Fidelity and Transversely Intersecting Lagrangians

Master Thesis

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Chapter 1

Introduction

Quantum mechanics has been a very active field of study for almost a century. To make a correspondence between classical dynamical systems and quantum mechanical systems, different approaches of quantization have been proposed by both mathematicians and physicists. The well-known correspondence principle suggests that in the semi-classical limit $\hbar \to 0$, where $\hbar$ is the Planck constant, a quantum mechanical system should transform to its classical counterpart. Note that the phase space of a classical dynamical system is either a symplectic manifold or it can be foliated by symplectic manifolds. A general definition of quantization of a symplectic manifold was proposed by Berezin in [2]. Berezin-Toeplitz operators appear in the study of the semi-classical limit of the quantization of compact symplectic manifolds [7]. In this thesis, we will concentrate on the quantization of compact Kahler manifolds. Note that one can see a detailed review of Berezin quantization of compact Kahler manifolds in [9].

In [5] Charles and Polterovich proposed the concept of fidelity as a measure of overlap between a pair of quantum states. To get a general idea about fidelity, one can note that fidelity between two orthogonal pure quantum states is zero, and thus they are maximally non-overlapping. They showed that computing fidelity between a quantized submanifold of a closed Kahler manifold and its quantum evolution can be translated as a footprint of symplectic rigidity in the classic side.

In this thesis, we will move according to the following outline. First, we start chapter two by defining the Berezin quantization. Indeed, we start with a compact Kahler manifold that one can consider as a phase space of a classical dynamical system. We move on to construct a Hilbert space as a space of quantum states, and we define Toeplitz operators to associate a quantum observable to each classical observable that is a smooth function on the Kahler manifold. As we move on through this chapter, we will elaborate on a semi-classical setting, and thus we need our computations to depend on the Planck constant. We define a family of Hilbert spaces and a family of Toeplitz operators, both depending on the Planck constant, and we call this family of Toeplitz operators a Berezin quantization of the Kahler manifold. In fact, the existence of quantization requires a special kind of holomorphic line bundle on the Kahler manifold. Then the Hilbert spaces will be the space of holomorphic sections of tensor power of this line bundle. In this case, the role of the Planck constant will be played by the inverse of a positive integer k, and the semi-classical limit will correspond to $k \to \infty$. The next parts of this chapter are devoted to some general ideas about Schwartz kernels that we will use intensively in the rest of this thesis, and a study of the asymptotic behavior of the projection from the space of square-integrable sections of the k-th tensor power of the aforementioned line

bundle to the space of its holomorphic sections. Then we move on to define very important elements of Hilbert spaces, called coherent states and obtain Toeplitz operators in terms of them. Moreover, we see how one can define the projection along with a coherent state. Note that we will use these projection operators to define quantized submanifolds in the next chapter. The material of this chapter up to now is mainly borrowed from [7], however, we have gone through computations in more detail whenever needed. The rest of chapter two concentrates on two important generalizations of the stationary phase approximation method to Morse and Morse-Bott functions. One can use the stationary phase method to study the asymptotic behavior of some special integrals of the form (2.40). This part is a generalization of results in [6] obtained for Euclidean spaces and smooth functions.

In the third chapter, we introduce the quantum state associated with an arbitrary submanifold of a compact Kahler manifold. We see that if the submanifold is a point, its quantization is a pure quantum state, and otherwise, it is a mixed state in the semi-classical limit. In addition, we obtain a formula for the trace of the density matrix of a quantized non-trivial submanifold in the semi-classical limit. We see that the trace of the density matrix only depends on the dimension of the submanifold, the Riemannian metric, and the Planck constant. In the rest of this chapter first, we introduce the concept of microsupport of a quantized submanifold, which is a subset of the underlying Kahler manifold. Intuitionally, a point in the phase space is in the microsupport of a quantum state if the probability of the existence of the quantum state in all of the neighborhoods of this point is considerable. We will finish this chapter by using the stationary phase method to prove that the microsupport of a quantized submanifold is the submanifold itself.

Chapter four is the most important part of these notes, and we start this chapter with the definition of fidelity between a pair of quantum states. In the first section we show that for two disjoint submanifolds of a compact Kahler manifold, fidelity between their quantization is negligible. This statement was proved in [5], but we have proposed a different proof here based on an application of the stationary phase method and integral kernels. Finally, in the last section, we compute fidelity between transversely intersecting Lagrangian submanifolds of a compact Kahler manifold. In general, two submanifolds L and K of an ambient space M intersect transversely if for all points $p \in L \cap K$,

$$T_pL \oplus T_pK = \{v + w | v \in T_pL, w \in T_pK\} = T_pM.$$

Note that if two submanifolds do not intersect, they are automatically transversal. Moreover, if the submanifolds L and K meet transversely, then $L \cap K$ is a smooth submanifold of dimension $\dim L + \dim K - \dim M$. Therefore, if $L, K \subset M$ are Lagrangian submanifolds of a compact Kahler manifold that intersect transversely, they either do not intersect at all or they intersect in a finite number of points. The following theorem is the main result we obtained in this thesis.

Theorem 1.0.1. *Let $L, K \subset M^{2n}$ be two transversely intersecting Lagrangian submanifolds of a quantizable compact Kahler manifold. Then in the semi-classical limit $k \to \infty$,*

i) If L and K do not intersect at all, fidelity between their quantization is of order $\mathcal{O}(k^{-\infty})$,

ii) If L and K intersect transversely in a finite number of points, fidelity between their quantization is bounded from below by $\mathcal{O}(k^{-n/2})$.

Hence, for two intersecting Lagrangian submanifolds, fidelity can detect the transverse intersection. Note that a sequence $a_k \in \mathbb{R}$ is $\mathcal{O}(k^{-N})$ if there exists a constant C_N such that $|a_k| \leq C_N k^{-N}$ for all sufficiently large k, and a sequence $a_k \in \mathbb{R}$ is $\mathcal{O}(k^{-\infty})$ if for every $N \geq 1$, there exists a constant C_N such that

$|a_k| \leq C_N k^{-N}$ for all sufficiently large k. Moreover, we have defined what the term quantizable means in the beginning of chapter two when we have defined the prequantum line bundle.

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Chapter 2

Preliminaries

2.1 Quantization

This section will be devoted to an introduction to the concept of Berezin quantization of a compact Kahler manifold (M^{2n}, J, ω). Then, using the notion of Schwartz kernels and coherent states, we see how one can obtain the Berezin quantization in terms of integral operators on the Kahler manifold. Our main references for this section are chapters four, five, six, seven, and nine of [7] and chapter two of [5].

2.1.1 Berezin Quantization

Let (M^{2n}, J, ω) be a compact Kahler manifold, and assume that $\mu = \frac{\omega^n}{n!}$ be the Liouville volume form.

Definition. A prequantum line bundle $(L, \nabla, h) \to M$ is a holomorphic Hermitian line bundle whose Chern connection has curvature $\text{curv}(\nabla) = -i\omega$.

Check Appendix A for definitions of a Hermitian line bundle and the Chern connection. One can see in chapter four of [7] that a prequantum line bundle does not always exist. Indeed, a prequantum line bundle $L \to M$ exists if and only if the cohomology class $[\omega/(2\pi)]$ lies in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$.

Define the scalar product of two sections of L by

$$\langle s, t \rangle = \int_M h(s, t)(x) d\mu(x) \tag{2.1}$$

Thus the norm of a section $s \in \Gamma(M, L)$ is

$$||s|| = \left(\int_M |s(x)|^2 d\mu(x) \right)^{1/2}, \tag{2.2}$$

where $|s(x)|^2 = h(s, s)(x)$.

Definition. We define the quantum space $\mathcal{H} = H^0(M, L)$ as the space of global holomorphic sections of $L \to M$.

Let $L^2(M, L)$ be the completion of the space of smooth sections of $L \to M$ with respect to the inner product $\langle \cdot, \cdot \rangle$, and let $\Pi : L^2(M, L) \to \mathcal{H}$ be the orthogonal projector.

Definition. Let $f \in C^0(M)$. The Toeplitz operator associated to f is the operator

$$T(f) = \Pi f : \mathcal{H} \to \mathcal{H} \tag{2.3}$$

where f stands for the operator of multiplication by f.

In fact, the Toeplitz operator in (2.3) is the composition of operators

$$\mathcal{H} \ni \phi \overset{f}{\longmapsto} f\phi \in L^2(M, L) \overset{\Pi}{\longmapsto} \Pi f\phi \in \mathcal{H}, \tag{2.4}$$

and one can see that

$$\forall s, t \in \mathcal{H} \quad ; \quad \langle T(f)s, t\rangle = \langle fs, t\rangle. \tag{2.5}$$

Now let $\Lambda \subset \mathbb{R}_{>0}$ be a subset having 0 as a limit point. For example one can take $\Lambda = \{k^{-1} | k \in \mathbb{N}\}$. Now let $\mathcal{H}_k$ be the space of holomorphic sections of $L^k \to M$, where $L^k := L^{\otimes k}$. Indeed, recalling the Hermitian metric h on L, for each $k \geq 1$, h induces a Hermitian metric h_k on the line bundle L^k defined by:

$$h_k(u_1 \otimes \cdots \otimes u_k, v_1 \otimes \cdots \otimes v_k) = \prod_{j=1}^{k} h(u_j, v_j) \tag{2.6}$$

for pure tensors and extending sesquilinearly. It also induces a Hermitian metric h_{-1} on the dual bundle L^* such that for any $x \in M$, any $\phi, \psi \in L_x^*$, and for all $s \in L_x$ one has

$$(h_{-1})(\phi, \psi)(x) = \frac{\phi(s)\psi(s)}{h(s, s)(x)}. \tag{2.7}$$

Now we can endow the space $C^\infty(M, L^k)$ of smooth sections with a Hermitian inner product

$$\langle \phi, \psi\rangle_k = \int_M h_k(\phi, \psi) d\mu \tag{2.8}$$

for every $\phi, \psi \in C^\infty(M, L^k)$.

Definition. The quantum state at level k is defined as the space $\mathcal{H}_k = H^0(M, L^k)$ of global holomorphic sections of $L^k \to M$.

Similar to above, consider $L^2(M, L^k)$ as the completion of the space of smooth sections of $L^k \to M$ with respect to the inner product $\langle \cdot, \cdot \rangle_k$, and let $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$ be the orthogonal projection.

Definition. Let $f \in C^0(M)$. The Berezin-Toeplitz operator associated to f is the operator

$$T_k(f) = \Pi_k f : \mathcal{H}_k \to \mathcal{H}_k. \tag{2.9}$$

Definition. We call the family $\{T_k\}_{k^{-1} \in \Lambda}$, a Berezin quantization of M.

Lemma 2.1.1. *Berezin-Toeplitz operators have the following properties*

i) $T_k(1) = id_{\mathcal{H}_k}$.

ii) For every $f \in C^0(M)$, $||T_k(f)|| \leq ||f||_\infty$.

iii) If $f \in C^0(M)$, then $T_k(f)^* = T_k(\overline{f})$. In particular, if f is real-valued, then $T_k(f)$ is self-adjoint. Moreover, if f takes its values in S^1, then $T_k(f)$ is a unitary operator.

iv) Let $f \in C^0(M, \mathbb{R})$ be such that $f(x) \geq 0$ for every $x \in M$. Then $T_k(f)$ in a non-negative operator, in the sense that for every $\phi \in \mathcal{H}_k$,

$$\langle T_k(f)\phi, \phi\rangle_k \geq 0. \tag{2.10}$$

Proof. i) Clear by definition.

ii) Since the norm of Π_k is smaller than one, we have the result.

iii) Let $\phi, \psi \in \mathcal{H}_k$ be arbitrary. We have to show that

$$\langle T_k(f)\phi, \psi\rangle_k = \langle \phi, T_k(f)^*\psi\rangle_k = \langle \phi, T_k(\overline{f})\psi\rangle_k.$$

Using the definition and since Π_k is self-adjoint, LHS of the above equation is

$$\langle T_k(f)\phi, \psi\rangle_k = \langle \Pi_k(f\phi), \psi\rangle_k = \langle f\phi, \Pi_k\psi\rangle_k = \langle f\phi, \psi\rangle_k.$$

Then,

$$\langle f\phi, \psi\rangle_k = \int_M h_k(f\phi, \psi)d\mu = \int_M f h_k(\phi, \psi)d\mu = \int_M h_k(\phi, \overline{f}\psi)d\mu.$$

Hence, one has $\langle \Pi_k(f\phi), \psi\rangle_k = \langle \phi, \overline{f}\psi\rangle_k$. The same computation as above shows that

$$\langle \phi, \Pi_k(\overline{f}\psi)\rangle_k = \langle \Pi_k\phi, \overline{f}\psi\rangle_k = \langle \phi, \overline{f}\psi\rangle_k,$$

and thus completes the proof.

iv) Similar to the computations in the proof of the previous part,

$$\langle T_k(f)\phi, \phi\rangle_k = \langle \Pi_k(f\phi), \phi\rangle_k = \int_M f h_k(\phi, \phi)d\mu \geq 0.$$

■

2.1.2 Schwartz Kernel

This subsection is based on chapter six of [7]. Consider a compact manifold X with Hermitian vector bundle $F \to X$ and a Hermitian form h^F. One can endow the space $C^\infty(X, F)$ of smooth sections with the inner product

$$\langle \phi, \psi\rangle_{L^2(X,F)} = \int_X h^F(\phi(x), \psi(x))dx, \tag{2.11}$$

where $\phi, \psi \in C^\infty(X, F)$. We then define the Hilbert space $L^2(X, F)$ of square-integrable sections as the completion of $C^\infty(X, F)$ with respect to this scalar product.

To endow $C^\infty(X, F)$ with a topology, take a finite cover $(U_i)_{1\leq i\leq p}$ of X, trivializing both X and F. For every i take an increasing sequence $(K_n^i)_{n\geq 1}$ of compact sets such that $\bigcup_{n\geq 1} K_n^i = U_i$. Consider the

countable family $(P_{j,n,i,r})_{j,n\geq 1,1\leq i\leq p,1\leq r\leq rank(F)}$ of seminorms defined by

$$P_{j,n,i,r}(\phi) = \max_{K_n^i} |\partial^j \phi_r^i| \tag{2.12}$$

for every $\phi \in C^\infty(X,F)$, where ϕ_r^i is the r-th coordinate of the image of ϕ in the trivialization associated with U_i. Endow $C^\infty(X,F)$ with the topology induced by these seminorms.

Let $D^{'}(X,F)$ be the space of distributions, which is the topological dual space of $C^\infty(X,F)$ with respect to the topology defined by the seminorms. Indeed, this is the space of continuous linear forms on $C^\infty(X,F)$. For every $\alpha \in D^{'}(X,F)$ and any $\phi \in C^\infty(X,F)$, there exists a pairing

$$(\alpha|\phi)_{D'(X,F),C^\infty(X,F)} = \alpha(\phi). \tag{2.13}$$

Moreover, any section $\psi \in L^2(X,F)$ can be seen as a section distribution by putting

$$\forall \phi \in C^\infty(X,F), \quad (\psi|\phi)_{D'(X,F),C^\infty(X,F)} = \langle \phi|\psi\rangle_{L^2(X,F)} = \int_X h_x^F(\phi(x),\psi(x))dx. \tag{2.14}$$

Remark. The map $\psi \in L^2(X,F) \overset{f}{\longmapsto} (\psi|\cdot)_{D'(X,F),C^\infty(X,F)}$ is injective and we get the inclusions

$$C^\infty(X,F) \subset L^2(X,F) \subset D^{'}(X,F). \tag{2.15}$$

Indeed, to show injectivity, assume that $\psi \in L^2(X,F)$ belongs to $\ker(f)$. Thus, for every smooth function $\phi \in C^\infty(X,F)$, we have

$$0 = \langle \phi|\psi\rangle_{L^2(X,F)} = \int_X h_x^F(\phi(x),\psi(x))dx.$$

Since smooth functions are dense in $L^2(X,F)$, there exists a sequence of smooth functions $(\phi_n)_{n\geq 1}$ converging to ψ in $L^2(X,F)$. Then by the dominated convergence theorem,

$$0 = \lim_{n\to\infty} \int_X h_x^F(\phi_n(x),\psi(x))dx = \int_X h_x^F(\psi(x),\psi(x))dx.$$

Now as Hermitian inner product has the property that $h_x^F(\psi(x),\psi(x)) \geq 0$ for every ψ, we can conclude that $h_x^F(\psi(x),\psi(x)) = 0$ for almost every $x \in X$. Hence, for almost every $x \in X$, $\psi(x) = 0$, again by a property of Hermitian inner product. Since ψ is an L^2 function, and $\psi = 0$ almost everywhere, by definition of L^2 functions, it should be identically zero, and thus the map f is injective.

Take another compact manifold Y with a Hermitian vector bundle $G \to Y$. Define the external tensor product $F \boxtimes G \to X \times Y$ by

$$F \boxtimes G := P_X^* F \otimes P_Y^* G, \tag{2.16}$$

for projections $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$. For sections $\phi \in C^\infty(X,F)$ and $\psi \in C^\infty(Y,G)$, define $\phi \boxtimes \psi := P_X^*\phi \otimes P_Y^*\psi \in C^\infty(X\times Y, F\boxtimes G)$, and call it a pure tensor. Define a Hermitian form $h^{F\boxtimes G}$ on $F \boxtimes G$ by considering its values on pure tensors

$$\forall (x,y) \in X\times Y, \forall f \in F_x, \forall g \in G_y, \quad h^{F\boxtimes G}_{(x,y)}(f\otimes g) := h_x^F(f) h_y^G(g), \tag{2.17}$$

and extending to $C^\infty(X\times Y, F\boxtimes G)$ sesquilinearly.

Given a section $K \in C^\infty(X \times Y, F \boxtimes \overline{G})$, the formula

$$\forall x \in X \quad ; \quad (\mathcal{K}\phi)(x) = \int_Y K(x,y).\phi(y)dy \tag{2.18}$$

defines an operator $\mathcal{K} : C^\infty(Y,G) \to C^\infty(X,F)$, where the dot stands for contraction with respect to the Hermitian product h^G on G:

$$\forall y \in Y, \forall f \in G_y, \forall \overline{g} \in \overline{G_y}, \quad \overline{g}.f := h^G_y(f,g). \tag{2.19}$$

Lemma 2.1.2. *If $K \in C^\infty(X \times Y, F \boxtimes \overline{G})$ and $\phi \in C^\infty(Y,G)$, then*

$$(\mathcal{K}(\phi)|\psi)_{D'(X,F),C^\infty(X,F)} = (K|\psi \boxtimes \overline{\phi})_{D'(X\times Y, F\boxtimes\overline{G}),C^\infty(X\times Y,F\boxtimes\overline{G})} \tag{2.20}$$

for every $\psi \in C^\infty(X,F)$. In this formula we view K as an element of $D'(X \times Y, F \boxtimes \overline{G})$ and $\mathcal{K}(\phi)$ as an element of $D'(X,F)$.

Proof. By linearity of both sides with respect to K, it is enough to prove for

$$K(x,y) = f(x,y)K_X(x) \boxtimes \overline{K_Y(y)} \tag{2.21}$$

with $K_X \in C^\infty(X,F)$ and $K_Y \in C^\infty(Y,G)$, as any other section $K \in C^\infty(X \times Y, F \boxtimes \overline{G})$ can be written as a finite sum of elements of the form (2.21). Using (2.14) and (2.15),

$$(\mathcal{K}(\phi)|\psi)_{D'(X,F),C^\infty(X,F)} = \int_X h^F_x(\psi(x), \mathcal{K}(\phi)(x))dx,$$

where by (2.18)

$$(\mathcal{K}\phi)(x) = \int_Y K(x,y).\phi(y)dy = \big(\int_Y f(x,y)h^G_y(\phi(y), K_Y(y))dy\big)K_X(x).$$

Thus,

$$\begin{aligned}(\mathcal{K}(\phi)|\psi)_{D'(X,F),C^\infty(X,F)} = & \\ & \int_{X\times Y} \overline{f(x,y)}h^F_x(\psi(x), K_X(x))\overline{h^G_y(\phi(y), K_Y(y))}dxdy = \\ & \int_{X\times Y} \overline{f(x,y)}h^F_x(\psi(x), K_X(x))h^{\overline{G}}_y(\overline{\phi}(y), \overline{K}_Y(y))dxdy = \\ & \int_{X\times Y} h^{F\boxtimes\overline{G}}_{(x,y)}\big((\psi \boxtimes \overline{\phi})(x,y), f(x,y)(K_X \boxtimes \overline{K_Y})(x,y)dxdy\big)\end{aligned}$$

which is equal to the RHS of (2.20). ∎

Theorem 2.1.3 (The Schwartz kernel theorem)**.** *Every $K \in D'(X\times Y, F\boxtimes\overline{G})$ defines according to (2.20) an operator $\mathcal{K} : C^\infty(Y,G) \to D'(X,F)$ which is continuous in the sense that for every sequence $(\phi_n)_{n\geq 1}$ of elements of $C^\infty(Y,G)$ converging to zero, the sequence $(\mathcal{K}(\phi_n))_{n\geq 1}$ converges to zero in $D'(X,F)$. Conversely, any continuous operator $\mathcal{K} : C^\infty(Y,G) \to D'(X,F)$ has a unique section distribution $K \in D'(X\times Y, F\boxtimes\overline{G})$ such that (2.20) holds. K is called the Schwartz kernel of $\mathcal{K}$.*

Now let $(\psi_i)_{i\geq 1}$ and $(\phi_j)_{j\geq 1}$ be orthonormal bases of the Hilbert spaces $L^2(X,F)$ and $L^2(Y,G)$ respectively. Then $(\psi_i \boxtimes \overline{\phi_j})_{i,j\geq 1}$ is an orthonormal basis for $L^2(X\times Y, F\boxtimes \overline{G})$. Given $K(\cdot,\cdot)\in L^2(X\times Y, F\boxtimes\overline{G})$, equation (2.18) defines an operator

$$\mathcal{K} : L^2(Y,G)\to L^2(X,F) \tag{2.22}$$

which is continuous in L^2-sense.

