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The Decay of Matrix Coefficients of Special Linear Groups over Local Fields

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Abstract

A non-compact topological group G is said to have the Howe-Moore Property if for each of its unitary representations without trivial subrepresentations the matrix coefficients vanish at infinity. In the first part of this thesis it is shown that, for every $d \geq 2$ and every local field F, $SL_d(F)$ has the Howe–Moore Property. In the second half the unitary dual of $SL_d(F)$ is studied, with the goal of calculating quantitative estimates and uniform pointwise upper bounds for matrix coefficients of $SL_d(F)$. This is done by using spherical functions and representations. In particular, if F is non-archimedean, with $\mathcal{O}$ its ring of integers, then, using homogeneous trees, the spherical functions of $(SL_2(F), SL_2(\mathcal{O}))$ are completely classified, together with an integral formula, and its spherical representations of are partly classified.

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Introduction

A local field is a non-discrete topological field which is locally compact. Let $d \geq 1$ be a positive integer, and let F be local field. Let $SL_d(F)$ be the group of all $d \times d$ matrices over F whose determinant is 1. It is called the *special linear group* of degree d over F. The goals of this thesis are

- to show that for $d \geq 2$ $SL_d(F)$ has the Howe–Moore Property,
- to *understand* the decay of matrix coefficients of $SL_d(F)$ and
- to find quantitative estimates or upper bounds for those matrix coefficients.

A unitary representation of a topological group G is a group homomorphism from G to the unitary group of a Hilbert space which is continuous with respect to the strong operator topology.

A non-compact topological group G is said to have the Howe–Moore Property if for every unitary representation of G without trivial subrepresentations the matrix coefficients vanish at infinity. The property is named after Roger E. Howe and Calvin C. Moore. It turns out that it is not easy to find groups which have the Howe–Moore Property. At this date there are only a few groups known to enjoy the property. The first ones were identified by Howe and Moore around 1977. In their paper [HM79] they showed that all connected non-compact simple real Lie groups with finite center have the Howe–Moore Property; e.g. $SL_d(\mathbb{R})$ and $SL_d(\mathbb{C})$ for $d \geq 2$. They also showed the property for every isotropic simple algebraic group over a non-archimedean local field. The most recent addition to this list of known examples was made by Corina Ciobotaru in [CC21]. See there for the currently complete list of groups known to admit the Howe–Moore Property. In this thesis it will be shown only for a few groups from this list that they have the property. In particular, in Chapter 2 it will be shown that, for every $d \geq 2$ and every local field F, $SL_d(F)$ has the Howe–Moore Property.

One approach to study the decay of the matrix coefficients of unitary representations of $SL_d(F)$ is to study its unitary dual. In particular, consider the question whether, for a topological group G which has the Howe–Moore Property, the matrix coefficients of unitary representations of G without trivial subrepresentations decay at some *minimal rate*. When G is locally compact, the diagonal matrix coefficients of those unitary representations of G decay arbitrarily slowly on compact subsets if and only if the trivial representation is not isolated in the unitary dual $\widehat{G}$ with respect to the Fell topology, or equivalently G does not have Kazhdan's Property (T). To answer whether a group G has Property (T), positive definite spherical functions are used throughout this thesis, which are a special kind of matrix coefficients of G. A function on G is spherical with respect to a compact subgroup $K < G$, and a positive definite spherical function is the matrix coefficient of an irreducible representation of G with respect to a K-invariant unit vector, which is unique up to a scalar multiple. It turns out that if (G, K) is a Gelfand pair, then G has Property (T) if and only if its spherical matrix coefficients decay arbitrarily slowly.

In the case where F is non-archimedean with $\mathcal{O}$ its ring of integers, in Chapter 5 an integral formula for the spherical functions of $(SL_2(F), SL_2(\mathcal{O}))$ is established, which is similar to the integral formulas for the spherical functions of $(SL_2(\mathbb{R}), SO(2))$ and of $(SL_2(\mathbb{C}), SU(2))$ in Chapter 3. In the same chapter also the positive definite spherical functions of $(SL_2(F), SL_2(\mathcal{O}))$

are patially identified. One conclusion of this incomplete classification is that $SL_2(F)$ does not have Property (T).

Using positive definite spherical functions and their quantitative estimates, also non-trivial uniform pointwise upper bounds for the matrix coefficients with respect to K-finite vectors of unitary representations can be calculated. This is accomplished by using a theorem proven by Michael Cowling, Uffe Haagerup and Howe in [CHH88]. The sharpest possible uniform pointwise upper bounds for such matrix coefficients of $SL_d(F)$ have been calculated by Hee Oh in [Oh98] and [Oh02]. A possible application of the quantitative estimates and upper bounds is to compute Kazhdan constants.

The prerequisites for reading this thesis are kept at a minimum. The reader is expected to have basic knowledge in functional analysis. Some familiarity with the theory of locally compact groups and local fields can be useful but is not necessary, and the unfamiliar reader is referred to the appendices A and B. Only in Section 2.4 is some knowledge about Lie groups required; however the reader unfamiliar with the theory of Lie groups can skip this section, as its content is not used for the remainder of the thesis.

Outline

The following is an overview of the content of the different chapters and should serve the reader as a guide through the thesis.

In Chapter 1 the theory of unitary representations of topological groups is introduced. In particular, it includes the Gelfand-Naimark-Segal construction, the Raikov Theorem and the inducing construction. The main references for that chapter are [Fol15] and [BdlHV08].

In Chapter 2, the Howe–Moore Property is defined, and an algebraic criterion is proven for determining that a group has the Howe–Moore Property. The criterion was introduced by Corina Ciobotaru in her paper [Cio15]. The criterion is used to prove that, for every $d \geq 2$ and every local field F, $SL_d(F)$ has the Howe–Moore Property. The chapter is finished by proving the Vanishing Theorem for semisimple Lie groups from [BM00].

In Chapter 3 the notions of spherical functions and representations are introduced. The main references for that are [Lan85] and [Mac71]. At the end of the chapter the integral formulas for the spherical functions of $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$ are stated, and the spherical representations of $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$ are classified without proof.

Chapter 4 introduces the notion of weak containment and the Fell topology on the unitary dual of a topological group. The two theorems in [CHH88] are stated and proven. In particular, a generalized version of the Harish-Chandra function is defined, and a generalized version of [CHH88, Thm. 2] is stated and proven. The chapter is finished by introducing Kazhdan's Property (T).

Chapter 5 is about the theory of unitary representations of groups of automorphisms on a homogeneous trees. It is shown that certain subgroups of the automorphism group of a homogeneous tree admit the Howe–Moore Property. Moreover, for every closed non-compact subgroup G, which acts transitively on the boundary of the tree, and for the stabilizer group $K < G$ of a vertex, all spherical functions of (G, K) are described in Corollary 5.6.3, and in Theorem 5.8.6 and Theorem 5.8.7 the spherical representations of (G, K) are classified. The main reference for that chapter is [FTN91].

Let F be a non-archimedean local field. In Chapter 6 the Bruhat–Tits building for $SL_d(F)$, for $d \geq 2$, is constructed, and it is shown that for $d = 2$ the building is a homogeneous tree. The main reference for that is [Ste97]. Using the results from the previous chapter the spherical representations of $(SL_2(F), SL_2(\mathcal{O})$ are classified in Theorem 6.5.3.

In Chapter 7 quantitative estimates for spherical matrix coefficients of $SL_2(F)$ are calculated, and for $d \geq 3$ effective uniform pointwise upper bounds for matrix coefficients of $SL_d(F)$ are established.

Appendix A is a short introduction into the theory of locally compact groups. In particular, the Weil Formula and its generalized version are stated, which are used in chapters 3, 4 and 5. Appendix B is a brief survey of the theory of local fields. Its first two sections are mostly a summary of chapters 2, 3 and 4 in [Cas86], and the last section is a survey of the results in [Wei67, Chap. I], which includes the classification of all local fields. In Appendix C integral formulas for the Iwasawa and Cartan decompositions of $SL_d(F)$ are proven, where $d \geq 2$ and F is a local field.

Chapter 1

Unitary Representations of Locally Compact Groups

Throughout this thesis a Hilbert space will be assumed to be over $\mathbb{C}$. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all continuous linear operators $T\colon \mathcal{H}_1 \to \mathcal{H}_2$. The *adjoint* of $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the unique $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that for all

$$\langle Tu, v\rangle_2 = \langle u, T^*v\rangle_1 \quad \forall u \in \mathcal{H}_1 \ \forall v \in \mathcal{H}_2,$$

where $\langle \cdot, \cdot \rangle_i$ is the inner product on $\mathcal{H}_i$, $i = 1, 2$. If $\mathcal{H}$ is a Hilbert space, write $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. The unitary group of $\mathcal{H}$ is

$$\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{L}(\mathcal{H})\colon U \textit{ is an isomorphism with } U^{-1} = U^*\}.$$

It is easy to check that $U \in \mathcal{U}(\mathcal{H})$ if and only if it is an invertible linear operator such that $\|Uv\| = \|v\|$ for all $v \in \mathcal{H}$, that is, $\mathcal{U}(\mathcal{H})$ is the set of all linear bijective isometries of the metric space $(\mathcal{H}, \|\cdot\|)$.

On $\mathcal{L}(\mathcal{H})$ three different topologies can be defined. Let $\{T_i\}_{i\in I}$ be a net in $\mathcal{L}(\mathcal{H})$, and let $T \in \mathcal{L}(\mathcal{H})$. Then:

- $T_i \to T$ in the *operator norm topology* if $\|T_i - T\| \to 0$;
- $T_i \to T$ in the *strong operator topology* if $\|T_iu - Tu\| \to 0$ for all $u \in \mathcal{H}$;
- $T_i \to T$ in the *weak operator topology* if $\langle T_iu, v\rangle \to \langle Tu, v\rangle$ for all $u, v \in \mathcal{H}$.

If $\{T_i\}_{i\in I}$ converges to T in the weak operator topology, write $T_i \xrightarrow{w} T$. Notice that if $\mathcal{H}$ is finite-dimensional, then all three topologies coincide. Moreover, all three topologies are Hausdorff, and, by the Banach-Alaoglu Theorem, $\mathcal{L}(\mathcal{H})$ is locally compact with respect to the weak operator topology.

Lemma 1.0.1. *The strong and the weak operator topologies coincide on $\mathcal{U}(\mathcal{H})$.*

Proof. Let $\{U_i\}_{i\in I}$ be a net in $\mathcal{U}(\mathcal{H})$, and let $U \in \mathcal{U}(\mathcal{H})$. By the Cauchy-Schwarz inequality, if $U_i \to U$ in the strong operator topology, then also $U_i \xrightarrow{w} U$. Suppose now $U_i \xrightarrow{w} U$. Then for all $v \in \mathcal{H}$

$$\begin{aligned}\|U_iv - Uv\|^2 &= \langle U_iv - Uv, U_iv\rangle - \langle U_iv - Uv, Uv\rangle \\ &= \langle U_iv, U_iv\rangle - \langle Uv, U_iv\rangle - \langle U_iv, Uv\rangle + \langle Uv, Uv\rangle \\ &= 2(\|v\|^2 - \mathrm{Re}(\langle U_iv, Uv\rangle)) \to 2(\|v\|^2 - \mathrm{Re}(\|Uv\|^2)) = 0,\end{aligned}$$

which implies that $U_i \to U$ in the strong operator topology. □

Next let G be a topological group, and let f be a function on G. For $g \in G$, define the left and right translates of f through g by

$$L_g f(x) \coloneqq f(g^{-1}x), \quad R_g f(x) \coloneqq f(xg).$$

Notice that for every $g, h \in G$ $L_g L_h = L_{gh}$ and $R_g R_h = R_{gh}$. Moreover, let $f^\vee(x) \coloneqq f(x^{-1})$ denote the *reflection* of f. Let $C_c(G)$ denote the space of all compactly supported continuous functions on G.

1.1 Basic Definitions

Henceforth let G be a topological group.

Definition 1.1.1. A unitary representation of G is a pair $(\pi, \mathcal{H}_\pi)$ where $\mathcal{H}_\pi$ is a Hilbert space and $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a group homomorphism which is continuous with respect to the strong operator topology on $\mathcal{U}(\mathcal{H}_\pi)$.

Given a unitary representation $(\pi, \mathcal{H}_\pi)$, π is said to be *strongly continuous* because the map $g \mapsto \pi(g)u$ is continuous for every $u \in \mathcal{H}_\pi$.

Example 1.1.2. An easy example of a unitary representation is the *trivial representation* $(1_G, \mathbb{C})$, where $1_G(G) = \{1\} \subset S^1 \cong U(\mathbb{C})$.

Example 1.1.3. Suppose G is locally compact. Let $L^2(G)$ be the space of all complex-valued functions on G that are square-integrable with respect to a left Haar measure dx on G. Then, together with the inner product

$$\langle f_1, f_2 \rangle_2 = \int_G f_1(x)\overline{f_2(x)}\, dx, \tag{1.1}$$

$L^2(G)$ is a Hilbert space. There are now two unitary representations of G on $L^2(G)$:

- the *left regular representation* $\lambda_G\colon G \to \mathcal{U}(L^2(G))$, $\lambda_G(g)f(x) \coloneqq f(g^{-1}x)$, and
- the *right regular representation* $\rho_G\colon G \to \mathcal{U}(L^2(G))$, $\rho_G(g)f(x) \coloneqq \Delta_G^{\frac{1}{2}}(g)f(xg)$.

It follows directly from the left-invariance of dx and the definition of the modular function Δ_G that λ and ρ are well-defined. Strong continuity follows from Proposition A.3.11.

Example 1.1.4. Suppose G is locally compact. Let $H < G$ be a closed subgroup, and let μ be a strongly quasi-invariant measure on G/H. Let ρ be the rho-function associated to μ. Let $L^2(G/H)$ be the set of all complex-valued functions on G/H that are square-integrable with respect to μ. Together with the inner product

$$\langle f_1, f_2 \rangle \coloneqq \int_{G/H} f_1(xH)\overline{f_2(xH)}\, d\mu(xH), \tag{1.2}$$

$L^2(G/H)$ is a Hilbert space. Define the map $\lambda_{G/H}\colon G \to \mathcal{U}(L^2(G/H))$ by

$$\lambda_{G/H} f(xH) \coloneqq \sqrt{\frac{\rho(g^{-1}x)}{\rho(x)}} f(g^{-1}xH).$$

Note that, by Theorem A.4.9,

$$\begin{aligned}
\langle \lambda_{G/H}(g)f_1, \lambda_{G/H}(g)f_2 \rangle &= \int_{G/H} \frac{\rho(g^{-1}x)}{\rho(x)} f_1(g^{-1}xH)\overline{f_2(g^{-1}xH)}\, d\mu(xH) \\
&= \int_{G/H} f_1(g^{-1}xH)\overline{f_2(g^{-1}xH)} \frac{d\mu_{g^{-1}}}{d\mu}(xH)\, d\mu(xH) \\
&= \int_{G/H} f_1(xH)\overline{f_2(xH)}\, d\mu(xH) = \langle f_1, f_2 \rangle.
\end{aligned}$$

Thus $\lambda_{G/H}$ is well-defined. Moreover, $\lambda_{G/H}$ is strongly continuous by Proposition A.3.11. This gives the unitary representation $(\lambda_{G/H}, L^2(G/H))$, which is called the *quasi-regular representation* of G associated to H. This construction foreshadows the induction construction in Section .

Next some basic definitions are introduced. Henceforth, let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G.

Definition 1.1.5. Suppose $H < G$ is a subgroup. An *H-invariant subspace* of $(\pi, \mathcal{H}_\pi)$ is a closed subspace $\mathcal{M} \subset \mathcal{H}_\pi$ such that $\pi(h)(\mathcal{M}) \subset \mathcal{M}$ for all $h \in G$.

Observe that $\{0\}$ and $\mathcal{H}_\pi$ are G-invariant subspaces.

Definition 1.1.6. A unitary representation $(\pi, \mathcal{H}_\pi)$ is called *irreducible* if $\{0\}$ and $\mathcal{H}_\pi$ are the only G-invariant subspaces.

Definition 1.1.7. A *subrepresentation* of $(\pi, \mathcal{H}_\pi)$ is a unitary representation $(\pi^{\mathcal{M}}, \mathcal{M})$ where $\mathcal{M}$ is a G-invariant subspace of $\mathcal{H}_\pi$ and $\pi^{\mathcal{M}} \colon G \to U(\mathcal{M})$ is given by $\pi^{\mathcal{M}}(g) \coloneqq \pi(g)|_{\mathcal{M}}$.

Note that a G-invariant subspace $\mathcal{M}$ of $(\pi, \mathcal{H}_\pi)$ induces the subrepresentation $(\pi^{\mathcal{M}}, \mathcal{M})$.

Let $\{\mathcal{H}_i\}_{i \in I}$ be a family of Hilbert spaces. Their *direct sum* $\bigoplus_{i \in I} \mathcal{H}_i$ is the completion of the set

$$\Big\{ \bigoplus_{i \in I} u_i \colon u_i \in \mathcal{H}_i,\ u_i = 0 \text{ for all but finitely many } i \in I \Big\}$$

with respect to the inner product

$$\langle u, v \rangle = \sum_{i \in I} \langle u_i, v_i \rangle,$$

for $u = \bigoplus_{i \in I} u_i, v = \bigoplus_{i \in I} v_i$ and where $\langle \cdot, \cdot \rangle_i$ is the inner product of $\mathcal{H}_i$. The unitary group is $\mathcal{U}\Big(\bigoplus_{i \in I} \mathcal{H}_i \Big) = \prod_{i \in I} \mathcal{U}(\mathcal{H}_i)$. Moreover, given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, their *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to the inner product

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle_1 \langle u_2, v_2 \rangle_2,$$

for $u_i, v_i \in \mathcal{H}_i$ and where $\langle \cdot, \cdot \rangle_i$ denotes the the inner product of $\mathcal{H}_i$, $i = 1, 2$. For more on the tensor product of Hilbert spaces, see [Fol15, Appx. 3].

Definition 1.1.8. The *direct sum* of a family $\{(\pi_i, \mathcal{H}_i)\}_{i \in I}$ of unitary representations of G is $\Big(\bigoplus_{i \in I} \pi_i, \bigoplus_{i \in I} \mathcal{H}_i \Big)$, where $\bigoplus_{i \in I} \pi_i \colon G \to \mathcal{U}\Big(\bigoplus_{i \in I} \mathcal{H}_i \Big)$ is the homomorphism induced by the homomorphisms $\pi_i \colon G \to \mathcal{U}(\mathcal{H}_i)$.

Observe that every $(\pi_i, \mathcal{H}_i)$ is a subrepresentation of $\Big(\bigoplus_{i \in I} \pi_i, \bigoplus_{i \in I} \mathcal{H}_i \Big)$.

Lemma 1.1.9 ([Fol15, Prop. 3.1])**.** *Suppose $(\pi^{\mathcal{M}}, \mathcal{M})$ is a subrepresentation of $(\pi, \mathcal{H}_\pi)$. Then, the orthogonal complement $\mathcal{M}^\perp$ is a closed G-invariant subspace of $\mathcal{H}_\pi$, and $(\pi, \mathcal{H}_\pi)$ is the direct sum of $(\pi^{\mathcal{M}}, \mathcal{M})$ and $(\pi^{\mathcal{M}^\perp}, \mathcal{M}^\perp)$.*

Proof. For all $g \in G$ and $u \in \mathcal{M}, v \in \mathcal{M}^\perp$

$$\langle \pi(g)v, u \rangle = \langle v, \pi(g^{-1})u \rangle = 0,$$

which implies that $\pi(g)(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$. Thus the direct sum $\mathcal{H}_\pi = \mathcal{M} \oplus \mathcal{M}^\perp$ is preserved by $\pi(g)$ for every $g \in G$. Hence the claim follows. □

Definition 1.1.10. Let G_1 ang G_2 be locally compact groups, and let $(\pi_i, \mathcal{H}_i)$ be a unitary representation of G_i for $i = 1, 2$. Their *outer product* is the unitary representation $(\pi_1 \times \pi_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\pi_1 \times \pi_2 \colon G_1 \times G_2 \to \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is given by

$$(\pi_1 \times \pi_2)(g_1, g_2)(u_1 \otimes u_2) \coloneqq \pi_1(g_1)u_1 \otimes \pi_2(g_2)u_2,$$

for $g_i \in G_i$ and $u_i \in \mathcal{H}_i$, $i = 1, 2$.

Definition 1.1.11. The *(inner) tensor product* of two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of G is $(\pi_1 \otimes \pi_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\pi_1 \otimes \pi_2 \colon G \to \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is given by

$$(\pi_1 \otimes \pi_2)(g)(u_1 \otimes u_2) \coloneqq \pi_1(g)u_1 \otimes \pi_2(g)u_2,$$

for $g \in G$ and $u_i \in \mathcal{H}_i$, $i = 1, 2$.

Definition 1.1.12. A subset of a Hilbert space $\mathcal{H}$ is *total* if its linear span is dense in $\mathcal{H}$.

Given a family $\{\mathcal{H}_i\}_{i \in I}$ of Hilbert spaces,

$$\Big\{ \bigoplus_{i \in I} u_i \coloneqq u_i \in \mathcal{H}_i,\ u_i = 0 \text{ for all but finitely many } i \in I \}$$

is total in $\bigoplus_{i \in I} \mathcal{H}_i$. Moreover, for each $i, j \in I$

$$\{u_i \otimes u_j \colon u_i \in \mathcal{H}_i,\ u_j \in \mathcal{H}_j\}$$

is total in $\mathcal{H}_i \otimes \mathcal{H}_j$.

For a subset S of $\mathcal{H}$, let $\langle S \rangle \subset \mathcal{H}$ denote the linear span of S.

Definition 1.1.13. A *cyclic vector* of $(\pi, \mathcal{H}_\pi)$ is a vector $u \in \mathcal{H}_\pi$ such that $\{\pi(g)u \colon g \in G\}$ is a total subset of $\mathcal{H}_\pi$, that is, $\langle \pi(g)u \colon g \in G \rangle$ is dense in $\mathcal{H}_\pi$. If $(\pi, \mathcal{H}_\pi)$ has a cyclic vector, then it is called *cyclic*.

fix $u \in \mathcal{H}_\pi$, and let $\mathcal{M}$ be the closure of $\langle \pi(g)u \colon g \in G \rangle$. Then $\mathcal{M}$ is G-invariant, and $(\pi^{\mathcal{M}}, \mathcal{M})$ is a cyclic subrepresentation of π. Hence every non-zero vector of $\mathcal{H}_\pi$ generates a cyclic subrepresentation of π. In particular, $(\pi, \mathcal{H}_\pi)$ is irreducible if and only if every non-zero vector in $\mathcal{H}_\pi$ is cyclic. In fact one has

Proposition 1.1.14 ([Fol15, Prop. 3.3])**.** *Every $(\pi, \mathcal{H}_\pi)$ is the direct sum of cyclic subrepresentations.*

Proof. Using Zorn's Lemma it can be shown that there exists a maximal family $\{(\pi^{\mathcal{M}_i}, \mathcal{M}_i)\}_{i \in I}$ of mutually orthogonal cyclic subrepresentations of π. Arguing by contradiction, suppose there exists a non-zero $u \in \left(\bigoplus_{i \in I} \mathcal{M}_i \right)^\perp$. As in the remarks preceding the proposition, let $\mathcal{N}$ be the closure of $\langle \pi(g)u \colon g \in G \rangle$. Then the cyclic representation $(\pi^{\mathcal{N}}, \mathcal{N})$ is contained in the orthogonal complement of $\bigoplus_{i \in I} \mathcal{M}_i$, which contradicts the maximality of $\{(\pi^{\mathcal{M}_i}, \mathcal{M}_i)\}_{i \in I}$. □

1.2 Equivalent unitary representations

Definition 1.2.1. An *interwining operator* between two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of G is a linear operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that, for all $g \in G$, $T\pi_1(g) = \pi_2(g)T$. Let $\mathcal{C}(\pi_1, \pi_2) \subset \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be the subset of all interwining operators between $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$.

Definition 1.2.2. Two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of G are *unitarily equivalent*, written $\pi_1 \simeq \pi_2$, if $\mathcal{C}(\pi_1, \pi_2)$ contains an invertible isometry.

Returning to Example 1.1.3 one has

Example 1.2.3. Let G and dx be as in Example 1.1.3, and let $(\lambda_G, L^2(G))$and $(\rho_G, L^2(G))$ be the left and right regular representations of G. Define on $L^2(G)$ the linear operator $Tf(x) \coloneqq (\Delta_G^{\frac{1}{2}} f)^\vee$. It is clear that T is invertible. By the properties of the modular function, for all $f \in L^2(G)$

$$\|Tf\|_2 = \left(\int_G \Delta_G(x^{-1}) |f(x^{-1})|^2 \, dx \right)^{\frac{1}{2}} = \left(\int_G |f(x)|^2 \, dx \right)^{\frac{1}{2}} = \|f\|_2.$$

Hence, $T \in \mathcal{L}(L^2(G))$ is an isometry. Moreover, for all $f \in L^2(G)$ and all $g, x \in G$

$$T(\lambda_G(g)f)(x) = \Delta_G^{\frac{1}{2}}(x^{-1}) f(g^{-1}x^{-1}) = \Delta_G^{\frac{1}{2}}(g)(\Delta_G^{\frac{1}{2}} f)^\vee(xg) = \rho_G(g)(Tf)(x).$$

It follows that $T \in \mathcal{C}(\lambda_G, \rho_G)$ and that $\lambda_G \simeq \rho_G$.

If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G, write $\mathcal{C}(\pi) = \mathcal{C}(\pi, \pi)$. It is called the *centralizer* or *commutant* of π.

Proposition 1.2.4. *For any two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$, $\mathcal{C}(\pi_1, \pi_2)$ is a closed subspace of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Moreover, for all $T \in \mathcal{C}(\pi_1, \pi_2)$ one has $T^* \in \mathcal{C}(\pi_2, \pi_1)$, and if $T \in \mathcal{C}(\pi_1, \pi_2)$ is invertible, then $T^{-1} \in \mathcal{C}(\pi_2, \pi_1)$.*

Proof. It is clear that $\mathcal{C}(\pi_1, \pi_2)$ is a linear subspace of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. To show that it is closed, suppose that $\{T_i\}_{i \in I}$ is a net in $\mathcal{C}(\pi_1, \pi_2)$ such that $T_i \to T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ with respect to the operator norm topology. Then for all $g \in G$

$$\|T\pi_1(g) - \pi_2(g)T\| = \lim_{i \in I} \|T_i \pi_1(g) - \pi_2(g) T_i\| = 0,$$

which implies that $T\pi_1(g) = \pi_2(g)T$ and thus that $T \in \mathcal{C}(\pi_1, \pi_2)$. Moreover, for all $T \in \mathcal{C}(\pi_1, \pi_2)$ and all $g \in G$

$$T^* \pi_2(g) = (\pi_2(g)^* T)^* = (T\pi_1(g^{-1}))^* = \pi_1(g) T^*,$$

and hence $T^* \in \mathcal{C}(\pi_2, \pi_1)$. Similarly, it can be shown that if $T \in \mathcal{C}(\pi_1, \pi_2)$ is invertible, then $T^{-1} \in \mathcal{C}(\pi_2, \pi_1)$. □

Corollary 1.2.5. *If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G, then $\mathcal{C}(\pi)$ is a unital $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\pi)$.*

Proof. By the laws of composition $\mathcal{C}(\pi)$ is closed under composition. Because it contains the identity operator, the corollary follows from Proposition 1.2.4. □

The next goal is to show that for two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ to be unitarily equivalent, it is enough that $\mathcal{C}(\pi_1, \pi_2)$ contains an invertible operator.

Recall from basic functional analysis that a linear operator $T \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is *positive* if $\langle Tu, u \rangle \geq 0$ for all $u \in \mathcal{H}$. A *partial isometry* is a map $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\|Uv\|_2 = \|v\|_1$ for all $v \in (\ker U)^\perp$. It is a well-known fact from functional analysis that for every $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ there exist a positive operator $|T| \in \mathcal{L}(\mathcal{H}_1)$ and a partial isometry $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $|T| = \sqrt{T^*T}$, $\ker T = \ker U$ and $T = U|T|$ (see [Rud91, p. 330−333]). This decomposition is called the *polar decomposition* of T. Moreover, by the Stone-Weierstrass Theorem and by [Fol15, Thm. 1.51], $|T| = \sqrt{T^*T}$ is a limit of polynomials in T^*T with respect to the strong operator topology.

Proposition 1.2.6 ([BdlHV08, Prop. A.1.4])**.** *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two unitary representations of G with $T \in \mathcal{C}(\pi_1, \pi_2)$. Let $\mathcal{M}_1 \coloneqq (\ker T)^\perp$ and $\mathcal{M}_2 \coloneqq \overline{\operatorname{im} T}$. Then $\mathcal{M}_i \subset \mathcal{H}_i$ is a closed G-invariant subspace, $i = 1, 2$, and $(\pi_1^{\mathcal{M}_1}, \mathcal{M}_1) \simeq (\pi_2^{\mathcal{M}_2}, \mathcal{M}_2)$.*

Proof. It is immediate that $\mathcal{M}_1$ and $\mathcal{M}_2$ are G-invariant. Let $T = U|T|$ be the polar decomposition of T, and notice that $\mathrm{im}|T| = \mathcal{M}_1$. Because, as remarked above, $|T|$ is a limit, in the strong operator topology, of polynomials in T^*T, it follows from Proposition 1.2.4 that $|T| \in C(\pi_1)$. Moreover, for all $g \in G$ and all $v \in \mathcal{H}_1$,

$$\pi_2(g)U|T|v = \pi_2(g)Tv = T\pi_1(g)v = U|T|\pi_1(g)v = U\pi_1(g)(|T|v).$$

Since $U = 0$ on $\ker T$ and because $\mathrm{im}|T| = \mathcal{M}_1$ and $\ker T$ are G-invariant, $U \in \mathcal{C}(\pi_1, \pi_2)$. Because U is an isometry from $\mathcal{M}_1$ to $\mathcal{M}_2$, $\pi^{\mathcal{M}_1} \simeq \pi^{\mathcal{M}_2}$. □

Corollary 1.2.7. *Two unitary representation $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of G are equivalent if and only if there exists an invertible $T \in \mathcal{C}(\pi_1, \pi_2)$.*

Proof. If $T \in \mathcal{C}(\pi_1, \pi_2)$ is invertible, then $\ker T = \{0\}$ and $\mathrm{im}\, T = \mathcal{H}_2$; whence $\mathcal{M}_1 = \mathcal{H}_1$ and $\mathcal{M}_2 = \mathcal{H}_2$. The corollary follows from Proposition 1.2.6. □

It is clear that unitary equivalence defines an equivalence relation on the set of all unitary representations of G. Let $\mathcal{R}(G)$ be the set of all equivalence classes of unitary representations of G, and denote the equivalence classes of $\mathcal{R}(G)$ by $[\pi]$. It is easy to check that if $[\pi_1] = [\pi_2]$, then π_1 is irreducible if and only if π_2 is irreducible.

Definition 1.2.8. The *unitary dual* of a topological group G is the set of equivalence classes of irreducible unitary representations of G. Denote it by $\widehat{G}$.

Definition 1.2.9. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G, and let $u, v \in \mathcal{H}_\pi$. The *matrix coefficient* of π with respect to u and v is the function $\phi_{u,v}\colon G \to \mathbb{C}$, $\phi_{u,v}(g) \coloneqq \langle \pi(g)u, v\rangle$. If $u = v$, then $\phi_{u,u}$ is called a *diagonal matrix coefficient.*

Concerning the matrix coefficients with respect to cyclic vectors, one has

Proposition 1.2.10 ([Fol15, Prop. 3.23])**.** *Suppose $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are both cyclic representations of G, and let $u \in \mathcal{H}_\pi$ and $v \in \mathcal{H}_\sigma$ be cyclic vectors. Then if $\phi_{u,u} = \phi_{v,v}$, that is, $\langle \pi(g)u, u\rangle = \langle \sigma(g)v, v\rangle$ for all $g \in G$, then $\pi \simeq \sigma$.*

Proof. Let T be the map from $\{\pi(g)u\colon g \in G\}$ to $\{\sigma(g)v\colon g \in G\}$ such that $T\pi(g)u = \sigma(g)v$ for all $g \in G$. By linearity and continuity, T extends to a bounded linear map from $\mathcal{H}_\pi$ to $\mathcal{H}_\sigma$. In particular, T is surjective and $T \in \mathcal{C}(\pi, \sigma)$. Moreover, for all $g_1, g_2 \in G$

$$\langle \pi(g_1)u, \pi(g_2)u\rangle = \phi_{u,u}(g_2^{-1}g_1) = \phi_{v,v}(g_2^{-1}g_1) = \langle \sigma(g_1)v, \sigma(g_2)v\rangle.$$

By the definition of T, $\phi_{u,u} = \phi_{Tu,Tu}$, and thus T is a surjective isometry. □

Definition 1.2.11. Let $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\pi)$ be two unitary representations of G. If π is unitarily equivalent to a subrepresentation of σ, π is said to be *(strongly) contained* in σ, written $\pi \subset \sigma$.

By Proposition 1.2.6, $\pi \subset \sigma$ if and only if there exists a partial isometry $T \in \mathcal{C}(\pi, \sigma)$.

It is clear that containment is preserved under unitary equivalence and that therefore it defines a preorder on $\mathcal{R}(G)$, that is, a binary relation on $\mathcal{R}(G)$ which is reflexive and transitive, but not necessarily antisymmetric. To show that it is a partial order, one needs to check that, for $[\pi], [\sigma] \in \mathcal{R}(G)$, if $\pi \subset \sigma$ and $\sigma \subset \pi$, then $\pi \simeq \sigma$. If π and σ are finite-dimensional, then this follows from an easy dimension counting argument. The proof for the general case is more complicated.

Proposition 1.2.12 ([BdlH20, Prop. 1.A.8])**.** *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be unitary representations of G, and suppose that $\pi_1 \subset \pi_2$ and $\pi_2 \subset \pi_1$. Then $\pi_1 \simeq \pi_2$.*

Proof. By assumption, there exist partial isometries $S \in \mathcal{C}(\pi_1, \pi_2)$ and $T \in \mathcal{C}(\pi_2, \pi_1)$. Then $\pi_2 \simeq \pi_1^{T(\mathcal{H}_2)}$, and $U \coloneqq TS \in \mathcal{C}(\pi_1)$ is a partial isometry. It is enough to show that $\pi_1 \simeq \pi_1^{T(\mathcal{H}_2)}$. For this set $\mathcal{M}_0 \coloneqq \mathcal{H}_1$ and $\mathcal{M}_1 \coloneqq T(\mathcal{H}_2)_1$, and define for $n \geq 1$

$$\mathcal{M}_{2n} \coloneqq U^n(\mathcal{H}_1), \quad \mathcal{M}_{2n+1} \coloneqq U^n(T(\mathcal{H}_2)).$$

Observe that $\mathcal{M}_{n+1} \subset \mathcal{M}_n$, and let $\mathcal{N}_n \coloneqq \mathcal{M}_{n+1}^{\perp} \cap \mathcal{M}_n$ for $n \geq 0$. Notice that each $\mathcal{M}_n$ and $\mathcal{N}_n$ are closed G-invariant subspaces of $\mathcal{H}_1$. Let $\mathcal{J} \coloneqq \bigcap_{n\geq 0} \mathcal{M}_n$, which is a closed G-invariant subspace. Then

$$\mathcal{H}_\pi = \mathcal{J} \oplus \Big(\bigoplus_{n\geq 0} \mathcal{N}_n\Big), \quad T(\mathcal{H}_\sigma) = \mathcal{J} \oplus \Big(\bigoplus_{n\geq 1} \mathcal{N}_n\Big).$$

Thus

$$\pi \simeq \pi^{\mathcal{J}} \oplus \Big(\bigoplus_{n\geq 0} \pi^{\mathcal{N}_n}\Big), \quad \pi^{T(\mathcal{H}_2)} \simeq \pi^{\mathcal{J}} \oplus \Big(\bigoplus_{n\geq 1} \pi^{\mathcal{N}_n}\Big).$$

By construction, for every $n \geq 1$, U induces a surjective isometry from $\mathcal{N}_{2n}$ to $\mathcal{N}_{2n+2}$, and hence $\pi^{\mathcal{N}_{2n}} \simeq \pi^{\mathcal{N}_{2n+2}}$. This implies that

$$\pi \simeq \pi^{\mathcal{J}} \oplus \Big(\bigoplus_{n\geq 0} \pi^{\mathcal{N}_n}\Big) \simeq \pi^{\mathcal{J}} \oplus \Big(\bigoplus_{n\geq 1} \pi^{\mathcal{N}_n}\Big) \simeq \pi^{T(\mathcal{H}_2)}.$$

As remarked above, this completes the proof. □

Concerning containment and operations on unitary representations, like taking the direct sum, one has

Proposition 1.2.13 ([BdlHV08])**.** *Let $\{(\pi_i, \mathcal{H}_{\pi_i})\}_{i\in I}$ and $\{(\sigma_i, \mathcal{H}_{\sigma_i})\}_{i\in I}$ be families of unitary representations of G such that $\pi_i \subset \sigma_i$. Then:*

1) $\pi_i \otimes \pi_j \subset \sigma_i \otimes \sigma_j$;

2) $\bigoplus_{i\in I} \pi_i \subset \bigoplus_{i\in I} \sigma_i$.

Proof. Let $\mathcal{M}_i \subset \mathcal{H}_{\sigma_i}$ be the closed G-invariant subspace such that $(\pi_i, \mathcal{H}_{\pi_i}) \simeq (\sigma^{\mathcal{M}_i}, \mathcal{M}_i)$.

1) $\mathcal{M}_i \otimes \mathcal{M}_j$ is a G-invariant subspace of $\mathcal{H}_{\sigma_i} \otimes \mathcal{H}_{\sigma_j}$, and for all $g \in G$ and all $u_i \in \mathcal{M}_i$, $u_j \in \mathcal{M}_j$

$$\begin{aligned}(\sigma_i^{\mathcal{M}_i} \otimes \sigma_j^{\mathcal{M}_j})(g)u_i \otimes u_j = \sigma_i(g)u_i \otimes \sigma_j(g)u_j = (\sigma_i \otimes \sigma_j)(g)u_i \otimes u_j \\ = (\sigma_i \otimes \sigma_j)^{\mathcal{M}_i \otimes \mathcal{M}_j}(g)u_i \otimes u_j.\end{aligned}$$

Thus, by the continuity of $\sigma_i^{\mathcal{M}_i} \otimes \sigma_j^{\mathcal{M}_j}$ and of $(\sigma_i \otimes \sigma_j)^{\mathcal{M}_i}$, $\sigma_i^{\mathcal{M}_i} \otimes \sigma_j^{\mathcal{M}_j} \simeq (\sigma_i \otimes \sigma_j)^{\mathcal{M}_i \otimes \mathcal{M}_j}$, where the identity map is the interwining operator.

2) $\bigoplus_{i\in I} \mathcal{M}_i$ is a G-invariant subspace of $\bigoplus_{i\in I} \mathcal{H}_{\sigma_i}$. Moreover for all $g \in G$ and all $\bigoplus_{i\in I} u_i \in \bigoplus_{i\in I} \mathcal{M}_i$

$$\Big(\bigoplus_{i\in I} \sigma_i^{\mathcal{M}_i}\Big)(g)\Big(\bigoplus_{i\in I} u_i\Big) = \bigoplus_{i\in I} \sigma_i(g)u_i = \Big(\bigoplus_{i\in I} \sigma_i\Big)^{\oplus \mathcal{M}_i}(g)\Big(\bigoplus_{i\in I} u_i\Big).$$

Hence, by the continuity of $\bigoplus_{i\in I} \sigma_i^{\mathcal{M}_i}$ and $\Big(\bigoplus_{i\in I} \sigma_i\Big)^{\oplus \mathcal{M}_i}$, $\bigoplus_{i\in I} \sigma_i^{\mathcal{M}_i} \simeq \Big(\bigoplus_{i\in I} \sigma_i\Big)^{\oplus \mathcal{M}_i}$.

□

An orthogonal projection on a Hilbert space $\mathcal{H}$ is *trivial* if its image is either $\{0\}$ or $\mathcal{H}$.
Whether a unitary representation of is irreducible, is determined by its commutant as follows:

Proposition 1.2.14 ([Fol15, Prop. 3.4])**.** *A unitary representation $(\pi, \mathcal{H}_\pi)$ of G is irreducible if and only if $\mathcal{C}(\pi)$ contains non-trivial orthogonal projections.*

Proof. Suppose $\mathcal{C}(\pi)$ contains a non-trivial orthogonal projection $P \in \mathcal{L}(\mathcal{H}_\pi)$. Then $\mathcal{M} \coloneqq \operatorname{im} P$ is a closed G-invariant non-trivial proper subspace of $\mathcal{H}_\pi$, and thus $(\pi^{\mathcal{M}}, \mathcal{M})$ is a non-trivial proper subrepresentation of $(\pi, \mathcal{H}_\pi)$. Conversely, if $(\pi, \mathcal{H}_\pi)$ has a non-trivial proper subrepresentation $(\pi^{\mathcal{M}}, \mathcal{M})$, then $P^{\mathcal{M}} \in \mathcal{C}(\pi)$ is a non-trivial orthogonal projection. □

Using the spectral theory of self-adjoint bounded linear operators, one can prove

Theorem 1.2.15 (Schur's Lemma, [Fol15, Thm. 3.5])**.** *1) A unitary representation $(\pi, \mathcal{H}_\pi)$ of G is irreducible if and only if every $T \in \mathcal{C}(\pi)$ is a scalar multiple of the identity operator, that is, there is $\lambda \in \mathbb{C}$ such that $T = \lambda I$.*

2) Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two irreducible unitary representations of G. Then if $\pi_1 \simeq \pi_2$, $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional, and otherwise $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.

Proof. 1) Suppose $(\pi, \mathcal{H}_\pi)$ is irreducible. Argue by contradiction, and suppose there exists $T \in \mathcal{C}(\pi)$ such that $T \neq \lambda I$ for all $\lambda \in \mathbb{C}$. Then $T = T_1 + T_2$ where $T_1 \coloneqq \frac{1}{2}(T + T^*)$ and $T_2 \coloneqq \frac{1}{2}(T - T^*)$ are self-adjoint interwining operators. Because at least one of them is not a scalar multiple of the identity operator, one can assume $T_1 \neq \lambda I$ for all $\lambda \in \mathbb{C}$. But, by [Fol15, Thm. 1.51], every operator in $\mathcal{L}(\mathcal{H}_\pi)$ that commutes with T_1 also commutes with the projections $\{\chi_B(T_1) \colon B \subset \mathbb{R} \text{ Borel}\}$. Then for every Borel subset $B \subset \mathbb{R}$ $\chi_B(T_1) \in \mathcal{C}(\pi)$. This implies that $\mathcal{C}(\pi)$ contains a non-trivial orthogonal projection, which yields a contradiction. Conversely, if $\mathcal{C}(\pi)$ is one-dimensional, it contains no non-trivial orthogonal projections and hence, by Proposition 1.2.14, $(\pi, \mathcal{H}_\pi)$ is irreducible.

2) Take $T \in \mathcal{C}(\pi_1, \pi_2)$. Then, by Proposition 1.2.4, $T^* \in \mathcal{C}(\pi_2, \pi_1)$ and hence $T^*T \in \mathcal{C}(\pi_1)$. But then by part 1), there is $\lambda \in \mathbb{C}$ such that $T^*T = \lambda I$. It follows that either $T = 0$ or $\sqrt{\lambda}T$ is a unitary operator. Hence, if $\pi_1 \not\simeq \pi_2$, then $\mathcal{C}(\pi_1, \pi_2) = \{0\}$. Otherwise, if $T_1, T_2 \in \mathcal{C}(\pi_1, \pi_2)$ and if T_1 is invertible, then $T_1^{-1}T_2 \in \mathcal{C}(\pi_1)$ and thus $T_1^{-1}T_2 = \lambda I$ for some $\lambda \in \mathbb{C}$, which implies $T_2 = \lambda T_1$.

□

Corollary 1.2.16 ([Fol15, Cor. 3.6])**.** *If G is abelian, then every irreducible unitary representation of G is one-dimensional.*

Proof. Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G. Because G is abelian, for all $g \in G$ $\pi(g) \in \mathcal{C}(\pi)$, and thus, by Theorem 1.2.15, $\pi(g)$ is a scalar multiple of the identity operator. This implies that every one-dimensional subspace of $\mathcal{H}_\pi$ is G-invariant, and therefore $\dim \mathcal{H}_\pi = 1$. □

The section is concluded with

Definition 1.2.17. Let $H < G$ be a closed subgroup, and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. The *restriction of π from G to H* is the unitary representation $(\operatorname{Res}^G_H(\pi), \mathcal{H}_\pi)$ of H, where $\operatorname{Res}^G_H(\pi)(h) \coloneqq \pi(h)$ for all $h \in H$.

The proof of the following proposition is trivial and is thus omitted.

Proposition 1.2.18. *Let $H < G$ be a closed subgroup, and let $\{(\pi_i, \mathcal{H}_i)\}_{i \in I}$ be a family of unitary representations of G.*

1) If $\pi_i \simeq \pi_j$, then $\operatorname{Res}^G_H(\pi_i) \simeq \operatorname{Res}^G_H(\pi_j)$.

2) $\operatorname{Res}^G_H\left(\bigoplus_{i \in I} \pi_i\right) \simeq \bigoplus_{i \in I} \operatorname{Res}^G_H(\pi_i)$.

1.3 Representation theory of $*$-algebras

Recall that an algebra $\mathcal{A}$ over a field F is a vector space over F equipped with a multiplication $\cdot\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ such that

- for all $a_1,a_2,a_3\in\mathcal{A}$ $(a_1+a_2)\cdot a_3 = a_1\cdot a_3 + a_2\cdot a_3$ and $a_1\cdot(a_2+a_3) = a_1\cdot a_2 + a_1\cdot a_3$,
- for all $x_1,x_2\in F$ and all $a_1,a_2\in\mathcal{A}$ $(x_1a_1)\cdot(x_2a_2) = (x_1x_2)(a_1\cdot a_2)$.

$\mathcal{A}$ is called a *Banach algebra* if $F=\mathbb{C}$ and if it is equipped with a norm $\|\cdot\|$ with respect to which $\mathcal{A}$ is a Banach space and such that $\|x\cdot y\|\leq\|x\|\|y\|$ for all $x,y\in\mathcal{A}$. Notice that in a Banach algebra addition and multiplication are continuous with respect to the topology induced by the norm. A Banach algebra is said to be *unital* if it contains a multiplicative identity element. Henceforth, to simplify notation, for $x,y\in\mathcal{A}$, write $xy = x\cdot y$.

Definition 1.3.1. Let $\mathcal{A}$ be a Banach algebra. An *involution* on $\mathcal{A}$ is a continuous map $x\mapsto x^*$ on $\mathcal{A}$ such that for every $x,y\in\mathcal{A}$ and every $\lambda\in\mathbb{C}$

1) $(x+y)^* = x^*+y^*$ and $(xy)^* = y^*x^*$,
2) $(\lambda x)^* = \overline{\lambda}x^*$,
3) $x^{**} = x$.

A $*$-algebra is a Banach algebra $\mathcal{A}$ with an involution. If in addition $\|x^*x\| = \|x\|^2$ for all $x\in\mathcal{A}$, $\mathcal{A}$ is called a *C^*-algebra*.

A *$*$-subalgebra* of a $*$-algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ which is closed under the involution of $\mathcal{A}$.

Definition 1.3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras. A *(Banach algebra) homomorphism* is a bounded linear map $\phi\colon\mathcal{A}\to\mathcal{B}$ such that $\phi(xy)=\phi(x)\phi(y)$ for all $x,y\in\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras and $\phi(x^*)=\phi(x)^*$ for all $x\in\mathcal{A}$, then ϕ is called a *$*$-homomorphism*.

C^*-algebras play an important role in the spectral theory of bounded linear operators on Hilbert spaces; see [Fol15, Chap. 1]. The following examples of $*$-algebras will be used throughout this thesis.

Example 1.3.3. Let $\mathcal{H}$ be a Hilbert space. It is a standard fact of functional analysis that $\mathcal{L}(\mathcal{H})$ is a Banach space with respect to the operator norm. Together with the composition as the multiplication and the identity operator $I\in\mathcal{L}(\mathcal{H})$ as the multiplicative identity element, it is a unital Banach algebra. The map $T\mapsto T^*$ is an involution on $\mathcal{L}(\mathcal{H})$ such that $\|T^*T\| = \|T\|$. Thus $\mathcal{L}(\mathcal{H})$ is a C^*-algebra. In particular, every closed $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ is a C^*-algebra; e.g. $\mathcal{U}(\mathcal{H})$.

Example 1.3.4. Recall that $L^1(G)$ is a Banach space with respect to $\|\cdot\|_1$. Equip it with the convolution

$$f_1 * f_2(g) = \int_G f_1(x)f_2(x^{-1}g)\,dx,$$

and note that

$$\|f_1*f_2\|_1 = \int_G\left|\int_G f_1(y)f_2(y^{-1}x)\,dy\right|dx \leq \int_G\int_G|f_1(y)f_2(x)|\,dy\,dx = \|f_1\|_1\|f_2\|_1.$$

Thus $L^1(G)$, together with the convolution as the multiplication, is a Banach algebra. Next define on $L^1(G)$ the map $f\mapsto f^*$, where

$$f^*(g) \coloneqq \Delta_G(g^{-1})\overline{f^\vee(g)},$$

for $g \in G$. It is immediate that the map is antilinear and that $(f^*)^* = f$. Moreover, for all $f \in L^1(G)$

$$\|f^*\|_1 = \int_G \Delta_G(x^{-1})|\overline{f(x^{-1})}|\,dx = \int_G |f(x)|\,dx = \|f\|_1,$$

and for every $g \in G$ and every $f_1, f_2 \in L^1(G)$

$$\begin{aligned}(f_1 * f_2)^*(g) &= \Delta_G(g^{-1})\overline{f_1 * f_2(g^{-1})} = \int_G \Delta_G(g^{-1})\overline{f_1(x)f_2(x^{-1}g^{-1})}\,dx \\ &= \int_G \Delta_G(g^{-1})\overline{f_1^\vee(x^{-1})f_2^\vee(gx)}\,dx \\ &= \int_G \Delta_G(g^{-1}x)\Delta_G(x^{-1})\overline{f_1^\vee(x^{-1}g)f_2^\vee(x)}\,dx = f_2^* * f_1^*(g).\end{aligned}$$

Hence $f \mapsto f^*$ is an involution on the Banach algebra $L^1(G)$. The resulting $*$-algebra is called the *group algebra* of G.

Throughout this thesis let $L^1(G)$ denote the group algebra of G.
Henceforth let $\mathcal{A}$ be a $*$-algebra.

Definition 1.3.5. A $*$-representation of $\mathcal{A}$ is a pair $(\pi, \mathcal{H}_\pi)$ where $\mathcal{H}_\pi$ is a Hilbert space and $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H}_\pi)$ is a $*$-homomorphism. $(\pi, \mathcal{H}_\pi)$ is called *non-degenerate* if there is no non-zero $u \in \mathcal{H}_\pi$ such that $\pi(x)u = 0$ for all $x \in \mathcal{A}$

Many of the definitions in the representation theory of topological groups can be reformulated for $*$-representations of $*$-algebras. Let $\mathcal{A}$ be a $*$-algebra, and let $(\pi, \mathcal{H}_\pi)$ be a $*$-representations of $\mathcal{A}$.

Definition 1.3.6. Suppose $\mathcal{B} \subset \mathcal{A}$ is a $*$-subalgebra. A *$\mathcal{B}$-invariant subspace* is a closed subspace $\mathcal{M} \subset \mathcal{H}_\pi$ such that $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in \mathcal{B}$.

A *subrepresentation* $(\pi^{\mathcal{M}}, \mathcal{M})$ of $(\pi, \mathcal{H}_\pi)$ consists of an $\mathcal{A}$-invariant subspace $\mathcal{M} \subset \mathcal{H}_\pi$ and of the $*$-homomorphism $\pi^{\mathcal{M}}\colon \mathcal{A} \to \mathcal{L}(\mathcal{M})$, $\pi^{\mathcal{M}}(x) \coloneqq \pi(x)|_{\mathcal{M}}$.

Any $\mathcal{A}$-invariant subspace of $\mathcal{M}$ of $(\pi, \mathcal{H}_\pi)$ gives a subrepresentation $(\pi^{\mathcal{M}}, \mathcal{M})$, with $\pi^{\mathcal{M}}$ as in Definition 1.3.6.

Definition 1.3.7. Let $\{(\pi_i, \mathcal{H}_i)\}_{i\in I}$ be a family of $*$-representations of $\mathcal{A}$. The *direct sum* is $\Big(\bigoplus_{i\in I}\pi_i, \bigoplus_{i\in I}\mathcal{H}_i\Big)$ where $\bigoplus_{i\in I}\pi_i\colon \mathcal{A} \to \mathcal{U}\Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)$ is induced by $\pi_i\colon G \to \mathcal{H}_i$.

Note that every π_i is a subrepresentation of $\bigoplus_{i\in I}\pi_i$. Conversely one has

Proposition 1.3.8 ([Dix77, 2.2.4])**.** *If $\mathcal{M}$ is an $\mathcal{A}$-invariant subspace of $(\pi, \mathcal{H}_\pi)$, then $(\pi, \mathcal{H}_\pi) = (\pi^{\mathcal{M}} \oplus \pi^{\mathcal{M}^\perp}, \mathcal{M} \oplus \mathcal{M}^\perp)$.*

Proof. It is enough to show that $\mathcal{M}^\perp$ is $\mathcal{A}$-invariant under $\pi(\mathcal{A})$. Indeed, for all $x \in \mathcal{A}$ and all $u \in \mathcal{M}$, $v \in \mathcal{M}^\perp$

$$0 = \langle \pi(x)u, v\rangle = \langle u, \pi(x^*)v\rangle.$$

Because the involution $*\colon \mathcal{A} \to \mathcal{A}$ is bijective, the proposition follows. □

Definition 1.3.9. A $*$-representation $(\pi, \mathcal{H}_\pi)$ of $\mathcal{A}$ is said to be *irreducible* if its only $\mathcal{A}$-invariant subspaces are $\{0\}$ and $\mathcal{H}_\pi$.

Definition 1.3.10. A $*$-representation $(\pi, \mathcal{H}_\pi)$ is *cyclic*, if there is $u \in \mathcal{H}_\pi$ such that the set $\{\pi(x)u\colon x \in \mathcal{A}\}$ is dense in $\mathcal{H}_\pi$.

It is immediate that in an irreducible $*$-representation every non-zero vector is cyclic. The following is the analogue to Proposition 1.1.14. Its proof is the same as the one of Proposition 1.1.14 and is therefore omitted.

Proposition 1.3.11 ([Dix77, Prop. 2.2.7])**.** *Every non-degenerate $*$-representation of $\mathcal{A}$ is the direct sum of cyclic representations of $\mathcal{A}$.*

Interwining operators and the equivalence of $*$-representations of $\mathcal{A}$ are defined as for unitary representations of topological groups.

Definition 1.3.12. Let $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ be two $*$-representations of $\mathcal{A}$. An interwining operator between π and σ is a continuous linear operator $T\colon \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that $\sigma \circ T = T \circ \pi$. Let $\mathcal{C}(\pi, \sigma)$ be the set of interwining operators between π and σ.

If $\pi = \sigma$, write $\mathcal{C}(\pi) = \mathcal{C}(\pi, \sigma)$.

Proposition 1.3.13. *The set $\mathcal{C}(\pi)$ constitutes a unital $*$-algebra if it is equipped with the composition as the multiplication, the operator norm as the Banach norm and the map $T \mapsto T^*$ as the involution, where T^* denotes the adjoint of T.*

Proof. The proof is the same as the one of Proposition 1.2.4. □

Definition 1.3.14. Two $*$-representations $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are said be *equivalent*, written $\pi \simeq \sigma$, if $\mathcal{C}(\pi, \sigma)$ contains an invertible operator.

It is clear that $\simeq$ defines an equivalence relation on the set of $*$-representations of $\mathcal{A}$. Direct sums of $*$-representations and thus irreducibility are preserved under $\simeq$. finally one has a version of Schur's Lemma for $*$-representations of $*$-algebras. Its proof is the same as the one of Theorem 1.2.15 and is therefore omitted.

Proposition 1.3.15. *A $*$-representation $(\pi, \mathcal{H}_\pi)$ of $\mathcal{A}$ is irreducible if and only if $\mathcal{C}(\pi)$ contains no non-trivial orthogonal projection.*

Theorem 1.3.16 (Schur's Lemma, [Dix77, Prop. 2.3.1])**.** *1) A $*$-representation $(\pi, \mathcal{H}_\pi)$ of $\mathcal{A}$ is irreducible if and only if $\mathcal{C}(\pi)$ is one-dimensional, that is, $\mathcal{C}(\pi) = \{\lambda I \colon \lambda \in \mathbb{C}\}$.*

2) Suppose $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are irreducible $*$-representations of $\mathcal{A}$. Then either $\mathcal{C}(\pi, \sigma)$ is one-dimensional, in which case $\pi \simeq \sigma$, or $\mathcal{C}(\pi, \sigma) = \{0\}$.

An algebra is said to be *abelian* if $xy = yx$ for all $x, y \in \mathcal{A}$.

Corollary 1.3.17. *The irreducible $*$-representations of an abelian $*$-algebra are all one-dimensional.*

Important examples of $*$-representations of $*$-algebras are the following:

Example 1.3.18. For any $*$-algebra $\mathcal{A}$ let $(1_\mathcal{A}, \mathbb{C})$ be the *trivial representation* where $1_\mathcal{A}(0) = 0$ and $1_\mathcal{A}(\mathcal{A} - \{0\}) = \{I\} \subset \mathcal{U}(\mathbb{C})$. It is clear that $1_\mathcal{A}$ is non-degenerate and irreducible.

Example 1.3.19. As seen in Example 1.3.4, $L^1(G)$ is a $*$-algebra with involution $f^*(x) = \Delta_G(x)^{-1}\overline{f^\vee(x)}$. Let $(\pi_G, L^2(G))$ be the $*$-representation of $L^1(G)$ defined by $\pi_G(f)g \coloneqq f * g$, for $g \in L^2(G)$. Observe that

$$\|\pi_G(f)g\|_2 = \|f * g\|_2 \leq \|f\|_1 \|g\|_2,$$

which implies that π_G is strongly continuous. Moreover, for every $g, h \in L^2(G)$

$$\begin{aligned}
\langle \pi_G(f)g, h\rangle &= \int_G f * g(x)\overline{h(x)}\,dx = \int_G \int_G f(y)g(y^{-1}x)\overline{h(x)}\,dy\,dx \\
&= \int_G \int_G g(y^{-1})f(xy)\overline{h(x)}\,dx\,dy = \int_G \int_G \Delta_G(y^{-1})g(y^{-1})f(x)\overline{h(xy^{-1})}\,dx\,dy \\
&= \int_G \int_G g(y)\overline{\Delta_G(x^{-1})\overline{f^\vee(x)}h(x^{-1}y)}\,dx\,dy = \int_G g(y)\overline{f^* * h(y)}\,dy = \langle g, \pi_G(f^*)h\rangle.
\end{aligned}$$

1.4 Representation theory of the group algebra

Henceforth let G be a locally compact group, and fix a left Haar measure dx on G (see Theorem A.3.4). The goal is to establish a bijective correspondence between the unitary representations of G and the non-degenerate $*$-representations of $L^1(G)$.

Definition 1.4.1. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Let $\pi^1 \colon L^1(G) \to \mathcal{L}(\mathcal{H}_\pi)$ be the map such that, for all $f \in L^1(G)$, $\pi^1(f) \in \mathcal{L}(\mathcal{H}_\pi)$ is the linear operator such that

$$\langle \pi^1(f)u, v\rangle = \int_G f(x)\langle \pi(x)u, v\rangle \, dx \quad \forall u, v \in \mathcal{H}_\pi.$$

The uniqueness of $\pi^1(f) \in \mathcal{L}(\mathcal{H}_\pi)$ is an immediate consequence of the Riesz Representation Theorem [Rud91, Thm. 12.5]. Moreover, for all $f \in L^1(G)$ and all $u \in \mathcal{H}_\pi$

$$\begin{aligned}\|\pi^1(f)u\|^2 &= \int_G f(x)\overline{\langle \pi^1(f)u, \pi(x)u\rangle}\, dx = \int_G\int_G f(x)\overline{f(y)}\langle \pi(x)u, \pi(y)u\rangle \, dy\, dx \\ &\leq \int_G\int_G |f(x)||f(y)|\langle \pi(y^{-1}x)u, u\rangle| \, dy\, dx \leq \|f\|_1^2\|u\|^2.\end{aligned}$$

Hence $\pi^1(f) \in \mathcal{L}(\mathcal{H}_\pi)$ with operator norm $\|\pi^1(f)\| \leq \|f\|_1$. Therefore π^1 is well-defined.

Theorem 1.4.2 ([Fol15, Thm. 3.9])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Then $(\pi^1, \mathcal{H}_\pi)$ is a non-degenerate $*$-representation of $L^1(G)$. Moreover, for all $g \in G$ and all $f \in L^1(G)$*

$$\pi(g)\pi^1(f) = \pi^1(L_g f), \quad \pi^1(f)\pi(g) = \Delta_G(g^{-1})\pi^1(R_{g^{-1}}f). \tag{1.3}$$

Proof. Let $f_1, f_2 \in L^1(G)$ and $u, v \in \mathcal{H}_\pi$. Then

$$\begin{aligned}\langle \pi^1(f_1 * f_2)u, v\rangle &= \int_G\int_G f_1(y)f_2(y^{-1}x)\langle \pi(x)u, v\rangle \, dx\, dy \\ &= \int_G\int_G f_1(y)f_2(x)\langle \pi(yx)u, v\rangle \, dx\, dy = \int_G f_1(y)\langle \pi^1(f_2)u, \pi(y^{-1})v\rangle \, dy \\ &= \int_G f_1(y)\langle \pi(y)\pi^1(f_2)u, v\rangle \, dy = \langle \pi^1(f_1)\pi^1(f_2)u, v\rangle.\end{aligned}$$

Hence $\pi^1(f_1 * f_2) = \pi^1(f_1)\pi^1(f_2)$. Moreover, for all $f \in L^1(G)$ and all $g \in G$

$$\begin{aligned}\langle \pi^1(f^*)u, v\rangle &= \int_G f^*(x)\langle \pi(x)u, v\rangle \, dx = \int_G \Delta_G(x^{-1})\overline{f(x^{-1})}\langle \pi(x)u, v\rangle \, dx \\ &= \int_G \overline{f(x)}\langle \pi(x^{-1})u, v\rangle \, dx = \overline{\int_G f(x)\langle \pi(x)v, u\rangle \, dx} \\ &= \overline{\langle \pi^1(f)v, u\rangle} = \langle u, \pi^1(f)v\rangle, \\ \langle \pi(g)\pi^1(f)u, v\rangle &= \int_G f(x)\langle \pi(g)\pi(x)u, v\rangle \, dx = \int_G f(g^{-1}x)\langle \pi(x)u, v\rangle \, dx \\ &= \langle \pi^1(L_g f)u, v\rangle, \\ \langle \pi^1(f)\pi(g)u, v\rangle &= \int_G f(x)\langle \pi(x)\pi(g)u, v\rangle \, dx = \Delta_G(g^{-1})\int_G f(xg^{-1})\langle \pi(x)u, v\rangle \, dx \\ &= \Delta_G(g^{-1})\langle \pi^1(R_{g^{-1}}f)u, v\rangle.\end{aligned}$$

It follows that $\pi^1(f)^* = \pi^1(f^*)$ and that (1.3) holds. Hence, $\pi^1 \colon L^1(G) \to \mathcal{L}(\mathcal{H}_\pi)$ is a $*$-homomorphism.
It remains to show that $(\pi, \mathcal{H}_\pi)$ is non-degenerate. For this take a unit vector $u \in \mathcal{H}_\pi$ and $\epsilon > 0$. Let $U \subset G$ be an open symmetric neighbourhood of $e_G \in G$ such that $\|\pi(g)u - u\| \leq 1 - \epsilon$ and

$|\Delta_G(g)| < 1 + \epsilon$ for all $g \in U$. Assume that U has compact closure, and let $f \coloneqq \frac{1}{|U|}\chi_U$. Then $|U| < \infty$ and

$$\begin{aligned}
\|\pi^1(f)u - u\|^2 &= \langle \pi^1(f^*)\pi^1(f)u, u\rangle - \langle \pi^1(f)u, u\rangle - \langle \pi^1(f^*)u, u\rangle + \langle u, u\rangle \\
&= \Bigg| \int_G \int_G \Delta_G(x^{-1})\overline{f(x)}f(y)\langle \pi(x)\pi(y)u, u\rangle - f(y)\langle \pi(y)u, u\rangle \\
&\qquad - \Delta_G(x^{-1})\overline{f(x)}\langle \pi(x)u, u\rangle + \langle u, u\rangle \, dx\, dy \Bigg| \\
&\leq \frac{1}{|U|^2}\int_U \int_U |\langle \Delta_G(x^{-1})\pi(x)\pi(y)u - \pi(y)u - \Delta_G(x^{-1})\pi(x)u + u, u\rangle| \, dx\, dy \\
&\leq \frac{1}{|U|^2}\int_U \int_U |\Delta_G(x^{-1})|\|\pi(y)u - u\| + \|\pi(y)u - u\| \, dx\, dy \\
&\leq (1+\epsilon)(1-\epsilon) = \|u\|^2 - \epsilon.
\end{aligned}$$

Hence, $\pi^1(f)u \neq 0$ and therefore $(\pi^1, \mathcal{H}_\pi)$ is non-degenerate. □

Example 1.4.3. The $*$-representation of the group algebra induced by the trivial representation 1_G is $(1_G)^1 = 1_{L^1(G)}$.

Example 1.4.4. Concerning the left regular representation $(\lambda_G, L^2(G))$ one has that for $f \in L^1(G)$ and for $g, h \in L^2(G)$

$$\begin{aligned}
\langle \lambda_G^1(f)g, h\rangle &= \int_G f(x)\langle \lambda_G(x)g, h\rangle \, dx = \int_G \int_G f(x)g(x^{-1}y)\overline{h(y)} \, dx\, dy \\
&= \int_G f * g(y)\overline{h(y)} \, dy = \langle f * g, h\rangle.
\end{aligned}$$

Hence, π_G in Example 1.3.19 is the $*$-representation of $L^1(G)$ induced by the regular representation λ_G.

The following proposition assures that a unitary representation is uniquely determined by the $*$-representation of its group algebra.

Proposition 1.4.5. *Let $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ be two unitary representations of G. Then $\pi = \sigma$ if and only if $\pi^1 = \sigma^1$.*

Proof. It follows directly from Definition 1.4.1 that if $\pi = \sigma$, then $\pi^1 = \sigma^1$. Next suppose that $\pi^1 = \sigma^1$. Then for all $f \in L^1(G)$ and all $u, v \in \mathcal{H}_\pi = \mathcal{H}_\sigma$

$$\int f(x)\langle \pi(x)u, v\rangle \, dx = \int f(x)\langle \sigma(x)u, v\rangle \, dx,$$

which implies $\langle \pi(x)u, v\rangle = \langle \sigma(x)u, v\rangle$ a.e. Thus, by continuity, the matrix coefficients of π and σ agree everywhere, and hence $\pi = \sigma$. □

The next goal is to show that every non-degenerate $*$-representation of the group algebra gives rise to a unitary representation of G. For this recall the notion of approximate identity (see Definition A.3.12 and Proposition A.3.13).

Theorem 1.4.6 ([Fol15, Thm. 3.11])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a non-degenerate $*$-representation of $L^1(G)$. Then there exists a unitary representation $(\sigma, \mathcal{H}_\pi)$ of G such that $\pi = \sigma^1$.*

Proof. Let $V := \{\pi(f)u : f \in L^1(G),\ u \in \mathcal{H}_\pi\}$. Because π is assumed to be non-degenerate, V is a dense subspace of $\mathcal{H}_\pi$. Indeed, if for some $u \in \mathcal{H}_\pi$ one has $\langle \pi(f)u, v\rangle = \langle u, \pi(f^*)v\rangle = 0$ for all $v \in \mathcal{H}_\pi$, then $\pi(f)u = 0$ and therefore $u = 0$.

fix now an approximate identity $\{\psi_U\}_{U\in\mathcal{U}}$. Then, for every $g \in G$ and every $f \in L^1(G)$, $L_g(\psi_U * f) = L_g\psi_U * f$ and hence $\pi(L_g(\psi_U * f)) = \pi(L_g\psi_U * f) \to \pi(L_g f)$ with respect to the norm topology on $\mathcal{L}(\mathcal{H}_\pi)$. For $g \in G$ let $\widetilde{\pi}(g)$ be the linear operator on V such that for every $f \in L^1(G)$ and every $u \in \mathcal{H}_\pi$

$$\pi(L_g f)u = \lim_U \pi(L_g(\psi_U * f))u = \lim_U \pi(L_g\psi_U)\pi(f)u = \widetilde{\pi}(g)\pi(f)u,$$

that is, $\widetilde{\pi}(g)$ is the limit of $\pi(L_g\psi_U)$ on V with respect to the strong operator topology. Since V is dense in $\mathcal{H}_\pi$, extend $\widetilde{\pi}(g)$ by continuity to all of $\mathcal{H}_\pi$. This continuous extension is unique and $\widetilde{\pi}(g)\pi(f) = \pi(L_g f)$ on all of $\mathcal{H}_\pi$. Since $\|L_g\psi_U\| \le \|L_g\psi_U\|_1 \le 1$, $\widetilde{\pi}(g)$ is a bounded linear operator with operator norm $\|\widetilde{\pi}(g)\| \le 1$.

The next goal is to show that $\widetilde{\pi}$ defines a unitary representation of G on $\mathcal{H}_\pi$. It is immediate that $\widetilde{\pi}(e_G) = I$. For all $g, h \in G$ and all $f \in L^1(G)$,

$$\widetilde{\pi}(gh)\pi(f) = \pi(L_{gh}f) = \pi(L_g L_h f) = \widetilde{\pi}(g)\pi(L_h f) = \widetilde{\pi}(g)\widetilde{\pi}(h)\pi(f).$$

Hence $\widetilde{\pi}(gh) = \widetilde{\pi}(g)\widetilde{\pi}(h)$ on V and thus, by continuity, on $\mathcal{H}_\pi$. Moreover, for all $u \in \mathcal{H}_\pi$

$$\|u\| = \|\widetilde{\pi}(g)\widetilde{\pi}(g^{-1})u\| \le \|\widetilde{\pi}(g^{-1})\| \le \|u\|.$$

Thus $\|u\| = \|\widetilde{\pi}(g)u\|$, which implies that $\widetilde{\pi}(g)$ is a unitary operator on $\mathcal{H}_\pi$. This gives a group homomorphism $\widetilde{\pi} : G \to \mathcal{U}(\mathcal{H}_\pi)$. The continuity of $\widetilde{\pi}$ follows from Proposition A.3.11 and the density of V in $\mathcal{H}_\pi$.

One needs to show now that $\pi = \widetilde{\pi}$. Indeed for all $f_1, f_2 \in L^1(G)$ and all $u, v \in \mathcal{H}_\pi$

$$\begin{aligned}\langle \pi(f_1)\pi(f_2)u, v\rangle = \langle \pi(f_1 * f_2)u, v\rangle &= \int f_1(x)\langle \pi(L_x f_2)u, v\rangle\, dx \\ &= \int f_1(x)\langle \widetilde{\pi}(x)\pi(f_2)u, v\rangle\, dx = \langle \widetilde{\pi}^1(f_1)\pi(f_2)u, v\rangle.\end{aligned}$$

Hence $\pi(f_1) = \widetilde{\pi}^1(f_2)$ on V and thus on $\mathcal{H}_\pi$. □

Proposition 1.4.5 and Theorem 1.4.6 establish a bijective correspondence between the unitary representations of G and the non-degenerate $*$-representations of $L^1(G)$. This correspondence is also preserved under unitary equivalence and the operations on unitary representations:

Proposition 1.4.7. *Let $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ be unitary representations of G, and let $(\pi^1, \mathcal{H}_\pi)$ and $(\sigma^1, \mathcal{H}_\sigma)$ be the corresponding representations of $L^1(G)$. The following holds:*

1) $\mathcal{C}(\pi, \sigma) = \mathcal{C}(\pi^1, \sigma^1)$.

2) $\pi \simeq \sigma$ if and only if $\pi^1 \simeq \sigma^1$.

3) A subspace $\mathcal{M} \subset \mathcal{H}_\pi$ is G-invariant if and only if it is $L^1(G)$-invariant.

Proof. To prove 1) take $u \in \mathcal{H}_\pi$ and $v \in \mathcal{H}_\sigma$. Observe that T lies in $\mathcal{C}(\pi, \sigma)$ or in $\mathcal{C}(\pi^1, \sigma^1)$ if and only if for all $f \in L^1(G)$

$$\langle \sigma^1(f)Tu, v\rangle_\sigma = \int_G f(x)\langle \sigma(x)Tu, v\rangle_\sigma\, dx = \int_G f(x)\langle T\pi(x)u, v\rangle_\sigma\, dx = \langle T\pi^1(f)u, v\rangle_\sigma.$$

Hence $\mathcal{C}(\pi, \sigma) = \mathcal{C}(\pi^1, \sigma^1)$.

2) and 3) follow from 1). □

Example 1.4.8. Suppose that G is unimodular. Then the right regular representation $(\rho_G, L^2(G))$ of G is given by $\rho_G(g)f(x) = f(xg)$ for $g \in G$ and $f \in L^2(G)$. Hence, for all $f \in L^1(G)$ and all $g, h \in L^2(G)$,

$$\begin{aligned}\langle \rho_G^1(f)g, h\rangle_2 &= \int_G f(x)\langle \rho_G(x)g, h\rangle\, dx = \int_G\int_G f(x)g(yx)\overline{h(y)}\, dy\, dx \\ &= \int_G\int_G f(y^{-1}x)g(x)\overline{h(y)}\, dx\, dy = \int_G\int_G g * f^\vee(y)\overline{h(y)}\, dy = \langle g * f^\vee, h\rangle_2.\end{aligned}$$

Thus the $*$-representation of $L^1(G)$ induced by the right regular representation is $\rho_G^1(f)g = g*f^\vee$. As seen in Example 1.2.3, $\lambda_G \simeq \rho_G$, where the interwining operator is $Tg := g^\vee$ for $g \in L^2(G)$. Hence, by Proposition 1.4.7, $\lambda_G^1 \simeq \rho_G^1$. Indeed,

$$T[\lambda_G^1(f)g] = (f * g)^\vee = g^\vee * f^\vee = \rho_G^1(f)(Tg) \quad \forall f \in L^1(G)\ \forall g \in L^2(G).$$

1.5 The Gelfand-Naimark-Segal construction

As in the previous sections, suppose G is a locally compact group, and fix a left Haar measure dx on G. The construction introduced here will be very useful when studying the matrix coefficients of a unitary representation of G and the unitary dual $\widehat{G}$.

Definition 1.5.1. A function *of positive type* of G is a function $\phi \in L^\infty(G)$ such that

$$\int_G (f^* * f)\phi \geq 0 \quad \forall f \in L^1(G).$$

Let $\mathcal{P}(G) \subset L^\infty(G)$ be the set of all continuous functions of positive type of G.

