \frac{1}{(s-1)!}\frac{d^{s-1}}{dx^{s-1}}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\binom{n+s-1}{s-1}x^n. \tag{3.4}$$

Therefore we get $r_{1,s}(n) = \binom{n+s-1}{s-1}$.

□

That's the proof for the case $k = 1$. This method of proof will be generalized. Exact formulas, like the one in (3.3), barely ever exist. That is why we will have to analyze the generating functions in a more general setting.

**Remark 3.1.** $r_{1,s}(n)$ denotes the number of representations of $n$ as the sum of $s$ non-negative integers. Let $R_{1,s}(n)$ denote the number of representations of $n$ as the sum of $s$ positive integers. If $n = a_1+\cdots+a_s$ is a decomposition of $n$ into $s$ non-negative integers, then $n+s = (a_1+1)+\cdots+(a_s+1)$ is a decomposition of $n$ into s positive integers. Therefore we get that $r_{1,s}(n) = R_{1,s}(n+s) = \binom{n+s-1}{s-1}$ and we conclude that $R_{1,s}(n) = \binom{n-1}{s-1}$.

For $k \geq 2$ we are talking about **Waring's Problem**. He stated that every natural number $n$ can be represented as the sum of at most $s$ perfect $k^{th}$-powers, where $s$ depends on $k$. He stated that every natural number $n \geq 1$ can be written as a sum of at most four squares ($s(2) = 4$), nine cubes, 19 fourth powers and so on. Waring's problem was first solved by Hilbert. From 1920 till 1928 Hardy and Littlewood wrote several papers on Waring's problem and they managed to prove that for any $k$ there is an $s$ such that every sufficiently large natural number is a sum of $s$ $k^{th}$-powers. Additionally they gave an approximation for the number of ways in which every natural number is a sum of $s$ $k^{th}$-powers. The circle-method is not only very useful to Waring's problem but also to many other additive number theoretic problems. [7, p. 303]

The circle method is often applied to problems of the following form:

**Problem 3.2.** *Let $A$ be a subset of $\mathbb{N}$ and $s \in \mathbb{N}$. Determine*

$$\{a_1 + a_2 + \cdots + a_s \mid a_1, a_2, \ldots, a_s \in A\} \cap \mathbb{N}$$

*In other words, determine which natural numbers can be written as the sum of $s$ elements of the set $A$ and in how many ways.* [7, p. 307]

We will consider the corresponding generating function:

$$F_A(x) = \sum_{a \in A} x^a \tag{3.5}$$

To avoid questions of convergence in the following, we shall assume that $A$ is a finite set. Then we write:

$$F_A(x)^s = \sum_{n=1}^{\infty} r(n; s, A) x^n \tag{3.6}$$

$r(n; s, A)$ is the number of ways of writing $n$ as a sum of $s$ elements of $A$.

Problem 3.2 could also be formulated as follows:

**Problem 3.3.** *Determine* $r(n; s, A)$. [7, p. 308]

In order to extract individual coefficients from power series and thereby solve problem 3.3 we need some complex analysis:

**Proposition 3.4.** *1.Let* $\gamma$ *be the unit circle oriented counter-clockwise. Then*

$$\frac{1}{2\pi i} \int_\gamma z^n dz = \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

*2. Let* $P(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a power series with radius of convergence larger than one. Then*

$$\frac{1}{2\pi i} \int_\gamma P(z) z^{-n-1} dz = a_n$$

[7, p. 308]

Applying this Proposition and ignoring convergence problems we now get:

$$r(n; s, A) = \frac{1}{2\pi i} \int_\gamma F_A(z)^s z^{-n-1} dz \tag{3.7}$$

Where $\gamma$ is the unit circle oriented counter-clockwise. [7, p. 308]

If we substitute $z = e^{2\pi i u}$ we get an equivalent but different generating function. Our new generating function $f_A(u)$ is defined as:

$$F_A(z) = \sum_{a \in A} z^a = \sum_{a \in A} e^{2\pi i u a} = \sum_{a \in A} e(au) = f_A(u) \tag{3.8}$$

Therefore, if we ignore convergence problems again, we get that:

$$r(n;s,A) = \int_0^1 f_A(u)^s e(-nu)du \tag{3.9}$$

If we now evaluate this integral we will know which $n$ can be written as the sum of $s$ elements of $A$, and we will also know in how many ways. [7, p. 309]

The problem 3.2 is only interesting if $A$ is infinite. If $A$ were finite we could just enumerate $a_1 + \cdots + a_s$ in a finite number of steps. If $A$ is infinite $f_A(u) = \sum_{a\in A} e(au)$ does not need to converge and could have a too small radius of convergence. To discuss the convergence issues we define:

$$A_N = \{a \in A : a \leq N\} = A \cap \{0, 1, \ldots, N\} \tag{3.10}$$

Note that this is an increasing sequence of subset as: $A_N \subset A_{N+1}$ and $\lim_{N\to\infty} A_N = A$.

For each $N$ we consider the truncated generating function attached to $A_N$:

$$f_N(u) = \sum_{a\in A_N} e(au) \tag{3.11}$$

[7, p. 310]

Due to the fact that $f_N(u)$ is a finite sum, we do not have convergence issues anymore and similarly to (3.6) we get:

$$f_N(u)^s = \sum_{n\leq sN} r_N(n;s,A)e(nu) \tag{3.12}$$

Now we have $r_N(n;s,A)$ instead of $r(n;s,A)$. But for $n \leq N$ those expressions are the same, because $r_N(n;s,A)$ gives us the number of ways in which we can write $n$ as the sum of $s$ elements at most $N$. The "at most $N$" part only matters, if $n$ is bigger than $N$. The generating function for the sum of $s$ elements (at most $N$) of A is $f_N(u)^s$. This proves the following:

**Lemma 3.5.** *If $n \leq N$ then*

$$r(n;s,A) = r_N(n;s,A) = \int_0^1 f_N(u)^s e(-nu)du \tag{3.13}$$

[7, p. 310]

The difficulty now is to evaluate this integral. The key idea of the circle method is to split $\gamma$ into two disjoint pieces, the Major arcs $\mathfrak{M}$ and the Minor arcs $\mathfrak{m}$.

Therefore we get:

$$r(n;s,A) = r_N(n;s,A) = \int_{\mathfrak{M}} f_N^s(u)e(-nu)du + \int_{\mathfrak{m}} f_N^s(u)e(-nu)du \quad (3.14)$$

We want to construct the Major arcs such that the function in the integral over the Major arcs can be evaluated asymptotically. We want the contribution of the Major arcs to be large and bounded away from zero. For the Minor arcs we want to show that if $N \to \infty$ the Minor arcs' contribution is of lower order than the Major arcs'. This implies that for $n$ large $r_N(n;s,A) > 0$, which proves that large $n$ can be written as the sum of $s$ elements of $A$. [7, pp. 310-311] [10, p. 32]

Summarizing we can say that we split up the circle method into 4 steps:

1. Given a set $A$ we construct $f_N(x)$ for $A_N$, and since $f_N(x)$ is a sum of complex exponentials of size 1, we expect significant cancellation.

2. We split $[0,1]$ into 2 disjoint pieces, the Major arcs $\mathfrak{M}$ and the Minor arcs $\mathfrak{m}$. The construction of the Major and the Minor arcs depends on $N$ and on the problem at hand.

3. We want to find a function that agrees with $f_N^s(u)$ up to lower-order terms on the Major arcs and is easy to integrate. Our goal is to get a large contribution from the Major arcs, which is bounded away from 0.

4. As $N \to \infty$ we want the Minor arcs contribution to be of lower-order than the Major arcs contribution. The Minor arcs cancellation is usually considered the most difficult step.

# 4 The Circle Method Applied to Goldbach's Ternary Conjecture

We will be taking a closer look at the application of the circle method on the following version of the previous problem 3.2:

**Problem 4.1.** *Let $A = \mathbb{P}$, where $\mathbb{P}$ denotes the set of all prime numbers, then for $s = 3$ the statement of problem 3.2 becomes: Determine the set*

$$O = \{p_1 + p_2 + p_3 \mid p_1, p_2, p_3 \in \mathbb{P}\} \cap \mathbb{N}$$

.

This problem is a reformulation of Vinogradov's Theorem (1.3). [10, p. 31] In that case $r(n; 3, \mathbb{P})$ is the number of ways of writing n as a sum of 3 prime numbers. Therefore we get:

$$r(n; 3, \mathbb{P}) = \sum_{n=p_1+p_2+p_3} 1 \tag{4.1}$$

The sum can be bounded by the following:

$$\begin{aligned} r(n, 3, \mathbb{P}) &= \sum_{n=p_1+p_2+p_3} 1 \\ &> \sum_{n=p_1+p_2+p_3} \frac{\log p_1 \cdot \log p_2 \cdot \log p_3}{\log^3 (p_1 + p_2 + p_3)} \\ &= \sum_{n=p_1+p_2+p_3} \frac{\log p_1 \cdot \log p_2 \cdot \log p_3}{\log^3 n} \end{aligned} \tag{4.2}$$

[10, p. 33]

so we have:

$$r(n, 3, \mathbb{P}) > \frac{1}{\log^3 n} \sum_{n=p_1+p_2+p_3} \log p_1 \cdot \log p_2 \cdot \log p_3 \tag{4.3}$$

Therefore we shall be working with $\sum_{n=p_1+p_2+p_3} \log p_1 \cdot \log p_2 \cdot \log p_3$ instead of the original sum $\sum_{n=p_1+p_2+p_3} 1$.

Vinogradov proved that

$$\sum_{n=p_1+p_2+p_3} \log p_1 \cdot \log p_2 \cdot \log p_3 \gg n^2,$$

which implies that $r(n,3,\mathbb{P}) \gg \frac{n^2}{\log^3 n}$. This means that there is a natural number $N$ such that for all $n \geq N$ $n$ can be written as the sum of three primes. [10, p. 34]

We now want to look at the use of the circle method within the proof of Vinogradov's Theorem (1.3). Due to the fact that we are working with a sum over logarithms, and because it simplifies several formulas, we will choose a weighted generating function. The weights are easily removed and we can pass to the unweighted function by partial summation.

If we want to write $m$ as the sum of $s$ primes at most $N$ we get:

$$R_N(m,s,\mathbb{P}) = \sum_{\substack{p_1+\cdots+p_s=m \\ p_i \leq N}} \log p_1 \cdots \log p_s \tag{4.4}$$

With $x \in \mathbb{R}$ the generating function of $R_N(m,s,\mathbb{P})$ is the exponential sum:

$$f_N(x) = \sum_{p \leq N} \log p \cdot e(px) \tag{4.5}$$

And working as before, using the orthogonality relation

$$\int_0^1 \exp(kx)dx = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

we can write $R_N(m,s,\mathbb{P})$ as an integral over $f_N(x)$ as follows:

$$R_N(m,s,\mathbb{P}) = \int_0^1 f_N^s(x)e(-mx)dx \tag{4.6}$$

[7, p. 314]

We are trying to write $m$ as the sum of three primes. So we will consider the case where $m = N$ and $s = 3$. As $m = N$ now, we can write $R(N,3,\mathbb{P})$ instead of $R_N(N,3,\mathbb{P})$. Then the weighted answer will be given by:

$$R(N,3,\mathbb{P}) = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \tag{4.7}$$

And in that case we get

$$R(N,3,\mathbb{P}) = \int_0^1 f_N^3(x)e(-Nx)dx \tag{4.8}$$

To be able to evaluate this integral we shall now construct the Major arcs and the Minor arcs. Our goal is to split the unit circle, or equivalently the interval

$[0,1]$, into two disjoint pieces. As mentioned before, we want the contribution of the Major arcs to be large. That is why we first have to look at the average size of $f_N(x)$ and then we will see where we have large values. We will choose our Major arcs where the value of $f_N(x)$ is large.

## 4.1 The Size of $f_N(x)$

**Lemma 4.2.** $|f_N(x)| \leq N + o(N)$

[7, p. 314]

*Proof.* Based on the prime number Theorem we get:

$$|f_N(x)| = \left| \sum_{p \leq N} \log p \cdot e(px) \right| \leq \sum_{p \leq N} \log p = N + o(N) \tag{4.9}$$

[7, p. 314]

□

**Lemma 4.3.** *The average value of* $|f_N(x)|^2$ *is* $N \log N + o(N \log N)$.

[7, p. 315]

*Proof.* Let $g(x)$ be a complex-valued function and let $\bar{g}(x)$ be its complex conjugate. Then $|g(x)|^2 = g(x)\bar{g}(x)$. In our case, as $\overline{f_N(x)} = f_N(-x)$ we have

$$\begin{aligned}
\int_0^1 |f_N(x)|^2 \, dx &= \int_0^1 f_N(x) f_N(-x) dx \\
&= \int_0^1 \sum_{p \leq N} \log p \cdot e(px) \sum_{q \leq N} \log q \cdot e(-qx) dx \\
&= \sum_{p \leq N} \sum_{q \leq N} \log p \log q \int_0^1 e((p-q)x) dx
\end{aligned}$$

Using the fact that for $m, n \in \mathbb{Z}$

$$\int_0^1 e(nx)e(-mx)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

the integral is 1 if $p = q$ and 0 otherwise.

Therefore the only pairs $(p, q)$ that contribute are when $p = q$, and we have

$$\int_0^1 |f_N(x)|^2 \, dx = \sum_{p \leq N} \log^2 p$$

Using partial summation, we can show

$$\sum_{p \leq N} \log^2 p = N \log N + o(N \log N).$$

Thus

$$\int_0^1 |f_N(x)|^2 \, dx = N \log N + o(N \log N).$$

[7, p. 315] □

The average value of $|f_N(x)|^2$ is only $N \log N + o\left(N \log N\right)$. It is a lot smaller than the maximum possible value of $N^2$. Thus, we have almost square root cancellation on average. [7, p. 315]

To define the Major arcs we want to look at the large values of $f_N(x)$. On average $f_N(x)$ is of size $\sqrt{N \log N}$.

$$f_N(0) = \sum_{p \leq N} \log p \cdot e(0) = \sum_{p \leq N} \log p = N + o(N)$$

by the prime number Theorem and

$$f_N(1) = f_N(0)$$

by periodicity.

$$f_N\left(\frac{1}{2}\right) = \sum_{p \leq N} \log p \cdot e\left(p \cdot \frac{1}{2}\right)$$

and

$$e\left(p \cdot \frac{1}{2}\right) = e^{\pi i p} = \begin{cases} -1 & \text{if } p \text{ is odd} \\ +1 & \text{if } p \text{ is even} \end{cases}$$

Thus,

$$f_N\left(\frac{1}{2}\right) = \log 2 - \sum_{3 \leq p \leq N} \log p \approx \log 2 - (N - \log 2) \approx 2 \log 2 - N = -N + o(N)$$

We see that $f_N(x)$ is occasionally as large as $N$. For a fixed $B$, let $Q = \log^B(N)$. Fix a $q \leq Q$ and an $a \leq q$ with $(a, q) = 1$ to evaluate $f_N(\frac{a}{q})$. We shall see, that for $x$ near $\frac{a}{q}$, $f_N(x)$ is large. [7, p. 315]

$$f_N\left(\frac{a}{q}\right) = \sum_{p\leq N} \log p \cdot e\left(p \cdot \frac{a}{q}\right) = \sum_{r=1}^{q} \sum_{\substack{p\equiv r(q)\\ p\leq N}} \log p \cdot e\left(\frac{ap}{q}\right)$$

[7, p. 316]

Due to the fact that $p \equiv r \bmod q$, we can write $p = \lambda q + r$, and using the periodicity of the exponential function we get:

$$e\left(\frac{pa}{q}\right) = e^{2\pi i \frac{pa}{q}} = e^{2\pi i \frac{(\lambda q + r)a}{q}} = e^{2\pi i \lambda a + 2\pi i \frac{ra}{q}} = \underbrace{e^{2\pi i \lambda a}}_{=1} e^{2\pi i \frac{ra}{q}} = e\left(\frac{ra}{q}\right)$$

Therefore we can rewrite our sum

$$\sum_{r=1}^{q} \sum_{\substack{p\equiv r(q)\\ p\leq N}} \log p \cdot e\left(\frac{ap}{q}\right) = \sum_{r=1}^{q} \sum_{\substack{p\equiv r(q)\\ p\leq N}} \log p \cdot e\left(\frac{ar}{q}\right) = \sum_{r=1}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p\equiv r(q)\\ p\leq N}} \log p$$

We now need the Siegel-Walfisz Theorem to evaluate the sum over $p \equiv r \bmod q$. To apply Siegel-Walfisz, $r$ and $q$ have to be relatively prime. We will show that we can restrict the sum to $(r, q) = 1$ because the terms in $f_N\left(\frac{a}{q}\right) = \sum_{r=1}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p\equiv r(q)\\ p\leq N}} \log p$, with $r$ and $q$ not being relatively prime, contribute lower-order terms. [7, p. 316]

*Proof.* Let $r$ and $q$ have a common factor $t$. If $p \equiv r \bmod q$, we write $p = \lambda q + r$, for some $\lambda \in \mathbb{N}$. As $p$ is prime and the sum of two integers with the same factor, one of those numbers must be zero. Otherwise $t$ and 2 would both be divisors of $p$. Therefore $\lambda q = 0$ and $p = r$. This means that if $r$ and $q$ have a common factor, there is only one $p \equiv r \bmod q$. Let us denote the contribution of those terms by $g_N$ and note that we have

$$g_N\left(\frac{a}{q}\right) = \sum_{\substack{r\text{prime}, r\leq N\\ r\mid q}} e\left(\frac{ar}{q}\right) \log r$$

This means that a term with $(r, q) \neq 1$ contributes a term on the order of $e\left(\frac{ar}{q}\right) \log r$ where a term with $(r, q) = 1$ contributes a term on the order of $e\left(\frac{ar}{q}\right) \frac{N}{\varphi(q)}$ plus an error term. As $\varphi(q) \log(r) << N$ we have $e\left(\frac{ar}{q}\right) \log r << e\left(\frac{ar}{q}\right) \frac{N}{\varphi(q)}$. We conclude that the terms with $(r, q) \neq 1$ contribute lower-order terms.