Remark. The bounded integral operator $\mathcal{K}$ as in (2.22) is called a Hilbert-Schmidt integral operator.

Proposition 2.1.4. *Let $\mathcal{K} : L^2(Y,G)\to L^2(X,F)$ be a bounded operator. Then $\mathcal{K}$ has a Schwartz kernel $K(\cdot,\cdot)\in L^2(X\times Y, F\boxtimes\overline{G})$ which can be computed as follows: For any two orthonormal bases $(\psi_i)_{i\geq 1}$ and $(\phi_j)_{j\geq 1}$ of $L^2(X,F)$ and $L^2(Y,G)$ respectively, we have*

$$K(x,y)=\sum_{i,j\geq 1}\langle \mathcal{K}\phi_j,\psi_i\rangle_{L^2(X,F)}\psi_i(x)\otimes\overline{\phi_j(y)} \tag{2.23}$$

in the L^2-sense.

Proof. Let $\mathcal{K} : L^2(Y,G)\to L^2(X,F)$ be a bounded operator, and let $i_1 : C^\infty(X,F)\to L^2(X,F)$, $i_2 : L^2(X,F)\to D'(X,F)$, $\tilde{i}_1 : C^\infty(Y,G)\to L^2(Y,G)$, and $\tilde{i}_2 : L^2(Y,G)\to D'(Y,G)$ be the canonical injective maps as in (2.15), and thus all of them are continuous. Since $\mathcal{K}$ is a bounded operator, it is a countinuous linear map. Hence the map $i_2\circ\mathcal{K}\circ\tilde{i}_1 : C^\infty(Y,G)\to D'(X,F)$ is also countinuous, and thus by the Schwartz kernel theorem 2.1.3 has a unique section distribution $K\in D'(X\times Y, F\boxtimes\overline{G})$ such that (2.20) holds. Then for every $\phi\in C^\infty(Y,G)$ and every $\psi\in C^\infty(X,F)$,

$$\langle\mathcal{K}(\phi)|\psi\rangle_{L^2(X,F)}=(i_2\circ\mathcal{K}\circ\tilde{i}_1)_{D'(X,F),C^\infty(X,F)}\overset{(2.20)}{=}(K|\psi\boxtimes\overline{\phi})_{D'(X\times Y,F\boxtimes\overline{G}),C^\infty(X\times Y,F\boxtimes\overline{G})}.$$

So, by the Schwartz kernel theorem, the kernel K exists for smooth functions. Now since smooth functions are dense in L^2, and since the L^2-inner product is continuous, the integral kernel K exists for L^2-functions and it is an element of $L^2(X\times Y, F\boxtimes\overline{G})$.

To obtain (2.23), let $\phi\in L^2(Y,G)$. Then

$$\mathcal{K}(\phi)=\sum_{i\geq 1}\langle\mathcal{K}\phi,\psi_i\rangle_{L^2(X,F)}\psi_i=\sum_{i,j\geq 1}\langle\phi,\phi_j\rangle_{L^2(Y,G)}\langle\mathcal{K}\phi_j,\psi_i\rangle_{L^2(X,F)}\psi_i. \tag{2.24}$$

Then using (2.11) and (2.18), for every $x\in X$ we have

$$\begin{aligned}\mathcal{K}(\phi)(x)=&\\ &\sum_{i,j\geq 1}\int_Y h^G(\phi(y),\phi_j(y))\langle\mathcal{K}\phi_j,\psi_i\rangle_{L^2(X,F)}\psi_i(x)dy=\\ &\sum_{i,j\geq 1}\int_Y(\overline{\phi_j(y)}.\phi(y))\psi_i(x)\langle\mathcal{K}\phi_j,\psi_i\rangle_{L^2(X,F)}dy=\\ &\sum_{i,j\geq 1}\int_Y(\psi_i(x)\otimes\overline{\phi_j(y)}).\phi(y)\langle\mathcal{K}\phi_j,\psi_i\rangle_{L^2(X,F)}dy\overset{(*)}{=}\\ &\int_Y\Big(\sum_{i,j\geq 1}\langle\mathcal{K}\phi_j,\psi_i\rangle_{L^2(X,F)}\psi_i(x)\otimes\overline{\phi_j(y)}\Big).\phi(y)dy.\end{aligned}$$

Note that swapping the sum and the integral in (*) is justified, as the integral kernel K exists by the arguments before equation (2.24), and thus Fubini's hypotheses are satisfied. ■

Lemma 2.1.5. *Let S be a closed subspace of $L^2(X,F)$ and let Π be the orthogonal projector $\Pi : L^2(X,F) \to S$. Let $(\phi_l)_{l\geq 1}$ be an orthonormal basis of S. Then the Schwartz kernel K of Π is given by*

$$K(x,y) = \sum_{l\geq 1} \phi_l(x) \otimes \overline{\phi_l}(y). \tag{2.25}$$

Proof. Note that $\Pi : L^2(X,F) \to S$ is a bounded operator, thus its Schwartz kernel K is given by (2.23). ■

Proposition 2.1.6 (Restriction to subspaces)**.** *Let $K : L^2(X,F) \to L^2(X,F)$ be a bounded operator with Schwartz kernel $K(\cdot,\cdot) \in L^2(X^2, F \boxtimes \overline{F})$. Moreover, let $S \subset L^2(X,F)$ be a closed subspace and let Π be the orthogonal projector from $L^2(X,F)$ to S. Then the operator $K\Pi$ possesses a Schwartz kernel $\tilde{K}$ which satisfies for any orthonormal bases $(\phi_l)_{l\geq 1}$ of S*

$$\tilde{K}(x,y) = \sum_{l\geq 1} (K\phi_l)(x) \otimes \overline{\phi_l(y)}. \tag{2.26}$$

Proof. Note that since both K and Π are bounded operators, $K\Pi : L^2(X,F) \to L^2(X,F)$ is bounded. Thus by proposition 2.1.4, it has a Schwartz kernel $\tilde{K} \in L^2(X^2, F \boxtimes \overline{F})$ as in (2.23). Let $(\tilde{\phi}_l)_{l\geq 1}$ be an extension of the basis $(\phi_l)_{l\geq 1}$ to an orthonormal basis of $L^2(X,F)$. Then by (2.23),

$$\tilde{K}(x,y) = \sum_{i,j\geq 1} \langle K\Pi\tilde{\phi}_j, \tilde{\phi}_i\rangle_{L^2(X,F)} \tilde{\phi}_i(x) \otimes \overline{\tilde{\phi}_j(y)}.$$

Let $\phi \in L^2(X,F)$. Since $\Pi\phi = \sum_{l\geq 1}\langle \phi, \phi_l\rangle_{L^2(X,F)}\phi_l$, and K is linear and bounded, $K\Pi\tilde{\phi}_j = \sum_{l\geq 1}\langle \tilde{\phi}_j, \phi_l\rangle_{L^2(X,F)} K\phi_l$. Thus

$$\langle K\Pi\tilde{\phi}_j, \tilde{\phi}_i\rangle_{L^2(X,F)} = \sum_{l\geq 1} \langle \tilde{\phi}_j, \phi_l\rangle_{L^2(X,F)} \langle K\phi_l, \tilde{\phi}_i\rangle_{L^2(X,F)}.$$

Now as $(\tilde{\phi}_l)_{l\geq 1}$ is an extended orthonormal basis of $L^2(X,F)$,

$$\begin{aligned} \tilde{K}(x,y) = \\ \sum_{i,l\geq 1} \langle K\phi_l, \tilde{\phi}_i\rangle_{L^2(X,F)} \tilde{\phi}_i(x) \otimes \overline{\phi_l(y)} = \\ \sum_{l\geq 1} \left(\sum_{i\geq 1} \langle K\phi_l, \tilde{\phi}_i\rangle_{L^2(X,F)} \tilde{\phi}_i(x) \right) \otimes \overline{\phi_l(y)} = \\ \sum_{l\geq 1} (K\phi_l)(x) \otimes \overline{\phi_l(y)}. \end{aligned}$$

■

Proposition 2.1.7 (Trace of an integral operator)**.** *Let $K : L^2(X,F) \to L^2(X,F)$ be a bounded operator with Schwartz kernel $K(\cdot,\cdot) \in L^2(X^2, F \boxtimes \overline{F})$. Then*

$$\mathrm{Tr}(K) = \int_X \mathrm{Tr}(K(x,x))dx, \tag{2.27}$$

where we see $K(x,x) \in F_x \otimes \overline{F_x}$ as an endomorphism of F_x.

Proof. By (2.23) we have that for $x \in X$

$$\mathrm{Tr}(K(x,x)) = \sum_{i,j\geq 1} \langle K\phi_i, \phi_j\rangle_{L^2(X,F)} \mathrm{Tr}(\phi_j(x) \otimes \overline{\phi_i(x)}).$$

But the endomorphism $\phi_j(x) \otimes \overline{\phi_i(x)}$ of F_x corresponds to $u \in F_x \mapsto h_x(u, \phi_i(x))\phi_j(x)$, hence its trace is equal to $h_x(\phi_j(x), \phi_i(x))$, and thus

$$\mathrm{Tr}(K(x,x)) = \sum_{i,j\geq 1} \langle K\phi_i, \phi_j\rangle_{L^2(X,F)} h_x(\phi_j(x), \phi_i(x)),$$

and

$$\int_X \mathrm{Tr}(K(x,x)) = \sum_{i,j\geq 1} \langle K\phi_i, \phi_j\rangle_{L^2(X,F)} \langle \phi_j(x), \phi_i(x)\rangle_{L^2(X,F)} = \sum_{i\geq 1} \langle K\phi_i, \phi_i\rangle_{L^2(X,F)} = \mathrm{Tr}(K).$$

■

2.1.3 Asymptotic behavior of the projector Π_k

Let (M^{2n}, J, ω) be a compact Kahler manifold, and let $L \to M$ be a Hermitian holomorphic complex line bundle. Consider the projection $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$. Our goal here is to compute the dimension of $\mathcal{H}_k$, the Hilbert space at level k. Indeed, we have to show that

$$\dim \mathcal{H}_k = \left(\frac{k}{2\pi}\right)^n \mathrm{Vol}(M) + \mathcal{O}(k^{n-1}). \tag{2.28}$$

Let $\overline{M}$ be the manifold M with symplectic form $-\omega$ and complex structure $-J$. Define $\Delta_M := \mathrm{diag}(M \times \overline{M})$ to be the diagonal of $M \times \overline{M}$, which is a Lagrangian submanifold. In order to prove (2.28), we need to understand the Schwartz kernel of the projection operator $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$. In order to do it we need to study a special section E of $L \boxtimes \overline{L} \to M \times \overline{M}$ which is related to Δ_M. In the following, first, we state Theorem 2.1.8, which is the most fundamental theorem in the theory of Berezin-Toeplitz operators. We then elaborate on the existence and uniqueness of the promised section E and use some properties of Schwartz kernel to conclude (2.28).

Definition. Let V be a smooth manifold. A function $f \in C^\infty(V)$ vanishes to order $N \geq 1$ along a submanifold $Y \subset V$ if for every $m \in \{0, \cdots, N-1\}$ and for any vector fields $X_1, \cdots, X_m$ one has

$$(\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_m} f)_{|Y} = 0.$$

If $K \to V$ is a complex line bundle with connection ∇, we say that a section $s \in C^\infty(V, K)$ vanishes to order $N \geq 1$ along Y if for every $m \in \{0, \cdots, N-1\}$ and for any vector fields $X_1, \cdots, X_m$ one has

$$(\nabla_{X_1} \cdots \nabla_{X_m} s)_{|Y} = 0.$$

A function or section is said to vanish to infinite order along Y, if it vanishes to order N along Y for every $N \geq 1$. Denote $\mathcal{I}_\infty(Y)$ for the set of functions or sections vanishing to infinite order along Y.

Theorem 2.1.8. *The projector $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$ has a Schwartz kernel of the form*

$$\Pi_k(x,y) = \left(\frac{k}{2\pi}\right)^n E^k(x,y)u(x,y,k) + R_k(x,y), \tag{2.29}$$

where the section E is as in proposition 2.1.9, and $u(\cdot,\cdot,k)$ is a sequence of functions in $C^\infty(M \times M, \mathbb{R})$ having an asymptotic expansion of the form

$$u(\cdot,\cdot,k) \sim \sum_{l\geq 0} k^{-l} u_l(\cdot,\cdot) \tag{2.30}$$

for the C^∞-topology, with $u_0(x,x) = 1$ where for any $Z \in C^\infty(M, T^{1,0}M)$, the functions $\mathcal{L}_{(\overline{Z},0)}u_l$ and $\mathcal{L}_{(0,Z)}u_l$ vanish to infinite order along the diagonal Δ_M of M^2, and $R_k = \mathcal{O}(k^{-\infty})$ uniformly in (x,y).

Proof. Proof of this theorem will be stated in the next subsection, after we have introduced the notion of coherent states. ■

Here the meaning of (2.30) is that for any $N \geq 0$, the function $u(\cdot,\cdot,k) - \sum_{l=0}^{N} k^{-l}u_l(\cdot,\cdot)$ and all its derivatives are uniformly $\mathcal{O}(K^{-(N+1)})$. Moreover, note that a sequence $a_k \in \mathbb{R}$ is $\mathcal{O}(k^{-\infty})$ if for every $N \geq 1$, there exists a constant C_N such that $|a_k| \leq C_N k^{-N}$ for all sufficiently large k [5].

Let us study the section E for a while.

Proposition 2.1.9. *There exists a section E of $L \boxtimes \overline{L} \to M \times \overline{M}$ such that*

i) $\forall x \in M$; $E(x,x) = 1$,

ii) For all $x, y \in M$ with $x \neq y$; $||E(x,y)|| < 1$,

iii) For all $x, y \in M$; $\overline{E(x,y)} = E(y,x)$,

iv) For any $Z \in C^\infty(M, T^{1,0}M)$ the sections $\tilde{\nabla}_{(\overline{Z},0)}E$ and $\tilde{\nabla}_{(0,Z)}E$ belong to $\mathcal{I}_\infty(\Delta_M)$.

Such a section is unique up to an element of $\mathcal{I}_\infty(\Delta_M)$. In this statement $\tilde{\nabla}$ is the connection induced by ∇ on $L \boxtimes \overline{L}$ and $||.||$ is the norm induced by h on $L \boxtimes \overline{L}$.

To prove Proposition 2.1.9 we need the following propositions and lemmas:

Proposition 2.1.10. *There exists a section E of $L \boxtimes \overline{L} \to M \times \overline{M}$ satisfying points (i) and (iv) of proposition 2.1.9. Such a section is unique up to an element of $\mathcal{I}_\infty(\Delta_M)$.*

Proof. Let s be a local non-vanishing holomorphic section on some open subset $U \subset M$ and let $\phi = -log(h(s,s))$. Thus $h(s,s) = \exp(-\phi)$. Then $u = \exp(\phi/2)s$ is such that $u(x) \otimes \overline{u}(x) = h_x(u(x), u(x)) = \exp(\phi)\exp(-\phi) = 1$ for every $x \in U$. So we look for some local section E of $L \boxtimes \overline{L}$ over $U \times U$ of the form

$$E(x,y) = \exp(i\psi(x,y))u(x) \otimes \overline{u}(y), \quad \psi(x,x) = 0.$$

Since $u(x)\otimes\overline{u}(y)=\exp(\frac{1}{2}(\phi(x)+\phi(y)))s(x)\otimes\overline{s}(y)$, we have

$$\begin{aligned}\tilde{\nabla}(u(x)\otimes\overline{u}(y))=\\ \nabla u(x)\otimes\overline{u}(y)+u(x)\otimes\nabla\overline{u}(y)=\\ \nabla(\exp(\phi(x)/2)s(x))\otimes\overline{u}(y)+u(x)\otimes\nabla(\exp(\phi(y)/2)\overline{s}(y))=\\ \frac{1}{2}d\phi(x)\exp(\phi(x)/2)s(x)\otimes\overline{u}(y)+\exp(\phi/2)\nabla s(x)\otimes\overline{u}(y)+\\ u(x)\otimes\frac{1}{2}d\phi(y)\exp(\phi(y)/2)\overline{s}(y)+u(x)\otimes\exp(\phi/2)\nabla\overline{s}(y)=\\ d\Phi\otimes(u(x)\otimes\overline{u}(y))+\exp(\Phi)\tilde{\nabla}(s(x)\otimes\overline{s}(y)),\end{aligned}$$

where $\Phi(x,y)=\frac{1}{2}(\phi(x)+\phi(y))$. Thus,

$$\tilde{\nabla}E=id\tilde{\psi}\otimes E+\exp(i\psi+\Phi)\tilde{\nabla}(s(x)\otimes\overline{s}(y)),$$

where $\tilde{\psi}(x,y)=\psi(x,y)-(i/2)(\phi(x)+\phi(y))$. Let $(z_1,\cdots,z_n)$ be a local holomorphic coordinate on U and let $(z_1^l,\cdots,z_n^l,z_1^r,\cdots,z_n^r)$ be the corresponding coordinates on $U\times U$. Now as $\psi(x,x)=0$, we have $\tilde{\psi}(x,x)=-i\phi(x)$. Besides, since s is holomorphic, $\nabla_{\overline{Z}}s=0$ for every $Z\in C^\infty(M,T^{1,0}M)$, and thus condition (iv) in the statement of proposition 2.1.9 is equivalent to

$$\tilde{\psi}(x,x)=-i\phi(x),\quad \frac{\partial\tilde{\psi}}{\partial\overline{z}_j^l}=0,\frac{\partial\tilde{\psi}}{\partial z_j^r}=0\quad \mathrm{mod}\mathcal{I}_\infty(\mathrm{diag}(U\times U));\quad 1\le j\le n. \tag{2.31}$$

Indeed, Every $\overline{Z}\in C^\infty(M,T^{0,1}M)$ can be written in local coordinates as $\overline{Z}=\sum_{i=1}^n a_i\frac{\partial}{\partial\overline{z}_i}$, and every $Z\in C^\infty(M,T^{1,0}M)$ can be written in local coordinates as $Z=\sum_{j=1}^n b_j\frac{\partial}{\partial z_j}$ where a_i and b_js are smooth functions on M. Hence, condition (iv) is equivalent to

$$i\frac{\partial\tilde{\psi}}{\partial\overline{z}_i}\otimes E+\exp(i\psi+\Phi)\tilde{\nabla}_{(\overline{z}_i,0)}(s\otimes\overline{s})=0,\quad \mathrm{mod}\mathcal{I}_\infty(\mathrm{diag}(U\times U));\quad 1\le i\le n,$$

and

$$i\frac{\partial\tilde{\psi}}{\partial z_j}\otimes E+\exp(i\psi+\Phi)\tilde{\nabla}_{(0,z_j)}(s\otimes\overline{s})=0,\quad \mathrm{mod}\mathcal{I}_\infty(\mathrm{diag}(U\times U));\quad 1\le j\le n.$$

Note that

$$\tilde{\nabla}_{(\overline{z}_i,0)}(s\otimes\overline{s})=\nabla_{\overline{z}_i}s\otimes\overline{s}+s\otimes\nabla_0\overline{s}=0,\quad \forall 1\le i\le n,$$

and similarly

$$\tilde{\nabla}_{(0,z_j)}(s\otimes\overline{s})=\nabla_0 s\otimes\overline{s}+s\otimes\nabla_{z_j}\overline{s}=0,\quad \forall 1\le j\le n,$$

so we can conclude (2.31).

Now we claim that there exists a function $\tilde{\psi}$, unique modulo $\mathcal{I}_\infty(\mathrm{diag}(U\times U))$ satisfying (2.31). Indeed, these equations force the Taylor expansion of $\tilde{\psi}$ along the diagonal to be of the form

$$\tilde{\psi}(w+z^l,w+z^r)=-i\sum_{\alpha,\beta\in\mathbb{N}^n}\frac{1}{\alpha!\beta!}\left(\frac{\partial^{\alpha+\beta}\phi}{\partial z_\alpha^l\partial\overline{z}_\beta^r}\right)(w)(z^l)^\alpha(\overline{z}^r)^\beta$$

Uniqueness of Taylor series gives us the uniqueness part of the proposition. For the existence part, Whit-

ney extension theorem (see next remark) implies that we can construct a function with this given Taylor expansion along the diagonal. Indeed, we can choose $U \subset M$ to be a trivialization chart open set, and thus homeomorphic to $\mathbb{R}^{2n}$, and thus $U \times U$ is homeomorphic to $\mathbb{R}^{4n}$. Let $K \subset \mathrm{diag}(U \times U)$ be a large enough compact set. Thus for any point $x \in K$, $x = (w, w)$ for some $w \in U$. To apply Whitney extension theorem, fix some positive integer k. Then for every $|p| \leq k$, let

$$u_p(x) := -i \left[\sum_{\alpha+\beta=p} \frac{1}{\alpha!\beta!} \left(\frac{\partial^p \phi}{\partial z_\alpha^l \partial \overline{z}_\beta^r} \right) (w)(z^l)^\alpha (\overline{z}^r)^\beta \right].$$

Then for every $|p| \leq k$, by the definition of u_p, and as ϕ is a smooth function, U_p is continuous on $K \times K$. Thus by the Whitney extension theorem, we can find some $v_k \in C^k(\mathbb{R}^{4n}) \cong C^k(U \times U)$ such that whenever $|p| \leq k$, $\partial^p v_k(x) = u_p(x)$, for every $x \in K$. Moreover, v_k can be chosen so that with C depending only on K,

$$\sum_{|p|\leq k} \sup |\partial^p v_k| \leq C \left(\sum_{|p|\leq k} \sup_{K\times K} U_p + \sum_{|p|\leq k} \sup_K |u_p| \right).$$

Now varying k, we will find a sequence of functions $\{v_k\}_{k\in\mathbb{N}}$, such that for each k, $v_k \in C^k(\mathbb{R}^{4n})$, with bounded derivatives as in the above inequality. Note that by the same inequality, each v_k is uniformly bounded. Moreover, as each v_k has bounded derivatives of all order, they are uniformly equicontinuous. Hence, we obtain a sequence of continuous functions $\{v_k\}_{k\in\mathbb{N}}$ that is uniformly bounded and uniformly equicontinuous. Thus, hypotheses of Arzela-Ascoli theorem are satisfied, and we obtain a uniformly convergent subsequent $\{v_{n_k}\}_{k\in\mathbb{N}}$. Define $\tilde{\psi} := \lim_{k\to\infty} v_{n_k}$. By Arzela-Ascoli theorem and since every v_k is continuous, $\tilde{\psi}$ exists and is a continuous function. Now as each v_k is a C^k-function, $\tilde{\psi}$ is a smooth function as it is a limit of C^k-functions, with the desired Taylor expansion along the diagonal.