Immediate examples of functions of positive type are 0 and 1.

Lemma 1.5.2. *Suppose $\phi \in L^\infty(G)$. Then*

$$\int (f^* * f)\phi = \int_G\int_G f(x)\overline{f(y)}\phi(y^{-1}x)\, dy\, dx \quad \forall f \in L^1(G).$$

and

$$\left|\int (f^* * f)\phi\right| \leq \|\phi\|_\infty \|f\|_1^2 \quad \forall f \in L^1(G).$$

Proof. Note first that for all $f \in L^1(G)$

$$\begin{aligned}\int_G (f^* * f)(x)\phi(x)\, dx &= \int_G\int_G \Delta_G(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\phi(x)\, dy\, dx \\ &= \int_G\int_G \Delta_G(y^{-1})\overline{f(y^{-1})}f(x)\phi(yx)\, dy\, dx \\ &= \int_G\int_G f(x)\overline{f(y)}\phi(y^{-1}x)\, dy\, dx.\end{aligned}$$

Moreover,

$$\left|\int_G (f^* * f)\phi\right| \leq \int_G\int_G |f(x)\overline{f(y)}\phi(y^{-1}x)|\, dy\, dx \leq \|\phi\|_\infty \|f\|_1^2.$$

□

Furthermore, one has

Proposition 1.5.3 ([Fol15, Prop. 3.15])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Then the diagonal matrix coefficients of π are of positive type, that is, for every $u \in \mathcal{H}_\pi$ $\phi_{u,u} \in \mathcal{P}(G)$.*

Proof. Observe that for all $f \in L^1(G)$, by Lemma 1.5.2,

$$\begin{aligned}\int_G \int_G f(x)\overline{f(y)}\phi_{u,u}(y^{-1}x)\,dx\,dy &= \int_G \int_G f(x)\overline{f(y)}\langle \pi(x)u, \pi(y)u\rangle\,dx\,dy \\ &= \langle \pi^1(f)u, \pi^1(f)u\rangle \geq 0.\end{aligned}$$

Because $\phi_{u,u}$ is continuous, $\phi_{u,u} \in \mathcal{P}(G)$. □

Corollary 1.5.4 ([Fol15, Cor. 3.16])**.** *For all $f \in L^2(G)$, $\overline{f} * f^\vee \in \mathcal{P}(G)$.*

Proof. Using the left regular representation one has that for all $g \in G$

$$\langle \lambda_G(g)f, f\rangle_2 = \int_G f(g^{-1}x)\overline{f(x)}\,dx = \int_G f^\vee(x^{-1}g)\overline{f(x)}\,dx = \overline{f} * f^\vee(g).$$

Thus, by Proposition 1.5.3, $\overline{f} * f^\vee \in \mathcal{P}(G)$. □

The goal is now to show that every function of positive type agrees a.e. with the diagonal matrix coefficient of a unitary representation of G. The following construction can be found in [Fol15, Sect. 3.3]. fix a function ϕ of positive type, and define on $L^1(G)$ the Hermitian form $\langle \cdot, \cdot \rangle_\phi$, given by

$$\langle f_1, f_2\rangle_\phi \coloneqq \int_G (f_2^* * f_1)\phi,$$

for $f_1, f_2 \in L^1(G)$. By construction, $\langle \cdot, \cdot \rangle_\phi$ is positive semi-definite. Because $(L_g f_2)^* * L_g f_1 = f_2^* * f_1$, $\langle \cdot, \cdot \rangle_\phi$ is G-invariant, that is,

$$\langle L_g f_1, L_g f_2\rangle_\phi = \langle f_1, f_2\rangle_\phi \quad \forall g \in G\ \forall f_1, f_2 \in L^1(G). \tag{1.4}$$

Set

$$\mathcal{N}_\phi \coloneqq \{f \in L^1(G) \colon \langle f, f\rangle_\phi = 0\},$$

which is obviously a closed subspace of $L^1(G)$. Then $\langle \cdot, \cdot \rangle_\phi$ is an inner product on $L^1(G)/\mathcal{N}_\phi$. Let $\mathcal{H}_\phi$ be the completion of $L^1(G)/\mathcal{N}_\phi$ with respect to $\langle \cdot, \cdot \rangle_\phi$, and let $\pi_\phi \colon G \to \mathcal{U}(\mathcal{H}_\phi)$ be the map given by

$$\pi_\phi(g)([f]) \coloneqq [L_g f],$$

where $[f] \in \mathcal{H}_\phi$ is the equivalence class of $f \in L^1(G)$. Then $\mathcal{H}_\phi$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\phi$, and by (1.4), π_ϕ is well-defined. Moreover, for every net $\{g_i\}_{i\in I} \subset G$ with $g_i \to g$ and every $f \in L^1(G)$, one has, by Lemma 1.5.2 and by Lemma A.3.11, that

$$\|[L_{g_i} f] - [L_g f]\|_\phi \leq \|\phi\|_\infty^{\frac{1}{2}} \|L_{g_i} f - L_g f\|_1 \to 0.$$

Therefore $(\pi_\phi, \mathcal{H}_\phi)$ is a unitary representation of G. This construction is called the *Gelfand-Naimark-Segal* construction.

Theorem 1.5.5 ([Fol15, Thm. 3.20])**.** *The unitary representation $(\pi_\phi, \mathcal{H}_\phi)$ is cyclic, with cyclic vector $u_\phi \in \mathcal{H}_\phi$ such that $\phi(g) = \langle \pi_\phi(g)u_\phi, u_\phi\rangle$ a.e.*

Proof. Take an approximate identity $\{\psi_U\}_{U\in\mathcal{U}}$. Then $\{\psi_U^*\}_{U\in\mathcal{U}}$ is also an approximate identity, and hence

$$\langle [f], [\psi_U]\rangle_\phi = \int_G (\psi_U^* * f)\phi \to \int_G f\phi \quad \forall f \in L^1(G).$$

Moreover, for every $f \in L^1(G)$, by the Cauchy-Schwarz inequality,

$$|\langle [f], [\psi_U]\rangle_\phi| \leq \|[f]\|_\phi \|[\psi_U]\|_\phi \leq \|\phi\|_\infty \|f\|_1 \|\psi_U\|_1 = \|\phi\|_\infty \|f\|_1.$$

It follows that there exists a unique $u_\phi \in \mathcal{H}_\phi$ such that $[\psi_U] \overset{w^*}{\rightharpoonup} u_\phi$ and

$$\langle [f], u_\phi \rangle_\phi = \int f\phi \quad \forall f \in L^1(G).$$

Next observe that for every $f \in L^1(G)$ and every $g \in G$

$$\langle [f], \pi_\phi(g)u_\phi \rangle_\phi = \langle [L_{g^{-1}} f], u_\phi \rangle_\phi = \int_G (L_{g^{-1}} f)(x)\phi(x)\, dx = \int_G f(x)\phi(g^{-1}x)\, dx,$$

and whence

$$\begin{aligned}\langle [f_1], [f_2] \rangle_\phi &= \int_G (f_2^* * f_1)\phi = \int_G \int_G f_1(x)\overline{f_2(y)}\phi(y^{-1}x)\, dx\, dy \\ &= \int_G \langle [f_1], \pi_\phi(y)u_\phi \rangle_\phi \overline{f_2(y)}\, dy = \langle [f_1], \pi_\phi^1(f_2)u_\phi \rangle_\phi.\end{aligned}$$

This implies that for all $f \in L^1(G)$ $[f] = \pi_\phi^1(f)u_\phi$. If $f \in L^1(G)$ is such that $\langle [f], \pi_\phi(g)u_\phi \rangle_\phi = 0$ for all $g \in G$, then $[f] = 0$. This implies that the linear span of $\{\pi_\phi(g)u_\phi : g \in G\}$ is dense in $\mathcal{H}_\phi$, and hence u_ϕ is a cyclic vector of π_ϕ. Moreover, for all $f \in L^1(G)$

$$\begin{aligned}\int_G \langle u_\phi, \pi_\phi(x)u_\phi \rangle_\phi \overline{f(x)}\, dx &= \lim_U \int_G \langle [\psi_U], \pi_\phi(x)u_\phi \rangle_\phi \overline{f(x)}\, dx = \lim_U \langle [\psi_U], [f] \rangle_\phi \\ &= \langle u_\phi, [f] \rangle_\phi = \overline{\langle [f], u_\phi \rangle_\phi} = \int \overline{f\phi}.\end{aligned}$$

Hence

$$\phi(g) = \overline{\langle u_\phi, \pi_\phi(g)u_\phi \rangle_\phi} = \langle \pi_\phi(g)u_\phi, u_\phi \rangle_\phi \quad \text{a.e.}$$

The last part of the theorem follows from Proposition 1.2.10. □

Theorem 1.5.5 establishes a correspondence between $\mathcal{P}(G)$ and the cyclic unitary representations of G. More precisely, $\phi \in \mathcal{P}(G)$ correspond uniquely to a triple $(\pi_\phi, \mathcal{H}_\phi, u_\phi)$ such that $u_\phi \in \mathcal{H}_\phi$ is a cyclic vector of π_ϕ and $\phi(g) = \langle \pi_\phi(g)u_\phi, u_\phi \rangle$.

Definition 1.5.6. The triple $(\pi_\phi, \mathcal{H}_\phi, u_\phi)$ is the *Gelfand triple* associated to ϕ.

Corollary 1.5.7 ([Fol15, Cor. 3.21])**.** *Every function of positive type agrees a.e. with a continuous function.*

Corollary 1.5.8 ([Fol15, Cor. 3.22])**.** *For all $\phi \in \mathcal{P}(G)$ $\|\phi\|_\infty = |\phi(e_G)|$ and $\phi^\vee = \overline{\phi}$.*

Proof. Let $(\pi_\phi, \mathcal{H}_\phi, u_\phi)$ be the Gelfand triple associated to ϕ. Then for every $g \in G$, by the Cauchy-Schwarz inequality,

$$|\phi(g)| = |\langle \pi_\phi(g)u_\phi, u_\phi \rangle_\phi| \leq \|u_\phi\|_\phi^2 = \phi(e_G).$$

Also

$$\phi^\vee(g) = \langle \pi_\phi(g^{-1})u_\phi, u_\phi \rangle_\phi = \overline{\langle \pi_\phi(g)u_\phi, u_\phi \rangle_\phi} = \overline{\phi(g)}.$$

□

Concerning subrepresentations one has

Proposition 1.5.9 ([BdlHV08, Prop. C.5.1])**.** *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Take $u \in \mathcal{H}_\phi$ and set $\phi(g) \coloneqq \langle \pi(g)u, u \rangle$. Then $\pi_\phi \subset \pi$.*

Proof. Let $(\pi_u, \mathcal{H}_u)$ be the cyclic subrepresentation of π generated by u. Then, because $\phi(g) = \langle \pi(g)u, u \rangle$, it follows from Proposition 1.2.10 that $\pi_\phi \simeq \pi_u$. □

Corollary 1.5.10 ([BdlHV08, Cor. C.4.16]). *For $\phi_1, \phi_2 \in \mathcal{P}(G)$ let $\psi := \phi_1\phi_2$. Then $\psi \in \mathcal{P}(G)$ and $\pi_\psi \subset \pi_{\phi_1} \otimes \pi_{\phi_2}$.*

Proof. Note that for every $g \in G$

$$\begin{aligned}\psi(g) = \phi_1(g)\phi_2(g) &= \langle \pi_{\phi_1}(g)u_{\phi_1}, u_{\phi_1}\rangle_{\phi_1}\langle \pi_{\phi_2}(g)u_{\phi_2}, u_{\phi_2}\rangle_{\phi_2} \\ &= \langle (\pi_{\phi_1} \otimes \pi_{\phi_2})(g)u_{\phi_1} \otimes u_{\phi_2}, u_{\phi_1} \otimes u_{\phi_2}\rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2}$. The corollary follows from Proposition 1.5.3 and Proposition 1.5.9. □

Proposition 1.5.11 ([BdlHV08, Prop. C.5.1]). *Suppose that for $\phi \in \mathcal{P}(G)$ one has that $\phi = \psi_1 + \psi_2$ for some $\psi_1, \psi_2 \in \mathcal{P}(G)$. Then $\pi_{\psi_1}, \pi_{\psi_2} \subset \pi_\phi$, and if π_ϕ is irreducible, then $\phi = c_1\psi_1 = c_2\psi_2$ for some real constants $c_1, c_2 > 0$.*

Proof. Note that

$$\langle f, f\rangle_{\psi_1} = \langle f, f\rangle_\phi - \langle f, f\rangle_{\psi_2} \leq \langle f, f\rangle_\phi.$$

Hence, $\mathcal{N}_\phi \subset \mathcal{N}_{\psi_1}$, and let $T \in \mathcal{L}(\mathcal{H}_\phi, \mathcal{H}_{\psi_1})$ be the bounded linear operator induced by the identity operator on $L^1(G)$. By construction, $Tu_\phi = u_{\psi_1}$, and the image of T is dense in $\mathcal{H}_{\psi_1}$. Moreover, by definition, $T \in \mathcal{C}(\pi_\phi, \pi_{\psi_1})$. Thus, by Proposition 1.2.6, the subrepresentation of π_ϕ on $(\ker T)^\perp$ is equivalent to π_{ψ_1}. Hence $\pi_{\psi_1} \subset \pi_\phi$. Also for every $g \in G$

$$\psi_1(g) = \langle \pi_{\psi_1}(g)Tu_\phi, Tu_\phi\rangle_{\psi_1} = \langle T\pi_\phi(g)u_\phi, Tu_\phi\rangle_{\psi_1} = \langle (T^*T)\pi_\phi(g)u_\phi, u_\phi\rangle_\phi. \tag{1.5}$$

If π_ϕ is irreducible, then, because $T^*T \in \mathcal{C}(\pi_\phi)$ and by Schur's Lemma, $T^*T = \lambda I$ for some constant $\lambda \in \mathbb{C} - \{0\}$. Since T^*T is self-adjoint, $\lambda \in \mathbb{R} - \{0\}$. By (1.5), $\psi_1 = \lambda\phi$. □

For the Gelfand-Naimark-Segal construction the section followed closely the outline in [Fol15, Sect. 3.3]. However, in other textbooks, like in [BdlHV08], the Gelfand-Naimark-Segal construction is given by using positive-definite functions instead of functions of positive type.

Definition 1.5.12. A continuous function $\phi\colon G \to \mathbb{C}$ is *positive definite* if for every $n \geq 1$

$$\sum_{1 \leq i,j \leq n} c_i\overline{c_j}\phi(g_j^{-1}g_i) \geq 0 \quad \forall c_1, c_2, \ldots c_n \in \mathbb{C}\ \forall g_1, g_2, \ldots, g_n \in G.$$

One has

Lemma 1.5.13. *If ϕ is positive definite, then:*

1) $\phi(g^{-1}) = \overline{\phi(g)}$ $\forall g \in G$.

2) $|\phi(g)| \leq \phi(e_G)$ $\forall g \in G$.

Proof. Because ϕ is positive definite, for every $g \in G$ the matrix

$$\begin{pmatrix} \phi(e_G) & \phi(g) \\ \phi(g^{-1}) & \phi(e_G) \end{pmatrix}$$

is positive semi-definite. Hence $\overline{\phi(g)} = \phi(g^{-1})$, and $\phi(e_G)^2 - \phi(g)\phi(g^{-1}) \geq 0$, which implies $|\phi(g)| \leq \phi(e_G)$. □

The following proposition asserts that every continuous positive definite function is also of positive type, and vice versa. Its proof is omitted here.

Proposition 1.5.14 ([Fol15, Prop. 3.35]). *Suppose ϕ is a bounded continuous function of G. The following are equivalent:*

1) ϕ is of positive type.

2) ϕ is positive definite.

By Proposition 1.5.14, $\mathcal{P}(G)$ can be identified with the set of all positive definite functions on G. From now the elements of $\mathcal{P}(G)$ will be referred to as positive definite functions. Henceforth let

$$\mathcal{P}_1(G) \coloneqq \{\phi \in \mathcal{P}(G) \colon \|\phi\|_\infty = 1\}$$

be the set of *normalised positive definite functions*. Note that for every $\phi \in \mathcal{P}_1(G)$ with Gelfand triple $(\pi_\phi, \mathcal{H}_\phi, u_\phi)$, u_ϕ is a unit vector. Moreover, set

$$\mathcal{P}_0(G) \coloneqq \{\phi \in \mathcal{P}(G) \colon \|\phi\|_\infty \leq 1\}.$$

Then both $\mathcal{P}_0(G)$ and $\mathcal{P}_1(G)$ are convex subsets contained in the unit ball of $L^\infty(G)$. Recall that the definition of extreme points of a convex set, and let $\operatorname{ext}(\mathcal{P}_0(G))$ and $\operatorname{ext}(\mathcal{P}_1(G))$ denote the set of extreme points of $\mathcal{P}_0(G)$ and $\mathcal{P}_1(G)$, respectively.

Theorem 1.5.15 ([Fol15, Thm. 3.25])**.** *Let $\phi \in \mathcal{P}_1(G)$. Then $(\pi_\phi, \mathcal{H}_\phi)$ is irreducible if and only if $\phi \in \operatorname{ext}(\mathcal{P}_1(G))$.*

Proof. Suppose $(\pi_\phi, \mathcal{H}_\phi)$ is not irreducible. Then there exists a proper non-zero G-invariant closed subspace $\mathcal{M} \subsetneq \mathcal{H}_\phi$ such that $\mathcal{H}_\phi = \mathcal{M} \oplus \mathcal{M}^\perp$. Because π_ϕ is cyclic with cyclic vector u_ϕ, there exist unit vectors $v_1 \in \mathcal{M}$, $v_2 \in \mathcal{M}^\perp$ and non-zero $a_1, a_2 \in \mathbb{R}$ such that $u_\phi = a_1 v_1 + a_2 v_2$. Set $\psi_i(g) \coloneqq \langle \pi_\phi(g) v_i, v_i \rangle$ for $i = 1, 2$. Then $\psi_1, \psi_2 \in \mathcal{P}_1(G)$ and $\phi = a_1\psi_1 + a_2\psi_2$. Observe that

$$\psi_1(g) - \psi_2(g) = \langle \pi_\phi(g) a_1 v_1, a_1^{-1} v_1 \rangle - \langle \pi_\phi(g) a_2 v_2, a_2^{-1} v_2 \rangle = \langle \pi_\phi(g) u_\phi, a_1^{-1} v_1 - a_2^{-1} v_2 \rangle.$$

Since $a_1^{-1} v_1 - a_2^{-1} v_2 \neq 0$ and because u_ϕ is a cyclic vector, $\psi_1 \neq \psi_2$. Moreover, for every $g \in G$,

$$\phi(g) = \langle a_1 \pi_\phi(g) v_1, a_1 v_1 \rangle_\phi + \langle a_2 \pi_\phi(g) v_2, a_2 v_2 \rangle_\phi = a_1^2 \psi_1(g) + a_2^2 \psi_2(g).$$

Since $a_1^2 + a_2^2 = \|u_\phi\|^2 = 1$, ϕ is not an extreme point of $\mathcal{P}_1(G)$. Hence, if $\phi \in \operatorname{ext}(\mathcal{P}_1(G))$, then π_ϕ is irreducible. The converse follows from Proposition 1.5.11. □

Theorem 1.5.15 establishes a correspondence between $\operatorname{ext}(\mathcal{P}_1(G))$ and the unitary dual $\widehat{G}$. To obtain more information about the extreme points of $\mathcal{P}_1(G)$ consider $\mathcal{P}_0(G)$ and $\mathcal{P}_1(G)$ as subspaces of $L^\infty(G)$.

Lemma 1.5.16 ([Fol15, p. 87])**.** *$\mathcal{P}_0(G)$ is closed with respect to the weak* topology on $L^\infty(G)$.*

Proof. Recall that every $f \in L^1(G)$ defines a linear form $\phi \mapsto \int f\phi$ on $L^\infty(G)$. It is immediate that if $\{\phi_i\}_{i \in I}$ is a net in $\mathcal{P}_0(G)$ converging to $\phi \in L^\infty(G)$, then $\phi \in \mathcal{P}(G)$. Moreover, because for every $\lambda \in L^\infty(G)^*$ $\|\lambda(\phi_i)\|_\infty \to \|\lambda(\phi)\|_\infty$, $\|\phi\|_\infty \leq 1$, that is, $\phi \in \mathcal{P}_0(G)$. □

Hence $\mathcal{P}_0(G)$ is a weak* closed subset of the closed unit ball of $L^\infty(G)$. By the Banach-Alaoglu Theorem [Rud91, Thm. 3.15], $\mathcal{P}_0(G)$ is compact with respect to the weak* topology, and by the Krein-Millman Theorem [Rud91, Thm. 3.23], $\mathcal{P}_0(G) = \overline{\operatorname{conv}(\operatorname{ext}(\mathcal{P}_0(G)))}$, where $\operatorname{conv}(\operatorname{ext}(\mathcal{P}_0(G)))$ denotes the convex hull of $\operatorname{ext}(\mathcal{P}_0(G))$, that is, the smallest convex subset of $L^\infty(G)$ containing $\operatorname{ext}(\mathcal{P}_0(G))$. On the other hand $\mathcal{P}_1(G)$ is not necessarily a weak* closed subset of $L^\infty(G)$. One still has however that $\mathcal{P}_1(G)$ is the closure of the convex hull of $\operatorname{ext}(\mathcal{P}_1(G))$ with respect to the subspace topology. To prove this, one needs the following lemma.

Lemma 1.5.17 ([Fol15, Lem. 3.26])**.** $\operatorname{ext}(\mathcal{P}_0(G)) = \operatorname{ext}(\mathcal{P}_1(G)) \cup \{0\}$.

Proof. Observe that $\operatorname{ext}(\mathcal{P}_0(G)) \subset \operatorname{ext}(\mathcal{P}_1(G)) \cup \{0\}$. Indeed, if $\phi \in \mathcal{P}_0(G)$ with $\phi \neq 0$ and $\phi(e_G) \leq 1$, then $\phi = \phi(e_G)\frac{\phi}{\phi(e_G)}$, where $\frac{\phi}{\phi(e_G)} \in \mathcal{P}_1(G)$.

For the converse inclusion take $\phi_1, \phi_2 \in \mathcal{P}_0(G)$ and $0 < t < 1$. If $t\phi_1 + (1-t)\phi_2 = 0$, then

$$0 = t\phi_1(e_G) + (1-t)\phi_2(e_G) = t\|\phi_1\|_\infty + (1-t)\|\phi_2\|_\infty,$$

and thus $\phi_1 = \phi_2 = 0$. On the other hand, if $t\phi_1 + (1-t)\phi_2 \in \mathcal{P}_1(G)$, then $t\phi_1(e_G) + (1-t)\phi_2(e_G) = 1$ and therefore $\phi_1(e_G) = \phi_2(e_G) = 1$, that is, $\phi_1, \phi_2 \in \mathcal{P}_1(G)$. It follows that $\operatorname{ext}(\mathcal{P}_0(G)) \cap \mathcal{P}_1(G) = \operatorname{ext}(\mathcal{P}_1(G))$. □

Theorem 1.5.18 ([Fol15, Thm. 3.27])**.** *The convex hull of* $\operatorname{ext}(\mathcal{P}_1(G))$ *is weak* dense in* $\mathcal{P}_1(G)$*.*

Proof. Recall that $\mathcal{P}_0(G)$ is a convex subset of $L^\infty(G)$ which is closed with respect to the weak* topology. As remarked above, $\operatorname{conv}(\operatorname{ext}(\mathcal{P}_0(G)))$ is weak* dense in $\mathcal{P}_0(G)$. Take $\phi \in \mathcal{P}_1(G)$. By Lemma 1.5.17, there exists a net $\{\phi_i\}_{i\in I}$ in $\operatorname{conv}(\operatorname{ext}(\mathcal{P}_0(G)))$ such that $\phi_i \to \phi$ and $\phi_i = \sum_{1\leq j_i\leq n_i+1} t_j\psi_j$, where $\sum_{1\leq j_i\leq n_i+1} t_j = 1$ and $\psi_j \in \operatorname{ext}(\mathcal{P}_1(G))$ for all $1 \leq j \leq n_i$, $\psi_{n_i+1} = 0$. Because $\phi(e_G) = \|\phi\|_\infty = 1$, $\lim_i \phi_i(e_G) = \lim_i \|\phi_i\|_\infty = 1$. Hence, by passing to a subnet if necessary, assume that $\phi_i(e_G) \neq 0$ for every $i \in I$. For each $i \in I$ set $\phi_i' \coloneqq \frac{\phi_i}{\phi_i(e_G)}$. Then

$$1 = \phi_i'(e_G) = \frac{1}{\phi_i(e_G)} \sum_{1\leq j_i\leq n_i} t_j\psi_j,$$

which implies that $\phi_i' \in \operatorname{conv}(\operatorname{ext}(\mathcal{P}_1(G)))$. Moreover, $\phi_i' \to \phi$. □

By Theorem 1.5.18, if G is non-trivial, then it has irreducible unitary representations which are non-zero and non-trivial.

Given a net $\{\phi_i\}_{i\in I} \subset \mathcal{P}_1(G)$ and $\phi \in \mathcal{P}_1(G)$, $\phi_i \to \phi$ with respect to the *topology of uniform convergence on compact subsets of* G if and only if for every compact subset $Q \subset G$

$$\sup_{g\in Q} \|\phi_i(g) - \phi(g)\| \to 0.$$

A basis of that topology on $\mathcal{P}_1(G)$ is constituted by the sets $\{N(\phi, Q, \epsilon) \colon \phi \in \mathcal{P}_1(G),\ Q \subset G \text{ compact},\ \epsilon > 0\}$, where

$$N(\phi, Q, \epsilon) \coloneqq \{\phi' \in \mathcal{P}_1(G) \colon |\phi(g) - \phi'(g)| < \epsilon\ \forall g \in Q\}.$$

Theorem 1.5.19 (Raikov's Theorem, [BdlHV08, Thm. C.5.6])**.** *The weak* topology and the topology of uniform convergence on compact subsets of* G *coincide on* $\mathcal{P}_1(G)$*.*

Raikov's Theorem will become very useful in Chapter 4, where a topology is introduced on the unitary dual. For completeness also the following theorem is stated here without proof.

Theorem 1.5.20 (Gelfand-Raikov Theorem, [Fol15, Thm. 3.34])**.** *The unitary dual* $\widehat{G}$ *separates points, that is, for every two distinct* $g, h \in G$ *there exists* $\pi \in \widehat{G}$ *such that* $\pi(g) \neq \pi(h)$*.*

1.6 Unitary representations of compact groups

This section is a short survey of the theory of unitary representations of compact groups. For a more detailed account see [Fol15, Chap. 5]. From now on suppose that G is compact, and let dx be the normalized Haar measure on G.

The following result will be used later again in this thesis.

Lemma 1.6.1. *Suppose that* (π, V) *is a finite-dimensional representation of* G*. Then there exists an inner product* $\langle\cdot,\cdot\rangle$ *on* V *which is* G*-invariant, that is,*

$$\langle \pi(g)u, \pi(g)v\rangle = \langle u, v\rangle \quad \forall g \in G\ \forall u, v \in V.$$

Proof. Let $\langle\cdot,\cdot\rangle_0$ be any inner product on V. Define the bilinear form $\langle\cdot,\cdot\rangle$ on V by

$$\langle u, v\rangle \coloneqq \int_G \langle \pi(x)u, \pi(x)v\rangle_0 \, dx.$$

It is obvious that $\langle\cdot,\cdot\rangle$ is an inner product. By the bi-G-invariance of dx, $\langle\cdot,\cdot\rangle$ is G-invariant. Because V is finite-dimensional, $\langle\cdot,\cdot\rangle$ induces the same topology on V as $\langle\cdot,\cdot\rangle_0$. □

The subsequent three theorems are stated here without proofs. For detailed proofs the reader is referred to the respective references in the header of the theorems.

Theorem 1.6.2 ([Fol15, Thm. 5.2])**.** *Every irreducible unitary representation of G is finite-dimensional. Moreover, every unitary representation of G is the direct sum of irreducible unitary representations.*

Let $[\pi] \in \widehat{G}$, and write $d_\pi = \dim \mathcal{H}_\pi$. fix an orthonormal basis $\{e_i\}_{1\le i\le d_\pi}$ of $\mathcal{H}_\pi$, and set

$$\pi_{ij}(g) \coloneqq \phi_{e_j,e_i}(g) = \langle \pi(g)e_j, e_i\rangle.$$

The matrix coefficients of π lie in $C(G)$, and, because G is compact, they are also contained in $L^p(G)$ for every $1 \le p \le \infty$. Let $\mathcal{E}_\pi \subset L^2(G)$ be the linear span of $\{\pi_{ij} \colon 1 \le i, j \le d_\pi\}$. Because $\mathcal{E}_\pi$ is finite-dimensional, it is closed. Observe that for every $g \in G$, every $f \in L^1(G)$ and every $u, v \in \mathcal{H}_\pi$

$$f * \phi_{u,v}(g) = \int_G f(x)\langle \pi(x^{-1}g)u, v\rangle \, dx = \int_G f(x)\langle \pi(g)u, \pi(x)v\rangle \, dx = \phi_{u,\pi^1(\overline{f})v}(g),$$

and

$$\phi_{u,v} * f(g) = \int_G \langle \pi(x)u, v\rangle f(x^{-1}g)\, dx = \int_G \langle \pi(g)\pi(x)u, v\rangle f(x^{-1})\, dx = \phi_{\pi^1(f^\vee)u,v}(g).$$

Hence $\mathcal{E}_\pi$ is a $L^1(G)$-invariant subspace of λ^1_G and ρ^1_G, that is, $\mathcal{E}_\pi$ can be identified with a closed two-sided ideal in $L^1(G)$.

Theorem 1.6.3 (Schur Orthogonality Relations, [Fol15, Thm. 5.8])**.** *1) If $[\pi], [\pi'] \in \widehat{G}$ with $[\pi] \ne [\pi']$, then $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$.*

2) $\{\sqrt{d_\pi}\pi_{ij} \colon 1 \le i, j \le d_\pi\}$ is an orthonormal basis of $\mathcal{E}_\pi$.

For $[\pi] \in \widehat{G}$ and $1 \le i, j \le d_\pi$, set

$$\mathcal{C}_{\pi,i} \coloneqq \langle \sqrt{d_\pi}\pi_{ji} \colon 1 \le j \le d_\pi\rangle \subset L^2(G)$$
$$\mathcal{R}_{\pi,j} \coloneqq \langle \sqrt{d_\pi}\pi_{ij} \colon 1 \le i \le d_\pi\rangle \subset L^2(G).$$

Theorem 1.6.4 ([Fol15, Thm. 5.9])**.** *Let $[\pi] \in \widehat{G}$. For every $1 \le i \le d_\pi$, $\mathcal{C}_{\pi,i} \subset L^2(G)$ is a G-invariant subspace of the left regular representation, and $\lambda_G^{\mathcal{C}_{\pi,i}} \simeq \pi$, where the unitary equivalence is given by*

$$u \mapsto \phi_{e_i,u}.$$

Similarly, for every $1 \le j \le d_\pi$, $\mathcal{R}_{\pi,j} \subset L^2(G)$ is a G-invariant subspace of the right regular representation, and $\rho_G^{\mathcal{R}_{\pi,j}} \simeq \pi$, where the unitary equivalence is given by

$$u \mapsto \phi_{u,e_j}.$$

Theorem 1.6.5 (Peter-Weyl Theorem, [Fol15, Thm. 5.12])**.** *One has that*

$$\mathcal{E}_\pi = \bigoplus_{1\le i\le d_\pi} \mathcal{C}_{\pi,i} = \bigoplus_{1\le j\le d_\pi} \mathcal{R}_{\pi,j},$$

and

$$L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi.$$

In particular,

$$\lambda_G = \bigoplus_{[\pi]\in\widehat{G}} \bigoplus_{1\le i\le d_\pi} \lambda_G^{\mathcal{C}_{\pi,i}}, \quad \rho_G = \bigoplus_{[\pi]\in\widehat{G}} \bigoplus_{1\le j\le d_\pi} \rho_G^{\mathcal{R}_{\pi,j}}.$$

It follows that $\{\sqrt{d_\pi}\pi_{ij}\colon [\pi]\in\widehat{G},\ 1\le i,j\le d_\pi\}$ is an orthonormal basis of $L^2(G)$. For $f\in L^2(G)$ write

$$f = \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{1\le i,j\le d_\pi} \langle f,\pi_{ij}\rangle_2 \pi_{ij},$$

where

$$\langle f,\pi_{ij}\rangle_2 = \int_G f(x)\overline{\pi_{ij}(x)}\,dx.$$

Also recall Example 1.4.4 and from Example 1.4.8.

Proposition 1.6.6. *Let $[\pi]\in\widehat{G}$, and take $1\le i\le d_\pi$. Then the orthogonal projection $P_i\colon L^2(G)\to\mathcal{R}_{\pi,i}$ is given by*

$$P_i(f) = d_\pi \pi_{ii} * f.$$

Moreover, if for $f\in L^2(G)$ $\lambda_G^1(f)$ is a orthogonal projection, then $\lambda_G^1(f)\in\mathcal{C}(\rho_G)$ and $\|f\|_2^2 = \dim(\operatorname{im}\lambda_G^1(f))$.

Proof. Observe that

$$\begin{aligned}
\pi_{ii} * f(g) &= \int_G \pi_{ii}(x) f(x^{-1}g)\,dx = \int_G \langle \pi(x^{-1})e_i, \pi(g^{-1})e_i\rangle f(x)\,dx \\
&= \int_G f(x)\Bigg(\sum_{1\le j\le d_\pi} \overline{\langle \pi(g^{-1})e_i,e_j\rangle}\langle \pi(x^{-1})e_i,e_j\rangle\Bigg)\,dx \\
&= \int_G f(x)\Bigg(\sum_{1\le j\le d_\pi} \pi_{ij}(g)\overline{\pi_{ij}(x)}\Bigg)\,dx = \sum_{1\le j\le d_\pi} \langle f,\pi_{ij}\rangle_2 \pi_{ij}(g).
\end{aligned}$$

Thus, $P_i(f) = d_\pi\pi_{ii} * f$.

Next suppose that $f\in L^2(G)$ is such that $\lambda_G^1(f)$ is an orthogonal projection on $L^2(G)$. Because $\lambda_G^1(f)^2 = \lambda_G^1(f*f) = \lambda_G^1(f)$, f is contained in the image of $\lambda_G^1(f)$. As seen in Example 1.4.8,

$$\lambda_G^1(f)g = f*g = \rho^1(g^\vee)(f) \quad \forall g\in L^2(G),$$

and therefore

$$\operatorname{im}\lambda_G^1(f) = \langle \rho_G(g)f\colon g\in G\rangle.$$

Hence $\lambda_G^1(f)\in\mathcal{C}(\rho_G)$, and there exist a finite subset $S\subset\widehat{G}$ and $1\le n_\pi\le d_\pi$ for each $[\pi]\in S$ such that

$$\rho_G^{\operatorname{im}\lambda_G^1(f)} \simeq \bigoplus_{[\pi]\in\widehat{G}} \bigoplus_{1\le i\le n_\pi} \rho_G^{\mathcal{R}_{\pi,i}}.$$

By the first part of the proposition and because $f\in\operatorname{im}\lambda_G^1(f)$, this implies that there is a bijective isometry $T\in\mathcal{C}(\rho_G)$ such that

$$T(f) = \sum_{[\pi]\in S} d_\pi \sum_{1\le i\le n_\pi} \pi_{ii}$$

Therefore $\|f\|_2^2 = \dim\operatorname{im}\lambda_G^1(f)$. □

The section is concluded with two results about cyclic unitary representations of G. The proof of the following proposition is omitted here.

Proposition 1.6.7 ([GM71, Thm. 1.10]). *Let $(\sigma, \mathcal{H}_\sigma)$ be a unitary representation of G, and suppose $\sigma = \bigoplus_{[\pi]\in\widehat{G}} \pi^{\oplus n_\pi}$ for $n_\pi \geq 0$. Then σ is cyclic if and only if $n_\pi \leq d_\pi$ for every $[\pi] \in \widehat{G}$ and $n_\pi = 0$ for all but countably many $[\pi] \in \widehat{G}$.*

The following proposition will be used in Section 4.4 to prove Proposition 4.4.3 and Theorem 4.4.6.

Proposition 1.6.8 ([CHH88, Prop.]). *Let $(\sigma, \mathcal{H}_\sigma)$ be a unitary representation of G, and let $u \in \mathcal{H}_\sigma$ be such that $\langle \sigma(g)u \colon g \in G\rangle$ is finite-dimensional. Then there exists a unique continuous function e_u on G such that $e_u = e_u * e_u = e_u^*$, $\sigma^1(e_u)u = u$ and $\|e_u\|_2^2 = \dim\langle \sigma(g)u \colon g \in G\rangle$.*

Proof. By passing to a subrepresentation if necessary, assume that u is a cyclic vector of $(\sigma, \mathcal{H}_\sigma)$. Then, $\dim \mathcal{H}_\sigma = \dim\langle \sigma(g)u \colon g \in G\rangle$. By Theorem 1.6.2, for every $[\pi] \in \widehat{G}$ there is an integer $n_\pi \geq 0$ such that $\sigma \simeq \bigoplus_{[\pi]\in\widehat{G}} \pi^{\bigoplus n_\pi}$, and by Proposition 1.6.7, $n_\pi \leq d_\pi$ for every $[\pi] \in \widehat{G}$ and $n_\pi = 0$ for all but countably many $[\pi] \in \widehat{G}$. Let $S \subset \widehat{G}$ be the finite subset of all $[\pi] \in \widehat{G}$ such that $n_\pi \neq 0$. Then, by Theorem 1.6.4, $\sigma \simeq \tau$, where τ is the subrepresentation of the left regular representation λ_G given by

$$\tau \coloneqq \bigoplus_{[\pi]\in S} \bigoplus_{1\leq i\leq n_\pi} \lambda_G^{\mathcal{C}_{\pi,i}},$$

with $\mathcal{H}_\tau = \bigoplus_{[\pi]\in S} \bigoplus_{1\leq i\leq n_\pi} \mathcal{C}_{\pi,i}$.

Henceforth assume that $\sigma = \tau$. Then, by Theorem 5.8.7, it can be assumed that $u = \sum_{[\pi]\in S} \sum_{1\leq i\leq n_\pi} \pi_{ii}$. By Proposition 1.6.6, $\lambda_G^1(u)u = u * u = u$. Moreover, it is clear that $u = u^*$ and that $\|u\|_2^2 = \dim\langle \lambda_G(g)u \colon g \in G\rangle$, which implies the existence of e_u.

Concerning uniqueness, recall from Example 1.4.8 that for all $f \in L^2(G)$ $\lambda^1(u)f = u * f = \rho_G^1(f^\vee)u$. Hence, by Proposition 1.6.6, $\lambda_G^1(e_u)$ needs to be the orthogonal projection onto

$$\langle \rho_G(g)u \colon g \in G\rangle = \bigoplus_{[\pi]\in S} \bigoplus_{1\leq i\leq n_\pi} \mathcal{R}_{\pi,i},$$

and , by Proposition 1.6.6, $e_u = u$. □

1.7 Induced Representations

The following construction will be quite helpful when spherical functions and spherical representations are introduced in Chapter 3. Let G be a locally compact group, and suppose $H < G$ is closed. Let $(\sigma, \mathcal{H}_\sigma)$ be a unitary representation of H. The goal is to construct from σ a unitary representation of G. Let $q \colon G \to G/H$ be the canonical quotient map, and define the set

$$\begin{aligned} \mathcal{F}_\sigma \coloneqq \{f \in C(G, \mathcal{H}_\sigma) \colon & q(\operatorname{supp} f) \text{ is compact}, \\ & f(xh) = \sigma(h^{-1})f(x)\ \forall x \in G,\ \forall h \in H\}. \end{aligned} \tag{1.6}$$

The following lemma asserts that $\mathcal{F}_\sigma$ is non-empty. Let $\langle \cdot, \cdot \rangle_\sigma$ be the inner product of $\mathcal{H}_\sigma$.

Proposition 1.7.1 ([Fol15, Prop. 6.1]). *Let $\alpha \colon G \in \mathcal{H}_\sigma$ be a continuous function with compact support. The unique map $f_\alpha \colon G \to \mathcal{H}_\sigma$ such that*

$$\langle f_\alpha(x), u\rangle_\sigma = \int_H \langle \sigma(h)\alpha(xh), u\rangle_\sigma \, dh \quad \forall u \in \mathcal{H}_\sigma,$$

lies in $\mathcal{F}_\sigma$. Moreover, f_α is left uniformly continuous, and for every $f \in \mathcal{F}_\sigma$ there is $\alpha \in C_c(G)$ such that $f = f_\alpha$.

Proof. It is clear that $q(\operatorname{supp} f_\alpha) \subset q(\operatorname{supp} \alpha)$. Moreover, for $h' \in H$

$$\begin{aligned}\langle f_\alpha(xh'), u\rangle_\sigma &= \int_H \langle \sigma(h)\alpha(xh'h), u\rangle_\sigma \, dh \\ &= \int_H \langle \sigma((h')^{-1}h)\alpha(xh), u\rangle_\sigma \, dh = \langle \sigma(h')^{-1} f_\alpha(x), u\rangle_\sigma.\end{aligned}$$

Next fix a relatively compact open neighbourhood N of e_G, and let $\epsilon > 0$. There is an open neighbourhood U_ϵ of e_G such that $\|\alpha(yx) - \alpha(x)\|_\sigma < \epsilon$ for every $x \in G$, $y \in U_\epsilon$, and set $J \coloneqq (\operatorname{supp}\alpha)^{-1}N^{-1}U_\epsilon$. Then for $x \in \operatorname{supp}\alpha$ and $y \in U_\epsilon$

$$\begin{aligned}&\|f_\alpha(yx) - f_\alpha(x)\|_\sigma^2 \\ &\quad= \left| \int_H \int_H \langle \sigma(h)\alpha(yxh) - \sigma(h)\alpha(yh), \sigma(h')\alpha(yxh') - \sigma(h')\alpha(yh')\rangle_\sigma \, dh \, dh' \right| \\ &\quad\leq \left(\int_{J\cap H} \|\alpha(yxh) - \alpha(yh)\|_\sigma^2 \, dh \right)^2 \leq (\epsilon |J \cap H|)^2.\end{aligned}$$

Moreover, for all $x \in \operatorname{supp}\alpha$ and all $h \in H$

$$\|f_\alpha(yxh) - f_\alpha(xh)\|_\sigma = \|\sigma(h^{-1})(f_\alpha(yx) - f_\alpha(x))\|_\sigma = \|f_\alpha(yx) - f_\alpha(x)\|_\sigma \leq \epsilon |J \cap H|.$$

Since $\operatorname{supp} f_\alpha = q^{-1}(\operatorname{supp}\alpha)$, $f_\alpha|_{G - q^{-1}(\operatorname{supp}\alpha)} = 0$, which implies that f_α is left uniformly continuous on G. Next, suppose $f \in \mathcal{F}_\sigma$. Then, by Lemma A.4.4, there exists $\psi \in C_c(G)$ such that $\int_H \psi(xh)\, dh = 1$ for every $x \in \operatorname{supp} f$. Hence, for $\alpha \coloneqq \psi f$ and $v \in \mathcal{H}_\sigma$,

$$\begin{aligned}\langle f_\alpha(x), v\rangle_\sigma &= \int_H \langle \sigma(h)(\psi f)(xh), v\rangle_\sigma \, dh = \int_H \psi(xh)\langle \sigma(h) f(xh), v\rangle_\sigma \, dh \\ &= \langle f(x), v\rangle_\sigma.\end{aligned}$$

This implies that $f_\alpha = f$. □

One can now use $\mathcal{F}_\sigma$ to construct a unitary representation of G. For this fix a strongly quasi-invariant measure μ on G/H, with rho-function $\rho\colon G \to \mathbb{C}$ (see Definition A.4.6). Define on $\mathcal{F}_\sigma$ the inner product

$$\langle f_1, f_2\rangle_\mu \coloneqq \int_{G/H} \langle f_1(x), f_2(x)\rangle_\sigma \, d\mu(xH),$$

and let

$$\pi_\mu(g) f(x) \coloneqq \sqrt{\frac{\rho(g^{-1}x)}{\rho(x)}} f(g^{-1}x).$$

Let $\mathcal{H}_\mu$ be the completion of $\mathcal{F}_\sigma$ under $\langle\cdot,\cdot\rangle_\mu$. Recall that, by Theorem A.4.9,

$$\frac{d\mu_{g^{-1}}}{d\mu}(xH) = \frac{\rho(g^{-1}x)}{\rho(x)},$$

and thus

$$\begin{aligned}\langle \pi_\mu(g) f_1, \pi_\mu(g) f_2\rangle_\mu &= \int_{G/H} \langle f_1(g^{-1}x), f_2(g^{-1}x)\rangle_\sigma \frac{d\mu_{g^{-1}}}{d\mu}(xH) \, d\mu(xH) \\ &= \int_{G/H} \langle f_1(x), f_2(x)\rangle_\sigma \, d\mu(xH) = \langle f_1, f_2\rangle_\mu.\end{aligned}$$

Hence $\pi_\mu\colon G \to \mathcal{U}(\mathcal{H}_\mu)$ is well-defined. Because, by Proposition A.1.14, every $f \in \mathcal{F}_\sigma$ is left uniformly continuous, $\pi_\mu\colon G \to \mathcal{U}(\mathcal{H}_\mu)$ is continuous with respect to the strong operator topology. Hence, $(\pi_\mu, \mathcal{H}_\mu)$ is a unitary representation of G.

Lemma 1.7.2 ([Fol15, p. 166])**.** *Suppose μ and μ' are two strongly quasi-invariant measures on G/H with rho-functions ρ and ρ'. Then $(\pi_\mu, \mathcal{H}_\mu) \simeq (\pi_{\mu'}, \mathcal{H}_{\mu'})$.*

Proof. Define the linear map $T\colon \mathcal{F}_\sigma \to \mathcal{F}_\sigma$ by

$$Tf \coloneqq \sqrt{\frac{\rho}{\rho'}} f.$$

Then, by Theorem A.4.10, $\langle f_1, f_2\rangle_\mu = \langle Tf_1, Tf_2\rangle_{\mu'}$ for all $f_1, f_2 \in \mathcal{F}_\sigma$. Hence, T extends uniquely to an isometric linear operator. Moreover, for all $f \in \mathcal{F}_\sigma$,

$$\begin{aligned} T(\pi_\mu(x)f)(y) &= \sqrt{\frac{\rho(y)}{\rho'(y)}}\sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f(x^{-1}y) = \sqrt{\frac{\rho(x^{-1}y)}{\rho'(x^{-1}y)}}\sqrt{\frac{\rho'(x^{-1}y)}{\rho'(y)}} f(x^{-1}y) \\ &= \pi_{\mu'}(Tf)(y). \end{aligned}$$

Hence, $T \in \mathcal{C}(\pi_{\mu'}, \pi_{\mu'})$. □

This gives way to

Definition 1.7.3. The *representation induced by* σ is $(\pi_\mu, \mathcal{H}_\mu)$, denoted by $\Big(\operatorname{ind}_H^G(\sigma), \mathcal{H}_{\operatorname{ind}_H^G(\sigma)}\Big)$.

Note that, by construction, $\mathcal{F}_\sigma$ is a dense subspace of $\mathcal{H}_{\operatorname{ind}_H^G(\sigma)}$.

Example 1.7.4. Let $H < G$ be a closed subgroup, and let $\sigma = 1_H$. Then $\mathcal{F}_\sigma = q^*(C_c(G/H))$ where $q^*\colon C_c(G/H) \to C(G)$ is given by $f \mapsto f \circ q$. By construction, the resulting Hilbert space $\mathcal{H}_{\operatorname{ind}_H^G(1_H)}$ is isomorphic to $L^2(G/H)$, where the isomorphism T is the continuous extension of q^*. Furthermore, by construction, $\lambda_{G/H} \simeq \operatorname{ind}_H^G(1_H)$ with interwining operator T.

Proposition 1.7.5 ([BdlHV08, Prop. E.2.1])**.** *Suppose $(\sigma_1, \mathcal{H}_1) \simeq (\sigma_2, \mathcal{H}_2)$ are unitary representations of H. Then $\operatorname{ind}_H^G(\sigma_1) \simeq \operatorname{ind}_H^G(\sigma_2)$.*

Proof. Let μ be a quasi-invariant Borel measure on G/H, with rho-function ρ. Let $\mathcal{F}_{\sigma_1}$ and $\mathcal{F}_{\sigma_2}$ be as in (1.6). For an invertible isometry $T \in \mathcal{C}(\sigma_1, \sigma_2)$ let $\widetilde{T}\colon \mathcal{F}_{\sigma_1} \to \mathcal{F}_{\sigma_2}$, $(\widetilde{T}f)(x) = T(f(x))$, $x \in G$. Observe that $\widetilde{T}$ is surjective and that for all $f_1, f_2 \in \mathcal{F}_{\sigma_1}$

$$\begin{aligned} \langle \widetilde{T}f_1, \widetilde{T}f_2\rangle_\mu &= \int_{G/H} \langle T(f_1(x)), T(f_2(x))\rangle_{\sigma_2}\, d\mu(xH) \\ &= \int_{G/H} \langle f_1(x), f_2(x)\rangle_{\sigma_1}\, d\mu(xH) = \langle f_1, f_2\rangle_\mu. \end{aligned}$$

Hence $\widetilde{T}$ preserves the inner product on $\mathcal{F}_{\sigma_1}$ and therefore can be extended to a linear isometry from the completion of $\mathcal{F}_{\sigma_1}$ to the completion of $\mathcal{F}_{\sigma_2}$. Moreover, for all $f \in \mathcal{F}_{\sigma_1}$ and all $g \in G$

$$\widetilde{T}[\operatorname{ind}_H^G(\sigma_1)(g)f](x) = \sqrt{\frac{\rho(gx)}{\rho(x)}} T(f(g^{-1}x)) = [\operatorname{ind}_H^G(\sigma_2)(g)](\widetilde{T}f)(x).$$

By continuity, $\widetilde{T} \in C\Big(\operatorname{ind}_H^G(\sigma_1), \operatorname{ind}_H^G(\sigma_2)\Big)$. □

Proposition 1.7.6 ([BdlHV08, Prop. E.2.2])**.** *Let $\{(\sigma_i, \mathcal{H}_i)\}_{i\in I}$ be a family of unitary representations of H. Then $\operatorname{ind}_H^G\Big(\bigoplus_{i\in I} \sigma_i\Big) \simeq \bigoplus_{i\in I} \operatorname{ind}_H^G(\sigma_i)$.*

Proof. The proof is only sketched, with the details left to the reader. Note that $\bigoplus_{i\in I}\mathcal{F}_{\sigma_i}$ is dense in $\bigoplus_{i\in I}\mathcal{H}_{\mathrm{ind}_H^G(\sigma_i)}$. Let $T\colon\bigoplus_{i\in I}\mathcal{F}_{\sigma_i}\to\mathcal{F}_{\bigoplus\sigma_i}$ be the map induced by the inclusions $\mathcal{F}_{\sigma_i}\to\mathcal{F}_{\bigoplus\sigma_i}$. Moreover, by construction, T is invertible and an isometry with respect to the inner products defined on $\bigoplus_{i\in I}\mathcal{F}_{\sigma_i}$ and $\mathcal{F}_{\bigoplus\sigma_i}$. Hence T can be extended to an invertible isometry between the respective completions. Furthermore, it is easy to verify that T interwines $\bigoplus_{i\in I}\mathrm{ind}_H^G(\sigma_i)$ and $\mathrm{ind}_H^G\left(\bigoplus_{i\in I}\sigma_i\right)$ on $\bigoplus_{i\in I}\mathcal{F}_{\sigma_i}$ and $\mathcal{F}_{\bigoplus\sigma_i}$. By continuity, $T\in C\Big(\bigoplus_{i\in I}\mathrm{ind}_H^G(\sigma_i),\mathrm{ind}_H^G\left(\bigoplus_{i\in I}\sigma_i\right)\Big)$. □

Corollary 1.7.7. *If* $\left(\mathrm{ind}_H^G(\sigma),\mathcal{H}_{\mathrm{ind}_H^G(\sigma)}\right)$ *is irreducible,* $(\sigma,\mathcal{H}_\sigma)$ *is irreducible.*

Proof. By Proposition 1.7.6, if σ has a non-trivial proper subrepresentation, $\mathrm{ind}_H^G(\sigma)$ has a non-trivial proper subrepresentation. □

For completeness the following result is stated here. Its proof can be found in [Fol15, Sect. 6.3]. However the proof is too involved to be reproduced here.

Theorem 1.7.8 (Induction in stages, [Fol15, Thm. 6.14])**.** *Let* $H,K<G$ *be closed subgroups such that that* $K<H$*. Suppose* $(\sigma,\mathcal{H}_\sigma)$ *is a unitary representation of* K*. Then* $\mathrm{ind}_H^G(\mathrm{ind}_K^H(\sigma))\simeq\mathrm{ind}_K^G(\sigma)$.

Chapter 2

The Howe–Moore Property

2.1 Definition and first results

Suppose that G is a topological group, and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Recall that the matrix coefficients of π are the maps $\phi_{u,v}\colon G \to \mathbb{C}$, $\phi_{u,v}(g) \coloneqq \langle \pi(g)u, v\rangle$ for $u, v \in \mathcal{H}_\pi$. Because π is defined to be strongly continuous, the maps $\phi_{u,v}$ are continuous for all $u, v \in \mathcal{H}_\pi$.

Definition 2.1.1. Suppose G is a non-compact topological group. G has the *Howe–Moore Property* if for every unitary representation of G without non-zero G-invariant vectors the matrix coefficients vanish at infinity.

A net in a topological space X is said to *go to infinity* if it has no accumulation points in X. A continuous function $f\colon X \to \mathbb{C}$ *vanishes at infinity* if for every net $\{x_i\}_{\in I}$ which goes to infinity, one has $f(x_i) \to 0$. If X is locally compact, this is equivalent to saying that for some $\epsilon > 0$ the set $\{x \in X\colon |f(x)| \geq \epsilon\}$ is compact.

Remark. A unitary representation $(\pi, \mathcal{H}_\pi)$ of G does not have non-zero G-invariant vectors if and only if none of its subrepresentations are unitarily equivalent to the trivial representation, that is, π has no trivial subrepresentations.

Remark. Notice that it makes no sense to define the Howe–Moore Property also for compact groups. Indeed, every continuous function on a compact topological space vanishes at infinity.

In the following some implications of the Howe–Moore property are explored. This leads to some easy criterions for determining that a group does not have the property.

Lemma 2.1.2 ([HT92, Chap. V, Prop. 1.2.6])**.** *Suppose G admits the Howe–Moore Property. Then, for every unitary representation $(\pi, \mathcal{H}_\pi)$ of G without non-zero G-invariant vectors, the closure of $\pi(G) \subset \mathcal{L}(\mathcal{H}_\pi)$ in the weak operator topology is $\pi(G) \cup \{0\}$.*

Proof. Let $\{g_i\}_{\in I}$ be a net in G, and suppose that it has no accumulation points. Then for every $u, v \in \mathcal{H}_\pi$ $\langle \pi(g_i)u, v\rangle \to 0$, that is, $\pi(g_i) \xrightarrow{w} 0$. On the other hand, if $g \in G$ is a limit point of $\{g_i\}_{i\in I}$, then, by the strong continuity of π, $\pi(g_i) \xrightarrow{w} \pi(g)$. This implies that the closure of $\pi(G) \subset \mathcal{L}(\mathcal{H}_\pi)$ in the weak operator topology is $\pi(G) \cup \{0\}$. □

Using this lemma one obtains

Proposition 2.1.3 ([CCL$^+$11, Prop. 2.7])**.** *Let G be a non-compact locally compact group, and suppose*

1) it admits a non-trivial finite-dimensional unitary representation, or

2) it has a proper closed non-compact normal subgroup.

Then G does not have the Howe–Moore Property.

Proof. Suppose first that G has a non-trivial finite-dimensional unitary representation $(\pi, \mathcal{H}_\pi)$. Recall that in a finite-dimensional Hilbert space the operator norm topology, the strong and the weak operator topologies coincide. Because $\mathcal{U}(\mathcal{H}_\pi)$ is closed in the former two topologies, and $0 \notin \mathcal{U}(\mathcal{H}_\pi)$, the closure of $\pi(G)$ cannot contain 0. By Lemma 2.1.2 this proves the first part of the proposition.

Next suppose that $N \lhd G$ is a proper closed non-compact normal subgroup. By the Gelfand-Raikov Theorem there is a non-trivial irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G/N, which gives a unitary representation $(\pi \circ q, \mathcal{H}_\pi)$ of G, where $q\colon G \to G/N$ is the canonical quotient map. Notice that $\pi \circ q$ is non-trivial and irreducible, and $N < \ker(\pi \circ q)$. But then there is a net $\{n_i\}_{\in I} \subset N$ which goes to infinity and such that for all non-zero $u \in \mathcal{H}_\pi$ $\phi_{u,u}(n_i) \to \phi_{u,u}(e_G) = \|u\| \neq 0$. □

Henceforth, for every $d \geq 1$, if F is a topological field, endow the groups $GL_d(F)$ and $SL_d(F)$ with the subspace topology they inherit from F^{d^2}. Then both are topological groups, and if F is local, then they are locally compact (see Lemma A.1.4).

Corollary 2.1.4. *If G has the Howe–Moore Property, then it cannot be solvable. In particular, $G = [G, G]$.*

Proof. Suppose that G is solvable. Then $G/[G, G]$ is non-trivial and abelian. Recall that the irreducible unitary representations of a locally compact abelian group are all one-dimensional. Hence, G has at least one non-trivial finite-dimensional unitary representation. □

Corollary 2.1.5. *For every $d \geq 1$ and every local field F, the group $GL_d(F)$ does not have the Howe–Moore Property.*

Proof. If $d = 1$, then $GL_1(F) = F^\times$ is abelian and thus solvable. Observe that for $d \geq 2$, $SL_d(F) = [GL_d(F), GL_d(F)]$ is a non-compact closed proper normal subgroup of $GL_d(F)$. The corollary follows from Corollary 2.1.4. □

2.2 An algebraic criterion for the Howe–Moore Property

In her paper [Cio15] Corina Ciobotaru presented an algebraic criterion for determining that a locally compact group has the Howe–Moore Property. In this section her proof is largely reproduced.

Theorem 2.2.1 ([Cio15, Thm. 1.2])**.** *Let G be a second countable non-compact locally compact group. Suppose that*

1) *there is decompositon $G = K_1 A^+ K_2$, where $K_1, K_2 < G$ are compact subgroups and $A^+ < G$ is an abelian subsemigroup, and that*
2) *for every sequence $\alpha = \{a_n\}_{\geq 1} \subset A^+$ going to infinity, $G = \overline{\langle U_\alpha^+, U_\alpha^- \rangle}$.*

Then G has the Howe–Moore Property.

Concerning U_α^+ and U_α^- in Theorem 2.2.1, see

Definition 2.2.2. Let G be a locally compact group, and let $\alpha = \{g_n\}_{n\geq 1} \subset G$ be a sequence.

- The *contraction groups* associated to α are

$$U_\alpha^+ := \{g \in G\colon g_n^{-1} g g_n \to e_G \text{ as } n \to \infty\},$$
$$U_\alpha^- := \{g \in G\colon g_n g g_n^{-1} \to e_G \text{ as } n \to \infty\}.$$

- The *parabolic groups* associated to α are

$$P_\alpha^+ \coloneqq \{g \in G \colon \{g_n^{-1} g g_n\}_{n\geq 1} \text{ is bounded}\},$$
$$P_\alpha^- \coloneqq \{g \in G \colon \{g_n g g_n^{-1}\}_{n\geq 1} \text{ is bounded}\}.$$

- The *Levi factor* attached to α is the group $M_\alpha \coloneqq P_\alpha^+ \cap P_\alpha^-$.

If $\alpha = \{g^n\}_{n\geq 1}$ for some $g \in G$, write $U_g^\pm \coloneqq U_\alpha^\pm$, $P_g^\pm \coloneqq P_\alpha^\pm$ and $M_g \coloneqq M_\alpha$.

A sequence in a locally compact space is said to be bounded if and only if it is contained in a compact subset. It can be shown that P_α^+ and P_α^- are always closed. Definition 2.2.2 will become useful again in Chapter 5. There also some results about the contraction and parabolic groups will be referenced from [BW04]. In this chapter only the contraction groups will be used.

Remark. Henceforth it will always be assumed that G is second countable, as it is done in [Cio15]. The advantage of this assumption is that instead of nets, one only needs to consider sequences.

Theorem 2.2.1 is proven using a sequence of auxiliary lemmas. The first one asserts that, given a decomposition $G = K_1 A^+ K_2$, for showing that G has the Howe–Moore Property it is sufficient to verify that the matrix coefficients on A^+ vanish at infinity.

Lemma 2.2.3 ([BM00, Chap. III, Lem. 1.3])**.** *Let G be a second countable topological group, and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Suppose that $G = K_1 A^+ K_2$ is a decomposition as in Theorem 2.2.1, and that $\phi_{u,v}(a_n) \to 0$ as $n \to \infty$, for every $u, v \in \mathcal{H}_\pi$ and every sequence $\{a_n\}_{n\geq 1} \subset A^+$ going to infinity. Then the matrix coefficients of π vanish at infinity.*

Proof. Let $\{g_n\}_{n\geq 1} \subset G$ be a sequence going to infinity, that is, having no accumulation points. For every $n \geq 1$ let $k_n^{(1)} \in K_1$, $k_n^{(2)} \in K_2$ and $a_n \in A^+$ be such that $g_n = k_n^{(1)} a_n k_n^{(2)}$. By choosing a subsequence if necessary, assume that, for $i = 1, 2$, there is $k^{(i)} \in K_i$ such that $k_n^{(i)} \to k^{(i)}$. Moreover, note that $\{a_n\}_{n\geq 1}$ goes to infinity. Fix $u, v \in \mathcal{H}_\pi$, and let $\epsilon > 0$. Then there is $N \geq 1$ such that for all $n \geq N$ $|\langle \pi(a_n)(\pi(k^{(2)})u), \pi((k^{(1)})^{-1})v\rangle| < \frac{\epsilon}{3}$ and $\|(\pi(k_n^{(2)}) - \pi(k^{(2)}))u\| < \frac{\epsilon}{3}$, $\|(\pi(k_n^{(1)})^{-1} - \pi(k^{(1)})^{-1})v\| < \frac{\epsilon}{3}$. By the Cauchy-Schwarz inequality, for all $n \geq N$

$$\begin{aligned}
|\langle \pi(g_n)u, v\rangle| &= |\langle \pi(a_n)(\pi(k_n^{(2)})u), \pi((k_n^{(1)})^{-1})v\rangle| \\
&\leq |\langle \pi(a_n)(\pi(k^{(2)})u), \pi(k^{(1)})^{-1}v\rangle| \\
&\quad + |\langle \pi(a_n)(\pi(k_n^{(2)}) - \pi(k^{(2)}))u, \pi(k^{(1)})^{-1}v\rangle| \\
&\quad + |\langle \pi(a_n)(\pi(k_n^{(2)})u), (\pi(k_n^{(1)})^{-1} - \pi(k^{(1)})^{-1})v\rangle| \\
&\leq |\langle \pi(a_n)(\pi(k^{(2)})u), \pi(k^{(1)})^{-1}v\rangle| + \|(\pi(k_n^{(2)}) - \pi(k^{(2)}))u\| \\
&\quad + \|(\pi(k_n^{(1)})^{-1} - \pi(k^{(1)})^{-1})v\| < \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ can be chosen arbitrarily small, the matrix coefficients of π vanish at infinity. □

Recall now that, given a Hilbert space $\mathcal{H}$, by the Banach-Alaoglu Theorem every sequence in $\mathcal{U}(\mathcal{H})$ has a subsequence which converges in the weak operator topology.

Lemma 2.2.4 (Mautner Phenomenon, [BM00, Chap. III, Thm. 1.4])**.** *Let G be a locally compact group, and let $\alpha = \{g_n\}_{n\geq 1}$ be a sequence. Suppose that $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G and that there are $u, v \in \mathcal{H}_\pi$ such that $\pi(g_n)u \overset{w^*}{\rightharpoonup} v$. Then for all $g \in \overline{U_\alpha^+}$ $\pi(g)v = v$.*

Proof. It is sufficient to show that $\pi(g)v = v$ for all $g \in U_\alpha^+$. Then for all $w \in \mathcal{H}_\pi$

$$\begin{aligned}|\langle \pi(g)v, w\rangle - \langle v, w\rangle| &= \lim_{n\to\infty} |\langle \pi(g)(\pi(g_n)u), w\rangle - \langle \pi(g_n)u, w\rangle| \\ &= \lim_{n\to\infty} |\langle \pi(g_n^{-1}gg_n)u, \pi(g_n^{-1})w\rangle - \langle u, \pi(g_n^{-1})w\rangle| \\ &\leq \lim_{n\to\infty} \|(\pi(g_n^{-1}gg_n) - \pi(e_G))u\|\|\pi(g_n^{-1})w\| \\ &\leq \lim_{n\to\infty} \|(\pi(g_n^{-1}gg_n) - \pi(e_G))u\|\|w\| = 0,\end{aligned}$$

where for the first inequality the Cauchy-Schwarz inequality was used. Since $w \in \mathcal{H}_\pi$ was chosen arbitrarily, $\pi(g)v = v$. □

Recall that a linear operator $T \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called *normal* if $TT^* = T^*T$. The next proposition is key to the proof of Theorem 2.2.1. For it the following lemma is needed:

Lemma 2.2.5 ([Cio15, Lem. 2.13])**.** *Let $\mathcal{H}$ be a Hilbert space, and suppose $\{U_n\}_{n\geq 1} \subset \mathcal{U}(\mathcal{H})$ is a sequence of unitary operators which commute pairwise. Suppose that the sequence converges to $T \in \mathcal{L}(\mathcal{H})$ in the weak operator topology. Then T is normal.*

Proof. Suppose that $U_n \overset{w}{\to} T$. Then, $U_n^* \overset{w}{\to} T^*$, and for all $u, v \in \mathcal{H}$,

$$\begin{aligned}\langle (TT^*)u, v\rangle &= \lim_{m\to\infty}\lim_{n\to\infty} \langle (U_mU_n^*)u, v\rangle = \lim_{m\to\infty}\lim_{n\to\infty} \langle (U_nU_m)(U_n^*u), U_nv\rangle \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \langle (U_mU_n)(U_n^*u), U_nv\rangle = \lim_{m\to\infty}\lim_{n\to\infty} \langle U_mu, U_nv\rangle \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \langle (U_n^*U_m)u, v\rangle = \langle (T^*T)u, v\rangle\end{aligned}$$

Hence $TT^* = T^*T$. □

Proposition 2.2.6 ([Cio15, Lem. 3.1])**.** *Let G be a second countable locally compact group, and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Let $\alpha = \{g_n\}_{n\geq 1} \subset G$ be a sequence of pairwise commuting elements. Suppose that α goes to infinity and that $\phi_{u,v}(g_n) \not\to 0$ for some $u, v \in \mathcal{H}_\pi$. Then there exists a non-zero vector in $\mathcal{H}_\pi$ which is $\overline{\langle U_\alpha^+, U_\alpha^-\rangle}$-invariant.*

Proof. By passing to a subsequence if necessary, assume that $\pi(g_n) \overset{w}{\to} T \in \mathcal{L}(\mathcal{H}_\pi)$ as $n \to \infty$. Recall that this is possible by the Banach-Alaoglu Theorem. By assumption $T \neq 0$. Notice that $\pi(g_n^{-1}) \overset{w}{\to} T^*$ and that, by Lemma 2.2.5, T is a normal operator. Moreover, by the Mautner Phenomena, for all $u_+ \in U_\alpha^+$ $\pi(u_+)T = T$ and for all $u_- \in U_\alpha^-$ $\pi(u_-)T^* = T^*$. Therefore every $v \in T(\mathcal{H}_\pi) \cap T^*(\mathcal{H}_\pi)$ is $\overline{\langle U_\alpha^+, U_\alpha^-\rangle}$-invariant. Observe now that for all $v \in \mathcal{H}_\pi$

$$\|Tv\|^2 = \langle (T^*T)v, v\rangle = \langle (TT^*)v, v\rangle = \|T^*v\|^2.$$

Since $T \neq 0$, $T^*T = TT^* \neq 0$, and thus $T(\mathcal{H}_\pi) \cap T^*(\mathcal{H}_\pi) \neq \{0\}$. □

Proof of Theorem 2.2.1. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G without trivial subrepresentations. By Lemma 2.2.3 it is enough to show that the matrix coefficients of π vanish on A^+ at infinity. For this fix a sequence $\alpha = \{a_n\}_{n\geq 1} \subset A^+$ which goes to infinity. Argue by contradiction and suppose $\phi_{u,v}(a_n) \not\to 0$. Then, because $G = \overline{\langle U_\alpha^+, U_\alpha^-\rangle}$ and by Proposition 2.2.6, there exists a non-zero G-invariant vector in $\mathcal{H}_\pi$, which yields a contradiction. Hence $\phi_{u,v}(a_n) \to 0$ for all $u, v \in \mathcal{H}_\pi$. □

2.3 The Howe–Moore Property for special linear groups over local fields

The goal of this section is to show that $SL_d(F)$ admits the Howe–Moore Property for every $d \geq 2$ and every local field F. This is a first application of Theorem 2.2.1. As a first step one needs to show that $SL_d(F)$ admits a decomposition as the one in Theorem 2.2.1 for every $d \geq 2$ and local field F. This decomposition is proven separately for archimedean and non-archimedean local fields.

For any local field F and $d \geq 1$, fix the notation as follows:

- Let $P_d^+(F), P_d^-(F) < GL_d(F)$ be the the subgroups of invertible upper and lower triangular $d \times d$ matrices, respectively. Let $\widetilde{P}_d^{\pm}(F) \coloneqq P_d^{\pm}(F) \cap SL_d(F)$.
- Let $N_d^+(F) < P_d^+(F)$ and $N_d^-(F) < P_d^-(F)$ be the subgroups of upper and lower triangular matrices, respectively, whose diagonal entries are 1. Note that $N_d^{\pm}(F) < SL_d(F)$.

Let $A_d < GL_d(\mathbb{R})$ be the subgroup of invertible diagonal matrices whose diagonal entries are in $\mathbb{R}_{>0}$, and set $\widetilde{A}_d \coloneqq A_d \cap SL_d(\mathbb{R})$. Also set

$$A_d^+ \coloneqq \{diag(a_1, a_2, \dots, a_d) \colon a_1 \geq a_2 \geq \cdots \geq a_d,\ a_i \in \mathbb{R}_{>0}\ \forall 1 \leq i \leq d\},$$

which is an abelian subsemigroup of $GL_d(\mathbb{R})$ and of $GL_d(\mathbb{C})$. Let $\widetilde{A}_d^+ \coloneqq A_d^+ \cap SL_d(\mathbb{R})$. Next suppose F is non-archimedean, and fix a uniformizing element $\mathfrak{p} \in F$. Let

$$T_d(F) \coloneqq \{diag(\mathfrak{p}^{k_1}, \mathfrak{p}^{k_2}, \dots, \mathfrak{p}^{k_d}) \colon k_i \in \mathbb{Z}\ \forall 1 \leq i \leq d\},$$

which is an abelian subgroup of $GL_d(F)$, and set $\widetilde{T}_d(F) \coloneqq T_d(F) \cap SL_d(F)$. Similarly to A_d^+ and $\widetilde{A}_d^+$, define

$$T_d^+(F) \coloneqq \{diag(\mathfrak{p}^{k_1}, \mathfrak{p}^{k_2}, \dots, \mathfrak{p}^{k_d}) \colon k_1 \leq k_2 \leq \cdots \leq k_d,\ k_i \in \mathbb{Z}\ \forall 1 \leq i \leq d\}$$

and $\widetilde{T}_d^+(F) \coloneqq T_d^+(F) \cap SL_d(F)$.

Finally for $d \geq 2$ and $1 \leq l \leq d-1$, define the subgroup

$$N_{d,l}^+(F) \coloneqq \left\{ \begin{pmatrix} n_l & x \\ & n_{d-l} \end{pmatrix} \colon n_l \in N_l^+(F),\ n_{d-l} \in N_{d-l}^+(F),\ x \in F^{l \times (d-l)} \right\} < N_d^+(F).$$

Similarly, define the subgroup $N_{d,l}^-(F) < N_d^-(F)$, that is,

$$N_{d,l}^-(F) \coloneqq \left\{ \begin{pmatrix} n_l & \\ x & n_{d-l} \end{pmatrix} \colon n_l \in N_l^+(F),\ n_{d-l} \in N_{d-l}^+(F),\ x \in F^{(d-l) \times l} \right\}.$$

The decompositions in Theorem 2.3.1 and Theorem 2.3.2 are called *Cartan decompositions.*

Theorem 2.3.1 (Cartan decomposition over $F \in \{\mathbb{R}, \mathbb{C}\}$, [Hal15, Thm. 2.17])**.** *For $d \geq 1$, the following holds:*

1) $GL_d(\mathbb{R}) = O(d)A_d^+O(d)$ *and* $SL_d(\mathbb{R}) = SO(d)\widetilde{A}_d^+SO(d)$,

2) $GL_d(\mathbb{C}) = U(d)A_d^+U(d)$ *and* $SL_d(\mathbb{C}) = SU(d)\widetilde{A}_d^+SU(d)$.

Proof. The decomposition is shown here only for the case of $F = \mathbb{C}$. The case of $F = \mathbb{R}$ is done similarly. Moreover it is enough to show the decomposition for $GL_d(\mathbb{C})$. For this let $x \in GL_d(\mathbb{C})$, and take $p \coloneqq x^*x$. Then p is a positive self-adjoint linear map. Therefore, by the spectral theorem for self-adjoint linear maps, there is $q \in U(d)$ such that

$$p = q^* \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix} q,$$

where $\lambda_i \in \mathbb{R}_{>0}$. Take the *square root*

$$p^{\frac{1}{2}} \coloneqq q^* \begin{pmatrix} \lambda_1^{\frac{1}{2}} & & & \\ & \lambda_2^{\frac{1}{2}} & & \\ & & \ddots & \\ & & & \lambda_d^{\frac{1}{2}} \end{pmatrix} q,$$

and set $u \coloneqq xp^{-\frac{1}{2}}$. Observe that

$$u^*u = p^{-\frac{1}{2}}(x^*x)p^{-\frac{1}{2}} = I_d,$$

that is, $u \in U(d)$. Hence, with $q \in U(d)$ and $a \in A_d$ such that $p^{-\frac{1}{2}} = q^*aq$, one has that $x = (uq^*)a^{-1}q \in U(d)A_dU(d)$. Since for every $a \in A_d$ there are permutation matrices $u_1, u_2 \in U(d)$ such that $u_1au_2 \in A_d^+$, the decomposition follows. □

The proof of the following theorem follows the ideas on [AB08, p. 353].