More precisely we get:

$$g_N\left(\frac{a}{q}\right) = \left|\sum_{\substack{r\text{prime},r\leq N\\ r|q}} \log r \cdot e\left(\frac{ra}{q}\right)\right| \leq \sum_{\substack{r\text{prime},r\leq N\\ r|q}} \left|\log r \cdot e\left(\frac{ra}{q}\right)\right|$$
$$= \sum_{\substack{r\text{prime},r\leq N\\ r|q}} \log r \cdot \left|e\left(\frac{ra}{q}\right)\right| = \sum_{\substack{r\text{prime},r\leq N\\ r|q}} \log r.$$

Further we get for r prime

$$\sum_{\substack{r\leq N\\ r|q}} \log r \leq \sum_{r|q} \log r$$

We can write:

$$\sum_{p|q} \log p = \log p_1 + \ldots + \log p_n = \log p_1 \ldots p_n$$

In the factorization of $q$ one or multiple primes could appear more than once and because $q \leq Q$ we get: $\log p_1 \ldots p_n \leq \log q \leq \log Q$

Finally we get:

$$g_N\left(\frac{a}{q}\right) = O(\log Q)$$

□

Knowing that we can restrict the sum to $(r, q) = 1$ we can apply the Siegel-Walfisz Theorem and for any $C > 0$ we get:

$$\sum_{\substack{p\equiv r(q)\\ p\leq N}} \log p = \frac{N}{\varphi(q)} + O\left(\frac{N}{\log^C N}\right)$$

Therefore we get:

$$f_N\left(\frac{a}{q}\right) = \sum_{\substack{r=1\\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right)\left(\frac{N}{\varphi(q)} + O\left(\frac{N}{\log^C N}\right)\right) + O(\log Q)$$

[7, p. 317]

With the definition of the Ramanujan sum we can rewrite $f_N\left(\frac{a}{q}\right)$ as follows:

$$\begin{aligned} f_N\left(\frac{a}{q}\right) &= c_q(a)\left(\frac{N}{\varphi(q)} + O\left(\frac{N}{\log^C N}\right)\right) + O(\log Q) \\ &= \frac{c_q(a)}{\varphi(q)}N + c_q(a)O\left(\frac{N}{\log^C N}\right) + O(\log Q) \end{aligned}$$

.

The second term leads to:

$$\begin{aligned} \left|c_q(a)O\left(\frac{N}{\log^C N}\right)\right| &= \left|\sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ra}{q}\right) O\left(\frac{N}{\log^C N}\right)\right| \\ &\leq \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \left|e\left(\frac{ra}{q}\right) \cdot K_C \cdot \frac{N}{\log^C N}\right| \\ &\leq K_C \cdot \frac{qN}{\log^C N} \end{aligned}$$

With $q \leq Q = \log^B N (B > 0)$ it follows that:

$$K_C \cdot \frac{qN}{\log^C N} \leq K_C \cdot \frac{QN}{\log^C N}$$

We finally get:

$$c_q(a)O\left(\frac{N}{\log^C N}\right) = O\left(\frac{QN}{\log^C N}\right)$$

Therefore we can write:

$$\begin{aligned} f_N\left(\frac{a}{q}\right) &= \frac{c_q(a)}{\varphi(q)}N + O\left(\frac{QN}{\log^C N}\right) + O(\log Q) \\ &= \frac{c_q(a)}{\varphi(q)}N + O\left(\max\left\{\frac{QN}{\log^C N}, \log Q\right\}\right) \end{aligned}$$

Now we need to determine the maximum.

$$\log^B N > \log\log^B N \Longrightarrow \log^B N \cdot \frac{N}{\log^C N} = \frac{QN}{\log^C N} > \log\log^B N = \log Q$$

summarizing we get:

$$O\left(\max\left\{\frac{QN}{\log^C N}, \log Q\right\}\right) = O\left(\frac{QN}{\log^C N}\right) = O\left(\frac{N}{\log^{C-B} N}\right)$$

whereas the implicit constant depends on the real numbers $B > 0$ and $C > 0$. Overall we get:

$$f_N\left(\frac{a}{q}\right) = \frac{c_q(a)}{\varphi(q)}N + O\left(\max\left\{\frac{QN}{\log^C N}, \log Q\right\}\right) = \frac{c_q(a)}{\varphi(q)}N + O\left(\frac{N}{\log^{C-B} N}\right)$$

Using Lemma 2.16 and the fact that $a$ and $q$ are relatively prime we get that $c_q(a) = \mu(q)$ and therefore:

$$f_N\left(\frac{a}{q}\right) = \frac{\mu(q)}{\varphi(q)}N + O\left(\frac{N}{\log^{C-B} N}\right) \tag{4.10}$$

Now that we investigated the size of $f_N(x)$, we can properly define the Major and Minor arcs. [8, p. 216]

## 4.2 Definition of the Major and Minor Arcs

The goal of this section is to split the interval $[0, 1]$ into two disjoint parts; namely into the Major and the Minor arcs. Since we are using the Siegel-Walfisz Theorem we need a positive constant $B$ and we set $Q = \log^B(N)$. For each $1 \leq q \leq Q$ and $0 \leq a \leq q$ with $(a, q) = 1$ we consider the set:

$$\mathfrak{M}(a, q) = \left\{x \in [0, 1] : \left|x - \frac{a}{q}\right| \leq \frac{Q}{N}\right\} \tag{4.11}$$

[7, p. 318]

This choice of the Major arcs has two main advantages.

1) When a denominator is small, relative to N, we can apply the Siegel-Walfisz Theorem to evaluate $f_N\left(\frac{a}{q}\right)$. That is why we required $q \leq Q = \log^B N$.

2) Each Major arc $\mathfrak{M}(a, q)$ has length $\frac{2Q}{N} = \frac{2\log^B N}{N}$. And as these intervals are small we expect $f_N(x) \approx f_N\left(\frac{a}{q}\right)$. For a fixed $\frac{a}{q}$ the size of the arc around it tends to zero as $N$ tends to infinity, but $f_N(x)$ is understood better and better in a smaller window around $\frac{a}{q}$. [7, p. 318]

**Lemma 4.4.** *Let $a, q$ be positive integers such that $0 \leq a \leq q, 1 \leq q \leq Q$ and $(a, q) = 1$. Then, for all sufficiently large $N$, the Major arcs $\mathfrak{M}$ can be expressed as a disjoint union of $\mathfrak{M}(a, q)$.* [10, p. 35]

*Proof.* Let $B > 0$ and $Q = \log^B(N)$. For each $1 \leq q \leq Q$ and $0 \leq a \leq q$ with $(a, q) = 1$,

the Major arc $\mathfrak{M}(a, q)$ is the interval consisting of all real numbers $x \in [0, 1]$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{Q}{N}.$$

If $x \in \mathfrak{M}(a_1, q_1) \cap \mathfrak{M}(a_2, q_2)$ and $\frac{a_1}{q_1} \neq \frac{a_2}{q_2}$, then $|a_1 q_2 - a_2 q_1| \geq 1$ and

$$\begin{aligned} \frac{1}{Q^2} \leq \frac{1}{q_1 q_2} \leq \frac{|a_1 q_2 - a_2 q_1|}{q_1 q_2} &= \left| \frac{a_1}{q_1} - \frac{a_2}{q_2} \right| \\ &\leq \left| \frac{a_1}{q_1} - x \right| + \left| x - \frac{a_2}{q_2} \right| \leq \frac{2Q}{N} \end{aligned}$$

or, equivalently,

$$N \leq 2Q^3 = 2\log^{3B}(N),$$

which is not true for large values of $N$. Hence, we have arrived at a contradiction. [10, p. 35-36] □

Thanks to this Lemma we can define the Major arcs to be the union of the arcs $\mathfrak{M}(a, q)$:

$$\mathfrak{M} = \bigcup_{q=1}^{Q} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q} \mathfrak{M}(a, q) \tag{4.12}$$

The Minor arcs $\mathfrak{m}$ are whatever is not in the Major arcs. Thus,

$$\mathfrak{m} = [0, 1] \setminus \mathfrak{M}$$

## 4.3 Contribution of the Major Arcs

We have already shown in 4.10, that $f_N(x)$ is large (of size $N$) for $x \in \mathfrak{M}(a, q)$. Now we need to approximate the integral

$$\int_{\mathfrak{M}} f_N^3(x) e(-xN)\, dx$$

**Lemma 4.5.** *Let $u(\beta) = \sum_{m=1}^{N} e(m\beta)$ and let $B$ and $C$ be positive real numbers with $C > 2B$. If $x \in \mathfrak{M}(a, q)$ and $\beta = x - a/q$, then*

$$f_N(x) = \frac{\mu(q)}{\varphi(q)} u(\beta) + O\left( \frac{Q^2 N}{\log^C N} \right) = \frac{\mu(q)}{\varphi(q)} u(\beta) + O\left( \frac{N}{\log^{C-2B} N} \right)$$

*and*

$$f_N(x)^3 = \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\frac{Q^2N^3}{\log^C N}\right) = \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\frac{N^3}{\log^{C-2B} N}\right)$$

*where the implied constants depend only on $B$ and $C$.* [8, p. 217]

*Proof.* If $x \in \mathfrak{M}(a,q)$, then $x = a/q + \beta$, where $|\beta| \leq Q/N$. Using 2.13 we get:

$$\begin{aligned} f_N(x) - \frac{\mu(q)}{\varphi(q)}u(\beta) &= \sum_{p\leq N}\log(p)e(px) - \frac{\mu(q)}{\varphi(q)}\sum_{m=1}^{N}e(m\beta) \\ &= \sum_{m=1}^{N}\Lambda(m)e\left(\frac{ma}{q}+m\beta\right) - \sum_{m=1}^{N}\frac{\mu(q)}{\varphi(q)}e(m\beta) \\ &= \sum_{m=1}^{N}\left(\Lambda(m)e\left(\frac{ma}{q}\right) - \frac{\mu(q)}{\varphi(q)}\right)e(m\beta) \end{aligned}$$

We will use 4.10 to simplify the following:

$$\begin{aligned} A(N) &= \sum_{m=1}^{N}\left(\Lambda(m)e\left(\frac{ma}{q}\right) - \frac{\mu(q)}{\varphi(q)}\right) \\ &= f_N\left(\frac{a}{q}\right) - \sum_{m=1}^{N}\frac{\mu(q)}{\varphi(q)} \\ &= \frac{\mu(q)}{\varphi(q)}N + O\left(\frac{QN}{\log^C N}\right) - \frac{\mu(q)}{\varphi(q)}N \quad = O\left(\frac{QN}{\log^C N}\right) \end{aligned}$$

Due to the fact that $e(m\beta)$ is holomorphic on $\mathbb{C}$, its derivative is continuous and by partial summation we get:

$$\begin{aligned} &\left|\sum_{m=1}^{N}\left(\Lambda(m)e\left(\frac{ma}{q}\right) - \frac{\mu(q)}{\varphi(q)}\right)e(m\beta)\right| = \left|A(N)e(N\beta) - \int_1^N A(x)(e(x\beta))'dx\right| \\ =&\left|A(N)e(N\beta) - 2\pi i\beta\int_1^N A(x)e(x\beta)dx\right| \\ \leq&|A(N)e(N\beta)| + \left|-2\pi i\beta\int_1^N A(x)e(x\beta)dx\right| \\ \leq&|A(N)| + 2\pi|\beta|\int_1^N |A(x)e(x\beta)|dx = |A(N)| + 2\pi|\beta|(N-1)|A(x)| \\ &\leq |A(N)| + 2\pi|\beta|\cdot N\cdot\max\{|A(x)| : 1\leq x\leq N\}. \end{aligned}$$

Therefore we get the following approximation:

$$A(N)e(N\beta)-2\pi i\beta\int_1^N A(x)e(x\beta)dx \ll |A(N)|+|\beta|\cdot N\cdot\max\{|A(x)| : 1 \leq x \leq N\}$$

Using the fact that

$$|\beta| \leq \frac{Q}{N} \text{ and } A(x) = O\left(\frac{QN}{\log^C N}\right)$$

we get:

$$\begin{aligned}|A(N)| + |\beta| \cdot N \cdot \max\{|A(x)| : 1 \leq x \leq N\} &\leq O\left(\frac{QN}{\log^C N}\right) + \frac{Q}{N}NO\left(\frac{QN}{\log^C N}\right) \\ &= O\left(\frac{QN}{\log^C N}\right) + O\left(\frac{Q^2N}{\log^C N}\right) \\ &= O\left(\frac{Q^2N}{\log^C N}\right)\end{aligned}$$

We can conclude that

$$f_N(x) - \frac{\mu(q)}{\varphi(q)}u(\beta) = O\left(\frac{Q^2N}{\log^C N}\right) = O\left(\frac{N}{\log^{C-2B} N}\right)$$

Using the definition of the Euler totient function and the definition of the Möbius function we get the following approximation:

$$\left|\frac{\mu(q)}{\varphi(q)}u(\beta)\right| \leq |u(\beta)| = \left|\sum_{m=1}^{N} e(m\beta)\right| \leq \sum_{m=1}^{N} |e(m\beta)| = \sum_{m=1}^{N} 1 = N.$$

Since $C > 2B$ we get:

$$\frac{Q^2N}{\log^C N} = \frac{N}{\log^{C-2B} N} < N$$

And we have our desired result:

$$f_N(x) = \frac{\mu(q)}{\varphi(q)}u(\beta) + O\left(\frac{N}{\log^{C-2B} N}\right)$$

To prove the second part of the Lemma we need to look at $f_N(x)^3$.

$$\begin{aligned} f_N(x)^3 =& \left(\frac{\mu(q)}{\varphi(q)}u(\beta) + O\left(\frac{Q^2N}{\log^C N}\right)\right)^3 \\ =& \frac{\mu(q)^3}{\varphi(q)^3}u(\beta)^3 + 3\cdot\frac{\mu(q)^2}{\varphi(q)^2}u(\beta)^2\cdot O\left(\frac{Q^2N}{\log^C N}\right) \\ &+ 3\cdot\frac{\mu(q)}{\varphi(q)}u(\beta)\cdot O\left(\left(\frac{Q^2N}{\log^C N}\right)^2\right) + O\left(\left(\frac{Q^2N}{\log^C N}\right)^3\right) \end{aligned}$$

The Möbius-function only has three possible values $0, 1$ and $-1$, therefore $\mu(q)^3 = \mu(q)$ and for the first term we get:

$$\frac{\mu(q)^3}{\varphi(q)^3}u(\beta)^3 = \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3$$

Furthermore we will use the fact that $u(\beta) \leq N$. The $\mu$-function is at most 1 and the $\varphi$-function is at least 1. That leads us to

$$\begin{aligned} f_N(x)^3 &= \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + 3\cdot\frac{\mu(q)^2}{\varphi(q)^2}u(\beta)^2\cdot O\left(\frac{Q^2N}{\log^C N}\right) \\ &\quad + 3\cdot\frac{\mu(q)}{\varphi(q)}u(\beta)\cdot O\left(\left(\frac{Q^2N}{\log^C N}\right)^2\right) + O\left(\left(\frac{Q^2N}{\log^C N}\right)^3\right) \\ &\leq \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + 3N^2O\left(\frac{Q^2N}{\log^C N}\right) + 3NO\left(\left(\frac{Q^2N}{\log^C N}\right)^2\right) + O\left(\left(\frac{Q^2N}{\log^C N}\right)^3\right) \\ &= \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\frac{Q^2N^3}{\log^C N}\right) + O\left(\frac{Q^4N^3}{\log^{2C} N}\right) + O\left(\frac{Q^6N^3}{\log^{3C} N}\right) \\ &= \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\max\left\{\frac{N^3}{\log^{C-2B} N}, \frac{N^3}{\log^{2(C-2B)} N}, \frac{N^3}{\log^{3(C-2B)} N}\right\}\right) \\ &= \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\frac{Q^2N^3}{\log^C N}\right) \end{aligned}$$

In summary we get

$$f_N(x)^3 = \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\frac{Q^2N^3}{\log^C N}\right) = \frac{\mu(q)}{\varphi(q)^3}u(\beta)^3 + O\left(\frac{N^3}{\log^{C-2B} N}\right)$$

where the implied constants only depend on B and C and $C > 2B$. [8, p. 217-218]

□

In the process of evaluating the integral over the Major arcs we will encounter the singular series and the singular integral.

**Definition 4.6.** The singular series for the ternary Goldbach problem is defined as

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)c_q(N)}{\varphi(q)^3}$$

where

$$c_q(N) = \sum_{\substack{a=1 \\ (q,a)=1}}^{q} e\left(\frac{aN}{q}\right)$$

is the Ramanujan sum. [8, p. 212]

**Theorem 4.7.** *The singular series $\mathfrak{S}(N)$ converges absolutely and uniformly in $N$ and has the Euler product*

$$\mathfrak{S}(N) = \prod_{p} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(1 - \frac{1}{p^2 - 3p + 3}\right)$$

*There exist positive constants $c_1$ and $c_2$ such that*

$$c_1 < \mathfrak{S}(N) < c_2$$

*for all positive integers $N$. Moreover, for any $\varepsilon > 0$,*

$$\mathfrak{S}(N, Q) = \sum_{q \leq Q} \frac{\mu(q)c_q(N)}{\varphi(q)^3} = \mathfrak{S}(N) + O\left(Q^{-(1-\varepsilon)}\right)$$

*where the implied constant depends only on $\varepsilon$.* [8, p. 212]

*Proof.* By the definition of $c_q(N)$ and $\varphi(q)$ we get

$$|c_q(N)| = \left| \sum_{\substack{a=1 \\ (q,a)=1}}^{q} e\left(\frac{aN}{q}\right) \right| \leq \sum_{\substack{a=1 \\ (q,a)=1}}^{q} 1 = \varphi(q).$$

Let $\varepsilon > 0$. Applying Theorem 2.3 we get $q^{1-\varepsilon} < \varphi(q)$, for sufficiently large $q$. Note that $|\mu(q)| \leq 1$. Therefore we get

$$\frac{\mu(q)c_q(N)}{\varphi(q)^3} \ll \frac{1}{\varphi(q)^2} < \frac{1}{(q^{1-\varepsilon})^2} \ll \frac{1}{q^{2-\varepsilon}}.$$

We can conclude that the singular series converges absolutely and uniformly in $N$.

Using Proposition 2.18 we further get:

$$\mathfrak{S}(N) - \mathfrak{S}(N,Q) = \sum_{q>Q} \frac{\mu(q)c_q(N)}{\varphi(q)^3} \ll \sum_{q>Q} \frac{1}{\varphi(q)^2} \ll \sum_{q>Q} \frac{1}{q^{2-\varepsilon}} \ll \frac{1}{Q^{1-\varepsilon}}$$

By Lemma 2.16, $c_q(N)$ is a multiplicative function of $q$ and for $p \mid N$ we get:

$$c_p(N) = \sum_{d|(p,N)} \mu\left(\frac{p}{d}\right) d = \sum_{d|p} \mu\left(\frac{p}{d}\right) d = \mu(1) \cdot p + \mu(p) = p - 1$$

If $p \nmid N$ we get $(p, N) = 1$ and therefore

$$c_p(N) = \mu(p) = -1$$

$$c_p(N) = \begin{cases} p-1 & \text{if } p \mid N \\ -1 & \text{if } p \nmid N \end{cases}$$

The arithmetic function

$$\frac{\mu(q)c_q(N)}{\varphi(q)^3}$$

is multiplicative in $q$ and $\mu\left(p^j\right) = 0$ for $j \geq 2$. Therefore it follows from Theorem 2.26 that the singular series has the Euler product

$$\begin{aligned}
\mathfrak{S}(N) &= \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\mu\left(p^j\right) c_{p^j}(N)}{\varphi\left(p^j\right)^3}\right) \\
&= \prod_p \left(1 - \frac{c_p(N)}{\varphi(p)^3}\right) \\
&= \prod_{p \nmid N} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right) \\
&= \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(\frac{1 - \frac{1}{(p-1)^2}}{1 + \frac{1}{(p-1)^3}}\right) \\
&= \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(1 - \frac{1}{p^2 - 3p + 3}\right),
\end{aligned}$$

In a last step we want to show that there exist positive constants $c_1$ and $c_2$, such that $c_1 < \mathfrak{S}(N) < c_2$ for all positive integers $N$.