So we have obtained a function $\tilde{\psi}$, unique modulo $\mathcal{I}_\infty(\mathrm{diag}(U\times U))$ satisfying (2.31). Now take a partition of unity subordinate to a finite cover of Δ_M by such open subsets $U \times U$. Local uniqueness implies that one can construct E globally and show that it is unique modulo an element of $\mathcal{I}_\infty(\Delta_M)$. This completes the proof. ■

Remark (Whitney extension theorem[6]). Let K be a compact set in $\mathbb{R}^n$ and u_α, $|\alpha| \leq k$, continuous functions on K. Set for $|\alpha| \leq k$

$$U_\alpha(x, y) = \left| u_\alpha(x) - \sum_{|\beta|\leq k-|\alpha|} \frac{1}{\beta!} u_{\alpha+\beta}(y)(x-y)^\beta \right| |x-y|^{|\alpha|-k}$$

when $x, y \in K$ and $x \neq y$, and $U_\alpha(x, x) = 0$ when $x \in K$. If U_α is continuous on $K \times K$ when $|\alpha| \leq k$, it is possible to find $v \in C^k(\mathbb{R}^n)$ with $\partial^\alpha v(x) = u_\alpha(x)$, $x \in K$, $|\alpha| \leq k$. One can then choose v so that with C depending only on K,

$$\sum_{|\alpha|\leq k} \sup |\partial^\alpha v| \leq C \left(\sum_{|\alpha|\leq k} \sup_{K\times K} U_\alpha + \sum_{|\alpha|\leq k} \sup_K |u_\alpha| \right).$$

This theorem states that for a compact subset K of an Euclidean space, a given function on K can be extended in such a way as to have prescribed derivatives at the points of K.

Lemma 2.1.11. *Let E be as in proposition 2.1.10, and let us introduce the 1-form α_E defined on a neigh-*

borhood of Δ_M by $\tilde{\nabla}E = -i\alpha_E \otimes E$. Then

i) α_E vanishes along Δ_M,

ii) There exists a section B_E of $(T^*(M \times \overline{M}) \otimes T^*(M \times \overline{M})) \otimes \mathbb{C} \to \Delta_M$ such that for any vector field X and Y of $M \times \overline{M}$, $\mathcal{L}_X(\alpha_E(Y)) = B_E(X, Y)$ along Δ_M. Moreover, twice the anti-symmetric part of B_E is equal to $\tilde{\omega} = p_1^*\omega - p_2^*\omega$,

iii) For every $x \in \Delta_M$ and for any $X, Y \in T_x(M \times \overline{M}) \otimes \mathbb{C}$

$$B_E(X, Y) = \tilde{\omega}(q(X), Y), \tag{2.32}$$

where q is the projection from $T_x(M \times \overline{M}) \otimes \mathbb{C}$ onto $T_x^{0,1}(M \times \overline{M})$ with kernel $T_x\Delta_M \otimes \mathbb{C}$.

Proof. Let $\tilde{J}$ be the complex structure on $M \times \overline{M}$ and let $\tilde{g} = \tilde{\omega}(\cdot, \tilde{J}\cdot)$ be the Kahler metric on $M \times \overline{M}$.

To show (i):

To begin, note that we can write $\tilde{\nabla}E = -i\alpha_E \otimes E$ by proposition A.2.1. For any $x \in \Delta_M$, $T_x^{0,1}(M \times \overline{M}) \oplus (T_x\Delta_M \otimes \mathbb{C}) = T_x(M \times \overline{M}) \otimes \mathbb{C}$. Indeed, let $Z \in T_x^{0,1}(M \times \overline{M}) \cap (T_x\Delta_M \otimes \mathbb{C})$. Since $T_x\Delta_M \otimes \mathbb{C}$ is Lagrangian and $\overline{Z}$ also belongs to $T_x\Delta_M \otimes \mathbb{C}$, we have $\tilde{\omega}(Z, \overline{Z}) = 0$. But since $Z \in T_x^{0,1}(M \times \overline{M})$, we can write $Z = X + i\tilde{J}X$ for some $X \in T_x(M \times \overline{M})$. Hence

$$0 = \tilde{\omega}(X + i\tilde{J}X, X - i\tilde{J}X) = -2i\tilde{\omega}(X, \tilde{J}X) = -2i\tilde{g}(X, X).$$

Thus, $\tilde{g}(X, X) = 0$. So $X = 0$ and $Z = 0$. Now for $X \in T_x\Delta_M \otimes \mathbb{C}$, $\alpha_E(X) = 0$ as $E_{|\Delta_M} = 1$. If $Y \in T_x^{0,1}(M \times \overline{M})$, by proposition 2.1.10, $\tilde{\nabla}_Y E = 0$ and thus $\alpha_E(Y) = 0$. Consequently, $\alpha_E = 0$ along Δ_M.

To show (ii):

B_E is well-defined because the value of $\mathcal{L}_X(\alpha_E(Y))$ at $x \in \Delta_M$ only depends on the values of X and Y at x. Indeed, it is enough to show that the map $(X, Y) \mapsto \mathcal{L}_X(\alpha_E(Y))$ is $C^\infty(M \times \overline{M})$-linear. Let $f, g \in C^\infty(M \times \overline{M})$ and X, Y be two vector fields as above. Then

$$\begin{aligned}
\mathcal{L}_{fX}(\alpha_E)(gY) = (fX)\alpha_E(gY) - \alpha_E([fX, gY]) = \\
fgX(\alpha_E(Y)) - fg\alpha_E([X, Y]) + fX(g)\alpha_E(Y) + gY(f)\alpha_E(X) = \\
fg\mathcal{L}_X(\alpha_E)(Y) + fX(g)\alpha_E(Y) + gY(f)\alpha_E(X) = \\
fg\mathcal{L}_X(\alpha_E)(Y),
\end{aligned}$$

because by part (i), α_E vanishes along Δ_M. Hence, we can conclude that however Lie derivative in general is not a point operator, the value of $\mathcal{L}_X(\alpha_E(Y))$ at $x \in \Delta_M$ only depends on the values of X and Y at the point x, and not on a whole neighborhood of x.

Note that E is a local non-vanishing section of $L \boxtimes \overline{L} \to M \times \overline{M}$, and since $\tilde{\nabla}E = -i\alpha_E \otimes E$, $-i\alpha_E$ is a primitive of $\mathrm{curv}(\nabla)$. That is, $\mathrm{curv}(\nabla) = -id\alpha_E$, by lemma A.3.1. Now given two vector fields X, Y of $M \times \overline{M}$, since the curvature of $\tilde{\nabla}$ is $-i\tilde{\omega}$ we have

$$\tilde{\omega}(X, Y) = d\alpha_E(X, Y) = \mathcal{L}_X(\alpha_E(Y)) - \mathcal{L}_Y(\alpha_E(X)) - \alpha_E([X, Y]).$$

Evaluating at a point on Δ_M and by part (i) we get

$$\tilde{\omega}(X,Y) = B_E(X,Y) - B_E(Y,X).$$

To show (iii):

Since α_E vanishes along Δ_M, we have that for $x \in \Delta_M$, $X \in T_x\Delta_M \otimes \mathbb{C}$ and $Y \in T_x(M \times \overline{M}) \otimes \mathbb{C}$, $B_E(X,Y) = 0$, hence

$$\forall x \in \Delta_M, \quad \forall X, Y \in T_x(M \times \overline{M}) \otimes \mathbb{C}; \quad B_E(X,Y) = B_E(q(X), Y),$$

where q is the projection from $T_x(M \times \overline{M}) \otimes \mathbb{C}$ onto $T_x^{0,1}(M \times \overline{M})$ with kernel $T_x\Delta_M \otimes \mathbb{C}$.

Thus, it is enough to prove that $B_E(X,Y) = \tilde{\omega}(X,Y)$ for $X \in T_x^{0,1}(M \times \overline{M})$. Now since for any $X \in T^{0,1}(M \times \overline{M})$ and for any $Y \in T(M \times \overline{M}) \otimes \mathbb{C}$ one has $\tilde{\nabla}_Y \tilde{\nabla}_X E = 0$ on Δ_M, we obtain

$$\forall x \in \Delta_M, \quad \forall X \in T_x^{0,1}(M \times \overline{M}), \quad \forall Y \in T_x(M \times \overline{M}) \otimes \mathbb{C}; \quad B_E(Y,X) = 0.$$

Now we can conclude (iii) by using that twice the anti-symmetric part of B_E is $\tilde{\omega}$. ■

Lemma 2.1.12. *We consider E as in proposition 2.1.10 and introduce the function $\phi_E := -2log||E||$. Then $d\phi_E$ vanishes along Δ_M and the Hessian of ϕ_E at $x \in \Delta_M$ is the bilinear symmetric form of $T_x(M \times \overline{M})$ with kernel $T_x\Delta_M$ and whose restriction to $\tilde{J}(T_x\Delta_M)$ is equal to $2\tilde{g}$.*

Proof. By the arguments in Appendix A, we know that if s is an arbitrary local section of $L \boxtimes \overline{L}$ over some open subset $U \times \overline{U} \subset M \times \overline{M}$ such that $h(s,s) = 1$, and if β is the local one-form such that $\tilde{\nabla} s = \beta \otimes s$, then $\tilde{\nabla}$ and h are compatible if and only if all the forms β are purely imaginary, i.e. of the form $\beta = i\alpha$ with $\alpha \in \Omega^1(U \times \overline{U})$. Moreover, by the arguments about Chern connections, we know that the curvature of the Chern connection of a Hermitian holomorphic line bundle is purely imaginary. That is, $\text{curv}(\tilde{\nabla}) = d\beta$ is purely imaginary, and thus β itself is purely imaginary. Hence, we can conclude that $\tilde{\nabla}$ and h are compatible.

Using the compatibility between the connection $\tilde{\nabla}$ and the Hermitian metric on $L \boxtimes \overline{L}$, we see that $d||E||^2 = d(h(E,E)) = h(\tilde{\nabla}E, E) + h(E, \tilde{\nabla}E) = h(-i\alpha_E \otimes E, E) + h(E, -i\alpha_E \otimes E) = -i(\alpha_E - \overline{\alpha_E})||E||^2$. So

$$d\phi_E = -2\frac{d||E||}{||E||} = i(\alpha_E - \overline{\alpha_E}).$$

Since α_E vanishes along Δ_M by Lemma 2.1.11, so does ϕ_E.

Now we compute for $x \in \Delta_M$ the Hessian of ϕ_E at $X, Y \in T_x(M \times \overline{M})$ which is defined as

$$Hess_{\phi_E}(x)(X,Y) = \mathcal{L}_{\hat{X}}(\mathcal{L}_{\hat{Y}}\phi_E)(x),$$

for any vector fields $\hat{X}, \hat{Y}$ with $\hat{X}(x) = X$ and $\hat{Y}(x) = Y$. By the above computations

$$Hess_{\phi_E}(x)(X,Y) = i\mathcal{L}_{\hat{X}}(\alpha_E(\hat{Y}) - \overline{\alpha_E}(\hat{Y}))(x) = i(B_E(X,Y) - \overline{B_E(X,Y)}).$$

By Lemma 2.1.11 we know that $B_E(X,Y) = 0$ whenever $x \in T_x\Delta_M$. Thus the kernel of this Hessian contains $T_x\Delta_M$. Moreover, since $T_x\Delta_M$ is Lagrangian, its orthogonal complement with respect to $\tilde{g}$ is $\tilde{J}(T_x\Delta_M)$. But if $x \in \tilde{J}(T_x\Delta_M)$, we have $q(X) = q(X + i\tilde{J}X - i\tilde{J}X) = X + i\tilde{J}X$, as $q : T_x(M \times \overline{M}) \otimes \mathbb{C} \to T_x^{0,1}(M \times \overline{M})$ is

the projection, $X + i\tilde{J}X \in T_x^{0,1}(M \times \overline{M})$ and $-i\tilde{J}X \in T_x\Delta_M \otimes \mathbb{C}$. Now using (2.32) we obtain

$$Hess_{\phi_E}(x)(X,Y) = -2\tilde{\omega}(\tilde{J}X,Y) = 2\tilde{g}(X,Y) \quad \text{for } X,Y \in \tilde{J}(T_x\Delta_M),$$

which finishes the proof. ■

Proof of Proposition 2.1.9. Pick any E as in Proposition 2.1.10. By Lemma 2.1.12, the Hessian of $\phi_E = -2log||E||$ is positive on the orthogonal complement of $T\Delta_M$. Hence, there exists a neighborhood U of $T\Delta_M$ such that ϕ_E itself is positive on $U - \Delta_M$. Thus modifying E outside U if necessary, we obtain that $||E|| < 1$ outside Δ_M. Now observe that if E satisfies conditions (i),(ii), and (iv), the section $(x,y) \mapsto \frac{1}{2}(E(x,y) + \overline{E(y,x)})$ satisfies conditions (i), (ii), (iii), and (iv). ■

Let us assume that Theorem 2.1.8 is true, and try to compute the dimension of the Hilbert space $\mathcal{H}_k$. Using (2.27) we have

$$\dim \mathcal{H}_K = \mathrm{Tr}(\Pi_k) = \int_M \mathrm{Tr}(\Pi_k(x,x))d\mu(x) = \left(\frac{k}{2\pi}\right)^n \mathrm{Vol}(M) + \mathcal{O}(k^{n-1}), \tag{2.33}$$

and thus we obtain our goal (2.28).

2.1.4 Coherent States

Let $P \subset L$ be the set of elements $u \in L$ with $||u|| = 1$, and denote by $\pi : P \to M$ the natural projection.

Lemma 2.1.13. *Fix $u \in P$. For every $k \geq 1$, there exists a unique vector $\xi_k^u \in \mathcal{H}_k$ such that*

$$\forall \phi \in \mathcal{H}_k, \quad \phi(\pi(u)) = \langle \phi, \xi_k^u \rangle_k u^k. \tag{2.34}$$

Proof. Consider the linear form F_k defined on $\mathcal{H}_k$ by

$$\forall \phi \in \mathcal{H}_k, \quad F_k(\phi) = h_k(\phi(\pi(u)), u^k).$$

Riesz representation theorem states that for a Hilbert space $\mathcal{H}$ whose inner product $\langle \phi, \psi \rangle$ is linear in its first argument and antilinear in its second argument, for every continuous linear functional $F \in \mathcal{H}^*$, there exists a unique $\xi_F \in \mathcal{H}$ such that

$$F(\phi) = \langle \phi, \xi_F \rangle$$

Now since $\mathcal{H}_k$ is finite-dimensional, F_k is continuous, and thus Riesz representation theorem implies that there exists a unique vector $\xi_k^u \in \mathcal{H}_k$ such that $F_k(\phi) = \langle \phi, \xi_k^u \rangle_k$ for all ϕ in $\mathcal{H}_k$. But since u^k is an orthonormal basis of $L^k_{\pi(u)}$, we have $\phi(\pi(u)) = F_k(\phi)u^k$. ■

Definition. The vector $\xi_k^u \in \mathcal{H}_k$ is called the coherent state at u.

Lemma 2.1.14. *Let T_k be an operator $C^\infty(M, L^k) \to C^\infty(M, L^k)$ with kernel $T_k(\cdot,\cdot)$, and such that $\Pi_k T_k \Pi_k = T_k$. Then*

i) $\forall x \in M$, $(T_k \xi_k^u)(x) = T_k(x, \pi(u)).u^k$,

ii) $\langle T_k \xi_k^u, \xi_k^v \rangle_k = \overline{v}^k . T_k(\pi(v), \pi(u)).u^k$,

where the dot stands for contraction with respect to h_k.

Proof. i) Let $(\phi_i)_{i\leq 1\leq d_k}$ be an orthonormal basis of $\mathcal{H}_k$. By proposition 2.1.6, one can write the Schwartz kernel of the restriction of T_k to $\mathcal{H}_k$ as

$$\forall x, y \in M, \quad T_k(x,y) = \sum_{i,j=1}^{d_k} \langle T_k\phi_i, \phi_j\rangle_k \phi_j(x) \otimes \overline{\phi_i(y)}.$$

Hence, for $x \in M$ we have

$$T_k(x, \pi(u)).u^k = \sum_{i,j=1}^{d_k} \langle T_k\phi_i, \phi_j\rangle_k h_k(u^k, \phi_i(\pi(u)))\phi_j(x).$$

Now since $h_k(u^k, \phi_i(\pi(u))) = \langle \xi_k^u, \phi_i\rangle_k$, one can rewrite the above equation as

$$T_k(x, \pi(u)).u^k = \sum_{j=1}^{d_k} \langle T_k \left(\sum_{i=1}^{d_k} \langle \xi_k^u, \phi_i\rangle_k \phi_i \right), \phi_j\rangle_k \phi_j(x),$$

which yields that

$$T_k(x, \pi(u)).u^k = \langle T_k\xi_k^u, \phi_j\rangle_k \phi_j(x) = (T_k\xi_k^u)(x).$$

ii) Let $x \in M$. Using the first part of Lemma, $h_k(T_k\xi_k^u(x), \xi_k^v(x)) = h_k(T_k(x, \pi(u)).u^k, \xi_k^v(x))$. Integrating this equality we obtain

$$\langle T_k\xi_k^u, \xi_k^v\rangle_k = \langle T_k(., \pi(u)), \xi_k^v\rangle_k.$$

But the RHS of this equation is equal to $h_k(T_k(\pi(v), \pi(u)).u^k, \xi_k^v(x))$ by definition of ξ_k^v, and this term is in turn equal to $\overline{v}^k.T_k(\pi(v), \pi(u)).u^k$, and we are done with the proof.

■

Now as $T_k = \Pi_k$ satisfies the hypothesis of Lemma 2.1.14, we can conclude that

Corollary 2.1.15. *For every $u, v \in P$,*

i) $\forall x \in M$, $\xi_k^u(x) = \Pi_k(x, \pi(u)).u^k$,

ii) $\langle \xi_k^u, \xi_k^v\rangle_k = \overline{v}^k.\Pi_k(\pi(v), \pi(u)).u^k$, *and thus* $\Pi_k(\pi(v), \pi(u)) = \langle \xi_k^u, \xi_k^v\rangle_k v^k \otimes \overline{u}^k$,

iii) $||\xi_k^u||_k^2 = \Pi_k(\pi(u), \pi(u))$.

where the dot stands for contraction with respect to h_k.

Now we show how coherent states can be used to describe Berezin-Toeplitz operators. Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{S}(\mathcal{H})$ be the space of bounded self-adjoint operators on $\mathcal{H}$. Let X be a set endowed with a σ-algebra $\mathcal{C}$.

Definition. A positive operator-valued measure on X with values in $\mathcal{S}(\mathcal{H})$ is a map $G : \mathcal{C} \to \mathcal{S}(\mathcal{H})$ which satisfies the following properties:

i) For every $A \in \mathcal{C}$, $G(A)$ is a positive operator, i.e. $\langle G(A)\xi, \xi\rangle \geq 0$ for every $\xi \in \mathcal{H}$,

ii) $G(\emptyset) = 0$, and $G(X) = id$,

iii) G is σ-additive. That is, for any sequence $(A_j)_{j\geq 1}$ of disjoint elements of $\mathcal{C}$, $G(\cup_{j\geq 1} A_j) = \sum_{j\geq 1} G(A_j)$.

One can use such a positive operator-valued measure to define a probability measure μ_ξ on X for every $\xi \in \mathcal{H}$, using $\mu_\xi(A) := \langle G(A)\xi, \xi\rangle$ for $A \in \mathcal{C}$.