Theorem 2.3.2 (Cartan decompositon over non-archimedean local fields)**.** *Let $d \geq 1$. Let F be a non-archimedean local field, and let $\mathcal{O}$ be its ring of integers. Then*

$$GL_d(F) = GL_d(\mathcal{O})T_d^+(F)GL_d(\mathcal{O})$$

and

$$SL_d(F) = SL_d(\mathcal{O})\widetilde{T}_d^+(F)SL_d(\mathcal{O}).$$

Proof. Fix a non-archimedean local field F, with valuation $|\cdot|$. As in the proof of Theorem 2.3.1 it is enough to just show that the decomposition holds for $GL_d(F)$. For this argue by induction on d. The base case of $d = 1$ is trivial. For the induction step, assume that the decomposition holds for some $d \geq 1$, and let $x \in GL_{d+1}(F)$. Then, by interchanging columns and rows, there are permutation matrices $y_1, y_2 \in GL_{d+1}(\mathcal{O})$ such that $|(y_1xy_2)_{11}| \geq |(y_1xy_2)_{ij}|$ for all $1 \leq i, j \leq d+1$. Hence for every $1 \leq i, j \leq d+1$ there is $\lambda_{ij} \in \mathcal{O}$ such that $(y_1xy_2)_{ij} = \lambda_{ij}(y_1xy_2)_{11}$. Therefore there are $y_1', y_2' \in GL_{d+1}(\mathcal{O})$ such that

$$(y_1'y_1)x(y_2y_2') = \begin{pmatrix} (y_1xy_2)_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x' & \\ 0 & & & \end{pmatrix},$$

where $x' \in GL_d(F)$. Let $k \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$ be such that $(y_1xy_2)_{11} = u\mathfrak{p}^k$. By the induction hypothesis there are $z, z' \in GL_d(\mathcal{O})$ and $t \in T_d^+(F)$ such that $x' = ztz'$. Thus

$$x = (y_1'y_1)^{-1} \begin{pmatrix} u & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & z & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \mathfrak{p}^k & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & t & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & z' & \\ 0 & & & \end{pmatrix} (y_2y_2'),$$

which implies $x \in GL_{d+1}(\mathcal{O})T_{d+1}(F)GL_{d+1}(\mathcal{O})$. Notice that by a similar argument as in the proof of Theorem 2.3.1,

$$GL_{d+1}(\mathcal{O})T_{d+1}(F)GL_{d+1}(\mathcal{O}) = GL_{d+1}(\mathcal{O})T^+_{d+1}(F)GL_{d+1}(\mathcal{O}),$$

which proves the decomposition for $d+1$. □

Theorem 2.3.1 and Theorem 2.3.2 show that, for $d \geq 2$ and for a local field F, $SL_d(F)$ satisfies the first condition in Theorem 2.2.1. For the second condition one needs the following two lemmas. The proof of Lemma 2.3.3 is inspired by [HT92, Chap. V, Lem. 2.0.2]

Lemma 2.3.3. *Let F be a local field, and let $d \geq 2$. Let $\alpha = \{a_k\}_{k\geq 1}$ be a sequence of diagonal matrices $a_k = diag(a_k^{(1)}, a_k^{(2)}, \ldots, a_k^{(d)})$ in $\widetilde{A}_d^+$ and $\widetilde{T}_d^+(F)$, respectively, and suppose that α goes to infinity. Then there exists $1 \leq l \leq d-1$ such that $N^+_{d,l}(F) \leq U^+_\alpha$ and $N^-_{d,l}(F) \leq U^-_\alpha$.*

Proof. By assumption, the sequences $\{a_k^{(i)}\}_{k\geq 1} \subset F$ go to infinity, $1 \leq i \leq d$. Then, because $|a_k^{(1)}| \geq |a_k^{(2)}| \geq \cdots \geq |a_k^{(d)}|$, there is $1 \leq l \leq d-1$ such that $a_k^{(l)}(a_k^{(1)})^{-1} \not\to 0$ and $a_k^{(l+1)}(a_k^{(1)})^{-1} \to 0$ as $k \to \infty$. Hence,

$$\frac{a_k^{(j)}}{a_k^{(i)}} = \frac{a_k^{(1)}}{a_k^{(i)}}\frac{a_k^{(j)}}{a_k^{(1)}} \to 0 \quad \text{as } k \to \infty,\ \forall 1 \leq i \leq l < j \leq d.$$

Next fix $n \in N^+_{d,l}(F)$. Then, for all $k \geq 1$ and all $1 \leq i, j \leq d$, $(a_k^{-1}na_k)_{ij} = (a_k^{(i)})^{-1}n_{ij}a_k^{(j)}$, and thus

$$\lim_{k\to\infty}(a_k^{-1}na_k)_{ij} = \begin{cases} \lim_{k\to\infty} n_{ij}(a_k^{(i)})^{-1}a_k^{(j)} = 0 \text{ if } 1 \leq i \leq l < j \leq d, \\ 1 \text{ if } i = j, \\ 0 \text{ otherwise,} \end{cases}$$

Hence $a_k^{-1}na_k \to I_d$. Similarly it can be shown that $N^-_{d,l}(F) \leq U^-_\alpha$. □

Lemma 2.3.4. *Let F be a field, and let $d \geq 2$. Then, for all $1 \leq l \leq d-1$, $SL_d(F) = \langle N^+_{d,l}(F), N^-_{d,l}(F)\rangle$.*

Proof. Argue by induction on d. The base case of $d = 2$ is immediate. For the induction step, suppose that the claim holds for some fixed $d \geq 2$. Fix $2 \leq l \leq d$ and take $x \in SL_{d+1}(F)$. Then, by using linear operations on the columns and the rows, there are $n_+ \in N^+_{d+1,l}(F)$ and $n_- \in N^-_{d+1,l}(F)$ such that

$$n_+xn_- = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x' & \\ 0 & & & \end{pmatrix},$$

where $x' \in SL_d(F)$. By the induction hypothesis $x' \in \langle N^+_{d,l-1}(F), N^-_{d,l-1}(F)\rangle$. Note that the mapping

$$N^\pm_{d,l-1}(F) \ni y \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & y & \\ 0 & & & \end{pmatrix} \in N^\pm_{d+1,l}(F)$$

defines a topological embedding and a group homomorphism. Hence n_+xn_- and therefore x are contained in $\langle N^+_{d+1,l}(F), N^-_{d+1,l}(F)\rangle$. The case of $l = 1$ can be treated similarly. □

Theorem 2.3.5. *For every local field F and every $d \geq 2$, the group $SL_d(F)$ has the Howe–Moore Property.*

Proof. As mentioned above, the decomposition in 1) of Theorem 2.2.1 is given in the archimedean case by Theorem 2.3.1 and in the non-archimedean case by Theorem 2.3.2. Let α be a sequence in the respective abelian subsemigroup. Then, by Lemma 2.3.3 and by Lemma 2.3.4, there exists $1 \leq l \leq d-1$ such that

$$SL_d(F) = \langle N^+_{d,l}(F), N^-_{d,l}(F) \rangle \leq \langle U^+_\alpha, U^-_\alpha \rangle \leq SL_d(F).$$

Therefore it follows from Theorem 2.2.1 that $SL_d(F)$ has the Howe–Moore Property. □

2.4 Vanishing theorem for semisimple Lie groups

The goal of this section is to show that all connected non-compact semisimple real Lie groups with finite center enjoy a generalized version of the Howe–Moore Property. It is a special case of that theorem that all connected non-compact simple real Lie groups with finite center have the Howe–Moore Property.

Suppose G is a Lie group, and let $\mathfrak{g}$ denote its Lie algebra. Recall that for every Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ there is an embedded Lie subgroup $H < G$ such that $\mathfrak{h}$ is the Lie algebra of H. In particular, if $\mathfrak{h}$ is an ideal, H is a normal subgroup of G. Moreover, every closed subgroup of G is an embedded Lie subgroup. Recall also that if $f\colon G \to H$ is a Lie group homomorphism with Lie groups G and H, then the differential of f at the identity element $e_G \in G$ defines a Lie algebra homomorphism $d_{e_G} f\colon \mathfrak{g} \to \mathfrak{h}$.

Lemma 2.4.1. *Suppose $\mathfrak{g} = \bigoplus_{1 \leq i \leq n} \mathfrak{h}_i$, where each $\mathfrak{h}_i \subset \mathfrak{g}$ is a Lie subalgebra, and let $H_i < G$ be the Lie subgroup with Lie algebra $\mathfrak{h}_i$. Then there are open neighbourhoods $U_i \subset H_i$ of e_G such that $U \coloneqq U_1 \cdot U_2 \cdots U_n$ is an open neighbourhood of e_G in G.*

Proof. Let $m\colon \prod_{1 \leq i \leq n} H_i \to G$, $(h_1, h_2, \ldots, h_n) \mapsto \prod_{1 \leq i \leq n} h_i$ be the multiplication map. Then the differential $d_{e_G} m\colon \bigoplus_{1 \leq i \leq n} \mathfrak{h}_i \to \mathfrak{g}$ is given by

$$d_{e_G} m(X_1, X_2, \ldots, X_n) = \sum_{1 \leq i \leq n} X_i.$$

Thus $d_{e_G} m$ is invertible, and hence, by the inverse function theorem of differential calculus, m is a local diffeomorphism at $(e_{H_1}, e_{H_2}, \ldots, e_{H_n}) \in \prod_{1 \leq i \leq n} H_i$. □

Henceforth suppose G is a connected non-compact semisimple Lie group with finite center. Then $\mathfrak{g} = \bigoplus_{1 \leq i \leq n} \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is a simple ideal of $\mathfrak{g}$. Let $S_i \lhd G$ be the closed normal subgroups such that $\mathfrak{g}_i$ is the Lie algebra of S_i. Observe that $S_i < Z_G(S_j)$ for all $i \neq j$, that is, every element in S_i commutes with every element in S_j. By Lemma 2.4.1, there is an open neighbourhood $U \subset G$ of e_G such that $U = U_1 \cdot U_2 \cdots U_n$, where each $U_i \subset S_i$ is an open neighbourhood of $e_G \in S_i$. Then for all $m \geq 1$ $U^m = U_1^m \cdot U_2^m \cdots U_n^m$, and because G is connected, by Lemma 2.4.1, this implies $G = S_1 \cdot S_2 \cdots S_n$ (see Lemma A.1.6). The normal subgroups $S_i \lhd G$ are called the *factors* of G. Because G is non-compact, there is at least one factor which is non-compact.

Theorem 2.4.2 (Vanishing Theorem for semisimple Lie groups, [BM00, Chap. III, Theorem 1.1])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G, and suppose that for every non-compact factor $S_i \lhd G$ π does not have non-zero S_i-invariant vectors. Then the matrix coefficients of π vanish at infinity.*

Because each factor of G is simple, the following corollary is immediate.

Corollary 2.4.3. *If G is simple, it has the Howe–Moore Property.*

Remark. Because for all $d \geq 2$ $SL_d(\mathbb{R})$ and $SL_d(\mathbb{C})$ are connected non-compact simple Lie groups with trivial center, the archimedean case of Theorem 2.3.5 is a special case of Corollary 2.4.3.

Remark. One can see from Theorem 2.4.2 how a generalized version of the Howe–Moore Property can be formulated; see [Lob19, Def. 3]. Indeed, the vanishing theorem is a special case of [Lob19, Cor. 1].

The proof of Theorem 2.4.2 that is given here is largely inspired by the proofs of [Lob19, Thm.] and [Lob19, Cor. 1]. It uses some of the auxiliary results in Section 2.2. In particular, one needs a Cartan decomposition for G. The following theorem is stated here without proof:

Theorem 2.4.4 (Cartan decomposition for semisimple Lie groups, [Hel78, Chap. IX, Thm. 1.1])**.** *There is a maximal compact subgroup $K < G$, which is unique up to conjugation, and an abelian subsemigroup $A^+ \subset G$ such that*

$$G = K\overline{A^+}K.$$

From now on let $G = K\overline{A^+}K$ be the Cartan decomposition of G, where $K < G$ is a maximal compact subgroup and $A^+ \subset G$ is an abelian subsemigroup. For the proof of Theorem 2.4.2 the following lemma is needed:

Lemma 2.4.5 ([BM00, Chap. III, Lem. 1.8])**.** *For every sequence $\alpha \subset \overline{A^+}$ going to infinity, $\overline{\langle U_\alpha^+, U_\alpha^- \rangle}$ is a non-discrete normal subgroup of G.*

The proof of Lemma 2.4.5 in [BM00] uses the structure theory of non-compact semisimple Lie groups. Since this would go beyond the frame of this thesis, the proof is omitted here.

Notice how, using Theorem 2.4.4 and Lemma 2.4.5, one can deduce from Theorem 2.2.1 the Howe–Moore Property in the case where G is simple.

Proof of Theorem 2.4.2. Suppose $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G such that for every non-compact factor $S_i \triangleleft G$ π does not have non-zero S_i-invariant vectors. Argue by contradiction, and assume there are a sequence $\alpha = \{g_k\}_{k\geq 1} \subset G$ going to infinity and vectors $u, v \in \mathcal{H}_\pi$ such that $\phi_{u,v}(g_k) \not\to 0$. For every $1 \leq i \leq d$, let $S_i = K_i\overline{A_i^+}K_i$ be the Cartan decomposition. Because the factors of G commute pairwise, one has

$$G = K'A'K',$$

where $A' \coloneqq \overline{A_1^+} \cdot \overline{A_2^+} \cdots \overline{A_n^+}$ is an abelian subsemigroup and $K' \coloneqq K_1 \cdot K_2 \cdots K_n$ is a compact subgroup of G. Then, by Lemma 2.2.3, there is a sequence $\beta = \{a_k\}_{k\leq 1} \subset A'$ going to infinity such that $\phi_{u,v}(a_k) \not\to 0$. For every $k \geq 1$, write $a_k = a_k^{(1)}a_k^{(2)} \cdots a_k^{(n)}$, where $a_k^{(i)} \in S_i$. Note that there is $1 \leq i \leq n$ such that a subsequence of $\{a_k^{(i)}\}_{k\geq 1}$ goes to infinity. Then, by Lemma 2.4.5 and because S_i is simple, $S_i = \overline{\langle U_{\beta^{(i)}}^+, U_{\beta^{(i)}}^- \rangle}$ and hence $S_i < \overline{\langle U_\beta^+, U_\beta^- \rangle}$. But then, by Proposition 2.2.6, there is a non-zero S_i-invariant vector in $\mathcal{H}_\pi$, which yields a contradiction. □

Chapter 3

Spherical Functions and Representations

Throughout this chapter suppose that G is a locally compact group and that $K < G$ is a compact subgroup. Fix a left Haar measure dx on G, and let dk be the normalized Haar measure on K. Let $L^1(G)$ be the group algebra of G.

3.1 Gelfand pairs

A function f on G is said to be *bi-K-invariant* if for all $k \in K$ $L_k f = f = R_k f$. An example of such a function would be the modular function Δ_G of G. Let $L^1(G, K)$ be the subset of all bi-K-invariant functions in $L^1(G)$. Similary, let $C(G, K)$ and $C_c(G, K)$ be the set of all continuous and the set of all compactly supported continuous complex-valued bi-K-invariant functions on G, respectively.

Proposition 3.1.1. *$L^1(G, K)$ is a closed $*$-subalgebra of $L^1(G)$. Moreover, the map $f \mapsto {}^K f^K$ on $L^1(G)$, given by*

$$ {}^K f^K(g) \coloneqq \int_K \int_K f(k_1 g k_2)\, dk_1\, dk_2, \tag{3.1} $$

is an isometric projection of $L^1(G)$ onto $L^1(G, K)$.

Proof. It is clear that $L^1(G, K)$ is a linear subspace of $L^1(G)$ and that $f \mapsto {}^K f^K$ is a projection. Because Δ_G is bi-K-invariant, one has that for all $f \in C_c(G)$

$$ \begin{aligned} \|f\|_1 &= \int_G |f(x)|\, dx \\ &= \int_K \int_K \Delta_G(k_2) \int_G |f(k_1 x k_2)|\, dx\, dk_1\, dk_2 \\ &= \int_G \int_K \int_K |f(k_1 x k_2)|\, dk_1\, dk_2\, dx = \|{}^K f^K\|_1. \end{aligned} $$

Since $C_c(G)$ is dense in $L^1(G)$, $f \mapsto {}^K f^K$ is an isometry. This can be used to show that $L^1(G, K)$ is closed in $L^1(G)$. Indeed, suppose that $\{f_n\}_{n \geq 1}$ is a sequence in $L^1(G, K)$ which converges to $f \in L^1(G)$. Then

$$ \|f_n - {}^K f^K\|_1 = \|{}^K f_n^K - {}^K f^K\|_1 = \|f_n - f\|_1 \to 0 $$

as $n \to \infty$. By the uniqueness of the limit, $f = {}^K f^K \in L^1(G, K)$. To show that $L^1(G, K)$ is a

subalgebra, take $f_1, f_2 \in L^1(G,K)$ and $g \in G$, $k_1, k_2 \in K$, and observe that

$$\begin{aligned} f_1 * f_2(k_1 g k_2) &= \int_G f_1(x) f_2(x^{-1} k_1 g k_2)\, dx \\ &= \int_G f_1(k_1 x) f_2(x^{-1} g)\, dx = f_1 * f_2(g); \end{aligned}$$

that is, $L^1(G,K)$ is closed under convolution. Moreover, by the bi-K-invariance of Δ_G, one has that for all $f \in L^1(G,K)$ and all $g \in G$, $k_1, k_2 \in K$

$$f^*(k_1 g k_2) = \Delta_G((k_1 g k_2)^{-1}) \overline{f((k_1 g k_2)^{-1})} = \Delta_G(g^{-1}) \overline{f(g^{-1})} = f^*(g).$$

It follows that $L^1(G,K)$ is a $*$-subalgebra of $L^1(G)$. □

Henceforth let $L^1(G,K)$ denote the closed $*$-subalgebra of $L^1(G)$ of bi-K-invariant functions.

Definition 3.1.2. A Gelfand pair is a pair (G,K) such that $L^1(G,K)$ is commutative, that is, for all $f_1, f_2 \in L^1(G,K)$ one has that $f_1 * f_2 = f_2 * f_1$.

Ways to determine that (G,K) is a Gelfand pair are given by Proposition 3.1.3 and Corollary 3.1.4. Recall that a bijective map $\theta\colon G \to G$ is called *involutive* if $\theta^2 = id_G$. Also recall the definition of the algebraic group homomorphism $\mathrm{mod}_G\colon \mathrm{Aut}(G) \to (\mathbb{R}_{>0}, \cdot)$ (see Corollary A.3.5).

Proposition 3.1.3 ([vD09, Prop. 6.1.3])**.** *Suppose there exists a continuous involutive automorphism θ of G such that $\theta(g) \in Kg^{-1}K$ for all $g \in G$. Then (G,K) is a Gelfand pair.*

Proof. Because $\theta^2 = id_G$, $(\mathrm{mod}_G \theta)^2 = \mathrm{mod}_G \theta^2 = 1$. Therefore, $\mathrm{mod}_G \theta = 1$ and for all $f_1, f_2 \in C_c(G,K)$ $(f_1 * f_2) \circ \theta = (f_1 \circ \theta) * (f_2 \circ \theta)$. Moreover, $f_i \circ \theta = f_i^\vee$, for $i = 1, 2$, and

$$f_1 * f_2 = ((f_1 \circ \theta) * (f_2 \circ \theta)) \circ \theta = (f_1^\vee \circ f_2^\vee)^\vee = f_2 * f_1.$$

For the last equality it was used that $(f_1 * f_2)^\vee = f_2^\vee * f_1^\vee$. The proposition follows from the density of $C_c(G,K)$ in $L^1(G,K)$. □

Recall that an anti-automorphism $\tau\colon G \to G$ is a bijective map such that $\tau(gh) = \tau(h)\tau(g)$ for all $g, h \in G$. The following corollary is a stronger version of [Lan85, Chap. IV, Thm. 1].

Corollary 3.1.4. *Suppose τ is a continuous involutive anti-automorphism on G such that $\tau(g) \in KgK$ for all $g \in G$. Then (G,K) is a Gelfand pair.*

Proof. Let $i_G\colon G \to G$, $g \mapsto g^{-1}$ denote the inverse map. Then $\theta := i_G \circ \tau$ is a continuous involutive automorphism on G where for all $g \in G$ $\theta(g) \in Kg^{-1}K$. The corollary follows from Proposition 3.1.3. □

A first application of this corollary is

Corollary 3.1.5. *Let $d \geq 1$. Then the following are Gelfand pairs:*

1) *$(GL_d(\mathbb{R}), O(d))$ and $(SL_d(\mathbb{R}), SO(d))$;*
2) *$(GL_d(\mathbb{C}), U(d))$ and $(SL_d(\mathbb{C}), SU(d))$;*
3) *$(GL_d(F), GL_d(\mathcal{O}))$ and $(SL_d(F), SL_d(\mathcal{O}))$, where F is a non-archimedean local field and $\mathcal{O}$ is its ring of integers.*

Proof. Let (G,K) be one of the above pairs. By Theorem 2.3.1 and by Theorem 2.3.2, there is an abelian subsemigroup $A^+ \subset G$ of diagonal matrices such that $G = KA^+K$. Let $\tau\colon G \to G, g \mapsto g^t$ be the transpose. It is an involutive anti-automorphism and fixes A^+ pointwise. Because K is invariant under τ, for all $g \in G$ $\tau(g) \in KgK$. Therefore, by Corollary 3.1.4, (G,K) is a Gelfand pair. □

Another interesting result concerning Gelfand pairs is

Proposition 3.1.6 ([vD09, Prop. 6.1.2])**.** *If (G, K) is a Gelfand pair, then G is unimodular.*

For the proof of Proposition 3.1.3 the following elementary lemma is used.

Lemma 3.1.7. *G is unimodular if and only if*

$$\int_G f(x)\,dx = \int_G f(x^{-1})\,dx \quad \forall f \in C_c(G, K). \tag{3.2}$$

Proof. By Proposition A.3.10,

$$\int_G f(x)\,dx = \int_G f(x^{-1})\Delta_G(x^{-1})\,dx \quad \forall f \in C_c(G).$$

Thus, if G is unimodular, the implication is clear.

For the converse argue by contradiction. Suppose that (3.2) holds for all $f \in C_c(G, K)$ and that G is not unimodular, that is, $\Delta_G \neq 1$. Then there are $\epsilon > 0$ and an open set $U \subset G$ with compact closure such that $\Delta_G(g) \geq 1 + \epsilon$ for all $g \in U$. By Urysohn's Lemma, there exists a function $f \in C_c(G)$ such that $f \geq 0$ and $\operatorname{supp}(f) \subset U$. Then, by the bi-$K$-invariance of the Haar measure dx,

$$\begin{aligned}\int_G f(x)\,dx &= \int_K \int_K \int_G f(k_1 x k_2)\,dx\,dk_1\,dk_2 \\ &= \int_G {}^K f^K(x)\,dx = \int_G {}^K f^K(x^{-1})\,dx \\ &= \int_K \int_K \int_G f(k_1 x^{-1} k_2)\,dx\,dk_1\,dk_2 = \int_G f(x^{-1})\,dx.\end{aligned}$$

This implies that

$$\int_G f(x)\,dx = \int_G f(x^{-1})\Delta_G(x^{-1})\,dx \geq (1+\epsilon)\int_G f(x^{-1})\,dx = (1+\epsilon)\int_G f(x)\,dx,$$

which yields a contradiction. □

Proof of Proposition 3.1.6. Let $f \in C_c(G, K)$. By Urysohn's Lemma, there exists $\chi \in C_c(G, K)$ such that $\chi|_{\operatorname{supp}(f)\cup\operatorname{supp}(f)^{-1}} = 1$. Then

$$\int_G f(x)\,dx = (f * \chi)(e_G) = (\chi * f)(e_G) = \int_G f(x^{-1})\,dx.$$

Therefore, by Lemma 3.1.7, G is unimodular. □

3.2 Spherical Functions

Throughout this section let (G, K) be a locally compact group G with compact subgroup K. It is not necessary to assume that it is a Gelfand pair.

Definition 3.2.1. A function $\phi \in C(G, K)$ is *spherical* with respect to K if

1) $\phi(e_G) = 1$ and

2) for all $f \in C_c(G, K)$ there is a constant $\lambda_f \in \mathbb{C}$ such that $f * \phi = \lambda_f \phi$.

Easy examples of spherical functions of a pair (G, K) are all continuous group homomorphisms $\psi\colon G \to \mathbb{C}^\times$ such that $K < ker(\psi)$, where $\mathbb{C}^\times$ denotes the multiplicative group of $\mathbb{C}$. In particular, consider the modular function Δ_G and recall that $\Delta_G|_K = 1$.

Examples 3.2.2. Let F be a non-archimedean local field, and let $\mathcal{O}$ be its ring of integers. By Lemma C.1.1, for $d \geq 2$, $\widetilde{P}_d^+(F)$ and $\widetilde{P}_d^-(F)$ are not unimodular. Thus the modular functions of $\widetilde{P}_d^+(F)$ and $\widetilde{P}_d^-(F)$ are non-trivial spherical functions of $(\widetilde{P}_d^+(F), \widetilde{P}_d^+(\mathcal{O}))$ and $(\widetilde{P}_d^-(F), \widetilde{P}_d^-(\mathcal{O}))$, respectively. The same is true for $(P_d^+(F), P_d^+(\mathcal{O}))$ and $(P_d^-(F), P_d^-(\mathcal{O}))$.

The following theorem gives equivalent definitions of spherical functions of (G, K). For its proof one needs

Lemma 3.2.3 ([Mac71, Lem. 1.1.2])**.** *Let $\phi \in C(G, K)$, and suppose that*

$$\int_G f\phi = 0 \quad \forall f \in L^1(G, K).$$

Then $\phi = 0$.

Proof. Because $\Delta_G|_K = 1$, the left Haar measure dx is bi-K-invariant. Thus for all $f \in L^1(G)$

$$\int_G f(x)\phi(x)\,dx = \int_G \int_K \int_K f(k_1 x k_2)\phi(k_1 x k_2)\,dk_1\,dk_2\,dx = \int_G {}^K f^K(x)\phi(x)\,dx = 0.$$

Hence, $\phi = 0$ almost everywhere. Because ϕ is continuous, $\phi = 0$. □

Theorem 3.2.4 ([Mac71, Prop. 1.2.5])**.** *For a continuous function $\phi\colon G \to \mathbb{C}$, the following are equivalent:*

1) *ϕ is a spherical function of (G, K).*

2) *ϕ is not identically zero, and*

$$\int_K \phi(gkh)\,dk = \phi(g)\phi(h) \quad \forall g, h \in G. \tag{3.3}$$

3) *ϕ is bi-K-invariant and the map $\mu\colon C_c(G, K) \to \mathbb{C}, \mu(f) := (f * \phi)(e_G)$, is a homomorphism of $\mathbb{C}$-algebras.*

Proof. 1) ⇒ 2): Because ϕ is spherical, $\phi \in C(G, K)$ and $\phi \neq 0$. It remains to prove (3.3). Notice first that for all $f \in C_c(G, K)$

$$\int_G f(x)\phi(x^{-1})\,dx = (f * \phi)(e_G) = \lambda_f.$$

Thus, for all $g \in G$ and all $f \in C_c(G, K)$,

$$\begin{aligned}\int_G f(x)\phi(x^{-1})\phi(g)\,dx = \lambda_f\phi(g) = (f * \phi)(g) &= \int_G f(x)\phi(x^{-1}g)\,dx \\ &= \int_G \int_K f(kx)\phi(x^{-1}k^{-1}g)\,dk\,dx \\ &= \int_G f(x)\left(\int_K \phi(x^{-1}kg)\,dk\right)dx.\end{aligned}$$

The last equality holds since K is unimodular. Note that $x \mapsto \int_K \phi(x^{-1}kg)\,dk$ is a continuous function on G and that, because ϕ and dk are bi-K-invariant, it lies in $C(G, K)$. It follows from Lemma 3.2.3 that

$$\int_K \phi(x^{-1}kg)\,dk = \phi(x^{-1})\phi(g) \quad \forall g, x \in G.$$

2) $\Rightarrow$ 3): It is enough to show that for all $f_1, f_2 \in C_c(G,K)$ $\mu(f_1 * f_2) = \mu(f_1)\mu(f_2)$. Indeed,

$$\begin{aligned}
\mu(f_1 * f_2) &= \int_G (f_1 * f_2)(x)\phi(x^{-1})\,dx = \int_G \int_G f_1(y) f_2(y^{-1}x)\phi(x^{-1})\,dy\,dx \\
&= \int_G \int_G f_1(y) f_2(x)\phi(x^{-1}y^{-1})\,dx\,dy \\
&= \int_G \int_G \int_K f_1(y) f_2(kx)\phi(x^{-1}k^{-1}y^{-1})\,dk\,dy\,dx \\
&= \int_G \int_G f_1(y) f_2(x) \int_K \phi(x^{-1}ky^{-1})\,dk\,dy\,dx \\
&= \left(\int_G f_1(y)\phi(y^{-1})\,dy\right)\left(\int_G f_2(x)\phi(x^{-1})\,dx\right) \\
&= \mu(f_1)\mu(f_2).
\end{aligned}$$

3) $\Rightarrow$ 1): Let $f_0 \in C_c(G,K)$ be such that $\mu(f_0) \neq 0$. Then for all $f \in C_c(G,K)$

$$\begin{aligned}
\int_G f(x) \int_G f_0(x^{-1}y)\phi(y^{-1})\,dy\,dx &= \int_G \left(\int_G f(x) f_0(x^{-1}y)\,dx\right)\phi(y^{-1})\,dy \\
&= \mu(f * f_0) = \mu(f)\mu(f_0).
\end{aligned}$$

Therefore set

$$\psi(g) \coloneqq \frac{1}{\mu(f_0)} \int_G f_0(gx)\phi(x^{-1})\,dx = \frac{(f_0 * \phi)(g)}{\mu(f_0)},$$

for $g \in G$. This defines a continuous bi-K-invariant function $\psi\colon G \to \mathbb{C}$ such that

$$\mu(f) = \int_G f(x)\psi(x^{-1})\,dx = \int_G f(x)\phi(x^{-1})\,dx,$$

for all $f \in C_c(G,K)$. By Lemma 3.2.3, $\phi = \psi$, that is, $\phi = \mu(f_0)^{-1} f_0 * \phi$. Thus $\phi \in C(G,K)$ with $\phi(e_G) = \frac{1}{\mu(f_0)} \int_G f_0(x)\phi(x^{-1})\,dx = 1$. Moreover, for all $f_1, f_2 \in C_c(G,K)$,

$$\int_G f_1(x)(f_2 * \phi)(x^{-1})\,dx = (f_1 * (f_2 * \phi))(e_G) = \mu(f_1 * f_2) = \mu(f_1)\mu(f_2)$$

and thus

$$\int_G f_1(x)((f_2 * \phi)(x^{-1}) - \mu(f_2)\phi(x^{-1}))\,dx = 0.$$

Since $f_1 \in C_c(G,K)$ is arbitrary, it follows from Lemma 3.2.3 that $f_2 * \phi = \mu(f_2)\phi$. □

Corollary 3.2.5. *If ϕ is a spherical function of (G,K), then so are $\overline{\phi}$, $\phi^\vee$ and ϕ^*.*

Proof. It is immediate from the definition that $\overline{\phi}$ is spherical. For ϕ^*, let $g, h \in G$ and note that, by part 2) of Theorem 3.2.4,

$$\begin{aligned}
\int_K \phi^*(gkh)\,dk &= \int_K \overline{\phi(h^{-1}kg^{-1})}\Delta_G(gh)^{-1}\,dk = \Delta_G(gh)^{-1}\overline{\int_K \phi(h^{-1}kg^{-1})\,dk} \\
&= \left[\overline{\phi(g^{-1})}\Delta_G(g^{-1})\right]\left[\overline{\phi(h^{-1})}\Delta_G(h^{-1})\right] = \phi^*(g)\phi^*(h).
\end{aligned}$$

Thus ϕ^* is spherical. A similar argument applies to $\phi^\vee$. □

3.3 Spherical Representations

As before let (G, K) be a pair consisting of a locally compact G and a compact subgroup $K < G$.

Definition 3.3.1. A unitary representation $(\pi, \mathcal{H}_\pi)$ of G is said to be *spherical* with respect to K if it is irreducible and has a non-zero K-invariant unit vector which is unique up to a scalar multiple.

Henceforth, given a unitary representation $(\pi, \mathcal{H}_\pi)$, let $\mathcal{H}_\pi^K$ denote the closed subspace of K-invariant vectors. Then π is spherical if and only if it is irreducible and $\dim \mathcal{H}_\pi^K = 1$. Recall that an orthogonal projection is a linear map $P\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ such that $\langle Pu, v\rangle = \langle u, Pv\rangle$ for all $u, v \in \mathcal{H}_\pi$.

Lemma 3.3.2. *The orthogonal projection P^K on $\mathcal{H}_\pi$ such that $P^K(\mathcal{H}_\pi) = \mathcal{H}_\pi^K$, is given by*

$$\langle P^K u, v\rangle = \int_K \langle \pi(k)u, v\rangle \, dk \quad \forall u, v \in \mathcal{H}_\pi. \tag{3.4}$$

Moreover, for every $f \in L^1(G, K)$ and every $u \in \mathcal{H}_\pi$, $\pi^1(f)u \in \mathcal{H}_\pi^K$.

Proof. For all $v \in \mathcal{H}_\pi$ the map $u \mapsto \int_K \langle \pi(k)u, v\rangle \, dk$ is a bounded linear functional on $\mathcal{H}_\pi$. By the Riesz Representation Theorem, there is a unique continuous linear map P^K on $\mathcal{H}_\pi$ such that for all $u, v \in \mathcal{H}_\pi$ (3.4) holds. Using the left-K-invariance of the Haar measure dk, it is immediate that $P^K(\mathcal{H}_\pi) = \mathcal{H}_\pi^K$. For all $u, v \in \mathcal{H}_\pi$,

$$\langle (P^K)^2 u, v\rangle = \int_K \langle \pi(k)(P^K u), v\rangle \, dk = \int_K \langle P^K u, v\rangle \, dk = \langle P^K u, v\rangle$$

and therefore $(P^K)^2 = P^K$. Moreover,

$$\langle P^K u, v\rangle = \int_K \langle u, \pi(k^{-1})v\rangle \, dk = \int_K \langle u, \pi(k)v\rangle \, dk = \langle u, P^K v\rangle,$$

which implies that P^K is an orthogonal projection.

For the second part of the lemma, take $f \in L^1(G, K)$ and note that for all $u \in \mathcal{H}_\pi$ and all $k \in K$

$$\begin{aligned}\langle \pi(k)(\pi^1(f)u), v\rangle &= \int_G f(x)\langle \pi(k)\pi(x)u, v\rangle \, dx = \int_G f(k^{-1}x)\langle \pi(x)u, v\rangle \, dx \\ &= \int_G f(x)\langle \pi(x)u, v\rangle \, dx = \langle \pi^1(f)u, v\rangle;\end{aligned}$$

whence $\pi^1(f)u \in \mathcal{H}_\pi^K$. □

By Proposition 1.4.7, a unitary representation $(\pi, \mathcal{H}_\pi)$ of G is irreducible if and only if so is the associated $*$-representation $(\pi^1, \mathcal{H}_\pi)$ of the group algebra $L^1(G)$.

Proposition 3.3.3 ([Lan85, Chap. IV, Thm. 2])**.** *Suppose $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G such that $\mathcal{H}_\pi^K \neq \{0\}$ and such that $\{\pi(g)u\colon g \in G,\ u \in \mathcal{H}_\pi^K\}$ is dense in $\mathcal{H}_\pi$. Then π is irreducible if and only if $(\mathrm{Res}^{L^1(G)}_{L^1(G,K)}(\pi^1), \mathcal{H}_\pi^K)$ is an irreducible $*$-representation of $L^1(G, K)$.*

Proof. Suppose π is irreducible. Then $(\pi^1, \mathcal{H}_\pi)$ is irreducible for $L^1(G)$. Write $\mathcal{H}_\pi^K = \mathcal{M}_\pi^K \oplus (\mathcal{M}_\pi^K)^\perp$, where $\mathcal{M}_\pi^K$ is $L^1(G, K)$-invariant. Assume that $u \in \mathcal{M}_\pi^K$ is non-zero. Then for all $v \in (\mathcal{M}_\pi^K)^\perp$

$$\begin{aligned}\langle \pi^1(f)u, v\rangle &= \int_G f(x)\langle \pi(x)u, v\rangle \, dx = \int_K \int_K \int_G f(x)\langle \pi(xk_1)u, \pi(k_2)v\rangle \, dx \, dk_1 \, dk_2 \\ &= \int_K \int_K \int_G f(x)\langle \pi(k_2^{-1}xk_1)u, v\rangle \, dx \, dk_1 \, dk_2 \\ &= \int_K \int_K \int_G f(k_2xk_1)\langle \pi(x)u, v\rangle \, dx \, dk_1 \, dk_2 = \langle \pi^1({}^K f^K)u, v\rangle = 0,\end{aligned}$$

which implies $(\mathcal{M}_\pi^K)^\perp = \{0\}$.

Conversely, suppose $(\mathrm{Res}^{L^1(G)}_{L^1(G,K)}(\pi^1), \mathcal{H}_\pi^K)$ is irreducible. It is enough to show that $(\pi^1, \mathcal{H}_\pi)$ is an irreducible $*$-representation of $L^1(G)$. Arguing by contradiction, assume that there is a non-trivial proper G-invariant closed subspace $\mathcal{M}_\pi \subset \mathcal{H}_\pi$ such that $\mathcal{H}_\pi = \mathcal{M}_\pi \oplus \mathcal{M}_\pi^\perp$. Then

$$\mathcal{H}_\pi^K = P^K(\mathcal{H}_\pi) = P^K(\mathcal{M}_\pi) \oplus P^K(\mathcal{M}_\pi^\perp) = \mathcal{M}_\pi^K \oplus (\mathcal{M}_\pi^K)^\perp.$$

By Lemma 3.3.2, for all $f \in L^1(G,K)$ $\pi^1(f) = P^K\pi^1(f)$. Hence both $\mathcal{M}_\pi^K$ and $(\mathcal{M}_\pi^K)^\perp$ are $L^1(G,K)$-invariant. But then $\mathcal{H}_\pi^K$ is contained in either $\mathcal{M}_\pi^K$ or $(\mathcal{M}_\pi^K)^\perp$, which implies that $\mathcal{H}_\pi = \mathcal{M}_\pi$ or $\mathcal{H}_\pi = (\mathcal{M}_\pi)^\perp$, yielding a contradiction. □

In the case where (G,K) is a Gelfand pair, one has

Corollary 3.3.4 ([Lan85, Chap. IV, Thm. 3])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G as in Propositon 3.3.3. If $\dim \mathcal{H}_\pi^K = 1$, then π is irreducible. If (G,K) is a Gelfand pair, then the converse is also true.*

Proof. If $\dim \mathcal{H}_\pi^K = 1$, then $(\mathrm{Res}^{L^1(G)}_{L^1(G,K)}(\pi^1), \mathcal{H}_\pi^K)$ is irreducible and hence, by Proposition 3.3.3, π is irreducible. Recall that, by Corollary 1.3.17, a $*$-representation of a commutative $*$-algebra is irreducible if and only if it is zero or it is one-dimensional. Hence, if (G,K) is Gelfand and π is irreducible, then $(\mathrm{Res}^{L^1(G)}_{L^1(G,K)}(\pi^1), \mathcal{H}_\pi^K)$ is one-dimensional. □

The following result establishes the connection between spherical functions of (G,K) and the dimension of $\mathcal{H}_\pi^K$.

Proposition 3.3.5 ([Lan85, Chap. IV, Thm. 8])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G, and suppose there exists a cyclic unit vector $u \in \mathcal{H}_\pi^K$. Then $\dim \mathcal{H}_\pi^K = 1$ if and only if the matrix coefficient $\phi_{u,u}(g) \coloneqq \langle \pi(g)u, u\rangle$ is a spherical function of (G,K).*

Proof. Suppose $\dim \mathcal{H}_\pi^K = 1$, that is $\mathcal{H}_\pi^K = \mathbb{C}u$. It is clear that $\phi_{u,u}$ is continuous and bi-K-invariant, with $\phi_{u,u}(e_G) = 1$. Note that, for every $f \in L^1(G,K)$, $\pi^1(f)(\mathcal{H}_\pi^K) = \mathcal{H}_\pi^K$ and hence there is a unique $\lambda_f \in \mathbb{C}$ such that $\pi^1(f)u = \lambda_f u$. It follows that

$$\begin{aligned}(f * \phi_{u,u})(g) &= \int_G f(x)\langle \pi(x^{-1}g)u, u\rangle\, dx = \int_G \langle \pi(g)u, \overline{f(x)}\pi(x)u\rangle\, dx \\ &= \langle \pi(g)u, \pi^1(\overline{f})u\rangle = \overline{\lambda_{\overline{f}}}\phi_{u,u}(g).\end{aligned}$$

Hence $\phi_{u,u}$ is a spherical function of (G,K).

For the converse suppose that $\phi_{u,u}$ is a spherical function of (G,K). Then for all $g, h \in G$

$$\begin{aligned}\langle P^K(\pi(g)u), \pi(h)u\rangle &= \int_K \langle \pi(k)\pi(g)u, \pi(h)u\rangle\, dk = \int_K \phi_{u,u}(h^{-1}kg)\, dk \\ &= \phi_{u,u}(h^{-1})\phi_{u,u}(g) = \langle \phi_{u,u}(g)u, \pi(h)u\rangle.\end{aligned}$$

Because u is assumed to be cyclic, this implies that for all $g \in G$ $P^K(\pi(g)u) = \phi_{u,u}(g)u$ and therefore that $\dim \mathcal{H}_\pi^K = 1$. □

Corollary 3.3.6. *A unitary representation of (G,K) is spherical if and only if exactly one of its diagonal matrix coefficients is a spherical function of (G,K).*

Proof. The first implication follows from Proposition 3.3.5, and the converse follows from Corollary 3.3.4 and from Proposition 3.3.5. □

Corollary 3.3.7. *Suppose (G,K) is a Gelfand pair. Then a unitary representation $(\pi, \mathcal{H}_\pi)$ of G is spherical if and only if $\mathcal{H}_\pi^K \neq \{0\}$.*

Proof. By Proposition 3.3.5, if π is spherical, then $\mathcal{H}_\pi^K \neq \{0\}$. The converse follows from Corollary 3.3.4. □

Recall from Section 1.5 the Gelfand-Naimark-Segal construction.

Proposition 3.3.8. *For every positive definite spherical function ϕ of (G, K) π_ϕ is spherical. Moreover, given two spherical functions ϕ and ψ of (G, K), $\pi_\phi \simeq \pi_\psi$ if and only if $\phi = \psi$.*

Proof. The first part follows immediately from Theorem 1.5.5 and from Corollary 3.3.6. The second part follows from Proposition 3.3.5 and from Proposition 1.2.10. □

3.4 Induction of spherical functions and representations

A useful tool for finding spherical functions is induction. In view of the Gelfand-Naimark-Segal construction in Section 1.5, the relationship between the induction of spherical functions and the induction of unitary representations is established in Proposition 3.4.3.

Proposition 3.4.1 ([Mac71, Prop. 1.3.1])**.** *Let $P < G$ be a closed subgroup, and suppose $G = KP$. Let ϕ be a spherical function of $(P, K \cap P)$. Extend ϕ to a continuous function on G by setting $\phi(kp) = \phi(p)$. Then the function $\phi_P^G \in C(G, K)$, given by*

$$\phi_P^G(g) \coloneqq \int_K \phi(gk)\, dk, \tag{3.5}$$

is a spherical function of (G, K).

Proof. Note first that the extension of ϕ to G is well-defined. Indeed, if $k_0 p_0 = k_1 p_1$, then $p_0 = (k_0^{-1} k_1) p_1$ with $k_0^{-1} k_1 \in K \cap P$ and hence $\phi(p_0) = \phi(p_1)$. Moreover, using that K is compact, it can be shown that ϕ and therefore ϕ_P^G are continuous on G. It follows that $\phi_P^G \in C(G, K)$ with $\phi_P^G(e_G) = 1$.

It remains to show that for all $f \in C_c(G, K)$, $f * \phi_P^G$ is a scalar multiple of ϕ_P^G. For this write $g = k(g)p(g)$, for $g \in G$ and where $k(g) \in K$, $p(g) \in P$. For all $g \in G$, $p \in P$

$$\int_{K \cap P} \phi(gkp)\, dk = \int_{K \cap P} \phi(p(g)kp)\, dk = \phi(g)\phi(p).$$

Thus for all $f \in C_c(G, K)$ and all $g \in G$

$$\begin{aligned}(f * \phi)(g) &= \int_G f(gx)\phi(x^{-1})\, dx = \int_G f(p(g)x)\phi(x^{-1})\, dx \\ &= \int_G f(x)\phi(x^{-1}p(g))\, dx = \int_G f(x)\left(\int_{K \cap P} \phi(x^{-1}kp(g))\, dk \right) dx \\ &= \int_G f(x)\phi(x^{-1})\phi(p(g))\, dx = \lambda_f \phi(g),\end{aligned}$$

where $\lambda_f = (f * \phi)(e_G)$. It follows that

$$\begin{aligned}(f * \phi_P^G)(g) &= \int_G f(x)\phi_P^G(x^{-1}g)\, dx = \int_G \int_K f(x)\phi(x^{-1}gk)\, dk\, dx \\ &= \int_K (f * \phi)(gk)\, dk = \lambda_f \phi_P^G(g).\end{aligned}$$

Therefore ϕ_P^G is a spherical function of (G, K). □

The function ϕ_P^G is said to be *induced* by ϕ. The following proposition follows from straightforward computations. It asserts that the induction of spherical functions is transitive.

Proposition 3.4.2 ([Mac71, Prop. 1.3.3])**.** *Let $P_1 < G$ and $P_0 < P_1$ be closed subgroups such that $G = KP_0$. Then $\phi_{P_0}^G = (\phi_{P_0}^{P_1})_{P_1}^G$.*

Proof. Notice that $G = KP_i$, $i = 0, 1$, and that $P_1 = (K \cap P_1)P_0$. Extend ϕ and $\phi_{P_0}^{P_1}$ to G as in Proposition 3.4.1. Then

$$\begin{aligned}\phi_{P_0}^G(g) &= \int_K \phi(gk)\,dk = \int_K \int_{K\cap P_1} \phi((gk)k')\,dk'\,dk \\ &= \int_K \phi_{P_0}^{P_1}(gk)\,dk = (\phi_{P_0}^{P_1})_{P_1}^G(g),\end{aligned}$$

for all $g \in G$. □

Recall now the notion of the rho-function associated to a strongly quasi-invariant measure on a homogeneous space (see Definition A.4.6 and Theorem A.4.9). Suppose $P < G$ is a closed subgroup such that $G = KP$. Extend the modular function Δ_P to G by setting $\Delta_P(kp) = \Delta_P(p)$ for $k \in K$ and $p \in P$. Then the map $\rho\colon G \to \mathbb{R}_{>0}$,

$$\rho(g) \coloneqq \frac{\Delta_P(g)}{\Delta_G(g)}, \tag{3.6}$$

is well-defined and a rho-function of (G, P) (see Lemma A.4.8 and Proposition A.4.13). The following proposition is a generalization of [Mac71, Prop. 1.4.7].

Proposition 3.4.3. *Suppose $P < G$ is a closed subgroup such that $G = KP$. Let ρ be as in* (3.6)*, and let ϕ be a spherical function of $(P, K \cap P)$. Suppose $p \mapsto \rho^{-\frac{1}{2}}(p)\phi(p)$ is positive definite. Then ϕ_P^G is a positive definite spherical function of (G, K), and $\pi_{\phi_P^G} \subset \operatorname{ind}_P^G\left(\sigma_{\rho^{-\frac{1}{2}}\phi}\right)$ is a spherical subrepresentation.*

Proof. Notice that the restriction $\rho^{-\frac{1}{2}}|_P$ is a bi-$K \cap P$-invariant continuous homomorphism to $(\mathbb{R}_{>0}, \cdot)$. Therefore $\rho^{-\frac{1}{2}}\phi$ is a spherical function of $(P, K \cap P)$. Let $u \in \mathcal{H}_{\rho^{-\frac{1}{2}}\phi}$ be a $(K \cap P)$-invariant unit vector of the corresponding cyclic unitary representation $\sigma_{\rho^{-\frac{1}{2}}\phi}$ such that $\rho^{-\frac{1}{2}}(p)\phi(p) = \langle \sigma_{\rho^{-\frac{1}{2}}\phi}(p)u, u\rangle$. Take $f_0 \in C(G, \mathcal{H}_{\Delta^{-\frac{1}{2}}\phi})$, given by $f_0(kp) \coloneqq \sigma_{\rho^{-\frac{1}{2}}\phi}(p^{-1})u$. This map is well-defined and lies in $\mathcal{F}_{\rho^{-\frac{1}{2}}\phi}$.

Recall from Section 1.7 that, for every $g, x \in G$ and every $f \in \mathcal{F}_{\rho^{-\frac{1}{2}}\phi}$,

$$\operatorname{ind}_P^G\left(\sigma_{\rho^{-\frac{1}{2}}\phi}\right)(g)f(x) = \sqrt{\frac{\rho(g^{-1}x)}{\rho(x)}}f(g^{-1}x).$$

Let μ be the strongly quasi-invariant measure on G/P associated to ρ. Because $G/P \cong K/(K \cap P)$ is compact and because $\rho|_K = 1$, μ is a probability measure. The inner product on $\mathcal{F}_{\rho^{-\frac{1}{2}}\phi}$ is given by

$$\begin{aligned}\langle f_1, f_2\rangle &= \int_{G/P} \langle f_1(x), f_2(x)\rangle_{\rho^{-\frac{1}{2}}\phi}\,d\mu(xP) \\ &= \int_{K/(K\cap P)} \langle f_1(x), f_2(x)\rangle_{\rho^{-\frac{1}{2}}\phi}\,d\nu(x(K\cap P)) = \int_K \langle f_1(k), f_2(k)\rangle_{\rho^{-\frac{1}{2}}\phi}\,dk,\end{aligned}$$

where ν is the K-invariant probability measure on $K/(K \cap P)$. Since μ is a K-invariant probability measure on G/P, ν is the pushforward of μ under the isomorphism $G/P \cong K/(K \cap P)$. The last equality follows from Weil Formula (see Corollary A.4.11).

By the left-K-invariance of f_0 and of ρ, f_0 is a K-invariant unit vector. For $g \in G$ write $g = k(g)p(g)$, with $k(g) \in K$, $p(g) \in P$. As in Proposition 3.4.2, extend ϕ to G by setting $\phi(g) = \phi(p(g))$. Then

$$\begin{aligned}\left\langle \operatorname{ind}_P^G\left(\sigma_{\rho^{\frac{1}{2}}\phi}\right)(g)f_0, f_0\right\rangle &= \int_K \rho^{\frac{1}{2}}(g^{-1}k)\langle f_0(g^{-1}k), f_0(k)\rangle_{\rho^{\frac{1}{2}}\phi}\,dk \\ &= \int_K \rho^{-\frac{1}{2}}(g^{-1}k)\left\langle\left(\sigma_{\rho^{-\frac{1}{2}}\phi}\right)(p(g^{-1}k)^{-1})u, u\right\rangle_{\rho^{-\frac{1}{2}}\phi}\,dk \\ &= \int_K \overline{\phi(g^{-1}k)}\,dk = \overline{\phi_P^G(g^{-1})} = \phi_P^G(g).\end{aligned}$$

Thus, ϕ_P^G is a matrix coefficient and is therefore positive definite, by Proposition 1.5.3.

By Proposition 3.3.8 and by Proposition 3.4.1, $\pi_{\phi_P^G}$ is spherical. By construction it is a subrepresentation of $\operatorname{ind}_P^G\left(\sigma_{\rho^{-\frac{1}{2}}\phi}\right)$. □

Corollary 3.4.4. *Suppose that G and P are unimodular. Suppose that ϕ is a spherical function of (G, K), and suppose that $\phi|_P$ is a spherical function of $(P, K \cap P)$. Then $\phi = (\phi|_P)_P^G$, and ϕ is positive definite if and only if $\phi|_P$ is positive definite.*

Proof. Extend $\phi|_P$ to G by setting $\phi|_P(kp) \coloneqq \phi|_P(p)$ for $k \in K$, $p \in P$. Then, for all $k \in K$ and all $p \in P$, $\phi|_P(kp) = \phi(p) = \phi(kp)$, and hence $\phi|_P(pk) = \phi(pk) = \phi(p)$. Thus $\phi = (\phi|_P)_P^G$.

By Proposition 3.4.3, if $\phi|_P$ is a positive definite function of P, then $\phi = (\phi|_P)_P^G$ is also positive definite. The converse is immediate. □

Henceforth let $P < G$ be a closed subgroup with $G = KP$, and let ρ be as in (3.6). As remarked in the proof of Proposition 3.4.3, $\rho|_P$ is a bi-$K \cap P$-invariant continuous character. For $t \in \mathbb{R}$, let $\sigma_{it}\colon P \to U(1) \cong S^1$ be the one-dimensional unitary representation of P given by $\sigma_{it}(p) \coloneqq \rho^{it}(p)$, $p \in P$. It is immediate that σ_{it} is spherical with $1 \in \mathbb{C}$ the $(K \cap P)$-invariant vector that is unique up to a scalar multiple.

Corollary 3.4.5. *For $z \in \mathbb{C}$,*

$$\phi_z(g) \coloneqq \int_K \left(\frac{\Delta_P(gk)}{\Delta_G(gk)}\right)^{\frac{1}{2}+z} dk$$

is a spherical function of (G, K). In particular, for all $t \in \mathbb{R}$, the unitary representation $\pi_{\phi_{it}}$ is a spherical representation of (G, K).

Proof. For $z \in \mathbb{C}$ set $\psi_z \coloneqq \rho^{\frac{1}{2}+z}|_P$. Then, by Proposition 3.4.1 and by the above remarks, $\phi_z = (\psi_z)_P^G$ is a spherical function of (G, K). The second part of the corollary is a consequence of Proposition 3.3.8. □

It will be seen that for some pairs (G, K) $\{\phi_z\colon z \in \mathbb{C}\}$ is the set of all spherical functions; for $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$ see Theorem 3.5.7 and for Gelfand pairs of tree automorphisms see Corollary 5.6.3. Concerning Gelfand pairs one has

Corollary 3.4.6. *Suppose (G, K) is a Gelfand pair. Then, for every $z \in \mathbb{C}$,*

$$\phi_z(g) = \int_K \Delta_P^{\frac{1}{2}+z}(gk)\,dk.$$

is a spherical function of (G, K).

Proof. By Proposition 3.1.6 G is unimodular, and the corollary follows from Corollary 3.4.5. □

Corollary 3.4.5 motivates

Definition 3.4.7. The *spherical principal series* of (G, K) in $\widehat{G}$ is $\{\pi_{\phi_{it}} : t \in \mathbb{R}\}$.

The following proposition will be useful in Section 6.5, when the spherical representations of $(SL_2(F), SL_2(\mathcal{O}))$ over a local field F, with $\mathcal{O}$ its ring of integers, are classified.

Proposition 3.4.8. *Let $K < G$ be a compact subgroup, and suppose that $Z(G) < K$. Then a function ϕ on $G/Z(G)$ is spherical with respect to $K/Z(G)$ if and only if $\phi \circ q$ is a spherical function of (G, K), where $q : G \to G/Z(G)$ is the canonical quotient map. Moreover, every spherical function of (G, K) is of the form $\phi \circ q$ for some spherical function ϕ of $(G/Z(G), K/Z(G))$.*

Proof. Let ϕ be a function on $G/Z(G)$. It is clear that ϕ is bi-$K/Z(G)$-invariant if and only if $\phi \circ q$ is bi-K-invariant. Moreover, by the Weil Formula, for all $g_1, g_2 \in G$

$$\int_K (\phi \circ q)(g_1 k g_2)\, dk = \int_{K/Z(G)} \int_{Z(G)} \phi(q(g_1) k z q(g_2))\, dz\, dk = \int_{K/Z(G)} \phi(q(g_1) k q(g_2))\, dk.$$

Hence, the first part of the proposition follows from Theorem 3.2.4.

Next suppose that ψ is a spherical function of (G, K), and set $\phi(q(g)) \coloneqq \psi(g)$. Because ψ is bi-$Z(G)$-invariant, ϕ is well-defined. Then, by the first part of the proposition, ϕ is a spherical function of $(G/Z(G), K/Z(G))$. □

3.5 Spherical functions and representations of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$

Recall the notation from Section 2.3, and recall the Cartan decomposition of $GL_d(F)$ and $SL_d(F)$ over a local field F. One also has the following decompositions, which are called *Iwasawa decompositions*.

Theorem 3.5.1 (Iwasawa decomposition over $F \in \{\mathbb{R}, \mathbb{C}\}$)**.** *Let $d \geq 1$. Then:*

1) *$GL_d(\mathbb{R}) = O(d) A_d N_d^+(\mathbb{R})$ and $SL_d(\mathbb{R}) = SO(d) \widetilde{A}_d N_d^+(\mathbb{R})$.*

2) *$GL_d(\mathbb{C}) = U(d) A_d N_d^+(\mathbb{C})$ and $SL_d(\mathbb{C}) = SU(d) \widetilde{A}_d N_d^+(\mathbb{C})$.*

Moreover, each decomposition is unique.

Proof. It is enough to prove the Iwasawa decomposition and its uniqueness for the case of $GL_d(F)$, $F \in \{\mathbb{R}, \mathbb{C}\}$. The proof is given here only for $GL_d(\mathbb{C})$ in detail. The proof works similarly in the case of $GL_d(\mathbb{R})$. Let $x \in GL_d(\mathbb{C})$, and for $1 \leq i \leq d$ set $v_i \coloneqq x \cdot e_i$. Then $v_1, v_2, \ldots, v_d$ form a basis of $\mathbb{C}^d$. By the Gram-Schmidt process there are $a \in A_d$ and $n \in N_d^+(\mathbb{C})$ such that $w_i \coloneqq (an) \cdot v_i$ form an orthonormal basis of $\mathbb{C}^d$. Hence, there is $u \in U(d)$ such that $e_i = (uan) \cdot v_i$. It follows that $x^{-1} = uan$, which proves that $GL_d(\mathbb{C}) = U(d) A_d N_d^+(\mathbb{C})$.

For the uniqueness, let $x \in GL_d(\mathbb{C})$, and suppose $x = u_1 a_1 n_1 = u_2 a_2 n_2$ for $u_i \in U(d)$, $a_i \in A_d^+$, $n_i \in N_d^+(\mathbb{C})$. Then $u_1 u_2^{-1} = (a_2 n_2)(a_1 n_1)^{-1} \in U(d) \cap P_d^+(\mathbb{C}) = \{I_d\}$. Thus $u_1 = u_2$ and $a_1 n_1 = a_2 n_2$. Since a_i is uniquely determined by the diagonal entries of $a_i n_i$, $a_1 = a_2$ and therefore $n_1 = n_2$. □

Corollary 3.5.2. *For every $d \geq 1$, one has that:*

1) *$GL_d(\mathbb{R}) = O(d) P_d^{\pm}(\mathbb{R})$ and $SL_d(\mathbb{R}) = SO(d) \widetilde{P}_d^{\pm}(\mathbb{R})$;*

2) *$GL_d(\mathbb{C}) = U(d) P_d^{\pm}(\mathbb{C})$ and $SL_d(\mathbb{C}) = SU(d) \widetilde{P}_d^{\pm}(\mathbb{C})$.*

Theorem 3.5.3 (Iwasawa decomposition over non-archimedean local fields)**.** *Let F be a non-archimedean local field, where $\mathcal{O}$ is its ring of integers and with uniformizing element $\mathfrak{p} \in \mathcal{O}$, and let $d \geq 1$. Then*

$$GL_d(F) = GL_d(\mathcal{O}) T_d(F) N_d^+(F)$$

and

$$SL_d(F) = SL_d(\mathcal{O}) \widetilde{T}_d(F) N_d^+(F).$$

Proof. As in the proof of Theorem 3.5.1 it is also here enough to prove the decomposition for $GL_d(F)$. The proof uses the same ideas as the proof of Theorem 2.3.2. Argue by induction on d. The base case of $d = 1$ is trivial. For the induction step, suppose that the decomposition holds for some $d \geq 1$, and take $x \in GL_{d+1}(F)$. Then there is a permutation matrix $u \in GL_{d+1}(\mathcal{O})$ such that $|(ux)_{11}| \geq |(ux)_{i1}|$ for all $1 \leq i \leq d+1$. Notice that there are $y_i \in \mathcal{O}$ such that $y_i(ux)_{11} = (ux)_{i1}$. Hence there exists $y \in GL_{d+1}(\mathcal{O})$ such that $((yu)x)_{i1} = 0$ for $2 \leq i \leq d+1$. Let $x' \in GL_d(F)$ be such that

$$(yu)x = \begin{pmatrix} (ux)_{11} & \cdots & \cdots & (ux)_{1d+1} \\ 0 & & & \\ \vdots & & x' & \\ 0 & & & \end{pmatrix}.$$

Let $k \in \mathbb{Z}$ be such that $q^k = |(ux)_{11}|$. By the induction hypothesis there are $z \in GL_d(\mathcal{O})$, $t \in T_d(F)$ and $n \in N_d^+(F)$ such that $x' = ztn$. Then

$$x = (yu)^{-1} \begin{pmatrix} \mathfrak{p}^{-k}(ux)_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & z & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \mathfrak{p}^k & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & t & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & v_1 & \cdots & v_d \\ 0 & & & \\ \vdots & & n & \\ 0 & & & \end{pmatrix},$$

where $v_i \in F$ for $1 \leq i \leq d$. Therefore the decomposition is proven for $GL_d(F)$. □

Corollary 3.5.4. *For every non-archimedean local field F and $d \geq 1$,*

$$GL_d(F) = GL_d(\mathcal{O})P_d^{\pm}(F)$$

and

$$SL_d(F) = SL_d(\mathcal{O})\widetilde{P}_d^{\pm}(F).$$

In particular one has

Lemma 3.5.5. *If $F \in \{\mathbb{R}, \mathbb{C}\}$ and if $K < GL_d(F)$ is a compact subgroup, then there is $g \in GL_d(F)$ such that $gKg^{-1} \leq O(d)$ or $gKg^{-1} \leq U(d)$, respectively. Hence $O(d)$ and $U(d)$ are maximal compact subgroups of $GL_d(\mathbb{R})$ and $GL_d(\mathbb{C})$, respectively, which are unique up to conjugation.*

Proof. The proof is given here in detail only for the case of $F = \mathbb{C}$. It works similarly for $F = \mathbb{R}$. Let $K < GL_d(\mathbb{C})$ be a compact subgroup, and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}$. Then, by Proposition 1.5.16, there is a K-invariant inner product $\langle \cdot, \cdot \rangle_K$ on $\mathbb{C}^d$. Thus there is a self-adjoint linear map $g \in GL_d(\mathbb{C})$ such that $\langle u, v \rangle_K = \langle g \cdot u, g \cdot v \rangle$ for all $u, v \in \mathbb{C}^d$. It follows that $gKg^{-1} \leq U(d)$. □

It follows from the above lemma that $SO(d) \leq SL_d(\mathbb{R})$ and $SU(d) \leq SL_d(\mathcal{R})$ are maximal compact subgroups, which are unique up to conjugation. If F is non-archimedean with $\mathcal{O}$ its ring of integers, then $SL_d(\mathcal{O})$ is a maximal compact subgroup of $SL_d(F)$, which is unique up to conjugation (see Lemma 6.5.1 for $d = 2$).

Theorem 3.5.1 is a special case of the Iwasawa decomposition of semisimple Lie groups.

Theorem 3.5.6 (Iwasawa decomposition for semisimple Lie groups, [Hel78, Chap. IX, Thm. 1.3])**.** *Let G be a connected semisimple Lie group, and let $K < G$ be a maximal compact subgroup. There exist an abelian Lie subgroup $A < G$ and a nilpotent Lie subgroup $N < G$ such that*

$$G = KAN.$$

Let F be a local field. By Corollary 3.1.5 and by Corollary 3.4.6, one has the spherical functions $\{\phi_{F,z}\colon z \in \mathbb{C}\}$ of $(SL_d(F), K_d(F))$, where

- $K_d(\mathbb{R}) \coloneqq SO(d)$ and
$$\phi_{\mathbb{R},z}(g) \coloneqq \int_{SO(d)} \Delta_{\widetilde{P}^+(\mathbb{R})}^{\frac{1}{2}+z}(gk)\,dk;$$
- $K_d(\mathbb{C}) \coloneqq SU(d)$ and
$$\phi_{\mathbb{C},z}(g) \coloneqq \int_{SU(d)} \Delta_{\widetilde{P}^+(\mathbb{C})}^{\frac{1}{2}+z}(gk)\,dk;$$
- if F is non-archimedean with integer ring $\mathcal{O}$, then $K_d(F) \coloneqq SL_d(\mathcal{O})$ and
$$\phi_{F,z}(g) \coloneqq \int_{SL_d(\mathcal{O})} \Delta_{\widetilde{P}^+(F)}^{\frac{1}{2}+z}(gk)\,dk.$$

Also recall from Section 3.4 the definition of the spherical principal series $\{\pi_{\phi_{F,it}}\colon t \in \mathbb{R}\}$.

The following theorem is a special case of [Hel84, Chap. IV, Thm. 4.3].

Theorem 3.5.7. *The spherical functions of $(SL_2(\mathbb{R}), SO(2))$ are exactly the functions $\{\phi_{\mathbb{R},z}\colon z \in \mathbb{C}\}$. Similarly the spherical functions of $(SL_2(\mathbb{C}), SU(2))$ are exactly the functions $\{\phi_{\mathbb{C},z}\colon z \in \mathbb{C}\}$. Moreover, for $F \in \{\mathbb{R}, \mathbb{C}\}$ and $z, w \in \mathbb{C}$, $\phi_{F,z} = \phi_{F,w}$ if and only if $z = \pm w$.*

The proof of [Hel84, Chap. IV, Thm. 4.3] uses the theory of Riemannian symmetric pairs. Since this would go beyond the frame of this thesis, the proof is omitted. For a direct computation of the spherical functions of $(SL_2(\mathbb{R}), SO(2))$ see [Lan85, Chap. X, Thm. 11].

Remark. Suppose G is a connected non-compact semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. Then [Hel84, Chap. IV, Thm. 4.3] provides integral formulas for the spherical functions of (G, K).

Concerning the question which of these spherical functions are positive definite, one has the following result. It is a special case of [FJK79, Thm. 1]. Its proof is omitted here.

Theorem 3.5.8. *If $F \in \{\mathbb{R}, \mathbb{C}\}$, then the spherical function $\phi_{F,z}$ is positive definite if and only if $z \in i\mathbb{R}$ or $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.*

Remark. [FJK79, Thm. 1] identifies the positive definite spherical functions of every semisimple Lie group of real rank one, with respect to their respective maximal compact subgroups.

Theorem 3.5.8 motivates

Definition 3.5.9. For $F \in \{\mathbb{R}, \mathbb{C}\}$, the *complementary series* of $SL_2(F)$ is
$$\left\{\pi_{\phi_{F,s}}\colon -\frac{1}{2} < s < \frac{1}{2},\ s \neq 0\right\}.$$

Remark. Theorem 3.5.7 identifies all positive definite spherical functions of $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$. However, both $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ have non-spherical irreducible unitary representations. In particular, both have a *non-spherical principal series*. The unitary dual of $SL_2(\mathbb{C})$ consists of the trivial representation, of the spherical and the non-spherical principal series and of the complementary series. The unitary dual of $SL_2(\mathbb{R})$ is completed by adding the so-called *discrete series* and its *limits*. For precise descriptions of $\widehat{SL_2(\mathbb{C})}$ and $\widehat{SL_2(\mathbb{R})}$ see [AWK16, Thm. 16.2] and [AWK16, Thm. 16.3], respectively.

The goal of the remainder of this section is to calculate more explicit formulas for the spherical functions of $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$. Recall for this that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \text{ with } |\alpha|^2 + |\beta|^2 = 1 \right\},$$

and that $SO(2) = GL_2(\mathbb{R}) \cap SU(2)$. One can show that $SO(2) \cong S^1$ and that $SU(2) \cong S^3$. Using the polar coordinates, one has $SO(2) = \{r(\theta) : 0 \leq \theta < 2\pi\}$, where $r(\theta) : S^1 \to S^1$ is the rotation defined by

$$r(\theta) \coloneqq \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Note that $r(\theta_1)r(\theta_2) = r(\theta_1+\theta_2)$, and the normalized Haar measure on $SO(2)$ is $\frac{1}{2\pi}\, d\theta$. Similarly, using the Hopf coordinates for S^3, write $SU(2) = \{k(\theta_1, \theta_2, \theta_3) : 0 \leq \theta_1 < \frac{\pi}{2}, 0 \leq \theta_2, \theta_3 < 2\pi\}$, where $k(\theta_1, \theta_2, \theta_3) : S^3 \to S^3$ is defined by

$$k(\theta_1, \theta_2, \theta_3) \coloneqq \begin{pmatrix} \cos(\theta_1)e^{i\theta_2} & -\sin(\theta_1)e^{-i\theta_3} \\ \sin(\theta_1)e^{i\theta_3} & \cos(\theta_1)e^{-i\theta_2} \end{pmatrix}.$$

The normalized Haar measure on $SU(2)$ is $\frac{1}{4\pi^2}\sin(2\theta_1)\, d\theta_1\, d\theta_2\, d\theta_3$ (see [Oh02, p. 20]).

Next recall from Theorem 2.3.1 and Theorem 3.5.1 the Iwasawa and Cartan decompositions of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. To simplify notation, write $d(a) = diag(a, a^{-1}) \in \widetilde{A}_d$ for $a \in \mathbb{R} - \{0\}$.

Proposition 3.5.10. *For $F \in \{\mathbb{R}, \mathbb{C}\}$, let $g \in SL_2(F)$. Then*

$$\Delta_{\widetilde{P}_2^+(F)}(g) = \Delta_{\widetilde{P}_2^+(F)}\Big(d(\sqrt{|g_{11}|^2 + |g_{21}|^2})\Big)$$

Proof. Suppose $F = \mathbb{C}$. By the Iwasawa decomposition, there are $0 \leq \theta_1, \theta_2, \theta_3 < 2\pi$, $a \geq 1$ and $z \in \mathbb{C}$ such that $g = k(\theta_1, \theta_2, \theta_3)d(a)n(z)$. Then, by the definition and properties of the modular function of $\widetilde{P}_2^+(\mathbb{C})$ (see Proposition C.1.3),

$$\Delta_{\widetilde{P}_2^+(\mathbb{C})}(g) = \Delta_{\widetilde{P}_2^+(\mathbb{C})}(d(a)).$$

To simplify notation set $\alpha = \cos(\theta_1)e^{i\theta_2}$ and $\beta = \sin(\theta_2)e^{i\theta_3}$. Then

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha a z - \overline{\beta} a^{-1} \\ \beta a & \beta a z + \overline{\alpha} a^{-1} \end{pmatrix}$$

Because $|\alpha|^2 + |\beta|^2 = 1$, $a = \sqrt{|g_{11}|^2 + |g_{21}|^2}$. □

Corollary 3.5.11. *1) For all $a \in \mathbb{R} - \{0\}$ and $z \in \mathbb{C}$,*

$$\phi_{\mathbb{R},z}(d(a)) = \frac{1}{2\pi}\int_0^{2\pi} (a^2\cos^2(\theta) + a^{-2}\sin^2(\theta))^{-\frac{1}{2}-z}\, d\theta.$$

2) For all $a \in \mathbb{R} - \{0\}$ and $z \in \mathbb{C}$,

$$\phi_{\mathbb{C},z}(d(a)) = \int_0^{\frac{\pi}{2}} (a^2\cos^2(\theta) + a^{-2}\sin^2(\theta))^{-1-2z}\sin(2\theta)\, d\theta.$$

Proof. For all $a \in \mathbb{R} - \{0\}$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, one has

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} a\alpha & -a\overline{\beta} \\ a^{-1}\beta & a^{-1}\overline{\alpha} \end{pmatrix}.$$

Thus, by Proposition 3.5.10, for all $0 \leq \theta_1, \theta_2, \theta_3 \leq 2\pi$

$$\Delta_{\widetilde{P}_2^+(\mathbb{C})}(d(a)k(\theta_1,\theta_2,\theta_3)) = \Delta_{\widetilde{P}_2^+(\mathbb{C})}(d(x)),$$

where

$$x = |a\alpha|^2 + |a^{-1}\beta|^2 = a^2\cos^2(\theta_1) + a^{-2}\sin^2(\theta_1).$$

The corollary follows from Proposition 3.5.10 and from Proposition C.1.3. □

Because the maps $\phi_{\mathbb{R},z}$ and $\phi_{\mathbb{C},z}$ are bi-$SO(2)$- and bi-$SU(2)$-invariant, respectively, they are determined by their values on $\widetilde{A}_2^+$. To obtain an even more explicit formula one can use the following proposition. For $g \in SL_d(\mathbb{C})$, let $\|g\|$ be the Hilbert-Schmidt norm of g, that is,

$$\|g\| = \Big(\sum_{1\leq i,j\leq d} |g_{ij}|^2\Big)^{\frac{1}{2}}.$$

Proposition 3.5.12. *Let $g \in SL_2(\mathbb{C})$. Then*

$$g = k_1 d\left(\sqrt{\frac{\|g\|^2 + \sqrt{\|g\|^4 - 4}}{2}}\right) k_2$$

for some $k_1, k_2 \in SU(2)$.

Proof. By the Cartan decomposition there are $k_1, k_2 \in SU(2)$ and $a \geq 1$ such that $g = k_1 d(a) k_2$. Write

$$k_i = \begin{pmatrix} \alpha_i & -\overline{\beta}_i \\ \beta_i & \overline{\alpha}_i \end{pmatrix},$$

for $i = 1, 2$. Then

$$\begin{aligned}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} &= \begin{pmatrix} \alpha_1 & -\overline{\beta}_1 \\ \beta_1 & \overline{\alpha}_1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \alpha_2 & -\overline{\beta}_2 \\ \beta_2 & \overline{\alpha}_2 \end{pmatrix} \\ &= \begin{pmatrix} a\alpha_1\alpha_2 - a^{-1}\overline{\beta}_1\beta_2 & -a\alpha_1\overline{\beta}_2 - a^{-1}\overline{\beta}_1\overline{\alpha}_2 \\ a\beta_1\alpha_2 + a^{-1}\overline{\alpha}_1\beta_2 & -a\beta_1\overline{\beta}_2 + a^{-1}\overline{\alpha}_1\overline{\alpha}_2 \end{pmatrix}.\end{aligned}$$

One computes

$$\begin{aligned}|g_{11}|^2 + |g_{12}|^2 &= a^2|\alpha_1|^2 + a^{-2}|\beta_1|^2, \\ |g_{21}|^2 + |g_{22}|^2 &= a^{-2}|\alpha_1|^2 + a^2|\beta_1|^2,\end{aligned}$$

Thus $\|g\|^2 = a^2 + a^{-2}$. Because $0 = a^4 - a^2\|g\|^2 + 1$ and $a \geq 1$, it follows that

$$a \in \left\{ \pm\sqrt{\frac{\|g\|^2 \pm \sqrt{\|g\|^4 - 4}}{2}} \right\}.$$

Because $a \geq 1$, $\|g\|^2 \geq 2$. Since $x - \sqrt{x^2-4} \leq 2$ for all $x \geq 2$, it follows that

$$a = \sqrt{\frac{\|g\|^2 + \sqrt{\|g\|^4 - 4}}{2}}.$$

□

Corollary 3.5.13. *1) For all $g \in SL_2(\mathbb{R})$ and $z \in \mathbb{C}$,*

$$\phi_{\mathbb{R},z}(g) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\|g\|^2 + \sqrt{\|g\|^4 - 4}}{2}\cos^2(\theta) + \left(\frac{\|g\|^2 + \sqrt{\|g\|^4 - 4}}{2}\right)^{-1}\sin^2(\theta)\right)^{-\frac{1}{2}-z} d\theta.$$

2) For all $g \in SL_2(\mathbb{C})$ and $z \in \mathbb{C}$,

$$\phi_{\mathbb{C},z}(g) = \int_0^{\frac{\pi}{2}} \left(\frac{\|g\|^2 + \sqrt{\|g\|^4 - 4}}{2}\cos^2(\theta) + \left(\frac{\|g\|^2 + \sqrt{\|g\|^4 - 4}}{2}\right)^{-1}\sin^2(\theta)\right)^{-1-2z} \sin(2\theta)\, d\theta.$$

Chapter 4

Almost Square-Integrable Matrix Coefficients

4.1 Weak containment

Let G be a topological group, and let $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ be unitary representations of G. Recall that $\pi \subset \sigma$ if π is equivalent to a subrepresentation of σ. Moreover, recall that containment defines a partial order on $\mathcal{R}(G)$.