Since $1 + \frac{1}{(p-1)^3} > 1$ and $0 < 1 - \frac{1}{(p-1)^2} < 1$ for $p \neq 2$,
$\prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right)$ can be taken as the lower bound.

And since $1 - \frac{1}{p^2-3p+3} < 1$ we can take $\prod_p \left(1 + \frac{1}{(p-1)^3}\right)$ to be our upper bound.

This completes the proof. [8, p. 213] □

**Definition 4.8.** Let

$$u(\beta) = \sum_{m=1}^{N} e(m\beta)$$

Then the singular integral for Goldbach's ternary Conjecture is defined as:

$$J(N) := \int_{-1/2}^{1/2} u(\beta)^3 e(-N\beta) d\beta$$

[8, p. 215]

**Lemma 4.9.**

$$J(N) = \int_{-1/2}^{1/2} u(\beta)^3 e(-N\beta) d\beta = \frac{N^2}{2} + O(N)$$

[8, p. 215]

*Proof.* As we have seen in remark 3.1

$$R_{1,s}(N) = \binom{N-1}{s-1} = \frac{N^{s-1}}{(s-1)!} + O\left(N^{s-2}\right)$$

for all positive integers $N$.

Therefore

$$R_{1,3}(N) = \binom{N-1}{2} = \frac{N^2}{2} + O\left(N\right)$$

We get

$$\begin{aligned} J(N) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\beta)^3 e(-N\beta) d\beta \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1 \leq N} e\left(m_1\beta\right) \sum_{m_2 \leq N} e\left(m_2\beta\right) \sum_{m_3 \leq N} e\left(m_3\beta\right) \cdot e(-N\beta) d\beta \\ &= \sum_{m_1, m_2, m_3 \leq N} \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\beta\left(m_1 + m_2 + m_3 - N\right)\right) d\beta \end{aligned}$$

Using the fact that for $m, n \in \mathbb{Z}$

$$\int_0^1 e(nx)e(-mx)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

the integral is 1 if $m_1 + m_2 + m_3 = N$ and 0 otherwise.

Therefore we have

$$J(N) = \sum_{m_1+m_2+m_3=N} 1 = R_{1,3}(N) = \binom{N-1}{2} = \frac{N^2}{2} + O(N)$$

[8, p. 215]

□

Now we can finally evaluate our integral

**Theorem 4.10.** *Let $B, C$, and $\varepsilon$ be positive numbers, with $C > 2B$. Then the integral over the Major arcs is*

$$\int_{\mathfrak{M}} f_N(x)^3 e(-Nx)dx = \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{\log^{(1-\varepsilon)B} N}\right) + O\left(\frac{N^2}{\log^{C-5B} N}\right)$$

*where the implied constants depend only on $B, C$, and $\varepsilon$.* [8, p. 218]

*Proof.* To compute $\int_{\mathfrak{M}} f_N(x)^3 e(-Nx)dx$ we will compute the following two integrals, where $\beta = x - \frac{a}{q}$.

1)

$$\int_{\mathfrak{M}} \left(f_N(x)^3 - \frac{\mu(q)}{\varphi(q)^3} u(\beta)^3\right) e(-Nx)dx$$

2)

$$\int_{\mathfrak{M}} \frac{\mu(q)}{\varphi(q)^3} u\left(\beta\right)^3 e(-Nx)dx$$

Summing the estimates of 1) and 2) will lead to the desired result.

Recall that $x \in \mathfrak{M}_{(a,q)}$ implies that $\beta = x - \frac{a}{q}$

1) Using Lemma 4.5 we get

$$\begin{aligned}
&\left|\int_{\mathfrak{M}} \left(f_N(x)^3 - \frac{\mu(q)}{\varphi(q)^3} u\left(x-\frac{a}{q}\right)^3\right) e(-Nx)dx\right| \\
=&\left|\sum_{q=1}^{Q}\sum_{\substack{a=0\\(a,q)=1}}^{q}\int_{\mathfrak{M}(a,q)} \left(f_N(x)^3 - \frac{\mu(q)}{\varphi(q)^3} u\left(x-\frac{a}{q}\right)^3\right) e(-Nx)dx\right| \\
\ll&\sum_{q=1}^{Q}\sum_{\substack{a=0\\(a,q)=1}}^{q}\int_{\mathfrak{M}(a,q)} \frac{Q^2N^3}{\log^C N}|e(-Nx)|dx \\
=&\sum_{q=1}^{Q}\sum_{\substack{a=0\\(a,q)=1}}^{q}\frac{Q^2N^3}{\log^C N}\int_{\mathfrak{M}(a,q)} 1dx
\end{aligned}$$

To estimate the integral $\int_{\mathfrak{M}(a,q)} 1dx$, we have to compute the length of the Major arcs. We have two different cases:

1. $q = 1 \implies a = 0$ or $a = 1$. In that case we have $\mathfrak{M}(0,1)$ and $\mathfrak{M}(1,1)$ with

$$|\mathfrak{M}(0,1)| = \frac{Q}{N} = |\mathfrak{M}(1,1)|$$

2. $q \geq 2$ with

$$|\mathfrak{M}(a,q)| = \frac{a}{q} + \frac{Q}{N} - \left(\frac{a}{q} - \frac{Q}{N}\right) = \frac{2Q}{N}$$

As $\frac{Q}{N} < \frac{2Q}{N}$ we can conclude that $\int_{\mathfrak{M}(a,q)} 1dx \ll \frac{Q}{N}$ for all choices of $a$ and $q$. Therefore our estimation continues as follows:

$$\begin{aligned}
\sum_{q=1}^{Q}\sum_{\substack{a=0\\(a,q)=1}}^{q}\frac{Q^2N^3}{\log^C N}\int_{\mathfrak{M}(a,q)} 1dx &\ll \sum_{q=1}^{Q}\sum_{\substack{a=0\\(a,q)=1}}^{q}\frac{Q^3N^2}{\log^C N} \\
&\leq \frac{Q^5N^2}{\log^C N} \\
&= \frac{N^2}{\log^{C-5B} N}
\end{aligned}$$

Therefore we get that

$$\int_{\mathfrak{M}} f_N(x)^3 e(-Nx)dx = \int_{\mathfrak{M}} \frac{\mu(q)}{\varphi(q)^3} u\left(x-\frac{a}{q}\right)^3 e(-Nx)dx + O\left(\frac{N^2}{\log^{C-5B} N}\right).$$

2)

$$\int_{\mathfrak{M}} \frac{\mu(q)}{\varphi(q)^3} u\left(\beta\right)^3 e(-Nx)dx = \int_{\mathfrak{M}} \frac{\mu(q)}{\varphi(q)^3} u\left(x-\frac{a}{q}\right)^3 e(-Nx)dx$$
$$= \sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\varphi(q)^3} \int_{\mathfrak{M}(a,q)} u\left(x-\frac{a}{q}\right)^3 e(-Nx)dx$$

The integrand is periodic with period 1. Therefore it does not matter whether we integrate over $[0, \frac{Q}{N}] \cup [1-\frac{Q}{N}, 1]$ or over $[1-\frac{Q}{N}, 1+\frac{Q}{N}]$. We can write

$$\sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\varphi(q)^3} \int_{\mathfrak{M}(a,q)} u\left(x-\frac{a}{q}\right)^3 e(-Nx)dx$$
$$= \sum_{q=1}^{Q} \frac{\mu(q)}{\varphi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\frac{a}{q}-\frac{Q}{N}}^{\frac{a}{q}+\frac{Q}{N}} u\left(x-\frac{a}{q}\right)^3 e(-Nx)dx$$

Substituting $x = \frac{a}{q} + \beta$ we get

$$\sum_{q=1}^{Q} \frac{\mu(q)}{\varphi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u\left(\beta\right)^3 e\left(-N\left(\frac{a}{q}+\beta\right)\right) d\beta$$
$$= \sum_{q=1}^{Q} \frac{\mu(q)}{\varphi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{Na}{q}\right) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u\left(\beta\right)^3 e\left(-N\beta\right) d\beta$$

Using the definition of the Ramanujan sum 2.15 leads to

$$\sum_{q=1}^{Q} \frac{\mu(q)}{\varphi(q)^3} c_q(-N) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u\left(\beta\right)^3 e\left(-N\beta\right) d\beta$$

Further we use the fact that $c_q(-N) = c_q(N)$ and Theorem 4.7 to obtain

$$\mathfrak{S}(N,Q) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u\left(\beta\right)^3 e\left(-N\beta\right) d\beta$$

.

We still have to evaluate the integral. In order to apply Lemma 4.9 we would like to integrate from $\frac{-1}{2}$ to $\frac{1}{2}$. In order to do so, we need to split up our integral. Due to the slow growth of the logarithm in the upper bound $\frac{Q}{N} = \frac{\log^B N}{N}$, the

inequality $\frac{Q}{N} < \frac{1}{2}$ holds for $N$ big enough and fixed $B > 0$. Therefore we can split up our integral as follows:

$$\int_{-\frac{Q}{N}}^{\frac{Q}{N}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{-\frac{Q}{N}} - \int_{\frac{Q}{N}}^{\frac{1}{2}}$$

The first integral is obviously the singular integral and with Lemma 4.9 we get:

$$J(N) = \int_{-1/2}^{1/2} u(\beta)^3 e(-N\beta) d\beta = \frac{N^2}{2} + O(N)$$

For the second integral $|\beta| \leq \frac{1}{2}$ for $\beta$ in the integration interval. Therefore $\|\beta\| = |\beta|$ and with Lemma 2.22 we get

$$u(\beta) = \sum_{m=1}^{N} e(m\beta) \ll \min\left(N - 0, \|\beta\|^{-1}\right) = \min\left(N, |\beta|^{-1}\right) = |\beta|^{-1} = \frac{1}{|\beta|}$$

Therefore we get

$$\begin{aligned}
\left| - \int_{-\frac{1}{2}}^{-\frac{Q}{N}} u(\beta)^3 e(-N\beta) d\beta \right| &\leq \int_{-\frac{1}{2}}^{-\frac{Q}{N}} \left| u(\beta)^3 e(-N\beta) \right| d\beta \\
&= \int_{-\frac{1}{2}}^{-\frac{Q}{N}} |u(\beta)|^3 d\beta \\
&\ll \int_{-\frac{1}{2}}^{-\frac{Q}{N}} \frac{1}{|\beta|^3} d\beta = \int_{-\frac{1}{2}}^{-\frac{Q}{N}} -\frac{1}{\beta^3} d\beta \\
&= \frac{1}{2}\frac{N^2}{Q^2} - 2 \leq \frac{1}{2}\frac{N^2}{Q^2}
\end{aligned}$$

We have:

$$\int_{-\frac{1}{2}}^{-\frac{Q}{N}} u(\beta)^3 e(-N\beta) d\beta \ll \frac{N^2}{Q^2}$$

Similarly, for the third integral, we get

$$\int_{\frac{Q}{N}}^{\frac{1}{2}} u(\beta)^3 e(-N\beta) d\beta \ll \frac{N^2}{Q^2}$$

Putting the estimates of the three integrals together we get:

$$\begin{aligned}
&\int_{-\frac{Q}{N}}^{\frac{Q}{N}} u(\beta)^3 e(-N\beta)d\beta \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\beta)^3 e(-N\beta)d\beta - \int_{-\frac{1}{2}}^{-\frac{Q}{N}} u(\beta)^3 e(-N\beta)d\beta - \int_{\frac{Q}{N}}^{\frac{1}{2}} u(\beta)^3 e(-N\beta)d\beta \\
&= \frac{N^2}{2} + O(N) + O\left(\frac{N^2}{Q^2}\right) + O\left(\frac{N^2}{Q^2}\right) \\
&= \frac{N^2}{2} + O\left(\frac{N^2}{Q^2}\right)
\end{aligned}$$

From Theorem 4.7 we know that

$$\mathfrak{S}(N,Q) = \mathfrak{S}(N) + O\left(Q^{-(1-\varepsilon)}\right)$$

and that $\mathfrak{S}(N)$ is bounded above by some positive constant $c_2$, which is why $\mathfrak{S}(N) = O(1)$.

Therefore we have

$$\begin{aligned}
\int_{\mathfrak{M}} \frac{\mu(q)}{\varphi(q)^3} u\left(\beta\right)^3 e(-Nx)dx &= \mathfrak{S}(N,Q) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u\left(\beta\right)^3 e\left(-N\beta\right) d\beta \\
&= \left(\mathfrak{S}(N) + O\left(Q^{-(1-\varepsilon)}\right)\right)\left(\frac{N^2}{2} + O\left(\frac{N^2}{Q^2}\right)\right) \\
&= \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{Q^2}\right) + O\left(\frac{N^2}{Q^{1-\varepsilon}}\right) + O\left(\frac{N^2}{Q^{3-\varepsilon}}\right) \\
&= \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{Q^{1-\varepsilon}}\right)
\end{aligned}$$

Putting the estimates from 1) and 2) together we get

$$\begin{aligned}
\int_{\mathfrak{M}} f_N(x)^3 e(-Nx)dx &= \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{Q^{1-\varepsilon}}\right) + O\left(\frac{N^2}{\log^{C-5B} N}\right) \\
&= \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{\log^{B(1-\varepsilon)} N}\right) + O\left(\frac{N^2}{\log^{C-5B} N}\right)
\end{aligned}$$

which completes the proof. [8, p. 218]

□

## 4.4 Contribution of the Minor Arcs

In a next step we will show, that $f_N(x)$ is much smaller on the Minor arcs. Showing that the Minor arcs contribute less than the Major arcs is considered the hardest part of the problem. To estimate the integral over the Minor arcs we will use Vinogradov's estimate for the exponential sum $f_N(x)$. In order to understand Vinogradov's estimate, Vaughan's identity will be needed.

**Lemma 4.11.** *Vaughan's identity*
*For $u \geq 1$, let*

$$M_u(k) = \sum_{\substack{d|k \\ d \leq u}} \mu(d)$$

*Let $\Phi(k, \ell)$ be an arithmetic function of two variables. Then*

$$\sum_{u<\ell\leq N} \Phi(1,\ell) + \sum_{u<k\leq N} \sum_{u<\ell\leq \frac{N}{k}} M_u(k)\Phi(k,\ell) = \sum_{d\leq u} \sum_{u<\ell\leq \frac{N}{d}} \sum_{m\leq \frac{N}{ld}} \mu(d)\Phi(dm,\ell)$$

[8, p. 220]

*Proof.* The goal is to evaluate the sum

$$S = \sum_{k=1}^{N} \sum_{u<\ell\leq \frac{N}{k}} M_u(k)\Phi(k,\ell)$$

in two different ways. Once for the left-hand side and once for the right-hand side.

For the left-hand side we need the fact that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$M_u(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } 1 < k \leq u \end{cases}$$

Therefore,

$$S = \underbrace{\sum_{u<\ell\leq N} \Phi(1,\ell)}_{\text{case where k=1}} + \sum_{u<k\leq N} \sum_{u<\ell\leq \frac{N}{k}} M_u(k)\Phi(k,\ell)$$

For the right-hand side we will interchange summations and let $k = dm$.

Then we get:

$$\begin{aligned}
S &= \sum_{k=1}^{N} \sum_{u<\ell\leq \frac{N}{k}} M_u(k)\Phi(k,\ell) \\
&= \sum_{k=1}^{N} \sum_{u<\ell\leq \frac{N}{k}} \sum_{\substack{d|k \\ d\leq u}} \mu(d)\Phi(k,\ell) \\
&= \sum_{d\leq u} \sum_{\substack{k=1 \\ d|k}}^{N} \sum_{u<\ell\leq \frac{N}{k}} \mu(d)\Phi(k,\ell) \\
&= \sum_{d\leq u} \sum_{m\leq \frac{N}{d}} \sum_{u<\ell\leq \frac{N}{dm}} \mu(d)\Phi(dm,\ell) \\
&= \sum_{d\leq u} \sum_{u<\ell\leq \frac{N}{d}} \sum_{m\leq \frac{N}{\ell d}} \mu(d)\Phi(dm,\ell).
\end{aligned}$$

[8, p. 220]

□

Using Vaughan's identity and setting $u = N^{\frac{2}{5}}$ and $\Phi(k,\ell) = \Lambda(\ell)e(xk\ell)$ we can split up $f_N(x)$ as follows.

**Lemma 4.12.** *Let $\Lambda(\ell)$ be the von Mangoldt function. For every real number $x$,*

$$f_N(x) = S_1 - S_2 - S_3 + O\left(N^{1/2}\right)$$

*where*

$$S_1 = \sum_{d\leq N^{2/5}} \sum_{\ell\leq \frac{N}{d}} \sum_{m\leq \frac{N}{ld}} \mu(d)\Lambda(\ell)e(xd\ell m)$$

$$S_2 = \sum_{d\leq N^{2/5}} \sum_{\ell\leq N^{2/5}} \sum_{m\leq \frac{N}{kd}} \mu(d)\Lambda(\ell)e(xd\ell m)$$

*and*

$$S_3 = \sum_{k>N^{2/5}} \sum_{N^{2/5}<\ell\leq N/k} M_{N^{2/5}}(k)\Lambda(\ell)e(xk\ell)$$

[8, p. 221]

*Proof.* We will apply Vaughan's identity with $u = N^{2/5}$ and $\Phi(k,\ell) = \Lambda(\ell)e(xk\ell)$.

Recalling the definition (2.13) of the von Mangoldt function the first term on the left-hand side in Vaughan's identity becomes:

$$\begin{aligned}
\sum_{u<\ell\leq N} \Phi(1,\ell) &= \sum_{N^{2/5}<\ell\leq N} \Lambda(\ell)e(x\ell) \\
&= \sum_{\ell=1}^{N} \Lambda(\ell)e(x\ell) - \sum_{l\leq N^{2/5}} \Lambda(\ell)e(x\ell) \\
&= \sum_{p^k\leq N} (\log p)e\left(xp^k\right) + O\left(N^{2/5}\log N\right) \\
&= \sum_{p\leq N} (\log p)e(xp) + \sum_{\substack{p^k\leq N \\ k\geq 2}} (\log p)e\left(xp^k\right) + O\left(N^{2/5}\log N\right) \\
&= f_N(x) + O\left(\sum_{\substack{p^k\leq N \\ k\geq 2}} \log p\right) + O\left(N^{2/5}\log N\right) \\
&= f_N(x) + O\left(\sum_{p^2\leq N} \left[\frac{\log N}{\log p}\right] \log p\right) + O\left(N^{2/5}\log N\right) \\
&= f_N(x) + O\left(\pi\left(N^{1/2}\right)\log N\right) + O\left(N^{2/5}\log N\right) \\
&= f_N(x) + O\left(N^{1/2}\right)
\end{aligned}$$

where in the last step we used the fact that

$$\pi\left(N^{1/2}\right) \ll \frac{N^{1/2}}{\log N}$$

by Theorem 2.11.