Definition. Given a bounded measurable function $f : X \to \mathbb{R}$, we define an operator $\int_X f dG \in \mathcal{S}(\mathcal{H})$ characterized by the following property:

$$\forall \xi \in \mathcal{H}, \quad \left\langle \left(\int_X f dG \right) \xi, \xi \right\rangle = \int_X f d\mu_\xi. \tag{2.35}$$

Now consider $X = M$ with the σ-algebra generated by its Borel sets, and let $\mathcal{H} = \mathcal{H}_k = H^0(M, L^k)$. For $x \in M$ and $u \in L_x$ with unit norm, let ξ_k^u be the coherent state at u. Using Corollary 2.1.15 and Theorem 2.1.8, we have that for every $u \in P$,

$$||\xi_k^u||_k^2 \sim \left(\frac{k}{2\pi}\right)^n \tag{2.36}$$

when k goes to infinity, the estimate being uniform in u. In particular, there exists $k_0 \geq 1$ such that for every $u \in P$, $\xi_k^u \neq 0$ when ever $k \geq k_0$. Considering $u \in L_x$, we claim that the function

$$R_k : M \to \mathbb{R}, \quad x \mapsto ||\xi_k^u||_k^2 \tag{2.37}$$

is well-defined, i.e. only depends on x. Indeed, let v be another unit vector in L_x. Then there exists some $\lambda \in S^1$ with $v = \lambda u$. Then $\xi_k^v = \lambda^k \xi_k^u$, and thus $||\xi_k^v||_k^2 = ||\xi_k^u||_k^2$. For $k \geq k_0$, R_k is a positive function.

Definition. The function R_k in (2.37) is known as the Rawnsley function.

Definition. We define the projection along the coherent state ξ_k^u as

$$P_{k,x} : \mathcal{H}_k \to \mathcal{H}_k, \quad \phi \mapsto \frac{\langle \phi, \xi_k^u\rangle_k}{||\xi_k^u||_k^2} \xi_k^u. \tag{2.38}$$

By the same argument as above, one can see that $P_{k,x}$ only depends on x.

Lemma 2.1.16. *For $k \geq k_0$, the map G_k such that $dG_k = R_k(x) P_{k,x} d\mu$ defines a positive operator-valued measure on M.*

Proof. To prove the Lemma, we have to show that G_k satisfies the properties of a positive operator-valued measure defined above.

The positivity and σ-additivity are immediate from how we have defined G_k. Indeed, let $\phi \in \mathcal{H}_k$, $x \in M$, $u \in L_x$, and $A \in \mathcal{A}(M)$ be a measurable function. Then, positivity follows from

$$\begin{aligned}\langle G_k(A)\phi, \phi\rangle_k = \langle \int_A R_k(x)(P_{k,x}\phi) d\mu(x), \phi\rangle_k = \int_A R_k(x) \langle P_{k,x}\phi, \phi\rangle_k d\mu(x) = \\ \int_A R_k(x) \frac{\langle \phi, \xi_k^u\rangle_k}{||\xi_k^u||_k^2} \langle \xi_k^u, \phi\rangle_k d\mu(x) = \int_A |\langle \phi, \xi_k^u\rangle_k|^2 d\mu(x) \geq 0.\end{aligned}$$

To show σ-additivity, let $(A_j)_{j\geq 1}$ be a family of disjoint measurable sets. Moreover, let $\phi \in \mathcal{H}_k$, and

$y \in M$. Then

$$(G_k(\cup_{j\geq 1} A_j)\phi)(y) = \int_{\cup_{j\geq 1} A_j} R_k(x)(P_{k,x}\phi)(y)d\mu(x) =$$
$$\sum_{j\geq 1} \int_{A_j} R_k(x)(P_{k,x}\phi)(y)d\mu(x) = \sum_{j\geq 1}(G_k(A_j)\phi)(y),$$

where in the second line we have used the dominated convergence theorem and the fact that the sets A_j are mutually disjoint.

To show that $G_k(M) = id$, let $\phi \in \mathcal{H}_k$, and $y \in M$. Then

$$(G_k(M)\phi)(y) = \int_M R_k(x)(P_{k,x}\phi)(y)d\mu(x).$$

Using $\xi_k^u(y) = \Pi_k(y,x).u^k$, we have

$$R_k(x)(P_{k,x}\phi)(y) = \langle \phi, \xi_k^u\rangle_k \xi_k^u(y) = \Pi_k(y,x).(\langle \phi, \xi_k^u\rangle_k u^k).$$

Recall that ξ_k^u satisfies the reproducing property (2.34). Hence $\langle \phi, \xi_k^u\rangle_k u^k = \phi(x)$. Thus

$$(G_k(M)\phi)(y) = \int_M \Pi_k(y,x).\phi(x)d\mu(x) = (\Pi_k\phi)(y) = \phi(y).$$

■

Proposition 2.1.17. *Let $k \geq k_0$. For any $f \in C^\infty(M,\mathbb{R})$, $T_k(f) = \int_M f dG_k$.*

Proof. Let $S_k(f) := \int_M f dG_k$, and let $\phi \in \mathcal{H}_k$ be arbitrary. Then by definition,

$$\langle S_k(f)\phi, \phi\rangle_k = \int_M f(x)\langle P_{k,x}\phi, \phi\rangle_k R_k(x)d\mu(x).$$

We claim that for every $x \in M$, $\langle P_{k,x}\phi, \phi\rangle_k R_k(x) = h_k(\phi(x), \phi(x))$. Firstly, since ξ_k^u satisfies the reproducing property (2.34), we have that $\phi(x) = \langle \phi, \xi_k^u\rangle_k u^k$. So

$$h_k(\phi(x), \phi(x)) = |\langle \phi, \xi_k^u\rangle_k|^2 h_k(u^k, u^k) = |\langle \phi, \xi_k^u\rangle_k|^2.$$

Secondly,

$$\langle P_{k,x}\phi, \phi\rangle_k = \frac{|\langle \phi, \xi_k^u\rangle_k|^2}{||\xi_k^u||_k^2} = \frac{|\langle \phi, \xi_k^u\rangle_k|^2}{R_k(x)},$$

which proves the claim. Therefore,

$$\langle S_k(f)\phi, \phi\rangle_k = \int_M f(x)h_k(\phi(x), \phi(x))d\mu(x) = \int_M h_k(f(x)\phi(x), \phi(x))d\mu(x) = \langle T_k(f)\phi, \phi\rangle_k,$$

which completes the proof. ■

Remark. As you can see, we obtained what we have promised. Berezin-Toeplitz operators in (2.9) can be

written in terms of coherent states as follows:

$$T_k(f) = \int_M f(x) R_k(x) P_{k,x} d\mu(x). \tag{2.39}$$

We are now ready to give a sketch of the proof of theorem 2.1.8, mostly based on chapter seven of [7].

Proof of Theorem 2.1.8. The main idea is that the projector $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$ is a reproducing kernel for $\mathcal{H}_k$. Indeed, let $x \in M$ and $u \in L_x$ be such that $h_x(u,u) = 1$. By corollary 2.1.15, $\xi_k^u(y) = \Pi_k(y,x).u^k$ is the coherent state at u which is an element of $\mathcal{H}_k$. It satisfies the reproducing property (2.34). That is, for any $\phi \in \mathcal{H}_k$,

$$\phi(x) = \langle \phi, \xi_k^u \rangle_k u^k.$$

The proof of theorem 2.1.8 is divided into two parts. Firstly, we will construct near each point of M a local section having the desired asymptotic expansion as in (2.29), and satisfying the same reproducing property up to some error. This section yields a local section of $L \boxtimes \overline{L}$. Secondly, we will show that Π_k must agree with this local section up to some error.

i) Local Reproducing Kernels:

Let $U \subset M$ be an open subset, and let s be a local non-vanishing holomorphic section of $L \to M$. Let $H = h(s,s)$ and $\phi = -\log H$. By definition, L is a prequantum line bundle whose Chern connection has curvature $-i\omega$. By computations before proposition A.4.2, $-i\omega = \overline{\partial}\partial(\log H)$, and thus

$$\omega = i\overline{\partial}\partial(\log H) = -i\overline{\partial}\partial(\log H) = i\partial\overline{\partial}\phi$$

on U, as $\partial\overline{\partial} + \overline{\partial}\partial = 0$. Any local holomorphic section of L^k is of the form fs^k, where f is a holomorphic function. For any two smooth local functions f and g, we define the scalar product

$$\langle f, g \rangle_{\phi,k} = \int_U f\bar{g} \exp(-k\phi) d\mu.$$

This is similar to the scalar product on $L^2(M, L^k)$. Indeed, $\exp(-k\phi) = \exp(k \log H) = \exp(\log H)^k = H^k = h(s,s)^k \overset{(2.6)}{=} h_k(s^k, s^k)$. Thus

$$\int_U f\bar{g} \exp(-k\phi) d\mu =$$
$$\int_U f\bar{g} h_k(s^k, s^k) d\mu = \int_U h_k(fs^k, gs^k) d\mu \overset{(2.8)}{=} \langle fs^k, gs^k \rangle_k.$$

Let $||\cdot||_{\phi,k}$ be the associated norm. We define the space

$$\mathcal{H}_{\phi,k}(U) = \{f : U \to \mathbb{C} \text{ holomorphic } \Big| ||f||_{\phi,k} < \infty\}.$$

Consider local holomorphic coordinates, so that the coordinate open set is the unit ball B of $\mathbb{C}^n$. Fix $\chi \in C_c^\infty(B)$ that is equal to 1 on the ball of radius $1/2$. Let $K_k : U^2 \to \mathbb{C}$ be a local smooth function. We associate to K_k the local function

$$\zeta_k^x : U \to \mathbb{C}, \quad y \mapsto K_k(y,x).$$

Such a function K_k is called a reproducing kernel modulo $\mathcal{O}(k^{-N})$ for $\mathcal{H}_{\phi,k}(U)$ if for any local holomorphic function f,

$$f(x) = \langle \chi f, \zeta_k^x \rangle_{\phi,k} + \mathcal{O}(k^{-N} \exp(k\phi(x)/2))||f||_{\phi,k}$$

uniformly near the origin.

Let $\tilde{\psi}$ be as in the proof of proposition 2.1.10. That is, $\tilde{\psi}$ is holomorphic (respectively anti-holomorphic) in the left variable (respectively in the right variable) up to a connection and satisfies $\tilde{\psi}(x,x) = -i\phi(x)$.

Proposition 2.1.18 ([3], Proposition 2.7)**.** *There exists smooth functions $(b_l)_{l\geq 0}$ such that for every $N \geq 0$, there exists a reproducing kernel $K_k^{(N)}$ modulo $\mathcal{O}(k^{n-N-1})$ for $\mathcal{H}_{\phi,k}(U)$ such that*

$$K_k^{(N)}(x,y) = \left(\frac{k}{2\pi}\right)^n \exp(ik\tilde{\psi}(x,y))(b_0(x,y) + k^{-1}b_1(x,y) + \cdots + k^{-N}b_N(x,y)).$$

ii) From Local to Global:

Let $K_k^{(N)}$ be as in the above proposition. Then we define a local section of $L \boxtimes \overline{L} \to U^2$ by

$$\overline{K_k^{(N)}}(x,y) = K_k^{(N)}(x,y)s^k(x) \otimes \bar{s}^k(y).$$

In particular, we have

$$\overline{K_k^{(N)}}(x,y) = \left(\frac{k}{2\pi}\right)^n E^k(x,y)(b_0(x,y) + k^{-1}b_1(x,y) + \cdots + k^{-N}b_N(x,y)).$$

Therefore, the following result implies theorem 2.1.8.

Theorem 2.1.19 ([3], Theorem 3.1)**.** *If $x, y \in U$ are close enough, then*

$$\Pi_k(x,y) = \overline{K_k^{(N)}}(x,y) + \mathcal{O}(k^{n-N-1}).$$

■

2.2 Stationary Phase Approximation Method

This section is devoted to a study of asymptotic behavior of integrals of the form

$$\int_M u e^{ikf} d\mu, \quad \text{as } k \to \infty, \tag{2.40}$$

where M is a n-dimensional manifold, and f and u are smooth functions with $\mathrm{Im}(f) \geq 0$. Our main goal here is to generalize the standard stationary phase method for Euclidean spaces as in theorem 2.2.1, and theorem 2.2.6, to the cases of Morse and Morse-Bott functions on arbitrary compact manifolds. See Appendix B for some backgrounds on Morse theory. The main idea behind the stationary phase method is that if f has no critical points in the support of u, the integral (2.40) is rapidly decreasing and goes to zero as $k \to \infty$, and in the case that the phase function f has some critical points, the integral is more and more concentrated around the critical points of f, as $k \to \infty$.

2.2.1 Stationary Phase Method for Morse Functions

This section is divided into two parts. First, in theorem 2.2.1, and corollary 2.2.2 we see how the stationary phase method works if the phase function f has one critical point and when the domain of integration is a compact subset of some Euclidean space. Then we try to generalize this method to Morse functions on arbitrary compact manifolds. Recall that for a manifold M, a Morse function $f : M \to \mathbb{R}$ is a smooth function with finitely-many isolated non-degenerate critical points.

Theorem 2.2.1. *Let $K \subset \mathbb{R}^n$ be a compact set, X an open neighborhood of K, and k a positive integer. If $u \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$, and $\operatorname{Im} f \geq 0$ in X, $\operatorname{Im} f(x_0) = 0$, $f'(x_0) = 0$, $\det f''(x_0) \neq 0$, and $f' \neq 0$ in $K - \{x_0\}$, then*

$$\left| \int u(x) e^{i\omega f(x)} dx - e^{i\omega f(x_0)} \left(\det(\omega f''(x_0)/2\pi i) \right)^{-1/2} \sum_{j<k} \omega^{-j} L_j u \right| \leq C\omega^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u|, \quad \omega > 0.$$

Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|x - x_0|/|f'(x)|$ has a uniform bound. With

$$g_{x_0}(x) = f(x) - f(x_0) - \langle f''(x_0)(x - x_0), x - x_0 \rangle / 2 \tag{2.41}$$

which vanishes of third order at x_0. We have

$$L_j u = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu} \langle f''(x_0)^{-1} D, D \rangle^\nu (g_{x_0}^\mu u)(x_0) / \mu! \nu!. \tag{2.42}$$

This is a differential operator of order $2j$ acting on u at x_0. The coefficients are rational homogeneous functions of degree $-j$ in $f''(x_0)$, $\cdots$, $f^{(2j+2)}(x_0)$ with denominator $(\det f''(x_0))^{3j}$. In every term the total number of derivatives of u and of f'' is at most $2j$.

Proof. For a proof, we refer to theorem 7.7.5 in [6]. ■

Now applying theorem 2.2.1 to smooth functions u, and f, we have:

Corollary 2.2.2. *Let $K \subset \mathbb{R}^n$ be a compact set, X an open neighborhood of K. If $u \in C_0^\infty(K)$, $f \in C^\infty(X)$, and $\operatorname{Im} f \geq 0$ in X, $\operatorname{Im} f(x_0) = 0$, $f'(x_0) = 0$, $\det f''(x_0) \neq 0$, and $f' \neq 0$ in $K - \{x_0\}$, then*

$$\int u(x) e^{ikf(x)} dx = e^{ikf(x_0)} \left(\frac{k}{2\pi} \right)^{-n/2} \det \left(\frac{f''(x_0)}{i} \right)^{-1/2} \sum_{j=0}^{\infty} k^{-j} L_j u + \mathcal{O}(k^{-\infty}). \tag{2.43}$$

Now considering the first term $j = 0$, one has $\mu = \nu = 0$ in equation (2.42), and we can write equation (2.43) as

$$\int u(x) e^{ikf(x)} dx = \left(\frac{2\pi}{k} \right)^{n/2} e^{ikf(x_0)} \det \left(\frac{f''(x_0)}{i} \right)^{-1/2} \left(u(x_0) + \mathcal{O}(k^{-1}) \right). \tag{2.44}$$

Now suppose that for a compact n-dimensional manifold M, there is a smooth function $f \in C^\infty(M; \mathbb{R})$, with

finitely-many isolated non-degenerate critical points. Our goal here is to compute the asymptotic integral

$$\int_M u e^{-kf} d\mu \tag{2.45}$$

as $k \to \infty$.

Lemma 2.2.3 (Morse Lemma)**.** *Let p be a non-degenerate critical point for $f : M \to \mathbb{R}$. Then there is a local coordinate system $(x^1, \cdots, x^n)$ in a neighborhood U of p with $x^i(p) = 0$ for all i, and such that the identity*

$$f = f(p) - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2 \tag{2.46}$$

holds throughout U, where λ is the index of $Hess(f)$ at p.

Proof. For a proof, we refer to Appendix B. ■

Theorem 2.2.4. *Let $f : M \to \mathbb{R}$ be a Morse function on a compact n-dimensional manifold M, with critical points $\{p_1, \cdots, p_r\}$, and assume that u is a smooth function on M. Let $\tilde{f} := if$, so that $\tilde{f} \in C^\infty(M, \mathbb{C})$. Assume that $\operatorname{Im} \tilde{f} \geq 0$ in M, so that $f \geq 0$. Moreover, let $\operatorname{Im} \tilde{f}(p_i) = 0$, so that $f(p_i) = 0$, for every $1 \leq i \leq r$. In addition, let for every $1 \leq i \leq r$, $\tilde{f}'(p_i) = 0$, $\det \tilde{f}''(p_i) \neq 0$, and $\tilde{f}' \neq 0$ in $M - \{p_1, \cdots, p_r\}$. Then*

$$\int_M u e^{ik\tilde{f}} d\mu = \sum_{i=1}^{r} \left(\frac{2\pi}{k}\right)^{n/2} e^{ik\tilde{f}(p_i)} \det\left(\frac{\tilde{f}''(p_i)}{i}\right)^{-1/2} \sqrt{\det g}\left(\lambda_i(p_i)u(p_i) + \mathcal{O}(k^{-1}))\right) \tag{2.47}$$

as $k \to \infty$, where λ_i is a partition of unity and g is the induced Riemannian metric in local coordinates.

Proof. Let $\{p_1, \cdots, p_r\}$ be the isolated, non-degenerate critical points of the Morse function f. Assume that $U_1, \cdots, U_r$ be the corresponding Morse charts, and extend them to $U_1, \cdots, U_r, \cdots, U_m$ to be a trivialization chart for the compact manifold M. Now take a partition of unity $\{\lambda_i | 1 \leq i \leq m\}$ subordinate to the open cover $\{U_i\}_{i=1}^m$. That is, $supp(\lambda_i) \subset U_i$, for each i.

The idea is by writing the integral (2.45) in local coordinates, and using partition of unity, one can apply (2.44) in corollary 2.2.2. Indeed,

$$\int_M u e^{ik\tilde{f}} d\mu = \sum_{i=1}^{m} \int_M \lambda_i u e^{ik\tilde{f}} d\mu = \sum_{i=1}^{m} \int_{U_i} \lambda_i u e^{ik\tilde{f}} d\mu. \tag{2.48}$$

Now let $\{h_i : \mathbb{R}^n \to U_i | 1 \leq i \leq m\}$ be the local coordinates induced by the Morse-lemma. Then one can write the RHS of the integral (2.48) as

$$\begin{aligned} RHS = \sum_{i=1}^{m} \int_{U_i := h_i(\mathbb{R}^n)} \lambda_i u e^{ik\tilde{f}} d\mu = \\ \sum_{i=1}^{m} \int_{\mathbb{R}^n} \lambda_i(h_i(x)) u(h_i(x)) e^{ik\tilde{f}(h_i(x))} \sqrt{\det g} dx^1 \cdots dx^n. \end{aligned} \tag{2.49}$$

Note that here we show both the Riemannian metric on M, and the induced Riemannian metric on $\mathbb{R}^n$ by g. Indeed, in the last line of the above equation, g is the induced Riemannian metric on $\mathbb{R}^n$, with elements

$g_{ij}(x) = g(Dh(x)[e_i], Dh(x)[e_j])$, where the RHS g is the Riemannian metric on M, and e_i and e_j are the standard basis elements of $\mathbb{R}^n$.

Here, recalling that the only critical points of f are $\{p_1, \cdots, p_r\}$, we can apply corollary 2.2.2, and thus the only contributions in the integral (2.48) are due to open sets $U_1, \cdots, U_r$, and we can conclude that

$$\begin{aligned} \int_M u e^{ik\tilde{f}} d\mu = \sum_{i=1}^{r} \int_{\mathbb{R}^n} \lambda_i(h_i(x)) u(h_i(x)) e^{ik\tilde{f}(h_i(x))} \sqrt{\det g} dx^1 \cdots dx^n = \\ = \sum_{i=1}^{r} \left(\frac{2\pi}{k}\right)^{n/2} e^{ik\tilde{f}(p_i)} \det \left(\frac{\tilde{f}''(p_i)}{i}\right)^{-1/2} \sqrt{\det g} \left(\lambda_i(p_i) u(p_i) + \mathcal{O}(k^{-1})\right). \end{aligned} \tag{2.50}$$

■

2.2.2 Stationary Phase Method for Morse-Bott Functions

Here we suppose that for a compact n-dimensional manifold M, there is a smooth function $f \in C^\infty(M, \mathbb{R})$, which is a Morse-Bott function, with non-degenerate connected critical submanifold N, of codimention q.

Lemma 2.2.5 (Morse-Bott Lemma)**.** *Let f be a function on a manifold of dimension n such that its critical space is a non-degenerate submanifold N. Then around each point $p \in N$, there exists a coordinate system $(x^i)_{i=1}^n$ on a neighborhood U with $p \in U$, in which f is expressed as*

$$f = f(p) + (x^1)^2 + \cdots + (x^\alpha)^2 - (x^{\alpha+1})^2 - \cdots - (x^q)^2 \tag{2.51}$$

where q is the codimension of N, and α is the index of Hessian on the normal bundle of N.