Definition 4.1.1. The unitary representation $(\pi, \mathcal{H}_\pi)$ is *weakly contained* in $(\sigma, \mathcal{H}_\sigma)$, written $\pi \prec \sigma$, if the diagonal matrix coefficients of π can be approximated by sums of diagonal matrix coefficients of σ uniformly on compact subsets of G, that is, for every $u \in \mathcal{H}_\pi$, $\epsilon > 0$ and compact subset $Q \subset G$ there exist $v_1, v_2, \ldots, v_n \in \mathcal{H}_\sigma$ such that

$$\left| \langle \pi(g)u, u \rangle - \sum_{1 \leq i \leq n} \langle \sigma(g)v_i, v_i \rangle \right| < \epsilon \quad \forall g \in Q.$$

Since matrix coefficients depend only on the equivalence class of unitary representations, weak containment is invariant under unitary equivalence. Moreover, weak containment is a weaker condition than containment. It defines only a preorder on $\mathcal{R}(G)$. Indeed, for any $\pi \in \mathcal{R}(G)$ and any non-empty index set I one has that $\pi \prec \pi^{\bigoplus i \in I}$ and $\pi^{\bigoplus i \in I} \prec \pi$. But while $\pi \subset \pi^{\bigoplus i \in I}$, π is not equivalent to $\pi^{\bigoplus i \in I}$, unless I has only one element.

Definition 4.1.2. If $\pi \prec \sigma$ and $\sigma \prec \pi$, then $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are *weakly equivalent*, written $\pi \sim \sigma$.

The following proposition is the analogue to Proposition 1.2.13. It is a combination of [BdlHV08, Prop. F.3.1] and [BdlHV08, Prop. F.3.2].

Proposition 4.1.3. *Let $\{(\pi_i, \mathcal{H}_{\pi_i})\}_{i \in I}$ and $\{(\sigma_i, \mathcal{H}_{\sigma_i})\}_{i \in I}$ be families of unitary representations of G such that $\pi_i \prec \sigma_i$. Then:*

1) $\pi_i \otimes \pi_j \prec \sigma_i \otimes \sigma_j$ $\forall i, j \in I$,

2) $\bigoplus_{i \in I} \pi_i \prec \bigoplus_{i \in I} \sigma_i$.

Proof. 1) Let $u \in \mathcal{H}_{\pi_i}$, $v \in \mathcal{H}_{\pi_j}$. Take $0 < \epsilon < 1$, and let $Q \subset G$ be a compact subset. Then there are $\xi_1, \xi_2, \ldots, \xi_m \in \mathcal{H}_{\sigma_i}$, $\eta_1, \eta_2, \ldots, \eta_n \in \mathcal{H}_{\sigma_j}$ such that

$$\left| \langle \pi_i(g)u, u \rangle - \sum_{1 \leq k \leq m} \langle \sigma_i(g)\xi_k, \xi_k \rangle \right| < \frac{\epsilon}{3} \quad \forall g \in Q,$$

$$\left| \langle \pi_j(g)v, v \rangle - \sum_{1 \leq l \leq n} \langle \sigma_j(g)\eta_l, \eta_l \rangle \right| < \frac{\epsilon}{3} \quad \forall g \in Q.$$

Then for all $g \in Q$

$$\begin{aligned}
&\left|\langle \pi_i \otimes \pi_j(g)u \otimes v, u \otimes v\rangle - \sum_{1\le k\le m}\sum_{1\le l\le n}\langle \sigma_i \otimes \sigma_j(g)\xi_k \otimes \eta_l, \xi_k \otimes \eta_l\rangle\right| \\
&\quad\le \left|\left(\langle \pi_i(g)u, u\rangle - \sum_{1\le k\le m}\langle \sigma_i(g)\xi_k, \xi_k\rangle\right)\langle \pi_j(g)v, v\rangle\right. \\
&\qquad\left. + \left(\sum_{1\le k\le m}\langle \sigma_i(g)\xi_k, \xi_k\rangle\right)\left(\langle \pi_j(g)v, v\rangle - \sum_{1\le l\le n}\langle \sigma_j(g)\eta_l, \eta_l\rangle\right)\right| \\
&\quad\le \left|\langle \pi_i(g)u, u\rangle - \sum_{1\le k\le m}\langle \sigma_i(g)\xi_k, \xi_k\rangle\right| \\
&\qquad + \left|\,|\langle \pi_i(g)u, u\rangle| + \frac{\epsilon}{3}\right|\left|\langle \pi_j(g)v, v\rangle - \sum_{1\le l\le n}\langle \sigma_j(g)\eta_l, \eta_l\rangle\right| \\
&\quad< \frac{\epsilon}{3} + \left(1 + \frac{\epsilon}{3}\right)\frac{\epsilon}{3} < \epsilon
\end{aligned}$$

The claim follows, because the set $\{u \otimes v \colon u \in \mathcal{H}_{\pi_i}, v \in \mathcal{H}_{\pi_j}\}$ is total in $\mathcal{H}_{\pi_i} \otimes \mathcal{H}_{\pi_j}$.

2) This follows immediately from the definition of weak containment and the construction of the direct sum of unitary representations.

□

Lemma 4.1.4 ([BdlHV08, Rem. F.1.2])**.** *Suppose $\pi \prec \sigma$. Then the diagonal matrix coefficients of π with respect to unit vectors can be approximated uniformly on compact subsets by convex combinations of diagonal matrix coefficient of σ with respect to unit vectors.*

Proof. Take a unit vector $u \in \mathcal{H}_\pi$, and let $0 < \epsilon < 1$. Take a compact subset $Q \subset G$, and assume that $e_G \in Q$. Then there exist $v_1, v_2, \ldots, v_n \in \mathcal{H}_\sigma$ such that

$$\left|\langle \pi(g)u, u\rangle - \sum_{1\le k\le n}\langle \sigma(g)v_k, v_k\rangle\right| < \epsilon \quad \forall g \in Q.$$

Set $\phi_k(g) \coloneqq \langle \sigma(g)v_k, v_k\rangle$. Then

$$\frac{1}{\sum_{1\le l\le n}\phi_l(e_G)}\sum_{1\le k\le n}\phi_k(g) = \sum_{1\le k\le n}\frac{\phi_k(e_G)}{\sum_{1\le l\le n}\phi_l(e_G)}\left(\frac{\phi_k(g)}{\phi_k(e_G)}\right).$$

Then, because $\langle \pi(e_G)u, u\rangle = 1$,

$$\left|1 - \sum_{1\le l\le n}\phi_k(e_G)\right| < \epsilon.$$

Thus for all $g \in Q$

$$\begin{aligned}
&\left|\langle \pi(g)u, u\rangle - \sum_{1\le k\le n}\frac{\phi_k(e_G)}{\sum_{1\le l\le n}\phi_l(e_G)}\left(\frac{\phi_k(g)}{\phi_k(e_G)}\right)\right| \\
&\quad= \frac{1}{\sum_{1\le l\le n}\phi_l(e_G)}\left|\left(\sum_{1\le l\le n}\phi_k(e_G)\right)\langle \pi(g)u, u\rangle - \sum_{1\le k\le n}\langle \sigma(g)v_k, v_k\rangle\right| \\
&\quad\le \frac{1}{1-\epsilon}\left(\left|\langle \pi(g)u, u\rangle - \sum_{1\le k\le n}\langle \sigma(g)v_k, v_k\rangle\right| + \left(1 - \sum_{1\le l\le n}\phi_k(e_G)\right)|\langle \pi(g)u, u\rangle|\right) \\
&\quad\le \frac{2\epsilon}{1-\epsilon}.
\end{aligned}$$

□

The following proposition asserts that if G is compact, then containment and weak containment are strongly related.

Proposition 4.1.5 ([BdlHV08, Prop. 1.21]). *Suppose that G is compact. Then $\pi \prec \sigma$ if and only if every irreducible subrepresentation of π is contained in σ.*

In his paper [Eym64] Pierre Eymard developed the following equivalent definitions of weak containment which will be used later on. Their proofs are omitted here since they would be too involved. In particular, the proof of Theorem 4.1.6 in [Eym64] uses the representation theory of C^*-algebras and the fact that the group algebra of G can be extended to a C^*-algebra.

Theorem 4.1.6 ([Eym64, Prop. 1.21]). *One has $\pi \prec \sigma$ if and only if*

$$\|\pi^1(f)\| \leq \|\sigma^1(f)\| \quad \forall f \in L^1(G).$$

Theorem 4.1.7 ([Eym64, Lem. 2.14]). *One has $\pi \prec \sigma$ if and only if for every $u, v \in \mathcal{H}_\pi$, every $\epsilon > 0$ and every compact subset $Q \subset G$ there exist $\xi_1, \xi_2, \ldots, \xi_n$, $\eta_1, \eta_2, \ldots, \eta_n \in \mathcal{H}_\sigma$ such that*

$$\left| \langle \pi(g)u, v \rangle - \sum_{1 \leq i \leq n} \langle \sigma(g)\xi_i, \eta_i \rangle \right| < \epsilon \quad \forall g \in Q$$

and

$$\sum_{1 \leq i \leq n} \|\xi_i\| \|\eta_i\| \leq \|u\| \|v\|.$$

4.2 Fell's topology

Recall that the unitary dual $\widehat{G}$ is the subset of $\mathcal{R}(G)$ which consists of all equivalence classes of irreducible unitary representations of G. Henceforth suppose that G is locally compact.

Proposition 4.2.1 ([BdlHV08, Prop. F.1.4]). *Suppose $\pi \prec \sigma$, and assume that π is irreducible. Then the normalised positive definite functions associated to π can be approximated by normalised positive definite functions associated to σ on compact subsets of G.*

Proof. Let $\phi \in \mathcal{P}_1(G)$ be associated to π. Let $\mathcal{P}_1(G, \sigma)$ be the set of normalised positive definite functions associated to σ, and let $K = \overline{\text{conv}(\mathcal{P}_1(G, \sigma))}$ be the closure of the convex hull with respect to the weak* topology, which is compact. Then, by Raikov's Theorem and by Lemma 4.1.4, $\phi \in K \subset \mathcal{P}_0(G)$. Because π is assumed to be irreducible, by Theorem 1.5.15 and by Lemma 1.5.17, ϕ lies in $\text{ext}(\mathcal{P}_0(G))$ and thus lies also in $\text{ext}(K)$. By Milman's Theorem (see [Rud91, Thm. 3.25]), this implies that ϕ is in the closure of $\mathcal{P}_1(G, \sigma)$ with respect to the weak* topology. The proposition follows, because, by Theorem 1.5.19, the weak* topology and the topology of uniform convergence on compact subsets coincide on $\mathcal{P}_1(G)$. □

Proposition 4.2.1 gives an interesting corollary.

Corollary 4.2.2 ([BdlHV08, Cor. F.1.5]). *One has that $1_G \prec \pi$ if and only if for every $\epsilon > 0$ and every compact subset $Q \subset G$ there exists a unit vector $u \in \mathcal{H}_\pi$ such that*

$$\|\pi(g)u - u\| < \epsilon \quad \forall g \in Q. \tag{4.1}$$

Proof. Suppose $1_G \prec \pi$. Because 1_G is irreducible, by Proposition 4.2.1, there exists a unit vector $u \in \mathcal{H}_\pi$ such that $|1 - \langle \pi(g)u, u \rangle| < \frac{\epsilon}{2}$ for all $g \in Q$. Then

$$\|\pi(g)u - u\|^2 = 2|1 - \text{Re}(\langle \pi(g)u, u \rangle)| \leq 2|1 - \langle \pi(g)u, u \rangle| < \epsilon.$$

Conversely suppose that $u \in \mathcal{H}_\pi$ is a unit vector which satisfies (4.1). Then, by the Cauchy-Schwarz inequality,

$$|1 - \langle \pi(g)u, u \rangle| = |\langle u - \pi(g)u, u \rangle| \leq \|\pi(g)u - u\| < \epsilon \quad \forall g \in Q.$$

□

Equip now the unitary dual with a topology. For this recall the Gelfand-Naimark-Segal construction from Section 1.5.

Definition 4.2.3. Let $[\pi] \in \widehat{G}$ be an irreducible unitary representation, and let $Q \subset G$ be a compact subset. Let $\epsilon > 0$, and take diagonal matrix coefficients $\phi_1, \phi_2, \ldots, \phi_n \in \mathcal{P}_1(G)$ of π. Let $W(\pi, \phi_1, \phi_2, \ldots, \phi_n, Q, \epsilon)$ be the set of all equivalence classes $[\sigma] \in \widehat{G}$ such that for every $1 \leq i \leq n$ there is a diagonal matrix coefficient $\psi_i \in \mathcal{P}_1(G)$ of σ such that

$$|\phi_i(g) - \psi_i(g)| < \epsilon \quad \forall g \in Q.$$

The *Fell topology* is the topology on $\widehat{G}$ generated by the basis

$$\{W(\pi, \phi_1, \phi_2, \ldots, \phi_n, Q, \epsilon) \colon [\pi] \in \widehat{G},\ \phi_i \in \mathcal{P}_1(G) \text{ associated to } \pi,\ Q \subset G \text{ compact},\ \epsilon > 0\}.$$

Henceforth endow $\widehat{G}$ with the Fell topology. One of the main features of the Fell topology is its connection with the Gelfand-Naimark-Segal construction.

Proposition 4.2.4 ([BdlH20, Prop. 1.C.8])**.** *The map $\mathcal{P}_1(G) \to \widehat{G}$, $\phi \mapsto [\pi_\phi]$ is continuous, surjective and open with respect to the weak* topology on $\mathcal{P}_1(G)$.*

Proof. By the Gelfand-Naimark-Segal construction, the map is surjective. Recall that by Raikov's Theorem the weak* topology and the topology of uniform convergence on compact subsets of G coincide on $\mathcal{P}_1(G)$. The preimage of a set $W(\pi, \phi_1, \phi_2, \ldots, \phi_n, Q, \epsilon) \subset \widehat{G}$, as defined in Definition 4.2.3, is

$$\bigcap_{1 \leq i \leq n} N(\phi_i, Q, \epsilon),$$

which is open. Hence the map is continuous. On the other hand, the image of $N(\phi, Q, \epsilon)$ is the open set $W(\pi, \phi, \epsilon) \subset \widehat{G}$. □

Concerning convergent nets of $\widehat{G}$ one has

Lemma 4.2.5 ([BdlHV08, Prop. F.2.2])**.** *If a net $\{[\pi_i]\}_{i \in I}$ in $\widehat{G}$ converges to $[\pi] \in \widehat{G}$, then $\pi \prec \bigoplus_{i \in I} \pi_i$.*

Proof. The lemma follows immediately from the definitions. □

Corollary 4.2.6 ([BdlH20, Prop. 1.D.8])**.** *If G is compact, then its unitary dual $\widehat{G}$ is discrete.*

Proof. This follows from Proposition 4.1.5 and from Lemma 4.2.5. □

4.3 Weak containment in the regular representation

Let G be a locally compact. Recall from Chapter 1 the definition of the regular representation λ_G. The goal of this section is to prove the following theorem.

Theorem 4.3.1 ([CHH88, Thm. 1])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Suppose there is a dense subset $U \subset \mathcal{H}_\pi$ such that the diagonal matrix coefficients $\{\phi_{u,u} \colon u \in U\}$ are contained in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$. Then $\pi \prec \lambda_G$.*

For the proof of Theorem 4.3.1 the following lemmas will be used. Fix a Haar measure dx on G. For a compact subset $Q \subset G$ write

$$|Q| = \int_Q dx,$$

and recall that $|Q| < \infty$.

Lemma 4.3.2 ([CHH88, p. 100])**.** *For every compact subset $Q \subset G$ with non-empty interior, the sequence $\{Q^n\}_{n\geq 1}$ grows at most exponentially, that is, there are constants $C, M > 0$ such that $|Q^n| \leq CM^{n-1}$.*

Proof. Note that for each $n \geq 1$ $Q^n \subset G$ is compact. Because Q has a non-empty interior, $\{g \operatorname{int} Q\}_{g\in G}$ is an open cover of Q^2, and thus there are finitely many $g_1, \ldots, g_M \in G$ such that

$$Q^2 \subset \bigcup_{1\leq i\leq M} g_i Q.$$

Moreover,

$$Q^n \subset \bigcup_{1\leq i\leq M} g_i Q^{n-1} \subset \bigcup_{1\leq i_1,i_2,\ldots,i_{n-1}\leq M} \Big(\prod_{1\leq j\leq n-1} g_{i_j}\Big) Q;$$

whence

$$|Q^n| \leq \sum_{1\leq i_1,i_2,\ldots,i_{n-1}\leq M} |Q| = |Q| M^{n-1}.$$

□

Henceforth, to simplify notation, for $n \geq 1$ and $f \in L^1(G)$ write

$$f^{(*n)} \coloneqq \underbrace{f * f * \ldots * f}_{n \text{ times}}.$$

By Minkowski's integral inequality, $f^{(*n)} \in L^1(G)$, and if $g \in L^p(G)$ for $1 \leq p \leq \infty$, $f*g \in L^p(G)$. Moreover, if $H < G$ is a closed subgroup with Haar measure dh, write

$$f_1 *_H f_2(g) \coloneqq \int_H f_1(h) f_2(h^{-1}g)\, dh$$

for $f_1, f_2 \in C_c(G)$ and $g \in G$.

Lemma 4.3.3. *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G, and suppose $f \in C_c(G)$ is self-adjoint, that is, $f^* = f$. Then for any dense subset $U \subset \mathcal{H}_\pi$*

$$\|\pi^1(f)\| = \sup_{u\in U} \lim_{n\to\infty} \langle \pi^1(f^{(*2n)})u, u\rangle^{\frac{1}{2n}}. \tag{4.2}$$

Proof. Because f is self-adjoint, so is $\pi^1(f) \in \mathcal{L}(\mathcal{H}_\pi)$. Then, by the spectral theorem for normal bounded operators (see [Rud91, Thm. 12.23]), there is a projection-valued measure P on the spectrum $\sigma(\pi^1(f)) \subset \mathbb{C}$ of $\pi^1(f)$ such that for every polynomial $p \in \mathbb{C}[X]$

$$p(\pi^1(f)) = \int_{\sigma(\pi^1(f))} p(t)\, dP(t).$$

Hence, by the generalized version of Hölder's inequality, for all $u \in U$

$$\begin{aligned}
\|\pi^1(f)u\|^2 = \langle \pi^1(f^{(*2)})u, u\rangle &= \int_{\sigma(\pi^1(f))} t^2\, d\langle P(t)u, u\rangle \\
&\leq \left(\int_{\sigma(\pi^1(f))} t^{2n}\, d\langle P(t)u, u\rangle\right)^{\frac{1}{n}} \left(\int_{\sigma(\pi^1(f))} d\langle P(t)u, u\rangle\right)^{1-\frac{1}{n}} \\
&\leq \langle \pi^1(f^{(*2n)})u, u\rangle^{\frac{1}{n}} \|u\|^{2(1-\frac{1}{n})}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\pi^1(f)\| &= \sup_{u\in U} \|\pi^1(f)u\| \|u\|^{-1} \\
&\leq \sup_{u\in U} \lim_{n\to\infty} \langle \pi^1(f^{(*2n)})u, u\rangle^{\frac{1}{2n}} \|u\|^{-\frac{1}{n}} = \sup_{u\in U} \lim_{n\to\infty} \langle \pi^1(f^{(*2n)})u, u\rangle^{\frac{1}{2n}}.
\end{aligned}$$

The converse inequality follows from the Cauchy-Schwarz inequality. □

Corollary 4.3.4. *Let $(\pi, \mathcal{H}_\pi)$ and $U \subset \mathcal{H}_\pi$ be as in Lemma 4.3.3. Then*

$$\|\pi^1(f)\| = \sup_{u\in U} \lim_{n\to\infty} \langle \pi^1(f^* * f)^{(*2n)} u, u\rangle^{\frac{1}{4n}} \quad \forall f \in C_c(G).$$

Proof. Note that $f^* * f \in C_c(G)$ is self-adjoint. Because

$$\|\pi^1(f)\|^2 = \|\pi^1(f^*)\pi^1(f)\| = \|\pi^1(f^* * f)\|,$$

the corollary follows from Lemma 4.3.3. □

In the special case of the regular representation one has

Proposition 4.3.5. *For every $f \in L^1(G)$*

$$\|\lambda^1_G(f)\| = \lim_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}}. \tag{4.3}$$

Proof. Recall that $C_c(G)$ is a dense subspace in $L^2(G)$. In particular, for any $h \in C_c(G)$ the map $g \mapsto \langle \lambda_G(g)h, h\rangle$ lies in $C_c(G)$. By the Cauchy-Schwarz inequality and by Corollary 4.3.4, for every $f \in L^1(G)$

$$\begin{aligned}\|\lambda^1_G(f)\| &= \sup_{h\in C_c(G)} \lim_{n\to\infty} \left(\int_G (f^* * f)^{(*2n)}(x) \langle \lambda_G(x)h, h\rangle \, dx \right)^{\frac{1}{4n}} \\ &\leq \sup_{h\in C_c(G)} \liminf_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}} \|\langle \lambda_G(\cdot)h, h\rangle\|_2^{\frac{1}{4n}} = \liminf_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}}.\end{aligned}$$

For the converse inequality, note that

$$\begin{aligned}\|(f^* * f)^{(*2n)}\|_2 &= \|\lambda^1_G((f^* * f)^{(*(2n-2))})(f^* * f)^{(*2)}\|_2 \\ &\leq \|\lambda^1_G(f^* * f)\|^{2n-2} \|(f^* * f)^{(*2)}\|_2 = \|\lambda^1_G(f)\|^{4n-4} \|(f^* * f)^{(*2)}\|_2.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \|(f^* * f)^{(*2n)}\|^{\frac{1}{4n}} \leq \|\lambda^1_G(f)\|,$$

which completes the proof. □

Proof of Theorem 4.3.1. Recall that any unitary representation of G can be written as the direct sum of cyclic subrepresentations. Thus by Proposition 1.1.14 and by Proposition 4.1.3 a unitary representation of G is weakly contained in λ_G if and only if each of its cyclic subrepresentations is as well. Hence assume π is cyclic, and fix a cyclic vector $u \in \mathcal{H}_\pi$.

By Theorem 4.1.6 it is enough to show that

$$\|\pi^1(f)\| \leq \|\lambda^1_G(f)\| \quad \forall f \in C_c(G).$$

Because π is assumed to be cyclic, π^1 is cyclic too. Hence, $\{\pi^1(f)u \colon f \in C_c(G)\}$ is a dense subspace of $\mathcal{H}_\pi$. For $g \in G$ set

$$\psi_f(g) \coloneqq \langle \pi(g)\pi^1(f)u, \pi^1(f)u\rangle.$$

Then

$$\begin{aligned}\psi_f(g) &= \int_G \int_G f(x)\overline{f(y)} \langle \pi(g)\pi(x)u, \pi(y)z\rangle \, dx \, dy \\ &= \int_G \int_G f(x)\overline{f(y)} \phi_{u,u}(y^{-1}gx) \, dx \, dy \\ &= \int_G f(x)(\overline{f} * \phi_{u,u})(gx) \, dx = (\overline{f} * \phi_{u,u} * f^\vee)(g).\end{aligned}$$

This implies that, for every $f \in C_c(G)$ and $\epsilon > 0$, $\psi_f \in L^{2+\epsilon}(G)$. Next let $h \coloneqq (f^* * f)^{(*2)}$ and set $Q \coloneqq \operatorname{supp}(h)$. By the Cauchy-Schwarz inequality and by Hölder's inequality,

$$\begin{aligned}\left| \int_G (f^* * f)^{(*n)}(x)\psi_f(x)\,dx \right|^2 &= \left| \int_Q h^{(*n)}(x)\psi_f(x)\,dx \right|^2 \\ &\leq \left(\int_Q |h^{(*n)}(x)|^2\,dx \right)\left(\int_Q |\psi_f(x)|^2\,dx \right) \\ &\leq \|h^{(*n)}\|_2^2 \left(\int_Q |\psi_f(x)|^{2+\epsilon}\,dx \right)^{\frac{2}{2+\epsilon}} \left(\int_Q dx \right)^{\frac{\epsilon}{2+\epsilon}} \\ &\leq \|h^{(*n)}\|_2^2 \|\psi_f\|_{2+\epsilon}^2 (CM^{n-1})^{\frac{\epsilon}{2+\epsilon}},\end{aligned}$$

where the last inequality follows from Lemma 4.3.2. Then, by Lemma 4.3.3,

$$\begin{aligned}\|\pi^1(f)\| &= \sup_{h\in C_c(G)} \lim_{n\to\infty} |\langle \pi^1((f^* * f)^{(*n)})\pi^1(h)u, \pi^1(h)u\rangle|^{\frac{1}{4n}} \\ &\leq \liminf_{n\to\infty} \|(f^* * f)^{(*n)}\|_2^{\frac{1}{4n}} M^{\frac{\epsilon}{8(2+\epsilon)}}.\end{aligned}$$

Since ϵ is arbitrary,

$$\|\pi^1(f)\| \leq \liminf_{n\to\infty} \|(f^* * f)^{(*n)}\|_2^{\frac{1}{4n}}.$$

Then, by Proposition 4.3.5, $\|\pi^1(f)\| \leq \|\lambda_G^1(f)\|$. □

4.4 The Harish-Chandra function

Henceforth suppose G has a compact subgroup K, and let dk be the normalized Haar measure on K. The following two definitions can be found in [Wan14].

Definition 4.4.1. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. For a vector $v \in \mathcal{H}_\pi$ let $\langle K \cdot v\rangle$ be the closed linear span of $\{\pi(k)v : k \in K\}$. If $\langle K \cdot v\rangle$ is finite-dimensional, then v is called *K-finite.*

Recall that, by the Peter-Weyl Theorem, every unitary representation of a compact group is the direct sum of irreducible unitary representations. Moreover, by Theorem 1.6.2, every irreducible unitary representation of a compact group is finite-dimensional. Thus, if $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G, then $\mathcal{H}_\pi$ contains a dense subspace of K-finite vectors.

Introduce also the following notion:

Definition 4.4.2. Let Ψ be a bi-K-invariant real-valued function on G such that $0 \leq \Psi \leq 1$ and $\Psi(e_G) = 1$. A unitary representation $(\pi, \mathcal{H}_\pi)$ of G is *(K, Ψ)-bounded* if for every K-finite vectors $u, v \in \mathcal{H}_\pi$

$$|\langle \pi(g)u, v\rangle| \leq \dim\langle K \cdot u\rangle^{\frac{1}{2}} \dim\langle K \cdot v\rangle^{\frac{1}{2}} \|u\|\|v\|\Psi(g).$$

Remark. Note that every unitary representation of G is $(K, 1)$-bounded.

The following proposition and its proof are generalized versions of the first step in the proof of [CHH88, Thm. 2].

Proposition 4.4.3. *Suppose $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are unitary representations of G, and suppose that σ is (K, Ψ)-bounded. If $\pi \prec \sigma$, then π is (K, Ψ)-bounded.*

Proof. Take $u, v \in \mathcal{H}_\pi$ and suppose that they are K-finite. Let $e_u, e_v \in C(K)$ be as in Proposition 1.6.8 with respect to $\operatorname{Res}_K^G(\pi)$. In particular, $\operatorname{Res}_K^G(\pi)^1(e_u)$ and $\operatorname{Res}_K^G(\pi)^1(e_v)$ are orthogonal projections, and

$$\operatorname{Res}_K^G(\pi)^1(e_u)u = u, \quad \operatorname{Res}_K^G(\pi)^1(e_v)v = v.$$

Then, since $\bar{e}_u = e_u^\vee$ and $\bar{e}_v = e_v^\vee$,

$$\begin{aligned}(\bar{e}_v *_K \phi_{u,v} *_K \bar{e}_u)(g) &= \int_K \int_K \bar{e}_v(k_1)\bar{e}_u(k_2^{-1}k_1^{-1}g)\langle \pi(k_2)u, v\rangle \, dk_1 \, dk_2 \\ &= \int_K \int_K \bar{e}_v(k_1)\bar{e}_u(k_2^{-1})\langle \pi(k_1^{-1}gk_2)u, v\rangle \, dk_1 \, dk_2 \\ &= \int_K \int_K \bar{e}_v(k_1)e_u(k_2)\langle \pi(gk_2)u, \pi(k_1)v\rangle \, dk_1 \, dk_2 \\ &= \left\langle \pi(g)\operatorname{Res}_K^G(\pi)^1(e_u)u, \operatorname{Res}_K^G(\pi)^1(e_v)v \right\rangle = \phi_{u,v}(g).\end{aligned}$$

By Theorem 4.1.7, $\phi_{u,v}$ can be approximated uniformly on compact subsets by sums $\sum_{1\le i\le n}\psi_i$ of matrix coefficients $\psi_i = \langle \sigma(\cdot)\xi_i, \eta_i\rangle$ of σ such that

$$\sum_{1\le i\le n} \|\xi_i\|\|\eta_i\| \le \|u\|\|v\|.$$

$\phi_{u,v}$ can be approximated uniformly on compact subsets of G by the sums $\sum_{1\le i\le n}\bar{e}_v *_K \psi_i *_K \bar{e}_u$. Note that

$$(\bar{e}_v *_K \psi_i *_K \bar{e}_u)(g) = \left\langle \sigma(g)\operatorname{Res}_K^G(\sigma)^1(e_u)\xi_i, \operatorname{Res}_K^G(\sigma)^1(e_v)\eta_i \right\rangle$$

and that, because $\operatorname{Res}_K^G(\sigma)^1(e_u)$ and $\operatorname{Res}_K^G(\sigma)^1(e_v)$ are non-zero orthogonal projections,

$$\sum_{1\le i\le n} \|\operatorname{Res}_K^G(\sigma)^1(e_u)\xi_i\|\|\operatorname{Res}_K^G(\sigma)^1(e_v)\eta_i\| \le \sum_{1\le i\le n} \|\xi_i\|\|\eta_i\| \le \|u\|\|v\|.$$

Hence it is enough to show that for all $\xi, \eta \in \mathcal{H}_\sigma$ with $\operatorname{Res}_K^G(\sigma)^1(e_u)\xi = \xi$ and $\operatorname{Res}_K^G(\sigma)^1(e_v)\eta = \eta$,

$$\langle \sigma(g)\xi, \eta\rangle \le \dim\langle K\cdot u\rangle^{\frac{1}{2}} \dim\langle K\cdot v\rangle^{\frac{1}{2}} \|\xi\|\|\eta\|\Psi(g) \quad \forall g \in G.$$

By construction, e_u and e_v are K-finite vectors of the regular representation $(\lambda_K, L^2(K))$, and, by Theorem 1.4.2, for all $k \in K$

$$\begin{aligned}\sigma(k)\operatorname{Res}_K^G(\sigma)^1(e_u) &= \operatorname{Res}_K^G(\sigma)^1(\lambda_K(k)e_u), \\ \sigma(k)\operatorname{Res}_K^G(\sigma)^1(e_v) &= \operatorname{Res}_K^G(\sigma)^1(\lambda_K(k)e_v).\end{aligned}$$

Hence, if $\sigma^1(e_u)\xi = \xi$ and $\sigma^1(e_v)\eta = \eta$, then ξ and η are K-finite vectors and

$$\begin{aligned}\dim\langle K\cdot\xi\rangle &\le \dim\langle K\cdot e_u\rangle = \|e_u\|_2^2 = \dim\langle K\cdot u\rangle, \\ \dim\langle K\cdot\eta\rangle &\le \dim\langle K\cdot e_v\rangle = \|e_v\|_2^2 = \dim\langle K\cdot v\rangle.\end{aligned}$$

Because σ is assumed to be (K,Ψ)-bounded, this completes the proof. □

Corollary 4.4.4. *Let Ψ be a bi-K-invariant function on G with $0 \le \Psi \le 1$. The (possibly empty) subset*

$$\widehat{G}_{(K,\Psi)} := \{[\pi] \in \widehat{G} : \pi \text{ is } (K,\Psi)\text{-bounded}\} \subset \widehat{G}$$

is closed with respect to the Fell topology.

Proof. The corollary is a consequence of Lemma 4.2.5 and of Proposition 4.4.3. □

From now on suppose that G also has a closed subgroup $P < G$ such that $G = KP$, which will be called an *Iwasawa decomposition*. By Proposition A.4.13,

$$\int_G f(x)\,dx = \int_K \int_P f(kp)\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{-1} dp\,dk \quad \forall f \in C_c(G), \tag{4.4}$$

where the modular function of P is extended to G by setting $\Delta_P(kp) = \Delta_P(p)$.

Definition 4.4.5. The *Harish-Chandra function* of (G, K) is the function $\Xi_{(G,K)}\colon G \to \mathbb{R}$,

$$\Xi_{(G,K)}(g) \coloneqq \int_K \left(\frac{\Delta_P(g^{-1}k)}{\Delta_G(g^{-1}k)}\right)^{\frac{1}{2}} dk.$$

It follows from the bi-K-invariance of the Haar measure dk on K that $\Xi_{(G,K)}$ is bi-K-invariance. In particular, by Proposition 3.4.3, $\Xi_{(G,K)}$ is a spherical function of (G, K) and it is a diagonal matrix coefficent of $\mathrm{ind}_P^G(1_P)$ with respect to a K-invariant unit vector of $\mathcal{H}_{\mathrm{ind}_P^G(1_P)}$.

Remark. The above definition of the Harish-Chandra function is quite general. In [CHH88] it assumed that G is a semisimple algebraic group over a local field, and $G = KP$ is the Iwasawa decomposition such that $K < G$ is a *good* maximal compact subgroup and $P < G$ is a minimal parabolic subgroup. However, the proof of [CHH88, Thm. 2] can be extended to the more general case of a group with just an Iwasawa decomposition $G = KP$.

Remark. If (G, K) is a Gelfand pair, then, by Proposition 3.1.6, G is unimodular and hence

$$\Xi_{(G,K)}(g) \coloneqq \int_K \Delta_P^{\frac{1}{2}}(g^{-1}k)\, dk.$$

Throughout this thesis every pair (G, K) for which quantitative estimates or explicit formulas of the Harish-Chandra function are computed, will be Gelfand.

The following theorem is the main result of this section, and it is a special case of [CHH88, Thm. 2].

Theorem 4.4.6. *The regular representation of G is $(K, \Xi_{(G,K)})$-bounded.*

Proof. The proof consists of two steps. In the first step the proof is reduced to the case of K-invariant vectors, and in the second step the theorem is proven to hold for K-invariant vectors of the regular representation.

Fix two K-finite vectors $f_1, f_2 \in L^2(G)$, and let $e_{f_1}, e_{f_2} \in C(K)$ be as in Corollary 1.6.8 with respect to $\mathrm{Res}_K^G(\lambda_G)$. Define now the functions $\widetilde{f_i}$ on G, for $i = 1, 2$, given by

$$\widetilde{f_i}(g) \coloneqq \sup_{k\in K} |f_i(kg)|.$$

Since $f_i = \mathrm{Res}_K^G(\lambda_G)^1(e_{f_i})f_i = e_{f_i} *_K f_i$, it follows from the Cauchy-Schwarz inequality that for $k' \in K$, $g \in G$,

$$\begin{aligned} |f_i(k'g)| = |e_{f_i} *_K f_i(k'g)| &\leq \|e_{f_i}\|_2 \bigg(\int_K |f_i(k^{-1}k'g)|^2\, dk\bigg)^{\frac{1}{2}} \\ &= \dim\langle K \cdot f_i\rangle^{\frac{1}{2}} \bigg(\int_K |f_i(kg)|^2\, dk\bigg)^{\frac{1}{2}}, \end{aligned}$$

and therefore

$$\bigg(\int_G |\widetilde{f_i}(x)|^2\, dx\bigg)^{\frac{1}{2}} \leq \dim\langle K \cdot f_i\rangle^{\frac{1}{2}} \bigg(\int_G \int_K |f_i(kx)|^2\, dk\, dx\bigg)^{\frac{1}{2}} = \dim\langle K \cdot f_i\rangle^{\frac{1}{2}} \|f_i\|_2;$$

whence $\widetilde{f_i} \in L^2(G)$. It is obvious that $\widetilde{f_i}$ is K-invariant, that is, $\lambda_G(k)\widetilde{f_i} = \widetilde{f_i}$ for all $k \in K$. Also

$$\begin{aligned} |\langle \lambda_G(g)f_1, f_2\rangle| &\leq \int_G |f_1(g^{-1}x)\overline{f}_2(x)|\, dx \\ &\leq \int_G \widetilde{f_1}(g^{-1}x)\widetilde{f_2}(x)\, dx = \langle \lambda_G(g)\widetilde{f_1}, \widetilde{f_2}\rangle. \end{aligned}$$

Hence it is enough to show that for all $\widetilde{f}_1, \widetilde{f}_2 \in L^2(G)$ which are K-invariant,

$$|\langle \lambda_G(g)\widetilde{f}_1, \widetilde{f}_2\rangle| \leq \|\widetilde{f}_1\|_2 \|\widetilde{f}_2\|_2 \Xi_{(G,K)}(g) \quad \forall g \in G.$$

As in Section 3.4 define now the function $\rho\colon G \to \mathbb{R}_{>0}$,

$$\rho(g) \coloneqq \frac{\Delta_P(p)}{\Delta_G(p)},$$

for $g \in G$ with $g = kp$ where $k \in K$ and $p \in P$, and recall that ρ is a rho-function of (G, P) (see Lemma A.4.8). By construction, the restriction $\rho|_P$ is a group homomorphism from P to $(\mathbb{R}_{>0}, \cdot)$. Using the integral formula (4.4) for the Iwasawa decomposition and the Cauchy-Schwarz inequality,

$$\begin{aligned} |\langle \lambda_G(g)\widetilde{f}_1, \widetilde{f}_2\rangle| &= \left| \int_K \int_P \widetilde{f}_1(g^{-1}kp)\widetilde{f}_2(kp)\rho(p)^{-1}\, dp\, dk \right| \\ &\leq \int_K \left(\int_P |\widetilde{f}_1(g^{-1}kp)|^2 \rho(p)^{-1}\, dp \right)^{\frac{1}{2}} \left(\int_P |\widetilde{f}_2(kp)|\rho(p)^{-1}\, dp \right)^{\frac{1}{2}} dk. \end{aligned} \tag{4.5}$$

Because $\widetilde{f}_2$ is K-invariant,

$$\int_P |\widetilde{f}_2(kp)|^2 \rho(p)^{-1}\, dp = \int_P |\widetilde{f}_2(p)|^2 \rho(p)^{-1}\, dp = \int_K \int_P |\widetilde{f}_2(kp)|^2 \rho(p)^{-1}\, dp\, dk = \|\widetilde{f}_2\|_2^2.$$

Next fix $g \in G$, $k \in K$ and write $g^{-1}k = k_0 p_0 \in KP$. Then

$$\begin{aligned} \int_P |\widetilde{f}_1(g^{-1}kp)|^2 \rho(p)^{-1}\, dp &= \int_P |\widetilde{f}_1(k_0 p_0 p)|^2 \rho(p)^{-1}\, dp = \int_P |\widetilde{f}_1(p)|^2 \rho(p_0^{-1}p)^{-1}\, dp \\ &= \int_K \int_P |\widetilde{f}_1(kp)|^2 \rho(kp)^{-1}\, dp\, dk \rho(p_0) = \|\widetilde{f}_1\|_2^2 \rho(p_0) = \|\widetilde{f}_1\|_2^2 \rho(g^{-1}k). \end{aligned}$$

It follows from (4.5) that

$$|\langle \lambda_G(g)\widetilde{f}_1, \widetilde{f}_2\rangle| \leq \|\widetilde{f}_1\|_2 \|\widetilde{f}_2\|_2 \int_K \rho(g^{-1}k)^{\frac{1}{2}}\, dk = \|\widetilde{f}_1\|_2 \|\widetilde{f}_2\|_2 \Xi_{(G,K)}(g).$$

□

Corollary 4.4.7 ([CHH88, Thm. 2])**.** *Every unitary representation which is weakly contained in the regular representation is $(K, \Xi_{(G,K)})$-bounded.*

Proof. This follows from Proposition 4.4.3 and from Theorem 4.4.6. □

Corollary 4.4.8 ([CHH88, Cor.])**.** *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G, and suppose there exists $n \geq 1$ such that, for some dense subset $U \subset \mathcal{H}_\pi$, for every $u \in U$ the diagonal matrix coefficient $\phi_{u,u}$ lies in $L^{2n+\epsilon}(G)$ for all $\epsilon > 0$. Then π is $(K, \Xi_{(G,K)}^{\frac{1}{n}})$-bounded.*

Proof. Consider the n-fold tensor product $(\pi^{\otimes n}, \mathcal{H}_\pi^{\otimes n})$. Then for every $u_1, u_2, \ldots, u_n \in U$, by the generalized Hölder inequality,

$$\begin{aligned} &\left(\int_G |\langle \pi^{\otimes n}(x) u_1 \otimes u_2 \otimes \cdots \otimes u_n, u_1 \otimes u_2 \otimes \cdots \otimes u_n\rangle|^{2+\epsilon}\, dx \right)^{\frac{1}{2+\epsilon}} \\ &\quad = \left(\int_G \left(\prod_{1 \leq i \leq n} |\phi_{u_i,u_i}(x)| \right)^{2+\epsilon} dx \right)^{\frac{1}{2+\epsilon}} \\ &\quad \leq \prod_{1 \leq i \leq n} \left(\int_G |\phi_{u_i,u_i}(x)|^{n(2+\epsilon)}\, dx \right)^{\frac{1}{n(2+\epsilon)}}. \end{aligned}$$

Hence the diagonal matrix coefficients $g \mapsto \prod_{1\leq i\leq n} \phi_{u_i,u_i}(g)$ lie in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$. Moreover, $\{u_1 \otimes u_2 \otimes \cdots \otimes u_n \colon u_1, u_2, \ldots, u_n \in U\}$ is total in $\mathcal{H}_\pi^{\otimes n}$. By Theorem 4.3.1, $\pi^{\otimes n}$ is weakly contained in the regular representation λ_G. If $v \in \mathcal{H}_\pi$ is K-finite, $\langle K \cdot v^{\otimes n}\rangle \subset \langle K \cdot v\rangle^{\otimes n}$, and therefore

$$\dim\langle K \cdot v^{\otimes n}\rangle \leq \dim\langle K \cdot v\rangle^{\otimes n} = \dim\langle K \cdot v\rangle^n.$$

This implies by Corollary 4.4.7 that for any K-finite vectors $v, w \in \mathcal{H}_\pi$ and every $g \in G$

$$\begin{aligned}|\langle \pi(g)v, w\rangle| &= |\langle \pi^{\otimes n}(g)v^{\otimes n}, w^{\otimes n}\rangle|^{\frac{1}{n}} \\ &\leq (\dim\langle K \cdot u^{\otimes n}\rangle \dim\langle K \cdot u^{\otimes n}\rangle)^{\frac{1}{2n}} \|v^{\otimes n}\|^{\frac{1}{n}} \|w^{\otimes n}\|^{\frac{1}{n}} \Xi_{(G,K)}^{\frac{1}{n}}(g) \\ &\leq \dim\langle K \cdot u\rangle^{\frac{1}{2}} \dim\langle K \cdot w\rangle^{\frac{1}{2}} \|v\|\|w\| \Xi_{(G,K)}^{\frac{1}{n}}(g).\end{aligned}$$

□

Corollary 4.4.7 and Corollary 4.4.8 motivate the search for explicit formulas or at least quantitative estimates for the Harish-Chandra function. If F is an archimedean local field, quantitative estimates for the Harish-Chandra function of $SL_2(F)$, with respect to a maximal compact subgroup, are computed in Proposition 7.1.2. If F is a non-archimedean local field, then Proposition 5.5.18 provides an explicit formula for the Harish-Chandra function.

The following is a quick exposition about the case where G is a connected semisimple real Lie group, e.g. $SL_d(\mathbb{R})$ and $SL_d(\mathbb{C})$. The proofs of the propositions below are omitted. Recall the Iwasawa decomposition $G = KAN$ from Theorem 3.5.6, where K is a maximal compact subgroup of G, and set $P \coloneqq AN$, which is a solvable Lie subgroup of G. The asymptotic behaviour of the Harish-Chandra function $\Xi_{(G,K)}$ is well-understood. In particular one has

Proposition 4.4.9 ([AWK16, Prop. 7.15])**.** *For every $\epsilon > 0$ there exist constants $c_1, c_2(\epsilon) > 0$ such that*

$$c_1 \Delta_P^{\frac{1}{2}}(p) \leq \Xi_{(G,K)}(p) \leq c_2(\epsilon)\Delta_P^{\frac{1}{2}+\epsilon}(p) \quad \forall p \in P. \tag{4.6}$$

Proposition 4.4.9 will be proven for $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ in Section 7.1 directly.

Proposition 4.4.10 ([AWK16, (7.48)])**.** *For every $\epsilon > 0$ $\Xi_{(G,K)} \in L^{2+\epsilon}(G)$.*

It follows from Proposition 4.4.10 and from Theorem 4.3.1 and Corollary 4.4.7 that a unitary representation of G is weakly contained in the regular representation of G if and only if it is $(K, \Xi_{(G,K)})$-bounded.

Using the explicit formulas and quantitative estimates for the Harish-Chandra function obtained in Chapter 5 and Chapter 7, it will be shown that for every local field F the Harish-Chandra function of $SL_2(F)$, with respect to a maximal compact subgroup, lies in $L^{2+\epsilon}(SL_2(F))$ for every $\epsilon > 0$. However, not every spherical function of $SL_2(F)$ is contained in $L^{2+\epsilon}(SL_2(F))$ for every $\epsilon > 0$

4.5 Kazhdan's Property (T)

At the beginning of this section suppose G is a topological group which is not necessarily locally compact. The following two definitions can be found in [BdlHV08, Chap. 1]

Definition 4.5.1. Let $Q \subset G$ be a compact subset, and let $\epsilon > 0$ be positive real number. A unitary representation $(\pi, \mathcal{H}_\pi)$ of G has a (Q, ϵ)-*invariant vector* if there exists a unit vector $u \in \mathcal{H}_\pi$ such that

$$\|\pi(g)u - u\| < \epsilon \quad \forall g \in Q.$$

Definition 4.5.2. A unitary representation $(\pi, \mathcal{H}_\pi)$ of G has *almost invariant vectors* if, for every compact subset $Q \subset G$ and every $\epsilon > 0$, π has a (Q, ϵ)-invariant vector.

Using weak containment, the notion of almost invariant vectors can be defined equivalently by

Lemma 4.5.3. *A unitary representation $(\pi, \mathcal{H}_\pi)$ of G has almost invariant vectors if and only if $1_G \prec \pi$.*

Proof. This is an immediate consequence of Corollary 4.2.2. □

Definition 4.5.4. A topological group G has *Kazhdan's Property (T)* if there exists a compact subset $Q \subset G$ and a constant $\epsilon > 0$ such that if a unitary representation of G has a (Q, ϵ)-invariant vector, then it has a non-zero G-invariant vector. If such a pair (Q, ϵ) exists, then it is called a *Kazhdan pair* with *Kazhdan set* Q and *Kazhdan constant* ϵ.

There is an easy way to construct Kazhdan pairs for groups having Property (T).

Definition 4.5.5. Suppose G has Property (T), and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation without non-zero G-invariant vectors. Let $Q \subset G$ be a compact subset. The *Kazhdan constant associated to Q and π* is

$$\kappa(G, Q, \pi) \coloneqq \inf\{\max_{g \in Q} \|\pi(g)u - u\| \colon \|u\| = 1\}.$$

Moreover, let

$$\kappa(G, Q) \coloneqq \inf \kappa(G, Q, \pi),$$

where the infimum is taken over all equivalence classes of unitary representations of G without non-zero G-invariant vectors.

The following proposition characterizes Property (T) using weak containment.

Proposition 4.5.6 ([BdlHV08, Prop. 1.2.1])**.** *The following are equivalent:*

1) G has Property (T).

2) If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G with $1_G \prec \pi$, then $1_G \subset \pi$.

Proof. It follows immediatley from Lemma 4.5.3 that 1) implies 2). For the converse argue by contradiction. Set $I \coloneqq \{(Q, \epsilon) \colon Q \subset G \text{ compact}, \ \epsilon > 0\}$. Assume that for every pair $(Q, \epsilon) \in I$ there exists a unitary representation $(\pi_{(Q,\epsilon)}, \mathcal{H}_{(Q,\epsilon)})$ which has a (Q, ϵ)-invariant vector $u_{(Q,\epsilon)}$, but no non-zero G-invariant vector. Consider their direct sum

$$\pi \coloneqq \bigoplus_{(Q,\epsilon) \in I} \pi_{(Q,\epsilon)}.$$

Then $1_G \prec \pi$, and

$$u \coloneqq \bigoplus_{(Q,\epsilon) \in I} u_{(Q,\epsilon)}$$

is an (Q, ϵ)-invariant vector for all $(Q, \epsilon) \in I$. This implies that u is a non-zero G-invariant vector, which yields a contradiction. □

One can show now how Property (T) gives information about the Fell topology on the unitary dual of a locally compact group, and vice versa.

Theorem 4.5.7 ([BdlHV08, Thm. 1.2.5])**.** *Suppose G is locally compact. The following are equivalent:*

1) G has Property (T);

2) 1_G is isolated in $\widehat{G}$.

Proof. Assume that there exists a net $\{\pi_i\}_{i\in I} \subset \widehat{G} - \{1_G\}$ such that $\pi_i \to 1_G$. Then $1_G \prec \pi \coloneqq \bigoplus_{i\in I} \pi_i$, and π does not contain 1_G. Hence 1) implies 2).

For the converse, argue by contradiction and assume that 1_G is isolated in $\widehat{G}$, but G does not have Property (T). Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G such that $1_G \prec \pi$, but 1_G is not contained in π. Fix a compact subset $Q \subset G$, and a constant $\epsilon > 0$. By Proposition 4.2.1, there exists a diagonal matrix coefficient $\phi \in \mathcal{P}_1(G)$ of π such that

$$|\phi(g) - 1| < \epsilon \quad \forall g \in Q.$$

By Theorem 1.5.18 there exists a net $\{\phi_i\}_{i\in I} \subset \mathcal{P}_1(G)$ of convex combinations of elements in $\mathrm{ext}(\mathcal{P}_1(G))$ such that $\phi_i \to \phi$ in the weak* topology on $L^\infty(G)$. Write $\phi_i = t_i\psi_i + (1 - t_i)$ where ψ_i is a convex combination of elements of $\mathrm{ext}(\mathcal{P}_1(G)) - \{1\}$ and where $0 \leq t_i \leq 1$. By passing to a subnet if necessary, assume that $t_i \to t$ and $\psi_i \to \psi \in \mathcal{P}_0(G)$. Then $\phi = t\psi + (1 - t)$. By Proposition 1.5.11, because 1_G is not contained in π, $\phi = \psi$. But this implies that for some subnet $J \subset I$

$$|\phi(g) - \psi_j(g)| < \epsilon \quad \forall j \in J \ \forall g \in Q.$$

Therefore 1 is contained in the closure of $\mathrm{conv}(\mathrm{ext}(\mathcal{P}_1(G) - \{1\}))$ with respect to the weak* topology on $\mathcal{P}_1(G)$. By Milman's Theorem and by Theorem 1.5.18, 1 can be approximated uniformly on compact subsets by elements in $\mathrm{ext}(\mathcal{P}_1(G)) - \{1\}$. Hence, 1_G is not isolated in $\widehat{G}$. □

Henceforth suppose that $K < G$ is a non-trivial compact subgroup. The following lemma is a useful tool to obtain more information about the structure of the unitary dual of a locally compact group and to compute Kazhdan constants.

Lemma 4.5.8. *Suppose G is locally compact, and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Suppose there exists a unit vector $u \in \mathcal{H}_\pi$ and a constant $0 < \epsilon < 1$ such that $\|\pi(k)u - u\| < \epsilon$ for every $k \in K$. Then π contains a non-zero K-invariant $v \in \mathcal{H}_\pi$ such that $\|u - v\| \leq \epsilon$ and $\|v\| \geq 1 - \epsilon$.*

Proof. Let $v \in \mathcal{H}_\pi$ be the unique vector such that

$$\langle v, w\rangle = \int_K \langle \pi(k)u, w\rangle \, dk \quad \forall w \in \mathcal{H}_\pi.$$

It is immediate that v is K-invariant. Moreover,

$$\begin{aligned}
\|u - v\|^2 &= \langle u, u\rangle - \int_K \langle u, \pi(k)u\rangle \, dk - \int_K \langle \pi(k)u, u\rangle \, dk + \int_K \int_K \langle \pi(k_1)u, \pi(k_2)u\rangle \, dk_1 \, dk_2 \\
&\leq \int_K \int_K |\langle u - \pi(k_1)u, u - \pi(k_2)u\rangle| \, dk_1 \, dk_2 \\
&\leq \left(\int_K \|\pi(k)u - u\| \, dk \right)^2 \leq \epsilon^2.
\end{aligned}$$

By the reversed triangle inequality,

$$\|v\| = \|u - (u - v)\| \geq 1 - \|u - v\| \leq 1 - \epsilon.$$

Because $0 < \epsilon < 1$, v is non-zero. □

This gives two useful propositions.

Proposition 4.5.9. *Suppose (G, K) is a Gelfand pair. G does not have Property (T) if and only if there exists a net $\{\phi_i\}_{i\in I}$ of positive definite spherical functions of (G, K) such that $\phi_i \to 1$ uniformly on compact subsets of G. In particular, if $\{[\pi_i]\}_{i\in I} \subset \widehat{G}$ is a net converging to 1_G, then there exists a subnet $J \subset I$ such that π_j is spherical for every $j \in J$.*

Proof. Suppose G does not have Property (T). Then, by Theorem 4.5.7, there exists a net $\{(\pi_i, \mathcal{H}_i)\}_{i\in I}$ of irreducible unitary representations of G such that $[\pi_i] \to 1_G$ with respect to the Fell topology. Let $Q \subset G$ be a compact subset, and fix $0 < \epsilon < 1$. By the definition of the Fell topology there exists a net $\{\phi_i\}_{i\in I}$ of positive definite functions such that ϕ_i is a diagonal matrix coefficient of π_i and such that

$$\sup_{g\in K\cup Q} |\phi_i(g) - 1| \to 0.$$

For each $i \in I$ let $u_i \in \mathcal{H}_i$ be the unit vector such that $\phi_i(g) = \langle \pi_i(g)u_i, u_i\rangle_i$ for every $g \in G$. By passing to a subnet if necessary, assume that $|\phi_i(g) - 1| < \frac{\epsilon^2}{32}$ for all $g \in K \cup Q$. Then

$$\|u_i - \pi_i(k)u_i\|_i^2 = 2(1 - \mathrm{Re}(\phi_i(k))) \leq \frac{\epsilon^2}{16} \quad \forall k \in K \ \forall i \in I$$

Then, by Lemma 4.5.8, there exists a non-zero K-invariant $v_i \in \mathcal{H}_i$ such that $\|u_i - v_i\|_i \leq \frac{\epsilon}{4}$. Let $\psi_i(g) \coloneqq \langle \pi_i(g)v_i, v_i\rangle_i$. Then, because (G, K) is Gelfand, by Corollary 3.3.4 and by Proposition 3.3.5, each ψ_i is a spherical function and each π_i is spherical. By the Cauchy-Schwarz inequality, for every $i \in I$ and every $g \in G$,

$$|\phi_i(g) - \psi_i(g)| = |\langle \pi_i(g)(u_i - v_i), u_i\rangle_i + \langle \pi_i(g)v_i, u_i - v_i\rangle_i| \leq 2\|u_i - v_i\|_i < \frac{\epsilon}{2},$$

and thus

$$|1 - \phi_i(g)| \leq |1 - \psi_i(g)| + |\psi_i(g) - \phi_i(g)| < \epsilon \quad \forall g \in K \cup Q.$$

Since ϵ is arbitrary, this implies that $\psi_i \to 1$ uniformly on compact subsets of G. □

Proposition 4.5.9 implies that for determining whether a locally compact group G, with a compact subgroup K such that (G, K) is Gelfand, has Property (T), it is enough to study the asymptotic behaviour of the positive definite spherical functions of (G, K). An application is

Corollary 4.5.10. *$SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ do not have Property (T).*

Proof. Recall from Theorem 3.5.7 the integral formulas for the spherical functions of the Gelfand pairs $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$. In particular, for $F \in \{\mathbb{R}, \mathbb{C}\}$, $(z, g) \mapsto \phi_{F,z}(g)$ defines a continuous function from $\mathbb{C} \times SL_2(F)$ to $\mathbb{C}$, and $\phi_{F,\frac{1}{2}} = 1$. The corollary follows from the existence of the complementary series, as seen in Theorem 3.5.8 □

For another applications of this criterion see Corollary 5.8.8.

The following proposition gives a method to compute Kazhdan constants.

Proposition 4.5.11 ([Oh02, Prop. 8.2])**.** *Let $H \subset G$ be a subset such that $H \cap K$ is a compact subgroup of K. Suppose there exists a function $\Phi\colon H \to \mathbb{R}$ such that*

1) $0 \leq \Phi(h) \leq 1$ for every $h \in H$;

2) $\Phi(h) = 1$ if and only if $h \in H \cap K$;

3) for every unitary representation $(\pi, \mathcal{H}_\pi)$ of G and every $H\cap K$-invariant unit vector $u \in \mathcal{H}_\pi$

$$|\langle \pi(h)u, u\rangle| \leq \Phi(h) \quad \forall h \in H.$$

Set

$$\chi(h) \coloneqq \frac{\sqrt{2(1 - \Phi(h))}}{\sqrt{2(1 - \Phi(h))} + 3}.$$

Then

1) $0 \leq \chi(h) \leq \frac{\sqrt{2}}{\sqrt{2}+3}$ *for all* $h \in H$,

2) $\chi(h) = 0$ *if and only if* $h \in H \cap K$,

3) $(K \cup \{h\}, \chi(h))$ *is a Kazhdan pair for every* $h \in H$ *and every unitary representation of* G *without non-zero* G*-invariant vectors.*

Proof. It is obvious that χ satisfies 1) and 2). To show 3), fix $h \in H$, and let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G without non-zero G-invariant vectors. One needs to show that for every unit vector $u \in \mathcal{H}_\pi$

$$\sup_{s \in \{h\} \cup K} \|\pi(s)u - u\| \geq \chi(h).$$

For this suppose that $\|\pi(k)u - u\| \leq \chi(h)$ for all $k \in K$. The goal is to show that $\|\pi(h)u - u\| \geq \chi(h)$. Because $0 \leq \chi(h) < 1$, by Lemma 4.5.8, there exists a non-zero K-invariant vector $v \in \mathcal{H}_\pi$ such that $\|u - v\| \leq \chi(h)$ and $\|v\| \geq 1 - \chi(h)$. Then

$$\|\pi(h)v - v\|^2 = 2(\|v\|^2 - \mathrm{Re}(\langle \pi(h)v, v\rangle)) \geq 2\|v\|^2(1 - \Phi(h)),$$

whence

$$\|\pi(h)v - v\| \geq \sqrt{2(1 - \Phi(h))}\|v\| \geq \sqrt{2(1 - \Phi(h))}(1 - \chi(h)).$$

Thus

$$\begin{aligned}\|\pi(h)u - u\| &= \|\pi(h)v - v + (\pi(h)u - \pi(h)v) + (v - u)\| \\ &\geq |\|\pi(h)v - v\| - 2\|u - v\|| \geq \sqrt{2(1 - \Phi(h))}(1 - \chi(h)) - 2\chi(h) = \chi(h).\end{aligned}$$

This completes the proof. □

Corollary 4.5.12. *Suppose that every unitary representation of* G *without non-zero* G*-invariant vectors is* (K, Ψ)*-bounded such that* $\Psi \neq 1$. *Then* G *has Property (T), and for every* $g \in G - \Psi^{-1}(\{1\})$, *the pair*

$$\left(\{g\} \cup K, \frac{\sqrt{2(1 - \Psi(g))}}{\sqrt{2(1 - \Psi(g))} + 3}\right)$$

is a Kazhdan pair.

Chapter 5

Automorphism Groups of Homogeneous Trees

5.1 Homogeneous Trees

One begins with the basic definitions of graphs and trees. The following definitions can be found in [Ser80, Chap. I, §2],

Definition 5.1.1. A graph Γ is a pair $(\mathrm{vert}(\Gamma), \mathrm{edg}(\Gamma))$, where $\mathrm{vert}(\Gamma)$ is the *set of vertices* of Γ and where $\mathrm{edg}(\Gamma)$ is the *set of edges* of Γ, consisting of pairs $\{x, y\}$ with $x, y \in \mathrm{vert}(T_{q+1})$. Two vertices x, y of Γ are *adjacent* if $\{x, y\} \in \mathrm{edg}(\Gamma)$.

Definition 5.1.2. A graph Δ is a *subgraph* of the graph Γ, written $\Delta \subset \Gamma$, if $\mathrm{vert}(\Delta) \subset \mathrm{vert}(\Gamma)$ and $\mathrm{edg}(\Delta) \subset \mathrm{edg}(\Gamma)$.

Remark. If Γ is a graph and $\{x, y\} \in \mathrm{edg}(\Gamma)$, then x and y do not need to be distinct.

Remark. Usually graphs are represented by a diagram. One can also consider the realization of a graph, which is a CW-complex of dimension smaller than 1. For details see [Ser80, p. 13 f.].

Definition 5.1.3. A *path* in a graph Γ is a sequence $\{x_k\}_{m<k<n} \subset \mathrm{vert}(\Gamma)$, with $-\infty \leq m < n \leq \infty$, such that $\{x_k, x_{k+1}\} \in \mathrm{edg}(\Gamma)$ for all k.

Definition 5.1.4. A graph Γ is called *connected* if for every two vertices $x, y \in \mathrm{vert}(\Gamma)$ there is a path $\{x_k\}_{0\leq k\leq n}$ such that $x = x_0$ and $y = x_n$.

Definition 5.1.5. A *circuit* in a graph Γ is a sequence $\{x_k\}_{0\leq k\leq n} \subset \mathrm{vert}(\Gamma)$, with $1 \leq n < \infty$ such that $x_0 = x_n$.

Definition 5.1.6. A *tree* is a connected non-empty graph with no circuits.

If T is a tree, then a vertex of T cannot be adjacent to itself. T is said to be *locally finite* if every vertex of T belongs to finitely many edges. If T is locally finite, the *degree* of a vertex of T is the number of edges it belongs to.

Definition 5.1.7. Suppose T is a locally finite tree, and let $q \geq 1$. T is said to be *homogeneous* of *degree* $q + 1$ if every vertex of T belongs to exactly $q + 1$ edges.

Remark. By denoting the degree of a homogeneous tree by $q + 1$, notations and computations in subsequent sections will be simplified.

Definition 5.1.8. Let Γ_1 and Γ_2 be graphs. An *isomorphism* between Γ_1 and Γ_2 is a bijective map $\phi\colon \mathrm{vert}(\Gamma_1) \to \mathrm{vert}(\Gamma_2)$ such that for all $\{x, y\} \in \mathrm{edg}(\Gamma_1)$ $\{\phi(x), \phi(y)\} \in \mathrm{edg}(\Gamma_2)$.

It is clear that for every $q \geq 1$ there exists a homogeneous tree of degree $q+1$ which is unique up to isomorphisms. Henceforth denote the homogeneous tree of degree $q+1$ by T_{q+1}.

Definition 5.1.9. Given a graph Γ, an *automorphism* of Γ is an isomorphism of Γ to itself.

Let Γ be a graph. It is immediate that the set of automorphisms of Γ constitute a group with the composition as the group law. Denote the resulting group by $\mathrm{Aut}(\Gamma)$, and call it the *automorphism group* of Γ.

Endow $\mathrm{vert}(\Gamma)$ with the discrete topology, and denote the resulting topological space by Γ. Endow $\mathrm{Aut}(\Gamma)$ with the compact-open topology (see Definition A.2.1). Then, by a famous theorem of Richard Arens (see Theorem A.2.2), $\mathrm{Aut}(\Gamma)$ is a topological group. The compact-open topology is generated by the basis $\{U_S(g)\colon S \subset \mathrm{vert}(\Gamma) \text{ finite},\ g \in \mathrm{Aut}(\Gamma)\}$, where

$$U_S(g) \coloneqq \{h \in \mathrm{Aut}(\Gamma)\colon g(x) = h(x)\ \forall x \in S\}, \tag{5.1}$$

(see Proposition A.2.3). Denote the stabilizer groups of a vertex $x \in \mathrm{vert}(\Gamma)$ and of an edge $\{x, y\} \in \mathrm{edg}(\Gamma)$ by $\mathrm{stab}(x)$ and $\mathrm{stab}(\{x, y\})$, respectively.

Definition 5.1.10. A *group action* of a topological group G on Γ is a continuous group homomorphism $\phi\colon G \to \mathrm{Aut}(\Gamma)$.

Using the above basis for the compact-open topology, it easy to check that the natural action of $\mathrm{Aut}(\Gamma)$ on the topological space Γ is continuous.

Remark. In graph theory homogeneous trees of degree $q+1$ are usually referred to as *regular trees of valence* $q+1$. As will be seen, every homogeneous tree is a homogeneous space of its automorphism group. In fact, as is readily seen, a tree is a homogeneous space of its automorphism group if and only if every two vertices have the same degree. Hence, throughout this thesis, the term *homogeneous* instead of *regular* will be used to describe trees as in Definition 5.1.7.

Examples 5.1.11. The homogeneous tree of degree 2 can be thought of as the set of integers where two integers are adjacent if their difference is 1. For diagrams of homogeneous trees of higher degrees see Figures 5.1, 5.2 and 5.3.

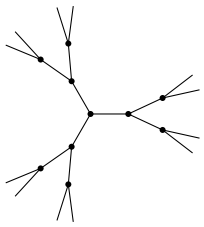

Figure 5.1: Homogeneous tree of degree $q+1=3$

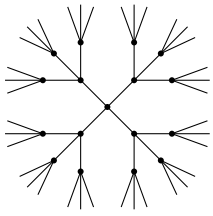

Figure 5.2: Homogeneous tree of degree $q+1=4$

Definition 5.1.12. A path $\{x_k\}_{m<k<n} \subset \mathrm{vert}(T_{q+1})$, for $-\infty \leq m < n \leq \infty$, is called a *geodesic* if $x_k \neq x_{k+2}$ for all $m < k < n-1$; it is said that $\{x_k\}_{m<k<n}$ has no *backtracking*. If $\{x_k\}_{m\leq k\leq n}$ is a geodesic with $-\infty < m \leq n < \infty$, the path is a *geodesic from x_m to x_n*. The *length* of a finite geodesic is the number of edges it contains.

A geodesic $\{x_k\}_{k\geq 0}$ is called a *one-sided infinite geodesic*, and a geodesic $\{x_k\}_{k\in\mathbb{Z}}$ is called a *two-sided infinite geodesic*.

Given two vertices $x, y \in \mathrm{vert}(T_{q+1})$, there exists a unique (finite) geodesic between x and y. Indeed, suppose $\{x_k\}_{0\leq k\leq m}$ and $\{y_l\}_{0\leq l\leq n}$ are both geodesics connecting $x_0 = y_0$ and $x_m = y_n$. Then the path $\{z_k\}_{0\leq k\leq m+n}$, with $z_k = x_k$ for $0 \leq k \leq m$ and $z_k = y_{k-m}$ for $m \leq k \leq m+n$,

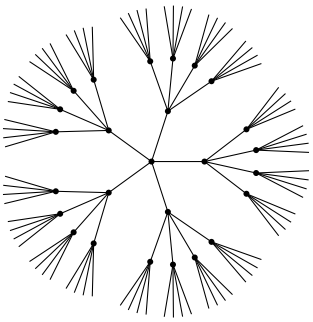

Figure 5.3: Homogeneous tree of degree $q+1=5$

is a path with starting and ending point x_0. Because T_{q+1} is a tree, for every $0 \leq k \leq m$ there is $0 \leq l(k) \leq n$ such that $x_k = z_k = z_{m+l(k)} = y_{l(k)}$. Since both paths are geodesics and have the same starting and ending points, for every $0 \leq k \leq m$ $l(k)$ is unique and $l(k+1) = l(k)+1$. Noticing that $l(0) = 0$, it follows that $l(k) = k$, that is, $x_k = y_k$ for all $0 \leq k \leq n$. From now denote the unique geodesic from x to y by $[x, y]$.

Remark. Observe that for all $x, y, z \in \mathrm{vert}(T_{q+1})$ with $z \in [x, y]$, it holds that $[x, y] = [x, z] \cup [z, y]$.

Using the geodesics in T_{q+1} introduce a metric on $\mathrm{vert}(T_{q+1})$.

Definition 5.1.13. For any two vertices $x, y \in \mathrm{vert}(T_{q+1})$ the *distance* $d(x, y)$ is the length of the geodesic $[x, y]$.

It is easy to verify that the distance function in Definition 5.1.6 is a metric on $\mathrm{vert}(T_{q+1})$. For $x \in \mathrm{vert}(T_{q+1})$ and a positive integer $r \geq 1$, let $S(x, r) \coloneqq \{y \in \mathrm{vert}(T_{q+1})\colon d(x, y) = r\}$ and $B(x, r) \coloneqq \{y \in \mathrm{vert}(T_{q+1})\colon d(x, y) \leq r\}$ be the *sphere* and the *ball* of radius r around x, respectively.

Next define the boundary of T_{q+1}.

Definition 5.1.14. On the set of one-sided infinite geodesics of T_{q+1} define the equivalence class $\sim_{\partial T_{q+1}}$ such that $\{x_k\}_{k\geq 0} \sim_{\partial T_{q+1}} \{y_l\}_{l\geq 0}$ if and only if there are $k_0, l_0 > 0$ with $x_{k_0+k} = y_{l_0+k}$ for all $k \geq 0$. The *boundary* ∂T_{q+1} of T_{q+1} is the set of equivalence classes of $\sim_{\partial T_{q+1}}$.

It should be clear that $\sim_{\partial T_{q+1}}$ is an equivalence class. Fix a vertex $x \in \mathrm{vert}(T_{q+1})$, and let $\omega \in \partial T_{q+1}$. Then let $\mathbf{y} = \{y_k\}_{k\geq 0}$ be a representative of the equivalence class ω. There is $k_0 \geq 0$ such that $\mathbf{y} \cap [x, y_{k_0}] = \{y_{k_0}\}$. Hence $\mathbf{x} = [x, y_{k_0}] \cup \{y_k\}_{k\geq k_0}$ defines a one-sided infinite geodesic with starting point x such that $\mathbf{x} \sim_{\partial T_{q+1}} \mathbf{y}$. On the other hand, suppose $\mathbf{x} = \{x_k\}_{k\geq 0}$ and $\mathbf{y} = \{y_k\}_{k\geq 0}$ are one-sided infinite geodesics with common starting point $x_0 = y_0$ and such that $\mathbf{x} \sim_{\partial T_{q+1}} \mathbf{y}$. There are $k_0, l_0 \geq 0$ such that $x_{k_0+k} = y_{l_0+k}$ for all $k \geq 0$. However, because $[x_0, x_{k_0}] = [x_0, y_{l_0}]$, $\mathbf{x} = \mathbf{y}$. Deduce that, for every $x \in \mathrm{vert}(T_{q+1})$, there is a one-to-one correspondence between ∂T_{q+1} and the set of one-sided infinite geodesics starting at x. For every $\omega \in \partial T_{q+1}$ denote by $[x, \omega)$ the unique one-sided infinite geodesic starting at x and belonging to the equivalence class ω.

A path $\gamma = \{x_k\}_{k\in\mathbb{Z}}$ is a geodesic if and only if both $\mathbf{x}^- = \{x_k\}_{k\leq 0}$ and $\mathbf{x}^+ = \{x_k\}_{k\geq 0}$ are geodesics and such that $\mathbf{x}^+ \cap \mathbf{x}^- = \{x_0\}$. For $\omega_0 = [\mathbf{x}^-], \omega_1 = [\mathbf{x}^+] \in \partial T_{q+1}$, write $\gamma = (\omega_0, \omega_1)$. Conversely for every two boundary points $\omega_0, \omega_1 \in T_{q+1}$, there exists a unique doubly infinite geodesics $\gamma \subset \mathrm{vert}(T_{q+1})$ such that $\gamma = (\omega_0, \omega_1)$.

One is now equipped to endow the disjoint union $\mathrm{vert}(T_{q+1}) \sqcup \partial T_{q+1}$ with a topology.

Definition 5.1.15. For every $x, y \in \mathrm{vert}(T_{q+1})$ let $U(x, y) \coloneqq \{t \in \mathrm{vert}(T_{q+1}), \omega \in \partial T_{q+1}\colon [y, \omega) \subset [x, \omega),\ t \in [y, \omega)\} \subset \mathrm{vert}(T_{q+1}) \sqcup \partial T_{q+1}$. Endow $\mathrm{vert}(T_{q+1}) \sqcup \partial T_{q+1}$ with the topology generated by the discrete topology on $\mathrm{vert}(T_{q+1})$ and the subsets $\{U(x, y)\colon x, y \in \mathrm{vert}(T_{q+1})\}$. Denote the resulting topological space by $\mathcal{T}_{q+1}$.

Note that $\{\{x\}\colon x \in \operatorname{vert}(T_{q+1})\}$ and $\{U(x,y)\colon x,y \in \operatorname{vert}(T_{q+1})\}$ constitute a basis of the topology of $\mathcal{T}_{q+1}$.

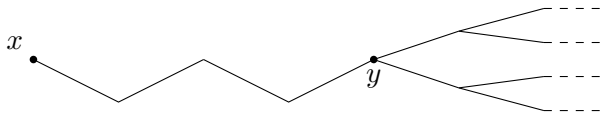

Figure 5.4: $U(x,y)$ for $x,y \in \operatorname{vert}(T_3)$

Proposition 5.1.16. *The topological space $\mathcal{T}_{q+1}$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $\mathcal{T}_{q+1}$. Let $I_0 \subset I$ be the subset of all i such that there are $x_i, y_i \in \operatorname{vert}(T_{q+1})$ with $U(x_i,y_i) \subset U_i$.