The second term on the left-hand side in Vaughan's identity is just

$$\sum_{N^{2/5}<k\leq N} \sum_{N^{2/5}<\ell\leq \frac{N}{k}} M_{N^{2/5}}(k)\Lambda(\ell)e(xk\ell) = S_3$$

The the term on the right-hand side in Vaughan's identity can be simplified to

$$\begin{aligned}
&\sum_{d\leq N^{2/5}} \sum_{N^{2/5}<\ell\leq \frac{N}{d}} \sum_{m\leq \frac{N}{\ell d}} \mu(d)\Lambda(\ell)e(xd\ell m) \\
&= \sum_{d\leq N^{2/5}} \sum_{\ell\leq \frac{N}{d}} \sum_{m\leq \frac{N}{\ell d}} \mu(d)\Lambda(\ell)e(xd\ell m) \\
&\quad - \sum_{d\leq N^{2/5}} \sum_{\ell\leq N^{2/5}} \sum_{m\leq \frac{N}{\ell d}} \mu(d)\Lambda(\ell)e(xd\ell m) \\
&= S_1 - S_2.
\end{aligned}$$

Therefore we get:

$$f_N(x) + O(N^{1/2}) + S_3 = S_1 - S_2$$

and this completes the proof. [8, pp. 221-222]

□

The three sums $S_1, S_2, S_3$ can all be estimated separately. The following three Lemmas will provide upper bounds for $S_1, S_2$ and $S_3$. The interested reader can find the proofs of these Lemmas in [8] on pages 222-227.

**Lemma 4.13.** *If*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

*where* $1 \leq q \leq N$ *and* $(a, q) = 1$, *then*

$$|S_1| \ll \left( \frac{N}{q} + N^{2/5} + q \right) (\log N)^2$$

[8, p. 222]

**Lemma 4.14.** *If*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

*where* $1 \leq q \leq N$ *and* $(a, q) = 1$, *then*

$$|S_2| \ll \left( \frac{N}{q} + N^{4/5} + q \right) (\log N)^2$$

[8, p. 224]

**Lemma 4.15.** *If*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

*where* $1 \leq q \leq N$ *and* $(a, q) = 1$, *then*

$$|S_3| \ll \left( \frac{N}{q^{1/2}} + N^{4/5} + N^{1/2} q^{1/2} \right) (\log N)^4$$

[8, p. 224]

Inserting the upper bounds we found into Lemma 4.12 leads us to the following Theorem:

**Theorem 4.16.** *(Vinogradov's estimate) If*

$$\left|x - \frac{a}{q}\right| \leq \frac{1}{q^2}$$

*where $a$ and $q$ are integers such that $1 \leq q \leq N$ and $(a, q) = 1$, then*

$$f_N(x) \ll \left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2}q^{1/2}\right)(\log N)^4$$

[8, p. 220]

Using Vinogradov's estimate we can now finally evaluate the integral over the Minor arcs.

**Theorem 4.17.** *For any $B > 0$, we have*

$$\int_{\mathfrak{m}} f_N(x)^3 e(-xN)dx \ll \frac{N^2}{\log^{(B/2)-5} N}$$

*where the implied constant depends only on $B$.* [8, p. 227]

*Proof.* Let $x \in \mathfrak{m} = [0, 1]\backslash\mathfrak{M}$. By Dirichlet's Theorem 2.24, for any real number $x$ there exists a fraction $\frac{a}{q} \in [0, 1]$ with $1 \leq q \leq \frac{N}{Q}$ and $(a, q) = 1$ such that

$$\left|x - \frac{a}{q}\right| \leq \frac{1}{q\frac{N}{Q}} = \frac{Q}{qN} \leq \min\left(\frac{Q}{N}, \frac{1}{q^2}\right)$$

If $q \leq Q$, then $x \in \mathfrak{M}(a, q) \subseteq \mathfrak{M}$, which leads to a contradiction. Therefore,

$$Q < q \leq \frac{N}{Q}$$

From Theorem 4.16 we know that

$$f_N(x) \ll \left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2}q^{1/2}\right)(\log N)^4$$

Recall that $Q = \log^B N$. Replacing q in the denominator with its lower bound and in the product with its upper bound yields

$$f_N(x) \ll \left(\frac{N}{\log^{B/2} N} + N^{4/5} + N^{1/2}\left(\frac{N}{\log^B N}\right)^{1/2}\right)(\log N)^4 \ll \frac{N}{(\log N)^{(B/2)-4}}.$$

As we have seen in Lemma 4.3

$$\int_0^1 |f_N(x)|^2 dx \ll N \log N$$

Applying this we conclude that

$$\begin{aligned}\int_{\mathfrak{m}} f_N(x)^3 e(-Nx)dx &\leq \int_{\mathfrak{m}} |f_N(x)|^3 dx \\ &\leq \sup\{|f_N(x)| : x \in \mathfrak{m}\} \int_{\mathfrak{m}} |f_N(x)|^2 dx \\ &\ll \frac{N}{\log^{(B/2)-4} N} \int_0^1 |f_N(x)|^2 dx \\ &\ll \frac{N^2}{\log^{(B/2)-5} N}\end{aligned}$$

[8, p. 228] □

## 4.5 Results for Vinogradov's Theorem

Recall that $r(N, 3, \mathbb{P})$, which was defined in 4.1, denotes the number of representations of $N$ as a sum of three primes. Whereas $R(N, 3, \mathbb{P})$, which was defined in 4.7, denotes the weighted sum over the representations of $N$ as a sum of three primes.

**Theorem 4.18.** *(Vinogradov's Formula for $R(N, 3, \mathbb{P})$)*
*Let $\mathfrak{S}(N)$ be the singular series for the ternary Goldbach problem. For all sufficiently large odd integers $N$ and for every $A > 0$,*

$$R(N, 3, \mathbb{P}) = \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{\log^A N}\right)$$

*where the implied constant depends only on $A$.* [8, p. 228]

*Proof.* It follows from 4.10 and 4.17 that, for any positive numbers $B, C$, and $\varepsilon$ with $C > 2B$

$$\begin{aligned}R(N, 3, \mathbb{P}) &= \int_0^1 F(\alpha)^3 e(-N\alpha)d\alpha = \int_{\mathfrak{M}} F(\alpha)^3 e(-N\alpha)d\alpha + \int_{\mathfrak{m}} F(\alpha)^3 e(-N\alpha)d\alpha \\ &= \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{(\log N)^{(1-\varepsilon)B}}\right) + O\left(\frac{N^2}{(\log N)^{C-5B}}\right) + O\left(\frac{N^2}{(\log N)^{(B/2)-5}}\right)\end{aligned}$$

where the implied constants depend only on $B, C$, and $\varepsilon$. For any $A > 0$, let $B := 2A + 10$ and $C := A + 5B$. Let $\varepsilon := 1/2$. Then

$$\min\left\{(1-\varepsilon)B, C - 5B, \frac{B}{2} - 5\right\} = A$$

and therefore

$$R(N,3,\mathbb{P}) = \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{\log^A N}\right)$$

As we have seen in 4.7 $c_1 < \mathfrak{S}(N) < c_2$ holds true for $c_1, c_2 > 0$. Therefore $R(N,3,\mathbb{P}) \ll N^2$, which is our desired result. [8, p. 228]

□

We can now derive Vinogradov's asymptotic formula for $r(N,3,\mathbb{P})$.

**Theorem 4.19.** *Vinogradov's asymptotic formula for $r(N,3,\mathbb{P})$*

*There exists an arithmetic function $\mathfrak{S}(N)$ (the singular series for the ternary Goldbach problem) and positive constants $c_1$ and $c_2$, such that*

$$c_1 < \mathfrak{S}(N) < c_2$$

*for all sufficiently large odd integers $N$, and*

$$r(N,3,\mathbb{P}) = \mathfrak{S}(N)\frac{N^2}{2\log^3(N)}\left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)$$

[8, p. 212]

*Proof.* In the first part of the proof our goal is to bound $R(N,3,\mathbb{P})$ from above and from below. In the second part of the proof the obtained estimates will be summarized and by some transformations the asymptotic expression for $r(N,3,\mathbb{P})$ can be derived. Recalling the definition of $R(N,3,\mathbb{P})$ 4.7 we get:

$$\begin{aligned} R(N,3,\mathbb{P}) &= \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \\ &\leq \sum_{p_1+p_2+p_3=N} \log^3(N) \\ &= \log^3(N) \sum_{p_1+p_2+p_3=N} 1 \\ &= \log^3(N) r(N,3,\mathbb{P}) \end{aligned}$$

This gives us our upper bound for $R(N,3,\mathbb{P})$. For the derivation of the lower bound we need some preliminary considerations. For $0 < \delta < \frac{1}{2}$ let $r_\delta(N,3,\mathbb{P})$

denote the number of representations of $N$ as a sum of three primes $N = p_1 + p_2 + p_3$, with $p_i \leq N^{1-\delta}$ for some $i = 1, 2, 3$.

$$r_\delta(N, 3, \mathbb{P}) := \sum_{\substack{p_1+p_2+p_3=N \\ p_1 \leq N^{1-\delta} \vee p_2 \leq N^{1-\delta} \vee p_3 \leq N^{1-\delta}}} 1$$

We can estimate $r_\delta(N, 3, \mathbb{P})$ as follows:

$$\sum_{\substack{p_1+p_2+p_3=N \\ p_1 \leq N^{1-\delta} \vee p_2 \leq N^{1-\delta} \vee p_3 \leq N^{1-\delta}}} 1 \leq 3 \cdot \sum_{\substack{p_1+p_2+p_3=N \\ p_1 \leq N^{1-\delta}}} 1 \ll \sum_{p_1 \leq N^{1-\delta}} \left( \sum_{p_1+p_2+p_3=N} 1 \right)$$

For the inner sum $p_1$ is fixed, therefore we get

$$\sum_{p_1 \leq N^{1-\delta}} \left( \sum_{p_1+p_2+p_3=N} 1 \right) = \sum_{p_1 \leq N^{1-\delta}} \left( \sum_{\substack{p_2,p_3: \\ p_2+p_3=N-p_1}} 1 \right)$$

Weakening the summation condition ($p_2 + p_3 = N - p_1 \Rightarrow p_2 < N$) the inner sum gets bigger and we have

$$\begin{aligned} \sum_{p_1 \leq N^{1-\delta}} \left( \sum_{\substack{p_2,p_3: \\ p_2+p_3=N-p_1}} \right) \quad &\leq \sum_{p_1 \leq N^{1-\delta}} \left( \sum_{p_2<N} 1 \right) = \left( \sum_{p_1 \leq N^{1-\delta}} 1 \right) \cdot \left( \sum_{p_2<N} 1 \right) \\ \leq \pi \left( N^{1-\delta} \right) \pi(N) \quad &\ll \frac{N^{1-\delta}}{\log(N^{1-\delta})} \frac{N}{\log(N)} = \frac{N^{2-\delta}}{(1-\delta)\log^2(N)} \end{aligned}$$

where we used the definition of the $\pi$-function 2.7 and the prime number Theorem 2.8 in the end.

And finally we get

$$r_\delta(N, 3, \mathbb{P}) \ll \frac{N^{2-\delta}}{\log^2(N)}$$

Now that we did those preliminary considerations we can find a lower bound for $R(N, 3, \mathbb{P})$.

$$\begin{aligned}
R(N,3,\mathbb{P}) &= \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \\
&\geq \sum_{\substack{p_1+p_2+p_3=N \\ p_i > N^{1-\delta}}} \log p_1 \log p_2 \log p_3 \\
&> \sum_{\substack{p_1+p_2+p_3=N \\ p_i > N^{1-\delta}}} (\log(N^{1-\delta}))^3 \\
&= (1-\delta)^3 \log^3(N) \sum_{\substack{p_1+p_2+p_3=N \\ p_i > N^{1-\delta}}} 1 \\
&= (1-\delta)^3 \log^3(N) \left(r(N,3,\mathbb{P}) - r_\delta(N,3,\mathbb{P})\right) \\
&\gg (1-\delta)^3 \log^3(N) \left(r(N,3,\mathbb{P}) - \frac{N^{2-\delta}}{\log^2(N)}\right)
\end{aligned}$$

We found our lower bound

$$(1-\delta)^3 \log^3(N) \left(r(N,3,\mathbb{P}) - \frac{N^{2-\delta}}{\log^2(N)}\right) \ll R(N,3,\mathbb{P})$$

Transfoming the equation we get

$$\log^3(N) r(N,3,\mathbb{P}) \ll \frac{R(N,3,\mathbb{P})}{(1-\delta)^3} + \log(N) N^{2-\delta}$$

.

Since $0 < \delta < \frac{1}{2}$, we know that $1 < \frac{1}{(1-\delta)^3} < 8$. Which leads to

$$0 < \frac{1-(1-\delta)^3}{(1-\delta)^3} < 8(1-(1-\delta)^3) = 24\delta - 24\delta^2 + 8\delta^3 < 24\delta$$

And finally we get

$$0 < \frac{1}{(1-\delta)^3} - 1 < 24\delta$$

Summarizing the lower and the upper bound for $R(N,3,\mathbb{P})$ we get

$$\begin{aligned}
0 &\leq \log^3(N) r(N,3,\mathbb{P}) - R(N,3,\mathbb{P}) \\
&\ll \frac{R(N,3,\mathbb{P})}{(1-\delta)^3} + \log(N) N^{2-\delta} - R(N,3,\mathbb{P}) \\
&= \left( \frac{1}{(1-\delta)^3} - 1 \right) R(N,3,\mathbb{P}) + \log(N) N^{2-\delta} \\
&< 24\delta R(N,3,\mathbb{P}) + \log(N) N^{2-\delta} \\
&\ll \delta R(N,3,\mathbb{P}) + \log(N) N^{2-\delta}
\end{aligned}$$

As we have seen in 4.18 $R(N,3,\mathbb{P}) \ll N^2$. Hence,

$$\delta R(N,3,\mathbb{P}) + \log(N) N^{2-\delta} \ll N^2 \left( \delta + \frac{\log(N)}{N^\delta} \right)$$

The above inequalities hold for all $\delta \in \left(0, \frac{1}{2}\right)$ and the implied constant does not depend on $\delta$. For $N$ big enough let $\delta := \frac{2\log(\log(N))}{\log(N)}$.

Then

$$\begin{aligned}
\delta + \frac{\log N}{N^\delta} &= \frac{2\log(\log N)}{\log N} + \frac{\log N}{N^{2\log_N(\log N)}} = \frac{2\log(\log N)}{\log N} + \frac{1}{\log N} \\
&\leq 2 \cdot \frac{\log(\log N)}{\log N} + \frac{\log(\log N)}{\log N} \ll \frac{\log(\log N)}{\log N}
\end{aligned}$$

and therefore

$$0 \leq \log^3(N) r(N,3,\mathbb{P}) - R(N,3,\mathbb{P}) \ll \frac{N^2 \log\log N}{\log N}$$

With $A \geq 1$ and by Theorem 4.18 we get

$$\begin{aligned}
\log^3(N) r(N,3,\mathbb{P}) &= R(N,3,\mathbb{P}) + O\left( \frac{N^2 \log\log N}{\log N} \right) \\
&= \mathfrak{S}(N) \frac{N^2}{2} + O\left( \frac{N^2}{\log^A N} \right) + O\left( \frac{N^2 \log\log N}{\log N} \right)
\end{aligned}$$

For $N$ big enough $1 < \log\log N$ and $A \geq 1$, hence

$$\log^3(N) r(N,3,\mathbb{P}) = \mathfrak{S}(N) \frac{N^2}{2} \left( 1 + O\left( \frac{\log\log N}{\log N} \right) \right)$$

Dividing by $\log^3 N$, we finally get

$$r(N,3,\mathbb{P}) = \mathfrak{S}(N) \frac{N^2}{2(\log N)^3} \left( 1 + O\left( \frac{\log\log N}{\log N} \right) \right),$$

which completes the proof.[8, pp. 228-230] □

# 5 The Circle Method Applied to Goldbach's Binary Conjecture

As we will see within this section the circle method unfortunately does not work for the binary case. We will show that the integral over the Minor arcs now contributes more than the integral over the Major arcs. We will approach the problem in the same way as in the previous section.

**Definition 5.1.**

$$r(N,2,\mathbb{P}) = \sum_{p_1+p_2=N} 1$$

and

$$R(N,2,\mathbb{P}) = \sum_{p_1+p_2=N} \log p_1 \log p_2$$

[7, p. 314]

Further we get

$$\begin{aligned} R(N,2,\mathbb{P}) &= \int_0^1 f_N(x)^2 e(-Nx)dx \\ &= \int_{\mathfrak{M}} f_N(x)^2 e(-Nx)dx + \int_{\mathfrak{m}} f_N(x)^2 e(-Nx)dx \end{aligned} \tag{5.1}$$

[7, p. 314]

## 5.1 Contribution of the Major arcs

**Definition 5.2.** The singular series for the binary Goldbach problem is defined as

$$\mathfrak{S}_2(N) = \sum_{q=1}^{\infty} \frac{\mu(q)^2 c_q(N)}{\varphi(q)^2}$$

where

$$c_q(N) = \sum_{\substack{a=1 \\ (q,a)=1}}^{q} e\left(\frac{aN}{q}\right)$$

is the Ramanujan sum.[4, p. 47]

As in Theorem 4.7 we have:

$$\mathfrak{S}_2(N,Q) = \sum_{q \leq Q} \frac{\mu(q)^2 c_q(N)}{\varphi(q)^2}$$

.

The singular series $\mathfrak{S}_2(N)$ also is a multiplicative function and it converges absolutely and uniformly in $N$. Again we let $\varepsilon > 0$ and we get:

$$\mathfrak{S}_2(N) - \mathfrak{S}_2(N,Q) = O\left(\frac{1}{Q^{1-\varepsilon}}\right).$$

The Euler product in this case is given by:

$$\begin{aligned}
\mathfrak{S}_2(N) &= \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\mu\left(p^j\right)^2 c_{p^j}(N)}{\varphi\left(p^j\right)^2}\right) \\
&= \prod_p \left(1 + \frac{c_p(N)}{\varphi(p)^2}\right) \\
&= \prod_{p \nmid N} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \left(1 + \frac{1}{p-1}\right) \\
&= \prod_p \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \left(\frac{1 + \frac{1}{p-1}}{1 - \frac{1}{(p-1)^2}}\right) \\
&= \prod_p \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \left(\frac{p-1}{p-2}\right).
\end{aligned}$$

[4, p. 47]

**Definition 5.3.** Let

$$u(\beta) = \sum_{m=1}^{N} e(m\beta)$$

Then the singular integral for Goldbach's binary Conjecture is defined as:

$$J(N) := \int_{-1/2}^{1/2} u(\beta)^2 e(-N\beta) d\beta$$

Proceeding as in Lemma 4.9 we get:

$$J(N) = \sum_{m_1+m_2=N} 1 = R_{1,2}(N) = \binom{N-1}{1} = N - 1 = N + O(1)$$

If we now want to evaluate our integral over the Major arcs we have to consider

$$\int_{\mathfrak{M}} f_N(x)^2 e(-Nx)dx$$

.