Proof. For a proof, see [1]. ■

Theorem 2.2.6. *Let $f(x, y)$ be a complex-valued C^∞ function in a neighborhood of $(0, 0)$ in $\mathbb{R}^{n+m}$ satisfying*

$$\operatorname{Im} f \geq 0, \quad \operatorname{Im} f(0,0) = 0, \quad f_x'(0,0) = 0, \quad \det f_{xx}''(0,0) \neq 0, \tag{2.52}$$

and let $u \in C_0^\infty(K)$ where K is a small neighborhood of $(0, 0)$ in $\mathbb{R}^{n+m}$. Then

$$\begin{aligned} \left| \int u(x,y) e^{i\omega f(x,y)} dx - \left((\det(\omega f_{xx}''/2\pi i))^0 \right)^{-1/2} e^{i\omega f^0} \sum_0^{N-1} (L_{f,j} u)^0 \omega^{-j} \right| \\ \leq C_N \omega^{-N-n/2} \end{aligned} \tag{2.53}$$

where for functions $G(x, y)$, the notation $G^0(y)$ stands for a function of y only which is in the same residue class modulo the ideal generated by $\partial f/\partial x_j$ for $j = 1, \cdots, n$.

Proof. For a proof we refer to arguments before theorem 7.7.12 of [6]. ■

Remark. One can see in [6] that $L_{f,j}$ in the above theorem is a differential operator in x of order $2j$ such that the coefficients multiplied by $(\det f_{xx}'')^{3j}$ are polynomials of degree $3nj - j$ in $D_x^\alpha f$ where $2 \leq |\alpha| \leq 2j + 2$.

Theorem 2.2.7. *Let M be a compact n-dimensional manifold, and let $f \in C^\infty(M, \mathbb{R})$ be a Morse-Bott function such that its critical space is a non-degenerate connected submanifold N of codimension q. Assume*

that u is a smooth function on M. Let $\tilde{f} := if$, so that $\tilde{f}$ be a complex-valued smooth function on M. Let $\operatorname{Im}\tilde{f} \geq 0$, so that $f \geq 0$ on M. Moreover, assume that for every point $x \in N$, $\operatorname{Im}\tilde{f}(x) = 0$. Then as $k \to \infty$

$$\int_M u e^{ik\tilde{f}} d\mu = \left(\frac{2\pi}{k}\right)^{q/2} \left(\int_N \left(\det(\tilde{f}''_{yy}/i)) \right)^{0}(z) \right)^{-1/2} u^0(z) e^{ik\tilde{f}^0(z)} dz + \mathcal{O}(k^{-1}) \Big) + \mathcal{O}(k^{-\infty}), \tag{2.54}$$

where z and y are local coordinates on N and the normal bundle of N respectively.

Proof. Recalling equation (2.51), in a Morse-Bott neighborhood, we can write f as $f = f(y, z)$ where $y = (x^1, \cdots, x^q)$, and $z = (x^{q+1}, \cdots, x^n)$. Thus if z is fixed, $f(\cdot, z)$ is a Morse function.

Without loss of generality assume that $(0, 0) \in N$. Here $\tilde{f}(y, z)$ is a complex-valued C^∞ function in a neighborhood of $(0, 0)$ in $\mathbb{R}^{q+(n-q)}$, satisfying

$$\operatorname{Im}\tilde{f} \geq 0, \quad \operatorname{Im}\tilde{f}(0,0) = 0, \quad \tilde{f}'_y(0,0) = 0, \quad \det \tilde{f}''_{yy}(0,0) \neq 0, \tag{2.55}$$

where the two RHS equality and inequality hold as f is a Morse-Bott function. Note that locally, coordinates on N are indicated by z, and coordinates of the normal bundle of N are indicated by y. Now let $u \in C_0^\infty(K)$, where K is a small neighborhood of $(0, 0) \in \mathbb{R}^{q+(n-q)}$. Then applying Theorem 2.2.6, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^q} u(y,z) e^{i\omega\tilde{f}(y,z)} dy - \left((\det(\omega\tilde{f}''_{yy}/2\pi i)) \right)^{0} \right)^{-1/2} e^{i\omega\tilde{f}^0(z)} \sum_0^{N-1} (L_{\tilde{f},j} u)^0(z) \omega^{-j} \Bigg| \\ \leq C_N \omega^{-N-q/2} \end{aligned} \tag{2.56}$$

where $\tilde{f}^0(z)$ is a function of z, which is in the same residue class modulo the ideal $I = I[\partial f/\partial x_1, \cdots, \partial f/\partial x_q]$.

Now in a Morse-Bott coordinate, and using the fact that a trivializing neighborhood in M is homeomorphic to $\mathbb{R}^n$, and a trivializing neighborhood in N is homeomorphic to $\mathbb{R}^{n-q}$, the thing we have to compute is:

$$\begin{aligned} &\int_{\text{neighborhood in } M} u(y,z) e^{i\omega\tilde{f}(y,z)} dy dz = \\ &= \int_{\text{neighborhood in } N} \left[\left((\det(\omega\tilde{f}''_{yy}/2\pi i)) \right)^{0} \right)^{-1/2} e^{i\omega\tilde{f}^0(z)} \sum_0^{N-1} (L_{\tilde{f},j} u)^0(z) \omega^{-j} \Bigg] dz + \mathcal{O}(\omega^{-\infty}) = \\ &\left(\frac{\omega}{2\pi}\right)^{-q/2} \left(\int_{\text{neighborhood in } N} \left(\det(\tilde{f}''_{yy}/i)) \right)^{0}(z) \right)^{-1/2} u^0(z) e^{i\omega\tilde{f}^0(z)} dz + \mathcal{O}(\omega^{-1}) \Big) + \mathcal{O}(\omega^{-\infty}). \end{aligned} \tag{2.57}$$

Where in the last line we have used only the first term $j = 0$.

What is left to do is to apply the same arguments as in the case of Morse function, using trivializing charts and partition of unity, to compute the integral on the entire manifold M.

Let $(U_i)_{i=1}^m$ be a trivializing chart for the compact manifold M, which is an extension of the collection of Morse-Bott neighborhoods to the entire manifold, and let $\{\lambda_i | 1 \leq i \leq m\}$ be a partition of unity subordinate to this open cover. Let $\{h_i : \mathbb{R}^n \to U_i | 1 \leq i \leq m\}$ be the local coordinate induced by the Morse-Bott lemma, and abbreviate both the Riemannian metrics on M and on $\mathbb{R}^n$ by g, similar to the arguments as in the proof

of theorem 2.2.4. Then by the same arguments as in equations (2.48), (2.49), and (2.50), we have

$$\begin{aligned}
\int_M u e^{ik\tilde{f}} d\mu = \sum_{i=1}^m \int_{U_i} \lambda_i u e^{ik\tilde{f}} d\mu = \sum_{i=1}^m \int_{U_i := h_i(\mathbb{R}^n)} \lambda_i u e^{ik\tilde{f}} d\mu = \\
\sum_{i=1}^m \int_{\mathbb{R}^n} \lambda_i(h_i(x)) u(h_i(x)) e^{ik\tilde{f}(h_i(x))} \sqrt{\det g} dx^1 \cdots dx^n = \\
\sum_{i=1}^m \int_{\mathbb{R}^n} \lambda_i(h_i(y,z)) u(h_i(y,z)) e^{ik\tilde{f}(h_i(y,z))} \sqrt{\det g} dy \wedge dz = \\
\sum_{i=1}^m \left(\frac{k}{2\pi}\right)^{-q/2} [\int_{\mathbb{R}^{n-q}} \left(\det(\tilde{f}''_{yy}/i)\right)^0(z)\Big)^{-1/2} \lambda_i(h_i^0(z)) u^0(h_i^0(z)) e^{ik\tilde{f}^0(h_i^0(z))} \\
\times \sqrt{\det g} dz + \mathcal{O}(k^{-1})] + \mathcal{O}(k^{-\infty}) = \\
\left(\frac{2\pi}{k}\right)^{q/2} \left(\int_N \left(\det(\tilde{f}''_{yy}/i)\right)^0(z)\Big)^{-1/2} u^0(z) e^{ik\tilde{f}^0(z)} dz + \mathcal{O}(k^{-1})\right) + \mathcal{O}(k^{-\infty}),
\end{aligned} \tag{2.58}$$

where in the last line dz is the induced volume form on N. ■

Chapter 3

Quantized Submanifolds and Microsupport

The first part of this chapter is devoted to quantum states associated with submanifolds. We will investigate conditions in which quantized submanifolds are pure and mixed quantum states. In the second part, we will define microsupport of a quantized submanifold of a compact Kahler manifold and will prove that it is equal to the submanifold itself.

3.1 Quantum States Associated with Submanifolds

Let (M^{2n}, ω, J) be a compact Kahler manifold, and let $N \subset M$ be a submanifold. For the induced Riemannian metric g_N on N, consider μ_N as the corresponding volume form on N. Take $a_N = \mathrm{Vol}(N, \mu_N)$.

Definition. The quantum state associated to a submanifold N is defined as

$$\theta_k(N) := \int_N P_{k,x} d\mu(x) \tag{3.1}$$

where $P_{k,x} : \mathcal{H}_k \to \mathcal{H}_k$ is the projection along the coherent state corresponding to $x \in N$ as in equation (2.38). Note that the volume form $d\mu$ is chosen such that $\int_N d\mu = 1$. Thus in the case where N is a point, μ is the Dirac delta function, and when N is not a point, $d\mu = \frac{1}{a_N} d\mu_N$.

Our goal in this section is to show that if N is a point, then $\theta_k(N)$ is a pure quantum state, and otherwise, it is a mixed quantum state.

Remark. Note that for local coordinates $(x^1, \cdots, x^{2n})$ on the manifold M, Riemannian volume form in local coordinates is of the form $\mu = \sqrt{|\det g|} dx^1 \wedge \cdots \wedge dx^{2n}$. So for a submanifold N with local coordinates $(x^1, \cdots, x^r)$, volume of N will be of the form

$$a_N = \mathrm{Vol}(N, \mu_N) = \int_N d\mu_N = \int_N \sqrt{|\det g_N|} dx^1 \wedge \cdots \wedge dx^r. \tag{3.2}$$

Here $g_N(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}) : U \to \mathbb{R}$ is the induced Riemannian metric on N, where $i, j \in \{1, \cdots, r\}$, and $(U, (x^1, \cdots, x^r))$ are the induced coordinate charts on N.

3.1.1 Quantum State Associated to a Point

Assume that $N = \{x_0\}$ is a point in M. Our goal here is to show that the quantization of $\{x_0\}$ is a pure quantum state. Indeed, by (3.1),

$$\theta_k(\{x_0\}) = \int_{\{x_0\}} P_{k,x} d\delta(x) = P_{k,x_0}. \tag{3.3}$$

Definition. The density matrix associated to the quantization of a submanifold N of M is $\theta_k(N)^2$, with trace

$$\tau_k(N) := \mathrm{Tr}\left(\theta_k(N)^2\right). \tag{3.4}$$

Remark. We can see in every standard quantum mechanics literature that a quantum state is a pure state if and only if the trace of the square of the density matrix associated with it is equal to one, and otherwise it is a mixed quantum state. Moreover, it is well-known that a density operator ρ describes a pure state if and only if $\rho^2 = \rho$.

So, the density matrix associated to the quantization of $\{x_0\}$ is $\rho = \theta_k(\{x_0\})^2 = P_{k,x_0}^2$. Let $\phi \in \mathcal{H}_k$ be an element, and $u \in L_{x_0}$ be of norm one. Then by the definition (2.38), one has

$$P_{k,x_0} P_{k,x_0} \phi = P_{k,x_0}\left(\frac{\langle \phi, \xi_k^u\rangle_k}{||\xi_k^u||_k^2}\xi_k^u\right) = \frac{\langle \phi, \xi_k^u\rangle_k}{||\xi_k^u||_k^2}\frac{\langle \xi_k^u, \xi_k^u\rangle_k}{||\xi_k^u||_k^2}\xi_k^u = P_{k,x_0}\phi,$$

and thus $P_{k,x_0}^2 = P_{k,x_0}$. Hence $\rho^2 = (P_{k,x_0}^2)^2 = P_{k,x_0}^2 = \rho$, and $\theta_k(\{x_0\})$ is a pure state.

3.1.2 Quantum States Associated to Submanifolds that are Not a Point

Let (M^{2n}, ω, J) be a compact Kahler manifold, and let $N \subset M$ be a submanifold with dimension greater than zero. Our goal in this section is to show that in the semi-classical limit, i.e., as $k \to \infty$, trace of the square of the density matrix associated to the quantization of a submanifold N is

$$\tau_k(N) = b_N(\frac{2\pi}{k})^{\dim N/2}(1 + \mathcal{O}(k^{-1})), \tag{3.5}$$

where b_N only depends on the Riemannian metric. Thus if $\dim N \geq 1$, $\tau_k(N) \neq 1$, and $\theta_k(N)$ is a mixed state.

Before proving (3.5), we need to see a remark on integral operators:

Remark. Let $L, K \subset M$ be two submanifolds, and let B be an integral operator on $L^2(L)$ with kernel $B(\cdot,\cdot)$, and C be another integral operator on $L^2(K)$ with kernel $C(\cdot,\cdot)$. Then for every ϕ,

$$B(\phi)(z) = \int_L B(x,z)\phi(x)d\mu(x), \quad \text{and} \quad C(\phi)(z) = \int_K C(y,z)\phi(y)d\mu(y). \tag{3.6}$$

Therefore,

$$\begin{aligned} BC(\phi)(z) &= B(C(\phi))(z) \\ &= \int_L B(x,z)C(\phi)(x)d\mu(x) \\ &= \int_L B(x,z)\left(\int_K C(y,x)\phi(y)d\mu(y)\right)d\mu(x) \\ &= \int_L\int_K B(x,z)C(y,x)\phi(y)d\mu(y)d\mu(x) \\ &\overset{(*)}{=} \int_K \left(\int_L B(x,z)C(y,x)d\mu(x)\right)\phi(y)d\mu(y) \\ &=: \int_K A(y,z)\phi(y)d\mu(y). \end{aligned}$$

Thus for $A := B \circ C$, the integral kernel of A is

$$A(y,z) = \int_L B(x,z)C(y,x)d\mu(x). \tag{3.7}$$

Note that in $(*)$ we have changed the order of integration. For $(*)$ to be justified, we have to make sure that the hypotheses of Fubini's theorem are satisfied. So we need to have either:

i) $B(\cdot,\cdot)$, $C(\cdot,\cdot)$, and ϕ are all non-negative, or

ii) $\int\int |B(x,z)C(y,x)\phi(y)|\, d\mu(x)d\mu(y) < \infty$, for every ϕ.

Now if the the Fubini's theorem holds,

$$\mathrm{Tr}(A) = \int_K\int_L B(x,y)C(y,x)d\mu(x)d\mu(y) = \int_K A(y,y)d\mu(y). \tag{3.8}$$

Proof of (3.5) to show that $\theta_k(N)$ is a mixed state. Recalling (3.1), for a submanifold $N \subset M$,

$$\tau_k(N) = \mathrm{Tr}\left(\theta_k(N)^2\right) = \mathrm{Tr}\left(\frac{1}{a_N^2}\int_N P_{k,x}d\mu(x)\int_N P_{k,y}d\mu(y)\right). \tag{3.9}$$

Using corollary 2.1.15, and equation (2.38), we can define

$$B := \int_N P_{k,x}d\mu(x) = \int_N \frac{1}{\Pi_k(x,x)}\Pi_k(\cdot,x)d\mu(x) = \int_N B(x,\cdot)d\mu(x), \tag{3.10}$$

and similarly

$$C := \int_N P_{k,y}d\mu(y) = \int_N \frac{1}{\Pi_k(y,y)}\Pi_k(\cdot,y)d\mu(y) = \int_N C(y,\cdot)d\mu(y). \tag{3.11}$$

Note that in the special case of computing $\tau_k(N)$, the operators B and C are the same. However, in the upcoming chapters, we need to compute more general cases where the domain of integration will be different submanifolds of M.

Note that both $B, C : \mathcal{H}_k \to \mathcal{H}_k$ are bounded operators on $L^2(M, L^k)$. Thus their composition $A := BC$ is also a bounded operator on $L^2(M, L^k)$. By proposition 2.1.4, the integral kernel $A(\cdot,\cdot) \in L^2(M \times M, L^k \boxtimes \overline{L}^k)$ exists, and thus hypotheses of Fubini's theorem in the above remark are satisfied.

Using equations (3.8), and (3.9), and taking $A := BC$ we can write

$$\tau_k(N) = \frac{1}{a_N^2}\int_N\int_N B(x,y)C(y,x)d\mu(x)d\mu(y) = \frac{1}{a_N^2}\int_N\int_N \frac{\Pi_k(y,x)\Pi_k(x,y)}{\Pi_k(x,x)\Pi_k(y,y)}d\mu(x)d\mu(y). \tag{3.12}$$

Hence,

$$\tau_k(N) = \frac{1}{a_N^2}\int_{N\times N} \frac{|\Pi_k(x,y)|^2}{\Pi_k(x,x)\Pi_k(y,y)}d\mu(x)d\mu(y). \tag{3.13}$$

Applying theorem 2.1.8, and considering the first order expansion we have

$$\Pi_k(x,y) = \left(\frac{k}{2\pi}\right)^n E^k(x,y)u_0(x,y) + \mathcal{O}(k^{-1}) \tag{3.14}$$

where E is a section of $L\boxtimes\overline{L}\to M\times\overline{M}$ as in proposition 2.1.9, which is equal to 1 on the diagonal, and $u_0(x,x)=1$. Thus

$$\Pi_k(x,x) = \left(\frac{k}{2\pi}\right)^n E^k(x,x)u_0(x,x) + \mathcal{O}(k^{-1}) = \left(\frac{k}{2\pi}\right)^n + \mathcal{O}(k^{-1}), \tag{3.15}$$

and similarly

$$\Pi_k(y,y) = \left(\frac{k}{2\pi}\right)^n + \mathcal{O}(k^{-1}). \tag{3.16}$$

Hence, equation (3.13) becomes

$$\tau_k(N) = \frac{1}{a_N^2}\left(\int_{N\times N} ||E^k(x,y)||^2|u_0(x,y)|^2 d\mu(x)d\mu(y) + \mathcal{O}(k^{-1})\right). \tag{3.17}$$

Take $\phi_E := -2\log||E||$. Then $||E(x,y)|| = e^{-k\phi_E(x,y)/2}$, and $||E^k||^2 = e^{-k\phi_E}$. Thus one can write equation (3.17) as

$$\tau_k(N) = \frac{1}{a_N^2}\left(\int_{N\times N} e^{-k\phi_E}|u_0(x,y)|^2 d\mu(x)d\mu(y) + \mathcal{O}(k^{-1})\right). \tag{3.18}$$

By lemma 2.1.12, ϕ_E is a Morse-Bott function with critical submanifold Δ_N. Thus to compute the integral (3.18), it is enough to apply the stationary phase method for Morse-Bott functions as in theorem 2.2.7. Indeed, take $\tilde{f} := i\phi_E$. Then $\tilde{f}$ satisfies the hypotheses of theorem 2.2.7. In fact, by definition of ϕ_E we have that $\phi_E \geq 0$ on M, and for every point $(x,x)\in\Delta_N$, $\operatorname{Im}\tilde{f}(x,x) = \phi_E(x,x) = 0$ as $E=1$ on the diagonal.

Thus applying theorem 2.2.7, in the semi-classical limit as $k \to \infty$ we have:

$$\begin{aligned}
\tau_k(N) &= \frac{1}{a_N^2}\left(\int_{N\times N} e^{-k\phi_E}|u_0(x,y)|^2 d\mu(x)d\mu(y) + \mathcal{O}(k^{-1})\right) \\
&= \frac{1}{a_N^2}\left(\int_{N\times N} e^{ik(i\phi_E)}|u_0(x,y)|^2 d\mu(x)d\mu(y) + \mathcal{O}(k^{-1})\right) \\
&= \frac{1}{a_N^2}\left(\int_{N\times N} e^{ik\tilde{f}}|u_0(x,y)|^2 d\mu(x)d\mu(y) + \mathcal{O}(k^{-1})\right) \\
&\overset{(*)}{=} \frac{1}{a_N^2}\left(\frac{2\pi}{k}\right)^{\dim N/2}\left(\int_{\Delta_N}|u_0(x,x)|^2 e^{ik\tilde{f}(x,x)}\left(\det(\tilde{f}''_{yy}/i)(x)\right)^{-1/2} d\mu(x) + \mathcal{O}(k^{-1})\right) + \mathcal{O}(k^{-\infty}) \\
&\overset{(**)}{=} \frac{1}{a_N^2}\left(\frac{2\pi}{k}\right)^{\dim N/2}\left(\int_{\Delta_N}\left(\det(\tilde{f}''_{yy}/i)(x)\right)^{-1/2} d\mu(x) + \mathcal{O}(k^{-1})\right) + \mathcal{O}(k^{-\infty}) \\
&\overset{(***)}{=} \frac{1}{a_N^2}\left(\frac{2\pi}{k}\right)^{\dim N/2}\left[\frac{a_N}{\sqrt{2|\det\tilde{g}|}} + \mathcal{O}(k^{-1})\right] \\
&= \frac{1}{a_N}\left(\frac{2\pi}{k}\right)^{\dim N/2}\left[\frac{1}{\sqrt{2|\det\tilde{g}|}} + \mathcal{O}(k^{-1})\right] \\
&= b_N\left(\frac{2\pi}{k}\right)^{\dim N/2}(1+\mathcal{O}(k^{-1}))
\end{aligned}$$

where in $(*)$ we abbreviate by $d\mu(x)$ as the induced volume form on Δ_N, and we have used that $\Delta_N \subset N\times N$ is the critical submanifold with codimension equal to $\dim N$, and the term $\tilde{f}''_{yy}$ is the Hessian of $\tilde{f}$ with respect to coordinates on the normal bundle of Δ_N. Since ϕ_E is a Morse-Bott function, $\left(\det(\tilde{f}''_{yy}/i)(x)\right)^{-1/2} = \left(\det(\phi''_{yy})(x)\right)^{-1/2}$ is non-zero. In $(**)$ we used the fact that $u_0(x,x) = 1$ and $ik\tilde{f}(x,x) = -k\phi_E(x,x) = 0$. Moreover, note that by lemma 2.1.12, $Hess(\phi_E)(x)$ at $x \in \Delta_N$ restricted to the normal bundle of Δ_N is equal to twice the Riemannian metric, and thus $(***)$ is justified. Note that here $\tilde{g}$ is the induced Riemannian metric on $M \times M$. ■

3.2 Microsupport of Quantized Submanifold

Let (M^{2n}, ω, J) be a compact Kahler manifold, and let $N \subset M$ be a submanifold. Let $\theta_k(N)$ be the quantization of the submanifold N as in (3.1).