Claim that there is a finite subset $J_0 \subset I_0$ such that $\partial T_{q+1} \subset \bigcup_{j\in J_0} U_j$. Fix $x \in \operatorname{vert}(T_{q+1})$. For every $i \in I_0$ let $y_i \in \operatorname{vert}(T_{q+1})$ be such that $U(x,y_i) \subset U_i$. Hence, it can be assumed that $U_i = U(x,y_i)$ for some $y_i \in \operatorname{vert}(T_{q+1})$. To prove the claim, it suffices to show that there is $n_0 \geq 1$ such that for all $\omega \in \partial T_{q+1}$ there is $i \in I_0$ with $d(x,y_i) \leq n_0$ and $\omega \in U(x,y_i)$, or equivalently

$$\{z \in \operatorname{vert}(T_{q+1})\colon d(x,z) \geq n_0\} \subset \bigcup_{\substack{i\in I_0,\\ d(x,y_i)\leq n_0}} U(x,y_i).$$

Arguing by contradiction, suppose that this is not true. Then for every $n \geq 0$ there is $z_n \in \operatorname{vert}(T_{q+1})$ with $d(x,z_n) = n$ and such that for all $i \in I_0$ $z_n \notin U(x,y_i)$. This gives a one-sided infinite geodesic $[x,\omega)$, which yields a contradiction.

Deduce that there exist $n_0 \geq 0$ and a finite subset $J_0 \subset I_0$ such that

$$V \coloneqq \{z \in \operatorname{vert}(T_{q+1})\colon d(x,z) \geq n_0\} \cup \partial T_{q+1} \subset \bigcup_{j\in J_0} U_j.$$

Because the complement of V is the finite subset $\{z \in \operatorname{vert}(T_{q+1})\colon d(x,z) < n_0\}$, there is a finite subset $J_1 \subset I$ such that $\{U_j\}_{j\in J_1}$ is a finite open cover of $\mathcal{T}_{q+1} - V$. Hence, $\{U_j\}_{j\in J_0\cup J_1}$ is a finite open cover of $\mathcal{T}_{q+1}$. □

Endow the boundary ∂T_{q+1} with the subspace topology from $\mathcal{T}_{q+1}$. For $x,y \in \operatorname{vert}(T_{q+1})$ define

$$\Omega(x,y) \coloneqq \{\omega \in \partial T_{q+1}\colon [y,\omega) \subset [x,\omega)\},$$

and note that $\Omega(x,y) = U(x,y) \cap \partial T_{q+1}$. Thus $\{\Omega(x,y)\colon x,y \in \operatorname{vert}(T_{q+1})\}$ is a basis of the subspace topology.

Corollary 5.1.17. *∂T_{q+1} is compact and totally disconnected, and the sets $\{\Omega(x,y)\colon x,y \in \operatorname{vert}(T_{q+1})\}$ form a basis of the subspace topology. In particular, ∂T_{q+1} is discrete if and only if $q+1 = 2$.*

Proof. By Proposition 5.1.16, ∂T_{q+1} is compact. For every two boundary points ω_0, ω_1 and every $x \in \operatorname{vert}(T_{q+1})$, there are $x, y_0, y_1 \in \operatorname{vert}(T_{q+1})$ such that $\omega_i \in \Omega(x,y_i)$ for $i = 0,1$ and $\Omega(x,y_0) \cap \Omega(x,y_1) = \emptyset$. Observe that for every $n \geq 1$

$$\partial T_{q+1} = \bigsqcup_{y\in S(x,n)} \Omega(x,y). \tag{5.2}$$

Hence, $\Omega(x,y)$ is open and closed for every $x,y \in \operatorname{vert}(T_{q+1})$. Therefore, because $\{\Omega(x,y)\colon x,y \in \operatorname{vert}(T_{q+1})\}$ is a basis of the subspace topology, ∂T_{q+1} is totally disconnected.

If $q+1 = 2$, then T_{q+1} has only two points on the boundary and thus ∂T_{q+1} is discrete. If $q+1 \geq 3$, then for every $x,y,z \in \operatorname{vert}(T_{q+1})$ with $[x,y] \subset [x,z]$, $\Omega(x,z) \subsetneq \Omega(x,y)$, which implies that no boundary point can be open. □

5.2 Automorphisms of homogeneous trees

As in the previous section, in this and subsequent sections let T_{q+1} be the homogeneous tree of degree $q+1$, with $q \geq 1$, and let ∂T_{q+1} be its boundary. Let $\mathrm{Aut}(T_{q+1})$ be the topological group of automorphisms of the tree T_{q+1}, endowed with the compact-open topology. As will be seen, $\mathrm{Aut}(T_{q+1})$ is locally compact and separable (see Corollary 5.3.2).

By construction, a map $g\colon \mathrm{vert}(T_{q+1}) \to \mathrm{vert}(T_{q+1})$ is an automorphism of T_{q+1} if and only if it is an isometry, with respect to the metric in Definition 5.1.13. Observe that each $g \in \mathrm{Aut}(T_{q+1})$ maps geodesics in T_{q+1} to geodesics. In particular, g preserves the equivalence classes of $\sim_{\partial T_{q+1}}$. This gives an action of $\mathrm{Aut}(T_{q+1})$ on the topological space $\mathcal{T}_{q+1}$ where for $x \in \mathrm{vert}(T_{q+1})$ $g \cdot x \coloneqq g(x)$ and for $\omega = [\{x_k\}_{k\geq 0}] \in \partial T_{q+1}$, $g \cdot \omega \coloneqq [\{g \cdot x_k\}_{k\geq 0}]$. Call this action the *natural action of* $\mathrm{Aut}(T_{q+1})$ on $\mathcal{T}_{q+1}$.

Proposition 5.2.1. *The natural action of* $\mathrm{Aut}(T_{q+1})$ *on the topological space* $\mathcal{T}_{q+1}$ *is continuous.*

Proof. Using the basis in (5.1), it is immediate that the natural action on $\mathrm{vert}(T_{q+1})$ is continuous. Next fix $g \in \mathrm{Aut}(T_{q+1})$ and $\omega \in \partial T_{q+1}$, and take $x, y \in \mathrm{vert}(T_{q+1})$ such that $g(\omega) \in \Omega(x, y)$. Then, the preimage of $\Omega(x, y)$ under $\mathrm{Aut}(T_{q+1}) \to \partial T_{q+1}, h \mapsto h(\omega)$ contains the open neighbourhood $U_{\{g^{-1}(x), g^{-1}(y)\}}(g)$ of g. Hence the action of $\mathrm{Aut}(T_{q+1})$ on the boundary ∂T_{q+1} is continuous too. □

Throughout this chapter the natural action of $\mathrm{Aut}(T_{q+1})$ and of its subgroups on $\mathcal{T}_{q+1}$ will be considered. Hence from now on write $g \cdot x = g(x)$ for $g \in \mathrm{Aut}(T_{q+1})$ and $x \in \mathrm{vert}(T_{q+1})$. It is clear that the natural actions of $\mathrm{Aut}(T_{q+1})$ on $\mathrm{vert}(T_{q+1})$ and on ∂T_{q+1} are transitive. Hence the tree T_{q+1} and its boundary are homogeneous spaces of $\mathrm{Aut}(T_{q+1})$ (see Theorem A.1.10).

One has the following types of automorphisms on T_{q+1}.

Definition 5.2.2. Let $g \in \mathrm{Aut}(T_{q+1})$.

1) If g fixes a vertex, it is a *rotation.*

2) If g stabilizes an edge by exchanging the vertices of the same edge, it is called an *inversion.*

3) Suppose there are a doubly infinite geodesic $\gamma = \{x_k\}_{k\in\mathbb{Z}}$ and an integer $s \geq 1$ such that $g \cdot x_k = x_{k+s}$ for all $k \in \mathbb{Z}$. Then g is called a *translation* of step s along γ.

Notice that the identity element of $Aut(T_{q+1})$ is a rotation. It also not difficult to show that inversions on any edge and translations of any step along any geodesic exist.

Remark. If $g \in \mathrm{Aut}(T_{q+1})$ is a translation along the geodesic $\gamma = (\omega_0, \omega_1)$ for $\omega_0, \omega_1 \in \partial T_{q+1}$, then for every $x \in \gamma$, $g^{-n} \cdot x \to \omega_0$ and $g^n \cdot x \to \omega_1$ as $n \to \infty$.

Theorem 5.2.4 asserts that every automorphism of T_{q+1} is of exactly one of the types in Definition 5.2.2. For the proof the following auxiliary lemma will be used.

Lemma 5.2.3 ([FTN91, Chap. I, Lem. 3.1])**.** *Let* $g \in \mathrm{Aut}(T_{q+1})$*, and let* $x_0 \in \mathrm{vert}(T_{q+1})$*. Suppose* $x_0 \neq g \cdot x_0$*, and write* $[x_0, g \cdot x_0] = \{x_k\}_{0\leq k\leq s}$*, with* $s = d(x_0, g \cdot x_0) \geq 2$*. If* $g \cdot x_1 \neq x_{s-1}$*, there is a doubly infinite geodesic* $\{x_k\}_{k\in\mathbb{Z}}$ *containing* $[x_0, g \cdot x_0]$ *and such that* $g \cdot x_k = x_{k+s}$ *for every* $k \in \mathbb{Z}$*.*

Proof. By an argument of induction a doubly infinite geodesic as described above can be constructed. It suffices to show that for all $n \geq 0$ and for every $1 \leq i \leq n$ there are $x_{-i}, x_{s+i} \in \mathrm{vert}(T_{q+1})$ such that $\{x_k\}_{-n\leq k\leq s+n}$ is a geodesic with $g \cdot x_k = x_{k+s}$ and $g \cdot x_{-n+1} \neq x_{s+n-1}$. Arguing by induction on n, the base case of $n = 0$ is covered by the assumptions about $\{x_k\}_{0\leq k\leq s}$. For the induction step, fix $n \geq 0$ and suppose the induction hypothesis holds for all $0 \leq m \leq n$. Because $g \cdot x_{n+1} \neq x_{s+n-1}$ and $g^{-1} \cdot x_{s-n-1} \neq x_{-n+1}$, one can set $x_{s+n+1} \coloneqq g \cdot x_{n+1}$ and $x_{-n-1} \coloneqq g^{-1} \cdot x_{s-n-1}$ such that $\{x_k\}_{-n-1\leq k\leq s+n+1}$ is a geodesic. Moreover, by construction, $g \cdot x_{n+2} \neq x_{s+n}$. □

Theorem 5.2.4 ([FTN91, Chap. I, Thm. 3.2]). *An automorphism $g \in \mathrm{Aut}(T_{q+1})$ is exactly one of the following: a rotation, an inversion or a translation.*

Proof. Let $n \coloneqq \min\{d(x, g\cdot x) \colon x \in \mathrm{vert}(T_{q+1})\}$. Let $x \in \mathrm{vert}(T_{q+1})$ be such that $d(x, g\cdot x) = n$. If $n = 0$, then $g \cdot x = x$ and g is a rotation. If $n = 1$, then $\{x, g\cdot x\}$ is an edge. In this case, if $g^2 \cdot x = x$, then it is an inversion and if $g^2 \cdot x \neq x$, then $\gamma = \{g^k \cdot x\}_{k\in\mathbb{Z}}$ is a geodesic and g is a translation.

For the last case suppose $n \geq 2$, and let $\{x_k\}_{0\leq k\leq n}$ be the geodesic from $x_0 = x$ to $x_n = g\cdot x$. By the definition of n, $g \cdot x_1 \neq x_{n-1}$. Hence, by Lemma 5.2.3, there is a doubly infinite geodesic $\gamma = \{x_k\}_{k\in\mathbb{Z}}$ with $x_0 = x$ and $x_n = g\cdot x$ and such that $g \cdot x_k = x_{k+n}$ for all $k \in \mathbb{Z}$, that is, g is a translation.

It remains to verify that every tree automorphism is of only one of the types from Definition 5.2.2. Suppose g is an inversion, and let $\{x, y\} \in \mathrm{edg}(T_{q+1})$ be the edge such that $g \cdot x = y$ and $g^2 \cdot x = x$. In this case, for all $z \in \mathrm{vert}(T_{q+1})$ and $k \in \mathbb{Z}$ $d(g^{2k} \cdot z, x) = d(z, x)$ and $d(g^{2k+1} \cdot z, y) = d(z, x)$. Hence, for $n = d(x, z)$, $\mathrm{orb}_{\langle g\rangle}(z) \not\subset S(x, n)$ and $\mathrm{orb}_{\langle g\rangle}(z) \subset S(x, n)\cup S(y, n)$. This implies that g does not fix a vertex, that is, it cannot be a rotation. Next suppose g is a translation along the doubly infinite geodesic $\gamma = \{x_k\}_{k\in\mathbb{Z}}$ of step n. In this case for all $y \in \mathrm{vert}(T_{q+1})$ there is no $n > 0$ such that $\mathrm{orb}_G(y) \subset B(x, n)$. It follows that g cannot be a rotation or an inversion. □

The following proposition is fairly technical, but it will be helpful in subsequent sections.

Proposition 5.2.5 ([FTN91, Chap. I, Prop. 3.4]). *1) The composition of two inversions on two distinct edges is a translation of even step.*

2) The composition of a rotation and an inversion on an edge whose vertices are not fixed by the rotation is a translation of odd step.

Proof. 1) Let $\{x, y\}, \{w, z\} \in \mathrm{edg}(T_{q+1})$ be two distinct edges of T_{q+1}, that is, they have at most one common vertex. Let $g, h \in \mathrm{Aut}(T_{q+1})$ be inversions on $\{x, y\}$ and $\{w, z\}$, respectively. Then $(gh) \cdot w = g \cdot z$. Suppose $[x, z] \subset [y, w]$ (see Figure 5.5). Then $d(y, g\cdot z) = d(x, z) < d(y, w)$, which implies $g\cdot z \neq w$. Let $[x, t] = [x, z]\cap[x, g\cdot z]\cap[x, g\cdot w]$. Then

$$\begin{aligned} d(z, (gh)\cdot z) &= d(z, g\cdot w) = d(x, g\cdot w) + d(z, x) - 2d(x, t) \\ &= d(y, w) + d(x, z) - 2d(x, t) = 2 + 2d(z, t). \end{aligned}$$

and, by a similar argument, $d(w, (gh)\cdot w) = 2 + 2d(z, t)$. Hence, $(gh)\cdot z \notin [w, (gh)\cdot w]$, and the claim follows from Lemma 5.2.3.

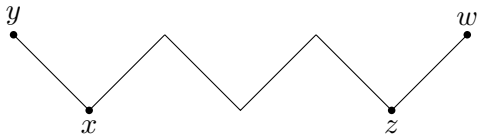

Figure 5.5

2) Fix $x \in \mathrm{vert}(T_{q+1})$ and $\{y, z\} \in \mathrm{edg}(T_{q+1})$. Suppose $[x, y] \subset [x, z]$. Let $g \in \mathrm{stab}(x)$ be a rotation which does not fix both y and z, and let $h \in \mathrm{Aut}(T_{q+1})$ be an inversion on $\{y, z\}$. Then $g\cdot z \neq y$ and $(gh)\cdot y = g\cdot z$. Let $[x, t] = [x, y]\cap[x, g\cdot z]$. Hence,

$$\begin{aligned} d(z, (gh)\cdot z) &= d(z, g\cdot y) = d(y, g\cdot z) + 1 \\ &= d(x, y) + d(x, g\cdot z) - 2d(x, t) + 1 = 2(d(x, z) - d(x, t)) + 1. \end{aligned}$$

Since $d((gh)\cdot y, x) = d((gh)\cdot z, x) + 1$, $(gh)\cdot y \notin [z, (gh)\cdot z]$, and hence, by Lemma 5.2.3, gh is a translation of odd step.

□

5.3 Aut(T_{q+1})

Henceforth consider the topological group Aut(T_{q+1}), endowed with the compact-open topology. Using the basis in (5.1), it is easy to check that the natural action of Aut(T_{q+1}) on the topological space $\mathcal{T}_{q+1}$ is continuous.

A first result about closed subgroups of Aut(T_{q+1}) is the following one, which will help to characterize all compact subgroups of Aut(T_{q+1}). Moreover, it can be deduced from this result that Aut(T_{q+1}) is locally compact.

Proposition 5.3.1 ([FTN91, Chap. I, p. 10 f.])**.** *For all $x \in \mathrm{vert}(T_{q+1})$ and all $\{y,z\} \in \mathrm{edg}(T_{q+1})$,* stab($x$) *and* stab($\{y,z\}$) *are compact open subgroups of* Aut(T_{q+1})*.*

Proof. Let $\{y,z\} \in \mathrm{edg}(T_{q+1})$, and let $g \in \mathrm{stab}(\{y,z\})$ be an inversion on the edge. Observe that

$$\mathrm{stab}(\{y,z\}) = (\mathrm{stab}(y) \cap \mathrm{stab}(z)) \cup g \cdot (\mathrm{stab}(y) \cap \mathrm{stab}(z)).$$

Hence it is enough to show that for all $x \in \mathrm{vert}(T_{q+1})$ the stabilizer group stab(x) is compact and open. That it is open, follows from the continuity of the natural action of Aut(T_{q+1}) on $\mathcal{T}_{q+1}$. To show compactness it is enough to prove that stab(x) is isomorphic to a closed subgroup of the topological group $\prod_{n\geq 1} S_{q^{n-1}(q+1)}$, where S_n denotes the symmetric group of order $n \geq 1$, endowed with the discrete topology, and where the product group is endowed with the usual product topology. For all $n \geq 1$ index the elements of $S(x,n)$ by $\{1,2,\dots,q^{n-1}(q+1)\}$. Then every $g \in \mathrm{stab}(x)$ maps $S(x,n)$ to itself, and thus g corresponds to an element $(\sigma_1,\sigma_2,\dots) \in \prod_{n\geq 1} S_{q^{n-1}(q+1)}$. The resulting map $i\colon \mathrm{stab}(x) \to \prod_{n\geq 1} S_{q^{n-1}(q+1)}$ is clearly an injective (algebraic) group homomorphism. To show that i is continuous, recall that for every $\sigma = (\sigma_1,\sigma_2,\dots) \in \prod_{n\geq 1} S_{q^{n-1}(q+1)}$ the open subsets $V_k(\sigma) = \left(\prod_{1\leq i\leq k}\{\sigma_i\}\right) \times \left(\prod_{n>k} S_{q^{n-1}(q+1)}\right)$ form a neighbourhood basis of σ. Then $i^{-1}(V_k(i(g))) = U_{B(x,k)}(g)$ is open, which implies the continuity of i. To show that i is an embedding, let $g \in stab(x)$ and $y \in \mathrm{vert}(T_{q+1})$, and suppose σ is in the complement of $i(\mathrm{stab}(x) - U_{\{x,y\}}(g))$. Then for $k = d(x,y)$ $V_k(\sigma) \cap i(\mathrm{stab}(x) - \mathrm{stab}(x) \cap U_{\{y\}}(g)) = \emptyset$. Hence $i(\mathrm{stab}(x) - U_{\{x,y\}}(g))$ is closed, and therefore $i(U_{\{x,y\}}(g))$ is open with respect to the relative subspace topology of $i(\mathrm{stab}(x))$. Deduce that i is an embedding, whose image is closed. Because, by Tychonoff's Theorem, $\prod_{n\geq 1} S_{q^{n-1}(q+1)}$ is compact, stab(x) is isomorphic to a compact group and thus is itself compact. □

Corollary 5.3.2. Aut(T_{q+1}) *is locally compact and separable.*

Proof. Let $g \in \mathrm{Aut}(T_{q+1})$, and let $x \in \mathrm{vert}(T_{q+1})$. Then $g \cdot \mathrm{stab}(x)$ is both open and compact, and it contains g. Furthermore, as seen in the proof of Proposition 5.3.1, stab(x) is separable. Since the natural action of Aut(T_{q+1}) on $\mathcal{T}_{\mathrm{II}+\infty}$ is transitive, stab(x) is a subgroup in Aut(T_{q+1}) of countable index, which implies that Aut(T_{q+1}) is separable too. □

Recall that a subset of a discrete topological space is compact if and only if it is finite.

Lemma 5.3.3 ([FTN91, Chap. I, Prop. 5.1])**.** *A subgroup $G <$ Aut(T_{q+1}) is compact if and only if for every $x \in \mathrm{vert}(T_{q+1})$ the orbit* orb$_G(x)$ *is finite.*

Proof. If G is compact, then orb$_G(x)$ is also compact and thus finite.

Conversely, suppose $\mathrm{orb}_G(x) = \{g_1 \cdot x, g_2 \cdot x, \dots, g_n \cdot x\}$ is finite with $g_i \in G$. Then $G \subset \bigcup_{1\leq i\leq n} g_i \cdot \mathrm{stab}_G(x)$. Using the fact that the finite union of compact sets is also compact, deduce that G is compact. □

The following theorem identifies the maximal compact subgroups of Aut(T_{q+1}). Given a subset $S \subset \mathrm{vert}(T_{q+1})$, the *subtree $D \subset T_{q+1}$ generated by S* is the smallest connected subtree of T_{q+1} which contains S.

Theorem 5.3.4 ([FTN91, Chap. I, Thm. 5.2])**.** *A compact subgroup $K < \mathrm{Aut}(T_{q+1})$ is contained in stab(x) for some $x \in \mathrm{vert}(T_{q+1})$ or in stab$(\{y,z\})$ for some $\{y,z\} \in \mathrm{edg}(T_{q+1})$. Moreover, the maximal compact subgroups of Aut$(T_{q+1})$are* stab(x) *and stab$(\{y,z\})$, for $x \in \mathrm{vert}(T_{q+1})$ and $\{y,z\} \in \mathrm{edg}(T_{q+1})$.*

Proof. Fix $x_0 \in \mathrm{vert}(T_{q+1})$. By Lemma 5.3.3, $\mathrm{orb}_K(x_0)$ is finite. Then the connected subtree $D \subset \mathcal{T}_{q+1}$ generated by $\mathrm{orb}_K(x_0)$ is finite. This subtree is K-invariant. If D contains more than two vertices, then there are $x \in \mathrm{vert}(T_0)$ and $k \in K$ such that $x = k \cdot x_0$ has degree 1 in D. Since the degree of a vertex in D is preserved under the action of K, every vertex in D needs to have degree 0 or 1. This implies that D is either a single vertex or an edge.

It remains to show that the stabilizer groups of vertices and edges are the maximal compact subgroups of $\mathrm{Aut}(T_{q+1})$. Indeed, for all $x, y, z \in \mathrm{vert}(T_{q+1})$, by Proposition 5.2.5, $\mathrm{stab}_G(x) \not\subset \mathrm{stab}_G(\{y,z\})$ and $\mathrm{stab}_G(\{y,z\}) \not\subset \mathrm{stab}_G(x)$, since otherwise $\mathrm{stab}_G(x)$ or $\mathrm{stab}_G(\{y,z\})$ would contain a translation. Moreover, it is clear that, for $x_0, x_1 \in \mathrm{vert}(T_{q+1})$ and $\{y_0,z_0\}, \{y_1,z_1\} \in \mathrm{edg}(T_{q+1})$, $\mathrm{stab}_G(x_0) \subset \mathrm{stab}_G(x_1)$ and $\mathrm{stab}_G(\{y_0,z_0\}) \subset \mathrm{stab}_G(\{y_1,z_1\})$ imply $x_0 = x_1$ and $\{y_0,z_0\} = \{y_1,z_1\}$. □

Next, one wants to investigate non-compact closed subgroups of $\mathrm{Aut}(T_{q+1})$. For the remainder of this chapter suppose that $q+1 \geq 3$.

Henceforth, closed non-compact subgroups of $\mathrm{Aut}(T_{q+1})$ will be considered. To better understand their natural action on $\mathcal{T}_{q+1}$ introduce the following notions. The notation follows here [LM92].

Definition 5.3.5. Let $x \in \mathrm{vert}(T_{q+1})$ and $\omega \in \partial T_{q+1}$. Then the *horocycle* around ω at x is

$$H(x,\omega) \coloneqq \{y \in \mathrm{vert}(T_{q+1}) \colon d(x,z) = d(y,z) \text{ where } [z,\omega) = [x,\omega) \cap [y,\omega)\}.$$

The *horoball* around ω at x is

$$C(x,\omega) \coloneqq \bigcup_{y \in [x,\omega)} H(y,\omega).$$

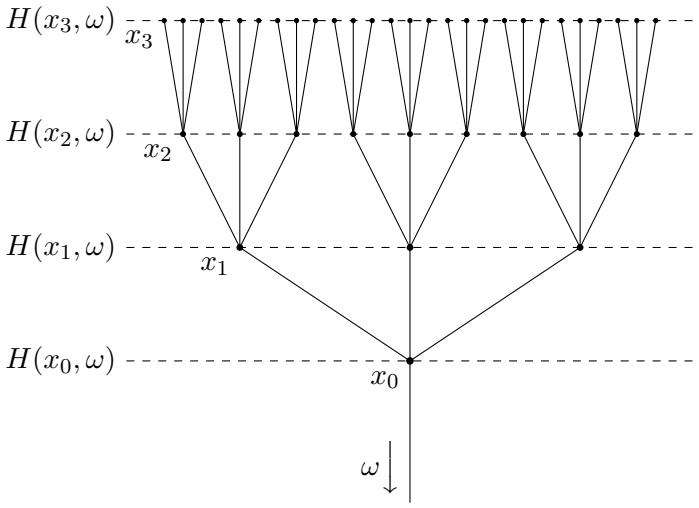

Figure 5.6: Horocycles around a boundary point $\omega \in \partial T_4$.

It is left to the reader to show that for every $g \in \mathrm{Aut}(T_{q+1})$

$$g \cdot H(x,\omega) = H(g \cdot x, g \cdot \omega) \quad \forall x \in \mathrm{vert}(T_{q+1})\ \forall \omega \in \partial T_{q+1}.$$

Lemma 5.3.6. *For every $\omega_0, \omega_1 \in \partial T_{q+1}$ and every $x \in \mathrm{vert}(T_{q+1})$, $H(x,\omega_0) = H(x,\omega_1)$ if and only if $\omega_0 = \omega_1$.*

Proof. Fix two distinct boundary points $\omega_0, \omega_1 \in \partial T_{q+1}$. Let $y \in H(x, \omega_0) \cap (\omega_0, \omega_1)$, and assume that $x \neq y$. Then $d(x,t) = d(y,t)$, where $t \in \mathrm{vert}(T_{q+1})$ is such that $[t, \omega_0) = [x, \omega_0) \cap [y, \omega_0)$. Then $[t, \omega_0) = [x, \omega_0) \cap (\omega_0, \omega_1)$. This implies that $[x, \omega_1) = [x, t] \cup [t, \omega_1)$, and because $[y, \omega_1) \subset [x, \omega_1)$, $x = y$, which yields a contradiction.

The converse is immediate. □

For a boundary point $\omega \in \partial T_{q+1}$, let $\mathcal{H}_\omega$ be the set of all horocycles around ω, that is,

$$\mathcal{H}_\omega \coloneqq \{H(x, \omega) \colon x \in \mathrm{vert}(T_{q+1})\}.$$

Notice that for every $x \in \mathrm{vert}(T_{q+1})$ and every $y \in H(x, \omega)$, $H(x, \omega) = H(y, \omega)$.

Definition 5.3.7. Let $G < \mathrm{Aut}(T_{q+1})$ be a subgroup, and fix $x \in \mathrm{vert}(T_{q+1})$, $\omega \in \partial T_{q+1}$. Let

$$N_G(\omega) \coloneqq \{g \in G \colon g \cdot H(x, \omega) = H(x, \omega)\ \forall x \in \mathrm{vert}(T_{q+1})\}.$$

Also let

$$U_G(x, \omega) \coloneqq \{g \in G \colon g \cdot y = y\ \forall y \in H(x, \omega)\},$$

and $U_G(\omega) \coloneqq \bigcup_{x \in \mathrm{vert}(T_{q+1})} U_G(x, \omega)$.

By Lemma 5.3.6, $N_G(\omega) < \mathrm{stab}_G(\omega)$.

If $G = \mathrm{Aut}(T_{q+1})$, write $N(\omega)$, $U(x, \omega)$, and $U(\omega)$. The definition of the stabilizer group of a horocycle can be relaxed as follows:

Proposition 5.3.8. *Let $G < \mathrm{Aut}(T_{q+1})$ be a subgroup, and let $\omega \in \partial T_{q+1}$. Then*

$$N_G(\omega) = \{g \in G \colon g \cdot H(x, \omega) = H(x, \omega) \text{ for some } x \in \mathrm{vert}(T_{q+1})\}, \tag{5.3}$$

$$= \{g \in G \colon g \cdot \omega = \omega \text{ and there is } x \in \mathrm{vert}(T_{q+1}) \text{ such that } g \cdot x = x\}. \tag{5.4}$$

Proof. Denote the sets on the right-hand side of (5.3) and (5.4) by N_1 and N_2, respectively. The inclusion $N_G(\omega) \subset N_1$ is immediate.

$N_1 \subset N_2$: Let $g \in N_1$ with $x, y \in \mathrm{vert}(T_{q+1})$ be such that $y, g \cdot y \in H(x, \omega)$. By Lemma 5.3.6, $g \cdot \omega = \omega$. Let $z \in \mathrm{vert}(T_{q+1})$ be such that $[z, \omega) = [y, \omega) \cap [g \cdot y, \omega)$. Because $g \cdot [y, \omega) = [g \cdot y, \omega)$ and $d(y, z) = d(g \cdot y, g \cdot z) = d(g \cdot y, z)$, $g \cdot z = z$.

$N_2 \subset N_G(\omega)$: Let $g \in N_0$ and $x \in \mathrm{vert}(T_{q+1})$ be such that $g \cdot x = x$, and let $y \in \mathrm{vert}(T_{q+1})$. Then, because $g \cdot \omega = \omega$, $g \cdot [x, \omega) = [x, \omega)$, that is, g fixes the one-sided infinite geodesic $[x, \omega)$ pointwise. Let $z \in \mathrm{vert}(T_{q+1})$ be such that $[x, \omega) \cap [y, \omega) = [z, \omega)$. Then $g \cdot z = z$ and hence $[z, \omega) \subset [t, \omega) = [y, \omega) \cap [g \cdot y, \omega)$, where $t \in \mathrm{vert}(T_{q+1})$. It follows that

$$d(y, t) = d(y, z) - d(t, z) = d(g \cdot y, z) - d(t, z) = d(g \cdot y, t),$$

which implies $g \cdot y \in H(y, \omega)$. Since $y \in \mathrm{vert}(T_{q+1})$ was arbitrary, $N_2 \subset N_G(\omega)$. □

Next consider

Lemma 5.3.9 ([FTN91, Chap. I, Lem. 5.3])**.** *Let $G < \mathrm{Aut}(T_{q+1})$ be a subgroup. If for some $x \in \mathrm{vert}(T_{q+1})$ $\mathrm{orb}_G(x)$ contains three distinct vertices on the same geodesic, then G contains a translation along the geodesic.*

Proof. Let $x \in \mathrm{vert}(T_{q+1})$, and suppose $x, y, z \in \mathrm{orb}_G(x) \cap \gamma$ are three distinct vertices on some geodesic $\gamma \subset \mathrm{vert}(T_{q+1})$. Let $g, h \in G$ with $y = g \cdot x$ and $z = h \cdot x$, and suppose $[y, z] = [y, x] \cup [x, z]$. Let $[y, x] = \{y_k\}_{0 \leq k \leq m}$ and $[z, x] = \{z_k\}_{0 \leq k \leq n}$ be the respective geodesics with $x = y_m = z_n$, $y = y_0$, $z = z_0$. If $g^{-1} \cdot y_1 \neq y_{m-1}$ or $h \cdot z_1 \neq z_{n-1}$, then $g \in G$ or $h \in G$ is a translation by Lemma 5.2.3. Hence, suppose $g^{-1} \cdot y_1 = y_{m-1}$ and $h \cdot z_1 = z_{n-1}$. Then $(hg^{-1}) \cdot y = z$ and $[y, z] = \{y_k\}_{0 \leq k \leq m} \cup \{z_k\}_{0 \leq k \leq n}$ is the geodesic from y to z. Notice that $(hg^{-1}) \cdot y_1 = h \cdot y_{m-1} \neq z_{n-1}$, since $y_{m-1} \neq z_1$. Therefore, by Lemma 5.2.3, $hg^{-1} \in G$ is a translation along the geodesic γ. □

Theorem 5.3.10 ([FTN91, Chap. I, Thm. 8.1])**.** *Let $G < \mathrm{Aut}(T_{q+1})$, and suppose G contains no translation. Then $G < \mathrm{stab}(x)$ for some vertex $x \in \mathrm{vert}(T_{q+1})$, $G < \mathrm{stab}(\{x,y\})$ for some edge $\{x,y\} \in \mathrm{edg}(T_{q+1})$ or $G < N(\omega)$ for some boundary point $\omega \in \partial T_{q+1}$.*

Proof. By Lemma 5.3.9, for every $x \in \mathrm{vert}(T_{q+1})$, $\mathrm{orb}_G(x)$ cannot contain three vertices on the same geodesic. By Theorem 5.3.4, if G has compact closure, then it is contained in $\mathrm{stab}_G(x)$ for some $x \in \mathrm{vert}(T_{q+1})$ or in $\mathrm{stab}_G(\{x,y\})$ for some edge $\{x,y\} \in \mathrm{edg}(T_{q+1})$. Henceforth, suppose G has non-compact closure. Then, by Lemma 5.3.3, for all $x \in \mathrm{vert}(T_{q+1})$ the orbit $\mathrm{orb}_G(x)$ is infinite. Therefore, since $\mathcal{T}_{q+1}$ is compact, for every $x \in \mathrm{vert}(T_{q+1})$, $\mathrm{orb}_G(x)$ has a limit point on ∂T_{q+1}.

The goal now is to show that all orbits of G in $\mathrm{vert}(T_{q+1})$ have a unique common limit point on ∂T_{q+1}. For this argue by contradiction, and let $x \in \mathrm{vert}(T_{q+1})$. Suppose $\mathrm{orb}_G(x)$ has two distinct limit points $\omega_0, \omega_1 \in \partial T_{q+1}$, and set $\gamma = (\omega_0, \omega_1)$. Claim that $\mathrm{orb}_G(x) \cap \gamma = \emptyset$. Indeed, assume $y \in \mathrm{orb}_G(x) \cap \gamma$. But, for $z_i \in [y, \omega_i)$, $i = 0, 1$, also $\mathrm{orb}_G(x) \cap U(y, z_i) \neq \emptyset$. Assume that $U(y, z_0) \cap U(y, z_1) = \emptyset$, and take $w_i \in \mathrm{orb}_G(x) \cap U(y, z_i)$, $i = 0, 1$. Then $\mathrm{orb}_G(x)$ contains three distinct vertices of $[w_1, w_2]$, and hence, by Lemma 5.3.9, G contains a translation, which yields a contradiction.

Since $\mathrm{vert}(T_{q+1})$ is countable, there are countably many $x_n \in \mathrm{vert}(T_{q+1})$, $n \geq 1$, such that $\mathrm{vert}(T_{q+1}) = \bigsqcup_{n\geq 1} \mathrm{orb}_G(x_n)$. Let $\Omega_n \subset \partial T_{q+1}$ be the subset of limit points of $\mathrm{orb}_G(x_n)$ on ∂T_{q+1}, and let $D_n \subset T_{q+1}$ be the subtree generated by the geodesics (ω_n, ω_n') for $\omega_n, \omega_n' \in \Omega_n$. Notice that all Ω_n and D_n are G-invariant, and that $\Omega_n = \overline{D_n} \cap \partial T_{q+1}$. Next claim that there is a sequence $\{n_k\}_{\geq 1}$ such that $D_{n_k} \cap \mathrm{orb}_G(x_{n_k}) = \emptyset$, $\mathrm{orb}_G(x_{n_k}) \subset D_{n_{k-1}}$ and $\bigcap_{k\geq 1} D_{n_k} = \emptyset$. This sequence can be constructed inductively. Indeed, fix $x_{n_1} \in \mathrm{vert}(T_{q+1})$. As seen above, $x_{n_1} \notin D_{n_1}$, and take $x_{n_2} \in D_{n_1}$. Proceeding inductively, it follows that $\mathrm{orb}_G(x_{n_k}) \subset D_{n_{k-1}}$, and thus $D_{n_k} \subset D_{n_{k-1}}$ and $\Omega_{n_k} \subset \Omega_{n_{k-1}}$. By the finite intersection property $\bigcap_{k\geq 1} \Omega_k \neq \emptyset$. On the other hand, because the closure of G is non-compact, $\bigcap_{k\geq 1} D_{n_k} = \emptyset$. Therefore, $\bigcap_{k\geq 1} \Omega_k$ contains exactly one point $\omega_0 \in \partial T_{q+1}$, which is fixed by G.

Finally, because no inversion can fix a boundary point, G consists only of rotations. Thus, by Proposition 5.3.8, $G < N(\omega_0)$. □

It follows from Theorem 5.3.10 that if $G < \mathrm{Aut}(T_{q+1})$ is a closed non-compact subgroup acting transitively on ∂T_{q+1}, then it contains a translation. Moreover, G acts transitively on ∂T_{q+1} if and only if $\mathrm{stab}_G(x)$ acts transitively on $S(x,n)$ for every $x \in \mathrm{vert}(T_{q+1})$ and $n \geq 1$. Thus, if G acts transitively on ∂T_{q+1} and is non-compact, there is a subset $Q \subset \mathbb{N}$ such that for all $x \in \mathrm{vert}(T_{q+1})$ $\mathrm{orb}_G(x) = \bigcup_{q\in Q} S(x,q)$. Using the greatest common divisor, one can show that there is $n \geq 1$ such that $\mathrm{orb}_G(x) = \bigcup_{k\geq 0} S(x, n\cdot k)$. Theorem 5.3.13 below asserts now that $n \in \{1, 2\}$. Henceforth fix a vertex $o \in \mathrm{vert}(T_{q+1})$, which will be called the *base point* of T_{q+1}.

Remark. It is known that every homogeneous tree is the Cayley graph of the free product of finitely many copies of $\mathbb{Z}$ and $\mathbb{Z}_2$, and vice versa (see [FTN91, Chap. I, Sect. 7]). For this reason, o is sometimes also called the *root* of T_{q+1}.

Set

$$\mathrm{vert}^+(T_{q+1}) \coloneqq \{x \in \mathrm{vert}(T_{q+1}) \colon d(o,x) \text{ is even}\},$$
$$\mathrm{vert}^-(T_{q+1}) \coloneqq \{x \in \mathrm{vert}(T_{q+1}) \colon d(o,x) \text{ is odd}\}.$$

Proposition 5.3.11 ([FTN91, Chap. I, Prop. 10.1])**.** *Let $G < \mathrm{Aut}(T_{q+1})$ be a closed non-compact subgroup. Then G acts transitively on ∂T_{q+1} if and only if $\mathrm{stab}_G(o)$ acts transitively on ∂T_{q+1}.*

Proof. Suppose that G acts transitively on ∂T_{q+1}. By Theorem 5.3.10, G contains a translation τ along a geodesic $\gamma = (\omega_0, \omega_1)$. Since $G/\mathrm{stab}_G(o)$ is homeomorphic to a subset of $\mathrm{vert}(T_{q+1})$, $\mathrm{stab}_G(o)$ has countable index in G. Hence, because G acts transitively on ∂T_{q+1}, for every

$\omega \in \partial T_{q+1}$ there exists a countable subset $S \subset G$ such that $\partial T_{q+1} = \bigcup_{g\in S} g \cdot \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega)$. By Baire's Category Theorem, this implies that for some $g \in S$ $g \cdot \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega)$ is open. Hence, $\mathrm{orb}_{\mathrm{stab}_G(o)}(\omega)$ is open for every $\omega \in \partial T_{q+1}$, and by the compactness of ∂T_{q+1}, there is a finite subset $Q \subset \partial T_{q+1}$ such that $\partial T_{q+1} = \bigcup_{\omega\in Q} \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega)$.

Assume that $\tau^n \cdot o \to \omega_0$ as $n \to \infty$, that is, τ moves o along γ towards ω_0. Let $\omega_2 \in \partial T_{q+1} - \{\omega_1\}$. Then, there is $N \geq 0$ such that $[o, \omega_1) \cap [o, \omega_2) = [\tau^{-N} \cdot o, o]$, and thus for all $n \geq N$ $[o, \tau^n \cdot \omega_2) = [o, \tau^{n-N} \cdot o] \cup [\tau^{n-N} \cdot o, \tau^n \cdot \omega_2)$. Because $\mathrm{orb}_{\mathrm{stab}_G(o)}(\omega_0)$ is open, it follows that for some $n \geq 0$ $\tau^{n+N} \cdot \omega_2 \in \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega_0)$ and that there is $k \in \mathrm{stab}_G(o)$ such that $\omega_0 = (k\tau^{n+N}) \cdot \omega_2$. Observe that

$$k \cdot [o, \tau^{n+N} \cdot \omega_2) = [o, (k\tau^n) \cdot o] \cup [(k\tau^n) \cdot o, \omega_0) = [o, \omega_0),$$

and thus $(k\tau^n) \cdot o = \tau^n \cdot o$. Deduce that $\tau^{-n}k\tau^n \in \mathrm{stab}_G(o)$ and $(\tau^{-n}k\tau^n)^{-1} \cdot \omega_0 = \tau^N \cdot \omega_2$, which proves $\tau^N \cdot \omega_2 \in \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega_0)$. Therefore

$$\partial T_{q+1} - \Omega(o, \tau^{-1} \cdot o) \subset \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega_0),$$

and it can be shown by a similar argument that

$$\partial T_{q+1} - \Omega(o, \tau \cdot o) \subset \mathrm{orb}_{\mathrm{stab}_G(o)}(\omega_1).$$

But since $q+1 \geq 3$, there are more than two boundary points, and therefore the two orbits need to intersect. Hence $\mathrm{stab}_G(\omega_0)$ acts transitively on ∂T_{q+1}.

The converse is immediate. □

Using Proposition 5.3.11 one can deduce an *Iwasawa* decomposition for a closed non-compact subgroup $G < \mathrm{Aut}(T_{q+1})$ acting transitively on the boundary.

Corollary 5.3.12. *Suppose $G < \mathrm{Aut}(T_{q+1})$ is a closed non-compact subgroup which acts transitively on ∂T_{q+1}. Then for every $\omega \in \partial T_{q+1}$ $G = \mathrm{stab}_G(o)\,\mathrm{stab}_G(\omega)$.*

Proof. Fix $\omega \in \partial T_{q+1}$. By Proposition 5.3.11, G and $\mathrm{stab}_G(o)$ act transitively on $\partial T_{q+1} \cong G/\mathrm{stab}_G(\omega)$. Thus $G = \mathrm{stab}_G(o)\,\mathrm{stab}_G(\omega)$. □

Proposition 5.3.13 ([FTN91, Chap. I, Prop. 10.2])**.** *Let $G < \mathrm{Aut}(T_{q+1})$ be a closed non-compact subgroup which acts transitively on ∂T_{q+1}. Then the action of G on $\mathrm{vert}(T_{q+1})$ is either transitive or its orbits are $\mathrm{vert}^+(T_{q+1})$ and $\mathrm{vert}^-(T_{q+1})$.*

Proof. As remarked above, by Proposition 5.3.11, $\mathrm{stab}_G(o)$ acts transitively $S(x, n)$ for every $x \in \mathrm{vert}(T_{q+1})$ and $n \geq 0$. Also observe that if $n = d(g \cdot o, h \cdot o)$ for some $g, h \in G$, then $n = d(o, (g^{-1}h) \cdot o)$ and hence $S(o, n) \subset \mathrm{orb}_G(o)$.

Notice that for $n \geq 1$ $\min\{d(x, y) \colon x, y \in S(o, n) \text{ with } x \neq y\} = 2$. Indeed, fix $x \in S(o, n)$ for some $n \geq 1$. Then let $z \in S(o, n-1)$ be such that $\{x, z\} \in \mathrm{edg}(T_{q+1})$. Hence, because T_{q+1} has degree $q + 1 \geq 3$, there is $y \in S(o, n)$ such that $x \neq y$ and $\{z, y\} \in \mathrm{edg}(T_{q+1})$ (see Figure 5.7). Moreover, since $\{x, y\} \notin \mathrm{edg}(T_{q+1})$, $d(x, y) = 2$ (see Figure 5.7).

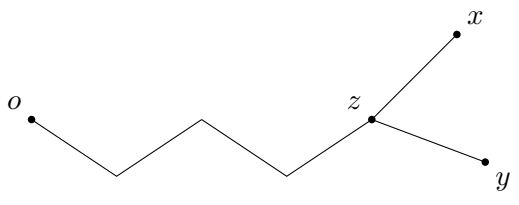

Figure 5.7

It follows that $n \coloneqq \min\{d(o, g \cdot o) \colon g \in G \text{ with } o \neq g \cdot o\}$ is either 1 or 2. It remains to show that $\mathrm{orb}_G(o) = \mathrm{vert}(T_{q+1})$ or $\mathrm{orb}_G(o) = \mathrm{vert}^+(T_{q+1})$. Because G is non-compact, by Lemma 5.3.3, $\mathrm{orb}_G(o)$ is not bounded, that is, there is an unbounded set $Q \subset \mathbb{N}$ such that $\mathrm{orb}_G(o) = \bigcup_{q \in Q} S(o, q)$. Notice that $Q \subset n \cdot \mathbb{N}$. Fix $g \in G$ such that $d(g \cdot o, o) = n$, and let $q \in Q$ with $q > n$. Let $h \in G$ be such that $h \cdot o \in S(o, q)$ and $d(g \cdot o, h \cdot o) = q - n$, that is, $[h \cdot o, g \cdot o] \subset [h \cdot o, o]$. Hence $d(o, (g^{-1}h) \cdot o) = q - n$ and $S(o, q - n) \subset \mathrm{orb}_G(o)$. By an argument of induction it follows that $\bigcup_{0 \leq k \leq \frac{q}{n}} S(o, k \cdot n) \subset \mathrm{orb}_G(o)$. Since $\mathrm{orb}_G(o)$ is not bounded, this implies $\mathrm{orb}_G(o) = \bigcup_{k \geq 0} S(o, k \cdot n)$. □

Notice that by the decomposition in Corollary 5.3.12, a closed non-compact subgroup $G < \mathrm{Aut}(T_{q+1})$, which acts transitively on the boundary, acts transitively on $\mathrm{vert}(T_{q+1})$ if and only if, for every $\omega \in \partial T_{q+1}$, $\mathrm{stab}_G(\omega)$ acts transitively on $\mathrm{vert}(T_{q+1})$.

Proposition 5.3.13 motivates

Definition 5.3.14. Let $\mathrm{Aut}_0(T_{q+1}) < \mathrm{Aut}(T_{q+1})$ be the subgroup of automorphisms $g \in \mathrm{Aut}(T_{q+1})$ such that $g \cdot \mathrm{vert}^+(T_{q+1}) \subset \mathrm{vert}^+(T_{q+1})$. The elements of $\mathrm{Aut}_0(T_{q+1})$ are called the *color-preserving automorphisms* of T_{q+1}.

It is clear that $\mathrm{Aut}_0(T_{q+1})$ is a closed normal subgroup of $\mathrm{Aut}(T_{q+1})$ with index 2. A closed subgroup $G < \mathrm{Aut}(T_{q+1})$ either is contained in $\mathrm{Aut}_0(T_{q+1})$ or has a closed normal subgroup $G \cap \mathrm{Aut}_0(T_{q+1})$ of index 2.

Henceforth the case will be considered where G is non-compact and acts transitively on the boundary ∂T_{q+1}. The goal now is to obtain a Cartan decomposition of G as in Theorem 2.3.1 and Theorem 2.3.2. Set

$$s(G) \coloneqq \min\{d(o, g \cdot o) \colon g \in G, \text{ such that } g \cdot o \neq o\}$$

A translation in G is said to be of *minimal step* if it is of step $s(G)$. By Proposition 5.3.13, $s(G) \in \{1, 2\}$. One has

Lemma 5.3.15. *If $G < \mathrm{Aut}(T_{q+1})$ is a closed non-compact subgroup acting transitively on the boundary ∂T_{q+1}, then G contains a translation of step $s(G)$, that is, a translation of minimal step.*

Proof. Fix three distinct vertices $x, y, z \in \mathrm{vert}(T_{q+1})$ such that $\{x, y\}, \{y, z\} \in \mathrm{edg}(T_{q+1})$. Then, $[x, y] = \{x, z, y\}$ and $d(x, y) = 2$.

Suppose that $G < \mathrm{Aut}_0(T_{q+1})$. By Proposition 5.3.13, G acts transitively on $\mathrm{vert}^+(T_{q+1})$ and on $\mathrm{vert}^-(T_{q+1})$. Hence there exists $g \in G$ such that $g \cdot x = y$. By Proposition 5.3.11, $\mathrm{stab}_G(x)$ acts transitively on ∂T_{q+1} and thus on $S(x, 1)$. Since $d(x, g \cdot z) = d(y, z) = 1$, there exists a rotation $k \in \mathrm{stab}_G(x)$ such that $(kg) \cdot z \neq z$. By Lemma 5.2.3 $\tau = kg$ is a translation of step 2.

Suppose now that G acts transitively on $\mathrm{vert}(T_{q+1})$. Then, $G \cap \mathrm{Aut}_0(T_{q+1})$ contains a translation τ of step 2 along a one-sided infinite geodesic $[x, \omega)$ for some $\omega \in \partial T_{q+1}$, that is, $\tau^n \cdot x \to \omega$ as $n \to \infty$. Because $\mathrm{stab}_G(o)$ acts transitively on ∂T_{q+1}, one can assume that $y, z \in [x, \omega)$, and by Corollary 5.3.12, that $\mathrm{stab}_G(\omega)$ acts transitively on $\mathrm{vert}(T_{q+1})$. Let $p \in \mathrm{stab}_G(\omega)$ be such that $z = p \cdot y$. Then, because $p \cdot [y, \omega) = [z, \omega)$, $\tau' = p\tau$ is a translation of step 1 along $[x, \omega)$. □

Using Lemma 5.3.15 the following decomposition of G can be established in addition to the Iwasawa decomposition in Corollary 5.3.12. It will be called the *Cartan decomposition*. To simplify notation, write $|x| \coloneqq d(x, o)$ for $x \in \mathrm{vert}(T_{q+1})$.

Proposition 5.3.16 ([LM92, Lem. 1])**.** *Let $G < \mathrm{Aut}(T_{q+1})$ be a closed non-compact subgroup acting transitively on ∂T_{q+1}. Let $\tau \in G$ be a translation of minimal step along a geodesic $\gamma = (\omega_0, \omega_1)$, and set $A \coloneqq \{\tau^k \colon k \in \mathbb{Z}\}$, $A^+ \coloneqq \{\tau^n \colon n \geq 0\}$.*

Suppose $o \in \gamma$. Then:

1) $G = \mathrm{stab}_G(o) A^+ \mathrm{stab}_G(o)$*;*

2) $\mathrm{stab}_G(\omega_0) = A \cdot N_G(\omega_0)$.

Proof. 1) Let $g \in G$. Then, because $\mathrm{stab}_G(o)$ acts transitively on $S(o,r)$ for every integer $r \geq 1$, there are $n \geq 0$ and $k_1 \in \mathrm{stab}_G(o)$ such that $\tau^n \cdot o \in S(o, |g \cdot o|)$ and $g \cdot o = (k_1 \tau^n) \cdot o$. Hence, since $\mathrm{orb}_G(o) \cong G/\mathrm{stab}_G(o)$, $(k_1\tau^n)^{-1} g = k_2 \in \mathrm{stab}_G(o)$, that is, $g = k_1 \tau^n k_2$.

2) Let $p \in \mathrm{stab}_G(\omega_0)$. Then let $k \in \mathbb{Z}$ such that $\tau^k \cdot o \in H(p \cdot o, \omega_0)$, that is, $p \cdot o \in H(\tau^k \cdot o, \omega_0)$. It follows that $(\tau^{-k} p) \cdot o \in H(o, \omega_0)$ and therefore, because $\tau^k p \in \mathrm{stab}_G(\omega_0)$, $\tau^{-k} p \in N_G(\omega_0)$.

□

Corollary 5.3.17. *Let the hypothesis be as in Proposition 5.3.19. Then*

$$G = \mathrm{stab}_G(o) A N_G(\omega_0).$$

Proof. This is an immediate consequence of Corollary 5.3.12 and part 1) of Proposition 5.3.16. □

Using the geometry of T_{q+1} one can deduce from Proposition 5.3.16 an integral formula for the Cartan decomposition. For this, set $c(n) \coloneqq |S(x,n)|$ for $n \geq 1$, and note that

$$c(n) \coloneqq \begin{cases} 1 & \text{if } n = 0, \\ q^{n-1}(q+1) & \text{if } n \geq 1. \end{cases}$$

A *function* f on T_{q+1} is a complex-valued function $f \colon \mathrm{vert}(T_{q+1}) \to \mathbb{C}$. Let $C(T_{q+1})$ be the set of all functions on T_{q+1}, and let $C_c(T_{q+1})$ be the set of all compactly supported functions on T_{q+1}, that is, $\mathrm{supp}(f)$ is finite.

Theorem 5.3.18 (Integral Formula for the Cartan decomposition)**.** *Suppose that* $G < \mathrm{Aut}(T_{q+1})$ *is a closed subgroup which acts transitively on* ∂T_{q+1}*. Let* A^+ *be as in Proposition 5.3.16. Then*

$$\int_G f(x)\, dx = \int_{\mathrm{stab}_G(o)} \int_{A^+} \int_{\mathrm{stab}_G(o)} f(k_1 a k_2) c(|a \cdot o|)\, dk_1\, da\, dk_2 \quad \forall f \in C_c(G),$$

where dk_1 *and* dk_2 *are the normalized Haar measure on* $\mathrm{stab}_G(o)$ *and* da *is the counting measure on the discrete semisubgroup* $A^+ < G$*.*

Proof. Because $\mathrm{stab}_G(o)$ is compact, by the Weil Formula, one has that for all compactly supported functions $f \in C_c(T_{q+1})$

$$\int_G f(x \cdot o)\, dx = \int_{G/\mathrm{stab}_G(o)} \int_{\mathrm{stab}_G(o)} f((xk) \cdot o)\, dk\, dx = \int_{\mathrm{vert}(T_{q+1})} f\, d\mu,$$

where μ denotes the counting measure on $\mathrm{vert}(T_{q+1})$. The last equality follows, because $G/\mathrm{stab}_G(o) \cong \mathrm{vert}(T_{q+1})$ and μ is G-invariant. It remains to show that for all $f \in C_c(T_{q+1})$

$$\int_{\mathrm{vert}(T_{q+1})} f\, d\mu = \int_{\mathrm{stab}_G(o)} \int_{A^+} f((ka) \cdot o) c(|a \cdot o|)\, da\, dk.$$

It is enough to show that this equality holds for the indicator functions $\chi_{\{x_0\}}$ for all $x_0 \in \mathrm{vert}(T_{q+1})$. Fix $x_0 \in \mathrm{vert}(T_{q+1})$, and let $a_0 \in A^+$, $k_0 \in \mathrm{stab}_G(o)$ such that $x_0 = (k_0 a_0) \cdot o$. Then

$$\begin{aligned}
&\int_{\mathrm{stab}_G(o)} \int_{A^+} \chi_{\{x_0\}}((ka) \cdot o) c(|a \cdot o|)\, da\, dk \\
&\quad = c(|a_0 \cdot o|) \int_{\mathrm{stab}_G(o)} \chi_{\{x_0\}}((ka_0) \cdot o)\, dk \\
&\quad = c(|a_0 \cdot o|) \sum_{y \in S(o, |a_0 \cdot o|)} \frac{1}{c(|a_0 \cdot o|)} \chi_{\{x_0\}}(y) \\
&\quad = 1 = \int_{\mathrm{vert}(T_{q+1})} \chi_{\{x_0\}}\, d\mu.
\end{aligned}$$

□

A subgroup $G < Aut(T_{q+1})$ is said to act 2-transitively on ∂T_{q+1} if for every two pairs (ω_0, ω_0'), (ω_1, ω_1') of distinct boundary points $\omega_i, \omega_i' \in \partial T_{q+1}$ there exists $g \in G$ such that $(g \cdot \omega_0, g \cdot \omega_0') = (\omega_1, \omega_1')$. It is easy to see that G acts 2-transitively on ∂T_{q+1} if and only if for every $\omega \in \partial T_{q+1}$ $\mathrm{stab}_G(\omega)$ acts transitively on $\partial T_{q+1} - \{\omega\}$.

Proposition 5.3.19. *Let $G < Aut(T_{q+1})$ be a closed subgroup which acts 2-transitively on ∂T_{q+1}. Then for every two distinct boundary points $\omega_0, \omega_1 \in \partial T_{q+1}$ one has $G = \langle \mathrm{stab}_G(\omega_0), \mathrm{stab}_G(\omega_1) \rangle$.*

Proof. For $i = 0, 1$ one has $\partial T_{q+1} \cong G/\mathrm{stab}_G(\omega_i)$. Let $g \in G$. If $g \cdot \omega_0 = \omega_0$, then $g \in \mathrm{stab}_G(\omega_0)$. Assume $g \cdot \omega_0 \neq \omega_0$, and let $\omega_2 \in \partial T_{q+1} - \{\omega_0, \omega_1\}$. Then, because $\mathrm{stab}_G(\omega_i)$ acts transitively on $\partial T_{q+1} - \{\omega_i\}$, there are $p_i \in \mathrm{stab}_G(\omega_i)$ such that $\omega_2 = p_1 \cdot \omega_0 = (p_0 g) \cdot \omega_0$. Thus $g \cdot \omega_0 = (p_0^{-1} p_1) \cdot \omega_0$ and therefore $(p_1^{-1} p_0) g \in \mathrm{stab}_G(\omega_0)$. □

Corollary 5.3.20. *Let the hypothesis be as in Proposition 5.3.19. Then for every two distinct boundary points $\omega_0, \omega_1 \in \partial T_{q+1}$, $G \cap \mathrm{Aut}_0(T_{q+1}) = \langle N_G(\omega_0), N_G(\omega_1) \rangle$.*

Proof. By Proposition 5.3.8, $N_G(\omega_i) < G \cap \mathrm{Aut}_0(T_{q+1})$, which implies that $N_G(\omega_i) = N_{G \cap \mathrm{Aut}_0(T_{q+1})}(\omega_i)$, for $i = 0, 1$. Hence it is enough to show the corollary for the case of $s(G) = 2$.

Assume that $G < \mathrm{Aut}_0(T_{q+1})$. By assumption, $P_G(\omega_i)$ and thus, by Corollary 5.3.17, $N_G(\omega_i)$ acts transitively on $\partial T_{q+1} - \{\omega_i\}$, for $i = 0, 1$. Hence the group $N := \langle N_G(\omega_0), N_G(\omega_1) \rangle$ acts transitively on ∂T_{q+1}. By Lemma 5.3.15 there is a translation $\tau \in N$ of step 2. By taking $k \tau k^{-1}$ for an appropriate $k \in \mathrm{stab}_G(o) \cap N$, assume that τ is a translation along the geodesic $[o, \omega_0)$. Then, setting $A = \{\tau^n : n \in \mathbb{Z}\}$, one has $\mathrm{stab}_G(\omega_0) = A \cdot N_G(\omega_0)$. This implies $\mathrm{stab}_G(\omega_0) < N$. By a similar argument one can show that $\mathrm{stab}_G(\omega_1) < N$. Hence $\langle \mathrm{stab}_G(\omega_0), \mathrm{stab}_G(\omega_1) \rangle < N$ and, by Proposition 5.3.19, $N = G$. □

5.4 The Howe–Moore Property for Tree Automorphisms

As in the previous section, let T_{q+1} be a homogeneous tree of degree $q + 1 \geq 3$. Using Theorem 2.2.1 one can prove

Theorem 5.4.1 ([Cio15, Thm. 4.19])**.** *Let $G < \mathrm{Aut}(T_{q+1})$ be a closed non-compact subgroup whose action on ∂T_{q+1} is 2-transitive. Then G has the Howe–Moore Property if and only if it is topologically simple and is contained in $Aut_0(T_{q+1})$.*

The first implication follows from Proposition 2.1.3. For the converse recall the definition of contraction and parabolic groups associated to a sequence in G. The following two results about those subgroups of G will be used here. The results are just stated here with minor modifications and without proofs, since this would not be in the frame of the thesis.

Proposition 5.4.2 ([BW04, Prop. 3.4])**.** *Suppose G is a locally compact group, and let $g \in G$. Then $U_g^{\pm} \triangleleft P_g^{\pm}$ and $U_g^{\pm} M_g < P_g^{\pm}$.*

Theorem 5.4.3 ([BW04, Cor. 3.17])**.** *Suppose G is a second countable totally disconnected locally compact group, and let $g \in G$. Then $U_g^{\pm} M_g = P_g^{\pm}$.*

Theorem 5.4.3 is actually a consequence of [BW04, Cor. 3.17], where G is assumed to be metrizable. However, every first countable locally compact group is metrizable (see Theorem A.1.12). Observe that every closed subgroup $G < \mathrm{Aut}(T_{q+1})$ is second countable, totally disconnected and locally compact.

Proposition 5.4.4 ([Cio15, Prop. 4.15])**.** *Let $G < \mathrm{Aut}_0(T_{q+1})$ be a closed subgroup, and suppose $\tau \in G$ is a translation along a geodesic $\gamma = (\omega_1, \omega_0)$. Then $P_\tau^+ = \mathrm{stab}_G(\omega_0)$ and $P_\tau^- = \mathrm{stab}_G(\omega_1)$.*

Proof. The proposition is proven here only for P_τ^+; the case of P_τ^- is done similarly. Let $g \in P_\tau^+$, and notice that for all $x \in \mathrm{vert}(T_{q+1})$ $\tau^n \cdot x \to \omega_0$ as $n \to \infty$. Because $\{(\tau^{-n} g \tau^n) \cdot x\}_{n \geq 0}$ is a bounded and thus finite subset of $\mathrm{vert}(T_{q+1})$, take a sequence $\{n_k\}_{k \geq 0} \subset \mathbb{N}$ going to infinity such that $(\tau^{-n_k} g \tau^{n_k}) \cdot x = y$ for all $k \geq 0$, for some $y \in \mathrm{vert}(T_{q+1})$. Then

$$g \cdot \omega_0 = \lim_{k \to \infty} (g \tau^{n_k}) \cdot x = \lim_{k \to \infty} \tau^{n_k} \cdot y = \omega_0.$$

This implies $P_\tau^+ \subset \mathrm{stab}_G(\omega_0)$.

For the converse inclusion take $g \in \mathrm{stab}_G(\omega_0)$, and fix $x \in (\omega_0, \omega_1) \cap \mathrm{vert}^+(T_{q+1})$. Then $\{\tau^n \cdot x\}_{n \geq 0} = [x, \omega_0) \cap \mathrm{vert}^+(T_{q+1})$ and $\{(g\tau^n) \cdot x\}_{n \geq 0} = [g \cdot x, \omega_0) \cap \mathrm{vert}^+(T_{q+1})$. This implies that there are $m, N \geq 0$ such that $(g\tau^n) \cdot x = \tau^{n+m} \cdot x$ for all $n \geq N$. Then for all $n \geq N$ $(\tau^{-n} g \tau^n) \cdot x = \tau^m \cdot x$, and therefore $\{(\tau^{-n} g \tau^n) \cdot x\}_{n \geq 0}$ is finite, that is, bounded in the metric space $(\mathrm{vert}(T_{q+1}), d)$. Hence $\mathrm{stab}_G(\omega_0) \subset P_\tau^+$. □

Corollary 5.4.5 ([Cio15, Cor. 4.17])**.** *Let the hypothesis be as in Proposition 5.4.4. Then $\mathrm{stab}_G(\omega_0) = U_\tau^+(\mathrm{stab}_G(\omega_0) \cap \mathrm{stab}_G(\omega_1))$ and $\mathrm{stab}_G(\omega_1) = U_\tau^-(\mathrm{stab}_G(\omega_0) \cap \mathrm{stab}_G(\omega_1))$.*

Proof. This is an immediate consequence of Theorem 5.4.3 and Proposition 5.4.4. □

Let $N_G(U_\tau^+)$ and $N_G(U_\tau^-)$ denote the normalizers of U_τ^+ and U_τ^- in G, respectively.

Lemma 5.4.6 ([Cio15, Cor. 4.18])**.** *Let the hypothesis be as in Proposition 5.4.4. Then $\overline{\langle U_\tau^+, U_\tau^- \rangle} \triangleleft G$.*

Proof. By continuity, it is enough to show that $\langle U_\tau^+, U_\tau^- \rangle \triangleleft G$. Recall that, by Proposition 5.3.19, $G = \overline{\langle \mathrm{stab}_G(\omega_0), \mathrm{stab}_G(\omega_1) \rangle}$. Moreover, by Proposition 5.4.4, $P_\tau^+ = \mathrm{stab}_G(\omega_0)$ and $P_\tau^- = \mathrm{stab}_G(\omega_1)$. Thus it suffices to show that, for every $p_0 \in \mathrm{stab}_G(\omega_0)$ and every $p_1 \in \mathrm{stab}_G(\omega_1)$, $p_i^{-1} U_\tau^{\pm} p_i \subset \langle U_\tau^+, U_\tau^- \rangle$. By Proposition 5.4.2 and by Proposition 5.4.4, $U_\tau^+ \triangleleft \mathrm{stab}_G(\omega_0)$ and $U_\tau^- \triangleleft \mathrm{stab}_G(\omega_1)$. By Corollary 5.4.5, there are $u_0 \in U_\tau^+$, $m_0 \in \mathrm{stab}_G(\omega_0) \cap \mathrm{stab}_G(\omega_1)$ and $u_1 \in U_\tau^-$, $m_1 \in \mathrm{stab}_G(\omega_0) \cap \mathrm{stab}_G(\omega_1)$ such that $p_i = m_i u_i$. Because $\mathrm{stab}_G(\omega_0) \cap \mathrm{stab}_G(\omega_1) < N_G(U_\tau^+) \cap N_G(U_\tau^-)$, one has that $p_i^{-1} U_\tau^{\pm} p_i = u_i^{-1} U_\tau^{\pm} u_i \in \langle U_\tau^+, U_\tau^- \rangle$. □

Proof of Theorem 5.4.1. Suppose G has the Howe–Moore Property. Because $G \cap \mathrm{Aut}_0(T_{q+1}) \triangleleft G$ is of finite index, G needs to be contained in $\mathrm{Aut}_0(T_{q+1})$. Moreover, by Proposition 2.1.3, all proper closed normal subgroups are compact. Let $K \triangleleft G$ be a normal compact subgroup. Then, by Theorem 5.3.4, there exists a vertex x_0 or an edge $\{x_0, y_0\} \in \mathrm{edg}(T_{q+1})$ such that $K < G \cap \mathrm{stab}_G(x_0)$ or $K < G \cap \mathrm{stab}_G(\{x_0, y_0\})$. Because $G < \mathrm{Aut}_0(T_{q+1})$, $\mathrm{stab}_G(\{x_0, y_0\}) = \mathrm{stab}_G(x_0) \cap \mathrm{stab}_G(y_0)$. This implies that

$$K \leq \bigcap_{g \in G} \mathrm{stab}_G(g \cdot x_0) = \bigcap_{x \in \mathrm{vert}^+(T_{q+1})} \mathrm{stab}_G(x) = \{e_G\},$$

or

$$K \leq \bigcap_{g\in G} \mathrm{stab}_G(g\cdot x_0)\cap \mathrm{stab}_G(g\cdot y_0) = \bigcap_{x\in \mathrm{vert}(T_{q+1})} \mathrm{stab}_G(x) = \{e_G\}.$$

Hence the trivial subgroup is the only proper closed normal subgroup of G, that is, G is topologically simple.

For the converse, suppose G is topologically simple and contained in $\mathrm{Aut}_0(T_{q+1})$. By Lemma 5.3.15, G contains a translation of step 2 along a geodesic $\gamma = (\omega_0, \omega_1)$. Assume that $o \in \gamma$. Setting $A^+ = \{\tau^n : n \geq 0\}$, one has, by Proposition 5.3.16, the decomposition $G = \mathrm{stab}_G(o)A^+\,\mathrm{stab}_G(o)$. Because G is topologically simple, by Lemma 5.4.6, $G = \overline{\langle U_\tau^+, U_\tau^-\rangle}$. It follows from Theorem 2.2.1 that G has the Howe–Moore Property. □

It is usually not immediate whether G is topologically simple or not; e.g. consider $\mathrm{Aut}_0(T_{q+1})$. The following is a more general version of [LM92, Thm. 1].

Proposition 5.4.7. *Let $G < \mathrm{Aut}_0(T_{q+1})$ be a closed subgroup which acts 2-transitively on the boundary ∂T_{q+1}. Suppose that for some $\omega \in \partial T_{q+1}$ $U_G(\omega)$ is dense in $N_G(\omega)$. Then G enjoys the Howe–Moore Property.*

Proof. Because G acts transitively on ∂T_{q+1} and because $gU_G(\omega)g^{-1} = U_G(g\cdot\omega)$ for all $g \in G$, $U_G(\omega)$ is dense in $N_G(\omega)$ for all $\omega \in \partial T_{q+1}$. Fix $\omega_0, \omega_1 \in \partial T_{q+1}$, and let $\tau \in G$ be the translation of step 2 along $\gamma = (\omega_1, \omega_0)$. By Proposition 5.3.16, one has $G = \mathrm{stab}_G(o)A^+\,\mathrm{stab}_G(o)$ and $A^+ = \{\tau^n : n \geq 0\}$. Hence, it remains to show that G also satisfies the second condition in Theorem 2.2.1. For this, claim that $U_G(\omega_0) < U_\tau^+$. Indeed, let $g \in U_G(\omega_0)$ and $x \in \mathrm{vert}(T_{q+1})$ be such that g fixes $C(x, \omega_0)$ pointwise. Let $y \in \mathrm{vert}(T_{q+1})$ be any vertex, and notice that for n large enough $\tau^n \cdot y \in C(x, \omega_0)$ and therefore $(\tau^{-n}g\tau^n)\cdot y = y$. This implies $U_G(\omega_0) < U_\tau^+$. Similarly, $U_G(\omega_1) < U_\tau^-$. By Corollary 5.3.20,

$$G = \langle N_G(\omega_0), N_G(\omega_1)\rangle = \overline{\langle U_G(\omega_0), U_G(\omega_1)\rangle} = \langle U_\tau^+, U_\tau^-\rangle.$$

It follows from Theorem 2.2.1 that G has the Howe–Moore Property. □

It is clear that, for $\omega \in \partial T_{q+1}$, $U_{\mathrm{Aut}_0(T_{q+1})}(\omega)$ is dense in $N_{\mathrm{Aut}_0(T_{q+1})}(\omega)$, implying that $\mathrm{Aut}_0(T_{q+1})$ has the Howe–Moore Property. In particular, by Theorem 5.4.1, every $G \leq \mathrm{Aut}_0(T_{q+1})$ as in the hypothesis of Proposition 5.4.7 is topologically simple. However, not for every closed subgroup $G < \mathrm{Aut}_0(T_{q+1})$ which has the Howe–Moore Property, is $U_G(\omega)$ dense in $N_G(\omega)$; indeed consider the case of $PSL_2(F) < \mathrm{Aut}_0(T_{q+1})$, where F is a non-archimedean field and q is the cardinality of the residue field of F (see Section 6.5).

5.5 Radial eigenfunctions of the Laplacian operator

Definition 5.5.1. For $n \geq 0$, $x \in \mathrm{vert}(T_{q+1})$ and a complex-valued function f on T, let

$$\mu_n f(x) \coloneqq \frac{1}{c(n)} \sum_{d(x,y)=n} f(y).$$

The *Laplacian operator* is $\Delta f \coloneqq \mu_1 f - f$.

Definition 5.5.2. A function f on T_{q+1} is an *eigenfunction* of μ_n, with $n \geq 0$, if there is $\lambda_n(f) \in \mathbb{C}$ such that $\mu_n(f) = \lambda_n(f)f$.

A function on T_{q+1} is an eigenfunction of the Laplacian operator if and only if it is an eigenfunction of μ_1.

Definition 5.5.3. A function f on T_{q+1} is called *radial* if for every $n \geq 1$ f is constant on $S(o, n)$.

If f is a radial function on T_{q+1}, then f can be interpreted as a complex-valued function on the set $\mathbb{N}_0$ of non-negative integers. Write $f(|x|) = f(x)$, where $|x| = d(o,x)$. The goal is now to determine all radial eigenfunctions of the Laplacian operator, that is, of μ_1.

Proposition 5.5.4. *One has that*

$$\mu_{n+1} = \frac{q+1}{q}\mu_n\mu_1 - \frac{1}{q}\mu_{n-1} \quad \forall n \geq 1,$$

and

$$\mu_{n+2} = \frac{q+1}{q}\mu_n\mu_2 - \frac{q-1}{q^2}\mu_n - \frac{1}{q^2}\mu_{n-2} \quad \forall n \geq 2.$$

Proof. Take a function f on T_{q+1}, and fix $x \in \mathrm{vert}(T_{q+1})$. Then for all $n \geq 1$

$$\begin{aligned}
\mu_{n+1}f(x) &= \frac{1}{c(n+1)} \sum_{y\in S(x,n+1)} f(y) \\
&= \frac{1}{c(n+1)}\Big(\sum_{y\in S(x,n)} \sum_{z\in S(y,1)} f(z) - q \sum_{w\in S(x,n-1)} f(w)\Big) \\
&= \frac{1}{c(n+1)}[c(n)c(1)\mu_n(\mu_1 f)(x) - qc(n-1)\mu_{n-1}f(x)] \\
&= \frac{q+1}{q}(\mu_n\mu_1)f(x) - \frac{1}{q}\mu_{n-1}f(x).
\end{aligned}$$

It was used here that $c(n+1) = \frac{1}{q}c(n)$ for all $k \geq 1$. Moreover,

$$\begin{aligned}
\mu_{n+2} &= \frac{q+1}{q}\mu_{n+1}\mu_1 - \frac{1}{q}\mu_n = \Big(\frac{q+1}{q}\Big)^2\mu_n\mu_1^2 - \frac{q+1}{q^2}\mu_{n-1}\mu_1 - \frac{1}{q}\mu_n \\
&= \frac{q+1}{q}\mu_n\Big(\frac{q+1}{q}\mu_1^2 - \frac{1}{q}\Big) - \frac{q+1}{q^2}\mu_{n-1}\mu_1 + \Big(\frac{q+1}{q^2} - \frac{1}{q}\Big)\mu_n \\
&= \frac{q+1}{q}\mu_n\mu_2 - \frac{1}{q}\Big(\mu_n + \frac{1}{q}\mu_{n-2}\Big) + \frac{1}{q^2}\mu_n \\
&= \frac{q+1}{q}\mu_n\mu_2 - \frac{q-1}{q^2}\mu_n - \frac{1}{q^2}\mu_{n-2}.
\end{aligned}$$

□

Corollary 5.5.5. *Suppose f is an eigenfunction of μ_1. Then f is an eigenfunction of μ_n for every $n \geq 0$, with eigenvalues $\{\lambda_n(f)\}_{n\geq 0}$ such that*

$$\lambda_{n+1}(f) = \frac{q+1}{q}\lambda_n(f)\lambda_1(f) - \frac{1}{q}\lambda_{n-1}(f) \quad \forall n \geq 1. \tag{5.5}$$

In particular, a radial function f on T_{q+1} is an eigenfunction of μ_1 if and only if

$$f(n+1) = \frac{q+1}{q}f(n)f(1) - \frac{1}{q}f(n-1) \quad \forall n \geq 1. \tag{5.6}$$

Proof. The first part of the corollary is an immediate consequence of Proposition 5.5.4.