As in Lemma 4.5 we let $u(\beta) = \sum_{m=1}^{N} e(m\beta)$ and let $B$ and $C$ be positive real numbers with $C > 2B$. If $x \in \mathfrak{M}(a,q)$ and $\beta = x - a/q$, then

$$f_N(x) = \frac{\mu(q)}{\varphi(q)} u(\beta) + O\left(\frac{Q^2 N}{\log^C N}\right) = \frac{\mu(q)}{\varphi(q)} u(\beta) + O\left(\frac{N}{\log^{C-2B} N}\right)$$

and therefore we get

$$f_N(x)^2 = \frac{\mu(q)^2}{\varphi(q)^2} u(\beta)^2 + O\left(\frac{Q^2 N^2}{\log^C N}\right) = \frac{\mu(q)^2}{\varphi(q)^2} u(\beta)^2 + O\left(\frac{N^2}{\log^{C-2B} N}\right)$$

where the implied constants depend only on $B$ and $C$. [8, p. 217]

As in Theorem 4.10 we will have to approximate two integrals.

1)

$$\int_{\mathfrak{M}} \left(f_N(x)^2 - \frac{\mu(q)^2}{\varphi(q)^2} u(\beta)^2\right) e(-Nx)dx$$

2)

$$\int_{\mathfrak{M}} \frac{\mu(q)^2}{\varphi(q)^2} u\left(\beta\right)^2 e(-Nx)dx$$

Summing the estimates of 1) and 2) will lead to the desired result.

Recall that $x \in \mathfrak{M}_{(a,q)}$ implies that $\beta = x - \frac{a}{q}$. For the first integral we get:

1)

$$\begin{aligned}
&\left|\int_{\mathfrak{M}} \left(f_N(x)^2 - \frac{\mu(q)^2}{\varphi(q)^2} u\left(x - \frac{a}{q}\right)^2\right) e(-Nx)dx\right| \\
&= \left|\sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}(a,q)} \left(f_N(x)^2 - \frac{\mu(q)^2}{\varphi(q)^2} u\left(x - \frac{a}{q}\right)^2\right) e(-Nx)dx\right| \\
&\ll \sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}(a,q)} \frac{Q^2 N^2}{\log^C N} |e(-Nx)|dx \\
&\ll \sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \frac{Q^2 N^2}{\log^C N} \underbrace{\int_{\mathfrak{M}(a,q)} 1dx}_{\ll \frac{Q}{N}} \ll \sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \frac{Q^3 N}{\log^C N} \\
&\leq \frac{Q^5 N}{\log^C N} = \frac{N}{\log^{C-5B} N}.
\end{aligned}$$

Therefore we get that

$$\int_{\mathfrak{M}} f_N(x)^2 e(-Nx)dx = \int_{\mathfrak{M}} \frac{\mu(q)^2}{\varphi(q)^2} u\left(\beta\right)^2 e(-Nx)dx + O\left(\frac{N}{\log^{C-5B} N}\right)$$

.

2) Proceeding as we did in the proof of Theorem 4.10 we get that

$$\int_{\mathfrak{M}} \frac{\mu(q)^2}{\varphi(q)^2} u\left(\beta\right)^2 e(-Nx)dx = \mathfrak{S}_2(N,Q) \int_{-Q/N}^{Q/N} u(\beta)^2 e(-N\beta)d\beta$$

The singular integral evaluated form $-\frac{Q}{N}$ to $\frac{Q}{N}$ yields:

$$\int_{-Q/N}^{Q/N} u(\beta)^2 e(-N\beta)d\beta = N + O\left(\frac{N}{Q}\right)$$

And since we know that

$$\mathfrak{S}_2(N,Q) = \mathfrak{S}_2(N) + O\left(\frac{1}{Q^{1-\varepsilon}}\right)$$

and that $\mathfrak{S}_2(N)$ is bounded above by some positive constant $c_2$
we get:

$$\mathfrak{S}_2(N,Q) \int_{-Q/N}^{Q/N} u(\beta)^2 e(-N\beta)d\beta = \left(\mathfrak{S}_2(N) + O\left(\frac{1}{Q^{1-\varepsilon}}\right)\right)\left(N + O\left(\frac{N}{Q}\right)\right)$$
$$= N\mathfrak{S}_2(N) + O\left(\frac{N}{Q}\right) + O\left(\frac{N}{Q^{1-\varepsilon}}\right) + O\left(\frac{N}{Q \cdot Q^{1-\varepsilon}}\right)$$
$$= N\mathfrak{S}_2(N) + O\left(\frac{N}{Q^{1-\varepsilon}}\right) = N \cdot \mathfrak{S}_2(N) + O\left(\frac{N}{\log^{(1-\varepsilon)B} N}\right)$$

We finally get that

$$\int_{\mathfrak{M}} f_N(x)^2 e(-Nx)dx = N \cdot \mathfrak{S}_2(N) + O\left(\frac{N}{\log^{(1-\varepsilon)B} N}\right) + O\left(\frac{N}{\log^{C-5B} N}\right).$$

We can see that the order of the main and the order of the error term decreased by 1.

## 5.2 Contribution of the Minor Arcs

Now we shall estimate the integral over the Minor Arcs. In the proof of Theorem 4.17 we have seen that

$$f_N(x) \ll \frac{N}{\log^{(B/2)-4} N}$$

and from Lemma 4.2 we know that

$$|f_N(x)| \leq N + o(N).$$

Therefore we can conclude that

$$\begin{aligned}
\int_{\mathfrak{m}} f_N(x)^2 e(-Nx)dx &\leq \int_{\mathfrak{m}} |f_N(x)|^2 dx \\
&\leq \sup\{|f_N(x)| : x \in \mathfrak{m}\} \int_{\mathfrak{m}} |f_N(x)|dx \\
&\ll \frac{N}{\log^{(B/2)-4} N} \int_0^1 |f_N(x)|dx \\
&\ll \frac{N^2}{\log^{(B/2)-4} N}
\end{aligned}$$

As we can see, the error term of the contribution of the Minor arcs still is of order $\frac{N^2}{\log^A N}$, for some $A > 0$, which means that the Minor arcs contribute more than the Major arcs.

Another approach would be to apply the Cauchy-Schwarz inequality to get a different bound for $\int_0^1 |f_N(x)|dx$. In this case we get:

$$\begin{aligned}
\int_{\mathfrak{m}} f_N(x)^2 e(-Nx)dx &\leq \sup\{|f_N(x)| : x \in \mathfrak{m}\} \int_0^1 |f_N(x)|dx \\
&\leq \sup\{|f_N(x)| : x \in \mathfrak{m}\} \left(\int_0^1 |f_N(x)|^2\right)^{\frac{1}{2}} \left(\int_0^1 |1|^2\right)^{\frac{1}{2}} \\
&\ll \frac{N}{\log^{(B/2)-4} N} \cdot (N \log N)^{\frac{1}{2}} \\
&= \frac{N^{3/2}}{\log^{(B/2)-9/2} N}
\end{aligned}$$

[7, p. 324]

Unfortunately the contribution of the Minor arcs is still larger than the contribution of the Major arcs. Therefore we have to conclude that the circle method does not work for Goldbach's binary Conjecture.

# 6 The Circle Method Applied to Counting Rational Points on Quartic Hypersurfaces

A Diophantine equation is an equation of the form

$$f(x_1, \ldots, x_n) = 0, \tag{6.1}$$

where f is a polynomial with integer coefficients. We want to study the set of integer solutions to this equation. More specifically we want to estimate the number of solutions to such an equation. To measure the size of the solution set we define a counting function as follows:

Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d$ and let

$$N(f; B) := \#\{\mathbf{x} \in \mathbb{Z}^n : f(\mathbf{x}) = 0, \|\mathbf{x}\| \leqslant B\}$$

for any $B \geqslant 1$.

Our goal is to understand how the counting function $N(f; B)$ behaves, as $B \to \infty$. Here $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{>0}$ is an arbitrary choice of norm. We will reserve $|\cdot|$ for the norm $|\mathbf{x}| := \max_{1 \leqslant i \leqslant n} |x_i|$, for any $\mathbf{x} \in \mathbb{R}^n$. [2, p. 1]

The study of the function $N(f; B)$ often helps to decide whether or not the equation $f = 0$ has any non-trivial integer solutions at all. Applying the Hardy-Littlewood circle method one is often able to prove that the set of non-zero integer solutions to the equation $f = 0$ is infinite by showing that $N(f; B) \to \infty$ as $B \to \infty$.[2, p.2]

## 6.1 A Naive Heuristic

Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geqslant 1$.

Then for the vectors $\mathbf{x} \in \mathbb{Z}^n$ counted by $N(f; B)$, the values of $f(\mathbf{x})$ will be of order $B^d$. Therefore the probability for a randomly chosen value of $f(\mathbf{x})$ to vanish might be expected to be of order $B^{-d}$. Due to the fact that the number of $\mathbf{x}$ to be considered has order $B^n$, this leads us to the following "naive heuristic":

**Heuristic** When $n \geqslant d$ we have

$$B^{n-d} \ll N(f; B) \ll B^{n-d}.$$

[2, p. 2]

From this heuristic we can conclude that we might expect polynomials whose degree is less than the number of variables to have infinitely many solutions.

Of course there are many reasons for this heuristic expectation not to hold. For example, if the set of non-zero integer solutions to the equation $f = 0$ is empty, then $N(f; B)$ would be identical to zero no matter the value of $d$ and $n$.[2, p. 2] We do not want to get into the details of when this heuristic fails. Instead we shall look at statements supporting it, such as the following Theorem based on Birch's work.[1]

**Theorem 6.1.** *Suppose $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is a non-singular homogeneous polynomial of degree $d$ in $n > (d-1)2^d$ variables. Assume that $f(\mathbf{x}) = 0$ has non-trivial solutions in $\mathbb{R}$ and each $p$ -adic field $\mathbb{Q}_p$. Then there is a constant $c_f > 0$ such that*

$$N(f; B) \sim c_f B^{n-d}$$

*as $B \to \infty$.* [2, p. 4]

This Theorem is established using the circle method. Therefore the number of variables $n$ needs to be large compared to the degree $d$.

## 6.2 The Relation of the Counting Functions

Phrasing things in the language of algebraic geometry, equation 6.1 defines a hypersurface $X$ in the affine space $\mathbb{A}^n$. The set of integral solutions of 6.1 can be viewed as the intersection of $X$ with the integral lattice $\mathbb{Z}^n$.

In many cases working with only one equation may be too restrictive. Therefore we want to define a counting function for arbitrary projective varieties $V \subseteq \mathbb{P}^{n-1}$. Those varieties are assumed to be cut out by a finite system of homogeneous equations defined over $\mathbb{Q}$. From now on irreducible will mean that the variety is geometrically reduced and irreducible. When a variety is cut out by a single equation this is equivalent to the underlying polynomial being irreducible over the complex numbers.

The main interest lies upon varieties $V$ for which we expect the set $V(\mathbb{Q}) = V \cap \mathbb{P}^{n-1}(\mathbb{Q})$ to be infinite. Let $x = [\mathbf{x}] \in \mathbb{P}^{n-1}(\mathbb{Q})$ be a projective rational point, with $\mathbf{x} \in \mathbb{Z}^n$ chosen such that $(x_1, \ldots, x_n) = 1$. Then we define the height of $x$ to be

$$H(x) := \|\mathbf{x}\|$$

This therefore defines a function $H : \mathbb{P}^{n-1}(\mathbb{Q}) \to \mathbb{R}_{>0}$, and is none other than the exponential height function metrized by the choice of norm $\|\cdot\|$. We may further define the counting function

$$N_V(B) := \#\{x \in V(\mathbb{Q}) : H(x) \leqslant B\}$$

for each $B \geqslant 1$.[2, p. 5]

Since our main interest is to count rational points on quartic hypersurfaces we will show how to relate $N(f;B)$ and $N_V(B)$, when $V \subset \mathbb{P}^{n-1}$ is a hypersurface with underlying form $f \in \mathbb{Z}[x_1,\ldots,x_n]$. Since $f$ is a homogeneous polynomial any multiple of a solution to 6.1 is also a solution to 6.1. But looking at $N_V(B)$ we are only interested in primitive integer solutions, which means that the components of the vector $\mathbf{x} \in \mathbb{Z}^n$ should share no common prime factors. The advantage of $N_V(B)$ is that all scalar multiples of a given non-zero integer solution are treated as a single point. To be able to relate $N(f;B)$ and $N_V(B)$ we will need the Möbius function, and more specifically Theorem 2.5. Due to the fact that $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in $\mathbb{P}^{n-1}$, we get

$$\begin{aligned} N_V(B) &= \frac{1}{2}\#\left\{\mathbf{x} \in \mathbb{Z}^n : f(\mathbf{x}) = 0, \|\mathbf{x}\| \leqslant B, (x_1,\ldots,x_n) = 1\right\} \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\mu(k)\#\left\{\mathbf{x} \in \mathbb{Z}^n : f(\mathbf{x}) = 0, \|\mathbf{x}\| \leqslant B, k \mid x_i \forall i\right\}. \end{aligned}$$

And through a change of variables we get:

$$\begin{aligned} N_V(B) &= \frac{1}{2}\sum_{k=1}^{\infty}\mu(k)\#\left\{\mathbf{x} \in \mathbb{Z}^n \backslash \{\mathbf{0}\} : f(\mathbf{x}) = 0, \|\mathbf{x}\| \leqslant k^{-1}B\right\} \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\mu(k)N\left(f;k^{-1}B\right) \end{aligned}$$

[2, p.6]

To get from the asymptotic formula for $N(f;B)$ to the asymptotic formula for $N_V(B)$ we need the following relation:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty}\frac{\mu(n)}{n^s} \tag{6.2}$$

[9, p. 97]

Let $V \subset \mathbb{P}^{n-1}$ be a hypersurface defined by a non-singular homogeneous polynomial $f$ of degree $d$, with $n > (d-1)2^d$.

Using Theorem 6.1 we get:

$$\begin{aligned} N_V(B) &= \frac{1}{2}\sum_{k=1}^{\infty}\mu(k)N\left(f;k^{-1}B\right) \\ &\sim \frac{1}{2}\sum_{k=1}^{\infty}\mu(k)c_f\left(\frac{B}{k}\right)^{n-d} \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{\mu(k)}{k^{n-d}}c_f B^{n-d} \\ &= \frac{1}{2}\frac{1}{\zeta(n-d)}c_f B^{n-d} \end{aligned}$$

Therefore we can deduce that:

$$\text{if } N(f;B)\sim c_f B^{n-d}, \text{then} \quad N_V(B)\sim \frac{1}{2\zeta(n-d)}c_f B^{n-d}. \tag{6.3}$$

[2, p. 15]

## 6.3 The Generating Function $S(\alpha)$

To be able to apply the Hardy-Littlewood circle method we will be working with the counting function $N(f;B)$ and as we have seen, it is a simple matter to switch from $N(f;B)$ to $N_V(B)$ as long as the exponent of $B$ in the asymptotic formula for $N(f;B)$ is bigger than 1.

Let $f\in\mathbb{Z}[x_1,\ldots,x_n]$ be a homogeneous polynomial of degree $d$ and let

$$N(f;B) := \#\left\{\mathbf{x}\in\mathbb{Z}^n : f(\mathbf{x})=0, |\mathbf{x}|\leqslant B\right\}$$

for any $B\geqslant 1$.

Furthermore let $e(z) := e^{2\pi i z}$ and $e_q(z) := e\left(\frac{z}{q}\right)$, for any $z\in\mathbb{R}$.

In this section we will allow our implied constants to depend on the equations that are being considered. To be able to apply the circle method we need a generating function.

**Definition 6.2.**

$$S(\alpha) = S_B(\alpha;f) := \sum_{\mathbf{x}\in\mathbb{Z}^n\cap[-B,B]^n} e(\alpha f(\mathbf{x})),$$

with $\alpha\in[0,1]$ being a real variable. [2, pp. 123-124]

Using the orthogonality relation

$$\int_0^1 e(\alpha n)d\alpha = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

we can write $N(f;B)$ as an integral over $S(\alpha)$ as follows:

$$N(f;B) = \int_0^1 S(\alpha)d\alpha \tag{6.4}$$

[2, p. 124]

## 6.4 The Size of $S(\alpha)$

Due to the central limit Theorem we expect square root cancellation for the **average size** of $S(\alpha)$, with $0 < \alpha < 1$. Therefore we assume that $S(\alpha)$ should roughly be of order $B^{n/2}$. But from the naive heuristic 6.1 we know that the main term in any asymptotic formula is supposed to be of order $B^{n-d}$. Hence we can only apply the circle method if $n > 2d$.

The cubic exponential sum $S(\alpha) = S_B(\alpha; f)$ can be large near rationals $\frac{a}{q}$ with small denominators.

We now define the complete exponential sums

$$S_{q,a} := \sum_{\mathbf{x}\in(\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x})) \tag{6.5}$$

[2, p. 125]

$e_q(af(\mathbf{x})) = e_q(af(\mathbf{y}))$ holds for $\mathbf{x} \equiv \mathbf{y}(\mathrm{mod}q)$, such that

$$S\left(\frac{a}{q}\right) = \sum_{\mathbf{x}\in(\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x}))\#\left\{\mathbf{y} \in \mathbb{Z}^n : |\mathbf{y}| \leqslant B, \mathbf{y} \equiv \mathbf{x}(\mathrm{mod}q)\right\}$$

Using the fact that for $b \geqslant a$ and $q \in \mathbb{N}$, we get

$$\#\left\{n \in \mathbb{Z} \cap [a,b] : n \equiv n_0(\mathrm{mod}q)\right\} = \frac{b-a}{q} + O(1).$$

We can conclude that the cardinality in the sum is

$$\left(\frac{2B}{q} + O(1)\right)^n = \frac{2^n B^n}{q^n} + O\left(B^{n-1}\right)$$

subsequently

$$\begin{aligned} S\left(\frac{a}{q}\right) &= \sum_{\mathbf{x}\in(\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x}))\#\left\{\mathbf{y}\in\mathbb{Z}^n : |\mathbf{y}|\leqslant B, \mathbf{y}\equiv\mathbf{x}(\mathrm{mod}q)\right\} \\ &= \left(\sum_{\mathbf{x}\in(\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x}))\right)\cdot\left(\frac{2^nB^n}{q^n}+O\left(B^{n-1}\right)\right) \\ &= S_{q,a}\cdot\left(\frac{2^nB^n}{q^n}+O\left(B^{n-1}\right)\right) \\ &= \frac{2^nB^n}{q^n}S_{q,a}+O\left(q^nB^{n-1}\right) \end{aligned}$$

[2, p. 125]

We can deduce that for any fixed fraction $\frac{a}{q}$, $S\left(\frac{a}{q}\right)$ will be of exact order $B^n$ if $S_{q,a}\neq 0$.