Definition. Abbreviating $\theta_k(N)$ by θ_k, we define the microsupport of θ_k to be the subset $MS(\theta_k) \subset M$ such that $x \notin MS(\theta_k)$ if and only if there exists $f \in C^\infty(M)$ with $f(x) \neq 0$ and $T_k(f)(\theta_k) = \mathcal{O}(k^{-\infty})$ [5].

Recall that the family $\{T_k\}_{k\in\mathbb{N}}$ is the Berezin quantization of M, as in (2.9) and (2.39).

Remark. Note that $T_k(f)(\theta_k) = \mathcal{O}(k^{-\infty})$ means that $||T_k(f)(\theta_k)||_{\text{op}} = \mathcal{O}(k^{-\infty})$. We have shown in (2.33) that $\mathcal{H}_k$ is $\mathcal{O}(k^n)$-dimensional, thus we can equivalently work with the Hilbert-Schmidt norm or the trace norm $||\rho||_{tr} = \operatorname{Tr}\sqrt{\rho\rho^*}$ [5], where ρ^* is the conjugate transpose of ρ. So $x \notin MS(\theta_k)$ if and only if there exists $f \in C^\infty(M)$ with $f(x) \neq 0$ and $||T_k(f)(\theta_k)||_{HS} = \mathcal{O}(k^{-\infty})$.

Remark. Note that since both $T_k(f)$ and θ_k are Hilbert-Schmidt operators, their composition $T_k(f)(\theta_k)$ is a Hilbert-Schmidt integral operator, and thus its Hilbert-Schmidt norm is equal to L^2-norm of its kernel.

Our goal in this section is to prove the following theorem:

Theorem 3.2.1. *Let (M^{2n}, ω, J) be a compact Kahler manifold, and let $N \subset M$ be a submanifold, with quantization $\theta_k(N)$. Then*

$$MS(\theta_k(N)) = N. \tag{3.19}$$

Proof. For simplicity, we abbreviate $\theta_k(N)$ by θ_k. Let us first assume that the submanifold N has dimension greater than zero, and write $||T_k(f)(\theta_k)||_{HS}$ in a form which is suitable to use stationary phase approximation method. Let

$$\begin{aligned} B := T_k(f) = \int_M f(x) R_k(x) P_{k,x} d\mu(x) = \\ \int_M f(x) \Pi_k(x,x) \frac{\Pi_k(\cdot, x)}{\Pi_k(x,x)} d\mu(x) = \\ \int_M f(x) \Pi_k(\cdot, x) d\mu(x) = \\ \int_M B(x, \cdot) d\mu(x), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} C := \frac{1}{a_N} \int_N P_{k,y} d\mu_N(y) = \\ \frac{1}{a_N} \int_N \frac{\Pi_k(\cdot, y)}{\Pi_k(y,y)} d\mu_N(y) = \\ \int_N C(y, \cdot) d\mu(y). \end{aligned} \tag{3.21}$$

Now let $A := T_k(f)(\theta_k) = BC$. Since B and C are Hilbert-Schmidt operators, hypotheses of Fubini's theorem are satisfied, and by (3.6) we have

$$\begin{aligned} A(z) = T_k(f)(\theta_k)(z) = \int_N A(y,z) d\mu(y) = \\ \int_N \int_M B(x,z) C(y,x) d\mu(x) d\mu(y) = \\ \frac{1}{a_N} \int_N \int_M f(x) \Pi_k(z,x) \frac{\Pi_k(x,y)}{\Pi_k(y,y)} d\mu(x) d\mu(y). \end{aligned} \tag{3.22}$$

Now, as A is a Hilbert-Schmidt operator, $||A||_{HS} = ||A(y,z)||_{L^2}$, and thus we have

$$\begin{aligned} ||T_k(f)(\theta_k)||^2_{HS} = ||A||^2_{HS} = ||A(y,z)||^2_{L^2} = \\ \int_M \int_N |A(y,z)|^2 d\mu(y) d\mu(z) = \\ \frac{1}{a_N^2} \int_M \int_N \left(f(x) \Pi_k(z,x) \frac{\Pi_k(x,y)}{\Pi_k(y,y)} d\mu(x) \right)^2 d\mu(y) d\mu(z) = \\ \frac{1}{a_N^2} \int_M \int_N \int_M \int_M \left| f(x) \Pi_k(z,x) \frac{\Pi_k(x,y)}{\Pi_k(y,y)} f(x') \Pi_k(z,x') \frac{\Pi_k(x',y)}{\Pi_k(y,y)} \right| d\mu(x) d\mu(x') d\mu(y) d\mu(z). \end{aligned} \tag{3.23}$$

Recalling theorem 2.1.8, the projector $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$ has a kernel of the form

$$\Pi_k(x,y) = \left(\frac{k}{2\pi}\right)^n E^k(x,y) \sum_{l\geq 0} k^{-l} u_l(x,y) + \mathcal{O}(k^{-\infty}).$$

Thus,

$$\begin{aligned}\Pi_k(y,y) = \left(\frac{k}{2\pi}\right)^n E^k(y,y) \sum_{l\geq 0} k^{-l} u_l(y,y) + \mathcal{O}(k^{-\infty}) = \\ \left(\frac{k}{2\pi}\right)^n + \mathcal{O}(k^{-1})\end{aligned} \tag{3.24}$$

as both E and u_0 are equal to 1 on the diagonal. Moreover,

$$\begin{aligned}\Pi_k(z,x)\Pi_k(x,y) = \\ \left(\frac{k}{2\pi}\right)^{2n} \left[E^k(z,x) \sum_{l\geq 0} k^{-l} u_l(z,x)\right] \left[E^k(x,y) \sum_{i\geq 0} k^{-i} u_i(x,y)\right] + \mathcal{O}(k^{-\infty}) = \\ \left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l\geq 0} k^{-l-i} E^k(z,x) E^k(x,y) u_l(z,x) u_i(x,y) + \mathcal{O}(k^{-\infty}).\end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) in (3.23) one has

$$\begin{aligned}||T_k(f)(\theta_k)||^2_{HS} = \\ \frac{1}{a_N^2} \left(\frac{k}{2\pi}\right)^{2n} \int_M \int_N \int_M \int_M \left| f(x) f(x') E^k(z,x) E^k(x,y) E^k(z,x') E^k(x',y) \right| \\ \times \left| \sum_{i,l,j,r\geq 0} k^{-l-i-j-r} u_l(z,x) u_i(x,y) u_j(z,x') u_r(x',y) \right| d\mu(x) d\mu(x') d\mu(y) d\mu(z) + \mathcal{O}(k^{-\infty}).\end{aligned} \tag{3.26}$$

Now define $\phi(x,y) := -2 \log ||E(x,y)||$. Thus $||E^k(x,y)|| = e^{\frac{-k}{2}\phi(x,y)}$. Then substituting in (3.26), we have

$$\begin{aligned}||T_k(f)(\theta_k)||^2_{HS} \leq \\ \frac{1}{a_N^2} \left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l,j,r\geq 0} k^{-l-i-j-r} \int_M \int_N \int_M \int_M |f(x)||f(x')| e^{\frac{-k}{2}[\phi(z,x)+\phi(x,y)+\phi(z,x')+\phi(x',y)]} \\ \times \left| u_l(z,x) u_i(x,y) u_j(z,x') u_r(x',y) \right| d\mu(x) d\mu(x') d\mu(y) d\mu(z) + \mathcal{O}(k^{-\infty}).\end{aligned} \tag{3.27}$$

Let $\psi(z,y,x,x') := \frac{1}{2}[\phi(z,x) + \phi(x,y) + \phi(z,x') + \phi(x',y)]$. Then $\psi : M \times N \times M \times M \to \mathbb{R}$ is a smooth function. Applying lemma 2.1.12, since $d\phi$ vanishes along Δ_M, $d\psi$ vanishes along $\Delta_N \times \Delta_N$. Moreover, as ϕ is a Morse-Bott function with critical submanifold Δ_M, ψ is a Morse-Bott function with critical submanifold $\Delta_N \times \Delta_N$, having codimension $2.\dim N$.

Now applying the stationary phase method as in theorem 2.2.7 with $\tilde{f} = i\psi$, equation (3.27) becomes

$$||T_k(f)(\theta_k)||^2_{HS} \leq$$
$$\frac{1}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l,j,r\geq 0} k^{-l-i-j-r} \int_M \int_N \int_M \int_M |f(x)||f(x')|e^{-k\psi(z,y,x,x')}$$
$$\times \left|u_l(z,x)u_i(x,y)u_j(z,x')u_r(x',y)\right| d\mu(x)d\mu(x')d\mu(y)d\mu(z) + \mathcal{O}(k^{-\infty}) =$$
$$\frac{1}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l,j,r\geq 0} k^{-l-i-j-r} \left(\frac{2\pi}{k}\right)^{\dim N} \left[\int_{\Delta_N\times\Delta_N} \left(\det(\tilde{f}''_{yy}/i)\right)^{-1/2} u(\tilde{z})e^{ik\tilde{f}(\tilde{z})}d\tilde{z} + \mathcal{O}(k^{-1})\right] + \mathcal{O}(k^{-\infty}), \tag{3.28}$$

where $\tilde{z}$ and y are local coordinates on $\Delta_N \times \Delta_N$ and it's normal bundle respectively. Note that since ψ is a Morse-Bott function, $\left(\det(\tilde{f}''_{yy}/i)\right)^{-1/2}$ is a constant function on all $\Delta_N \times \Delta_N$. Since we are only interested in the order of the above integral, we will denote it by the constant $I := \left(\det(\tilde{f}''_{yy}/i)\right)^{-1/2}$. Moreover, in the above integral $u(\tilde{z}) = |u_l(y,y)u_i(y,y)u_j(y,y)u_r(y,y)|$ for $y \in N$, and $ik\tilde{f}(\tilde{z}) = -2k\phi(y,y)$ for $y \in N$, which is equal to zero by definition of ϕ being zero on the diagonal.

We are now ready to show that $MS(\theta_k) \subset N$. Indeed, we have to show that if $z \notin N$, then $z \notin MS(\theta_k)$. So let $z \in M - N$, and by contradiction assume that $z \in MS(\theta_k)$. Thus by the definition of microsupport, for every function $f \in C^\infty(M)$ with $f(z) \neq 0$, we have $T_k(f)(\theta_k) \neq \mathcal{O}(k^{-\infty})$. Take a small neighborhood U of z which is not overlapping the submanifold N. Take $f \in C_c^\infty(U)$ with $f(z) \neq 0$. Then for each $i, l, j, r \geq 0$,

$$\int_{\Delta_N\times\Delta_N} I|f(y)|^2 e^{-2k\phi(y,y)} \left|u_l(y,y)u_i(y,y)u_j(y,y)u_r(y,y)\right| d\mu_{\Delta_N\times\Delta_N} = 0 \tag{3.29}$$

as u_l's are smooth functions on M and thus bounded on the closed submanifold N, and moreover as f is not supported on N.

Note that in order to conclude this part we have to view $\mathcal{O}(k^{-1})$ in inequality (3.28) in more details. Recalling theorem 2.2.6 and theorem 2.2.7, to obtain equation (2.54) we have only considered the zero term in $\sum_{j=0}^{\infty}(L_{\tilde{f},j}u)\omega^{-j}$ which resulted in the term $\mathcal{O}(k^{-1})$ in (2.54). By the remark after theorem 2.2.6, $L_{\tilde{f},j}$ is a differential operator in the coordinates on the normal bundle of $\Delta_N \times \Delta_N$. In our case $u = |f|^2 u_l u_i u_j u_r$, and thus $L_{\tilde{f},j}u = 0$ on $\Delta_N \times \Delta_N$ as we have chosen f to be not supported on N.

Applying these arguments, (3.28) becomes

$$||T_k(f)\theta_k||^2_{HS} \leq \frac{1}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l,j,r\geq 0} \left(\frac{2\pi}{k}\right)^{\dim N} [0+0] + \mathcal{O}(k^{-\infty}), \tag{3.30}$$

and therefore, $||T_k(f)(\theta_k)||_{HS} = \mathcal{O}(k^{-\infty})$, which is a contradiction. Hence we can conclude that $MS(\theta_k) \subset N$.

We are left to show that $N \subset MS(\theta_k)$. By contradiction assume that there exists a point $z \in N$ with $z \notin MS(\theta_k)$. So there is a function $f \in C^\infty(M)$ with $f(z) \neq 0$ and $||T_k(f)(\theta_k)||_{HS} = \mathcal{O}(k^{-\infty})$. Note that in (3.27) we have found an upper bound for $||T_k(f)(\theta_k)||_{HS}$, but now we are going to obtain a lower bound,

using only the first order expansion, . Indeed, one can write (3.26) as

$$||T_k(f)(\theta_k)||^2_{HS} = \frac{1}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n}\int_M\int_N\int_M\int_M |f(x)f(x')|e^{-k\psi(z,y,x,x')}\left|u_0(z,x)u_0(x,y)u_0(z,x')u_0(x',y)+\mathcal{O}(k^{-1})\right| d\mu(x)d\mu(x')d\mu(y)d\mu(z)+\mathcal{O}(k^{-\infty}). \tag{3.31}$$

Then by the same argument as after (3.28), this is equal to

$$||T_k(f)(\theta_k)||^2_{HS} = \frac{1}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n}\left(\frac{2\pi}{k}\right)^{\dim N}\left[\int_{\Delta_N\times\Delta_N} I|f(y)|^2\left(u_0(y,y)u_0(y,y)u_0(y,y)u_0(y,y)+\mathcal{O}(k^{-1})\right)d\mu_{\Delta_N\times\Delta_N}\right]+\mathcal{O}(k^{-\infty}) \tag{3.32}$$

where $u_0(y,y)=1$. Now since $f(z)\neq 0$, there exists a neighborhood U of z such that $|f|\geq C>0$ on U. Therefore,

$$\begin{aligned} ||T_k(f)(\theta_k)||^2_{HS} = \\ \frac{I}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n}\left(\frac{2\pi}{k}\right)^{\dim N}\left[\int_{\Delta_N\times\Delta_N}|f(y)|^2d\mu+\mathcal{O}(k^{-1})\right]+\mathcal{O}(k^{-\infty})\geq \\ \frac{I}{a_N^2}\left(\frac{k}{2\pi}\right)^{2n}\left(\frac{2\pi}{k}\right)^{\dim N}C^2a_N^2+\mathcal{O}(k^{-\infty}) = \\ \mathcal{O}(k^{2n-\dim N}). \end{aligned} \tag{3.33}$$

Thus $||T_k(f)(\theta_k)||_{HS}\geq\mathcal{O}(k^{n-\dim N/2})$, and hence not equal to $\mathcal{O}(k^{-\infty})$, as $n-\dim N/2\geq 0$. We can now conclude that every point $z\in N$ is contained in $MS(\theta_k)$.

Consequently, as $MS(\theta_k)\subset N$ and $N\subset MS(\theta_k)$, we have $MS(\theta_k(N))=N$, which completes the proof for a non-trivial submanifold.

Now we are left to show (3.19) when $N=\{x_0\}$ is a point. One can see in [5] that the microsupport of the normalized coherent state at x is $\{x\}$. In fact, using a similar computation as in the above proof we can conclude the same result. Let us briefly go thorough computations. Let $B:=T_k(f)$ be as in (3.20) and let

$$C:=\int_{\{x_0\}}P_{k,y}d\delta(y)=\int_{\{x_0\}}\frac{\Pi_k(\cdot,y)}{\Pi_k(y,y)}d\delta(y)=\int_{\{x_0\}}C(y,\cdot)d\delta(y). \tag{3.34}$$

Then by the same arguments as in (3.22)-(3.27) we can write

$$\begin{aligned} ||T_k(f)(\theta_k)||^2_{HS}\leq \\ \frac{1}{a_N}\left(\frac{k}{2\pi}\right)^{2n}\sum_{i,l,j,r\geq 0}k^{-l-i-j-r}\int_M\int_{\{x_0\}}\int_M\int_M |f(x)||f(x')|e^{\frac{-k}{2}[\phi(z,x)+\phi(x,y)+\phi(z,x')+\phi(x',y)]} \\ \times\left|u_l(z,x)u_i(x,y)u_j(z,x')u_r(x',y)\right|d\mu(x)d\mu(x')d\delta(y)d\mu(z)+\mathcal{O}(k^{-\infty}). \end{aligned} \tag{3.35}$$

One can see similarly that $\psi(z,y,x,x'):=\frac{1}{2}[\phi(z,x)+\phi(x,y)+\phi(z,x')+\phi(x',y)]$ is a Morse function with the critical point (x_0,x_0,x_0,x_0). Then applying the stationary phase method as in theorem 2.2.4, we can

obtain an upper bound for $||T_k(f)(\theta_k(\{x_0\}))||_{HS}$ as in (3.28). Indeed,

$$||T_k(f)(\theta_k)||^2_{HS} \leq$$
$$\frac{1}{a_N}\left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l,j,r\geq 0} k^{-l-i-j-r} \int_M \int_{\{x_0\}} \int_M \int_M |f(x)||f(x')| e^{-k\psi(z,y,x,x')}$$
$$\times \left| u_l(z,x)u_i(x,y)u_j(z,x')u_r(x',y) \right| d\mu(x)d\mu(x')d\delta(y)d\mu(z) + \mathcal{O}(k^{-\infty}) =$$
$$\frac{1}{a_N}\left(\frac{k}{2\pi}\right)^{2n} \sum_{i,l,j,r\geq 0} k^{-l-i-j-r} \left(\frac{2\pi}{k}\right)^{3n} \left[\int_{\{(x_0,x_0,x_0,x_0)\}} \left(\det(\tilde{f}''_{yy}/i)\right)^{-1/2} u(\tilde{z}) e^{ik\tilde{f}(\tilde{z})} d\tilde{z} + \mathcal{O}(k^{-1}) \right] + \mathcal{O}(k^{-\infty}). \tag{3.36}$$

Now by the same argument as in (3.29) and (3.30) we can conclude that $MS(\theta_k(\{x_0\})) \subset \{x_0\}$. Moreover, it has been shown in [5] that the microsupport cannot be empty. Thus, $MS(\theta_k(x_0)) = \{x_0\}$, and we are done with the proof.

■

Remark. To show that the microsupport is non-empty, it has been shown in [5] that for any family $(\theta_k)_{k\in\mathbb{N}}$, and a function $f \in C^\infty(M)$ such that $MS(\theta_k)$ and the support of f are disjoint, $T_k(f)(\theta_k) = \mathcal{O}(k^{-\infty})$. This statement implies that $MS(\theta_k)$ is non-empty. Indeed, by contradiction assume that $MS(\theta_k) = \emptyset$. Then taking $f = 1$ on M, we obtain $\theta_k = \mathcal{O}(k^{-\infty})$ by lemma 2.1.1, which contradicts $||\theta_k||_{tr} = 1$.

Chapter 4

Fidelity

Given a finite-dimensional complex Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ be the space of all Hermitian operators on $\mathcal{H}$, and let $\mathcal{S}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ be the subset of all positive trace 1 operators. In the matrix model of quantum mechanics, elements $A \in \mathcal{L}(\mathcal{H})$ and $\theta \in \mathcal{S}(\mathcal{H})$ represent observables and (mixed) states of the quantum system respectively [5].

Definition. Let $\theta, \sigma \in \mathcal{S}(\mathcal{H})$ be two quantum states. The fidelity $\Phi(\theta, \sigma)$ is a measure of overlap between this pair of quantum states, and is defined by [5]

$$\Phi(\theta, \sigma) = ||\sqrt{\theta}\sqrt{\sigma}||_{tr}. \tag{4.1}$$

The first section of this chapter is devoted to the computation of fidelity between quantization of disjoint submanifolds of a compact Kahler manifold. We will see in theorem 4.1.1 that in this case fidelity is negligible. In the next section, we compute fidelity between quantized versions of Lagrangian submanifolds that intersect transversely in a finite number of points. One can see in theorem 4.2.1 and proposition 4.2.2 that fidelity, in this case, is positive, and thus it can detect transverse intersections of Lagrangians.