Next let f be a radial function on T_{q+1}. Suppose f is an eigenfunction of μ_1. Then $f(n) = \mu_n(f)(o) = f(n)f(o)$ and thus $\lambda_n(f) = f(n)$ for all $n \geq 1$. Hence, (5.6) follows from (5.5). For the converse suppose that equation (5.5) holds. Then, for all $x \in \mathrm{vert}(T_{q+1})$, with $n = d(o,x)$,

$$\begin{aligned}
\mu_1 f(x) &= \frac{1}{q+1} \sum_{d(x,y)=1} f(y) = \frac{q}{q+1}f(n+1) + \frac{1}{q+1}f(n-1) \\
&= \Big(f(1)f(n) - \frac{1}{q+1}f(n-1)\Big) + \frac{1}{q+1}f(n-1) = f(1)f(n),
\end{aligned}$$

that is, f is a radial eigenfunction of μ_1 with eigenvalue $f(1)$. □

Concerning radial eigenfunctions of μ_2 one has

Corollary 5.5.6. *A radial eigenfunction f of μ_2 is an eigenfunction of μ_1 if and only if*

$$f(2n+1) = \frac{q+1}{q} f(2n) f(1) - \frac{1}{q} f(2n-1) \quad \forall n \geq 1.$$

Proof. This follows directly from Corollary 5.5.5. □

Observe that for every non-zero radial eigenfunction f of μ_1 one needs to have $f(o) \neq 0$. Indeed, if $f(o) = 0$, then $0 = \mu_1(f)(o) = f(1)$ and, by an argument of induction on n and by Corollary 5.5.5, it follows that $f(n) = 0$ for all $n \geq 1$, which yields a contradiction. Notice here that every radial eigenfunction of the operator μ_1 is uniquely determined by its value at o and its value on the sphere $S(o,1)$.

The goal is now to calculate all radial eigenfunctions of μ_1. The key ingredient for this will be the Radon-Nykodim derivative of probability measures on ∂T_{q+1}. For this fix a boundary point $\omega_0 \in \partial T_{q+1}$, and recall that $\mathrm{Aut}(T_{q+1}) = \mathrm{stab}(o)\,\mathrm{stab}(\omega_0)$, and extend the modular function $\Delta_{\mathrm{stab}(\omega_0)}$ to $\mathrm{Aut}(T_{q+1})$ by setting $\Delta_{\mathrm{stab}(\omega_0)}(kp) = \Delta_{\mathrm{stab}(\omega_0)}(p)$ for $k \in \mathrm{stab}(o)$, $p \in \mathrm{stab}(\omega_0)$. Then, $\rho\colon G \to \mathbb{R}_{>0}$, $\rho(g) \coloneqq \Delta_{\mathrm{stab}(\omega_0)}(g)$ is a rho-function of the pair $(\mathrm{Aut}(T_{q+1}), \mathrm{stab}(\omega_0))$ (see Lemma A.4.8, and it is a consequence of Theorem 5.6.1 that $\mathrm{Aut}(T_{q+1})$ is unimodular). Let ν be the associated strongly quasi-invariant Radon measure on $\mathrm{Aut}(T_{q+1})/\mathrm{stab}(\omega) \cong \partial T_{q+1}$. Because $\partial T_{q+1} \cong \mathrm{stab}(o)/\mathrm{stab}_{\mathrm{stab}(o)}(\omega_0)$ is compact and because $\rho|_{\mathrm{stab}(o)} = 1$, ν is a probability measure. For $g \in \mathrm{Aut}(T_{q+1})$ let ν_g be the translate of ν defined by

$$\nu_g(B) \coloneqq \nu(gB) \quad \forall B \subset \mathrm{Aut}(T_{q+1})/\mathrm{stab}(\omega_0) \text{ Borel}.$$

Then ν and ν_g are mutually absolutely continuous, and their Radon-Nikodym derivate is

$$\frac{d\nu_g}{d\nu}(h\,\mathrm{stab}(\omega)) = \frac{\rho(gh)}{\rho(h)} \quad \forall h \in G,$$

(see Theorem A.4.9). Observe that $\nu = \nu_k$ for every $k \in \mathrm{stab}(o)$, and for all $g \in \mathrm{Aut}(T_{q+1})$, ν_g is $\mathrm{stab}(g^{-1} \cdot o)$-invariant, where $g^{-1}\,\mathrm{stab}(o)g = \mathrm{stab}(g^{-1} \cdot o)$. Thus, by Corollary A.4.11, ν_g is the pushforward of the the unique $\mathrm{stab}(g^{-1} \cdot o)$-invariant probability measure on $\mathrm{stab}(g^{-1} \cdot o)/\mathrm{stab}_{\mathrm{stab}(g^{-1} \cdot o)}(\omega_0) \cong \mathrm{Aut}(T_{q+1})/\mathrm{stab}(\omega_0)$.

For $x, t \in \mathrm{vert}(T_{q+1})$ recall the definition of $\Omega(x,t)$ from Section 5.1. By Corollary 5.1.17, those sets form a basis of the subspace topology on ∂T_{q+1}. Recall that for all $n > 0$

$$\partial T_{q+1} = \bigsqcup_{t \in S(x,n)} \Omega(x,t),$$

that is, $\{\Omega(x,t)\colon t \in S(x,n)\}$ is a partition of ∂T_{q+1}, and observe that the partition is preserved by the action of $\mathrm{stab}(x)$. For $g \in \mathrm{Aut}(T_{q+1})$ with $x = g \cdot o$, let ν_x be the pushforward of $\nu_{g^{-1}}$ under the isomorphism $\mathrm{Aut}(T_{q+1})/\mathrm{stab}(\omega_0) \cong \partial T_{q+1}$. Then, by the above remarks, ν_x is the unique $g\,\mathrm{stab}(o)g^{-1}$-invariant Borel probability measure on ∂T_{q+1}, and thus

$$\nu_x(\Omega(x,t)) = \frac{1}{q+1} q^{1-d(x,t)} \quad \forall t \in \mathrm{vert}(T_{q+1}).$$

By construction, the definition of ν_x is independent from the choice of ω_0, and ν_x and ν_y are absolutely continuous to each other for all $x, y \in \mathrm{vert}(T_{q+1})$. To compute their Radon-Nikodym derivative explicitly and independent from the above construction, introduce a metric on the space of horocycles around a boundary point.

Definition 5.5.7. Let $x, y \in \mathrm{vert}(T_{q+1})$ be distinct, and let $\omega \in \partial T_{q+1}$. The *distance between the horocycles $H(x,\omega)$ and $H(y,\omega)$* is

$$d(x,y,\omega) \coloneqq \min\{d(x,z)\colon z \in H(y,\omega)\}.$$

Next define the map $\delta\colon \mathrm{vert}(T_{q+1}) \times \mathrm{vert}(T_{q+1}) \times \partial T_{q+1} \to \mathbb{R}$,

$$\delta(x,y,\omega) \coloneqq \begin{cases} d(x,y,\omega) & \text{if } \omega \in \Omega(x,z) \text{ for some } z \in H(y,\omega), \\ -d(x,y,\omega) & \text{otherwise.} \end{cases}$$

Note that $\delta(x,y,\omega)$ is positive if and only if the horocycle $H(y,\omega)$ is *closer* to ω than $H(x,\omega)$. Moreover, consider

Proposition 5.5.8. *For all $x, y \in \mathrm{vert}(T_{q+1})$ and $\omega \in \partial T_{q+1}$*

$$\delta(x,y,\omega) = d(x,t) - d(y,t)$$

where $t \in [x,\omega) \cap [y,\omega)$.

Proof. Let $x, y \in \mathrm{vert}(T_{q+1})$, and fix $\omega \in \partial T_{q+1}$. Suppose $\omega \in \Omega(x,z)$ for some $z \in H(y,\omega)$, that is, $H(y,\omega)$ is closer to ω than $H(x,\omega)$. The other case is treated similarly. Note that $[x,z] \subset [x,\omega)$, and hence $d(x,z,\omega) = d(x,z)$. Let $t \in [x,\omega) \cap [y,\omega) = [z,\omega) \cap [y,\omega)$. Then

$$\delta(x,y,\omega) = d(x,y,\omega) = d(x,z) = d(x,t) - d(z,t) = d(x,t) - d(y,t).$$

For the last equality use that $z \in H(y,\omega)$ and that therefore $d(y,t) = d(z,t)$ for all $t \in [y,\omega) \cap [z,\omega)$. □

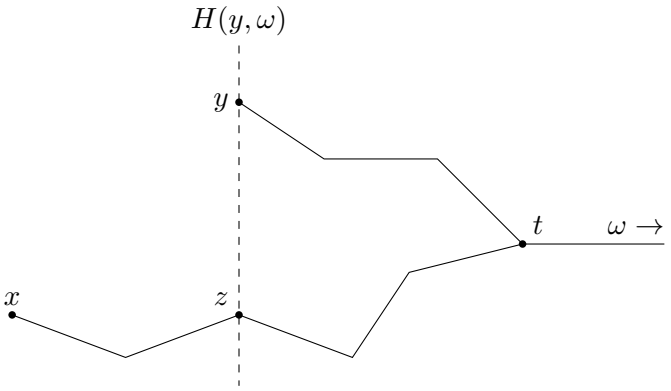

Figure 5.8: $\delta(x,y,\omega) = d(x,t) - d(y,t)$

Corollary 5.5.9. *δ is G-invariant.*

Proof. Let $x, y \in \mathrm{vert}(T_{q+1})$ and $\omega \in \partial T_{q+1}$. Let $t \in [x,\omega) \cap [y,\omega)$. Then for all $g \in G$ $g \cdot t \in [g \cdot x, g \cdot \omega) \cap [g \cdot y, g \cdot \omega)$ and thus

$$\begin{aligned} \delta(g \cdot x, g \cdot y, g \cdot \omega) &= d(g \cdot x, g \cdot t) - d(g \cdot y, g \cdot t) \\ &= d(x,t) - d(y,t) = \delta(x,y,\omega), \end{aligned}$$

which implies that δ is G-invariant. □

Proposition 5.5.10 ([FTN91, p. 35])**.** *For all $x, y \in \mathrm{vert}(T_{q+1})$*

$$\frac{d\nu_x}{d\nu_y}(\omega) = q^{-\delta(x,y,\omega)}.$$

Proof. Fix two distinct vertices $x, y \in \mathrm{vert}(T_{q+1})$, and let $\phi\colon \partial T_{q+1} \to \mathbb{R}$, $\phi(\omega) \coloneqq q^{-\delta(x,y,\omega)}$. To prove that $\phi = \frac{d\nu_x}{d\nu_y}$ one needs to show that for all $f \in C(\partial T_{q+1})$

$$\int_{\partial T_{q+1}} f \, d\nu_x = \int_{\partial T_{q+1}} \phi f \, d\nu_y.$$

Using the description of the topology on ∂T_{q+1}, it is enough to show that for all $t \in \mathrm{vert}(T_{q+1})$

$$\nu_x(\Omega(x,t)) = \int_{\Omega(x,t)} \phi \, d\nu_y.$$

Let $t \in \mathrm{vert}(T_{q+1})$ be such that $\Omega(x,t) = \Omega(y,t)$, that is, $[t,\omega) = [x,\omega) \cap [y,\omega)$ for all $\omega \in \Omega(x,t)$. Then $\delta(x,y,\omega) = d(x,t) - d(y,t)$ for all $\omega \in \Omega(x,t)$, and hence

$$\nu_x(\Omega(x,t)) = \frac{1}{q+1} q^{1-d(x,t)} = q^{d(y,t)-d(x,t)} \nu_y(\Omega(y,t)) = \int_{\Omega(y,t)} \phi \, d\nu_y.$$

Notice now that for all $t \in \mathrm{vert}(T_{q+1})$ there is a finite subset $S \subset \mathrm{vert}(T_{q+1})$ such that for all $s \in S$ $\Omega(x,s) = \Omega(y,s)$ and

$$\Omega(x,t) = \bigsqcup_{s \in S} \Omega(x,s) = \bigsqcup_{s \in S} \Omega(y,s).$$

Deduce that $\phi = \frac{d\nu_x}{d\nu_y}$. □

Define now the map $P\colon \mathrm{vert}(T_{q+1}) \times \partial T_{q+1} \to \mathbb{R}$, $P(x,\omega) \coloneqq \frac{d\nu_x}{d\nu_o}(\omega)$. One has the following properties of P.

Corollary 5.5.11. *1) For every $g \in \mathrm{Aut}(T_{q+1})$ and every $\omega \in \partial T_{q+1}$*

$$P(g \cdot o, g \cdot \omega) = P^{-1}(g^{-1} \cdot o, \omega). \tag{5.7}$$

2) For every $g, h \in \mathrm{Aut}(T_{q+1})$ and every $\omega \in \partial T_{q+1}$

$$P((gh) \cdot o, \omega) = P(h \cdot o, g^{-1} \cdot \omega) P(g \cdot o, \omega).$$

3) For every $k \in \mathrm{stab}(o)$ and every $\omega \in \partial T_{q+1}$

$$P(k \cdot x, \omega) = P(x, k^{-1} \cdot \omega).$$

Proof. 1) Let $t \in [o, g \cdot \omega) \cap [g \cdot o, g \cdot \omega)$. Then $g^{-1} \cdot t \in [g^{-1} \cdot o, \omega) \cap [o, \omega)$, and hence

$$\begin{aligned} \delta(g \cdot o, o, g \cdot \omega) &= d(g \cdot o, t) - d(o, t) = d(o, g^{-1} \cdot t) - d(g^{-1} \cdot o, g^{-1} \cdot t) \\ &= -\delta(g^{-1} \cdot o, o, \omega). \end{aligned}$$

Thus (5.7) follows from Proposition 5.5.10.

2) By the properties of the Radon-Nikodym derivative,

$$\begin{aligned} P((gh) \cdot o, \omega) &= \frac{d\nu_{(gh)\cdot o}}{d\nu_{g\cdot o}}(\omega) \frac{d\nu_{g\cdot o}}{d\nu_o}(\omega) = \frac{d\nu_{h\cdot o}}{d\nu_o}(g^{-1} \cdot \omega) \frac{d\nu_{g\cdot o}}{d\nu_o}(\omega) \\ &= P(h \cdot o, g^{-1} \cdot \omega) P(g \cdot o, \omega). \end{aligned}$$

3) Let $g \cdot \mathrm{Aut}(T_{q+1})$ be such that $x = g \cdot o$. Then, by part 2),

$$P(k \cdot x, \omega) = P((kg) \cdot o, \omega) = P(g \cdot o, k^{-1} \cdot \omega) P(k \cdot o, \omega) = P(x, k^{-1} \cdot \omega).$$

The last equality follows from the fact that $P(k \cdot o, \omega) = P(o, \omega) = 1$.

□

Proposition 5.5.12 ([FTN91, p. 35 f.])**.** *For every $\omega \in \partial T_{q+1}$ and every $z \in \mathbb{C}$, the map $P^z(\cdot, \omega)$ on $\mathrm{vert}(T_{q+1})$ is an eigenfunction of μ_1 with eigenvalue $\kappa(z) := \frac{q^z + q^{1-z}}{q+1}$.*

Proof. Fix $x \in \mathrm{vert}(T_{q+1})$, and write $S(x,1) = \{y_0, y_1, \ldots, y_q\}$. Suppose $[y_0, \omega) \subset [x, \omega)$. Then $[x, \omega) \subset [y_i, \omega)$ for $i = 1, \ldots, q$. It follows that $\delta(y_0, x, \omega) = -1$ and $\delta(y_i, x, \omega) = 1$ for $i = 1, \ldots, q$. Hence, using the properties of Radon-Nykodim derivatives,

$$\begin{aligned}\mu_1(P^z(\cdot,\omega))(x) &= \frac{1}{q+1}\sum_{0\le i\le q}\left(\frac{d\nu_{y_i}}{d\nu_o}\right)^z(\omega) = \frac{1}{q+1}\left(\frac{d\nu_x}{d\nu_o}\right)^z(\omega)\sum_{0\le i\le q}\left(\frac{d\nu_{y_i}}{d\nu_x}\right)^z(\omega)\\ &= P^z(x,\omega)\frac{1}{q+1}(q^z + q^{1-z}) = \kappa(z)P^z(x,\omega).\end{aligned}$$

□

To simplify notation write $\Omega(x) = \Omega(o, x)$ for $x \in \mathrm{vert}(T_{q+1})$.

Definition 5.5.13. Let $\mathcal{K}(\partial T_{q+1})$ be the set of linear combinations of characteristic functions $\{\chi_{\Omega(x)} : x \in \mathrm{vert}(T_{q+1})\}$. Let $\mathcal{K}(\partial T_{q+1})^*$ be the set of finitely additive complex-valued set functions defined on the Borel σ-algebra of ∂T_{q+1}. The elements of $\mathcal{K}(\partial T_{q+1})^*$ are called *finitely additive measures.*

Both $\mathcal{K}(\partial T_{q+1})$ and $\mathcal{K}(\partial T_{q+1})^*$ are complex vector spaces. Moreover, because $\{\Omega(x) : x \in \mathrm{vert}(T_{q+1})\}$ is a basis of the subspace topology on ∂T_{q+1}, they generate the Borel σ-algebra on ∂T_{q+1}. Thus $\mathcal{K}(\partial T_{q+1})^*$ is the dual space of $\mathcal{K}(\partial T_{q+1})$.

Definition 5.5.14. For $z \in \mathbb{C}$ the *Poisson transform* $\mathcal{P}_z$ is the map from $\mathcal{K}(\partial T_{q+1})^*$ to the set of functions on T_{q+1}, given by

$$\mathcal{P}_z m(x) := \int_{\partial T_{q+1}} P^z(x, \omega)\, dm(\omega)$$

for $m \in \mathcal{K}(\partial T_{q+1})$ and $x \in \mathrm{vert}(T_{q+1})$.

Because, by Proposition 5.5.12, for every $\omega \in \partial T_{q+1}$ $x \mapsto P^z(x, \omega)$ is a $\kappa(z)$-eigenfunction of μ_1, $\mathcal{P}_z m$ is also a $\kappa(z)$-eigenfunction of μ_1. For a subset of $z \in \mathbb{C}$, the converse is also true.

Theorem 5.5.15 ([FTN91, Chap. II, Thm- 1.2])**.** *Let $z \in \mathbb{C}$ with $q^{2z} \neq 1$. The $\kappa(z)$-eigenfunctions of μ_1 are exactly $\{\mathcal{P}_z m : m \in \mathcal{K}(\partial T_{q+1})^*\}$.*

The proof of Theorem 5.5.15 is omitted. There is a more straightforward way to determine all radial eigenfunctions of μ_1. For this define the functions $\psi_z := \mathcal{P}_z \nu_o$, for $z \in \mathbb{C}$.

Lemma 5.5.16. *The function $\kappa : \mathbb{C} \to \mathbb{C}$ in Proposition 5.5.12 is surjective.*

Proof. It is clear that $z \mapsto q^z$ is a surjective map from $\mathbb{C}$ onto $\mathbb{C} - \{0\}$. Thus it is enough to show that $\kappa' : \mathbb{C} - \{0\} \to \mathbb{C}$,

$$\kappa'(z) := \frac{z + qz^{-1}}{q+1}$$

is surjective. Indeed, by the fundamental theorem of algebra, for every $w \in \mathbb{C}$ there exists $z \in \mathbb{C}$ such that

$$(q+1)zw = z^2 + q,$$

that is, $w = \kappa'(z)$. □

Proposition 5.5.17. *The radial eigenfunctions of μ_1 are exactly $\{c\psi_z : c, z \in \mathbb{C}\}$.*

Proof. By Proposition 5.5.12, each $c\psi_z$ is an eigenfunction of μ_1. The next goal is to show that $c\psi_z$ is radial, which is equivalent to showing that $c\psi_z$ is left-stab(o)-invariant. Indeed, by the stab(o)-invariance of ν_o and by Corollary 5.5.11, one has that for every $k \in \mathrm{stab}(o)$ and every $x \in \mathrm{vert}(T_{q+1})$

$$\begin{aligned} c\psi_z(k \cdot x) &= c \int_{\partial T_{q+1}} P^z(k \cdot x, \omega)\, d\nu_o(\omega) \\ &= c \int_{\partial T_{q+1}} P^z(x, k^{-1} \cdot \omega)\, d\nu_o(\omega) = c\psi_z(x). \end{aligned}$$

Hence, each $c\psi_z$ is a radial eigenfunction of μ_1. It remains to show that every radial eigenfunction f of μ_1 is of this form. For this recall that, by Corollary 5.5.5, f is uniquely determined by its values at o and on the sphere $S(o,1)$. Set $c = f(o)$, and by Lemma 5.5.16 there exists $z \in \mathbb{C}$ such that $c\kappa(z) = f(1)$. Thus $f = c\psi_z$. □

Using Corollary 5.5.5, one can obtain explicit expressions for the radial eigenfunctions of μ_1.

Proposition 5.5.18 ([FTN91, Chap. II, Prop. 2.4])**.** *Let $z \in \mathbb{C}$, and define $h_z(x) := q^{-z|x|}$, $x \in \mathrm{vert}(T_{q+1})$.*

1) If $q^{2z-1} \neq 1$, then

$$\psi_z(x) = c(z)h_z(x) + c(1-z)h_{1-z}(x), \tag{5.8}$$

where

$$c(z) = \frac{q^{1-z} - q^{z-1}}{(q+1)(q^{-z} - q^{z-1})}.$$

2) If $q^{2z-1} = 1$, then

$$\psi_z(x) = \Big(1 + \frac{q-1}{q+1}|x|\Big)h_z(x). \tag{5.9}$$

Proof. For $z \in \mathbb{C}$, denote by ω_z the function on the right-hand side of the equation (5.8) or (5.9), respectively. To show that $\psi_z = \omega_z$, it is enough to show that $\omega_z(o) = 1$, $\omega_z(1) = \psi_z(1)$ and that ω_z satisfies equation (5.6).

1) If $q^{2z-1} \neq 1$, then $q^{-z} - q^{z-1} \neq 0$ and hence $c(z)$ is well-defined. Moreover,

$$\begin{aligned} \omega_z(o) = c(z) + c(1-z) &= \frac{1}{q+1}\left(\frac{q^{1-z} - q^{z-1}}{q^{-z} - q^{z-1}} + \frac{q^z - q^{-z}}{q^{z-1} - q^{-z}}\right) \\ &= \frac{1}{q+1}\frac{(1+q)(q^{-z} - q^{z-1})}{q^{-z} - q^{z-1}} = 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} \omega_z(1) = c(z)q^{-z} + c(1-z)q^{z-1} &= \frac{1}{q+1}\left(\frac{q^{1-2z} - q^{-1}}{q^{-z} - q^{z-1}} - \frac{q^{2z-1} - q^{-1}}{q^{-z} - q^{z-1}}\right) \\ = \frac{q(q^{-2z} - q^{2z-2})}{(q+1)(q^{-z} - q^{z-1})} &= \frac{q(q^{-z} + q^{z-1})}{q+1} = \frac{q^z + q^{1-z}}{q+1} = \psi_z(1) \end{aligned}$$

Observe that, for all $k, n \geq 0$, $h_z(n+k) = q^{-kz} h_z(n)$. Notice that

$$\begin{aligned}
&\frac{q+1}{q}\omega_z(1)\omega_z(n) - \frac{1}{q}\omega_z(n-1) \\
&\quad = \frac{q+1}{q}\frac{q^z + q^{1-z}}{q+1}[c(z)q^z h_z(n+1) + c(1-z)q^{1-z}h_{1-z}(n+1)] \\
&\qquad - \frac{1}{q}[c(z)q^{2z}h_z(n+1) + c(1-z)q^{2(1-z)}h_{1-z}(n+1)] \\
&\quad = \left(\frac{q^{2z}+q}{q} - \frac{q^{2z}}{q}\right)c(z)h_z(n+1) \\
&\qquad + \left(\frac{q+q^{2(1-z)}}{q} - \frac{q^{2(1-z)}}{q}\right)c(1-z)h_{1-z}(n+1) \\
&\quad = \omega_z(n+1).
\end{aligned}$$

This proves the first part of the proposition.

2) It is immediate that $\omega_z(o) = 1$. Observe that $q^z = q^{1-z}$, and that therefore

$$\omega_z(1) = \left(1 + \frac{q-1}{q+1}\right)h_z(1) = \frac{2q}{q+1}q^{-z} = \frac{q^z + q^{1-z}}{q+1} = \psi_z(1).$$

Recall from above that $h_z(n+k) = q^{-kz}h_z(n)$, for $k, n \geq 0$. It follows that

$$\begin{aligned}
&\frac{q+1}{q}\omega_z(1)\omega_z(n) - \frac{1}{q}\omega_z(n-1) \\
&\quad = \frac{q^z + q^{1-z}}{q}\left(1 + \frac{q-1}{q+1}n\right)q^z h_z(n+1) \\
&\qquad - \frac{1}{q}\left(1 + \frac{q-1}{q+1}(n-1)\right)q^{2z}h_z(n+1) \\
&\quad = (q^{2z-1}+1)\left(1 + \frac{q-1}{q+1}n\right)h_z(n+1) \\
&\qquad - q^{2z-1}\left(1 + \frac{q-1}{q+1}(n-1)\right)h_z(n+1) \\
&\quad = \left[2\left(1 + \frac{q-1}{q+1}(n+1) - \frac{q-1}{q+1}\right)\right. \\
&\qquad \left. - \left(1 + \frac{q-1}{q+1}(n+1) - 2\frac{q-1}{q+1}\right)\right]h_z(n+1) \\
&\quad = \left(1 + \frac{q-1}{q+1}(n+1)\right)h_z(n+1) = \omega_z(n+1).
\end{aligned}$$

This completes the proof.

□

Corollary 5.5.19. *A radial eigenfunction ψ_z vanishes at infinity if and only $0 < \mathrm{Re}(z) < 1$.*

Proof. Note that h_z vanishes at infinity if and only if $0 < \mathrm{Re}(z) < 1$. Next, observe that $c(z) = 0$ if and only if $q^{1-z} = q^{1-z}$, that is, $q^{2z} = q^2$. Hence, $c(z) = 0$ if and only if $z = 1 + \frac{k\pi i}{\log(q)}$ for some $k \in \mathbb{Z}$. The corollary follows from Proposition 5.5.18. □

5.6 Gelfand pairs of tree automorphisms

Recall the definition of Gelfand pairs from Section 3.1. Given a closed subgroup $G < \mathrm{Aut}(T_{q+1})$ acting transitively on ∂T_{q+1}, the following proposition gives an easy way to construct a Gelfand pair (G, K).

Proposition 5.6.1 ([FTN91, Chap. II, Lem. 4.1])**.** *Let $G < \mathrm{Aut}(T_{q+1})$ be a closed subgroup, and suppose that G acts transitively on ∂T_{q+1}. Then for every $x \in \mathrm{vert}(T_{q+1})$, $(G, \mathrm{stab}_G(x))$ is a Gelfand pair.*

Proof. Recall from Proposition 5.3.11 that G acts transitively on ∂T_{q+1} if and only if $\mathrm{stab}_G(x)$ does too. Hence $\mathrm{stab}_G(x)$ acts transitively on $S(x, n)$ for every $n \geq 0$. Observe that for all $g \in G$, $d(x, g \cdot x) = d(g^{-1} \cdot x, x)$, that is, $g \cdot x$ and $g^{-1} \cdot x$ belong to the same sphere around x. Thus, for every $g \in G$ one has that $g^{-1} \in \mathrm{stab}_G(x) g \,\mathrm{stab}_G(x)$. It follows that the inverse map $i_G \colon G \to G, g \mapsto g^{-1}$ is an anti-automorphism such that $i_G(g) \in \mathrm{stab}_G(x) g \,\mathrm{stab}_G(x)$. By Corollary 3.1.4, this implies that $(G, \mathrm{stab}_G(x))$ is a Gefland pair. □

The goal is now to calculate all spherical functions of a Gelfand pair (G, K) of tree automorphisms such that both G and K act transitively on the boundary ∂T_{q+1}. For this fix henceforth a closed non-compact subgroup $G < \mathrm{Aut}(T_{q+1})$, and suppose that it acts transitively on ∂T_{q+1}. For K take $\mathrm{stab}_G(o)$, and let $s(G) \in \{1, 2\}$ be as in Section 5.3.

By Proposition 5.3.19 $(G, \mathrm{stab}_G(o))$ is a Gelfand pair. Observe that if $s(G) = 1$, every right-$\mathrm{stab}_G(o)$-invariant function $f \in C(G)$ corresponds uniquely to a function $\widetilde{f} \colon \mathrm{vert}(T_{q+1}) \to \mathbb{C}$ given by $\widetilde{f}(g \cdot o) \coloneqq f(g)$. Similarly, if $s(G) = 2$, then every right-$\mathrm{stab}_G(o)$-invariant $f \in C(G)$ corresponds uniquely to a function $\widetilde{f} \colon \mathrm{vert}(T_{q+1}) \to \mathbb{C}$ given by $\widetilde{f}(g \cdot o) \coloneqq f(g)$ and $\widetilde{f}(x) \coloneqq 0$ for $x \in \mathrm{vert}^-(T_{q+1})$. If $f \in C(G, \mathrm{stab}_G(o))$, then $\widetilde{f}$ is constant on the $S(o, n)$ for every $n \geq 1$, that is, $\widetilde{f}$ is radial. To simplify notation, for $g \in \mathrm{Aut}(T_{q+1})$ write $|g| \coloneqq d(o, g \cdot o)$.

Proposition 5.6.2 ([FTN91, Chap. II, Lem. 4.3])**.** *Let the notation be as above. A function $\phi \in C(G, \mathrm{stab}_G(o))$ is spherical if and only if $\widetilde{\phi}$ is a radial eigenfunction of $\mu_{s(G)}$ with $\widetilde{\phi}(o) = 1$.*

Proof. Let ϕ be a continuous bi-$\mathrm{stab}_G(o)$-invariant function, and let $\widetilde{\phi}$ be the corresponding radial function on T_{q+1}, defined as in the above remarks. Then, for all $g_1, g_2 \in G$

$$\begin{aligned}\int_{\mathrm{stab}_G(o)} \phi(g_1 k g_2)\, dk &= \int_{\mathrm{stab}_G(o)} \phi((g_1 k g_1^{-1})(g_1 g_2))\, dk = \int_{\mathrm{stab}_G(g_1 \cdot o)} \widetilde{\phi}(k(g_1 g_2 g_1^{-1}) g_1 \cdot o)\, dk \\ &= \frac{1}{c(|g_2|)} \sum_{x \in S(g_1 \cdot o, |g_2|)} \widetilde{\phi}(x) = \mu_{|g_2|} \widetilde{\phi}(|g_1|).\end{aligned}$$

If ϕ is a spherical function of $(G, \mathrm{stab}_G(o))$, then for every $g_1, g_2 \in G$

$$\mu_{|g_2|} \widetilde{\phi}(|g_1|) = \int_{\mathrm{stab}_G(o)} \phi(g_1 k g_2)\, dk = \phi(g_1)\phi(g_2) = \widetilde{\phi}(|g_1|)\widetilde{\phi}(|g_2|),$$

which implies that $\widetilde{\phi}$ is a radial eigenfunction of $\mu_{s(G)}$. Conversely, if $\widetilde{\phi}$ is a radial eigenfunction of $\mu_{s(G)}$, then

$$\phi(g_1)\phi(g_2) = \widetilde{\phi}(|g_1|)\widetilde{\phi}(|g_2|) = \mu_{|g_2|} \widetilde{\phi}(|g_1|) = \int_{\mathrm{stab}_G(o)} \phi(g_1 k g_2)\, dk,$$

and thus ϕ is spherical with respect to $\mathrm{stab}_G(o)$. □

From Proposition 5.6.2 one can deduce the following integral formula for the spherical functions of $(G, \mathrm{stab}_G(o))$, which is similar to the integral formulas for the spherical functions of

$(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$ in Theorem 3.5.7. For this fix $\omega_0 \in \partial T_{q+1}$, and recall from Corollary 5.3.12 that $G = \mathrm{stab}_G(o)\,\mathrm{stab}_G(\omega_0)$. As was done for the proof of Proposition 3.4.3, extend the modular function of $\mathrm{stab}_G(\omega_0)$ to G by setting $\Delta_{\mathrm{stab}_G(\omega_0)}(pk) \coloneqq \Delta_{\mathrm{stab}_G(\omega_0)}(p)$ for $p \in \mathrm{stab}_G(\omega_0)$, $k \in \mathrm{stab}_G(o)$.

Corollary 5.6.3. *The spherical functions of* $(G, \mathrm{stab}_G(o))$ *are exactly the functions* $\{\phi_z : z \in \mathbb{C}\}$ *with*

$$\phi_z(g) \coloneqq \int_{\mathrm{stab}_G(o)} \Delta^{\frac{1}{2}+z}_{\mathrm{stab}_G(\omega_0)}(gk)\,dk.$$

Proof. Because G is unimodular, $\rho \coloneqq \Delta_{\mathrm{stab}_G(\omega_0)}$ is a rho-function of $(G, \mathrm{stab}_G(\omega_0))$. Let ν be the associated strongly quasi-invariant measure on $G/\mathrm{stab}_G(\omega_0)$. Because $\rho|_{\mathrm{stab}_G(o)} = 1$, ν is the unique $\mathrm{stab}_G(o)$-invariant probability measure on $G/\mathrm{stab}_G(\omega_0) \cong \mathrm{stab}_G(o)/\mathrm{stab}_G(o, \omega_0)$. Hence, by construction, $\nu_{g\cdot o}$ is the pushforward of the translate $\nu_{g^{-1}}$ under the isomorphism $G/\mathrm{stab}_G(\omega_0) \cong \partial T_{q+1}$. Therefore, by Theorem A.4.9, for all $g \in G$ and all $k \in \mathrm{stab}_G(o)$

$$\frac{d\nu_{g\cdot o}}{d\nu_o}(k \cdot \omega_0) = \frac{d\nu_{g^{-1}}}{d\nu_{e_g}}(k \cdot \mathrm{stab}_G(\omega_0)) = \frac{\rho(g^{-1}k)}{\rho(k)} = \Delta_{\mathrm{stab}_G(\omega_0)}(g^{-1}k).$$

It follows that for all $z \in \mathbb{C}$

$$\begin{aligned}\psi_z(g \cdot o) &= \int_{\partial T_{q+1}} P^z(g \cdot o, \omega)\,d\nu_o(\omega) = \int_{\mathrm{stab}_G(o)/\mathrm{stab}_G(o,\omega_0)} P^z(g \cdot o, k \cdot \omega_0)\,d\nu_{eG}(k) \\ &= \int_{\mathrm{stab}_G(o)} P^z(g \cdot o, k \cdot \omega_0)\,dk = \int_{\mathrm{stab}_G(o)} \Delta^z_{\mathrm{stab}_G(\omega_0)}(g^{-1}k)\,dk = \phi_{-\frac{1}{2}+z}(g).\end{aligned}$$

From Proposition 5.5.17 and Proposition 5.6.2 deduce the corollary. □

By Proposition 1.2.10 and by the Gelfand-Naimark-Segal construction, it is of interest what are sufficient and necessary conditions for $z, w \in \mathbb{C}$ such that $\phi_z = \phi_w$.

Lemma 5.6.4. *For* $z, w \in \mathbb{C}$, $\kappa(\frac{1}{2}+z) = \kappa(\frac{1}{2}+w)$ *if and only if* $z \in \left\{ \pm w + \frac{2k\pi i}{\log(q)} : k \in \mathbb{Z} \right\}$.

Proof. Note that

$$\kappa\Big(\frac{1}{2}+z\Big) = \frac{q^{\frac{1}{2}}(q^z + q^{-z})}{q+1}.$$

Thus $\kappa\Big(\frac{1}{2}+z\Big) = \kappa\Big(\frac{1}{2}+w\Big)$ if and only if $q^z+q^{-z} = q^w+q^{-w}$. Suppose now that $z+z^{-1} = w+w^{-1}$. Then

$$z^2 - (w + w^{-1})z + 1 = 0,$$

and thus

$$z = \frac{w + w^{-1} \pm \sqrt{(w+w^{-1})^2 - 4}}{2} = \frac{w + w^{-1} \pm (w - w^{-1})}{2}.$$

Hence $z + z^{-1} = w + w^{-1}$ if and only if $z \in \{w, w^{-1}\}$. The lemma follows because $q^z = q^w$ if and only if $z = w + \frac{2k\pi i}{\log(q)}$ for some $k \in \mathbb{Z}$. □

Lemma 5.6.5. *For* $z, w \in \mathbb{C}$, $\kappa\Big(\frac{1}{2}+z\Big) = \pm\kappa\Big(\frac{1}{2}+w\Big)$ *if and only if* $z = \pm w + \frac{i\pi k}{\log(q)}$ *for some* $k \in \mathbb{Z}$.

Proof. Note that $q^z = \pm q^w$ if and only if $z = w + \frac{i\pi k}{\log(q)}$ for some $k \in \mathbb{Z}$. Moreover, as seen in the proof of Lemma 5.7.4, $\kappa\Big(\frac{1}{2}+z\Big) = -\kappa\Big(\frac{1}{2}+w\Big)$ if and only if

$$z = \pm w + \frac{i\pi(2k+1)}{\log(q)}$$

for some $k \in \mathbb{Z}$. □

Proposition 5.6.6. *$\phi_z = \phi_w$ if and only if $z = \pm w + \frac{2\pi k i}{s(G)\log(q)}$ for some $k \in \mathbb{Z}$.*

Proof. Recall that, for $z \in \mathbb{C}$, $\phi_z(g) = \psi_{\frac{1}{2}+z}(|g \cdot o|)$ for all $g \in G$. If $s(G) = 1$, the proposition follows directly from Lemma 5.6.4. Suppose $s(G) = 2$ and that $\phi_z = \phi_w$. Then $\psi_{\frac{1}{2}+z}(2n) = \psi_{\frac{1}{2}+w}(2n)$ for all $n \geq 0$, and by Corollary 5.5.5,

$$\psi_{\frac{1}{2}+z}(1)^2 = \frac{1 + q\psi_{\frac{1}{2}+z}(2)}{q+1} = \frac{1 + q\psi_{\frac{1}{2}+w}(2)}{q+1} = \psi_{\frac{1}{2}+w}(1)^2.$$

Hence, $\psi_{\frac{1}{2}+z}(1) = \kappa(\frac{1}{2}+z) = \pm\kappa(\frac{1}{2}+w) = \pm\psi_{\frac{1}{2}+w}(1)$. Conversely, if $\kappa\Big(\frac{1}{2}+z\Big) = \pm\kappa\Big(\frac{1}{2}+w\Big)$, then $\phi_z = \phi_w$. Thus the proposition follows from Lemma 5.6.5. □

Because $(G, \mathrm{stab}_G(o))$ is a Gelfand pair, as seen in Chapter 4, the Harish-Chandra function of $(G, \mathrm{stab}_G(o))$ is $\Xi_{(G,\mathrm{stab}_G(o))} = \phi_{\frac{1}{2}}$. Write $\Xi_G = \Xi_{(G,\mathrm{stab}_G(o))}$. By using the explicit expressions of the radial eigenfunctions of μ_1 in Proposition 5.5.18, and by applying the integral formula for the Cartan decompositon in Proposition 5.3.18, one can deduce some useful properties of the Harish-Chandra function Ξ_G.

Proposition 5.6.7. *Let $G < \mathrm{Aut}(T_{q+1})$ be a closed subgroup acting transitively on ∂T_{q+1}. Let $\tau \in G$ be a translation of minimal step, and let $G = \mathrm{stab}_G(o)A^+\,\mathrm{stab}_G(o)$ be the Cartan decomposition, with $A^+ = \{\tau^n : n \geq 0\}$.*

1) For all $\epsilon > 0$ there is a constant $c(\epsilon) > 0$ such that, for all $n \geq 0$,

$$q^{-\frac{1}{2}ns(G)} \leq \Xi_G(\tau^n) \leq c(\epsilon)q^{(-\frac{1}{2}+\epsilon)ns(G)}. \tag{5.10}$$

2) For all $\epsilon > 0$ $\Xi_G \in L^{2+\epsilon}(G)$.

Proof. 1) By Corollary 5.6.3, $\Xi_G(\tau^n) = \phi_{\frac{1}{2}}(\tau^n)$. Thus, by Proposition 5.5.18,

$$\Xi_G(\tau^n) = \left(1 + \frac{q-1}{q+1}ns(G)\right)q^{-\frac{1}{2}ns(G)}.$$

Notice that for all $\epsilon > 0$ there is a constant $d(\epsilon) > 0$ such that

$$1 \leq 1 + \frac{q-1}{q+1}n \leq c(\epsilon)q^{\epsilon n} \quad \forall n \geq 0,$$

and thus

$$q^{-\frac{1}{2}ns(G)} \leq \Xi_G(\tau^n) \leq c(\epsilon)q^{-(\frac{1}{2}-\epsilon)ns(G)} \quad \forall n \geq 0.$$

2) Using the integral formula for the Cartan decomposition and (5.10),

$$\begin{aligned}
\int_G \Xi_G^{2+\epsilon}(x)\,dx &= \int_{A^+} \Xi_G^{2+\epsilon}(a)(q+1)q^{d(a\cdot o,o)-1}\,da \\
&= \sum_{n\geq 0} \Xi_G^{2+\epsilon}(\tau^n)(q+1)q^{ns(G)-1} \\
&\leq c\left(\frac{\epsilon}{4(2+\epsilon)}\right)^{2+\epsilon}\frac{(q+1)}{q}\sum_{n\geq 0} q^{-(2+\epsilon)\left(\frac{1}{2}-\frac{\epsilon}{4(2+\epsilon)}\right)ns(G)}q^{ns(G)} \\
&= c\left(\frac{\epsilon}{4(2+\epsilon)}\right)^{2+\epsilon}\frac{(q+1)}{q}\sum_{n\geq 0} q^{-\left(\frac{1}{2}\epsilon-\frac{\epsilon}{4(2+\epsilon)}(2+\epsilon)\right)ns(G)} \\
&= c\left(\frac{\epsilon}{4(2+\epsilon)}\right)^{2+\epsilon}\frac{(q+1)}{q}\sum_{n\geq 0} q^{-\frac{1}{4}\epsilon ns(G)} < \infty.
\end{aligned}$$

□

Corollary 5.6.8. *A unitary representation $(\pi, \mathcal{H}_\pi)$ of G is weakly contained in the regular representation if and only if for a dense subset $U \subset \mathcal{H}_\pi$ one has that $\phi_{u,u} \in L^{2+\epsilon}(G)$.*

Proof. This follows directly from Propositon 5.6.7 and from Theorem 4.3.1 and Corollary 4.4.7. □

5.7 Interwining Operators for the Poisson transform

It remains to determine which spherical functions of $(G, \mathrm{stab}_G(o))$ are positive definite. For this, one introduces the interwining operators for the Poisson transform. To simplify notation, for $n \geq 1$, let $\mathcal{K}_n(\partial T_{q+1})$ be the linear subspace of $\mathcal{K}(\partial T_{q+1})$ generated by the functions $\{\chi_{\Omega(x)} \colon x \in B(o,n)\}$. Notice that

$$\mathcal{K}(\partial T_{q+1}) = \bigcup_{n \geq 1} \mathcal{K}_n(\partial T_{q+1}).$$

From now on suppose $z \in \mathbb{C}$ is such that $q^{kz} \neq 1$ and $q^{k(1-z)} \neq 1$ for all $k \in \mathbb{Z}$, that is, such that $k \mapsto q^{kz}$ and $k \mapsto q^{k(1-z)}$ are injective functions on $\mathbb{Z}$.

Proposition 5.7.1. *Let $n \geq 1$. Then for all $\xi \in \mathcal{K}_n(\partial T_{q+1})$ there is a unique function f on T_{q+1} with its support contained in $S(o,n)$ such that*

$$\xi(\omega) = \sum_{x \in S(o,n)} f(x) P^z(x, \omega) \quad \forall \omega \in \partial T_{q+1}.$$

Proof. Note that ξ and $P^z(x, \cdot)$ are constant on $\Omega(y)$ for all $x, y \in S(o,n)$. Write $S(o,n) = \{x_1, x_2, \ldots, x_{c(n)}\}$, and fix $\omega_i \in \Omega(x_i)$ for every $1 \leq i \leq c(n)$. Then the problem of finding f is reduced to finding a solution $v = (f(x_1), f(x_2), \ldots, f(x_{c(n)})) \in \mathbb{C}^{c(n)}$ to the system of linear equations

$$(\xi(\omega_1), \xi(\omega_2), \ldots, \xi(\omega_{c(n)})) = \Big(P^z(x_i, \omega_j)\Big)_{1 \leq i,j \leq c(n)} v^T. \tag{5.11}$$

Because, by assumption, $k \mapsto q^{kz}$ is an injective function on $\mathbb{Z}$, the columns of the matrix $\Big(P^z(x_i, \omega_j)\Big)_{1 \leq i,j \leq c(n)}$ are linearly independent, and thus the matrix is invertible. Hence there is a unique solution to (5.11). □

Definition 5.7.2. Let the map $I_z \colon \mathcal{K}(\partial T_{q+1}) \to \mathcal{K}(\partial T_{q+1})$ be given by

$$I_z \xi(\omega) \coloneqq \sum_{x \in S(o,n)} f(x) P^{1-z}(x, \omega),$$

where $\xi(\omega) = \sum_{x \in S(o,n)} f(x) P^z(x, \omega)$.

The map I_z is well-defined by Proposition 5.7.1. Observe that it maps $K_n(\partial T_{q+1})$ to itself.

Proposition 5.7.3. *1) For all $\xi \in \mathcal{K}_n(\partial T_{q+1})$ $\mathcal{P}_z(I_z \xi \, d\nu_o) = \mathcal{P}_{1-z}(\xi \, d\nu_o)$.*

2) For all $n \geq 1$, $I_z(\mathcal{K}_n(\partial T_{q+1})) = \mathcal{K}_n(\partial T_{q+1})$.

3) $I_z \colon \mathcal{K}(\partial T_{q+1}) \to \mathcal{K}(\partial T_{q+1})$ is an isomorphism where the inverse is $I_z^{-1} = I_{1-z}$.

Proof. 1) Take $\xi \in \mathcal{K}_n(\partial T_{q+1})$, with $\xi = \sum_{x\in B(o,n)} f(x)P^z(x,\cdot)$. Then

$$\begin{aligned}\mathcal{P}_z(I_z\xi\, d\nu_0)(x) &= \int_{\partial T_{q+1}} P^z(x,\omega)\Big(\sum_{y\in S(o,n)} f(y)P^{1-z}(y,\omega)\Big)\, d\nu_o(\omega)\\ &= \sum_{y\in S(o,n)} f(y)\int_{\partial T_{q+1}} P^z(x,\omega)P^{1-z}(y,\omega)\, d\nu_o(\omega)\\ &= \sum_{y\in S(o,n)} f(y)\int_{\partial T_{q+1}} P^{1-z}(x,\omega)P^{z}(y,\omega)\, d\nu_o(\omega)\\ &= \int_{\partial T_{q+1}} P^{1-z}(x,\omega)\xi(\omega)\, d\nu_o(\omega) = \mathcal{P}_{1-z}(\xi\, d\nu_0)(x)\end{aligned}$$

It was used here that the two measure $\left(\frac{d\nu_x}{d\nu_y}\right)^z d\nu_y = P^z(x,\cdot)P^{1-z}(y,\cdot)\, d\nu_o$ and $\left(\frac{d\nu_y}{d\nu_x}\right)^z d\nu_x = P^{1-z}(x,\cdot)P^z(y,\cdot)\, d\nu_o$ have the same distribution.

2) and 3) follows directly from the definition of I_z. □

Next define the functions $\{\xi_x : x \in \mathrm{vert}(T_{q+1})\}$ with $\xi_o := 1$, and for $x \neq o$

$$\xi_x := \frac{q}{q+1}\Big(q^{|x|}\chi_{\Omega(x)} - q^{|x'|}\chi_{\Omega(x')}\Big),$$

where $x' \in \mathrm{vert}(T_{q+1})$ is the unique vertex such that $x' \in [o,x]$ with $d(x,x') = 1$. For the next proposition the following simple computations will be needed.

Lemma 5.7.4. *Let $z \in \mathbb{C}$.*

1) For every $y \in B(o,|x'|)$ $\mathcal{P}_z(\xi_x\, d\nu_o)(y) = 0$.

2) One has that

$$\mathcal{P}_z(\xi_x\, d\nu_o)(x) = \Big(\frac{q}{q+1}\Big)q^{(z-1)|x'|}\frac{q-1}{q}(q^z - q^{-z}).$$

Proof. 1) Fix $y \in B(o,|x'|)$. By Proposition 5.5.10 one has that $P(y,\cdot)$ is constant on $\Omega(z)$ for every $z \in S(o,|y|)$. Moreover, $\Omega(x') \subset \Omega(y)$ or $\Omega(x') \cap \Omega(y) = \emptyset$. Hence $P(y,\cdot)$ is constant on $\Omega(x')$. Therefore for $\omega \in \Omega(x)$

$$\mathcal{P}_z(\xi_x\, d\nu_o)(y) = P^z(y,\omega)\Big(q^{|x|}\nu_o(\Omega(x)) - q^{|x'|}\nu_o(\Omega(x'))\Big) = 0.$$

2) By Proposition 5.5.10 one has that

$$P(x,\omega) = \begin{cases} q^{|x|} & \text{if } \omega \in \Omega(x),\\ q^{|x'|-1} & \text{if } \omega \in \Omega(x') - \Omega(x).\end{cases}$$

Thus

$$\mathcal{P}_z(\chi_{\Omega(x)}\, d\nu_o) = q^{z|x|}\nu_o(\Omega(x)) = \frac{q}{q+1}q^{(z-1)|x|},$$

and

$$\begin{aligned}\mathcal{P}_z(\chi_{\Omega(x')}\, d\nu_o)(x) &= \mathcal{P}_z(\chi_{\Omega(x)}\, d\nu_o)(x) + \mathcal{P}_z(\chi_{\Omega(x')-\Omega(x)}\, d\nu_o)(x)\\ &= \nu_o(\Omega(x))\Big(q^{z|x|} + (q-1)q^{z(|x'|-1)}\Big)\\ &= \frac{q}{q+1}q^{(z-1)|x'|}(q^{z-1} + (q-1)q^{-1-z}).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{P}_z(\xi_x\,d\nu_o) &= \frac{q}{q+1}\Big(q^{|x|}\mathcal{P}_z(\chi_{\Omega(x)}\,d\nu_o) - q^{|x'|}\mathcal{P}_z(\chi_{\Omega(x')}\,d\nu_o)\Big)\\ &= \Big(\frac{q}{q+1}\Big)^2\Big(q^{z|x|} - q^{z|x'|}(q^{z-1}+(q-1)q^{-1-z})\Big)\\ &= \Big(\frac{q}{q+1}\Big)^2 q^{z|x'|}(q^z - q^{z-1} - (q-1)q^{-1-z})\\ &= \Big(\frac{q}{q+1}\Big)^2 q^{z|x'|}\frac{q-1}{q}(q^z - q^{-z}).\end{aligned}$$

□

Proposition 5.7.5. *The functions $\{\xi_x : x \in B(o,n)\}$ span $\mathcal{K}_n(\partial T_{q+1})$. Moreover, each ξ_x is an eigenfunction of I_z such that*

$$I_z\xi_x = \frac{q^z - q^{-z}}{q^{1-z} - q^{z-1}}q^{(2z-1)|x'|}\xi_x. \tag{5.12}$$

Proof. By an easy of argument of induction on n one can show that $\{\xi_x : x \in B(o,n)\}$ spans $\mathcal{K}_n(\partial T_{q+1})$.

For the second part note that for all $x \in \mathrm{vert}(T_{q+1})$ and all $z \in \mathbb{C}$ one has $P^z(x,\cdot) \in K_{|x|}(\partial T_{q+1})$. In particular, there exist $\eta_1 \in \mathcal{K}_{|x'|}(\partial T_{q+1})$ and $c_1 \in \mathbb{C}$ such that $P^z(x,\cdot) = \eta_1 + c_1\xi_x$. Thus $\xi_x = \eta_2 + c_2P^z(x,\cdot)$ for some $\eta_2 \in \mathcal{K}_{|x'|}(\partial T_{q+1})$ and $c_2 \in \mathbb{C}$. By the definition of I_z, there exist $\eta_3 \in \mathcal{K}_{|x'|}(\partial T_{q+1})$ and $c_3 \in \mathbb{C}$ such that $I_z\xi_x = \eta_3 + c_3\xi_x$. By Lemma 5.7.4 one has that for all $y \in B(o,|x'|)$

$$\mathcal{P}_z(\eta_3\,d\nu_o)(y) = c_3\mathcal{P}_z(\xi_x\,d\nu_o)(y) - \mathcal{P}_z(I_z\xi_x\,d\nu_o)(y) = -\mathcal{P}_{1-z}(\xi_x\,d\nu_o)(y) = 0.$$

Because $\eta_3 \in \mathcal{K}_{|x'|}(\partial T_{q+1})$, this implies $\eta_3 = 0$. Hence there exists $c \in \mathbb{C}$ such that $I_z\xi_x = c\xi_x$. The proof is completed by calculating c. Observe that

$$\mathcal{P}_{1-z}(\xi_x) = \mathcal{P}_z(I_z\xi_x) = c\mathcal{P}_z(\xi_x\,d\nu_o).$$

Hence, by Lemma 5.7.4,

$$c = \frac{\mathcal{P}_{1-z}(\xi_x\,d\nu_o)}{\mathcal{P}_z(\xi_x\,d\nu} = \frac{q^{1-z} - q^{z-1}}{q^z - q^{-z}}q^{-(2z-1)|x'|}.$$

□

On $\mathcal{K}(\partial T_{q+1})$ define the inner product

$$\langle f_1, f_2\rangle \coloneqq \int_{\partial T_{q+1}} f_1\overline{f_2}\,d\nu,$$

and let $\mathcal{H}(\partial T_{q+1})$ be the completion of $\mathcal{K}(\partial T_{q+1})$ with respect to the inner product. Then I_z is a bounded linear operator on $\mathcal{H}(\partial T_{q+1})$. Using part 1) of Proposition 5.7.3, one can show that $I_z^* = I_{\overline{z}}$. The following is the main result of this section.

Proposition 5.7.6. *Suppose G acts transitively on ∂T_{q+1}. The spherical functions $\{\phi_s : -\frac{1}{2} \le s \le \frac{1}{2}\}$ are positive definite.*

Proof. By using the properties of P established in Corollary 5.5.11, it follows that for all $z \in \mathbb{C}$ and all $g, h \in G$,

$$\begin{aligned}\phi_{-\frac{1}{2}+z}(g^{-1}h) &= \int_{\partial T_{q+1}} P^z((g^{-1}h)\cdot o, \omega)\, d\nu_o(\omega) = \int_{\partial T_{q+1}} P^z(g^{-1}\cdot o, \omega) P^z(h\cdot o, g\cdot\omega)\, d\nu_o(\omega)\\ &= \int_{\partial T_{q+1}} P^z(g^{-1}\cdot o, g^{-1}\cdot\omega) P^z(h\cdot o, \omega)\, d\nu_{g\cdot o}(\omega)\\ &= \int_{\partial T_{q+1}} P^z(g^{-1}\cdot o, g^{-1}\cdot\omega) P^z(h\cdot o, \omega) P(g\cdot o, \omega)\, d\nu_o(\omega)\\ &= \int_{\partial T_{q+1}} P^{1-z}(g\cdot o, \omega) P^z(h\cdot o, \omega)\, d\nu_o(\omega).\end{aligned}$$

Hence, $\phi_{\frac{1}{2}+z}$ is positive definite if and only if for every $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and every $g_1, g_2, \ldots, g_n \in G$

$$\begin{aligned}\sum_{1\le i,j\le n} c_i\overline{c_j}\phi(g_j^{-1}g_i) &= \sum_{1\le i,j\le n}\int_{\partial T_{q+1}} c_i\overline{c_j} P^{1-z}(g_j\cdot o, \omega) P^z(g_i\cdot o, \omega)\, d\nu_o(\omega)\\ &= \sum_{1\le i,j\le n}\int_{\partial T_{q+1}} c_i P^z(g_i\cdot o, \omega) I_z(\overline{c_j} P^z(g_j\cdot o, \omega))\, d\nu_o(\omega) \ge 0.\end{aligned}$$

Hence, to show that $\phi_{-\frac{1}{2}+z}$ is positive definite, it is enough to prove that

$$\langle I_z\xi, \xi\rangle = \int_{\partial T_{q+1}} I_z(\xi)\overline{\xi}\, d\nu_o \ge 0 \quad \forall \xi \in \mathcal{K}(\partial T_{q+1}).$$

By Proposition 5.7.5, for all irrational $0 < s < 1$ the functions $\{\xi_x : x \in \mathrm{vert}(T_{q+1})\}$ form a generating subset of eigenfunctions of I_s of $\mathcal{K}(\partial T_{q+1})$ with positive eigenvalues, where

$$I_s\xi_x = \frac{q^s - q^{-s}}{q^{1-s} - q^{s-1}} q^{(2s-1)|x'|}\xi_x.$$

Thus, for every $0 < s < 1$ in $\mathbb{R} - \mathbb{Q}$, I_s is a self-adjoint linear operator on $\mathcal{H}(\partial T_{q+1})$ with its spectrum contained in $[0, \infty)$. By [Rud91, Thm. 12.32], I_s is a positive operator on $\mathcal{H}(\partial T_{q+1})$ and therefore $\langle I_s\xi, \xi\rangle \ge 0$ for all $\xi \in \mathcal{K}(\partial T_{q+1})$.

It follows that ϕ_s is positive definite for every $-\frac{1}{2} < s < \frac{1}{2}$ with $s \in \mathbb{R} - \mathbb{Q}$. Because, by Corollary 5.6.3, for every $g \in G$ the map $z \mapsto \phi_z(g)$ is continuous, ϕ_s is positive definite for every $-\frac{1}{2} \le s \le \frac{1}{2}$. □

5.8 Classification of spherical representations

In this final section all spherical representations of a Gelfand pair $(G, \mathrm{stab}_G(o))$ are classified. For this one needs

Lemma 5.8.1. *$\kappa(\frac{1}{2} + z) \in \mathbb{R}$ if and only if*

1) $z \in i\mathbb{R}$ or

2) $z \in \left\{s + \frac{k\pi i}{\log(q)} : s \in \mathbb{R},\ k \in \mathbb{Z}\right\}$.

Moreover, $\kappa(\frac{1}{2} + z) \in i\mathbb{R}$ if and only if

$$z \in \left\{s + \frac{(2k+1)\pi i}{2\log(q)} : s \in \mathbb{R},\ k \in \mathbb{Z}\right\}.$$

Furthermore, $\kappa(\frac{1}{2} + z) = -\kappa(\frac{1}{2} + z + \frac{\pi i}{\log(q)})$.

Proof. Note that for all $z \in \mathbb{C}$

$$\kappa\Big(\frac{1}{2}+z\Big) = \frac{q^{\frac{1}{2}}(q^z+q^{-z})}{q+1}.$$

Thus $\kappa(\frac{1}{2}+z) \in \mathbb{R}$ if and only if $q^z+q^{-z} \in \mathbb{R}$. Note that $q^z+q^{-z} \in \mathbb{R}$ if and only if $q^z \in \mathbb{R}$ or $\overline{q^z} = q^{-z}$. It is immediate that $\overline{q^z} = q^{\overline{z}} = q^{-z}$ if and only if $\mathrm{Re}(z) = 0$. Moreover, $q^z \in \mathbb{R}$ if and only if $z = s + \frac{k\pi i}{\log(q)}$ for some $s \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Furthermore, $\kappa(\frac{1}{2}+z) \in i\mathbb{R}$ if and only if $\mathrm{Re}(q^z+q^{-z}) = 0$, which is equivalent to

$$0 = (q^{\mathrm{Re}(z)} + q^{-\mathrm{Re}(z)})\cos(\log(q)\,\mathrm{Im}(z)).$$

Hence, $\kappa(\frac{1}{2}+z) \in i\mathbb{R}$ if and only if $z = s + \frac{(2k+1)\pi i}{2\log(q)}$ for some $s \in \mathbb{R}$ and $k \in \mathbb{Z}$. □

Next consider

Example 5.8.2. Let $\sigma\colon G \to S^1$ be the map given by

$$\sigma(g) \coloneqq (-1)^{|g\cdot o|}.$$

Because, by Proposition 5.3.13, $G \cap \mathrm{Aut}_0(T_{q+1}) \triangleleft G$ has index 2, σ is a unitary character of G. Hence $(\sigma, \mathbb{C})$ is a unitary representation of G. Moreover, because $\sigma(k) = 1$ for all $k \in \mathrm{stab}_G(o)$, it is a spherical representation of $(G, \mathrm{stab}_G(o))$. By Proposition 5.6.6 and by Lemma 5.8.1,

$$\sigma \simeq \pi_{\phi_{\frac{1}{2}+\frac{i\pi}{\log(q)}}} \simeq \pi_{\phi_{-\frac{1}{2}+\frac{i\pi}{\log(q)}}}.$$

In particular, if $s(G) = 1$, then $\sigma \not\simeq 1_G$, and if $s(G) = 2$, $\sigma \simeq 1_G$.

Proposition 5.8.3. *Suppose that $s(G) = 1$. Then a spherical function ϕ_z of $(G, \mathrm{stab}_G(o))$ is positive definite if and only if*

1) $z \in i\mathbb{R}$ or

2) $z \in \Big\{s + \frac{i\pi k}{\log(q)} \colon -\frac{1}{2} \le s \le \frac{1}{2},\ k \in \mathbb{Z}\Big\}$.

Proof. Suppose that the spherical function ϕ_z is positive definite. Then, because, as seen in the proof of Proposition 5.6.1, $g^{-1} \in \mathrm{stab}_G(o)g\,\mathrm{stab}_G(o)$ for all $g \in G$, $\phi_z(g) = \phi_z(g^{-1}) = \overline{\phi_z(g)}$, and hence ϕ_z is a real-valued function such that $|\phi_z(g)| \le 1$ for all $g \in G$. Since $\psi_{\frac{1}{2}+z}(g\cdot o) = \phi_z(g)$, $\psi_{\frac{1}{2}+z}$ is real-valued with

$$-1 \le \psi_{\frac{1}{2}+z}(1) = \frac{q^{\frac{1}{2}}(q^z+q^{-z})}{q+1} \le 1.$$

By Lemma 5.8.1, this implies that $z \in i\mathbb{R}$ or $z = s + \frac{i\pi k}{\log(q)}$ for some $-\frac{1}{2} \le s \le \frac{1}{2}$ and some $k \in \mathbb{Z}$. Note now the following:

- If $z \in i\mathbb{R}$, then ϕ_z is positive definite by Corollary 3.4.6.
- If $z = s + \frac{i\pi k}{\log(q)}$ for some $\frac{1}{2} \le s \le \frac{1}{2}$ and some $k \in \mathbb{Z}$, then ϕ_z is positive definite by Proposition 5.6.7 and by Proposition 5.7.6.

This completes the proof. □

For the case of $s(G) = 2$, one has

Lemma 5.8.4. *If $s(G) = 2$, then the spherical function ϕ_z of $(\mathrm{Aut}_0(T_{q+1}), \mathrm{stab}_0(o))$ is positive definite if and only if $\phi_z|_G$ is positive definite.*

Proof. Observe that, because $G/\mathrm{stab}_G(o) \cong \mathrm{vert}^+(T_{q+1}) \cong \mathrm{Aut}_0(T_{q+1})/\mathrm{stab}_0(o)$, $\mathrm{Aut}_0(T_{q+1}) = \mathrm{stab}_G(o)G$. Since both $\mathrm{Aut}_0(T_{q+1})$ and G are unimodular, the lemma follows from Corollary 3.4.4. □

Proposition 5.8.5. *Suppose that $s(G) = 2$. Then there exists a (possibly empty) closed subset $S \subset \left(-\frac{1}{2}, \frac{1}{2}\right)$ such that a spherical function ϕ_z of $(G, \mathrm{stab}_G(o))$ is positive definite if and only if*

1) $z \in i\mathbb{R}$ or

2) $z \in \left\{s + \frac{k\pi i}{\log(q)} : -\frac{1}{2} \leq s \leq \frac{1}{2},\ k \in \mathbb{Z}\right\}$ or

3) $z \in \left\{\pi_{\phi_{s+\frac{(2k+1)\pi i}{2\log(q)}}} : s \in S,\ s \neq 0,\ k \in \mathbb{Z}\right\}$.

Moreover, the subset S is independent from G.

Proof. Suppose ϕ_z is positive definite. As seen in the proof of Proposition 5.8.3, $\phi_z(g) \in \mathbb{R}$ and $|\phi_z(g)| \leq 1$ for all $g \in G$. Then $\psi_{\frac{1}{2}+z}(g \cdot o) = \phi_z(g)$ for all $g \in G$, and

$$\psi_{\frac{1}{2}+z}(1) = \sqrt{\frac{1 + q\psi_{\frac{1}{2}+z}(2)}{q+1}}.$$

Because $|\psi_{\frac{1}{2}+z}(2)| \leq 1$,

$$|\psi_{\frac{1}{2}+z}(1)| \leq \sqrt{\frac{1 + q|\psi_{\frac{1}{2}+z}(2)|}{q+1}} \leq 1.$$

Let $\widetilde{\phi}_z(g)$ be the spherical function of $(\mathrm{Aut}(T_{q+1}), \mathrm{stab}(o))$ given by $z \in \mathbb{C}$, as in Corollary 5.6.3, and note that $\widetilde{\phi}_z(g) \coloneqq \psi_{\frac{1}{2}+z}(g \cdot o)$. Suppose that $-\frac{1}{q} \leq \psi_{\frac{1}{2}+z}(2) \leq 1$. Then, $\widetilde{\phi}_z$ is real-valued with $|\widetilde{\phi}_z(g)| \leq 1$. Hence, as seen in the proof of Proposition 5.8.3, $\widetilde{\phi}_z$ is positive definite, and therefore ϕ_z is positive definite. By Proposition 5.8.3, $z \in i\mathbb{R}$ or $z \in \left\{s + \frac{k\pi i}{\log(q)} : -\frac{1}{2} \leq s \leq \frac{1}{2},\ k \in \mathbb{Z}\right\}$.

Next suppose that $-1 \leq \psi_{\frac{1}{2}+z}(2) < -\frac{1}{q}$. Then $\widetilde{\phi}_z(1) \in i\mathbb{R}$ and thus, by Proposition 5.8.3, $z \in \left\{s + \frac{(2k+1)\pi i}{2\log(q)} : s \in \mathbb{R},\ k \in \mathbb{Z}\right\}$. Since $\mathrm{Aut}_0(T_{q+1})$ has the Howe–Moore Property, ϕ_z vanishes at infinity, and by Corollary 5.5.19, $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$. Hence $z = s + \frac{(2k+1)\pi i}{2\log(q)}$ for some $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ with $s \neq 0$ and $k \in \mathbb{Z}$.

Let S be the subset of all $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ such that $\phi_{s+\frac{(2k+1)\pi i}{2log(q)}}$ is positive definite for all $k \in \mathbb{Z}$. Observe that, because for every $g \in G$ the map $z \mapsto \phi_z(g)$ is a continuous function on $\mathbb{C}$, the set S is closed, and it is independent from G by Lemma 5.8.4. □

One is now equipped to classify spherical representations of $(G, \mathrm{stab}_G(o))$ for any closed non-compact subgroup $G < \mathrm{Aut}(T_{q+1})$, up to unitary equivalence. If $s(G) = 1$, the classification is complete.

Theorem 5.8.6. *Suppose $s(G) = 1$. The spherical representations of $(G, \mathrm{stab}_G(o))$ are exactly the following:*

1) the trivial representation $(1_G, \mathbb{C})$;

2) $(\sigma, \mathbb{C})$;

3) the spherical principal series $\{\pi_{\phi_{it}} : t \in \mathbb{R}\}$, where $\pi_{\phi_{it_1}} \simeq \pi_{\phi_{it_2}}$ if and only if $t_1 = \pm t_2 + \frac{2k\pi i}{\log(q)}$ for some $k \in \mathbb{Z}$;

4) the complementary series $\left\{\pi_{\phi_{s+\frac{k\pi i}{\log(q)}}} : -\frac{1}{2} < s < \frac{1}{2},\ k \in \mathbb{Z}\right\}$, *where* $\pi_{\phi_{s+\frac{k_1\pi i}{\log(q)}}} \simeq \pi_{\phi_{s+\frac{k_2\pi i}{\log(q)}}}$ *if and only if* $s_1 = \pm s_2$ *and* $k_1 - k_2$ *is even.*

Proof. By Corollary 3.3.6, the classification follows from Proposition 5.8.3. The unitary equivalence follows from Corollary 3.3.8 and from Proposition 5.6.7. □

Theorem 5.8.7. *Suppose* $s(G) = 2$. *Then there exists a (possibly empty) closed subset* $S \subset \left(-\frac{1}{2}, \frac{1}{2}\right)$ *such that the spherical representations of* $(G, \mathrm{stab}_G(o))$ *are exactly the following:*

1) the trivial representation $(1_G, \mathbb{C})$*;*

2) the spherical principal series $\{\pi_{\phi_{it}} : t \in \mathbb{R}\}$, *where* $\pi_{\phi_{it_1}} \simeq \pi_{\phi_{it_2}}$ *if and only if* $t_1 = \pm t_2 + \frac{k\pi i}{\log(q)}$ *for some* $k \in \mathbb{Z}$*;*

3) the complementary series $\left\{\pi_{\phi_{s+\frac{i\pi k}{\log(q)}}} : -\frac{1}{2} < s < \frac{1}{2},\ s \neq 0,\ k \in \mathbb{Z}\right\}$, *where* $\pi_{\phi_{s_1+\frac{k_1\pi i}{\log(q)}}} \simeq \pi_{\phi_{s_2+\frac{k_2\pi i}{\log(q)}}}$ *if and only if* $s_1 = \pm s_2 + \frac{l\pi i}{\log(q)}$ *for some* $l \in \mathbb{Z}$*;*

4) the series $\left\{\pi_{\phi_{s+\frac{(2k+1)\pi i}{2\log(q)}}} : s \in S,\ s \neq 0,\ k \in \mathbb{Z}\right\}$, *where* $\pi_{\phi_{s_1+\frac{(2k_1+1)\pi i}{2\log(q)}}} \simeq \pi_{\phi_{s_2+\frac{(2k_2+1)\pi i}{2\log(q)}}}$ *if and only if* $s_1 = \pm s_2 + \frac{l\pi i}{\log(q)}$ *for some* $l \in \mathbb{Z}$.

Moreover, the subset S *is independent from* G.

Proof. As for Theorem 5.8.6 the classification follows from Proposition 5.8.3. The unitary equivalence follows from Corollary 3.3.8 and from Proposition 5.6.7. □

Corollary 5.8.8. *If* $G < Aut(T_{q+1})$ *is a closed non-compact subgroup which acts transitively on the boundary* ∂T_{q+1}*, then* G *does not have Property (T).*

Proof. By the continuity of the map $\mathbb{C} \times G \to \mathbb{C}$, $(z, g) \mapsto \phi_z(g)$, the corollary follows from the existence of the complementary series. □

Chapter 6

The Bruhat–Tits Building for $SL_d(F)$ over a non-archimedean local field

Throughout this chapter let F be a non-archimedean local field, and let $\mathcal{O}$ denote the ring of integers of F. Let q be the cardinality of the residue field of F, and let $\mathbb{F}_q$ be the finite field of cardinality q. Fix a uniformizing element $\mathfrak{p} \in \mathcal{O}$, and let $|\cdot|$ be the unique valuation on F such that $q = |\mathfrak{p}|$. Let $F^\times$ denote the multiplicative group of F. The goal is to associate to $SL_d(F)$ a graph whose geometry can help to obtain information about the spherical functions and spherical representations of $SL_d(F)$. This graph is known as the *Bruhat–Tits building* of $SL_d(F)$. The analogue of this approach for Lie groups is the theory of symmetric spaces.

6.1 The space of lattices

Recall that $\mathcal{O}$ is a subring of F. Thus the d-dimensional vector space F^d can be seen as a $\mathcal{O}$-module.

Definition 6.1.1. A *d-dimensional lattice* L is a free d-dimensional $\mathcal{O}$-submodule of F^d, that is, there is a basis $\{v_i\}_{1\leq i\leq n}$ of F^n such that $L = \bigoplus_{1\leq i\leq n} \mathcal{O}v_i$. Let $\mathcal{L}_d$ denote the set of all d-dimensional lattices.

Let $GL_d(F)\times\mathcal{L}_d \to \mathcal{L}_d$, $(g, L) \mapsto g\cdot L$ be the natural action of $GL_d(F)$ on $\mathcal{L}_d$. It is immediate that the action is transitive.

Next define on $\mathcal{L}_d$ the equivalence relation $\sim_{F^\times}$ where $L_1 \sim_{F^\times} L_2$ if and only if there is $\lambda \in F^\times$ such that $L_1 = \lambda L_2$. Set $X_d \coloneqq \mathcal{L}_d/\sim_{F^\times}$, and let $[L] \in X_d$ denote the equivalence class of a lattice $L \in \mathcal{L}_d$. The space X_d can now be given the structure of a metric space. Recall for this from Chapter 2 the Cartan decomposition of $GL_d(F)$.