Now we want to conclude that $S(\alpha)$ is still of exact order $B^n$ when $\alpha$ is near such a rational $\frac{a}{q}$ with small denominator.

To do so we let $a,q\in\mathbb{Z}$ such that $1\leqslant a\leqslant q\leqslant B$ and $(a,q)=1$. We write $\alpha=\frac{a}{q}+z$ and, as above, we break the sum into congruence classes modulo $q$ to deduce that

$$S\left(\frac{a}{q}+z\right)=\sum_{\mathbf{x}\in(\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x}))\sum_{\substack{\mathbf{y}\in\mathbb{Z}^n\cap[-B,B]^n\\ \mathbf{y}\equiv\mathbf{x}(\mathrm{\ mod\ }q)}} e(zf(\mathbf{y})) \tag{6.6}$$

Our goal is to replace the discrete variable $\mathbf{y}$ by a continuous one in the inner sum, and the summation over $\mathbf{y}$ by an integral. For this we will need the following Lemma.

**Lemma 6.3.** *Let $P\geqslant 1$, let $\mathbf{a}\in\mathbb{Z}^n$ and let $r\in\mathbb{N}$ such that $r\leqslant P$. Let $F$ be a function on $\mathbb{R}^n$ all of whose first order partial derivatives exist and are continuous on $\mathscr{R}:=[-P,P]^n$. Define*

$$M_F:=\sup_{\mathbf{x}\in\mathscr{R}}\max_{1\leqslant i\leqslant n}\left|\frac{\partial F}{\partial x_i}(\mathbf{x})\right|$$

*Then we have*

$$\sum_{\substack{\mathbf{x}\in\mathbb{Z}^n\cap\mathscr{R}\\ \mathbf{x}\equiv\mathbf{a}(\mathrm{\ mod\ }r)}} e(F(\mathbf{x}))=\frac{1}{r^n}\int_{\mathscr{R}} e(F(\mathbf{t}))\mathrm{d}\mathbf{t}+O\left(\frac{P^{n-1}\left(1+PM_F\right)}{r^{n-1}}\right)$$

[2, p. 126]

The proof of this Lemma is based on the Euler-Maclaurin summation formula and can be found in [2] on page 126.

To apply Lemma 6.3 to equation 6.6 we let $\mathscr{R} := [-B, B]^n$ and $F = zf$, where $f$ is a homogeneous polynomial of degree $d$. In that case $M_F = |z|B^{d-1}$. Therefore we get

$$\begin{aligned} S\left(\frac{a}{q}+z\right) &= \sum_{\mathbf{x}\in(\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x})) \sum_{\substack{\mathbf{y}\in\mathbb{Z}^n\cap[-B,B]^n \\ \mathbf{y}\equiv\mathbf{x}(\bmod q)}} e(zf(\mathbf{y})) \\ &= S_{q,a}\left(\frac{1}{q^n}\int_{[-B,B]^n} e(zf(\mathbf{t}))\mathrm{d}\mathbf{t} + O\left(\frac{B^{n-1}\left(1+B|z|B^{d-1}\right)}{q^{n-1}}\right)\right), \end{aligned} \tag{6.7}$$

where $S_{q,a}$ is given by 6.5. Further we substitute $\mathbf{t}$ by $B\mathbf{u}$ and we get:

$$\begin{aligned} &= S_{q,a}\left(\frac{1}{q^n}\int_{[-1,1]^n} e(zf(B\mathbf{u}))B^n\mathrm{d}\mathbf{u} + O\left(\frac{B^{n-1}\left(1+B|z|B^{d-1}\right)}{q^{n-1}}\right)\right) \\ &= S_{q,a}\left(\frac{B^n}{q^n}\int_{[-1,1]^n} e(zB^df(\mathbf{u}))\mathrm{d}\mathbf{u} + O\left(\frac{B^{n-1}\left(1+|z|B^{d}\right)}{q^{n-1}}\right)\right) \\ &= \frac{B^n}{q^n}S_{q,a}\int_{[-1,1]^n} e(zB^df(\mathbf{u}))\mathrm{d}\mathbf{u} + O\left(q^n\frac{B^{n-1}\left(1+|z|B^{d}\right)}{q^{n-1}}\right) \end{aligned}$$

Due to these calculations we finally get the asymptotic formula for $S(\alpha)$, when $\alpha$ is close to a rational number with small denominator.

**Lemma 6.4.** *Let $a, q \in \mathbb{Z}$ such that $1 \leqslant a \leqslant q \leqslant B$ and $(a, q) = 1$. Then we have*

$$S\left(\frac{a}{q}+z\right) = q^{-n}B^nS_{q,a}I\left(zB^d\right) + O\left(qB^{n-1}\left(1+|z|B^d\right)\right)$$

*where*

$$I(\gamma) := \int_{\mathbf{t}\in[-1,1]^n} e(\gamma f(\mathbf{t}))\mathrm{d}\mathbf{t}$$

[2, p. 127]

Therefore we can conclude that for fixed $\frac{a}{q}$, with $S_{q,a} \neq 0$, $S(\frac{a}{q}+z)$ has exact order $B^n$ for sufficiently small $z$. Due to that we decide to split our integral over $[0, 1]$ in 6.4 into the "Major arcs" and the "Minor arcs". In the Major arcs $\alpha$ shall be close to a rational number with small denominator. And the Minor arcs will be the rest of $[0, 1]$. Our goal is to show that $S(\alpha)$ is small over the Minor arcs.

## 6.5 Definition of the Major and Minor Arcs

For homogeneous polynomials of degree $d$ we define the Major arcs as follows:

$$\mathfrak{M}_{q,a}(\Delta) = \left[\frac{a}{q} - B^{-d+\Delta}, \frac{a}{q} + B^{-d+\Delta}\right] \tag{6.8}$$

[2, p. 128]

for $1 \leqslant a \leqslant q$ with $(a, q) = 1$ and $q \leqslant B^{\Delta}$.

With $\Delta$ being a small positive parameter.

The full set of Major arcs is given by

$$\mathfrak{M}(\Delta) = \bigcup_{1 \leqslant q \leqslant B^{\Delta}} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_{q,a}(\Delta) \tag{6.9}$$

[2, p. 128]

We define the Minor arcs modulo 1 as: $\mathfrak{m}(\Delta) = [0, 1] \backslash \mathfrak{M}(\Delta)$.

**Lemma 6.5.** *Assume that $\Delta < \frac{d}{3}$. Then the intervals $\mathfrak{M}_{q,a}(\Delta)$ are disjoint if $B > 2^{\frac{1}{d-3\Delta}}$.*

[2, p. 128]

This Lemma shows that the intervals in $\mathfrak{M}(\Delta)$ do not overlap for sufficiently small $\Delta$.

## 6.6 Quartic Hypersurfaces

We now let $d = 4$ and we let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a non-singular, quartic, homogeneous polynomial. Assume that

$$f(\mathbf{x}) = \sum_{i,j,k,\ell=1}^{n} f_{ijk\ell} x_i x_j x_k x_\ell$$

for $f_{ijk\ell} \in \mathbb{Z}$ that are symmetric in the indices. If we want $f_{ijk\ell} \in \mathbb{Z}$ to hold, it is possible that we might have to replace $f \in \mathbb{Z}[x_1, \ldots, x_n]$ by a multiple.

We may then define the trilinear forms

$$L_i(\mathbf{w}; \mathbf{x}; \mathbf{y}) := 4! \sum_{j,k,\ell=1}^{n} f_{ijk\ell} w_j x_k y_\ell \tag{6.10}$$

for $1 \leqslant i \leqslant n$.[2, p. 128]

The goal of this section will be to show that

$$N(f;B) = c_f B^{n-4}(1+o(1)) \tag{6.11}$$

is the asymptotic formula for $N(f;B)$, when $n > 48$ and to define the constant $c_f$.

**Lemma 6.6.** *Define the affine algebraic variety*

$$V = \left\{(\mathbf{w},\mathbf{x},\mathbf{y}) \in \mathbb{A}^{3n} : L_i(\mathbf{w};\mathbf{x};\mathbf{y}) = 0 \forall i \leqslant n\right\}$$

*Then we have* $\dim V \leqslant 2n$.

Our Major arcs are now defined as:

$$\mathfrak{M}_{q,a}(\Delta) = \left[\frac{a}{q} - B^{-4+\Delta}, \frac{a}{q} + B^{-4+\Delta}\right] \tag{6.12}$$

for $1 \leqslant a \leqslant q$ with $(a,q) = 1$ and $q \leqslant B^{\Delta}$.

We will assume that $0 < \Delta < 1$ and $B \gg 1$, so that Lemma 6.5 ensures the intervals $\mathfrak{M}_{q,a}(\Delta)$ do not overlap.

And as before the full set of Major arcs is given by

$$\mathfrak{M}(\Delta) = \bigcup_{1 \leqslant q \leqslant B^{\Delta}} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_{q,a}(\Delta) \tag{6.13}$$

We now want to compute the volume of the Major arcs.

$$|\mathfrak{M}_{q,a}(\Delta)| = 2 \cdot B^{-4+\Delta}$$

$$\begin{aligned}
|\mathfrak{M}(\Delta)| &= \left| \bigcup_{1 \leqslant q \leqslant B^{\Delta}} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_{q,a}(\Delta) \right| \leq \sum_{q=1}^{B^{\Delta}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |\mathfrak{M}_{q,a}(\Delta)| \\
&= \sum_{q=1}^{B^{\Delta}} q \cdot 2 \cdot B^{-4+\Delta} \\
&= 2 \cdot B^{-4+\Delta} \cdot \frac{1}{2} B^{\Delta}\left(B^{\Delta}+1\right) \\
&= B^{-4+3\Delta} + B^{-4+2\Delta}
\end{aligned}$$

Therefore the volume of $\mathfrak{M}(\Delta)$ is $O\left(B^{-4+3\Delta}\right)$.

## 6.7 Contribution of the Minor Arcs

We will handle the Minor arcs in the same way as Birch did it, by applying Weyl-differencing.

We start with a bound for $|S(\alpha)|^2$.

$$|S(\alpha)|^2 = \overline{S(\alpha)} \cdot S(\alpha) = \sum_{\mathbf{x} \in \mathbb{Z}^n \cap [-B,B]^n} e(-\alpha f(\mathbf{x})) \sum_{\mathbf{x}' \in \mathbb{Z}^n \cap [-B,B]^n} e(\alpha f(\mathbf{x}'))$$

For $\mathbf{x}' = \mathbf{x} + \mathbf{w}$ we get:

$$\begin{aligned} |S(\alpha)|^2 &= \sum_{|\mathbf{w}| \leqslant B} \sum_{\substack{\mathbf{x} \in [-B,B]^n \\ \mathbf{x}+\mathbf{w} \in [-B,B]^n}} e(\alpha(f(\mathbf{w}+\mathbf{x}) - f(\mathbf{x}))) \\ &\leq \sum_{|\mathbf{w}| \leqslant B} \left| \sum_{\mathbf{x} \in \mathscr{R}(\mathbf{w})} e(\alpha(f(\mathbf{w}+\mathbf{x}) - f(\mathbf{x}))) \right|, \end{aligned}$$

where $\mathscr{R}(\mathbf{w}) := \{\mathbf{x} \in [-B,B]^n : \mathbf{w} + \mathbf{x} \in [-B,B]^n\}$.[2, p. 129]

From this we would like to get an estimation for $|S(\alpha)|^4$, using Weyl-differencing and Cauchy's inequality.

$$\begin{aligned} |S(\alpha)|^4 = \left(|S(\alpha)|^2\right)^2 &\leq \left( \sum_{|\mathbf{w}| \leqslant B} \left| \sum_{\mathbf{x} \in \mathscr{R}(\mathbf{w})} e(\alpha(f(\mathbf{w}+\mathbf{x}) - f(\mathbf{x}))) \right| \right)^2 \\ &\leq 2^n B^n \sum_{|\mathbf{w}| \leqslant B} \left| \sum_{\mathbf{x} \in \mathscr{R}(\mathbf{w})} e(\alpha(f(\mathbf{w}+\mathbf{x}) - f(\mathbf{x}))) \right|^2 \end{aligned}$$

We define:

$$S_{\mathbf{w}}(\alpha) := \sum_{\mathbf{x} \in \mathscr{R}(\mathbf{w})} e(\alpha(f(\mathbf{w}+\mathbf{x}) - f(\mathbf{x})))$$

Then we get:

$$\begin{aligned} &\left| \sum_{\mathbf{x} \in \mathscr{R}(\mathbf{w})} e(\alpha(f(\mathbf{w}+\mathbf{x}) - f(\mathbf{x}))) \right|^2 = |S_{\mathbf{w}}(\alpha)|^2 = \overline{S_{\mathbf{w}}(\alpha)} \cdot S_{\mathbf{w}}(\alpha) \\ = &\sum_{\mathbf{y} \in \mathscr{R}(\mathbf{w})} e(-\alpha(f(\mathbf{w}+\mathbf{y}) - f(\mathbf{y}))) \cdot \sum_{\mathbf{y}' \in \mathscr{R}(\mathbf{w})} e(\alpha(f(\mathbf{w}+\mathbf{y}') - f(\mathbf{y}'))) \end{aligned}$$

Substituting $\mathbf{y}'$ by $\mathbf{x}+\mathbf{y}$ we get:

$$
\begin{aligned}
&= \sum_{\mathbf{x},\mathbf{y}\in\mathscr{R}(\mathbf{w})} e\left(\alpha\left(f(\mathbf{w}+\mathbf{x}+\mathbf{y})-f(\mathbf{x}+\mathbf{y})-f(\mathbf{w}+\mathbf{y})+f(\mathbf{y})\right)\right)\\
&= \sum_{|x|\leq B}\left|\sum_{\mathbf{y}\in\mathscr{S}(\mathbf{w},\mathbf{x})} e\left(\alpha\left(f(\mathbf{w}+\mathbf{x}+\mathbf{y})-f(\mathbf{x}+\mathbf{y})-f(\mathbf{w}+\mathbf{y})+f(\mathbf{y})\right)\right)\right|
\end{aligned}
$$

with $\mathscr{S}(\mathbf{w},\mathbf{x}) := \{\mathbf{y}\in[-B,B]^n : \mathbf{w}+\mathbf{x}+\mathbf{y}\in[-B,B]^n\}$

Therefore we get to:

$$
|S(\alpha)|^4 \ll B^n \sum_{|\mathbf{w}|,|\mathbf{x}|\leqslant B}\left|\sum_{\mathbf{y}\in\mathscr{S}(\mathbf{w},\mathbf{x})} e(\alpha f(\mathbf{w},\mathbf{x};\mathbf{y}))\right| \tag{6.14}
$$

where

$$
f(\mathbf{w},\mathbf{x};\mathbf{y}) := f(\mathbf{w}+\mathbf{x}+\mathbf{y})-f(\mathbf{w}+\mathbf{y})-f(\mathbf{x}+\mathbf{y})+f(\mathbf{y})
$$

[2, p. 130]

Further we define:

$$
S_{\mathbf{w},\mathbf{x}}(\alpha) := \sum_{\mathbf{y}\in\mathscr{S}(\mathbf{w},\mathbf{x})} e(\alpha f(\mathbf{w},\mathbf{x};\mathbf{y}))
$$

And we get:

$$
\begin{aligned}
|S(\alpha)|^8 = (|S(\alpha)|^4)^2 &\ll \left(B^n \sum_{|\mathbf{w}|,|\mathbf{x}|\leqslant B}|S_{\mathbf{w},\mathbf{x}}(\alpha)|\right)^2\\
&\leq B^{2n}(2B)^n(2B)^n\,|S_{\mathbf{w},\mathbf{x}}(\alpha)|^2 \ll B^{4n}\,|S_{\mathbf{w},\mathbf{x}}(\alpha)|^2
\end{aligned}
$$

We will apply the Weyl-differencing one more time to the inner sum in 6.14. Hence:

$$
\begin{aligned}
|S_{\mathbf{w},\mathbf{x}}(\alpha)|^2 &= \overline{S_{\mathbf{w},\mathbf{x}}(\alpha)}\cdot S_{\mathbf{w},\mathbf{x}}(\alpha)\\
&= \sum_{\mathbf{z}\in\mathscr{S}(\mathbf{w},\mathbf{x})} e(-\alpha f(\mathbf{w},\mathbf{x};\mathbf{z}))\cdot\sum_{\mathbf{z}'\in\mathscr{S}(\mathbf{w},\mathbf{x})} e(\alpha f(\mathbf{w},\mathbf{x};\mathbf{z}'))\\
&= \sum_{\mathbf{z}',\mathbf{z}\in\mathscr{S}(\mathbf{w},\mathbf{x})} e(\alpha(f(\mathbf{w},\mathbf{x};\mathbf{z}')-f(\mathbf{w},\mathbf{x};\mathbf{z})))
\end{aligned}
$$

$$f(\mathbf{w},\mathbf{x};\mathbf{z}') - f(\mathbf{w},\mathbf{x};\mathbf{z}) = f(\mathbf{w}+\mathbf{x}+\mathbf{z}') - f(\mathbf{w}+\mathbf{z}') - f(\mathbf{x}+\mathbf{z}') + f(\mathbf{z}') \\ -f(\mathbf{w}+\mathbf{x}+\mathbf{z}) + f(\mathbf{w}+\mathbf{z}) + f(\mathbf{x}+\mathbf{z}) - f(\mathbf{z})$$

Substituting $\mathbf{z}' = \mathbf{y}+\mathbf{z}$ we get:

$$f(\mathbf{w},\mathbf{x},\mathbf{y};\mathbf{z}) := f(\mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z}) - f(\mathbf{w}+\mathbf{y}+\mathbf{z}) - f(\mathbf{x}+\mathbf{y}+\mathbf{z}) + f(\mathbf{y}+\mathbf{z}) \\ -f(\mathbf{w}+\mathbf{x}+\mathbf{z}) + f(\mathbf{w}+\mathbf{z}) + f(\mathbf{x}+\mathbf{z}) - f(\mathbf{z})$$

Therefore we can conclude that:

$$|S(\alpha)|^8 \ll B^{4n} \sum_{|\mathbf{w}|,|\mathbf{x}|,|\mathbf{y}|\leqslant B} \left| \sum_{\mathbf{z}\in\mathscr{T}(\mathbf{w},\mathbf{x},\mathbf{y})} e(\alpha f(\mathbf{w},\mathbf{x},\mathbf{y};\mathbf{z})) \right| \tag{6.15}$$

with $\mathscr{T}(\mathbf{w},\mathbf{x},\mathbf{y}) := \{\mathbf{z}\in[-B,B]^n : \mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z}\in[-B,B]^n\}$

To be able to relate $f(\mathbf{w},\mathbf{x},\mathbf{y};\mathbf{z})$ to the trilinear forms 6.10 we need to define the following:

$$F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i,j,k,\ell=1}^{n} f_{ijkl} w_i x_j y_k z_\ell$$

Then $f(\mathbf{x}) = F(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x})$ and since
$F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}) = F(\text{any permutation of } \mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z})$ we get:

$$4!F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i=1}^{n} z_i L_i(\mathbf{w};\mathbf{x};\mathbf{y}),$$

by pulling out the $z$ coefficient in each multinomial.