4.1 Fidelity and Non-intersecting Submanifolds

Let (M^{2n}, ω, J) be a compact Kahler manifold, with a prequantum line bundle $L \to M$. Let $\mathcal{H}_k$ be the quantum space at level k, as defined in chapter two. Recalling the quantized submanifold (3.1), our goal here is to prove the following theorem:

Theorem 4.1.1. *Let $L, K \subset M$ be two disjoint submanifolds. Then $\Phi(\theta_k(L), \theta_k(K)) = \mathcal{O}(k^{-\infty})$.*

Before proving the above theorem, we need to see the following lemma:

Lemma 4.1.2. *Let M be a $n \times n$ matrix, with $||M||_{tr}$ its trace norm, and $||M||_{HS}$ its Hilbert-Schmidt norm. Then*

$$||M||_{HS} \leq ||M||_{tr} \leq \sqrt{\text{rank}(M)}||M||_{HS}. \tag{4.2}$$

Proof. For the LHS inequality note that

$$||M||_{tr} = \text{Tr}\,\sqrt{M^*M} \geq \sqrt{\text{Tr}(M^*M)} = ||M||_{HS}.$$

Now, for the RHS inequality, consider the inner product on the space of $m \times n$ matrices

$$\langle M, N\rangle = \mathrm{Tr}(MN^*) = \mathrm{Tr}(N^*M).$$

Then by the Cauchy-Schwartz inequality,

$$|\,\mathrm{Tr}(MN^*)| = |\langle M, N\rangle| \leq ||M||_{HS}||N||_{HS} = \sqrt{\mathrm{Tr}(M^*M)\,\mathrm{Tr}(N^*N)} = \sqrt{\mathrm{Tr}((\sqrt{M^*M})^2)\,\mathrm{Tr}((\sqrt{N^*N})^2)}. \tag{4.3}$$

Applying the polar decomposition to the square complex matrix M, one can write $M = U\sqrt{M^*M}$ where U is a unitary matrix. Then applying (4.3), and since U is unitary and $\sqrt{M^*M} = U^*M$, we have

$$||M||_{tr}^2 =$$
$$\left(\mathrm{Tr}\,\sqrt{M^*M}\right)^2 = (\mathrm{Tr}(U^*M))^2 = |\langle M, U\rangle|^2 \leq$$
$$\langle M, M\rangle\langle U, U\rangle = \mathrm{Tr}(M^*M).\,\mathrm{rank}(M) = ||M||_{HS}^2.\,\mathrm{rank}(M).$$

■

Remark. Let θ and σ be pure quantum states. Then, they are given by rank 1 projectors, and can be identified with unit vectors $\xi, \eta \in \mathcal{H}$ up to a phase factor respectively. By the above lemma and since both θ and σ have rank 1, their trace norm is equal to their Hilbert-Schmidt norm. Moreover since the square root of a pure state is equal to the state itself, we have $\Phi(\theta, \sigma) = ||\theta\sigma||_{HS} = |\langle\xi, \eta\rangle|$. Thus, orthogonal quantum states are maximally non-overlapping [5].

Proof of theorem 4.1.1. By definition (4.1),

$$\Phi(\theta_k(L), \theta_k(K)) = ||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{tr}, \tag{4.4}$$

and by (4.2) we have

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS} \leq \Phi(\theta_k(L), \theta_k(K)) \leq \sqrt{\mathrm{rank}(\sqrt{\theta_k(L)}\sqrt{\theta_k(K)})}.||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS}. \tag{4.5}$$

So, let us first compute the Hilbert-Schmidt norm.

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS} =$$
$$\sqrt{\mathrm{Tr}\left[(\theta_k(L)^{1/2}\theta_k(K)^{1/2})^*(\theta_k(L)^{1/2}\theta_k(K)^{1/2})\right]} = \sqrt{\mathrm{Tr}[\theta_k(K)^{1/2}\theta_k(L)\theta_k(K)^{1/2}]} = \sqrt{\mathrm{Tr}[\theta_k(L)\theta_k(K)]} \overset{(3.1)}{=}$$
$$\left(\mathrm{Tr}[\frac{1}{a_L a_K}\int_L P_{k,x}d\mu_L(x)\int_K P_{k,y}d\mu_K(y)]\right)^{1/2}.$$

In the last equality we have assumed that the submanifolds $L, K \subset M$ are not zero-dimensional. We will argue about the cases where L and K are points after we are done with this part.

Now, similar to the arguments in the proof of theorem 3.2.1, let

$$B := \frac{1}{a_L}\int_L P_{k,x}d\mu_L(x) = \frac{1}{a_L}\int_L \frac{\Pi_k(\cdot, x)}{\Pi_k(x, x)}d\mu_L(x) = \int_L B(x, \cdot)d\mu(x), \tag{4.6}$$

and

$$C := \frac{1}{a_K}\int_K P_{k,y}d\mu_K(y) = \frac{1}{a_K}\int_K \frac{\Pi_k(\cdot,y)}{\Pi_k(y,y)}d\mu_K(y) = \int_K C(y,\cdot)d\mu(y), \tag{4.7}$$

where μ in both cases is such that $\int_L d\mu = 1$, and $\int_K d\mu = 1$. Let $A := BC$. Since B and C are Hilbert-Schmidt operators, by (3.8) we have

$$\mathrm{Tr}(A) = \int_K\int_L B(x,y)C(y,x)d\mu(x)d\mu(y), \tag{4.8}$$

and thus

$$\mathrm{Tr}[\frac{1}{a_La_K}\int_L P_{k,x}d\mu_L(x)\int_K P_{k,y}d\mu_K(y)] = \int_K\int_L \frac{\Pi_k(y,x)}{\Pi_k(x,x)}\frac{\Pi_k(x,y)}{\Pi_k(y,y)}d\mu(x)d\mu(y). \tag{4.9}$$

Hence,

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||^2_{HS} = \int_K\int_L \frac{|\Pi_k(x,y)|^2}{\Pi_k(x,x)\Pi_k(y,y)}d\mu(x)d\mu(y). \tag{4.10}$$

Recall that by theorem 2.1.8, the projector $\Pi_k : L^2(M,L^k) \to \mathcal{H}_k$ has a Schwartz kernel of the form

$$\Pi_k(x,y) = \left(\frac{k}{2\pi}\right)^n E^k(x,y)\sum_{l\geq 0}k^{-l}u_l(x,y) + \mathcal{O}(k^{-\infty}).$$

Thus by (3.24),

$$\Pi_k(x,x) = \Pi_k(y,y) = \left(\frac{k}{2\pi}\right)^n + \mathcal{O}(k^{-1}), \tag{4.11}$$

and

$$|\Pi_k(x,y)|^2 = \left(\frac{k}{2\pi}\right)^{2n}||E^k(x,y)||^2\sum_{l,j\geq 0}k^{-l-j}u_l(x,y)u_j(x,y) + \mathcal{O}(k^{-\infty}). \tag{4.12}$$

Now define $\phi(x,y) := -2\log||E(x,y)||$. Therefore, we can write

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||^2_{HS} = \sum_{l,j\geq 0}k^{-l-j}\int_{K\times L}e^{-k\phi(x,y)}u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) + \mathcal{O}(k^{-\infty}). \tag{4.13}$$

By lemma 2.1.12, ϕ is a Morse-Bott function with the critical submanifold Δ_M. Now since $L\cap K = \emptyset$, ϕ has no critical points in $K\times L$. Taking $\tilde{f} := i\phi$, $\tilde{f}$ satisfies the hypotheses of theorem 2.2.7 by the same arguments as the ones after equation (3.18). Thus applying the stationary phase method for Morse-Bott functions as in theorem 2.2.7, for every $l,j\geq 0$,

$$\int_{K\times L}e^{-k\phi(x,y)}u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) = \mathcal{O}(k^{-\infty}), \tag{4.14}$$

and thus

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||^2_{HS} = \mathcal{O}(k^{-\infty}). \tag{4.15}$$

Recall from (2.28) that the Hilbert space $\mathcal{H}_k$ is finite-dimensional with dimension of order $\mathcal{O}(k^n)$. Since rank of any operator on $\mathcal{H}_k$ is bounded by dimension of $\mathcal{H}_k$, we must have

$$\sqrt{\mathrm{rank}(\sqrt{\theta_k(L)}\sqrt{\theta_k(K)})} \leq \mathcal{O}(k^{n/2}). \tag{4.16}$$

Then by (4.5),

$$\mathcal{O}(k^{-\infty}) \leq \Phi(\theta_k(L), \theta_k(K)) \leq \mathcal{O}(k^{-\infty+n/2}) = \mathcal{O}(k^{-\infty}). \tag{4.17}$$

Therefore, we can conclude finally that if $L \cap K = \emptyset$,

$$\Phi(\theta_k(L), \theta_k(K)) = \mathcal{O}(k^{-\infty}). \tag{4.18}$$

The only thing left is to prove theorem 4.1.1 for the case that $L = \{x_0\}$ and $K = \{x_1\}$ are disjoint points in M. By basically the same arguments,

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS} = \left(\mathrm{Tr}[\int_{\{x_0\}} P_{k,x} d\delta(x) \int_{\{x_1\}} P_{k,y} d\delta(y)]\right)^{1/2}. \tag{4.19}$$

Then similar to equations (4.9) and (4.10), and (4.13), by defining $\phi = -2\log||E||$, one can write (4.19) as

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS} = \sum_{l,j\geq 0} k^{-l-j} \int_{\{x_1\}\times\{x_0\}} e^{-k\phi(x,y)} u_l(x,y) u_j(x,y) d\delta(x) d\delta(y) + \mathcal{O}(k^{-\infty}). \tag{4.20}$$

Now the rest of the proof is similar to what we have done for the case of disjoint non-trivial submanifolds. Indeed, since (x_1, x_0) is not on the diagonal, it is not a critical point of the function ϕ, and similar to (4.15), (4.16), and (4.17), we have

$$\Phi(\theta_k(\{x_0\}), \theta_k(\{x_1\})) = \mathcal{O}(k^{-\infty}), \tag{4.21}$$

and we are done with the proof. ■

4.2 Fidelity and Transversely Intersecting Lagrangians

Let (M^{2n}, ω, J) be a compact Kahler manifold, with a prequantum line bundle $L \to M$. Let $\mathcal{H}_k$ be the quantum space at level k. Recalling the quantized submanifold (3.1), our goal here is to prove the following theorem:

Theorem 4.2.1. *Let $L, K \subset M^{2n}$ be two Lagrangian submanifolds intersecting transversely in a point $p \in M$. That is, $p \in L \pitchfork K$. Then*

$$\mathcal{O}(k^{-n/2}) \leq \Phi(\theta_k(L), \theta_k(K)) \leq \mathcal{O}(1). \tag{4.22}$$

Proof. The idea to prove the above theorem is basically the same as what we did in the proof of theorem 4.1.1. Indeed, by the definition (4.1),

$$\Phi(\theta_k(L), \theta_k(K)) = ||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{tr}, \tag{4.23}$$

and we have shown in (4.5) that

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS} \leq \Phi(\theta_k(L), \theta_k(K)) \leq \sqrt{\mathrm{rank}(\sqrt{\theta_k(L)}\sqrt{\theta_k(K)})}.||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||_{HS}. \tag{4.24}$$

We saw in (4.13) that taking $\phi(x,y) := -2\log||E(x,y)||$,

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||^2_{HS} = \sum_{l,j\geq 0} k^{-l-j}\int_{K\times L} e^{-k\phi(x,y)}u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) + \mathcal{O}(k^{-\infty}). \tag{4.25}$$

In fact,

$$\phi = -log(h(E,E)) : L\times K \to \mathbb{R} \tag{4.26}$$

where h is the Hermitian metric. Here we claim that the point $(p,p)\in L\times K$ is the single critical point of the function ϕ. Indeed,

$$\begin{aligned} d\phi = \frac{-1}{h(E,E)}dh(E,E) = \frac{-1}{h(E,E)}[h(\tilde{\nabla}E,E) + h(E,\tilde{\nabla}E)] = \\ \frac{-1}{h(E,E)}[h(-i\alpha_E\otimes E,E) + h(E,-i\alpha_E\otimes E)] = i(\alpha_E - \overline{\alpha_E}). \end{aligned} \tag{4.27}$$

Using lemma 2.1.11, as α_E vanishes along Δ_M, $d\phi$ vanishes at the point (p,p). Thus $(p,p)\in L\times K$ is the critical point of ϕ. Hence, Hessian of ϕ is well-defined on (p,p). To compute the Hessian of ϕ at the point $(p,p)\in L\times K$, first consider the more general picture. Indeed, let $(p,p)\in\Delta_M$ and let $X,Y\in T_{(p,p)}M\times\overline{M}$. Then

$$Hess_\phi(p,p)(X,Y) = \mathcal{L}_{\hat{X}}(\mathcal{L}_{\hat{Y}}\phi)(p,p) \tag{4.28}$$

where $\hat{X}$ and $\hat{Y}$ are extensions of X and Y to vector-fields. Thus using lemma 2.1.11 we obtain

$$Hess_{\phi_E}(p,p)(X,Y) = i\mathcal{L}_{\hat{X}}(\alpha_E(\hat{Y}) - \overline{\alpha_E}(\hat{Y}))(p,p) = i(B_E(X,Y) - \overline{B_E(X,Y)})(p,p). \tag{4.29}$$

Using part (iii) of lemma 2.1.11, whenever $X\in T_{(x,x)}\Delta_M$, $B_E(X,Y)=0$. Let us have a deeper look on the tangent space of diagonal at the point (p,p). Indeed, the tangent space of diagonal is the diagonal of the product of tangent spaces. So,

$$T_{(p,p)}\Delta_M = \Delta(T_pM\times T_pM) = \{(v,v)|v\in T_pM\} \tag{4.30}$$

and thus in our case the kernel of Hessian is $\{(v,v)|v\in T_pM\}$. However, $T_{(p,p)}L\times K = \{(v,w)|v\in T_pL, w\in T_pK\}$. Since $p\in L\pitchfork K$, one has $T_pM = T_pL\oplus T_pK$, and thus v and w are never the same except the trivial case of $v,w=0$. So, kernel of Hessian of ϕ is zero on $T_{(p,p)}L\times K$.

Here we again look at the embedding of $T_{(p,p)}\Delta_M$ inside $T_{(p,p)}M\times\overline{M}$. Since $T_{(p,p)}\Delta_M$ is Lagrangian, its orthogonal complement with respect to $\tilde{g}$ is $\tilde{J}(T_{(p,p)}\Delta_M)$. If $X\in\tilde{J}(T_{(p,p)}\Delta_M)$, we have $q(X) = q(X + i\tilde{J}X - i\tilde{J}X) = X + i\tilde{J}X$. So,

$$\begin{aligned} Hess_\phi(p,p)(X,Y) = i[\tilde{\omega}(X+i\tilde{J}X,Y) - \overline{\tilde{\omega}(X+i\tilde{J}X,Y)}] = \\ i[2i\tilde{\omega}(\tilde{J}X,Y)] = -2\tilde{\omega}(\tilde{J}X,Y) = 2\tilde{g}(X,Y) \quad ; \quad X\in\tilde{J}(T_{(p,p)}\Delta_M). \end{aligned} \tag{4.31}$$

Now take a generic $X\in T_{(p,p)}L\times K$. By the same argument as in the previous paragraph, $X\in\tilde{J}(T_{(p,p)}\Delta_M)$. Indeed, if the intersection of the tangent spaces of the product Lagrangians and diagonal at the point (p,p) is nonzero, then there is a curve on each of them with the same direction. So, there must be curves on both Lagrangians intersecting and having the same direction. This cannot happen since the two Lagrangians intersect transversely.

Thus Hessian of ϕ at the point (p,p) where $p \in L \pitchfork K$, is positive-definite and thus non-degenerate. So, the point $(p,p) \in L \times K$ is in fact the minimum of the function ϕ.

Thus $\phi : L \times K \to \mathbb{R}$ is a Morse function on a compact $2n$-dimensional manifold $L \times K$, with a critical point (p,p), and for any $l, j \geq 0$, $u_l u_j$ is a smooth function. Taking $\tilde{f} := i\phi$, and since $\phi \geq 0$ and moreover, $\phi(p,p) = -2\log||E(p,p)|| = 0$, and using the above argument on Hessian of ϕ, $\tilde{f}$ satisfies the hypotheses of theorem 2.2.4, and one can apply the stationary phase method (2.47) to compute the integral (4.25). Indeed,

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||^2_{HS} = \sum_{l,j\geq 0} k^{-l-j} \int_{K\times L} e^{-k\phi(x,y)} u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) + \mathcal{O}(k^{-\infty}) =$$

$$\sum_{l,j\geq 0} k^{-l-j} \int_{K\times L} e^{ik\tilde{f}(x,y)} u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) + \mathcal{O}(k^{-\infty}) \overset{(2.47)}{=}$$

$$\sum_{l,j\geq 0} k^{-l-j} \left(\frac{2\pi}{k}\right)^n e^{-k\phi(p,p)} \det(Hess_\phi(p,p))^{-1/2} \sqrt{\det \tilde{g}} \left(\lambda(p,p)u_l(p,p)u_j(p,p) + \mathcal{O}(k^{-1})\right) =$$

$$\left(\frac{2\pi}{k}\right)^n \frac{\sqrt{\det \tilde{g}}}{\sqrt{2\det \tilde{g}}} \left[\lambda(p,p)|u_0(p,p)|^2 + \mathcal{O}(k^{-1})\right] + \sum_{l+j\geq 1} \left(\frac{2\pi}{k}\right)^n \frac{\sqrt{\det \tilde{g}}}{\sqrt{2\det \tilde{g}}} \left[\lambda(p,p)u_l(p,p)u_j(p,p) + \mathcal{O}(k^{-1})\right] =$$

$$\left(\frac{2\pi}{k}\right)^n .(Const.) + \mathcal{O}(k^{-(n+1)}) = \mathcal{O}(k^{-n}).$$

Thus by (4.24),

$$\Phi(\theta_k(L)\theta_k(K)) \geq \mathcal{O}(k^{-n/2}). \tag{4.32}$$

Substituting (4.16) in (4.5), we have

$$\mathcal{O}(k^{-n/2}) \leq \Phi(\theta_k(L), \theta_k(K)) \leq \mathcal{O}(k^{n/2-n/2}) = \mathcal{O}(1), \tag{4.33}$$

that completes the proof. ■

Remark. Note that non of the computations in the proof of theorem 4.2.1 actually depend on L and K being Lagrangian submanifolds. The only thing we have used is that the two submanifolds are intersecting transversely in a point such that $\dim L + \dim K = \dim M = 2n$.

Now, one can generalize theorem 4.2.1 to the case of a finite number of intersections.

Proposition 4.2.2. *Let $L, K \subset M^{2n}$ be two Lagrangian submanifolds intersecting transversely in more than one point. That is, $p_1, \cdots, p_r \in L \pitchfork K$. Then*

$$\mathcal{O}(k^{-n/2}) \leq \Phi(\theta_k(L), \theta_k(K)) \leq \mathcal{O}(1). \tag{4.34}$$

Proof. The idea behind the proof is basically using the same arguments as in the proof of the previous theorem, but applying the stationary phase method for finite numbers of critical points. Indeed, $(p_1,p_1), \cdots, (p_r,p_r)$ are isolated non-degenerate critical points of the Morse function ϕ, as defined in (4.26). Now by the same arguments as before the inequality (4.32), one can apply the stationary phase method (2.47), and

therefore we obtain

$$||\sqrt{\theta_k(L)}\sqrt{\theta_k(K)}||^2_{HS} = \sum_{l,j\geq 0} k^{-l-j} \int_{K\times L} e^{-k\phi(x,y)} u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) + \mathcal{O}(k^{-\infty}) =$$

$$\sum_{l,j\geq 0} k^{-l-j} \int_{K\times L} e^{ik\tilde{f}(x,y)} u_l(x,y)u_j(x,y)d\mu(x)d\mu(y) + \mathcal{O}(k^{-\infty}) \overset{(2.47)}{=}$$

$$\sum_{i=1}^{r}\sum_{l,j\geq 0} k^{-l-j}\left(\frac{2\pi}{k}\right)^n e^{-k\phi(p_i,p_i)} \det(Hess_\phi(p_i,p_i))^{-1/2}\sqrt{\det\tilde{g}}\left(\lambda(p_i,p_i)u_l(p_i,p_i)u_j(p_i,p_i) + \mathcal{O}(k^{-1})\right) =$$

$$\sum_{i=1}^{r}\left(\frac{2\pi}{k}\right)^n \frac{\sqrt{\det\tilde{g}}}{\sqrt{2\det\tilde{\tilde{g}}}}\left[\lambda(p_i,p_i)|u_0(p_i,p_i)|^2 + \mathcal{O}(k^{-1})\right] +$$

$$\sum_{i=1}^{r}\sum_{l+j\geq 1}\left(\frac{2\pi}{k}\right)^n \frac{\sqrt{\det\tilde{\tilde{g}}}}{\sqrt{2\det\tilde{g}}}\left[\lambda(p_i,p_i)u_l(p_i,p_i)u_j(p_i,p_i) + \mathcal{O}(k^{-1})\right] =$$

$$\left(\frac{2\pi}{k}\right)^n .(Const.) + \mathcal{O}(k^{-(n+1)}) = \mathcal{O}(k^{-n}).$$

The rest of the proof is similar to what we have done in theorem 4.2.1. ■

Appendices

Appendix A

Complex Line Bundles with Connections

This section is based on chapter three of [7]. First we define complex line bundles, and then we try to elaborate on connections on line bundles, and their curvature. The last part is devoted to the Chern connection of a Hermitian holomorphic line bundle. Chern connection is an important concept, which we have used in the beginning of thesis when we have defined prequantum line bundles.

Let M be a smooth manifold.

A.1 Complex Line Bundles

Definition. A complex line bundle over M is the data of a smooth manifold L and a smooth map $\pi : L \to M$ such that

i) For every $m \in M$, the fiber $L_m := \pi^{-1}(m)$ over m is a one-dimensional complex vector space,

ii) M is covered by local trivializations $(U_i, \tau_i)_{i \in I}$, where $U_i \subset M$ is an open set and

$$\tau_i : U_i \times \mathbb{C} \to \pi^{-1}(U_i)$$

is a diffeomorphism which, when restricted to $\{m\} \times \mathbb{C}$, $m \in U_i$, induces a linear isomorphism $\tau_i^m = \tau_i(m, \cdot) : \mathbb{C} \to L_m$.