Lemma 6.1.2 ([AB08, p. 353])**.** *Let $L_1, L_2 \in \mathcal{L}_d$ with*

$$L_1 = \bigoplus_{1\leq i\leq d} \mathcal{O}u_i, \quad L_2 = \bigoplus_{1\leq i\leq d} \mathcal{O}v_i,$$

where $\{u_i\}_{1\leq i\leq d}$ and $\{v_i\}_{1\leq i\leq d}$ are each a basis of F. Then there exist a basis $\{w_i\}_{1\leq i\leq d}$ and a unique tuple $\{k_i\}_{1\leq i\leq d} \subset \mathbb{Z}$ such that $-\infty < k_1 \leq k_2 \leq \cdots \leq k_d < \infty$ and

$$L_1 = \bigoplus_{1\leq i\leq d} \mathcal{O}w_i, \quad L_2 = \bigoplus_{1\leq i\leq d} \mathfrak{p}^{k_i}\mathcal{O}w_i.$$

Proof. Because the action of $GL_d(F)$ on $\mathcal{L}_d$ is transitive, it can be assumed that $\{v_i\}_{1\leq i\leq d} = \{e_i\}_{1\leq i\leq d}$ is the standard basis of F^d. Then $k \cdot L_1 = L_1$ if and only if $k \in GL_d(\mathcal{O})$. Let

$g \in GL_d(F)$ be such that $L_2 = g \cdot L_1$. By Theorem 2.3.2, there are $k_1, k_2 \in GL_d(\mathcal{O})$ and a unique $t = diag(\mathfrak{p}^{k_1}, \mathfrak{p}^{k_2}, \ldots, \mathfrak{p}^{k_d}) \in T_d^+(F)$ such that $g = k_1 t k_2$. Then $L_2 = (k_1 t) \cdot L_1$ and

$$L_1 = \bigoplus_{1 \le i \le d} \mathcal{O} w_i, \quad L_2 = \bigoplus_{1 \le i \le d} \mathfrak{p}^{k_i} \mathcal{O} w_i,$$

where $w_i = k_1 \cdot e_i$. □

Proposition 6.1.3 ([FTN91, Lem. 5.2])**.** *Let $L_1, L_2 \in \mathcal{L}_d$. Then there exists a unique $k \in \mathbb{Z}$ such that $\mathfrak{p}^k L_1 \subset L_2$ and $\mathfrak{p}^k L_1 \not\subset \mathfrak{p} L_2$.*

Proof. By Lemma 6.1.2 there are a basis $\{v_i\}_{1 \le i \le d}$ of F^d and a unique tuple $\{k_i\}_{1 \le i \le d} \subset \mathbb{Z}$ such that $-\infty < k_1 \le k_2 \le \cdots \le k_d < \infty$ and such that

$$L_1 = \bigoplus_{1 \le i \le d} \mathcal{O} v_i, \quad L_2 = \bigoplus_{1 \le i \le d} \mathfrak{p}^{k_i} \mathcal{O} v_i.$$

Then $\mathfrak{p}^{k_d} L_1$ is the unique representative of $[L_1]$ such that $\mathfrak{p}^{k_d} L_1 \subset L_2$ and $\mathfrak{p}^{k_d} L_1 \not\subset \mathfrak{p} L_2$. □

By Proposition 6.1.3, given two lattices $L_1, L_2 \in \mathcal{L}_d$, there is a unique representative $L_1' \in [L_1]$ such that $L_1' \subset L_2$ and $L_1' \not\subset \mathfrak{p} L_2$. Call L_1' the *maximal lattice of* $[L_1]$ *contained in* L_2.

Definition 6.1.4. Let $d\colon X_d \times X_d \to \mathbb{N}$, $d([L_1], [L_2]) = |k_1 - k_2|$, where $k, n \in \mathbb{Z}$ are the unique maximal integers such that $\mathfrak{p}^{k_1} L_1$ and $\mathfrak{p}^{k_2} L_2$ are the maximal lattices of $[L_1]$ and of $[L_2]$ contained in L_2 and $\mathfrak{p}^{k_1} L_1$, respectively.

It is not difficult to show that d is well-defined and a distance function. Because it only assumes values in $\mathbb{N}$, the topology induced by d on X_d is discrete. Henceforth, let X_d be endowed with the topology induced by d.

Proposition 6.1.5. *The action $GL_d(F) \times X_d \to X_d$, $(g, [L]) \mapsto [g \cdot L]$ is well-defined and continuous. The action is transitive.*

Proof. Because for every $g \in GL_d(F)$ and every $\lambda \in F$ $\lambda(g \cdot L) = g \cdot (\lambda L)$, the action of $GL_d(F)$ on $\mathcal{L}_d$ preserves the equivalence classes of $\sim_{F^\times}$ and hence the induced action on X_d is well-defined. It is clear that the action on $\mathcal{L}_d$ is transitive and that the stabilizer group of $L_0 = \bigoplus_{1 \le i \le d} \mathcal{O} e_i$ is $GL_d(\mathcal{O})$. Continuity follows because $GL_d(\mathcal{O})$ is an open subgroup and thus for every $g \in GL_d(F)$, $g \cdot GL_d(\mathcal{O})$ is an open neighbourhood of g. □

Let $PGL_d(F) \coloneqq GL_d(F)/\{\lambda I_d \colon \lambda \in F^\times\}$. Let $\mathrm{Iso}(X_d)$ be the group of all isometries of X_d, and endow it with the compact-open topology. Then, by Proposition A.2.3, the sets $\{U_S(g) \colon g \in G, S \subset X_d \text{ finite}\}$ form a basis of the compact-open topology, where

$$U_S(g) \coloneqq \bigcap_{x \in S} \{h \in G \colon g \cdot x = h \cdot x\}.$$

Proposition 6.1.6. *$PGL_d(F)$ is isomorphic to a closed subgroup of $Iso(X_d)$.*

Proof. Let $g \in GL_d(F)$. Then, for $k \in \mathbb{Z}$, one has that $\mathfrak{p}^k L_1 \subset L_2$ if and only if $\mathfrak{p}^k (g \cdot L_1) \subset g \cdot L_2$. Hence, there is $n \in \mathbb{Z}$ such that $\mathfrak{p}^n L_1$ and $\mathfrak{p}^n (g \cdot L_1)$ are the maximal sublattices of $[L_1]$ and $[g \cdot L_1]$ contained in L_2 and $g \cdot L_2$, respectively. Thus $d(L_1, L_2) = d(g \cdot L_1, g \cdot L_2)$, that is, $GL_d(F)$ acts via isometries on X_d. This induces a group homomorphism $i\colon GL_d(F) \to \mathrm{Iso}(X_d)$. It is immediate that $\ker i = \{\lambda I_d \colon \lambda \in F^\times\}$. Hence $PGL_d(F)$ is isomorphic to a subgroup of $\mathrm{Iso}(X_d)$. Then every sequence $\{g_n\}_{n \ge 1} \subset PGL_d(F)$ which is Cauchy with respect to the subspace topology induced by $Iso(X_d)$, is also Cauchy with respect to the quotient topology of $PGL_d(F)$. Because $GL_d(F)$ is complete, so is $PGL_d(F)$ and thus $\{g_n\}_{n \ge 1}$ converges to a point in $PGL_d(F)$, which implies that $PGL_d(F)$ is isomorphic to a closed subgroup of $\mathrm{Iso}(X_d)$. □

Henceforth, let $L_0 \coloneqq \bigoplus_{1\le i\le d}\mathcal{O}e_i$ be the *standard lattice*. By Proposition 6.1.6, $PGL_d(F)$ acts transitively on X_d and the stabilizer group of $[L_0]$ is $PGL_d(\mathcal{O})$. Thus $X_d \cong PGL_d(F)/PGL_d(\mathcal{O})$.

Example 6.1.7. Note that, for $g,h \in GL_d(F)$, $g\cdot[L_0] = h\cdot[L_0]$ if and only if $q(g)q(h)^{-1} \in PGL_d(\mathcal{O})$, where $q\colon GL_d(F) \to PGL_d(F)$ is the canonical quotient map. Then $|det(g)| = q^{kd}|det(h)|$ for some $k\in\mathbb{Z}$. Take the lattice

$$L_1 = \left(\bigoplus_{1\le i\le d-1}\mathcal{O}e_i\right)\oplus \mathfrak{p}^1\mathcal{O}e_d.$$

Then $g\cdot L_0 = L_1$ if and only if $|det(g)| = q^{1+kd}$ for some $k\in\mathbb{Z}$. Consider now the closed subgroup $PSL_d(F) = SL_d(F)/\{\pm I_d\}$. For $g\in SL_d(F)$ and $h\in GL_d(F)$, $g\cdot[L_0] = h\cdot[L_0]$ if and only if $|det(h)| = q^{kd}$ for some $k\in\mathbb{Z}$. Hence, there is no $g\in SL_d(F)$ such that $g\cdot[L_0] = [L_1]$, and therefore $SL_d(F)$ does not act transitively on X_d.

6.2 The graph $\widetilde{X}_d$

The metric space X_d can be given the structure of a graph.

Definition 6.2.1. Let $\widetilde{X}_d$ be the graph such that X_d is the set of vertices and

$$\mathrm{edg}(\widetilde{X}_d) \coloneqq \{\{x,y\}\subset X_d\colon d(x,y)=1\}$$

is the set of edges.

Recall the basic definition about graphs from Section 5.1. Then a *path* in $\widetilde{X}_d$ is a sequence $\mathbf{x} = \{x_k\}_{m<k<n}\subset \mathrm{vert}(\widetilde{X}_d)$, with $-\infty\le m<n\le\infty$, such that for all k $d(x_k,x_{k+1})=1$. $\mathbf{x}$ is said be *without backtracking* if $x_k\neq x_l$ for all $k\neq l$. If $\mathbf{x}$ is finite and without backtracking, then the *length* of $\mathbf{x}$ is $l(\mathbf{x}) = n-m-2$.

Proposition 6.2.2 ([Ste97, p. 91])**.** *Let $\boldsymbol{x} = \{x_k\}_{1\le k\le n}\subset \mathrm{vert}(\widetilde{X}_d)$ be a path without backtracking in $\widetilde{X}_d$. Then there are lattices $\{L_k\}_{1\le k\le n}$ such that $[L_k] = x_k$ and*

$$\mathfrak{p}^n L_1 \subset L_n \subset \cdots \subset L_2 \subset L_1,$$

where the inclusions are strict and $\mathfrak{p}L_k \subset L_{k+1}\subset L_k$.

Proof. Argue by induction on n. The base case of $n=1$ is immediate. For the induction step suppose that the hypothesis holds for $n\ge 1$. Then because $d(x_n,x_{n+1})=1$, there is a lattice $L_{n+1}\in x_{n+1}$ such that $\mathfrak{p}L_n \subsetneq L_{n+1}\subsetneq L_n$. Since $\mathbf{x}$ is assumed to be without backtracking, $L_{n+1}\neq L_{n-1}$. Moreover, $\mathfrak{p}^{n+1}L_1 \subset \mathfrak{p}L_n \subset L_{n+1}$, which proves the induction hypothesis for $n+1$. □

Proposition 6.2.2 gives an equivalent definition of the distance on X_d.

Corollary 6.2.3 ([FTN91, Lem. 5.2])**.** *Let $x,y\in X_d$. Then*

$$d(x,y) = \min\{l(\boldsymbol{x})\colon \boldsymbol{x}\subset X_d \text{ is a path with endpoints } x \text{ and } y\}.$$

Proof. Let $n = d(x,y)$, and take representatives $L_1\in x$, $L_n\in y$ such that $\mathfrak{p}^nL_1\subset L_n\subset L_1$ and $\mathfrak{p}^{n-1}L_1\not\subset L_n$. By Lemma 6.1.2 and by Proposition 6.2.2,there is a basis $\{v_i\}_{1\le i\le d}$ such that

$$L_1 = \bigoplus_{1\le i\le d}\mathcal{O}v_i, \quad L_n = \bigoplus_{1\le i\le d}\mathfrak{p}^{k_i}\mathcal{O}v_i,$$

where $0 \leq k_1 \leq k_2 \leq \cdots \leq k_d < \infty$ and $k_d = n$. Then set $L_k \coloneqq \left(\bigoplus_{1 \leq i \leq d-1} \mathcal{O}v_i \right) \oplus \mathfrak{p}^k \mathcal{O}v_d$. This gives a sequence $\mathbf{x} = \{x_k\}_{1 \leq k \leq n}$ with $x_k = [L_k]$ and with endpoints $x_0 = x$, $x_n = y$ and such that $d(x, y) = l(\mathbf{x})$.

Conversely, suppose $\mathbf{x} = \{x_k\}_{1 \leq k \leq n}$ is a path with endpoints $x_0 = x$, $x_n = y$. One can assume that $\mathbf{x}$ has no backtracking. Then, by Lemma 6.1.2 and by Proposition 6.2.2, there are representatives $L_k \in x_k$ such that

$$\mathfrak{p}^n L_1 \subset L_n \subset \cdots \subset L_2 \subset L_1.$$

Thus $d(x, y) \leq n$. □

Definition 6.2.4. A *geodesic* in $\widetilde{X}_d$ is a path $\gamma = \{x_k\}_{m<k<n}$, with $-\infty \leq m < n \leq \infty$, such that for every finite path $\mathbf{y} = \{y_l\}_{1 \leq l \leq k} \subset \gamma$, $l(\mathbf{y}) = d(y_1, y_k)$.

Observe that a geodesic in $\widetilde{X}_d$ is a path without backtracking; however the converse is not ture if $d > 2$.

Remark ([Ste97, p. 93]). As seen in the proof of Corollary 6.2.3 between any two vertices a geodesic always exists. If $d > 2$, then the geodesic is not unique. Indeed, take

$$L_0 = \bigoplus_{1 \leq i \leq d} \mathcal{O}e_i, \quad L_n \coloneqq \left(\bigoplus_{1 \leq i \leq d-2} \mathcal{O}e_i \right) \bigoplus \pi \mathcal{O}v_{d-1} \bigoplus \mathfrak{p}^n \mathcal{O}v_d,$$

and for $1 < k < n$ let

$$L_k \coloneqq \left(\bigoplus_{1 \leq i \leq d-1} \mathcal{O}v_i \right) \oplus \mathfrak{p}^k \mathcal{O}v_d, \quad L'_k \coloneqq \left(\bigoplus_{1 \leq i \leq d-2} \mathcal{O}v_i \right) \oplus \mathfrak{p}\mathcal{O}v_{d-1} \oplus \mathfrak{p}^k \mathcal{O}v_d.$$

Let $\mathbf{x} = \{x_k\}_{0 \leq k \leq d}$ and $\mathbf{x}' = \{x'_k\}_{0 \leq k \leq n}$ with $x_0 = x'_0 = [L_0]$, $x_n = x'_n = [L_n]$ and $x_k = [L_k]$, $x'_k = [L'_k]$ for $1 \leq k \leq n-1$. Because $d(x_1, x_n) = n$, both $\mathbf{x}$ and $\mathbf{x}'$ are two distinct geodesics connecting x_0 and x_n. This implies that $\widetilde{X}_d$ contains a circuit and is therefore not a tree.

Recall that the residue field $\mathcal{O}/\mathfrak{p}\mathcal{O}$ is assumed to have cardinality q. If $L_1, L_2 \in \mathcal{L}_d$ are lattices with $\mathfrak{p}^n L_1 \subset L_2 \subset L_1$ and $n \geq 1$, then $L_2/\mathfrak{p}^n L_1$ is a submodule of the free $\mathcal{O}$-module $L_1/\mathfrak{p}^n L_1 \cong (\mathcal{O}/\mathfrak{p}^n\mathcal{O})^d$. Because every submodule of a free module is also free, there is a bijective correspondence between the lattices $\{L \in \mathcal{L} \colon \mathfrak{p}^n L_1 \subset L \subset L_1\}$ and the submodules of $L_1/\pi^n L_1$.

If $n = 1$, then $L_1/\mathfrak{p}L_1 \cong \mathcal{O}/\mathfrak{p}\mathcal{O} \cong \mathbb{F}_q^d$. Thus there is a bijective correspondence between the edges a vertex in $\widetilde{X}_d$ belongs to and the proper subspaces of $\mathbb{F}_q^d$. Moreover, this implies that all vertex in $\widetilde{X}_d$ belong to equally many edges.

Proposition 6.2.5 ([FTN91, Thm. 5.3])**.** *$\widetilde{X}_2$ is the homogeneous tree of degree $q+1$. In particular, for every infinite one-sided geodesic $\gamma = \{x_k\}_{k \geq 0}$ there is a basis v_1, v_2 of F^2 such that $x_k = [\mathcal{O}v_1 \oplus \mathfrak{p}^k \mathcal{O}v_2]$.*

Proof. Let $\mathbf{x} = \{x_k\}_{1 \leq k \leq n}$ be an infinite one-sided geodesic in $\widetilde{X}_2$. Then there are representatives $L_k \in x_k$ such that

$$\mathfrak{p}^n L_1 \subset L_n \subset \cdots \subset L_2 \subset L_1.$$

By Proposition 6.2.2 there is a basis v_1, v_2 of F^2 such that $L_1 = \mathcal{O}v_1 \oplus \mathcal{O}v_2$ and $L_n = \mathcal{O}v_1 \oplus \mathfrak{p}^n \mathcal{O}v_2$. Then $v_1 \in L_k$ for all $1 \leq k \leq n$. This implies that $L_k = \mathcal{O}v_1 \oplus \mathfrak{p}^{l_k} \mathcal{O}v_2$ for some $l_k \geq 0$. But, because γ is a chain without backtracking,

$$\mathfrak{p}L_k = \mathfrak{p}\mathcal{O}v_1 \oplus \mathfrak{p}^{l_k+1} \mathcal{O}v_2 \subsetneq L_{k+1} = \mathcal{O}v_1 \oplus \mathfrak{p}^{l_{k+1}} \mathcal{O}v_2 \subsetneq L_k, \tag{6.1}$$

and thus $l_{k+1} = l_k + 1$. Hence, because $l_0 = 0$ and $l_n = n$, $l_k = k$. Moroever, it follows that γ is the unique chain without backtracking that connects x_0 and x_n. Hence, $\widetilde{X}_2$ contains no circuits and is therefore a tree. As seen above, $\widetilde{X}_2$ is therefore a homogeneous tree of degree $|\mathbb{P}(\mathbb{F}_q^2)| = q+1$, where $\mathbb{P}(\mathbb{F}_q^2)$ is the projective space of $\mathbb{F}_q^2$, that is, the space of all one-dimensional subspaces of $\mathbb{F}_q^2$ (see Definition 6.3.6). □

If the tree $\widetilde{X}_2$ has degree $q+1$, by Proposition 6.2.5 the metric defined on the tree T_{q+1} coincides with the one defined on $\widetilde{X}_2$ in Definition 6.1.4. Hence, the metric spaces T_{q+1} and $\widetilde{X}_2$ are isometric. In particular, by Proposition 6.1.6, $\mathrm{Aut}(T_{q+1})$ contains copies of $PGL_2(F)$ and $PSL_2(F)$ as closed non-compact subgroups.

Henceforth, identify $PGL_2(F)$ and $PSL_2(F)$ with the subgroups of $\mathrm{Aut}(T_{q+1})$ such that, for $g \in PGL_2(F)$ and for $x = [L] \in T_{q+1}$, $g \cdot x = [g \cdot L]$

Remark. Note that a homogeneous tree T_{q+1} is isomorphic to the graph $\widetilde{X}_2$ over a non-archimedean local field F only if there are a prime number $p \geq 2$ and $n \geq 1$ such that $q = p^n$. Indeed, because q is the cardinality of the finite residue field of F, it needs to be the positive integer power of a prime number. Hence, there are homogeneous trees, e.g. T_7 and T_{12}, whose automorphism groups do not contain closed subgroups isomorphic to $PGL_2(F)$ or $PSL_2(F)$ for a non-archimedean local field F.

6.3 The boundary and horocycles of $\widetilde{X}_d$

The notion of appartements and Weyl chambers in $\widetilde{X}_d$ are generalizations of the notions of two- and one-sided infinite geodesics in a homogeneous tree. Let $\{v_i\}_{1\leq i\leq d}$ be a basis of the vector space F^d.

Definition 6.3.1. The *appartement* of $\widetilde{X}_d$ associated to the basis $\{v_i\}_{1\leq i\leq d}$ of F^d is

$$A(v_1, v_2, \ldots, v_d) \coloneqq \left\{ \left[\bigoplus_{1\leq i\leq d} \mathfrak{p}^{k_i} \mathcal{O} v_i \right] : k_1, k_2, \ldots, k_d \in \mathbb{Z} \right\}.$$

Definition 6.3.2. The set

$$W(v_1, \ldots, v_d) \coloneqq \left\{ \left[\bigoplus_{1\leq i\leq d} \pi^{k_i} \mathcal{O} v_i \right] : -\infty < k_1 \leq k_2 \leq \cdots \leq k_d < \infty \right\}$$

is a *Weyl chamber* of $\widetilde{X}_d$. The equivalence class $\left[\bigoplus_{1\leq i\leq d} \mathcal{O} v_i\right]$ is the *base point* of $W(v_1, v_2, \ldots, v_d)$.

By Proposition 6.2.5 the appartements in the homogeneous tree $\widetilde{X}_2$ are exactly the two-sided infinite geodesics, and the Weyl chambers of $\widetilde{X}_2$ are exactly the one-sided infinite geodesics. Let $\mathcal{W}_d$ be the set of all Weyl chambers in $\widetilde{X}_d$. Analogously to the definition of the boundary of $\widetilde{X}_2$ one has

Definition 6.3.3. Let $\sim_{\mathcal{W}_d}$ be the relation on $\mathcal{W}$ such that $W_1 \sim_{\mathcal{W}_d} W_2$ if and only if $W_1 \cap W_2$ contains a Weyl chamber, that is, $W(v_1, v_2, \ldots, v_d) \sim_{\mathcal{W}_d} W(w_1, w_2, \ldots, w_d)$ if and only if there are integers $-\infty < k_1 \leq k_2 \leq \cdots \leq k_d < \infty$ and $-\infty < l_1 \leq l_2 \leq \cdots \leq l_d < \infty$ such that $W(\mathfrak{p}^{k_1} v_1, \mathfrak{p}^{k_2} v_2, \ldots, \mathfrak{p}^{k_d} v_d) = W(\mathfrak{p}^{l_1} w_1, \mathfrak{p}^{l_2} w_2, \ldots, \mathfrak{p}^{l_d} w_d)$.

It is easy to check that $\sim_{\mathcal{W}_d}$ defines an equivalence relation on $\mathcal{W}_d$. Moreover, it is immediate that in the case of $\widetilde{X}_2$, $\sim_{\mathcal{W}_d}$ on $\mathcal{W}$ agrees with the equivalence relation $\sim_{\partial T_{q+1}}$.

Definition 6.3.4. The *boundary* $\partial\widetilde{X}_d$ of $\widetilde{X}_d$ is the set of equivalence classes $\mathcal{W}_d/\sim_{\mathcal{W}_d}$. Denote the equivalence classes of $\sim_{\mathcal{W}_d}$ by $[W]_{\mathcal{W}_d}$ for $W \in \mathcal{W}_d$.

Proposition 6.3.5. *The action of $PGL_d(F) < \mathrm{Iso}(\widetilde{X}_d)$ on $\mathcal{W}_d$ preserves the equivalence classes of $\sim_{\mathcal{W}_d}$. The induced action of $PGL_d(F)$ on $\partial\widetilde{X}_d$ is transitive.*

Proof. It is clear that for all $g \in PGL_d(F)$ $g \cdot W(v_1, v_2, \ldots, v_d) = W(g \cdot v_1, g \cdot v_2, \ldots, g \cdot v_d)$. Hence g maps Weyl chambers to Weyl chambers. In particular, if $-\infty < k_1 \leq k_2 \leq \cdots \leq k_d < \infty$ are integers, then $g \cdot W(\mathfrak{p}^{k_1} v_1, \mathfrak{p}^{k_2} v_2, \ldots, \mathfrak{p}^{k_d} v_d) = W(\mathfrak{p}^{k_1} g \cdot v_1, \mathfrak{p}^{k_2} g \cdot v_2, \ldots, \mathfrak{p}^{k_d} g \cdot v_d)$. Hence, the action of $PGL_d(F)$ on $\mathcal{W}_d$ preserves the equivalence classes of $\sim_{\mathcal{W}_d}$. Because $PGL_d(F)$ acts transitively on the set of Weyl chambers, the induced action of $PGL_d(F)$ on $\partial\widetilde{X}_d$ is transitive as well. □

Something interesting happens for $\widetilde{X}_2$, with useful implications for $PGL_2(F)$ and $PSL_2(F)$. Recall for this the notion of the projective space of a vector space:

Definition 6.3.6. Let V be a vector space over a field K. Let $\sim$ be the equivalence relation on $V - \{0\}$ such that for $v, w \in V$ $v \sim w$ if and only if there is $\lambda \in K^\times$ such that $\lambda v = w$. The *projective space* of V is the set of equivalence classes of $\sim$, denoted by $\mathbb{P}(V)$.

Endow $\mathbb{P}(F^d)$ with the quotient topology. Observe that the natural action $GL_d(F)$ on F^d preserves the equivalence classes of $\mathbb{P}(F^d)$. This induces a group action of $GL_d(F)$ and thus of $PGL_d(F)$ on $\mathbb{P}(F^d)$, which is given by $(g, [v]) \mapsto [g \cdot v]$ for $g \in GL_d(F)$ and $v \in F^d$.

Proposition 6.3.7 ([FTN91, Prop. 5.4])**.** *The map*

$$\begin{aligned} \partial\widetilde{X}_2 &\to \mathbb{P}(F^2) \\ [W(v_1, v_2)]_{\mathcal{W}_2} &\mapsto [v_1] \end{aligned} \tag{6.2}$$

is well-defined, bijective and $PGL_2(F)$-equivariant.

Proof. Let $o = [L_0] \in X_2$ be the equivalence class of the standard lattice. Recall from Section 5.1 that for every one-sided infinite geodesic γ in $\widetilde{X}_2$ there is always a unique one-sided infinite geodesic $\gamma' \subset \mathrm{vert}(\widetilde{X}_2)$ with base point o such that $\gamma \sim_{\partial T_{q+1}} \gamma'$, where $\widetilde{X}_2$ is identified with T_{q+1}. Thus if $W(v_1, v_2) \in \mathcal{W}$, there is a basis w_1, w_2 of F^2 such that $o = [\mathcal{O}v_1 \oplus \mathcal{O}v_2]$ and $W(v_1, v_2) \sim_{\mathcal{W}_2} W(w_1, w_2)$. This implies that for some $k_0 \in \mathbb{Z}$ and for all $k \geq 0$ $\mathcal{O}v_1 \oplus \mathfrak{p}^k\mathcal{O}v_2 = \mathcal{O}w_1 \oplus \mathfrak{p}^{k+k_0}\mathcal{O}w_2$. Hence,

$$\mathcal{O}v_1 = \bigcap_{k\geq 0} \mathcal{O}v_1 \oplus \mathfrak{p}^k\mathcal{O}v_2 = \bigcap_{k\geq 0} \mathcal{O}w_1 \oplus \mathfrak{p}^{k+k_0}\mathcal{O}w_2 = \mathcal{O}w_1,$$

and thus $[v_1] = [w_1] \in \mathbb{P}(F^2)$. It follows that the map (6.2) is well-defined. Surjectivity is immediate. To prove injectivity, it is enough to show that if v_1, v_2 and v_1, w_2 are each a basis of F^2, then $[W(v_1, v_2)]_{\mathcal{W}_2} = [W(v_1, w_2)]_{\mathcal{W}_2}$. Indeed, write $w_2 = av_1 + bv_2$, with $a, b \in F$. Then for $k \geq 0$ large enough

$$\mathfrak{p}^k w_2 = (a\mathfrak{p}^k)v_1 + (b\mathfrak{p}^k)v_2 \in \mathcal{O}v_1 \oplus \mathcal{O}v_2,$$

and thus $[\mathcal{O}v_1 \oplus \mathfrak{p}^k\mathcal{O}w_2] \in W(v_1, v_2)$. Hence, $W(v_1, v_2) \sim_{\mathcal{W}_2} W(v_1, w_2)$. □

Corollary 6.3.8. *The action $PSL_2(F) \times \partial\widetilde{X}_2 \to \partial\widetilde{X}_2$, $(g, [W]) \mapsto [g \cdot W]$ is 2-transitive. Moreover, the subgroup $PSL_2(\mathcal{O})$ acts also transitively on $\partial\widetilde{X}_2$.*

Proof. Observe that the above action of $PSL_2(F)$ on $\partial\widetilde{X}_2$ is induced by $SL_2(\mathcal{O}) \times \mathbb{P}_2(F^2) \to \mathbb{P}_2(F^2)$, $(g, [v]) \mapsto [g \cdot v]$. Let e_1, e_2 be a basis of F^2, and let $v \in F^2$ be such that $[v] \neq [e_1]$. Then $v = (x, y)$ with $y \neq 0$, and let $g \in SL_2(F)$ be the matrix whose first column is scalar multiple of e_1 and whose second column is v. Then $g \cdot [e_1] = [e_1]$ and $g \cdot [e_2] = [v]$. Thus $SL_2(F)$ and therefore $PSL_2(F)$ act 2-transitively on $\partial\widetilde{X}_2$. Because $PSL_2(\mathcal{O})$ is the subgroup of $PSL_2(F)$ which fixes the base point $o = [L_0]$, $PSL_2(\mathcal{O})$ acts transitively on $\partial\widetilde{X}_2$ by Proposition 5.3.11. □

Corollary 6.3.9. *Identifying $\widetilde{X}_2$ with T_{q+1}, one has that $PSL_2(F) \leq \mathrm{Aut}_0(T_{q+1})$.*

Proof. Note that $PSL_2(\mathcal{O}) = \mathrm{stab}_{PSL_2(F)}(o)$. Because $PSL_2(\mathcal{O})$ acts transitively on $\partial\widetilde{X}_2$, the corollary follows from Proposition 5.3.13. □

Next introduce horocycles on $\widetilde{X}_d$.

Definition 6.3.10. Let $\sim_{\mathcal{H}_d}$ be the relation on $\mathcal{W}_d$ such that $W(v_1, v_2, \ldots, v_d) \sim_{\mathcal{H}_d} W(w_1, w_2, \ldots, w_d)$ if and only if there are $-\infty < k_1 \leq k_2 \leq \cdots \leq k_d < \infty$ such that $W(\mathfrak{p}^{k_1}v_1, \mathfrak{p}^{k_2}v_2, \ldots, \mathfrak{p}^{k_d}v_d) = W(\mathfrak{p}^{k_1}w_1, \mathfrak{p}^{k_2}w_2, \ldots, \mathfrak{p}^{k_d}w_d)$.

It is easy to check that $\sim_{\mathcal{H}_d}$ defines an equivalence relation on $\mathcal{W}$.

Definition 6.3.11. The *horocycles* of $\widetilde{X}_d$ are the equivalence classes of $\mathcal{H}_d/\sim_{\mathcal{H}_d}$.

Observe that $\sim_{\mathcal{H}_d}$ is a stronger relation than $\sim_{\mathcal{W}_d}$. If $W_1 \sim_{\mathcal{H}_d} W_2$, then also $W_1 \sim_{\mathcal{W}_d} W_2$. However, the converse is not true; as a counterexample take $W_1 = W(e_1, e_2, \ldots, e_d)$ and $W_2 = W(e_1, e_2, \ldots, e_{d-1}, \mathfrak{p}e_d)$.

Proposition 6.3.12. *The action of $PGL_2(F)$ on $\mathcal{W}_d$ preserves the equivalence classes of $\sim_{\mathcal{H}_d}$.*

Proof. The proof is the same as the one of Proposition 6.3.5. □

6.4 Stabilizer Groups of vertices, Weyl chambers and horocycles

In this section, to simplify notation, write

$$[k_1, k_2, \ldots, k_d] = \left[\bigoplus_{1 \le i \le d} \mathfrak{p}^{k_1} \mathcal{O} e_i \right] \in X_d,$$

and

$$W[k_1, k_2, \ldots, k_n] = W(\mathfrak{p}^{k_1} v_1, \mathfrak{p}^{k_2} v_2, \ldots, \mathfrak{p}^{k_d} v_d).$$

To further simplify notation write $G = PGL_d(F)$.

For a horocycle $h \in \mathcal{H}_d$ set

$$\mathrm{fix}(h) \coloneqq \{g \in G \colon g \text{ fixes each Weyl chamber } W \in h \text{ pointwise}\}.$$

One computes the following:

1) (See [Ste97, p. 97].) The stabilizer group of a vertex $[k_1, k_2, \ldots, k_d]$ is

$$\begin{aligned}
&\mathrm{stab}_G([k_1, k_2, \ldots, k_d]) \\
&\quad = diag(\mathfrak{p}^{k_1}, \mathfrak{p}^{k_2}, \ldots, \mathfrak{p}^{k_d}) \, \mathrm{stab}_G([0, 0, \ldots, 0]) diag(\mathfrak{p}^{-k_1}, \mathfrak{p}^{-k_2}, \ldots, \mathfrak{p}^{-k_d}) \\
&\quad = \{g \in GL_d(F) \colon g_{ij}, (g^{-1})_{ij} \in \mathfrak{p}^{k_i - k_j} \mathcal{O}\} / \{\lambda I_d \colon \lambda \in \mathcal{O}^\times\}.
\end{aligned}$$

2) (See [Ste97, p. 99].) The stabilizer group of a Weyl chamber $W[k_1, k_2, \ldots, k_d]$ is

$$\begin{aligned}
&\mathrm{stab}_G(W[k_1, k_2, \ldots, k_d]) \\
&\quad = \bigcap_{l_1 \le l_2 \le \ldots \le l_d} \mathrm{stab}_G([k_1 + l_1, k_2 + l_2, \ldots, k_d + l_d]) \\
&\quad = \bigcap_{l_1 \le l_2 \le \ldots \le l_d} \{g \in GL_d(F) \colon g_{ij}, (g^{-1})_{ij} \in \mathfrak{p}^{(k_i + l_i) - (k_j + l_j)} \mathcal{O}\} / \{\lambda I_d \colon \lambda \in F^\times\} \\
&\quad = \{g \in GL_d(F) \colon g_{ii} \in \mathcal{O}^\times, \ g_{ij} = 0 \text{ if } i > j, \\
&\qquad\qquad g_{ij} \in \mathfrak{p}^{k_i - k_j} \mathcal{O} \text{ if } i < j\} / \{\lambda I_d \colon \lambda \in \mathcal{O}^\times\}.
\end{aligned}$$

3) (See [Ste97, p. 101].) Let $h = [W[k_1, k_2, \ldots, k_d]]_{\mathcal{H}_d}$ be a horocycle. Then

$$\begin{aligned}
&\mathrm{stab}_G(h) \\
&\quad = \bigcup_{l_1 \le l_2 \le \ldots \le l_d} \mathrm{stab}_G(W[k_1 + l_1, k_2 + l_2, \ldots, k_d + l_d]) \\
&\quad = \bigcup_{l_1 \le l_2 \le \ldots \le l_d} \{g \in GL_d(F) \colon g_{ii} \in \mathcal{O}^\times, \ g_{ij} = 0 \text{ if } i > j, \\
&\qquad\qquad g_{ij} \in \mathfrak{p}^{(k_i - k_j) + (l_i - l_j)} \mathcal{O} \text{ if } i < j\} / \{\lambda I_d \colon \lambda \in F^\times\} \\
&\quad = \{g \in GL_d(F) \colon g_{ii} \in \mathcal{O}^\times, \ g_{ij} = 0 \text{ if } i > j, \\
&\qquad\qquad g_{ij} \in F \text{ if } i < j\} / \{\lambda I_d \colon \lambda \in F^\times\}.
\end{aligned}$$

Because G acts transitively on the set $\mathcal{W}_d$ of Weyl chambers, by Proposition 6.3.12, $\mathrm{stab}_G(h)$ acts transitively on the representatives of h. Thus

$$\begin{aligned}\mathrm{fix}_G(h) &= \bigcap_{g\in \mathrm{stab}_G(h)} \mathrm{stab}_G(g\cdot W[k_1,k_2,\dots,k_d])\\ &= \bigcap_{g\in \mathrm{stab}_G(h)} g\cdot \mathrm{stab}_G(W[k_1,k_2,\dots,k_d])\cdot g^{-1}\\ &= \{g\in GL_d(F)\colon g_{ii}=1,\ g_{ij}=0 \text{ if } i>j,\\ &\qquad g_{ij}\in \mathfrak{p}^{k_i-k_j}\mathcal{O} \text{ if } i<j\}/\{\lambda I_d\colon \lambda\in\mathcal{O}^\times\}.\end{aligned}$$

Identify $\widetilde{X}_2$ with T_{q+1}. Then there are now two notions of horocycles in T_{q+1}. While a horocycle as in Definition 5.3.5 is an equivalence class of an equivalence relation on $\mathrm{vert}(T_{q+1})$, $\sim_{\mathcal{H}_2}$ is an equivalence relation on the set of one-sided infinite geodesics. The following proposition gives a bijective correspondence between those two different sets of horocycles.

Proposition 6.4.1. *The map*

$$\begin{aligned}\mathcal{H}_2 &\to \bigsqcup_{\omega\in\partial T_{q+1}} \mathcal{H}_\omega\\ [W(v_1,v_2)]_{\mathcal{H}_2} &\mapsto H([\mathcal{O}v_1\oplus\mathcal{O}v_2],[W(v_1,v_2)]_{\mathcal{W}_2})\end{aligned}$$

is well-defined, bijective and G-equivariant.

Proof. To show that the above map is well-defined, suppose that $W(v_1,v_2)\sim_{\mathcal{H}_2} W(w_1,w_2)$. Then, by definition, there exists $k>0$ such that $[\mathcal{O}v_1\oplus\mathfrak{p}^k\mathcal{O}v_2]=[\mathcal{O}w_1\oplus\mathfrak{p}^k\mathcal{O}w_2]$. Note that

$$d([\mathcal{O}v_1\oplus\mathcal{O}v_2],[\mathcal{O}v_1\oplus\mathfrak{p}^k\mathcal{O}v_2])=k=d([\mathcal{O}w_1\oplus\mathcal{O}w_2],[\mathcal{O}w_1\oplus\mathfrak{p}^k\mathcal{O}w_2]).$$

Since $W(v_1,v_2)\sim_{\mathcal{W}} W(w_1,w_2)$, this implies that there exists $t\in[[\mathcal{O}v_1\oplus\mathcal{O}v_2],[W(v_1,v_2)]_{\mathcal{W}_2})\cap[[\mathcal{O}w_1\oplus\mathcal{O}w_2],[W(w_1,w_2)]_{\mathcal{W}_2})$ such that $d([\mathcal{O}v_1\oplus\mathcal{O}v_2],t)=d([\mathcal{O}w_1\oplus\mathcal{O}w_2],t)$. Hence

$$H([\mathcal{O}v_1\oplus\mathcal{O}v_2],[W(v_1,v_2)]_{\mathcal{W}_2})=H([\mathcal{O}w_1\oplus\mathcal{O}w_2],[W(w_1,w_2)]_{\mathcal{W}_2}),$$

which implies that the above map is well-defined.

It remains to show that the map is bijective. It is clear that it is injective. To show that it is surjective, fix a horocycle $H(x,\omega)$ at $x\in\mathrm{vert}(T_{q+1})$ around $\omega\in\partial T_{q+1}$. Then, by Proposition 6.2.5, there exists a basis v_1,v_2 of F^2 such that $x=[\mathcal{O}v_1\oplus\mathcal{O}v_2]$ and $\omega=[W(v_1,v_2)]_{\mathcal{W}_2}$. Then $H(x,\omega)$ is the image of $[W(v_1,v_2)]_{\mathcal{H}_2}$ under the above map. Finally, G-equivariance can be verified by direct computations. □

This gives

Corollary 6.4.2. *Let $\omega\in\partial T_{q+1}$, and let $W(v_1,v_2)$ be a representative of ω. Let $x=[\mathcal{O}v_1\oplus\mathcal{O}v_2]$, and let $h=[W(v_1,v_2)]_{\mathcal{H}_2}$. The following holds:*

1) $\mathrm{stab}_G(\omega)=\mathrm{stab}_G([v_1])$, *where* $[v_1]\in\mathbb{P}(F^2)$.

2) $N_G(\omega)=\mathrm{stab}_G(h)$.

3) $U_G(\omega)=\mathrm{fix}_G(h)$.

Proof. Part 1) follows from Proposition 6.3.7. Part 2) and 3) follow from Proposition 6.4.1. □

It follows that for every $\omega\in\partial T_{q+1}$ $U_G(\omega)$ is not dense in $N_G(\omega)$. Thus, by Theorem 2.3.5, $PSL_2(F)<\mathrm{Aut}_0(T_{q+1})$ is a topologically simple subgroup which has the Howe–Moore Property, but such that for all $\omega\in\partial T_{q+1}$ $U_{PSL_2(F)}(\omega)$ is not dense in $N_{PSL_2(F)}(\omega)$. This proves that the hypothesis in Proposition 5.4.7 is not necessary.

6.5 Application to $GL_2(F)$ and $SL_2(F)$

The goal of this section is to apply the knowledge from the previous sections to $GL_2(F)$ and $SL_2(F)$. As above identify $\widetilde{X}_2$ with T_{q+1}, and consider $PGL_2(F)$ and $PSL_2(F)$ as closed subgroups of $\mathrm{Aut}(T_{q+1})$. This gives

Proposition 6.5.1. *$GL_2(\mathcal{O})$ and $SL_2(\mathcal{O})$ are maximal compact subgroups of $GL_2(F)$ and $SL_2(F)$, respectively.*

Proof. Observe that if $K < GL_2(F)$ is compact, then $\{|\det(k)|\colon k \in K\} < (\mathbb{R}_{>0}, \cdot)$ is compact. Because the trivial subgroup is the only compact subgroup of $(\mathbb{R}_{>0}, \cdot)$, $K < \det^{-1}(\mathcal{O}^\times)$. Next recall that $PGL_2(\mathcal{O}) = \mathrm{stab}_{PGL_2(\mathcal{O})}(o)$ and $PSL_2(\mathcal{O}) = \mathrm{stab}_{PSL_2(\mathcal{O})}(o)$. Thus, by Theorem 5.3.4, $PGL_2(\mathcal{O})$ and $PSL_2(\mathcal{O})$ are maximal compact subgroups of $PGL_2(F)$ and $PSL_2(F)$, respectively. The proposition follows because their preimages $GL_2(\mathcal{O})$ and $SL_2(\mathcal{O})$ under the quotient map $q\colon \det^{-1}(\mathcal{O}^\times) \to PGL_2(F)$ are compact. □

By similar arguments as in the proof of Theorem 5.3.4 and in the proof of Proposition 6.5.1, using the respective Bruhat–Tits building, it can be shown that for every $d \geq 1$ $SL_d(\mathcal{O})$ is a maximal compact subgroup of $SL_d(F)$.

By Proposition 5.6.1 and Corollary 6.3.8 one obtains another proof that $(GL_2(F), GL_2(\mathcal{O}))$ and $(SL_2(F), SL_2(\mathcal{O}))$ are Gelfand pairs. Moreover, by the work done in Chapter 5 and by Proposition 3.4.8, one knows now all spherical functions and representations of $PGL_2(F)$ and $PSL_2(F)$. Henceforth only the case of $SL_2(F)$ is considered; the case of $PGL_2(F)$ is treated similarly.

Theorem 6.5.2. *The spherical functions of $(SL_2(F), SL_2(\mathcal{O}))$ are exactly the functions $\{\phi_z\colon z \in \mathbb{C}\}$, where for $g \in G$*

$$\phi_{F,z}(g) \coloneqq \int_{SL_2(\mathcal{O})} \Delta_{\widetilde{P}_2^+(F)}^{\frac{1}{2}+z}(gk)\, dk.$$

Furthermore, each ϕ_z is given explicitly as follows:

1) If $q^{2z} \neq 1$, then for $n \geq 0$

$$\phi_z(diag(\mathfrak{p}^n, \mathfrak{p}^{-n})) = c(z)q^{-(1+2z)n} + c(1-z)q^{(1-2z)n},$$

where

$$c(z) = \frac{q^{1-z} - q^z}{(q+1)(q^{-z} - q^z)}.$$

2) If $q^{2z} = 1$, then for $n \geq 0$

$$\phi_z(diag(\mathfrak{p}^n, \mathfrak{p}^{-n})) = \Big(1 + \frac{q-1}{q+1}2n\Big)q^{-(1+2z)n}.$$

Proof. By Proposition 3.4.8 every spherical function of $(SL_2(F), SL_2(\mathcal{O}))$ is of the form $\psi \circ q$, where $q\colon SL_2(F) \to PSL_2(\mathcal{O})$ is the canonical quotient map and ψ is a spherical function of $(PSL_2(F), PSL_2(\mathcal{O}))$. Notice that $q(\widetilde{T}_2(F)) < \mathrm{Aut}(T_{q+1})$ is a subgroup of translations of even step along the two-sided infinite geodesic (ω_2, ω_1), where ω_i corresponds to $[e_i] \in \mathbb{P}(F^2)$ under the bijection (6.2). Moreover, $\mathrm{stab}_{PSL_2(F)}(\omega_1) = q(\widetilde{P}_2^+(F))$ and $\mathrm{stab}_{PSL_2(F)}(\omega_2) = q(\widetilde{P}_2^-(F))$. Then the integral formula follows from Corollary 5.6.3 and the explicit formulas follow from Proposition 5.5.18. □

Notice that Proposition 6.5.2 provides the same integral formula for the spherical functions of $(SL_2(F), SL_2(\mathcal{O}))$ as Theorem 3.5.7 does for $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$. Concerning the unitary dual of $SL_2(F)$ one has

Theorem 6.5.3. *There exists a (possibly empty) closed subset $S \subset \left(0, \frac{1}{2}\right)$ such that the spherical representations of $(SL_2(F), SL_2(\mathcal{O}))$ are exactly the following:*

1) the trivial representation $1_{SL_2(F)}$,

2) the principal series $\left\{\pi_{\phi_{F,it}} : 0 \leq t < \frac{\pi}{\log(q)}\right\}$,

3) the complementary series $\left\{\pi_{\phi_{F,s}} : 0 < s < \frac{1}{2}\right\}$ and

4) the series $\left\{\pi_{\phi_{F,s+\frac{\pi i}{2\log(q)}}} : s \in S\right\}$.

Proof. This follows from Proposition 3.4.8 and Theorem 5.8.7. □

Corollary 6.5.4. *$SL_2(F)$ does not have Property (T).*

Proof. By the continuity of the map $\mathbb{C} \times SL_2(F) \to \mathbb{C}$, $(z, g) \mapsto \phi_{F,z}(g)$, the corollary follows from Theorem 4.5.7 and from the existence of the complementary series. □

Chapter 7

Quantitative Estimates and Upper Bounds for Matrix Coefficients

7.1 Quantitative estimates for $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$

The goal of this section is to obtain quantitative estimates for the decay of matrix coefficients of the non-trivial spherical representations of $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$. Recall from Chapter 3 that the positive definite spherical functions of $(SL_2(\mathbb{R}), SO(2))$ are $\left\{\phi_{\mathbb{R},z} \colon z \in i\mathbb{R} \text{ or } z \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}$, where

$$\phi_{\mathbb{R},z}(d(a)) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} (a^2\cos^2(\theta) + a^{-2}\sin^2(\theta))^{-\frac{1}{2}-z}\, d\theta.$$

As in Chapter 3, write $d(a) = diag(a, a^{-1}) \in \widetilde{A}_2$ for $a \in \mathbb{R} - \{0\}$. The positive definite spherical functions of $(SL_2(\mathbb{C}), SU(2))$ are $\left\{\phi_{\mathbb{C},z} \colon z \in i\mathbb{R} \text{ or } z \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}$, where

$$\phi_{\mathbb{C},z}(d(a)) \coloneqq \int_0^{\frac{\pi}{2}} (a^2\cos^2(\theta) + a^{-2}\sin^2(\theta))^{-1-2z}\sin(2\theta)\, d\theta.$$

Recall that, for $F \in \{\mathbb{R}, \mathbb{C}\}$, $\phi_{F,\frac{1}{2}} = \phi_{F,-\frac{1}{2}} = 1$ and that $\phi_{F,z} = \phi_{F,w}$ if and only if $z = -w$. Because $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ have the Howe–Moore Property, $\phi_{\mathbb{R},z}$ and $\phi_{\mathbb{C},z}$ vanish at infinity for every $z \in i\mathbb{R}$ and every $z \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Moreover, since the spherical representations are irreducible, every matrix coefficient of $\pi_{\phi_{\mathbb{R},z}}$ and of $\pi_{\phi_{\mathbb{C},z}}$ can be approximated, uniformly on compact subsets, by linear combinations of left and right translates of $\phi_{\mathbb{R},z}$ and $\phi_{\mathbb{C},z}$, respectively. Hence to obtain estimates on the decay of the matrix coefficients of the spherical representations of $(SL_2(\mathbb{R}), SO(2))$ and $(SL_2(\mathbb{C}), SU(2))$, it is enough to compute estimates for $\phi_{\mathbb{R},z}$ and $\phi_{\mathbb{C},z}$.

The following lemma and the subsequent proposition are generalizations of [HT92, Prop. 3.1.5].

Lemma 7.1.1. *For $a \geq 1$, $n \geq 0$ and $0 < s \leq \frac{n+1}{2}$*

$$\int_0^{\frac{\pi}{2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-s} \theta^n\, d\theta \leq \begin{cases} \pi\log(a) & \text{for } a \text{ large enough if } s = \frac{n+1}{2}, \\ \frac{\pi^{n+1}}{2^{n+1}(n+1-2s)} & \text{for } a \text{ large enough if } 0 < s < \frac{n+1}{2}. \end{cases}$$

Proof. By using the Taylor series of the cosinus one can show that $\cos(\theta) \geq 1 - \frac{\theta^2}{2}$ for all $\theta \in \mathbb{R}$. Hence

$$\cos^2(\theta) = \frac{1}{4}(e^{i\theta} + e^{-i\theta})^2 = \frac{1}{4}(2 + 2\cos(2\theta)) \geq \frac{1}{2} + \frac{1}{2}(1 - 2\theta^2) = 1 - \theta^2. \tag{7.1}$$

Moreover, one has the inequality

$$\frac{\theta}{\sin(\theta)} \leq \frac{\pi}{2} \quad \forall 0 < \theta \leq \frac{\pi}{2}. \tag{7.2}$$

Thus

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-s} \theta^n \, d\theta \\
&\quad = \int_0^{\frac{1}{a^2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-s} \theta^n \, d\theta + \int_{\frac{1}{a^2}}^{\frac{\pi}{2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-s} \theta^n \, d\theta \\
&\quad \leq \int_0^{\frac{1}{a^2}} \left|\frac{\cos^2(\theta)}{a^4}\right|^{-s} \theta^n \, d\theta + \int_{\frac{1}{a^2}}^{\frac{\pi}{2}} |\sin^2(\theta)|^{-s} \theta^n \, d\theta \\
&\quad \leq (1-a^{-4})^{-s} a^{4s} \int_0^{\frac{1}{a^2}} \theta^n \, d\theta + \left(\frac{\pi}{2}\right)^{2s} \int_{\frac{1}{a^2}}^{\frac{\pi}{2}} \theta^{n-2s} \, d\theta \\
&\quad = (1-a^{-4})^{-s} \frac{1}{n+1} a^{2(2s-n-1)} + \left(\frac{\pi}{2}\right)^{2s} \int_{\frac{1}{a^2}}^{\frac{\pi}{2}} \theta^{n-2s} \, d\theta \\
&\quad = \begin{cases} \frac{1}{n+1}(1-a^{-4})^{-\frac{n+1}{2}} + \left(\frac{\pi}{2}\right)^{n+1} \Big(\log(\frac{\pi}{2}) + 2\log(a) \Big) & \text{if } s = \frac{n+1}{2}, \\ \frac{1}{n+1}(1-a^{-4})^{-s} a^{2(2s-n-1)} + \frac{1}{n+1-2s} \left(\frac{\pi}{2}\right)^{2s} \Big((\frac{\pi}{2})^{n+1-2s} - a^{2(2s-n-1)} \Big) & \text{if } 0 < s < \frac{n+1}{2}. \end{cases} \\
&\quad \sim \begin{cases} \frac{\pi^{n+1}}{2^n} \log(a) & \text{for } a \text{ large enough if } s = \frac{n+1}{2}, \\ \frac{\pi^{n+1}}{2^{n+1}(n+1-2s)} & \text{for } a \text{ large enough if } 0 < s < \frac{n+1}{2}. \end{cases}
\end{aligned}$$

For the last step it was used that $2(2s-n+1) < 0$ for all $0 < s < \frac{n+1}{2}$. □

Proposition 7.1.2. *1) For all $a \geq 1$ large enough*

$$|\phi_{\mathbb{R},it}(d(a))| \leq 2\frac{\log(a)}{a} \quad \forall t \in \mathbb{R}, \tag{7.3}$$

and

$$|\phi_{\mathbb{R},s}(d(a))| \leq \frac{1}{2|s|} \frac{1}{a^{1-2|s|}} \quad \forall -\frac{1}{2} < s < \frac{1}{2},\ s \neq 0. \tag{7.4}$$

2) For all $a \geq 1$ large enough

$$|\phi_{\mathbb{C},it} d((a))| \leq \pi^2 \frac{\log(a)}{a^2} \quad \forall t \in \mathbb{R}, \tag{7.5}$$

and

$$|\phi_{\mathbb{C},s}(d(a))| \leq \frac{\pi^2}{8|s|} \frac{1}{a^{2(1-2|s|)}} \quad \forall -\frac{1}{2} < s < \frac{1}{2},\ s \neq 0. \tag{7.6}$$

Proof. Recall that, for $F \in \{\mathbb{R}, \mathbb{C}\}$ and $z, w \in \mathbb{C}$, $\phi_{F,z} = \phi_{F,w}$ if and only if $z = -w$.

1) Suppose that $0 \leq \mathrm{Re}(z) < \frac{1}{2}$. Then

$$\begin{aligned}
|\phi_{\mathbb{R},z}(d(a))| = |\phi_{\mathbb{R},-z}(a)| &\leq \frac{1}{2\pi a^{1-2\,\mathrm{Re}(z)}} \int_0^{2\pi} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-\frac{1}{2}+\mathrm{Re}(z)} d\theta \\
&\leq \frac{2}{\pi a^{1-2\,\mathrm{Re}(z)}} \int_0^{\frac{\pi}{2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-\frac{1}{2}+\mathrm{Re}(z)} d\theta.
\end{aligned}$$

Then (7.3) and (7.4) follow from Lemma 7.1.1.

2) By the Taylor expansion of sinus, for all $\theta \geq 0$ $\sin(\theta) \leq \theta$. Then, for $0 \leq \mathrm{Re}(z) < \frac{1}{2}$,

$$|\phi_{\mathbb{C},z}(d(a))| = |\phi_{\mathbb{C},-z}(a)| \leq \frac{1}{a^{2-4\,\mathrm{Re}(z)}} \int_0^{\frac{\pi}{2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-1+2\,\mathrm{Re}(z)} \sin(2\theta)\, d\theta$$
$$\leq \frac{2}{a^{2-4\,\mathrm{Re}(z)}} \int_0^{\frac{\pi}{2}} \left|\frac{\cos^2(\theta)}{a^4} + \sin^2(\theta)\right|^{-1+2\,\mathrm{Re}(z)} \theta\, d\theta.$$

Then (7.5) and (7.6) follow from Lemma 7.1.1.

□

For quantitative estimates for the matrix coefficients of spherical representations of $(SL_2(F), SL_2(\mathcal{O}))$ over a local field F, see Proposition 5.5.18.

7.2 Upper Bounds for $SL_d(F)$ for $d \geq 3$

The goal of this section is to find uniform pointwise upper bounds for the matrix coefficients of unitary representations of $SL_d(F)$ where $d \geq 3$. Let $SL_2(F) \ltimes F^2$ be the semidirect product of $SL_2(F)$ and F^2, where the group law is defined by

$$(a, u) \cdot (b, v) = (ab, u + a \cdot v) \quad \forall a, b \in SL_2(F)\ \forall u, v \in F^2.$$

The following result is a special case of [Oh02, Prop. 4.4]. Its proof is fairly involved in the more advanced theory of unitary representations. Hence the proof is omitted here.

Theorem 7.2.1. *Suppose $(\pi, \mathcal{H}_\pi)$ is a unitary representation of $SL_2(F) \ltimes F^2$ without non-zero F^2-invariant vectors. Then $\left(\mathrm{Res}^{SL_2(F)\ltimes F^2}_{SL_2(F)}(\pi), \mathcal{H}_\pi\right)$ is weakly contained in the regular representation $\lambda_{SL_2(F)}$.*

Remark. A beautiful proof of Theorem 7.2.1 for the special case of $F = \mathbb{R}$ can be found in [HT92, Thm. 3.3.1]. There basic elements of the spectral theory of self-adjoint bounded linear operators and the geometry of $\mathbb{R}^2$ are used to obtain estimates on the diagonal matrix coefficients of $\left(\mathrm{Res}^{SL_2(F)\ltimes F^2}_{SL_2(F)}(\pi), \mathcal{H}_\pi\right)$ on $\widetilde{A}_2$. Theorem [HT92, Thm. 3.3.1] is then obtained by using the Cartan integral formula for $SL_2(\mathbb{R})$ (see Theorem C.2.1).
[Wan14, Prop. 7.1] is a generalized version of [HT92, Thm. 3.3.1] and Theorem 7.2.1 can be given as a corollary of it. The proof of [Wan14, Prop. 7.1] however is too long and involved for it to be reproduced here.

Henceforth, let F be any local field. If $F \in \{\mathbb{R}, \mathbb{C}\}$, write $K_d(\mathbb{R}) \coloneqq SO(d)$ or $K_d(\mathbb{C}) \coloneqq SU(d)$, respectively. If F is non-archimedean with integer ring $\mathcal{O}$, write $K_d(F) \coloneqq SL_d(\mathcal{O})$. Moreover, if $F \in \{\mathbb{R}, \mathbb{C}\}$, write $D_d(F) \coloneqq \widetilde{A}_d$ and $D_d^+(F) \coloneqq \widetilde{A}_d^+$. If F is non-archimedean, write $D_d(F) \coloneqq \widetilde{T}_d(F)$ and $D_d^+(F) \coloneqq \widetilde{T}_d^+(F)$. By the Cartan decomposition in Theorem 2.3.1 and Theorem 2.3.2, $SL_d(F) = K_d(F)D_d^+(F)K_d(F)$. To simplify notation, write for the Harish-Chandra function $\Xi_{SL_d(F)} = \Xi_{(SL_d(F),K_d(F))}$, and write

$$d(a_1, a_2, \ldots, a_{d-1}) = diag\Big(a_1, a_2, \ldots, a_{d-1}, \prod_{1\leq i\leq d-1} a_i^{-1}\Big) \in D_d(F)$$

for $a_1, a_2, \ldots, a_{d-1} \in F - \{0\}$.

The following corollary is an immediate consequence of Theorem 7.2.1.

Corollary 7.2.2. *Suppose $(\pi, \mathcal{H}_\pi)$ is a unitary representation of $SL_d(F) \ltimes F^2$ such that $\mathcal{H}_\pi$ does not have non-zero F^2-invariant vectors. Then $\left(\mathrm{Res}^{SL_d(F)\ltimes F^2}_{SL_2(F)}(\pi), \mathcal{H}_\pi\right)$ is $(K_2(F), \Xi_{SL_2(F)})$-bounded.*

The key step in obtaining uniform pointwise upper bounds for matrix coefficients of unitary representations of $SL_d(F)$ for $d \geq 3$ is the following proposition. It is a generalized version of [HT92, Thm. 3.3.12] and its proof works in the same way. For $1 \leq i, j \leq d$ let $E_{ij} \in F^{d\times d}$ be the elementary matrix given by

$$(E_{ij})_{kl} \coloneqq \begin{cases} 1 & \text{if } i = k, j = l; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.2.3. *Let $d \geq 3$, and suppose that $(\pi, \mathcal{H}_\pi)$ is a unitary representation of $SL_d(F)$ without non-zero $SL_d(F)$-invariant vectors. Then for all $K_d(F)$-finite vectors $u, v \in \mathcal{H}_\pi$ and $a = diag(a_1, a_2, \ldots, a_d) \in D_d(F)$,*

$$\begin{aligned} &|\langle \pi(a)u, v\rangle| \\ &\quad \leq \dim\langle K_d(F)\cdot u\rangle^{\frac{1}{2}} \dim\langle K_d(F)\cdot v\rangle^{\frac{1}{2}} \|u\|\|v\| \min_{i\neq j} \Xi_{SL_2(F)}\left(d\left(\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}}\right)\right). \end{aligned} \tag{7.7}$$

Thus $(\pi, \mathcal{H}_\pi)$ is $(K_d(F), \Psi_d)$-bounded, where $\Psi_d \colon SL_d(F) \mapsto \mathbb{R}_+$ is given by

$$\Psi_d(k_1 a k_2) \coloneqq \min_{i\neq j} \Xi_{SL_2(F)}\left(d\left(\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}}\right)\right).$$

Proof. For $1 \leq i, j \leq d$, let $m_{ij}^{\pm} \colon SL_2(F) \ltimes F^2 \to SL_d(F)$ be the embeddings given by

$$m_{ij}^{+}(g, x) \coloneqq I_d + (g_{11} - 1)E_{ii} + (g_{22} - 1)E_{jj} + g_{12}E_{ij} + g_{21}E_{ji} + x_1 E_{ij+1} + x_2 E_{jj+1},$$

and

$$m_{ij}^{-}(g, x) \coloneqq I_d + (g_{11} - 1)E_{ii} + (g_{22} - 1)E_{jj} + g_{12}E_{ij} + g_{21}E_{ji} + x_1 E_{ii-1} + x_2 E_{ji-1},$$

where $g \in SL_2(F)$ and $x \in F^2$. By straightforward computations, one can check that m_{ij}^{+} and m_{ij}^{-} are group homomorphisms. Denote the embedded copies of $SL_2(F) \ltimes F^2$ and of $SL_2(F)$ in $SL_d(F)$ by $(SL_2(F) \ltimes F^2)_{ij}^{\pm}$ and $SL_2(F)_{ij}$. Recall that $SL_d(F)$ has the Howe–Moore Property (see Theorem 2.3.5). Because $(\pi, \mathcal{H}_\pi)$ does not have non-zero $SL_d(F)$-invariant vectors, its matrix coefficients vanish at infinity, and hence the unitary representations $\left(\mathrm{Res}^{SL_d(F)}_{(SL_2(F)\ltimes F^2)_{ij}^{\pm}}(\pi), \mathcal{H}_\pi\right)$ of $SL_2(F) \ltimes F^2$ have no non-zero F^2-invariant vectors. By Corollary 7.2.2, this implies that the unitary representations $\left(\mathrm{Res}^{SL_d(F)}_{SL_2(F)_{ij}^{\pm}}(\pi), \mathcal{H}_\pi\right)$ of $SL_2(F)$ are $(K_2(F), \Xi_{SL_2(F)})$-bounded.

Next take $a = diag(a_1, a_2, \ldots, a_d) \in D_d(F)$. Write $a = a^{(ij)}\widehat{a^{(ij)}}$, with

$$a^{(ij)} = I_d + \left(\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}} - 1\right)E_{ii} + \left(\left[\frac{a_j}{a_i}\right]^{\frac{1}{2}} - 1\right)E_{jj} \in D_2(F)_{ij}$$

and

$$\widehat{a^{(ij)}} = a + ((a_i a_j)^{\frac{1}{2}} - a_i)E_{ii} + ((a_i a_j)^{\frac{1}{2}} - a_j)E_{jj}.$$

Note that for every $k \in K_2(F)_{ij}$, $k\widehat{a^{(ij)}} = \widehat{a^{(ij)}}k$. Moreover, because $K_2(F)_{ij} < K_d(F)$, one has that $\dim\langle K_2(F)_{ij}\cdot v\rangle \leq \dim\langle K_d(F)\cdot v\rangle$ for every $K_d(F)$-finite $v \in \mathcal{H}_\pi$. Thus, for every $K_d(F)$-finite $u, v \in \mathcal{H}_\pi$

$$\begin{aligned} \langle \pi(a)u, v\rangle &= \langle \pi(a^{(ij)})\pi(\widehat{a^{(ij)}})u, v\rangle \\ &\leq \dim\langle K_2(F)_{ij}\cdot u\rangle^{\frac{1}{2}} \dim\langle K_2(F)_{ij}\cdot v\rangle^{\frac{1}{2}} \|\pi(\widehat{a^{(ij)}})u\|\|v\| \Xi_{SL_2(F)}(d(a^{(ij)})) \\ &\leq \dim\langle K_d(F)\cdot u\rangle^{\frac{1}{2}} \dim\langle K_d(F)\cdot v\rangle^{\frac{1}{2}} \|u\|\|v\| \Xi_{SL_2(F)}\left(d\left(\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}}\right)\right). \end{aligned} \tag{7.8}$$

Since (7.8) holds for every $1 \leq i, j \leq d$, (7.7) follows.

Finally observe that the function Ψ_d is well-defined due to the Cartan decomposition. □

By Proposition 5.6.7 and Proposition 7.1.2, for every $\epsilon > 0$

$$|a|^{-1} \leq \Xi_{SL_2(F)}(d(a)) \leq |a|^{-1+\epsilon} \quad \forall a \in F.$$

Hence for every $\epsilon > 0$

$$\Psi_d(diag(a_1, a_2, \ldots, a_d)) \leq \min_{i \neq j} \left(\frac{a_i}{a_j}\right)^{\frac{-1+\epsilon}{2}} \quad \forall diag(a_1, a_2, \ldots, a_d) \in D_d(F).$$

By Lemma C.1.1 the Haar measure on $D_d(F)$ is

- $\frac{da_1}{|a_1|}\frac{da_2}{|a_2|}\cdots\frac{da_{d-1}}{|a_{d-1}|}$, for $d(a_1, a_2, \ldots, a_{d-1}) \in D_d(F)$ and with da_i the Lebesgue measure on $\mathbb{R}$, if $F \in \{\mathbb{R}, \mathbb{C}\}$;
- the $(d-1)$-fold product $\mu^{\times(d-1)}(k_1, k_2, \ldots, k_{d-1})$ of the counting measure on $\mathbb{Z}$, for $d(a_1, a_2, \ldots, a_{d-1}) = d(\mathfrak{p}^{k_1}, \mathfrak{p}^{k_2}, \ldots, \mathfrak{p}^{k_{d-1}}) \in D_d(F)$, if F is non-archimedean with uniformizing element $\mathfrak{p}$.

For $1 \leq p < \infty$ let $L^p(D_d^+(F))$ be the space of all functions on $D_d^+(F)$ which are L^p with respect to the restriction of the Haar measure of $D_d(F)$ to $D_d^+(F)$.

Lemma 7.2.4. *For $s \in \mathbb{R}$ with $s \neq 0$ and for $a_0 \in F$ with $|a_0| > 1$, define the functions $f_{s,a_0}^{\pm}: D_2^+(F) \to \mathbb{R}$,*

$$f_{s,a_0}^+(d(a)) \coloneqq |a|^s \chi_{\{x \in F : 1 \leq |x| \leq |a_0|\}}(a), \quad f_{s,a_0}^-(d(a)) \coloneqq |a|^s (1 - \chi_{B(0,|a_0|)}(a)).$$

Then f_{s,a_0}^+ lies in $L^1(D_d^+)$, and there is a constant $c(s) > 0$ such that

$$\int_{D_2^+(F)} f_{s,a_0}^+(x)\, dx \leq c(s) f_{s,a_0}^+(a_0).$$

Moreover, f_{s,a_0}^- lies in $L^1(D_d^+(F))$ if and only if $s < 0$.

Proof. If $F \in \{\mathbb{R}, \mathbb{C}\}$, then

$$\int_{D_2^+(F)} f_{s,a_0}^+(x)\, dx = \int_1^{a_0} |a|^s \frac{da}{|a|} = \int_1^{a_0} a^{s-1}\, da = \frac{1}{s}(a_0^s - 1) \leq \frac{1}{s} f_{s,a_0}^+(a_0),$$

and

$$\int_{D_2^+(F)} f_{s,a_0}^-(x)\, dx = \int_{a_0}^{\infty} |a|^s \frac{da}{|a|} = \int_{a_0}^{\infty} a^{s-1}\, da \begin{cases} = \infty & \text{if } s \geq 0, \\ < \infty & \text{if } s < 0. \end{cases}$$

If F is non-archimedean, then

$$\int_{D_2^+(F)} f_{s,a_0}^+(x)\, dx = \sum_{0 \leq k \leq \log_q(|a_0|)} q^{sk} = \frac{q^{s(\log_q(|a_0|)+1)} - 1}{q^s - 1} \leq \frac{q^s}{q^s - 1} f_{s,a_0}^+(a_0),$$

and

$$\int_{D_2^+(F)} f_{s,a_0}^-(x)\, dx = \sum_{\log_q(|a_0|) \leq k} q^{sk} \begin{cases} = \infty & \text{if } s \geq 0, \\ < \infty & \text{if } s < 0. \end{cases}$$

This completes the proof. □

To give uniform upper bounds for the matrix coefficients introduce the following notions:

Definition 7.2.5. Let G be locally compact. For $p \geq 1$, a unitary representation $(\pi, \mathcal{H}_\pi)$ is said to be *strongly L^p* if there is a dense subset $U \subset \mathcal{H}_\pi$ such that for every $u \in U$ $\phi_{u,u} \in L^p(G)$.

Definition 7.2.6. Let $p(G)$ be the smallest non-negative real number such that for all $q > p(G)$ a unitary representation of G without non-zero G-invariant vectors is strongly L^q.

Suppose that $K < G$ is a compact subgroup. Let $p_K(G)$ be the non-negative real number such that if $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G without non-zero G-invariant vectors, the matrix coefficients with respect to K-finite vectors of π lie in $L^q(G)$ for all $q > p_K(G)$.

By the Peter-Weyl Theorem, if $K < G$ is compact, then $p(G) \leq p_K(G)$. In view of Kazhdan's Property (T), one has

Proposition 7.2.7. *Let G be a locally compact group with a compact subgroup K and an Iwasawa decomposition $G = KP$. Suppose $\Xi_{(G,K)} \neq 1$. If $p(G) < \infty$, then G has Property (T).*

Proof. Let $n \geq 1$ be such that $p(G) \leq 2n$. Then, by Corollary 7.2.2, every unitary representation without trivial subrepresentations is $\left(K, \Xi_{(G,K)}^{\frac{1}{n}}\right)$-bounded. By Corollary 4.5.12, G has Property (T). □

One is now equipped to prove

Proposition 7.2.8. *For every $d \geq 3$ $p_{K_d(F)} \leq 2(d-1)$.*

Proof. Let $s > 0$ be a positive real number. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of $SL_d(F)$ without trivial subrepresentations. By Proposition 7.2.3, for every $a = diag(a_1, a_2, \ldots, a_d) \in D_d^+(F)$ and every K_d-finite vectors $u, v \in \mathcal{H}_\pi$

$$|\langle \pi(a)u, v\rangle| \leq \dim\langle K_d(F)\cdot u\rangle^{\frac{1}{2}} \langle K_d(F)\cdot v\rangle^{\frac{1}{2}} \|u\|\|v\|\Psi_d(a).$$

By the Peter-Weyl Theorem, the space of $K_d(F)$-finite vectors in $\mathcal{H}_\pi$ is dense. Thus it is enough to find the minimal real number $s > 0$ such that $\Psi_d \in L^{s+\epsilon}(SL_d(F))$ for every $\epsilon > 0$. For this note that for all $a = diag(a_1, a_2, \ldots, a_d) \in D_d^+(F)$ and $0 < \epsilon < 1$

$$\begin{aligned}\Psi_d(a) &= \min_{i\neq j} \Xi_{SL_2(F)}\left(\left[\frac{a_i}{aj}\right]^{\frac{1}{2}}\right) \leq \min_{i\neq j} \left|\frac{a_i}{a_j}\right|^{\frac{1}{2}(-1+\epsilon)} \\ &= \left|\frac{a_1}{a_d}\right|^{\frac{1}{2}(-1+\epsilon)} = \left(|a_1|^2 \prod_{2\leq i\leq d-1} |a_i|\right)^{\frac{1}{2}(-1+\epsilon)}.\end{aligned}$$

Recall from Theorem C.2.1 the integral formulas for the Cartan decomposition and the estimates for the density $\delta_{F,d}\colon D_d^+(F) \to \mathbb{R}$:

$$\delta_{F,d}(a) \leq c \prod_{1\leq i<j\leq d} |a_i a_j^{-1}| = c \prod_{1\leq i\leq d-1} |a_i|^{2(d-i)},$$

where $c > 0$ is some constant. It follows that

$$\begin{aligned}\int_{SL_2(F)} \Psi_d^s(x)\,dx &= \int_{K_d(F)} \int_{D_d^+(F)} \int_{K_d(F)} \Psi_d^s(k_1 a k_2)\delta_{F,d}(a)\,dk_1\,da\,dk_2 = \int_{D_d^+(F)} \Psi_d^s(a)\delta_{F,d}(a)\,da \\ &\leq c\int_1^\infty \int_0^{a_1} \int_0^{a_2} \cdots \int_0^{a_{d-1}} \left(|a_1|^2 \prod_{2\leq i\leq d-1} |a_i|\right)^{\frac{s}{2}(-1+\epsilon)} \left(\prod_{1\leq i\leq d-1} |a_i|^{2(d-i)}\right) da_1\,da_2\cdots da_{d-1} \\ &\leq c(s,d)\int_1^\infty \left(|a_1|^2 \prod_{2\leq i\leq d-1} |a_1|\right)^{\frac{s}{2}(-1+\epsilon)} \left(\prod_{1\leq i\leq d-1} |a_1|^{2(d-i)}\right) da_1 \\ &= c(s,d)\int_1^\infty |a_1|^{\frac{ds}{2}(-1+\epsilon)} |a_1|^{2d(d-1)-d(d-1)}\,da_1 = c(s,d)\int_1^\infty |a_1|^{d(\frac{s}{2}(-1+\epsilon)+d-1)}\,da_1,\end{aligned}$$

where $c(s,d) > 0$ is a constant depending on s and d. Hence, by Lemma 7.2.4, $\Psi_d \in L^s(SL_d(F))$ if and only if

$$\frac{s}{2}(1-\epsilon) > d-1 \quad \forall \epsilon > 0,$$

or equivalently

$$s > \frac{2(d-1)}{1-\epsilon}.$$

Therefore, for every $\epsilon > 0$, $\Psi_d \in L^{2(d-1)+\epsilon}(SL_d(F))$. □

By Corollary 4.4.8 and by Proposition 7.2.8, every unitary representation of $SL_d(F)$ without trivial subrepresentations is $\left(K_d(F), \Xi_{SL_d(F)}^{\frac{1}{d-1}}\right)$-bounded.

Corollary 7.2.9. *For every local field F and every $d \geq 3$, $SL_d(F)$ has Property (T).*

Proof. This follows from Proposition 7.2.7 and from Proposition 7.2.8. □

It is true that $p_{K_d(F)}(SL_d(F)) = 2(d-1)$. By Proposition 7.2.8, $p_{K_d(F)}(SL_d(F)) \leq 2(d-1)$. For the converse it is enough to find unitary representations of $SL_d(F)$ without trivial subrepresentations such that its matrix coefficients with respect to $K_d(F)$-finite vectors are not $L^{2(d-1)}$-integrable. For the archimedean case see [Oh98, Prop. 4.4] for those representations of slowest decay, and for the non-archimedean case see [Oh02, Thm. 6.3].

In the archimedean case, Proposition 7.2.8 and Corollary 7.2.9 are special cases of more general results. They are stated here for completeness. Henceforth let G be a connected simple real Lie group with finite center. Recall that the real rank of G is the dimension of a maximal abelian subgroup of G. The real rank of G is 0 if and only if G is compact. The following theorem is a combination of Theorem 2.4.2 and Theorem 2.5.3 in [Cow79].

Theorem 7.2.10. *1) If the real rank of G is at least 2, then $p(G) < \infty$.*

2) If the real rank of G is 1, then:

i) if G is locally isomorphic to $SO(n,1)$ or $SU(n,1)$ for some $n \geq 1$, then $p(G) = \infty$, and

ii) otherwise $p(G) < \infty$.

Moreover, one has

Theorem 7.2.11. *G has Property (T) if and only if $p(G) < \infty$.*

Proof. Suppose G is locally isomorphic to $SO(n,1)$ or $SU(n,1)$ for some $n \geq 1$, and let G be a maximal compact subgroup. Using [FJK79, Thm. 1], one can show that there exists a *complementary series* of spherical representations of (G,K) whose closure in $\widehat{G}$ contains the trivial representation, that is, G does not have Property (T). The converse follows from Proposition 7.2.7 and Theorem 7.2.10. □

Appendix A

Locally Compact Groups and Homogeneous Spaces

The following is a short introduction into the theory of locally compact groups and homogeneous spaces. For a more detailed account the reader is referred to [Fol15, Chap. 2].