Due to the fact that $F$ is multi-linear we can evaluate

$$f(\mathbf{w},\mathbf{x},\mathbf{y};\mathbf{z}) = f(\mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z}) - \dots \\ =F(\mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z},\mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z},\mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z},\mathbf{w}+\mathbf{x}+\mathbf{y}+\mathbf{z}) - \dots$$

by expanding multi-linearly. The first term would expand into $4^4$ terms. $f(\mathbf{w}+\mathbf{x}+\mathbf{y})$, $f(\mathbf{w}+\mathbf{y}+\mathbf{z})$ and $f(\mathbf{x}+\mathbf{y}+\mathbf{z})$ would each expand into $3^4$ terms etc. Due to the fact that $\mathbf{w},\mathbf{x}$ and $\mathbf{y}$ play a symmetric role we only have left

terms of the form $F(\mathbf{w},\mathbf{w},\mathbf{x},\mathbf{y}), F(\mathbf{w},\mathbf{x},\mathbf{x},\mathbf{y}), F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{y})$ and $F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z})$ after expanding all of $f(\mathbf{w},\mathbf{x},\mathbf{y};\mathbf{z})$.

Since $F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}) = F(\text{any permutation of } \mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z})$ we get:

$$\begin{aligned} f(\mathbf{w},\mathbf{x},\mathbf{y};\mathbf{z}) &= 4!F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}) + \frac{4!}{2!}F(\mathbf{w},\mathbf{w},\mathbf{x},\mathbf{y}) + \frac{4!}{2!}F(\mathbf{w},\mathbf{x},\mathbf{x},\mathbf{y}) + \frac{4!}{2!}F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{y}) \\ &= \sum_{i=1}^{n} z_i L_i(\mathbf{w};\mathbf{x};\mathbf{y}) + 12\left(F(\mathbf{w},\mathbf{w},\mathbf{x},\mathbf{y}) + F(\mathbf{w},\mathbf{x},\mathbf{x},\mathbf{y}) + F(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{y})\right) \\ &= \sum_{i=1}^{n} z_i L_i(\mathbf{w};\mathbf{x};\mathbf{y}) + g(\mathbf{w},\mathbf{x},\mathbf{y}) \end{aligned}$$

where $g(\mathbf{w},\mathbf{x},\mathbf{y})$ is independent of $\mathbf{z}$.

Let $\|x\|$ denote the distance from the real number $x$ to the nearest integer. Using Lemma 2.22 and combining this with 6.15 we finally get:

$$\begin{aligned} |S(\alpha)|^8 &\ll B^{4n} \sum_{|\mathbf{w}|,|\mathbf{x}|,|\mathbf{y}|\leqslant B} \left| \sum_{\mathbf{z}\in\mathscr{T}(\mathbf{w},\mathbf{x},\mathbf{y})} e\left(\alpha \sum_{i=1}^{n} z_i L_i(\mathbf{w};\mathbf{x};\mathbf{y})\right)\right| \\ &\ll B^{4n} \sum_{|\mathbf{w}|,|\mathbf{x}|,|\mathbf{y}|\leqslant B} \prod_{i=1}^{n} \min\left\{B, \|\alpha L_i(\mathbf{w};\mathbf{x};\mathbf{y})\|^{-1}\right\} \end{aligned} \tag{6.16}$$

[2, p. 130]

Furthermore we define:

$$N(\alpha,B) := \#\left\{(\mathbf{w},\mathbf{x},\mathbf{y}) \in (\mathbb{Z}\cap[-B,B])^{3n} : \|\alpha L_i(\mathbf{w};\mathbf{x};\mathbf{y})\| < B^{-1} \forall i \leqslant n\right\}$$

[2, p. 130]

Similar to the proof of Lemma 13.2 in [3], we can conclude that

$$\sum_{|\mathbf{w}|,|\mathbf{x}|,|\mathbf{y}|\leqslant B} \prod_{i=1}^{n} \min\left\{B, \|\alpha L_i(\mathbf{w};\mathbf{x};\mathbf{y})\|^{-1}\right\} \ll (B\log B)^n N(\alpha,B)$$

Putting this together with 6.16, we can follow that:

$$\begin{aligned} |S(\alpha)|^8 &\ll B^{4n} N(\alpha,B)\,(B\log B)^n \\ &= B^{5n}(\log B)^n N(\alpha,B) \end{aligned} \tag{6.17}$$

[2, p. 131]

From the trivial upper bound $N(\alpha,B) \ll B^{3n}$, it is evident that we have lost very little in the formulation of 6.17. We need the following result of Davenport to handle the set $N(\alpha,B)$.

**Lemma 6.7.** *Let* $\mathbf{L}$ *be a real symmetric* $n \times n$ *matrix. Let* $A > 0$ *be real, and let*

$$N(Z) := \#\left\{\mathbf{u} \in \mathbb{Z}^n : |\mathbf{u}| \leqslant AZ, \|(\mathbf{Lu})_i\| < A^{-1}Z \forall i \leqslant n\right\}$$

*Then, if* $0 < Z_1 \leqslant Z_2 \leqslant 1$, *we have*

$$N\left(Z_2\right) \ll \left(\frac{Z_2}{Z_1}\right)^n N\left(Z_1\right)$$

[2, p. 131]

Our goal is to apply this result three times, such that we can reduce the analysis of $N(\alpha, B)$ to a problem involving the system of equations $L_i(\mathbf{w}; \mathbf{x}; \mathbf{y}) = 0$,for $1 \leqslant i \leqslant n$.

As a first step we choose the matrix $\mathbf{L}$ in Lemma 6.7 to be given by $(\mathbf{Ly})_i = L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})$. Choosing $A = B$, $Z_2 = 1$ and $Z_1 = Z$ we have:

$$N(1) = \#\left\{\mathbf{y} \in \mathbb{Z}^n : |\mathbf{y}| \leqslant B, \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < B^{-1} \forall i \leqslant n\right\}$$

$$N(Z) = \#\left\{\mathbf{y} \in \mathbb{Z}^n : |\mathbf{y}| \leqslant BZ, \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < B^{-1}Z \forall i \leqslant n\right\}$$

With Lemma 6.7 we can conclude that

$$N\left(1\right) \ll \left(\frac{1}{Z}\right)^n N\left(Z\right) = Z^{-n} N\left(Z\right),$$

for any $0 < Z_1 \leq 1$.

It follows that:

$$N(\alpha, B) \ll Z^{-n} \# \left\{(\mathbf{w}, \mathbf{x}, \mathbf{y}) \in \mathbb{Z}^{3n} : \begin{array}{l} |\mathbf{w}|, |\mathbf{x}| \leqslant B, |\mathbf{y}| \leqslant ZB, \\ \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < ZB^{-1} \forall i \leqslant n \end{array}\right\}$$

[2, p. 131]

Birch chose to not use this estimate directly but to permute the roles of $\mathbf{w}, \mathbf{x}$ and $\mathbf{y}$.

We choose L to be the matrix given by $(\mathbf{Lx})_i = L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})$, $A = BZ^{-\frac{1}{2}}, Z_2 = Z^{\frac{1}{2}}$ and $Z_1 = Z^{\frac{3}{2}}$.

Therefore we get:

$$N(Z_2) = N(Z^{\frac{1}{2}}) = \#\left\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| \leqslant B, \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < B^{-1}Z, \forall i \leqslant n\right\}$$

$$N(Z_1) = N(Z^{\frac{3}{2}}) = \#\left\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| \leqslant BZ, \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < B^{-1}Z^2, \forall i \leqslant n\right\}$$

And again with Lemma 6.7 we can conclude that

$$N\left(Z_2\right) \ll \left(\frac{Z^{1/2}}{Z^{3/2}}\right)^n N\left(Z_1\right) = Z^{-n} N\left(Z_1\right)$$

Therefore it holds that:

$$N(\alpha, B) \ll Z^{-2n} \# \left\{ (\mathbf{w}, \mathbf{x}, \mathbf{y}) \in \mathbb{Z}^{3n} : \begin{array}{l} |\mathbf{w}| \leqslant B, |\mathbf{x}|, |\mathbf{y}| \leqslant ZB \\ \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < Z^2 B^{-1} \forall i \leqslant n \end{array} \right\}$$

[2, p. 131]

In a last step we choose L to be the matrix given by $(\mathbf{Lw})_i = L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})$, $A = BZ^{-1}, Z_2 = Z$ and $Z_1 = Z^2$. Our goal is to shrink the size of the box wherein lies $\mathbf{w}$. We can define:

$$N(Z_2) = N(Z) = \#\left\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| \leqslant B, \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < B^{-1} Z^2, \forall i \leqslant n\right\}$$

$$N(Z_1) = N(Z^2) = \#\left\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| \leqslant BZ, \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < B^{-1} Z^3, \forall i \leqslant n\right\}$$

Applying Lemma 6.7 we get:

$$N\left(Z_2\right) \ll \left(\frac{Z}{Z^2}\right)^n N\left(Z_1\right) = Z^{-n} N\left(Z_1\right)$$

We define:

$$S(R, Q) := \left\{(\mathbf{w}, \mathbf{x}, \mathbf{y}) \in (\mathbb{Z} \cap [-R, R])^{3n} : \|\alpha L_i(\mathbf{w}; \mathbf{x}; \mathbf{y})\| < Q^{-1} \forall i \leqslant n\right\}$$

for any $R, Q > 0$.

From this definition and our previous calculations it follows that

$$N(\alpha, B) \ll Z^{-3n} \# S\left(ZB, Z^{-3}B\right).$$

Out next goal is to find conditions on $\alpha$ and $Z$ under which

$$L_i(\mathbf{w}; \mathbf{x}; \mathbf{y}) = 0, \quad (1 \leqslant i \leqslant n)$$

for every $(\mathbf{w}, \mathbf{x}, \mathbf{y}) \in S\left(ZB, Z^{-3}B\right)$.

To be able to do so we will need the following Lemma, which is a result due to Heath-Brown [5]. A proof of this Lemma can be found in [2] on page 132.

**Lemma 6.8.** *Let $M > 0$ and let $\alpha = \frac{a}{q} + z$, with $|z| \leqslant (2qM)^{-1}$. Suppose that $m \in \mathbb{Z}$ is such that $|m| \leqslant M$ and $\|\alpha m\| < Q^{-1}$ for some $Q \geqslant 2q$. Then $m = 0$ if in addition we have either $M < q$ or $|z| > (qQ)^{-1}$.*[2, p. 132]

We will use this Lemma to reduce our considerations to the system of trilinear equations

$$L_i(\mathbf{w};\mathbf{x};\mathbf{y}) = 0, \quad (1 \leqslant i \leqslant n)$$

for every $(\mathbf{w},\mathbf{x},\mathbf{y}) \in S\left(ZB, Z^{-3}B\right)$.

To do so we write $\varphi = 24\sum |f_{ijk\ell}|$, where $f_{ijk\ell}$ are the coefficients of $f$, and we suppose that $\alpha = \frac{a}{q} + z$. We know $Q = Z^{-3}B$, $R = ZB$ and we set: $m = L_i(\mathbf{w};\mathbf{x};\mathbf{y})$. Therefore we get:

$$|m| = |L_i(\mathbf{w};\mathbf{x};\mathbf{y})| = \left|4! \sum_{j,k,\ell=1}^{n} f_{ijk\ell} w_j x_k y_\ell\right| \leq \varphi B^3 Z^3 = M$$

We have to choose $Z$ appropriately, such that the previous Lemma is applicable. Therefore we choose $Z$ to satisfy the following conditions:

From Lemma 6.7 we know that $0 < Z \leq 1$.

Since $Q = Z^{-3}B$ and $Q \geq 2q$ must hold for Lemma 6.8 we get

$$Z^3 = \frac{B}{Q} \leq \frac{B}{2q}.$$

And since $|z| \leqslant (2qM)^{-1} = (2q\varphi B^3 Z^3)^{-1}$ must hold as well in Lemma 6.8 we get:

$$Z^3 \leq (2q\varphi B^3 |z|)^{-1}$$

And for $m = L_i(\mathbf{w};\mathbf{x};\mathbf{y}) = 0$ we need either $M = \varphi B^3 Z^3 < q$ or $|z| > (qQ)^{-1} = (qZ^{-3}B)^{-1}$. Therefore we get the bound:

$$Z^3 < \max\left\{\frac{q}{\varphi B^3}, q|z|B\right\}.$$

If $Z$ satisfies all of these conditions, then we can apply Lemma 6.8. And we can conclude that:

$$N(\alpha, B) \ll Z^{-3n} \# T(ZB),$$

where

$$T(R) := \left\{(\mathbf{w},\mathbf{x},\mathbf{y}) \in (\mathbb{Z} \cap [-R,R])^{3n} : L_i(\mathbf{w};\mathbf{x};\mathbf{y}) = 0 \forall i \leqslant n\right\}$$

for any $R > 0$. [2, p. 133]

Now we can study the density of integer solutions to the system of equations

$$L_i(\mathbf{w};\mathbf{x};\mathbf{y}) = 0, \quad (1 \leqslant i \leqslant n)$$

Doing so will require the following Theorem and definitions from within the proof of it:

**Theorem 6.9.** *Let $V \subset \mathbb{P}^{n-1}$ be a variety of degree $d$. Then we have*

$$N_V(B) \ll_{d,n} B^{\dim V+1}$$

[2, p. 47]

We will write $m$ for the dimension of $V$, and for any affine variety $T \subset \mathbb{A}^\nu$ of degree $\delta$ and dimension $\mu$, we define the quantity

$$M_T(B) := \#\{\mathbf{t} \in T \cap \mathbb{Z}^\nu : |\mathbf{t}| \leqslant B\}$$

.

$$M_T(B) = O_{\delta,\nu}(B^\mu) \tag{6.18}$$

[2, p. 47]

By Lemma 6.6 our system of equations

$$L_i(\mathbf{w};\mathbf{x};\mathbf{y}) = 0, \quad (1 \leqslant i \leqslant n)$$

has affine dimension at most $2n$. Therefore its projective dimension is at most $2n-1$. Relating this to Theorem 6.9 we get $m = 2n-1$ and therefore $\mu = m+1 = 2n$. Thanks to 6.18 we can now conclude that $\#T(ZB) \ll (ZB)^{2n}$, if $ZB \geqslant 1$. We can deduce that:

$$N(\alpha, B) \ll Z^{-n}B^{2n}, \tag{6.19}$$

which holds trivially if $ZB < 1$. [2, p. 133]

We now want to choose $Z$ as large as possible but we have to take the previously made constraints into consideration. Since $Z^3 < \max\left\{\frac{q}{\varphi B^3}, q|z|B\right\}$ is a strict inequality we choose

$$Z = \frac{1}{2}\min\left\{1, \frac{1}{2\varphi q|z|B^3}, \frac{B}{2q}, \max\left\{\frac{q}{\varphi B^3}, q|z|B\right\}\right\}^{\frac{1}{3}}.$$

[2, p.133]

Putting together 6.17 and 6.19 we get

$$\begin{aligned}
|S(\alpha)|^8 &\ll B^{5n}(\log B)^n N(\alpha, B) \\
&\ll B^{7n}(\log B)^n Z^{-n} \\
&= B^{7n}(\log B)^n (BZ^{-1}B^{-1})^n \\
&= B^{8n}(\log B)^n (ZB)^{-n} \\
&\ll_\varepsilon B^{8n+\varepsilon}(ZB)^{-n}
\end{aligned}$$

We now have to estimate $(ZB)^{-n}$. We have: $(ZB)^{-n} = \left(\frac{1}{Z^3B^3}\right)^{n/3}$

$$\frac{1}{Z^3} = 8\max\left\{1, 2\varphi q|z|B^3, \frac{2q}{B}, q^{-1}\min\left\{\varphi B^3, \frac{1}{|z|B}\right\}\right\}$$

Now we can estimate the maximum, using the sum of the terms over which we take the maximum, and $\frac{1}{B^3}$ is drawn into the sum. Therefore we get:

$$|S(\alpha)|^8 \ll_\varepsilon B^{8n+\varepsilon}\left(B^{-3} + q|z| + qB^{-4} + q^{-1}\min\left\{1, \frac{1}{|z|B^4}\right\}\right)^{\frac{n}{3}},$$

for any $\varepsilon > 0$, whence

$$S(\alpha) \ll_\varepsilon B^{n+\varepsilon}\left(B^{-3} + q|z|\max\left\{1, \frac{1}{|z|B^4}\right\} + q^{-1}\min\left\{1, \frac{1}{|z|B^4}\right\}\right)^{\frac{n}{24}} \quad (6.20)$$

[2, p. 133]

From this bound we can now derive three basic estimates.

1) The first one involves the complete exponential sum $S_{q,a}$ defined in 6.5. We will take $z = 0$ and $B = q$ in the definition of $S(\alpha)$. Therefore we have $\alpha = \frac{a}{q}$, and looking at Lemma 6.8 we get the following bounds for $Z$:

$0 < Z \leq 1$ and $Z^3 < \frac{1}{\varphi q^2}$ and $Z^3 \leq \frac{1}{2}$.

Therefore

$$Z = \frac{1}{2}\min\left\{1, \frac{1}{\varphi q^2}, \frac{1}{2}\right\}^{1/3}$$

We get

$$Z = \frac{1}{2}\left(\frac{1}{\varphi q^2}\right)^{1/3} \to \frac{1}{Z^3} = 8\varphi q^2 \to \frac{1}{Z^3B^3} = 8\varphi q^{-1}$$

Looking at 6.20 we get:

$$S(\alpha) \ll_\varepsilon q^{n+\varepsilon}(q^{-1})^{n/24} = q^{\frac{23n}{24}+\varepsilon} \quad (6.21)$$

[2, p. 133]

for any coprime integers $a, q$ such that $1 \leqslant a \leqslant q$.

2) The trivial estimate for $S(\alpha)$ is $S(\alpha) \approx B^n$ and if $|\alpha| \leq B^{-4}$, then $(|\alpha|B^4)^{-n/24}$ is large. Therefore

$$S(\alpha) \ll_\varepsilon B^{n+\varepsilon}(|\alpha|B^4)^{-n/24}$$

will hold trivially for $|\alpha| \leq B^{-4}$. For $B^{-4} < |\alpha| < B^{-2}$ we look at $S(\alpha)$ with $a = 0, q = 1$ and $z = \alpha$. We get $S(\alpha) = S(z)$ and from 6.20 we have:

$$\begin{aligned} S(\alpha) &\ll_\varepsilon B^{n+\varepsilon}\left(B^{-3} + |\alpha| \max\left\{1, \frac{1}{|\alpha|B^4}\right\} + \min\left\{1, \frac{1}{|\alpha|B^4}\right\}\right)^{\frac{n}{24}} \\ &= B^{n+\varepsilon}(B^{-3} + |\alpha| + \frac{1}{|\alpha|B^4})^{n/24} \end{aligned}$$

Since $|\alpha| < B^{-2}$ we have $\frac{1}{|\alpha|} > B^2$ and $\frac{1}{|\alpha|B^4} > B^{-2}$.