Definition. A complex line bundle $L \to M$ is said to be holomorphic if L is a complex manifold, $\pi : L \to M$ is a holomorphic map, and the trivializations maps τ_i are biholomorphisms.

A.2 Connections on Line Bundles

Definition. A connection on a complex line bundle $L \to M$ is a linear map $\nabla : C^\infty(M, L) \to C^\infty(M, T^*M \otimes L)$ satisfying the Leibniz rule:

$$\nabla(fs) = f\nabla s + df \otimes s$$

for every $f \in C^\infty(M)$ and every $s \in C^\infty(M, L)$. If X is a smooth vector field on M, the map $\nabla_X : C^\infty(M, L) \to C^\infty(M, L)$ obtained by contracting ∇ and X is called the covariant derivative along X.

Proposition A.2.1. *Let $L \to M$ be a complex line bundle. The space of connections on L is non-empty and of the form*

$$\{\nabla + \beta | \beta \in \Omega^1(M) \otimes \mathbb{C}\}$$

Proof. For the existence part, consider a locally finite open cover $(U_i)_{i\in I}$ of M, together with maps $\tau_i : U_i \times \mathbb{C} \to \pi^{-1}(U_i)$, such that (U_i, τ_i) is a local trivialization. Let $s_i = \tau_i(\cdot, 1)$ be the associated unit section. Let $(\lambda_i)_{i\in I}$ be a partition of unity subordinate to the open cover $(U_i)_{i\in I}$. Let $t \in C^\infty(M, L)$ and for $i \in I$, let $f_i \in C^\infty(M)$ be such that $t = f_i s_i$ over U_i. Then define

$$\nabla t = \sum_{i\in I} \lambda_i df_i \otimes s_i.$$

By using the Leibliz rule for the differential of functions, it is easy to see that ∇ is indeed a connection.

Now, let ∇, ∇' be two connections on $L \to M$. Then, by the Leibniz rule, for any $s \in C^\infty(M, L)$ and any $f \in C^\infty(M)$, we have

$$\nabla(fs) - \nabla'(fs) = f(\nabla s - \nabla' s).$$

This implies that $\nabla - \nabla'$ is the multiplication by some one-form β. ■

Remark. i) If $L \to M$ and $L' \to M$ are two line bundles with connections ∇ and ∇' respectively, then $\nabla \otimes \nabla'$ defined by

$$(\nabla \otimes \nabla')(s \otimes t) = (\nabla s) \otimes t + s \otimes (\nabla' t), \quad \forall s \in C^\infty(M, L), \forall t \in C^\infty(M, L')$$

and extended linearly is a connection on $L \otimes L' \to M$.

ii) Let M, N be smooth manifolds, let $L \to N$ be a complex line bundle with connection ∇, and let $f : M \to N$ be a smooth function. There exists a unique connection $f^*\nabla$ on the pullback bundle $f^*L \to M$ such that

$$(f^*\nabla_X)(f^*s) = f^*(\nabla_{df(X)} s)$$

for every $s \in C^\infty(N, L)$ and every vector field X on M.

The previous remark shows that if $L \to M$ and $K \to N$ are two line bundles with connections ∇, ∇' respectively, we can obtain an induced connection $\tilde{\nabla}$ on the line bundle $L \boxtimes K \to M \times N$.

A.3 Curvature of a Connection

Definition. Let ∇ be a connection on a line bundle $L \to M$. The curvature of ∇ is the differential form $\mathrm{curv}(\nabla) \in \Omega^2(M) \otimes \mathbb{C}$ defined by

$$\mathrm{curv}(\nabla)(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for every pair of vector fields $X, Y \in C^\infty(M, TM)$.

Lemma A.3.1. *If $\nabla = d + \beta$ locally, then* $\mathrm{curv}(\nabla) = d\beta$. *In particular, the curvature of a connection is a closed form.*

Proof. We have the equality

$$\nabla_X \nabla_Y 1 - \nabla_Y \nabla_X 1 - \nabla_{[X,Y]} 1 = \nabla_X(\beta(Y)) - \nabla_Y(\beta(X)) - \beta([X,Y]).$$

Thus,

$$\mathrm{curv}(\nabla)(X,Y) = \mathcal{L}_X(\beta(Y)) - \mathcal{L}_Y(\beta(X)) - \beta([X,Y]) = d\beta(X,Y).$$

■

This shows us how to construct local primitives of the curvature: take a local non-vanishing section s of $L \to M$ and compute $\beta = \nabla s/s$.

Lemma A.3.2. *Let $\beta \in \Omega^1(M) \otimes \mathbb{C}$ be a primitive of* $\mathrm{curv}(\nabla)$. *Then for every $m \in M$, there exists a local non-vanishing section s defined on a neighborhood of m such that $\nabla s = \beta \otimes s$.*

Proof. Let $m \in M$ and take any non-vanishing local section s_0 near m. We have $d\beta_0 = \mathrm{curv}(\nabla)$ where $\beta_0 = \nabla s_0/s_0$. Since $\beta - \beta_0$ is closed, by restricting the neighborhood of m if necessary, the Poincare lemma implies that there exists a smooth function f with $\beta - \beta_0 = df$ near m. The local section $s = \exp(f)s_0$ satisfies the desired properties. ■

A.4 The Chern Connection

Definition. Let M be a smooth manifold. A Hermitian line bundle is a complex line bundle $L \to M$ endowed with a Hermitian metric h, that is the data of a Hermitian inner product h_m on each fiber L_m, such that the function $h(s,s)$ is smooth for any smooth section s of $L \to M$.

Note that every complex line bundle $L \to M$ is Hermitian. Indeed, let $(U_i, \tau_i)_{i\in I}$ be a cover of M by local trivializations and let $(\lambda_i)_{i\in I}$ be a partition of unity subordinate to this cover. Given $m \in U_i$, define

$$h^i_m(u,v) = (\tau_i^m)^{-1}(u)\overline{(\tau_i^m)^{-1}(v)}$$

for any $u, v \in L_m$, where τ_i^m is the linear isomorphism from $\mathbb{C}$ to L_m induced by τ_i. Then $h = \sum_{i\in I} \lambda_i h^i$ is a Hermitian metric on L.

Definition. A connection ∇ on a Hermitian line bundle $L \to M$ is said to be compatible with the Hermitian structure if

$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t)$$

for every $s, t \in C^\infty(M, L)$.

Let s be a local section of L over some open subset $U \subset M$ such that $h(s,s) = 1$, and let β be the local one-form such that $\nabla s = \beta \otimes s$. Then, for $f, g \in C^\infty(U)$

$$h(\nabla(fs), gs) + h(fs, \nabla(gs)) = d(h(fs, gs)) + (\beta + \overline{\beta})h(fs, gs).$$

Consequently, ∇ and h are compatible if and only if all the forms β obtained in this way are purely imaginary, i.e. of the form $\beta = i\alpha$ with $\alpha \in \Omega^1(U)$.

Definition. A connection ∇ on a holomorphic line bundle $L \to M$ over a complex manifold M is said to be compatible with the holomorphic structure if

$$s \text{ is a local holomorphic section of } L \to M \iff \forall Z \in C^\infty(M, T^{1,0}M), \quad \nabla_{\overline{Z}} s = 0.$$

If s is a local non-vanishing holomorphic section over some open subset $U \subset M$ and $\nabla s = \beta \otimes s$, then $\nabla_{\overline{Z}} s = 0$ for every Z of type $(1,0)$ if and only if the form β belongs to $\Omega^{1,0}(U)$.

Proposition A.4.1. *Let M be a complex manifold and $L \to M$ be a Hermitian holomorphic line bundle. Then there exists a unique connection ∇ on M which is compatible with both the Hermitian and holomorphic structures. This connection ∇ is called the Chern connection of $L \to M$.*

Proof. Assume that such a connection ∇ exists. Let $s \in C^\infty(U, L)$ be a local non-vanishing holomorphic section of L defined on some open set $U \subset M$, and consider the non-vanishing function $H = h(s, s)$. Let $\beta \in \Omega^1(U) \otimes \mathbb{C}$ be such that $\nabla s = \beta \otimes s$. Since s is holomorphic and ∇ is compatible with the holomorphic structure, we have that $\beta \in \Omega^{1,0}(U)$. Using the compatibility of ∇ with the Hermitian structure, we also have that

$$dH = h(\nabla s, s) + h(s, \nabla s) = \beta H + \overline{\beta} H.$$

By identifying the $(1,0)$ parts, this yields

$$\beta = \frac{\partial H}{H} = \partial(\log H),$$

which means that β, hence ∇, is uniquely determined on U. To prove the existence part, we follow the same reasoning backwards. The fact that the local connections that we obtain glue together to form a global connection comes from the local uniqueness. ∎

Remark. Observe that the curvature of the Chern connection of a Hermitian holomorphic line bundle lies in $\Omega^{1,1}(M)$ and is purely imaginary in the sense that it is of the form $i\alpha$ for some $\alpha \in \Omega^2(M)$. Indeed, from the proof of the uniqueness of the Chern connection, the local forms $\beta_i \in \Omega^1(U_i) \otimes \mathbb{C}$ such that $\nabla s_i = \beta_i s_i$ are given by $\beta_i = \partial(\log H_i)$, where $H_i = h(s_i, s_i)$. Therefore,

$$\mathrm{curv}(\nabla)_{|U_i} = d\beta_i = \overline{\partial}\beta_i = \overline{\partial}\partial(\log H_i)$$

which is a purely imaginary form. Indeed, $\mathrm{curv}(\nabla)$ is closed, because using $\partial^2 = 0$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$, and $\overline{\partial}^2 = 0$, we have

$$d(\overline{\partial}\partial(\log H_i)) = (\partial + \overline{\partial})(\overline{\partial}\partial(\log H_i)) = 0,$$

and

$$\overline{\mathrm{curv}(\nabla)_{|U_i}} = \partial\overline{\partial}(\overline{\log H_i}) = -\overline{\partial}\partial(\log H_i),$$

thus $\mathrm{curv}(\nabla)_{|U_i}$ is a purely imaginary form.

Proposition A.4.2. *Let (M, ω) be a complex manifold, and let $L \to M$ be a complex line bundle over M, endowed with a connection ∇ such that* $\mathrm{curv}(\nabla) \in \Omega^{1,1}(M)$. *Then there exists a unique holomorphic*

structure on $L \to M$ which is compatible with ∇.

Proof. See [7] for the proof. ■

Note that in general, a differential form in $\Omega^{1,1}(M) \cap i\Omega^2(M)$ may not be the curvature of a Chern connection. In the compact Kahler case, however, we have the following result.

Proposition A.4.3. *Let (M, ω) be a compact Kahler manifold, and let $L \to M$ be a holomorphic line bundle over M, with curvature class $-i\omega$. Then there exists a Hermitian metric on $L \to M$, unique up to a multiplicative constant, such that the Chern connection associated with this metric and the initial holomorphic structure has curvature $-i\omega$.*

Proof. Let h_0 be any Hermitian metric on $L \to M$, and let ∇_0 be the corresponding Chern connection. We look for another Hermitian metric, which will be of the form $h = \exp(f)h_0$ for some real-valued smooth function f on M, with Chern connection ∇. Remembering the proof of the uniqueness of the Chern connection, we have that if s is a local non-vanishing holomorphic section and $H = h(s, s)$, then

$$\nabla s = \partial(logH) \otimes s = (\partial f + \partial(\log h_0(s, s))) \otimes s,$$

which implies that $\mathrm{curv}(\nabla) = \mathrm{curv}(\nabla_0) + \overline{\partial}\partial f$. Consequently, we want to find some real-valued f solving the equation

$$-i\omega = \mathrm{curv}(\nabla_0) + \overline{\partial}\partial f.$$

But $i\mathrm{curv}(\nabla_0)$ and ω belong to the same cohomology class, hence $i\mathrm{curv}(\nabla_0) = \omega + \alpha$ for some exact form α. Furthermore, α is a real-valued form of type $(1, 1)$. So by the global $i\partial\overline{\partial}$-lemma, there exists a real-valued function f such that $\alpha = i\partial$; this function solves the above equation. Moreover, f is unique up to adding a constant, so h is unique up to multiplication by a constant. ■

Lemma A.4.4 (The global $i\partial\overline{\partial}$-lemma)**.** *Let (M, ω) be a compact Kahler manifold. Let α be an exact, real-valued form of type $(1, 1)$ on M. Then there exists a function $\phi \in C^\infty(M, \mathbb{R})$ such that $\alpha = i\partial\overline{\partial}\phi$. This function is unique up to the addition of a constant.*

Appendix B

A Quick Introduction to Morse Theory

First part of this section is devoted to introducing Morse theory mainly based on chapter one of [8]. In the next part we try to generalize Morse theory to the case of non-degenerate critical submanifolds. In this part we will use mostly [4] and [1].

B.1 Critical points and the Morse Lemma

Considering a smooth function on a manifold, in this section we introduce some basic properties of the function such as critical points and Hessian. Then we go forward to introduce Morse functions and some properties of their critical points.

Let f be a smooth real-valued function on a manifold M. A point $p \in M$ is a critical point of f if $Df(p) : T_pM \to \mathbb{R}$ is zero. That is, for a neighborhood U of p with local coordinates $(x^1, \cdots, x^n)$ we have $\frac{\partial f}{\partial x^i} = 0$ for all $i \in \{1, \cdots, n\}$. A critical point p is non-degenerate if and only if the matrix $(\frac{\partial^2 f}{\partial x^i \partial x^j}(p))$ is non-singular. For a critical point p the Hessian of f at p is a symmetric bilinear form on the tangent space

$$Hess_p(f) : T_pM \times T_pM \to \mathbb{R},$$

such that $Hess_p(f)(v, w) = \tilde{v}_p(\tilde{w}(f))$ where $\tilde{v}$ and $\tilde{w}$ are extensions of $v, w \in T_pM$ to vector fields. We can see that in the local coordinates $(x^1, \cdots, x^n)$ Hessian has the familiar form

$$Hess_p(f) = (\frac{\partial^2 f}{\partial x^i \partial x^j}(p)) \ .$$

Thus a critical point $p \in M$ of f is non-degenerate if $Hess_p(f)$ is non-singular. Index of a bilinear form H on a vector space V is defined to be the maximal dimension of a subspace of V on which H is negative-definite. We define the index of f at a critical point p to be the index of $Hess_p(f)$. By linear algebra we know that the the index of a bilinear form does not depend on the choice of diagonalization. Thus, index is well-defined.

Lemma B.1.1 (Lemma of Morse)**.** *Let p be a non-degenerate critical point of f. Then there is a local*

coordinate system $(y^1, \cdots, y^n)$ *in a neighborhood* U *of* p *with* $y^i(p) = 0$ *for all* i *and such that*

$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2 \tag{B.1}$$

on U*, where* λ *is the index of* f *at* p*.*

Proof. We first show that if we can write f as in (B.1), then λ is the index of f at p. Suppose that f is written as above. Then

$$\frac{\partial^2 f}{\partial y^i \partial y^j}(p) = \begin{cases} -2; & i = j \leq \lambda \\ 2; & i = j > \lambda \\ 0; & \text{otherwise} \end{cases}$$

Hence, with respect to the basis $\{\frac{\partial}{\partial y^1}|_p, \cdots, \frac{\partial}{\partial y^n}|_p\}$, Hessian of f at the point p has the form:

$$Hess_p(f) = \begin{bmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{bmatrix}$$

Thus there exists a subspace of T_pM of dimension λ where $Hess_p(f)$ is negative-definite, and a subspace V of dimension $n - \lambda$ where $Hess_p(f)$ is positive-definite. If there were a subspace of T_pM of dimension greater than λ on which $Hess_p(f)$ were negative-definite, then this subspace would intersect V, which is impossible. Thus λ is the index of f at p. Now we need to show that the corresponding local coordinate $(y^1, \cdots, y^n)$ exists. Assuming p being the origin of $\mathbb{R}^n$ and that $f(p) = f(0) = 0$, we can write

$$f(x^1, \cdots, x^n) = \sum_{j=1}^{n} x^j g_j(x^1, \cdots, x^n)$$

in some convex neighborhood of zero, where the functions g_j are smooth such that $g_j(0) = \frac{\partial f}{\partial x^j}(0)$. Now since 0 is assumed to be the critical point, $g_j(0) = 0$. Therefore, we can write $g_j(x^1, \cdots, x^n) = \sum_{i=1}^{n} x^i h_{ij}(x^1, \cdots, x^n)$ for some smooth functions h_{ij}. It follows that:

$$f(x^1, \cdots, x^n) = \sum_{i,j=1}^{n} x^i x^j h_{ij}(x^1, \cdots, x^n)$$

and $h_{ij}(0) = (\ \frac{\partial^2 f}{\partial x^i \partial x^j}(0))$. Replacing h_{ij} by $\frac{1}{2}(h_{ij} + h_{ji})$ if necessary, we can assume that h_{ij} is symmetric. Suppose by induction that there exist coordinates $(u^1, \cdots, u^n)$ in a neighborhood U of 0 such that

$$f = \pm(u^1)^2 \pm \cdots \pm (u^{r-1})^2 + \sum_{i,j \geq r} u^i u^j H_{ij}(u^1, \cdots, u^n),$$

where the matrices H_{ij} are symmetric. After a permutation of the last $n - r + 1$ coordinates, we can assume that $H_{rr}(0) \neq 0$. Define $g(u^1, \cdots, u^n) := \sqrt{|H_{rr}(u^1, \cdots, u^n)|}$ which is a smooth non-zero function on some smaller neighborhood $U_2 \subset U_1$ of zero. Define a new coordinate system $(v^1, \cdots, v^n)$ by setting $v^i = u^i$ for

$i \neq r$, and $v^r(u^1, \cdots, u^n) = g(u^1, \cdots, u^n)[\ u^r + \sum_{i>r} u^i H_{ir}(u^1, \cdots, u^n)/H_{rr}(u^1, \cdots, u^n)]$. By the Inverse Function Theorem, $(v^1, \cdots, v^n)$ are coordinate functions in some smaller neighborhood U_3 of zero, and f can be represented by

$$f = \sum_{i \leq r} \pm (v^i)^2 + \sum_{i,j>r} v^i v^j H'_{ij}(v^1, \cdots, v^n),$$

which concludes the lemma. ■

Corollary B.1.2. *Non-degenerate critical points are isolated. Considering compact manifolds, there are only finitely many critical points.*

Definition. A Morse function on a smooth manifold M is a smooth function $f : M \to \mathbb{R}$ such that all of its critical points are non-degenerate.

B.2 Morse-Bott Theory

In this section we will see a generalization of Morse theory to the case of functions with non-degenerate critical submanifolds, called Morse-Bott functions. In the following we see that one can view a Morse-Bott function as a parameterized Morse function.

Let $f : M \to \mathbb{R}$ be a smooth function.

Definition. Let $N \subset M$ be a submanifold such that

i) All points of N are critical points of the smooth function $f : M \to \mathbb{R}$,

ii) The nullity of all $x \in N$ equals the dimension of N.

Then N will be called a non-degenerate critical submanifold of f on M.

Now let $f : M \to \mathbb{R}$ be a smooth function whose critical set is a disjoint union $\sqcup_i C_i$ of connected smooth submanifolds of dimensions $d_i \geq 0$, called critical submanifolds. Taking a Riemannian metric on M, we can split the tangent bundle over each C_i as

$$TM_{|C_i} = TC_i \oplus \mathcal{N}C_i \tag{B.2}$$

where $\mathcal{N}C_i$ is the normal bundle of C_i. Let $p \in C_i$ and $v, w \in T_pC_i$. Then

$$Hess_p(f)(v, w) = v_p(\tilde{w}(f)) = 0 \tag{B.3}$$

since $v_p \in T_pC_i$, and any extension $\tilde{w}$ of w to a vectorfield on C_i satisfies $df(\tilde{w}) = 0$. Thus $Hess_p(f)$ induces a bilinear symmetric form $Hess_p(f)$ on $\mathcal{N}_pC_i \cong T_pM/T_pC_i$.

Definition. A smooth function $f : M \to \mathbb{R}$ is called a Morse-Bott function if the set of critical points of f is a disjoint union of connected smooth submanifolds $Crit(f) = \sqcup_i C_i$, and for each C_i and $p \in C_i$, the Hessian $Hess_p(f)$ is non-degenerate in the direction normal to the critical submanifold.

Lemma B.2.1 (Morse-Bott Lemma)**.** *Let M be a n-dimensional smooth manifold, and $f : M \to \mathbb{R}$ be a Morse-Bott function. Assume that C is a connected component of $Crit(f)$ of dimension d, and let $p \in C$. Then there exists an open neighborhood U of p and a smooth chart $\phi : U \to \mathbb{R}^d \times \mathbb{R}^{n-d}$ such that*

i) $\phi(p) = 0$,

ii) $\phi(U \cap C) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{n-d} | y = 0\}$,

iii) $(f \circ \phi^{-1})(x, y) = f(C) - y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_{n-d}^2$,

where $k \leq n - d$ *is the index of* $Hess_p(f)$ *(on the normal bundle), and* $f(C)$ *is the common value of* f *on* C.

Proof. For a proof we refer to [1]. ■

Corollary B.2.2. *The index* k *of* $Hess_p(f)$ *is locally constant. So, it is the same for all points* $p \in C$. *We say that* C *is a critical submanifold of index* k.

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