Therefore we have:

$$S(\alpha) \ll_\varepsilon B^{n+\varepsilon}\left(|\alpha|B^4\right)^{-\frac{n}{24}}, \quad \text{if } |\alpha| < B^{-2} \tag{6.22}$$

[2, p. 133]

3) The third estimate is stated as the following Lemma:

**Lemma 6.10.** *Let $\varepsilon > 0$ and let $a, q, z$ be such that*

$$1 \leqslant a \leqslant q \leqslant B^2, \quad (a, q) = 1, \quad |z| \leqslant \frac{1}{q^2}$$

*Then we have*

$$S\left(\frac{a}{q} + z\right) \ll_\varepsilon B^{n+\varepsilon} q^{-\frac{n}{24}} \min\left\{1, |z|B^4\right\}^{-\frac{n}{24}}$$

[2, p. 133]

*Proof.* We have to distinguish two cases.

1) Let $1 \leq \frac{1}{|z|B^4}$, then

$$\begin{aligned} S\left(\frac{a}{q} + z\right) &\ll_\varepsilon B^{n+\varepsilon}\left(B^{-3} + q|z|\frac{1}{|z|B^4} + q^{-1}\right)^{\frac{n}{24}} \\ &\leq B^{n+\varepsilon}\left(q^{-1}\underbrace{B^{-1}}_{\leq 1} + q^{-1}\underbrace{\frac{1}{|z|B^4}}_{\geq 1} + q^{-1}\right) \ll_\varepsilon B^{n+\varepsilon}\left(q^{-1}\frac{1}{|z|B^4}\right)^{\frac{n}{24}} \end{aligned}$$

2) Let $1 > \frac{1}{|z|B^4}$

$$S\left(\frac{a}{q}+z\right) \ll_\varepsilon B^{n+\varepsilon}\left(B^{-3}+q|z|+q^{-1}\frac{1}{|z|B^4}\right)^{\frac{n}{24}}$$
$$\leq B^{n+\varepsilon}\left(q^{-1}\underbrace{B^{-1}}_{\leq 1}+q^{-1}+q^{-1}\underbrace{\frac{1}{|z|B^4}}_{<1}\right)^{\frac{n}{24}} \ll_\varepsilon B^{n+\varepsilon}\left(q^{-1}\right)^{\frac{n}{24}}$$

Therefore we get:

$$S\left(\frac{a}{q}+z\right) \ll_\varepsilon B^{n+\varepsilon}q^{-\frac{n}{24}}\max\left\{1,\frac{1}{|z|B^4}\right\}^{\frac{n}{24}}$$
$$S\left(\frac{a}{q}+z\right) \ll_\varepsilon B^{n+\varepsilon}q^{-\frac{n}{24}}\min\left\{1,|z|B^4\right\}^{-\frac{n}{24}}$$

□

We can now finally estimate our Minor arcs. So let $\alpha \in \mathfrak{m}(\Delta)$. By Dirichlet's Approximation Theorem 2.24 we can find integers $a$ and $q$ such that $(a,q)=1$ and $1 \leq a \leq q \leq B^2$. Therefore $|\alpha - \frac{a}{q}| \leq \frac{1}{B^2 q}$. Recalling the definition of the Major arcs 6.8 we see that for $\alpha \in \mathfrak{m}(\Delta) = [0,1] \setminus \mathfrak{M}(\Delta)$ we may assume that $q > B^\Delta$ or $|q\alpha - a| > qB^{-4+\Delta}$.

Writing $\alpha = \frac{a}{q} + z$ we get that $|z| \leq \frac{1}{B^2 q} \leq \frac{1}{q^2}$ and it therefore follows from Lemma 6.10 that

$$\int_{\mathfrak{m}(\Delta)} |S(\alpha)|\mathrm{d}\alpha = \sum_{q\leq B^2}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{m}(a,q)}|S(\alpha)|d\alpha$$
$$\ll_\varepsilon \sum_{q\leq B^2}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{B^2q}}^{\frac{1}{B^2q}} B^{n+\varepsilon}q^{-\frac{n}{24}}\min\{1,|z|B^4\}^{-\frac{n}{24}}dz$$
$$\ll_\varepsilon B^{n+\varepsilon}\sum_{q\leq B^2} q^{1-\frac{n}{24}}\int_0^{\frac{1}{B^2q}}\min\{1,|z|B^4\}^{-\frac{n}{24}}dz$$

We have to consider two cases:

1) Given that $q \leq B^\Delta$ we follow that $|z| > B^{-4+\Delta}$. In this case the minimum is 1 and therefore we can estimate the integral using the maximum $|z|B^4$.

$$\begin{aligned}\int_{\mathfrak{m}(\Delta)} |S(\alpha)| \mathrm{d}\alpha &\ll_\varepsilon B^{n+\varepsilon} \sum_{q \leq B^\Delta} q^{1-\frac{n}{24}} \int_{B^{-4+\Delta}}^{\frac{1}{qB^2}} (zB^4)^{-\frac{n}{24}} dz \\ &\ll_\varepsilon B^{n+\varepsilon} \sum_{q \leq B^\Delta} q^{1-\frac{n}{24}} \left(B^{-4+\Delta}\right)^{(1-\frac{n}{24})} B^{-\frac{n}{6}}\end{aligned}$$

In the last estimation we used the fact that

$$\int_a^b z^{-k} dz \ll a^{1-k}$$

for $a < b$ and $k > 1$.

Since $q^{1-\frac{n}{24}} \leq 1$ we know that

$$\sum_{q \leq B^\Delta} q^{1-\frac{n}{24}} \leq B^\Delta$$

. Therefore we can conclude that:

$$\int_{\mathfrak{m}(\Delta)} |S(\alpha)| \mathrm{d}\alpha \ll_\varepsilon B^{n+\varepsilon-4-\Delta(\frac{n}{24}-2)}$$

2) For $q > B^\Delta$ we get:

$$\begin{aligned}\int_{\mathfrak{m}(\Delta)} |S(\alpha)| \mathrm{d}\alpha &\ll_\varepsilon B^{n+\varepsilon} \sum_{B^\Delta < q \leq B^2} q^{1-\frac{n}{24}} \int_0^{\frac{1}{qB^2}} \min\{1, |z| B^4\}^{-\frac{n}{24}} dz \\ &\leq B^{n+\varepsilon} \sum_{B^\Delta < q} q^{1-\frac{n}{24}} \left( \int_0^{B^{-4}} 1 dz + \int_{B^{-4}}^{\frac{1}{qB^2}} (zB^4)^{-\frac{n}{24}} \right) \\ &\leq B^{n+\varepsilon} \sum_{B^\Delta < q} q^{1-\frac{n}{24}} \left( B^{-4} + (B^{-4})^{(1-\frac{n}{24})} B^{-\frac{n}{6}} \right) \\ &\ll_\varepsilon B^{n+\varepsilon} \sum_{B^\Delta < q} q^{1-\frac{n}{24}} B^{-4} \\ &\ll_\varepsilon B^{n+\varepsilon-\Delta(\frac{n}{24}-2)-4}\end{aligned}$$

In the first inequality we again estimated the two integrands with their respective maxima. For the second inequality we again used the fact that

$$\int_a^b z^{-k} dz \ll a^{1-k}$$

for $a < b$ and $k > 1$ and for the last estimation we used Proposition 2.18.

We can finally conclude that

$$\int_{\mathfrak{m}(\Delta)} |S(\alpha)| \mathrm{d}\alpha \ll_\varepsilon B^{n+\varepsilon-\Delta(\frac{n}{24}-2)-4},$$

for $n > 48$. If we now choose $\varepsilon$ sufficiently small in terms of $\Delta$, then this shows that the contribution of the Minor arcs is $o(B^{n-4})$, as $B \to \infty$, which is satisfactory for 6.11.[2, p. 134]

## 6.8 Contribution of the Major Arcs

Recall the definition 6.5 of the complete exponential sum $S_{q,a}$.

$$S_{q,a} := \sum_{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n} e_q(af(\mathbf{x})),$$

for $a, q$ coprime such that $1 \leqslant a \leqslant q$

**Definition 6.11.** (singular series)

Let

$$\mathfrak{S}(R) := \sum_{q \leqslant R} \frac{1}{q^n} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S_{q,a}$$

for any $R > 1$. We define $\mathfrak{S} = \lim_{R \to \infty} \mathfrak{S}(R)$, when it exists.

[2, p. 134]

This is the singular series for the problem. It encodes the information whether the quartic hypersurface has $p$-adic solutions for all primes $p$ or not.

The singular series continued to infinity is given by:

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-n} S_{q,a}.$$

From 6.21 we know that

$$q^{-n}|S_{q,a}| \ll_\varepsilon q^{\frac{-n}{24}+\varepsilon}$$

.

Therefore we get:

$$\mathfrak{S} \ll_\varepsilon \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{\frac{-n}{24}+\varepsilon} = \sum_{q=1}^{\infty} q^{1-\frac{n}{24}+\varepsilon}$$

And we can conclude that the singular series continued to infinity converges absolutely for $n > 48$.

Using Proposition 2.18 we further get:

$$\begin{aligned}\mathfrak{S} - \mathfrak{S}(R) &= \sum_{q>R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-n} S_{q,a} \\ &\ll_\varepsilon \sum_{q>R} q^{1-\frac{n}{24}+\varepsilon} \\ &= \sum_{q>R} \frac{1}{q^{\frac{n}{24}-1-\varepsilon}} \\ &\ll_\varepsilon R^{1-(\frac{n}{24}-1-\varepsilon)} \\ &= R^{2-\frac{n}{24}+\varepsilon}\end{aligned}$$

Hence we conclude that:

$$\mathfrak{S}(R) = \mathfrak{S} + O_\varepsilon\left(R^{2-\frac{n}{24}+\varepsilon}\right) \tag{6.23}$$

for any $\varepsilon > 0$.[2, p. 134]

The real solubility is part of the singular integral.

**Definition 6.12.** (singular integral)

$$\Im(R) := \int_{-R}^{R} \int_{[-1,1]^n} e(zf(\mathbf{x}))\mathrm{d}\mathbf{x}\mathrm{d}z$$

We define: $\mathfrak{I} := \lim_{R\to\infty} \mathfrak{I}(R)$, when it exists.

[2, p. 134]

For the convergence of the singular integral we first need the following Lemma:

**Lemma 6.13.** *We have* $I(\gamma) \ll_\varepsilon \min\left\{1, |\gamma|^{-\frac{n}{24}+\varepsilon}\right\}$ *for any* $\varepsilon > 0$.

*Proof.* The estimate $I(\gamma) \ll 1$ is trivial. To prove the second estimate we can assume that $|\gamma| > 1$. Taking $a = 0$ and $q = 1$ in Lemma 6.4 we get:

$$S(z) = S(\alpha) = B^n I\left(\alpha B^4\right) + O\left(B^{n-1}\left(1 + |\alpha| B^4\right)\right)$$

for any $B \geqslant 1$.

From 6.22 we have

$$S(\alpha) \ll_\varepsilon B^{n+\varepsilon} \left(|\alpha| B^4\right)^{-\frac{n}{24}},$$

if $|\alpha| < B^{-2}$ and $a = 0$,$q = 1$ and $\alpha = z$.

Now we want to combine these estimates, so we write $\alpha B^4 = \gamma$ and we get:

$$\begin{aligned} S(\alpha) &= B^n I(\gamma) + O\left(B^{n-1}(1+|\gamma|)\right) \\ &\ll_\varepsilon B^{n+\varepsilon} |\gamma|^{-n/24} \end{aligned}$$

Therefore we can conclude that:

$$I(\gamma) \ll_\varepsilon |\gamma|^{-\frac{n}{24}} B^\varepsilon + |\gamma| B^{-1}$$

when $|\gamma| < B^2$.

Due to the fact that $I(\gamma)$ is independent of $B$ we can choose $B = |\gamma|^{1+\frac{n}{24}}$ and we get

$$I(\gamma) \ll_\varepsilon |\gamma|^{-\frac{n}{24}} (B^\varepsilon + 1)$$

which leads to the desired result. □

[2, p. 135]

From Lemma 6.13 we can now deduce that

$$\mathfrak{I} - \mathfrak{I}(R) = \int_{|\gamma| \geqslant R} I(\gamma) \mathrm{d}\gamma \ll_\varepsilon \int_R^\infty \min\left\{1, \gamma^{-\frac{n}{24}+\varepsilon}\right\} \mathrm{d}\gamma \ll_\varepsilon R^{-\frac{n}{24}+\varepsilon+1} \tag{6.24}$$

[2, p. 135]

and this shows that $\mathfrak{I}$ is absolutely convergent for $n > 48$.

We can now finally complete the analysis of the exponential sum $S(\alpha)$ on the Major arcs. Let $\alpha = \frac{a}{q} + z \in \mathfrak{M}(\Delta)$. Recall the definition 6.8 of the Major arcs.

As the full set of Major arcs is given by

$$\mathfrak{M}(\Delta) = \bigcup_{1 \leqslant q \leqslant B^\Delta} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_{q,a}(\Delta)$$

We get:

$$\int_{\mathfrak{M}(\Delta)} S(\alpha) d\alpha = \sum_{q=1}^{B^\Delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_{q,a}(\Delta)} S(\alpha) d\alpha$$

And from Lemma 6.4 we know that

$$S(\alpha) = q^{-n} B^n S_{q,a} I\left(zB^4\right) + O\left(qB^{n-1}\left(1 + |z|B^4\right)\right)$$

where

$$I(\gamma) := \int_{\mathbf{t}\in[-1,1]^n} e(\gamma f(\mathbf{t}))\mathrm{d}\mathbf{t}$$

Since $\alpha \in \mathfrak{M}(\Delta)$ we know that $q \leq B^{\Delta}$ and that $|z| \leq B^{-4+\Delta}$.

Therefore we get:

$$S(\alpha) = q^{-n} B^n S_{q,a} I\left(zB^4\right) + O\left(B^{n-1+2\Delta}\right)$$

Since the Major arcs have measure $O\left(B^{-4+3\Delta}\right)$ our new error term will be $O\left(B^{n-5+5\Delta}\right)$.

Now we have to estimate the main term:

$$\begin{aligned}\int_{\mathfrak{M}(\Delta)} S(\alpha)d\alpha &= \sum_{q=1}^{B^{\Delta}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{q,a}(\Delta)} q^{-n} B^n S_{q,a} I\left(zB^4\right) d\alpha \\ &= B^n \sum_{q=1}^{B^{\Delta}} q^{-n} \sum_{\substack{a=1\\(a,q)=1}}^{q} S_{q,a} \int_{\frac{a}{q}-B^{-4+\Delta}}^{\frac{a}{q}+B^{-4+\Delta}} I\left(zB^4\right) d\alpha\end{aligned}$$

By definition 6.11, and since $\alpha = \frac{a}{q} + z$, we further get

$$= B^n \mathfrak{S}(B^{\Delta}) \int_{-B^{-4+\Delta}}^{B^{-4+\Delta}} I\left(zB^4\right) dz$$

and substituting $\tilde{z} = zB^4$ we get:

$$\begin{aligned}&= B^n \mathfrak{S}(B^{\Delta}) B^{-4} \int_{-B^{\Delta}}^{B^{\Delta}} I\left(B^{-4}B^4\tilde{z}\right) d\tilde{z} \\ &= B^{n-4} \mathfrak{S}(B^{\Delta}) \int_{-B^{\Delta}}^{B^{\Delta}} I\left(\tilde{z}\right) d\tilde{z}\end{aligned}$$

And by definition 6.12 we get

$$= B^{n-4} \mathfrak{S}\left(B^{\Delta}\right) \mathfrak{I}\left(B^{\Delta}\right)$$

as an estimation of the main term.

Putting the estimations of the main and the error term together we get:

$$\int_{\mathfrak{M}(\Delta)} S(\alpha)\mathrm{d}\alpha = B^{n-4}\mathfrak{S}\left(B^{\Delta}\right)\mathfrak{I}\left(B^{\Delta}\right) + O\left(B^{n-5+5\Delta}\right)$$

Due to 6.23 we know that $\mathfrak{S}(R) = \mathfrak{S}+O(R^{2-\frac{n}{24}})$ and from 6.24 we can conclude that $J(B^{\Delta}) = O(1)$.

Therefore we get

$$\int_{\mathfrak{M}(\Delta)} S(\alpha)\mathrm{d}\alpha = \mathfrak{S}B^{n-4}\mathfrak{I}\left(B^{\Delta}\right) + O\left(B^{n-5+5\Delta} + B^{n-4-\Delta\left(\frac{n}{24}-2\right)}\right).$$

And for $n > 48$ note that $n - 48 \geq 1$ therefore $B^{-\Delta(\frac{n}{24}-2)} \leq B^{-\frac{\Delta}{24}}$.

Hence we get:

$$\mathfrak{S}\mathfrak{I}B^{n-4} + O_{\varepsilon}\left(B^{n-5+5\Delta} + B^{n-4-\frac{\Delta}{24}+\varepsilon}\right)$$

if $n > 48$.

If we now take $0 < \Delta < \frac{1}{5}$ we can deduce that the contribution from the Major arcs is

$$\int_{\mathfrak{M}(\Delta)} S(\alpha)\mathrm{d}\alpha = \mathfrak{S}\mathfrak{I}B^{n-4} + o\left(B^{n-4}\right).$$

And since the Minor arc contribution is only $o(B^{n-4})$ we get

$$N(f;B) = \int_0^1 S(\alpha)d\alpha = \int_{\mathfrak{M}(\Delta)} S(\alpha)d\alpha + \int_{\mathfrak{m}(\Delta)} S(\alpha)d\alpha = \mathfrak{S}\mathfrak{I}B^{n-4} + o\left(B^{n-4}\right).$$

We have now finally found the constant $c_f = \mathfrak{S}\mathfrak{I}$ to establish 6.11 the asymptotic formula for $N(f;B)$.

As we have seen before $N_V(B) \sim \frac{1}{2\zeta(n-d)}c_f B^{n-d}$.

Therefore we get the following asymptotic formula for $N_V(B)$:

$$N_V(B) = \frac{1}{2\zeta(n-4)}\mathfrak{S}\mathfrak{I}B^{n-4}.$$

To make a statement about the existence of rational points on the quartic hypersurfaces, $c_f$ must be positive. It is sufficient for the equation $f = 0$ to have solutions over $\mathbb{R}$ and over any $p$-adic field $\mathbb{Q}_p$, to guarantee that $c_f > 0$. The methods to ensure this are not discussed here. The interested reader can consult [1] for this.

# 7 References

# References

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