A.1 Basic definitions and results

Henceforth, given a group G, let $m_G: G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ be the multiplication map of G, and let $i_G: G \to G$, $g \mapsto g^{-1}$ be the inverse map of G. Notice that m_G and i_G are group homomorphisms if and only if G is abelian. Let e_G denote the identity element of G.

Definition A.1.1. A *topological group* is a group G endowed with a Hausdorff topology with respect to which the multiplication and the inverse maps are continuous. A *locally compact group* is a topological group which is locally compact as a topological space.

Recall that a topological space X is *locally compact* if every point in X has an open neighbourhood whose closure is compact.

Definition A.1.2. A *topological field* is a field endowed with a topology with respect to which the additive and the multiplicative groups are topological groups. A *local field* is a non-discrete topological field which is locally compact as a topological space.

Given a topological group G and $g \in G$, let $l_g: G \to G$, $x \mapsto gx$, and $r_g: G \to G$, $x \mapsto xg$. Note that for each $g \in G$, l_g and r_g are homeomorphisms on G.

From now on if $\phi: G \to H$ is a group homomorphism between two topological groups G and H, then ϕ is assumed to be continuous. An isomorphism is assumed to be an invertible group homomorphism which is a homeomorphism. As will be seen in Theorem A.1.10, every continuous invertible group homomorphism between two locally compact seperable groups is an isomorphism.

Examples A.1.3. 1) Every finite group is a locally compact group with respect to the discrete topology.

2) Suppose F is a topological field. The general linear group $GL_d(F)$, for $d \geq 1$, endowed with the topology it inherits as an open subset of the vector space F^{d^2}, is a topological group. Moreover, every subgroup of $GL_d(F)$ is a topological group with respect to the inherited subspace topology; e.g. $SL_d(F)$. Concerning the question whether $GL_d(F)$ or $SL_d(F)$ are locally compact, see Lemma A.1.4 below.

3) If G is a topological group, then every subgroup $H < G$ is a topological group with respect to the subspace topology. Moreover, if G is locally compact, then every closed subgroup $H < G$ is locally compact with respect to the subspace topology.

4) If $H \triangleleft G$ is a normal subgroup of a topological group G, then G/H is a topological group with respect to the quotient topology induced by the canonical quotient map $q\colon G \to G/H$. For more on the properties of G/H, see Lemma A.1.7 below.

Concerning the general linear groups and the special linear groups one has

Lemma A.1.4. *Suppose F is topological field. Then, for $d \geq 1$, $GL_d(F)$ and $SL_d(F)$ are locally compact if and only if F is locally compact.*

Proof. Note that $GL_1(F) = F^\times$ and $SL_1(F) = \{1\}$. Next suppose $d \geq 2$, and consider the closed subgroup

$$H := \left\{ \begin{pmatrix} 1 & 0 & \\ x & 1 & \\ & & I_{d-2} \end{pmatrix} : x \in F \right\}$$

of $SL_d(F)$, and note that it is homeomorphic to F. Observe that the vector space F^d is locally compact if and only if F is locally compact. The lemma follows because a closed subset of a locally comapct topological space is locally compact with respect to the subspace topology. □

Henceforth let G be a topological group. Concerning subgroups of G one has

Lemma A.1.5. *An open subgroup $H < G$ is closed.*

Proof. Take a set $R \subset G$ of representatives of the equivalence classes of G/H such that $e_G \in R$. Then $\{gH\colon g \in R\}$ is a partition of G. Because l_g is a homeomorphism on G, gH is open for every $g \in R$. This implies that

$$H = G - \bigcup_{g \in R - \{e_G\}} gH$$

is closed. □

The following are two useful facts about open neighbourhoods of the identity element in G.

Lemma A.1.6. *1) For every open neighbourhood $U \subset G$ of e_G there is an open neighbourhood $V \subset G$ of e_G such that $V \subset U$ and $V^{-1} = V$.*

2) If G is connected, then for every open neighbourhood $U \subset G$ of e_G one has $G = \bigcup_{k \in \mathbb{Z}} U^k$.

Proof. 1) Because i_G is a homeomorphism and fixes e_G, $U^{-1} = i_G(U)$ is open and contains e_G. The claim is proven by setting $V := U \cap U^{-1}$.

2) Note that $H := \bigcup_{k \in \mathbb{Z}} U^k$ is an open subgroup of G. Hence, by Lemma A.1.5, H is also closed and is therefore a connected component of G. Because G is assumed to be connected, $G = H$.

□

If $U \subset G$ is an open neighbourhood of e_G such that $U^{-1} = U$, then it is called *symmetric*.

Given a subgroup $H < G$, endow the quotient space G/H with the quotient topology, that is, the topology such that $U \subset G/H$ is open if and only if $q^{-1}(U) \subset G$ is open, where $q\colon G \to G/H$ denotes the canonical quotient map.

Lemma A.1.7. *Suppose $H < G$.*

1) The canonical quotient map $q\colon G \to G/H$ is open.

2) If H is closed, then G/H is Hausdorff.

3) If G is locally compact, then G/H is also locally compact.

4) If G is locally compact, then for every compact subset $K \subset G/H$ there is a compact subset $L \subset G$ such that $K = q(L)$.

Proof. For parts 1), 2) and 3) see [Fol15, Prop. 2.2], and for 4) see[Fol15, Lem. 2.48]. □

Recall the definition of a (algebraic) group action.

Definition A.1.8. A *(continuous) group action* of G on a topological space X is a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ such that for every $x \in X$ $e_G \cdot x = x$ and for every $g_1, g_2 \in G$ and every $x \in X$ $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$. If (X, d) is a metric space, then an action of G on (X, d) is *isometric* if $d(g \cdot x, g \cdot y) = d(x, y)$ for every $g \in G$ and every $x, y \in G$.

Notice that if one has an action of G on a topological space X, then for every $g \in G$ the map $x \mapsto g \cdot x$ is a homeomorphism on X, with inverse map $x \to g^{-1} \cdot x$. Hence an action of G on X gives an algebraic group homomorphism $G \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ is the group of homeomorphisms on X with the composition as the group law. An isometric action of G on a metric space (X, d) gives an algebraic group homomorphism $G \to \mathrm{Iso}(X)$, where $\mathrm{Iso}(X) < \mathrm{Homeo}(X)$ is the subgroup of invertible isometries of (X, d), that is,

$$\mathrm{Iso}(X) \coloneqq \{f \colon X \to X \colon f \text{ is invertible, } d(f(x), f(y)) = d(x, y)\ \forall x, y \in X\}.$$

Concerning a topology on $\mathrm{Homeo}(X)$ and $\mathrm{Iso}(X)$ with respect to which they are topological group, see Section A.2.

Definition A.1.9. An action of G on a topological space X is called *transitive* if for every $x, y \in X$ there exists $g \in G$ such that $x = g \cdot y$.

Theorem A.1.10 ([Fol15, Prop. 2.46])**.** *Suppose X is a locally compact Hausdorff space, and suppose that G is a locally compact separable group, which acts transitively on X. Then for every $x_0 \in X$ one has that $G/\mathrm{stab}_G(x_0)$ and X are homeomorphic via $[g] \mapsto g \cdot x_0$.*

Proof. The map $[g] \mapsto g \cdot x_0$ is continuous by the construction of $G/\mathrm{stab}_G(x_0)$. It remains to show that the map is open. For this it is enough to show that for every open neighbourhood $U \subset G$ of $e_G \in G$ the subset $U \cdot x_0$ is open in X. One can assume that $\overline{U} \subset G$ is compact. Then, by the continuity of the action and because X is Hausdorff, $\overline{U} \cdot x_0$ is a compact and thus closed subset of X. Since G acts transitively on X, there is a countable subset $S \subset G$ such that $X = \bigcup_{g \in S} (g\overline{U}) \cdot x_0$. By the Baire Category Theorem (see [Rud91, Thm. 2.2]), this implies that $(g\overline{U}) \cdot x_0$ has a non-empty interior for some and thus for all $g \in S$. Next let $W \subset U$ be an open neighbourhood of e_G such that $W^{-1} = W$ and $W^2 \subset U$. By the above argument, both $\mathrm{int}(U \cdot x_0)$ and $\mathrm{int}(W \cdot x_0)$ are non-empty. Take $h \in W$. Then

$$x_0 \in h^{-1} \cdot \mathrm{int}(W \cdot x_0) \subset \mathrm{int}(W^2 \cdot x_0) \subset \mathrm{int}(U \cdot x_0),$$

which completes the proof. □

Corollary A.1.11. *If G and H are locally compact separable groups and if $\phi \colon G \to H$ is an invertible group homomorphism, then it is an isomorphism.*

Proof. Note that ϕ gives a group action of G on H via $G \times H \to H$, $(g, h) \to \phi(g)h$. Because $\ker \phi = \mathrm{stab}_G(e_H) = \{e_G\}$ and because the action is transitive, it follows from Theorem A.1.10 that $g \mapsto \phi(g)e_H$ and thus ϕ are homeomorphims. □

The following results is recorded here for use in Chapter 5. Recall that a topological space X is first-countable if every element of X has a countable neighbourhood basis. If X is second countable, then it is also first countable.

Theorem A.1.12 ([CdlH16, Birkhoff-Kakutani, Thm. 2.B.2])**.** *A topological group is metrizable if and only if it is first countable.*

The section is concluded with

Definition A.1.13. A function f on G is said to be *left (respectively right) uniformly continuous* if $\|L_g f - f\|_{\sup} \to 0$ (respectively $\|R_g f - f\|_{\sup} \to 0$) as $g \to e_G$.

Let $C_c(G)$ denote the set of all compactly supported continuous complex-valued functions on G. The proof of the following proposition is omitted.

Proposition A.1.14 ([Fol15, Prop. 2.6])**.** *Every $f \in C_c(G)$ is left and right uniformly continuous.*

A.2 The compact-open topology

Suppose that X is a topological space, and let $\mathrm{Homeo}(X)$ be the group of homeomorphisms on X, together with the composition as the group law. The goal is to endow the abstract group $\mathrm{Homeo}(X)$ with a topology with respect to which it is a topological group. Let $C(X)$ be set of all continuous functions $f\colon X \to X$.

Definition A.2.1. The *compact-open topology* on $C(X)$ is the topology generated by the subbasis $\{S(K,U)\colon K \subset X \text{ compact},\ U \subset X \text{ open}\}$, where

$$S(K,U) \coloneqq \{f \in C(X)\colon f(K) \subset U\}.$$

The following famous theorem is stated here without proof.

Theorem A.2.2 ([Are46, Thm. 3])**.** *If X is a locally compact Hausdorff space, then* $\mathrm{Homeo}(X)$ *is a topological group with respect to the compact-open topology.*

Henceforth, endow $\mathrm{Homeo}(X)$ with the compact-open topology. Moreover, if (X,d) is a metric space, endow the subgroup $\mathrm{Iso}(X)$ with the subspace topology it inherits from $\mathrm{Homeo}(X)$.

Proposition A.2.3. *Suppose X is a discrete topological space. Then for every $g \in \mathrm{Homeo}(X)$ the sets $\{U_F(g)\colon F \subset X$ is finite$\}$ form a neighbourhood basis for g, where*

$$U_F(g) \coloneqq \{h \in G\colon g(x) = h(x)\ \forall x \in F\}.$$

Proof. Note that a subset of X is compact if and only if it is finite. Thus, for every $g \in G$ and every finite subset F,

$$U_F(g) = \bigcap_{x \in F} S(\{x\}, \{g(x)\})$$

is open. Moreover, if $K \subset X$ is compact and if $U \subset V$ are open subsets of X, then $S(K,V) \subset S(K,U)$. For every finite and thus compact set $F \subset X$, $U_F(g) \subset S(F, g(F))$, and hence if $g \in S(F,U)$ with $U \subset X$ open, then $U_F(g) \subset S(F,U)$. □

A.3 The Haar measure

Henceforth let G be a locally compact group.

Definition A.3.1. A *left (respectively right) Haar measure* on G is a non-zero Radon measure μ on G such that $\mu(g \cdot B) = \mu(B)$ (respectively $\mu(B \cdot g) = \mu(B)$) for every Borel set $B \subset G$ and every $g \in G$.

The following is an easy lemma whose proof is omitted.

Lemma A.3.2 ([Fol15, Prop. 2.9])**.** *Suppose μ is a Radon measure on G, and define on G the Radon measure $\widetilde{\mu}(B) \coloneqq \mu(B^{-1})$, for Borel subsets $B \subset G$. The following are equivalent:*

1) μ is a left Haar measure on G.

2) $\widetilde{\mu}$ is a right Haar measure on G.

3) For every $f \in C_c(G)$ and every $g \in G$ one has $\int L_g f \, d\mu = \int f \, d\mu$.

The following theorem is fundamental to the theory of locally compact groups.

Theorem A.3.3 ([Fol15, Thm. 2.10])**.** *On every locally compact group there exists a left Haar measure.*

Moreover, one has

Theorem A.3.4 ([Fol15, Thm. 2.20])**.** *The left Haar measure on a locally compact group is unique up to a positive constant factor.*

Henceforth let μ denote a fixed left Haar measure on G. If G is compact, then the Haar measure μ is finite, and μ is *normalised* if $\mu(G) = 1$, that is, if μ is a probability measure.

Corollary A.3.5. *For every automorphism $\phi \in \mathrm{Aut}(G)$ the pushforward $\phi_*\mu$ is a left Haar measure on G. This gives an algebraic group homomorphism $\mathrm{mod}_G \colon \mathrm{Aut}(G) \to (\mathbb{R}_{>0}, \cdot)$ such that for all $\phi \in \mathrm{Aut}(G)$ $\mu = \mathrm{mod}_G(\phi)\phi_*\mu$.*

Proof. For every $g \in G$ and every Borel subset $B \subset G$, one has that

$$\phi_*\mu(gB) = \mu(\phi^{-1}(gB)) = \mu(\phi^{-1}(g)\phi^{-1}(B)) = \mu(\phi^{-1}(B)) = \phi_*\mu(B).$$

By Theorem A.3.4 there exists a constant $\mathrm{mod}_G(\phi) > 0$ such that $\mu = \mathrm{mod}_G(\phi)\phi_*\mu$. In particular, if $\phi, \psi \in \mathrm{Aut}(G)$, then

$$\begin{aligned} \mathrm{mod}_G(\phi \circ \psi)\mu(B) &= (\phi \circ \psi)_*\mu(B) = \psi_*(\phi_*\mu(B)) \\ &= \mathrm{mod}_G(\phi)\psi_*\mu(B) = \mathrm{mod}_G(\phi)\,\mathrm{mod}_G(\psi)\mu(B), \end{aligned}$$

for every Borel subset $B \subset G$. □

It follows from the definition of the Haar measure μ that $\mathrm{mod}_G(L_g) = 1$ for every $g \in G$. Moreover, by Theorem A.3.4, the definition of mod_G is independent from the choice of μ.

Definition A.3.6. The *(left) modular function* of G is the algebraic group homomorphism $\Delta_G \colon G \to (\mathbb{R}_{>0}, \cdot)$, $\Delta_G(g) \coloneqq \mathrm{mod}_G(R_g)$, that is,

$$\Delta_G(g) \int R_g f \, d\mu = \int f \, d\mu \quad \forall g \in G \; \forall f \in C_c(G).$$

If $\Delta_G = 1$, then G is said to be *unimodular*.

It follows from the uniqueness of the Haar measure up to positive constant factors that the definition of the modular function on G is independent from the choice of the left Haar measure. If G is unimodular, then any left Haar measure on G is also a right Haar measure.

Proposition A.3.7 ([Fol15, Prop. 2.24])**.** *The modular function of G is continuous.*

Proof. Recall that every $f \in C_c(G)$ is left and right uniformly continuous. By the continuity of the inverse map, if $g_n \to g$ in G, then $g_n^{-1} \to g^{-1}$. Let $U \subset G$ be a relatively compact open neighbourhood of g, and let $f \in C_c(G)$ with $\operatorname{supp}(f) \subset Ug^{-1}$, $f \geq 0$ and $\int f \, d\mu = 1$. Then for all $n \geq 1$ large enough, $\operatorname{supp}(f) \subset Ug_n^{-1}$ and

$$\begin{aligned} |\Delta_G(g_n) - \Delta_G(g)| \int_G f(x)\, d\mu(x) &\leq \left| \int_G f(xg_n^{-1}) - f(xg^{-1})\, d\mu(x) \right| \\ &\leq |U| \| R_{g_n^{-1}} f - R - g^{-1} f \|_\infty \to 0. \end{aligned}$$

This proves the proposition. □

Proposition A.3.8. *If G is abelian or compact, then it is unimodular.*

Proof. It is immediate that if G is abelian, then it is unimodular. If G is compact, then, by Proposition A.3.7, Δ_G is a compact subgroup of $(\mathbb{R}_{>0}, \cdot)$. The proposition follows because the trivial subgroup $(\{0\}, \cdot)$ is the only compact subgroup of $(\mathbb{R}_{>0}, \cdot)$. □

By a similar argument as in the proof of Proposition A.3.8, one can show that if $K < G$ is compact, then $\Delta_G|_K = 1$.

Examples A.3.9.

1) For every discrete group G the counting measure is a left Haar measure on G. Because the counting measure is also right-invariant, G is unimodular.

2) Suppose F is a topological field, and let μ be a left Haar measure of the additive group, which is said to be a *translation-invariant measure* on F. Because the additive group of F is abelian, it is unimodular. If $F = \mathbb{R}$, then $\mu = \mathcal{L}$ is the Lebesgue measure, and if $F = \mathbb{C}$, then $\mu = \mathcal{L} \times \mathcal{L}$. More generally, a left Haar measure of the group $(F^d, +)$ is $\mu^{\times d}$.

Henceforth let dx be a fixed left Haar measure on G.

Proposition A.3.10 ([Fol15, Prop. 2.31])**.** *For all $f \in C_c(G)$*

$$\int_G f(x) d\, x = \int_G f(x^{-1}) \Delta_G(x^{-1})\, dx. \tag{A.1}$$

Proof. Note that for all $g \in G$ and all $f \in C_c(G)$

$$\begin{aligned} \int_G L_g f(x^{-1}) \Delta_G(x^{-1})\, dx &= \int_G f((xg)^{-1}) \Delta_G(x^{-1})\, dx \\ &= \Delta_G(g^{-1}) \int_G f(x^{-1}) \Delta_G(gx^{-1})\, dx = \int_G f(x^{-1}) \Delta_G(x^{-1})\, dx. \end{aligned}$$

This implies that the functional $f \mapsto \int_G f(x^{-1}) \Delta_G(x^{-1})\, dx$ on $C_c(G)$ defines a left Haar measure on G. Thus, by Theorem A.3.4, there exists a constant $C > 0$ such that

$$\int_G f(x^{-1}) \Delta_G(x^{-1})\, dx = C \int_G f(x)\, dx \quad f \in C_c(G).$$

To show that $C = 1$, take $\epsilon > 0$ and suppose that $U \subset G$ is an open neighbourhood of e_G such that $U = U^{-1}$ and such that $|1 - \Delta_G(g)| < \epsilon$ for all $g \in G$. Let $f \in C_c(G)$ be such that $f \geq 0$, $\int_G f(x)\, dx = 1$ and $\operatorname{supp}(f) \subset U$. By taking $\frac{1}{2}(f + f^\vee)$ if necessary, suppose $f = f^\vee$. Then

$$|1 - C| = \left| \int_G f(x)(1 - \Delta_G(x^{-1}))\, dx \right| \leq \epsilon \int_G |f(x)|\, dx = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $C = 1$. □

For $1 \leq p < \infty$, let $L^p(G)$ be the space of all functions $f: G \to \mathbb{C}$ such that $\int_G |f(x)|^p \, dx < \infty$. Define on $L^p(G)$ the norm

$$\|f\|_p \coloneqq \left(\int_G |f|^p \right)^{\frac{1}{p}}.$$

For $p = \infty$, let $L^\infty(G)$ be the space of all essentially bounded functions $f: G \to \mathbb{C}$, with norm

$$\|f\|_\infty \coloneqq \inf\{C > 0 \colon |f(g)| \leq C \text{ a.e.}\}.$$

It is a well-known fact of functional analysis that for every $1 \leq p \leq \infty$ $(L^p(G), \|\cdot\|_p)$ is a Banach space.

For every $1 \leq p \leq \infty$ and every $g \in G$, L_g and R_g are linear operators on $L^p(G)$. Moreover, for every $f \in L^p(G)$ $\|L_g f\|_p = \|f\|_p$ and $\|R_g f\|_p = \Delta_G^{-\frac{1}{p}}(g)\|f\|_p$. Thus $L_g, R_g \in \mathcal{L}(L^p(G))$.

Proposition A.3.11 ([Fol15, Prop. 2.42])**.** *Suppose that $\{g_n\}_{n\geq 1} \subset G$ is a sequence converging to $g \in G$. Then for every $1 \leq p \leq \infty$ and every $f \in L^1(G)$, $\|L_{g_n} f - L_g f\|_p \to 0$ and $\|R_{g_n} f - R_g f\|_p \to 0$.*

Proof. Recall that $C_c(G)$ is dense in $L^p(G)$. Since L_{g_n}, L_g and R_{g_n}, R_g are continuous linear operators on $L^p(G)$, it is enough to show the proposition for the case of $f \in C_c(G)$. Moreover, by the same argument, it is enough to show the proposition for the case of $g = e_G$. Hence fix $f \in C_c(G)$ and suppose $g_n \to e_G$. Let U be an open neighbourhood of e_G whose closure is compact, and assume that $g_n \in U$ for all $n \geq 1$. Let $K \coloneqq \operatorname{supp}(f)\overline{U}^{-1} \cup \overline{U}\operatorname{supp}(f)$. Then, by Proposition A.2.3,

$$\|L_{g_n} f - f\|_p \leq |K|^{\frac{1}{p}} \|L_{g_n} f - f\|_\infty \to 0,$$

and

$$\|R_{g_n} f - f\|_p \leq |K|^{\frac{1}{p}} \|R_{g_n} f - f\|_\infty \to 0.$$

□

The section is concluded with introducing the notion of approximate identity, which is used in the proof of Theorem 1.4.6 and of Theorem 1.5.5.

Definition A.3.12. Let $\{U_i\}_{i\in I}$ be a basis of neighbourhoods of e_G such that $\bigcap_{i\in I} U_i = \{e_G\}$. An *approximate identity* is a collection $\{\psi_i\}_{i\in I}$ of functions $\psi_i: G \to \mathbb{C}$ such that for all $i \in I$

1) $\operatorname{supp}(\psi_i)$ is compact and contained in U_i,

2) $\psi_i \geq 0$ and $\int_G \psi_i(x)\, dx = 1$,

3) $\psi_i^\vee = \psi_i$.

The following is a nice application of Proposition A.3.11.

Proposition A.3.13 ([Fol15, Prop. 2.44])**.** *Let $\{\psi_U\}_{U\in\mathcal{U}}$ be an approximate identity. Then for every $1 \leq p \leq \infty$ and every $f \in L^p(G)$, $\psi_U * f \to f$ in $L^p(G)$.*

Proof. The case of $p = \infty$ is immediate. Suppose $p \neq \infty$. Then, by Minkowski's inequality for integrals,

$$\begin{aligned}\|\psi_U * f - f\|_p &= \left(\int_G \left| \int_G \psi_U(y) f(y^{-1}x) - \psi_U(y) f(x)\, dy \right|^p dx \right)^{\frac{1}{p}} \leq \int_G \psi_U(y) \|L_y f - f\|_p \, dy \\ &\leq \sup_{y\in U} \|L_y f - f\|_p.\end{aligned}$$

Since $U \to \{e_G\}$, the proposition follows from Proposition A.3.11. □

A.4 Homogeneous Spaces

Henceforth let G be a locally compact group.

Definition A.4.1. A *G-space* is a locally compact Hausdorff space X with an action of G. A G-space X is called *transitive* if the action of G is transitive.

Definition A.4.2. If X and Y are G-spaces, then a map $f\colon X \to Y$ is *G-equivariant* if

$$g \cdot f(x) = f(g \cdot x) \quad \forall g \in G \; \forall x \in X.$$

X and Y are *isomorphic*, written $X \cong Y$, if there is a G-equivariant homeomorphism between them.

Definition A.4.3. A *homogeneous space* is a transitive G-space such that $G/\mathrm{stab}_G(x_0) \cong X$ for every $x_0 \in X$.

Henceforth let X be a G-space. By the continuity of the action of G, for all $x \in X$ the stabilizer groups

$$\mathrm{stab}_G(x) \coloneqq \{g \in G\colon g \cdot x = x\}$$

are closed subgroups of G. By Theorem A.1.10, if G is separable, then every transitive G-space is a homogeneous space.

Remark. Notice that if X is a homogeneous space, then for every $x, y \in X$ with $g \in G$ such that $g \cdot x = y$, one has that $\mathrm{stab}_G(x) = g^{-1}\,\mathrm{stab}_G(y)g$. In particular, for every $x, y \in X$, $G/\mathrm{stab}_G(x) \cong G/\mathrm{stab}_G(y)$.

Henceforth, let $H < G$ be a closed subgroup, and let $q\colon G \to G/H$ be the canonical quotient map. Let dx and dh denote left Haar measure on G and H, respectively. The goal now is to find G-invariant Radon measures on G/H. For this define the map $P^H\colon C_c(G) \to C_c(G/H)$ by

$$P^H f(gH) \coloneqq \int_H f(gh)\, dh.$$

By the left-H-invariance of dh, $P^H f$ is well-defined. Moreover, it is easy to see that $P^H f$ is continuous with compact support contained $q(\mathrm{supp}(f))$. One has

Lemma A.4.4 ([Fol15, Prop. 2.50])**.** *For every $\phi \in C_c(G/H)$ there exists $f \in C_c(G)$ such that $\phi = P^H f$ and $\mathrm{supp}(\phi) = q(\mathrm{supp}(f))$, and if $\phi \geq 0$, then $f \geq 0$.*

Next recall the definition of absolute continuity between measures. Suppose μ is a Radon measure on G/H, and for $g \in G$ define the translate μ_g of μ by

$$\mu_g(B) \coloneqq \mu(gB) \quad \forall B \subset G/H \text{ Borel}.$$

Definition A.4.5. A Radon measure μ on G/H is *quasi-invariant* if for every $g \in G$ μ and μ_g are equivalent, that is, they are mutually strongly continuous. μ is called *strongly quasi-invariant* if it is quasi-invariant and if the map

$$(g, xH) \mapsto \frac{d\mu_g}{d\mu}(xH)$$

is continuous for every $g \in G$ and every $xH \in G/H$.

Definition A.4.6. A *rho-function* for (G, H) is a continuous function $\rho\colon G \to \mathbb{R}_+$ such that

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g) \quad \forall g \in G \; \forall h \in H.$$

Proposition A.4.7. *The pair (G, H) has a rho-function.*

Proof. See [Fol15, Prop. 2.56]. □

The following lemma gives an easy method to construct a rho-function for certain pairs (G, H).

Lemma A.4.8. *Suppose that $K, P < G$ are closed subgroups such that $G = KP$ and K is compact. Then the function $\rho\colon G \to \mathbb{R}_+$,*

$$\rho(kp) = \frac{\Delta_P(p)}{\Delta_G(p)}$$

is a rho-function.

Proof. Suppose $g = kp = k'p'$. Then $p'p^{-1} = (k')^{-1}k \in K \cap P$. Because $K \cap P$ is a compact subgroup of G, $\Delta_G|_{K\cap P} = \Delta_P|_{K\cap P} = 1$, and thus $\Delta_G(p) = \Delta_G(p')$, $\Delta_P(p) = \Delta_P(p')$. Therefore ρ is well-defined.

For $g \in G$ write $g = k(g)p(g)$, where $k(g) \in K$ and $p(g) \in P$. To show that ρ is continuous, suppose that $\{g_i\}_{i\in I}$ is a net in G which converges to $g \in G$. Then there exists a subnet $J \subset I$ such that $k(g_j) \to k_J \in K$. Then $p(g_j) = k(g_j)^{-1}g_j \to k_J^{-1}g$, and because P is closed, $k_J^{-1}g \in P$. Because the modular functions of G and P are continuous on P,

$$\rho(g_j) = \frac{\Delta_P(p(g_j))}{\Delta_G(p(g_j))} \to \frac{\Delta_P(k_J^{-1}g)}{\Delta_G(k_J^{-1}g)} = \rho(g).$$

Because every subnet of I contains a subsubnet of J such that $k(g_j) \to k_J \in K$, it follows that $\rho(g_i) \to \rho(g)$, that is, ρ is continuous. □

The main results of this section are the following two theorems.

Theorem A.4.9 (Generalized Weil Formula, [Fol15, Thm. 2.58])**.** *Suppose ρ is a rho-function of the pair (G, H). There is a strongly quasi-invariant Radon measure μ on G/H such that*

$$\int_G f(x)\rho(x)\,dx = \int_{G/H}\int_H f(xh)\,dh\,d\mu(xH) \quad \forall f \in C_c(G).$$

Moreover,

$$\frac{d\mu_g}{d\mu}(xH) = \frac{\rho(gx)}{\rho(x)} \quad \forall g, x \in G.$$

Theorem A.4.10 ([Fol15, Thm. 2.61])**.** *Every strongly quasi-invariant measure on G/H is associated to a rho-function as in Theorem A.4.9. Moreover, every two strongly quasi-invariant measures μ and μ' on G/H with associated rho-functions ρ and ρ' are strongly equivalent with Radon-Nikodym derivative*

$$\frac{d\mu}{d\mu'} = \frac{\rho}{\rho'}.$$

Corollary A.4.11 (Weil Formula, [Fol15, Thm. 2.51])**.** *A G-invariant Radon measure μ exists on G/H if and only if $\Delta_G|_H = \Delta_H$. Moreover, if such a μ exists, then it is unique up to a positive constant factor and, with the suitable constant factor, one has that*

$$\int_G f(x)\,dx = \int_{G/H}\int_H f(xh)\,dh\,d\mu(xH) \quad \forall f \in C_c(G/H).$$

Proof. Note that μ is a G-invariant Radon measure μ if and only if its associated rho-function ρ is constant. Thus, if μ is G-invariant, then

$$1 = \frac{d\mu_h}{d\mu}(e_G H) = \frac{\rho(h)}{\rho(e_G)} = \frac{\Delta_H(h)}{\Delta_G(h)}.$$

Conversely, if $\Delta_G|_H = \Delta_H$, then $\rho = 1$ is rho-function and the associated measure on G/H is G-invariant.

By Theorem A.4.10, if μ and μ' are G-invariant measures with associated rho-functions ρ and ρ', then $\frac{d\mu}{d\mu'} = \frac{\rho}{\rho'}$, which is constant because ρ and ρ are constant. Therefore every G-invariant measure on G/H is unique up to a positive constant factor. The formula follows from Theorem A.4.10. □

Corollary A.4.12. *Suppose $H \triangleleft G$. Then $\Delta_G|_H = \Delta_H$.*

Proof. By Lemma A.1.7, G/H is a locally compact group. Hence there exists a G-invariant Radon measure on G/H, and, by Corollary A.4.11, $\Delta_G|_H = \Delta_H$. □

The section is concluded with the special case where G admits an Iwasawa decomposition.

Proposition A.4.13. *Let G be a locally compact group, and suppose $G = KP$ for a compact subgroup $K < G$ and a closed subgroup $P < G$. Then with the suitable factors*

$$\int_G f(x)\,dx = \int_K \int_P f(kp)\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{-1} dp\,dk \quad \forall f \in C_c(G), \tag{A.2}$$

where dk and dp are left Haar measures on K and P, respectively.

Proof. By Lemma A.4.8, $\rho\colon G \to \mathbb{R}_+$,

$$\rho(kp) \coloneqq \frac{\Delta_P(p)}{\Delta_G(p)}$$

defines a *rho*-function on G with respect to P. Let μ be the strongly quasi-invariant measure on G/P associated to ρ. Because $\rho|_K = 1$, μ is the K-invariant measure on $G/P \cong K/(K \cap P)$, which, by Corollary A.4.11, is unique up to a constant factor. Let dk' be the normalized Haar measure on $K \cap P$. By the Generalized Weil Formula,

$$\begin{aligned}
\int_G f(x)\,dx &= \int_{G/P} \int_P f(xp)\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{-1} dp\,d\mu(x) \\
&= \int_{K/(K\cap P)} \int_P f(kp)\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{-1} dp\,d\mu(k) \\
&= \int_{K/(K\cap P)} \int_{K\cap P} \int_P f(k(k'p))\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{-1} dp\,dk'\,d\mu(k) \\
&= \int_K \int_P f(kp)\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{-1} dp\,dk,
\end{aligned}$$

which proves (A.2). □

Appendix B

Local Fields

Recall from Appendix A that a topological field F is called *local* if it is locally compact and non-discrete. If F is local, then both the additive group $(F,+)$ and the multiplicative group $F^\times = (F-\{0\},\cdot)$ are locally compact. A Haar measure of $(F,+)$ is said to be *translation-invariant*.

B.1 Fields with valuations

The main purpose of this section is to give examples of local fields. For this the notion of valuations on a field is introduced. Henceforth let F denote a field.

Definition B.1.1. A *valuation* is a real-valued fucntion $|\cdot|\colon F \to \mathbb{R}$ such that

1) $|x| \geq 0$ for all $x \in F$, and $|x| = 0$ if and only if $x = 0$,

2) $|xy| = |x||y|$ for all $x, y \in F$,

3) there exists a constant $C > 0$ such that for all $x \in F$ with $|x| \leq 1$ one has $|1 + x| \leq C$.

Some important examples of fields with a valuation are:

Examples B.1.2. 1) For any field F define the map $|\cdot|_0\colon F \to \mathbb{R}$,

$$|x|_0 \coloneqq \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that $|\cdot|_0$ is a valuation; it is called the *trivial valuation*.

2) The ordinary absolute value is a valuation on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

3) Let $p > 1$ be a prime number. For every rational number $q \in \mathbb{Q}$ write $q = \frac{m}{n}p^k$ where $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$, $n > 0$ are such that $\gcd(m,p) = \gcd(n,p) = 1$. The *p-adic valuation* is defined by $|q|_p \coloneqq p^{-k}$. It is left to the reader to verify that $|\cdot|_p$ is a valuation on $\mathbb{Q}$.

The following properties are easily verified. The following proposition is a combination of [Cas86, Chap. 2, Cor. 1], [Cas86, Chap. 2, Cor. 3] and [Cas86, Chap. 2, Lem. 1.1]. Its proof is trivial.

Proposition B.1.3. *Suppose $|\cdot|$ is a valuation on F. Then the following holds:*

1) $|\pm 1| = 1$.

2) $|x^n| = 1$ *for every* $n \geq 1$ *if and only if* $|x| = 1$.

3) For every $\lambda > 0$ $|\cdot|_\lambda \coloneqq |\cdot|^\lambda$ is a valuation on F.

Corollary B.1.4. *The trivial valuation is the only valuation on a finite field.*

Proof. The corollary follows from part 2) of Proposition B.1.3. □

Definition B.1.5. Two valuations $|\cdot|_1$, $|\cdot|_2$ on F are *equivalent* if there exists $\lambda > 0$ such that $|\cdot|_1 = |\cdot|_2^\lambda$.

Definition B.1.6. A valuation $|\cdot|$ on F satisfies the *triangle inequality* if

$$|x + y| \leq |x| + |y| \quad \forall x, y \in F.$$

Lemma B.1.7 ([Cas86, Chap. 2, Lem. 1.2])**.** *A valuation $|\cdot|$ on F satisfies the triangle inequality if and only if for all $x \in F$ with $|x| \leq 1$ one has that $|1 + x| \leq 2$.*

Corollary B.1.8. *Every valuation is equivalent to one satisfying the triangle inequality.*

If $|\cdot|$ is a valuation on F, then it induces a topology on F, whose basis are the sets $\{B(x, r) \colon x \in F, r > 0\}$, where

$$B(x, r) \coloneqq \{y \in F \colon |x - y| < r\}.$$

Note that the induced topology is Hausdorff and second countable. Moreover, it is easy to check that F is a topological field with respect to that topology.

If $|\cdot|$ satisfies the triangle inequality, then $d(x, y) \coloneqq |x - y|$ defines a metric on F.

Lemma B.1.9 ([Cas86, Lem. 3.2])**.** *Two valuations on F are equivalent if and only if they induce the same topology on F.*

Examples B.1.10. 1) The topology induced by the trivial valuation is the discrete one.

2) The absolute value on $\mathbb{R}$ and on $\mathbb{C}$ induces the Euclidean topology on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively.

There are two kinds of valuations.

Definition B.1.11. A valuation $|\cdot|$ on F is called *non-archimedean* if

$$|x + y| \leq \max\{|x|, |y|\} \quad \forall x, y \in F. \tag{B.1}$$

If $|\cdot|$ does not satisfy (B.1), then $|\cdot|$ is said to be *archimedean*.

Inequality (B.1) is called the *ultrametric inequality*. Moreover, every valuation which is equivalent to a non-archimedean valuation is also non-archimedean.

Lemma B.1.12 ([Cas86, Chap. 2, Lem. 1.4])**.** *If $|\cdot|$ is non-archimedean, then for all $x, y \in F$ with $|y| < |x|$ one has that $|x + y| = |x|$.*

Proof. Note that

$$|x| = |(x + y) - y| \leq \max\{|x + y|, |y|\} \leq \max\{|x|, |y|\} = |x|.$$

□

Examples B.1.13. 1) The trivial valuation is non-archimedean.

2) The ordinary absolute value on $\mathbb{Q}$ and $\mathbb{R}$ is archimedean. Indeed, for all $x, y \geq 0$ $|x + y| = |x| + |y|$. A similar argument shows that the ordinary value on $\mathbb{C}$ is also archimedean.

3) The p-adic valuation is non-archimedean. Indeed, suppose $|x| = p^s$ and $|y| = p^t$ for $s, t \in \mathbb{Z}$. Then there exist $m_1, m_2 \in \mathbb{Z}$ and $n_1, n_2 \geq 0$ and $k_1, k_2, l_1, l_2 \geq 0$ such that $\gcd(m_i, p) = \gcd(n_i, p) = 1$ and such that $x = \frac{m_1 p^{k_1}}{n_1 p^{l_1}}$ and $y = \frac{m_2 p^{k_2}}{n_2 p^{l_2}}$. Then $s = l_1 - k_1$ and $t = l_2 - k_2$. It follows that

$$x + y = \frac{m_1 n_2 p^{k_1+l_2} + m_2 n_1 p^{k_2+l_1}}{n_1 n_2 p^{l_1+l_2}} = p^{\min\{k_1+l_2, k_2+l_1\}-l_1-l_2} \frac{m'}{n_1 n_2},$$

where $m' \in \mathbb{Z}$ is such that $\gcd(m', p) = 1$. Because $\gcd(n_1 n_2, p) = 1$,

$$|x + y| = p^{l_1+l_2-\min\{k_1+l_2, k_2+l_1\}} \leq \max\{p^{l_1-k_1}, p^{l_2-k_2}\} = \max\{|x|, |y|\}.$$

The proof of the following lemma is omitted.

Lemma B.1.14 ([Cas86, Chap. 2, Lem. 1.5])**.** *A valuation $|\cdot|$ is non-archimedean if, for every x in the ring generated by 1, one has $|x| \leq 1$.*

Concerning field extensions this gives

Corollary B.1.15. *Let $F \subset K$ be fields, and let $|\cdot|$ be a valuation on K. Then $|\cdot|$ is non-archimedean if and only if the restriction of $|\cdot|$ to F is non-archimedean.*

Corollary B.1.16. *Any valuation on a field of non-zero characteristic is non-archimedean.*

Recall the basic notions of convergent and Cauchy sequences in a metric space.

Definition B.1.17. A field F is *complete* with respect to a valuation $|\cdot|$ if the induced topology is complete, that is, every Cauchy sequence converges.

Definition B.1.18. Let F be a field with valuation $|\cdot|_F$. A *completion* of F is a field K with a valuation $|\cdot|_K$ such that

1) the restriction of $|\cdot|_K$ to F agrees with $|\cdot|_F$,

2) K is complete and

3) F is dense in K.

Definition B.1.19. Let F_1 and F_2 be two fields with valuations $|\cdot|_1$ and $|\cdot|_2$. An *isomorphism* between F_1 and F_2 is a (algebraic) field isomorphism $\phi\colon F_1 \to F_2$ such that $|x|_1 = |\phi(x)|_2$ for all $x \in F_1$.

One has

Theorem B.1.20 ([Cas86, Chap. 2, Thm. 4.1])**.** *Any field F with a valuation $|\cdot|$ has a completion. In particular, for any two completions K_1 and K_2 of F there exists an isomorphism $\phi\colon K_1 \to K_2$ such that $\phi|_F = id_F$.*

Examples B.1.21. 1) It is a standard fact of basic analysis that the completion of $\mathbb{Q}$ with respect to the ordinary absolute value is $\mathbb{R}$.

2) Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$. It is called the *field of p-adic numbers.*

Definition B.1.22. A field which is complete with respect to a valuation, is called *archimedean* if the valuation is archimedean. Similarly, if a field is complete with respect to a non-archimedean valuation, it is called *non-archimedean.*

The following theorem implies that $\mathbb{R}$ and $\mathbb{C}$, together with the absolute value, are the only archimedean fields, up to isomorphisms.

Theorem B.1.23 ([Cas86, Chap. 3, Thm. 1.1])**.** *Every archimedean field is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.*

Corollary B.1.24. *Every field which is complete with respect to an archimedean valuation is locally compact.*

B.2 Fields with non-archimedean valuations

Henceforth let F be a field with a non-archimedean valuation $|\cdot|$. Then, by the ultrametric inequality, the set

$$\mathcal{O} \coloneqq \{x \in F \colon |x| \leq 1\}$$

is a ring, and

$$P \coloneqq \{x \in F \colon |x| < 1\}$$

is a maximal ideal of $\mathcal{O}$. Then $\mathcal{O}/P$ is a field.

Definition B.2.1. The ring $\mathcal{O}$ is the *ring of integers* of F, and $\mathcal{O}/P$ is called the *residue field* of F.

If $\overline{F}$ is the completion of F with respect to $|\cdot|$, then $\overline{\mathcal{O}}$ and $\overline{P}$ are the corresponding ring of integers and its maximal ideal, respectively. In particular, the embedding $F \hookrightarrow \overline{F}$ induces a field homomorphism $\mathcal{O}/P \to \overline{\mathcal{O}}/\overline{P}$. This map is also an isomorphism (see [Cas86, Chap. 2, Lem. 1.1]). Indeed, if $x \in \overline{\mathcal{O}}$, then there is $y \in F$ such that $|x - y| < 1$, which implies $y \in \mathcal{O}$ and thus $x - y \in P$.

Recall that, by definition, the restriction of $|\cdot|$ to the multiplicative group $F^\times$ defines a group homomorphism to $(\mathbb{R}_{>0}, \cdot)$.

Definition B.2.2. The image $\{|x| \colon x \in F^\times\}$ of $F^\times$ under $|\cdot|$ is called the *valuation group*. If the valuation group is a discrete subgroup of $(\mathbb{R}_{>0}, \cdot)$, then $|\cdot|$ is said to be *discrete*.

Note that a subgroup $G < (\mathbb{R}_{>0}, \cdot)$ is discrete if and only if there exists a positive $q \in \mathbb{R}_{>0}$ such that $G = \{q^k \colon k \in \mathbb{Z}\}$, written $G = \langle q \rangle$.

Lemma B.2.3 ([Cas86, Chap. 4, Lem. 1.2])**.** *The valuation is discrete if and only if there exists $\mathfrak{p} \in P$ such that $P = \mathfrak{p}\mathcal{O}$.*

Proof. Suppose $|\cdot|$ is discrete. Then there exists $\mathfrak{p} \in \mathcal{O}$ such that $|x| \leq |\mathfrak{p}| < 1$ for all $x \in P$. It follows that for all $x \in \mathcal{O}$ $|\mathfrak{p}x| < 1$, and hence $P = \mathfrak{p}\mathcal{O}$.

Conversely, suppose $P = \mathfrak{p}\mathcal{O}$. Then for all $x \in \mathcal{O}$

$$|x| \leq |\mathfrak{p}| < 1 < |\mathfrak{p}^{-1}| \leq |x^{-1}|.$$

This implies that 1 is isolated in the valuation group, and therefore the valuation group is discrete. □

If an element $\mathfrak{p} \in P$ exists such that $P = \mathfrak{p}\mathcal{O}$, it is called *uniformizing element*. As seen in the proof of Lemma B.2.3, the valuation group is $\langle |\mathfrak{p}| \rangle = \{|x| \colon x \in F\}$, that is, $|\mathfrak{p}|$ generates the valuation group. Because $|x^{-1}| = |x|^{-1}$, for all $x \in F$ either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Let $\mathcal{O}^\times$ be the set of all elements of $\mathcal{O}$ whose multiplicative inverse lies in $\mathcal{O}$. Since $|x^{-1}| = |x|^{-1}$, $\mathcal{O}^\times = \{x \in F \colon |x| = 1\}$.

Lemma B.2.4. *Suppose F is a non-archimedean field with discrete valuation. For every $x \in F$ there exist $k \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$ such that $x = \mathfrak{p}^k u$.*

Proof. Let $k \in \mathbb{Z}$ such that $|x| = |\mathfrak{p}^k|$. Then $|x\mathfrak{p}^{-k}| = 1$, which implies $x\mathfrak{p}^{-1} \in \mathcal{O}^\times$. □

Suppose the residue field $\mathcal{O}/P$ has cardinality p, and let $A \subset \mathcal{O}^\times$ be a set of representatives. Using Lemma B.2.4, one can show that for every $x \in F^\times$ there exist $s \in \mathbb{Z}$ and $\{a_n\}_{n \geq s} \subset A$ such that $x = \sum_{n \geq s} a_n \mathfrak{p}^n$ (see [Cas86, Chap. 4, Lem. 1.4] and its corollary). Moreover, with $q = |\mathfrak{p}|$, for every $k \in \mathbb{Z}$

$$B(x, q^k) \coloneqq \{x \in F \colon |x| \leq q^k\} = \{x \in F \colon |x| < q^{k+1}\}$$

and thus $\{x \in F \colon |x| = q^k\}$ are both closed and open.

Lemma B.2.5. *If the valuation $|\cdot|$ is discrete, then F is totally disconnected, that is, the only connected components of F are the one-point sets.*

Proof. By the definition of the topology on F, the sets $\{x \in F : |x| \leq q^k\}$, $k \in \mathbb{Z}$, form a neighbourhood basis for 0. Because, by the previous remarks, these sets are all open and closed, $\{0\}$ is a connected component of F. Since for every $x \in F$ the map $y \mapsto x+y$ is a homeomorphism on F, the lemma follows. □

Concerning the question under which conditions F is locally compact, one has

Lemma B.2.6. *F is locally compact if and only if $\mathcal{O}$ is compact.*

Proof. Note that for every $r \in \{|x| : x \in F^\times\}$ one has that, for $x \in F^\times$ with $|x| = r$,

$$\{y \in F : |y| \leq r\} = x \cdot \{y \in F : |y| \leq 1\} = x\mathcal{O}.$$

Thus $\{x\mathcal{O} : x \in F\}$ forms a neighbourhood basis of 0. Because for every $x \in F$ $y \mapsto xy$ is a homeomorphism on F, $x\mathcal{O}$ is compact for some $x \in F$ if and only if $\mathcal{O}$ is compact. Therefore F is locally compact if and only if $\mathcal{O}$ is compact. □

Proposition B.2.7. *Suppose F is a field with non-archimedean valuation $|\cdot|$. Then F is locally compact if and only if the following holds:*

1) F is complete.

2) $|\cdot|$ is discrete.

3) The residue field $\mathcal{O}/P$ is finite.

Proof. Suppose F is locally compact. Then, by Lemma B.2.6, $\mathcal{O}$ is compact. Moreover it is immediate that F is complete. Since $P \subset \mathcal{O}$ is an open subset, $\mathcal{O}/P$ is a compact, discrete and thus finite field.

The converse follows from [Cas86, Chap. 4, Lem. 1.5]. □

If the residue field is finite, its cardinality is the positive integer power of a prime number.

Turn your attention now to the example of the field $\mathbb{Q}_p$ of p-adic numbers. Let $\mathbb{Z}_p$ denote its ring of integers. The following follows closely [FTN91, p. 119 f.].

Lemma B.2.8. *$\mathbb{Z}$ is dense in $\mathbb{Z}_p$.*

Proof. Because $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, it is enough to show that $\mathbb{Z}$ is dense in $\{x \in \mathbb{Q} : |x|_p \leq 1\}$. Let $\epsilon > 0$, and let $x \in \mathbb{Q}$ with $|x|_p \leq 1$. Write $x = \frac{m}{n}p^k$ where $m \in \mathbb{Z}$, $n > 0$ and $k \geq 0$ are such that $\gcd(m,n) = \gcd(m,p) = \gcd(n,p) = 1$. Let $l \geq 0$ be such that $p^{-l} < \epsilon$. There are non-zero $a, b \in \mathbb{Z}$ such that $ap^l + bn = 1$. Then

$$|x - bp^k m|_p = \left|p^k m \frac{1-bn}{n}\right|_p = \left|p^{k+l}\frac{am}{n}\right|_p = p^{-k-l} < \epsilon.$$

□

Proposition B.2.9. *$\mathbb{Z}_p$ is compact, and $p \in \mathbb{Z}_p$ is a uniformizing element. Furthermore, $\mathbb{F}_p$ is the residue field.*

Proof. Show first that $\mathbb{Z}_p$ is totally bounded. For this fix $\epsilon > 0$, and let $n \geq 1$ be such that $p^{-n} < \epsilon$. Observe that for every $x \in \mathbb{Z}$ there is $k \in \{0, 1, \ldots, p^n - 1\}$ such that $x - k \equiv 0 \bmod p^n$, which implies that $|x-k|_p \leq p^{-n} < \epsilon$. It follows that

$$\mathbb{Z} \subset \bigcup_{0 \leq k \leq p^n - 1} B(k, \epsilon);$$

whence $\mathbb{Z}$ is totally bounded. Because $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, this implies that $\mathbb{Z}_p$ is totally bounded and thus is compact.

It is immediate that $p \in \mathbb{Z}_p$ is a uniformizing element. Finally because $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, the residue field of $\mathbb{Q}_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$. □

The section is concluded by introducing examples of local fields of non-zero characteristic. Fix $q > 1$ a positive integer power of a prime number p, and let $\mathbb{S}_q$ be the ring of formal Laurent series over $\mathbb{F}_q$, that is,

$$\mathbb{S}_q \coloneqq \left\{ \sum_{n \geq s} a_n X^n \colon s \in \mathbb{Z},\ a_n \in \mathbb{F}_q\ \forall n \geq s \right\}.$$

Notice that $\mathbb{S}_q$ has characteristic p. Endow it with the valuation $|\cdot|_q$, given by

$$\left| \sum_{n \geq s} a_n X^n \right|_q \coloneqq q^{-s},$$

where $a_s \neq 0$. It is easy to check that $|\cdot|_q$ is a valuation and that it satisfies the ultrametric inequality. Endow $\mathbb{S}_q$ with the topology induced by $|\cdot|_p$, and observe that $\mathbb{S}_q$ is complete and that the ring operations are continuous. Moreover, one has

Proposition B.2.10. *$\mathbb{S}_q$ is a topological field with respect to $|\cdot|_q$.*

Proof. Note that for all $a \in F$ and all $n \in \mathbb{Z}$, $aX^n \in \mathbb{S}_q$ is invertible whose multiplicative inverse is $a^{-1}X^{-n}$. It remains to show that any element of the form $x = 1 + \sum_{n \geq 1} a_n X^n$ is invertible. Then $|1 - x|_q \leq q^{-1}$ and therefore the series $\sum_{n \geq 0}(1-x)^n$ converges in $\mathbb{S}_q$. Because for every $n \geq 0$

$$\left| x \left(\sum_{0 \leq k \leq n} (1-x)^k \right) - 1 \right|_q = |(1-x)^{n+1}|_q \leq q^{-n-1},$$

$\sum_{n \geq 0}(1-x)^n$ is the multiplicative inverse of x. Hence $\mathbb{S}_q$ is a field.

To show that $\mathbb{S}_q$ is a topological field, it is enough to show that the inverse map is continuous. Let $\{x_n\}_{n \geq 0} \subset \mathbb{S}_q$ be a sequence such that $x_n \to x \in \mathbb{S}_q$. If $x = aX^k$ for some $a \in F$ and $k \in \mathbb{Z}$, then for n large enough $x_n = x$, which implies $x_n^{-1} \to x^{-1}$. Hence, it is enough to assume that $x = 1 + \sum_{k \geq 1} a_k X^k$. Then for n large enough, $x_n = 1 + \sum_{k \geq 1} a_k^{(n)} X^k$, and, as seen above, $x^{-1} = \sum_{k \geq 0}(1-x)^k$ and $x_n^{-1} = \sum_{k \geq 0}(1-x_n)^k$. Note that $|x|_q, |x_n|_q \leq 1$ and that, by the ultrametric inequlaity, $|(1-x)^k - (1-x_n)^k|_q \leq |x - x_n|_q$ for all $k \geq 0$. Thus, by the ultrametric inequality, for all $k \geq 0$

$$\left| \sum_{0 \leq i \leq k} (1-x)^i - \sum_{0 \leq i \leq k} (1-x_n)^i \right|_q \leq |x - x_n|_q,$$

and hence $|x^{-1} - x_n^{-1}|_q \leq |x - x_n|_q$. It follows that the inverse map is continuous. □

It follows that $\mathbb{S}_q$ is a complete non-archimedean field with discrete valuation. The ring of integers is $\mathcal{O} = \left\{ \sum_{n \geq 0} a_n X^n \colon a_n \in \mathbb{F}_q\ \forall n \geq 0 \right\}$ and the maximal ideal of $\mathcal{O}$ is $P = \left\{ \sum_{n \geq 1} a_n X^n \colon a_n \in \mathbb{F}_q \right\}$. Then $X \in \mathbb{S}_q$ is a uniformizing element. In particular, the residue field of $\mathbb{S}_q$ is $\mathbb{F}_q$. By Proposition B.2.7, $\mathbb{S}_q$ is locally compact.

B.3 Classification of local fields

Let F be a local field, and let μ be a translation-invariant measure on F, that is, a Haar measure of the additive group of F. For every $x \in F$ the map $l_x \colon F \to F$, $y \mapsto xy$ is linear with respect to the structure of F as a one-dimensional vector space over itself. In particular, if $x \neq 0$, l_x is a group isomorphism from the additive group of F to itself, with the inverse map being $l_{x^{-1}}$. Set $|x|_\mu \coloneqq \mathrm{mod}_{(F,+)}(l_x)$. Note that for every Borel subset $B \subset F$ with $0 < \mu(B) < \infty$, one has that

$$|x|_\mu = \frac{\mu(xB)}{\mu(B)} \quad \forall x \in F. \tag{B.2}$$

The goal is now to show that $|\cdot|_\mu$ defines a valuation on F and that the topology induced by $|\cdot|_\mu$ coincides with the topology of F.

Proposition B.3.1 ([Wei67, Chap. I, Prop. 1])**.** *The map $|\cdot|_\mu \colon F \to \mathbb{R}$ is continuous. Moreover, $|\cdot|_\mu \colon F^\times \to (\mathbb{R}_{>0}, \cdot)$ is a continuous group homomorphism.*

Proof. Let $\{x_n\}_{n\geq 1} \subset F$ be a sequence converging to $x \in F$, and fix $\epsilon > 0$. Fix an open neighbourhood $U \subset F$ of 0 with compact closure. Then, because μ is a Radon measure, there exist an open subset $V \subset F$ and compact subset $K \subset F$ such that $K \subset xU \subset V$ and $\mu(V) < \mu(xU) + \frac{\epsilon}{2} < \mu(K) + \epsilon$. Because U has compact closure, there exists $N(\epsilon) \geq 1$ such that for all $n \geq N(\epsilon)$ $K \subset x_n U \subset V$. Hence

$$||x_n|_\mu - |x|_\mu| = \frac{1}{\mu(U)} |\mu(x_n U) - \mu(xU)| < \frac{\epsilon}{\mu(U)} \quad \forall n \geq N(\epsilon).$$

Since U is independent from $\epsilon > 0$, the proposition follows. □

From now on suppose F is non-discrete.

Corollary B.3.2. *F is not compact.*

Proof. If F is compact, then $\{|x|_\mu \colon x \in F\}$ is bounded. But $|0|_\mu = 0$ and $|1|_\mu = 1$, and because F is assumed to be non-discrete, there exists $x \in F$ such that $0 < |x|_\mu < 1$. Then $|x^{-1}|_\mu > 1$ and $|x^{-n}|_\mu \to \infty$ as $n \to \infty$. □

Proposition B.3.3 ([Wei67, Chap. I, Prop. 2])**.** *For every $r > 0$ the set $B(0,r) \coloneqq \{x \in F \colon |x|_\mu < r\}$ has compact closure.*

Proof. Fix $r > 0$. By Proposition B.3.1, $\overline{B(0,r)} \subset \{x \in F \colon |x|_\mu \leq r\}$. Let $U \subset F$ be an open neighbourhood of 0 with compact closure. Because F is not discrete and $|0|_\mu = 0$, by Proposition B.3.1, there exists $x \in U$ such that $0 < |x|_\mu < 1$. Then $|x^n|_\mu \to 0$ and hence $x^n \to 0$ as $n \to \infty$. Therefore assume that $x^n \in U$ for all $n \geq 1$. If now $B(0,r) \subset U$, then it is compact and the proof is finished. Hence, assume that $y \in B(0,r)$ such that $y \notin U$. Because $x^n y \to 0$ as $n \to \infty$, let $m \geq 0$ be the smallest integer such that $x^m y \in U$ and $x^{m-1} y \notin U$. Then $x^m y \in U - xU$. Let $K \coloneqq \overline{U - xU}$, which is compact, and $0 \notin K$. Set $c \coloneqq \min_{z \in K} |z|_\mu$, and let $N \geq 0$ be such that $|x^N|_\mu \leq \frac{c}{r}$ for all $n \geq N$. Then

$$|x^N y|_\mu \leq c \leq |x^m y|_\mu \leq |x|_\mu^m r,$$

which implies $m \leq N$. Thus

$$B(0,r) \subset \bigcup_{0 \leq i \leq N} x^{-i} U,$$

and therefore $B(0,r)$ has compact closure. □

Corollary B.3.4 ([Wei67, Chap. I, Cor. 1])**.** *The sets $\{B(0,r) \colon r > 0\}$ form a neighbourhood basis of 0.*

Proof. Let $U \subset F$ be any open neighbourhood of 0. Because F is locally compact, assume that $\overline{U}$ is compact and set $r \coloneqq \max_{x\in\overline{U}}|x|_\mu$. Then $U \subset B(0,r)$, and set $V \coloneqq \overline{B(0,r)} - \overline{U}$. Then, by Proposition B.3.3, $\overline{V}$ is compact as well, and let $r' \coloneqq \min_{x\in\overline{V}}|x|_\mu$. Since $0 \notin \overline{V}$, $r' > 0$. Then $0 < r' \leq r$ and for every $0 < s < r'$ one has that $B(0,s) \subset \overline{U} \cup \overline{V} = \overline{B(0,r)}$, which implies that $B(0,s) \subset U$. □

Corollary B.3.5. *The sets* $\{B(x,r)\colon x \in F,\ r > 0\}$*, given by*

$$B(x,r) \coloneqq \{y \in F\colon |x-y|_\mu < r\},$$

form a basis of the topology on F*.*

Proof. This follows from Corollary B.3.4 and the fact that, for every $x \in$ and every $r > 0$, $B(x,r) = B(0,r) + x$. □

This gives

Theorem B.3.6. *The map* $|\cdot|_\mu$ *is a valuation on* F*, and the topology induced by* $|\cdot|_\mu$ *coincides with the topology on* F*.*

Proof. By Proposition B.3.1, $|\cdot|_\mu$ induces a continuous group homomorphism from the multiplicative group $F^\times$ to $(\mathbb{R}_{>0},\cdot)$. Moreover, $|0|_\mu = 0$. Let $C \coloneqq \max_{x\in\overline{B(0,1)}}|1+x|_\mu$. Because $\overline{B(0,1)}$ is compact, $0 < C < \infty$, and hence for all $x \in \overline{B(0,1)}$, $|1+x|_\mu \leq C$. Therefore $|\cdot|_\mu$ is a valuation on F.

The second part of the Theorem follows from Corollary B.3.5. □

Corollary B.3.7. *Every local field is a complete field with respect to a valuation. The valuation is determined uniquely, up to equivalence, by a Haar measure of the additive group of* F *as described in* (B.2)*.*

Corollary B.3.8. *If* F *is a locally compact field with valuation* $|\cdot|$*, then* $|\cdot|$ *is equivalent to* $|\cdot|_\mu$*, where* μ *is the Haar measure of the additive group of* F*.*

Concerning finite algebraic field extensions one has

Proposition B.3.9. *Suppose* K *is a finite algebraic field extension of* F *with dimension* d*. Then* K *is locally compact. In particular, if* μ_F *and* μ_K *are Haar measures of the additive groups of* F *and* K *respectively, then* $|x|_{\mu_K} = |x|^d_{\mu_F}$ *for all* $x \in F$*. Moreover,* F *is non-archimedean if and only if* K *is non-archimedean.*

Recall that if F is a non-archimedean local field, the valuation group of $|\cdot|_\mu$ is discrete. Moreover, by the ultrametirc inequality, for every $r > 0$ $(B(0,r),+)$ is a closed subgroup of the additive group $(F,+)$.

Proposition B.3.10. *Suppose* F *is non-archimedean, and let* q *be the cardinality of the residue field of* F*. Then* $\langle q\rangle = \{|x|_\mu\colon x \in F\}$*.*

Proof. Let $\mathfrak{p} \in F$ be a uniformizing element of F. Then $\langle|\mathfrak{p}|_\mu\rangle = \{|x|_\mu\colon x \in F\}$. Because $\mathcal{O}/P$ is finite and because

$$\mathcal{O} = \bigsqcup_{[x]\in\mathcal{O}/P} x + P,$$

one has that

$$|\mathfrak{p}|_\mu^{-1}\mu(P) = \mu(\mathfrak{p}^{-1}P) = \mu(\mathcal{O}) = \sum_{[x]\in\mathcal{O}/P}\mu(x+P) = |\mathcal{O}/P|\mu(P),$$

where $|\mathcal{O}/P|$ is the cardinality $\mathcal{O}/P$. Thus $\mathcal{O}/P$ is the finite field of order $|\mathfrak{p}^{-1}|_\mu$. □

Corollary B.3.11. *If F is an d-dimensional field extension over a non-archimedean local field whose residue field has cardinality q, then its residue field has cardinality q^d.*

Proof. This follows from Proposition B.3.9 and from Proposition B.3.10. □

Examples B.3.12. 1) A translation-invariant measure on $\mathbb{R}$ and on $\mathbb{C}$ is the Lebesgue measure $\mathcal{L}$ and its two-fold product $\mathcal{L}\times\mathcal{L}$, respectively. The induced valuations are the ordinary absolute value on $\mathbb{R}$ and $|z|^2 = z\overline{z}$ on $\mathbb{C}$.

2) By Proposition B.3.10, the valuation on $\mathbb{Q}_p$ induced by a translation-invariant measure is the p-adic valuation $|\cdot|_p$.

The final result of this section is the classification of all local fields. It is a consequence of [Wei67, Chap. I, Thm. 5] and of [Wei67, Chap. I, Thm. 8].

Theorem B.3.13. *Let F be a local field.*

1) If F is archimedean, then it is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$.

2) If F is non-archimedean and if $\mathrm{char}(F) = 0$, then it is isomorphic to a finite-dimensional algebraic extension of $\mathbb{Q}_p$ for some prime number $p > 1$.

3) If F is non-archimedean and if $\mathrm{char}(F) = p > 1$, then it is isomorphic to $\mathbb{S}_{p^n}$ for some $n \geq 1$.

Appendix C

Integral Formulas for Special Linear Groups

Let F be a local field, and let $d \geq 1$. Let μ be a translation-invariant measure on F, and let $|\cdot|$ be the valuation on F induced by μ (see Section B.3). As seen in Chapter 2 and Chapter 3, $SL_d(F)$ has the Cartan and the Iwasawa decomposition. The goal of this appendix is to obtain integral formulas of the Haar measure of $SL_d(F)$ with respect to those decompositions. For clarity adopt the notation from the beginning of section 7.2. Then the Iwasawa decomposition is

$$SL_d(F) = K_d(F)D_d(F)N_d^+(F),$$

and the Cartan decomposition is

$$SL_d(F) = K_d(F)D_d^+(F)K_d(F).$$

C.1 Integral formula for the Iwasawa decomposition

Using Proposition A.4.13 it remains to find an integral formula for the Haar measure of the group $\widetilde{P}_d^+(F)$ of upper triangular matrices in $SL_d(F)$ with respect to the decomposition $\widetilde{P}_d^+(F) = D_d(F)N_d^+(F)$. Concerning the abelian subgroup $D_d(F)$ one has the following obvious isomorphisms:

1) if $F \in \{\mathbb{R}, \mathbb{C}\}$, then $D_d(F) = \widetilde{A}_d \cong (\mathbb{R}^\times)^{d-1}$, and

2) if F is non-archimedean, then $D_d(F) = \widetilde{T}_d(F) \cong \mathbb{Z}^{d-1}$.

Lemma C.1.1. *1) If $F \in \{\mathbb{R}, \mathbb{C}\}$, then the Haar measure on $D_d(F) = \widetilde{A}_d$ is*

$$\frac{da_1}{|a_1|}\frac{da_2}{|a_2|}\cdots\frac{da_{d-1}}{|a_{d-1}|}$$

for $diag(a_1, a_2, \ldots, a_d) \in \widetilde{A}_d$ and where da_i is the Lebesgue measure on $\mathbb{R}$.

2) If F is non-archimedean, then the Haar measure on $D_d(F) = \widetilde{T}_d(F)$ is the pushforward of the $(d-1)$-fold product $\mu^{\times(d-1)}$ of the counting measure μ on $\mathbb{Z}$ under the isomorphism $\widetilde{T}_d(F) \cong \mathbb{Z}^{d-1}$, that is,

$$d\mu(k_1)\, d\mu(k_2)\cdots d\mu(k_{d-1})$$

for $diag(\mathfrak{p}^{k_1}, \mathfrak{p}^{k_2}, \ldots, \mathfrak{p}^{k_d}) \in \widetilde{T}_d(F)$.

Proof. In both cases the formula of the Haar measure follow from the respective isomorphism in the previous remarks. Using variable change it is easy to show that $\frac{dx}{|x|}$ defines a Haar measure

on the multiplicative group $\mathbb{R}^\times$. Thus part 1) follows from the isomorphism $\widetilde{A}_d \cong (\mathbb{R}^\times)^{d-1}$. Moreover, because $\mathbb{Z}$ is discrete, the counting measure μ is a Haar measure on the group $(\mathbb{Z}, +)$. Hence, part 2) follows from the isomorphism $\widetilde{T}_d(F) \cong \mathbb{Z}^{d-1}$. □

For the next result consider the identification $N_d^+(F) \approx F^{\frac{1}{2}d(d-1)}$, given by

$$n \mapsto (n_{12}, n_{13}, \ldots, n_{1d}, n_{23}, \ldots, n_{d-1d}). \tag{C.1}$$

Moreover, for every $a \in D_d(F)$ one has the automorphism $c_a \colon N_d(F) \to N_d(F), n \mapsto a^{-1}na$. In the case of $F = \mathbb{R}$, μ is, up to a constant factor, the usual Lebesgue measure.

Lemma C.1.2. *$N_d^+(F)$ is unimodular and its Haar measure is the pushforward of the $\frac{1}{2}d(d-1)$-fold product of μ under the homeomorphism* (C.1)*. Moreover,*

$$\mathrm{mod}_{N_d^+(F)}(c_a) = \prod_{1 \le i < j \le d} |a_i^{-1} a_j|.$$

Proof. Note that, for all $n, n' \in N_d^+(F)$ and all $1 \le i < j \le d$,

$$(nn')_{ij} = \sum_{i \le k \le j} n_{ik} n'_{kj} = n_{ij} + \sum_{i < k < j} n_{ik} n'_{kj} + n'_{ij}.$$

Thus, the pushforward of $\mu^{\frac{1}{2}d(d-1)}$ under the homeomorphism $N_d^+(F) \approx F^{\frac{1}{2}d(d-1)}$ defines a Haar measure on $N_d^+(F)$, which is both left- and right-invariant.

For the second part of the lemma note that, for all $x \in F$, by the definition of the valuation $|\cdot|$ on F, $\mu(xB) = |x|\mu(B)$ for every Borel subset $B \subset F$. Under the identification $N_d^+(F) \approx F^{\frac{1}{2}d(d-1)}$, c_a corresponds to the map

$$\begin{aligned}(n_{12}, n_{13}, \ldots, n_{1d}, n_{23}, \ldots, n_{d-1d}) \\ \mapsto (a_1^{-1} n_{12} a_2, a_1^{-1} n_{13} a_3, \ldots, a_1^{-1} n_{1d} a_d, a_2^{-1} n_{23} a_3, \ldots, a_{d-1}^{-1} n_{d-1d} a_d).\end{aligned}$$

Hence, by the first part of the lemma and by the definition of $|\cdot|$,

$$\mathrm{mod}_{N_d^+(F)}(c_a) = \prod_{1 \le i < j \le d} |a_i^{-1} a_j|.$$

□

One is now equipped to prove the main result of this section. Let da and dn denote the Haar measures on $D_d(F)$ and on $N_d^+(F)$, respectively.

Proposition C.1.3. *For every $f \in C_c(\widetilde{P}_d^+(F))$,*

$$\int_{\widetilde{P}_d^+(F)} f(p)\, dp = \int_{D_d(F)} \int_{N_d^+(F)} f(an)\, dn\, da, \tag{C.2}$$

where dp denotes a Haar measure on $\widetilde{P}_d^+(F)$. Moreover, the modular function of $\widetilde{P}_d^+(F)$ is

$$\Delta_{\widetilde{P}_d^+(F)}(an) = \prod_{1 \le i < j \le d} |a_i^{-1} a_j|.$$

Proof. For the first part of the proposition it is enough to show that $dn\, da$ defines a left Haar measure on $\widetilde{P}_d^+(F)$. For this fix $b \in D_d^+(F)$ and $m \in N_d^+(F)$. Then

$$\begin{aligned}\int_{D_d(F)} \int_{N_d^+(F)} f((bm)(an))\, dn\, da &= \int_{D_d(F)} \int_{N_d^+(F)} f(bac_a(m)n)\, dn\, da \\ &= \int_{D_d(F)} \int_{N_d^+(F)} f(an)\, dn\, da\end{aligned}$$

Moreover,

$$\int_{D_d(F)} \int_{N_d^+(F)} f((an)(bm))\, dn\, da = \int_{D_d(F)} \int_{N_d^+(F)} f(ac_b(n)m)\, dn\, da$$
$$= \mathrm{mod}_{N_d^+(F)}(c_b)^{-1} \int_{D_d(F)} \int_{N_d^+(F)} f(an)\, dn\, da.$$

Hence, $\Delta_{\widetilde{P}_d^+(F)}(an) = \mathrm{mod}_{N_d^+(F)}(c_a)$. The explicit formula for $\Delta_{\widetilde{P}_d^+(F)}$ follows from by Lemma C.1.2. □

Theorem C.1.4 (Integral Formula for the Iwasawa decomposition)**.** *For every $f \in C_c(\widetilde{P}_d^+(F))$*

$$\int_{SL_d(F)} f(x)\, dx = \int_{K_d(F)} \int_{D_d(F)} \int_{N_d^+(F)} f(kan)\, dn \frac{da}{\prod_{1\le i<j\le d} |a_i^{-1}a_j|} dk.$$

Proof. This follows directly from Lemma C.1.2 and Proposition C.1.3. □

C.2 Integral formulas for the Cartan decomposition

For the integral formulas for the Cartan decomposition no detailed proofs are given here. Instead the reader will be given the respective references.

Let da also denote the restriction of the Haar measure of $D_d(F)$ to the subsemigroup $D_d^+(F)$.

Theorem C.2.1 (Integral Formula for the Cartan decomposition)**.** *Let F be a local field, and let $d \ge 2$. Then there exists a continuous function $\delta_{F,d}\colon D_d^+(F) \to \mathbb{R}$ such that*

$$\int_{SL_d(F)} f(x)\, dx = \int_{K_d(F)} \int_{D_d^+(F)} \int_{K_d(F)} f(k_1 a k_2)\delta_{F,d}(a)\, dk_1\, da\, dk_2 \quad \forall f \in C_c(G).$$

In particular, the following holds:

1. *If $F \in \{\mathbb{R}, \mathbb{C}\}$, then*

$$\delta_{F,d}(a) = \prod_{1\le i<j\le d} (a_i^{-1}a_j - a_i a_j^{-1}) \quad \forall a = diag(a_1, a_2, \ldots, a_d) \in \widetilde{A}_d^+.$$

2. *If F is non-archimedean, then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \Delta^{-1}_{\widetilde{P}_d^+(F)}(a) \le \delta_{F,d}(a) \le c_2 \Delta^{-1}_{\widetilde{P}_d^+(F)}(a).$$

Proof. The archimedean case is a direct application of [Hel84, Chap. I, Thm. 5.8].

Suppose that F is non-archimedean. Recall that $K_d(F) < SL_d(F)$ is both open and compact. Therefore the homogeneous space $SL_d(F)/K_d(F)$ is discrete, and the counting measure μ is a $SL_d(F)$-invariant measure. Because $SL_d(F)$ and $K_d(F)$ are unimodular,

$$\int_{SL_d(F)} f(x)\, dx = \int_{SL_d(F)/K_d(F)} \int_{K_d(F)} f(xk)\, dk\, d\mu(x) \quad \forall f \in C_c(SL_d(F)).$$

Hence, for $q\colon SL_d(F) \to SL_d(F)/K_d(F)$ the canonical quotient map, it is enough to show that there exists $\delta_{F,d}\colon D_d^+(F) \to \mathbb{R}$ such that for all $f \in C_c(SL_d(F)/K_d(F))$

$$\int_{SL_d(F)/K_d(f)} f(x)\, d\mu(x) = \int_{K_d(F)} \int_{D_d^+(F)} f(q(ka))\delta_{F,d}(a)\, da\, dk.$$

Claim that $\delta_{F,d}(a) = \mu(q(K_d(F)a))$. Indeed, fix $x_0 \in SL_d(F)/K_d(F)$, and set $f = \chi_{\{x_0\}}$. Then $\int f\, d\mu = 1$. Let $a_0 \in D_d^+(F)$ be the unique element such that $x_0 \in K_d(F)a_0K_d(F)$. Notice that $q(K_d(F)a_0)$ is discrete and compact, and hence finite. In particular, $|q(K_d(F)a_0)| = \mu(q(K_d(F)a_0))$, where $|q(K_d(F)a_0)|$ denotes the cardinality of $q(K_d(F)a_0)$. Thus,

$$\begin{aligned}
&\int_{K_d(F)} \int_{D_d^+(F)} f(q(ka))\mu(q(K_d(F)a_0))\, da\, dk \\
&\quad = \mu(q(K_d(F)a_0)) \int_{K_d(F)} f(q(ka_0))\, dk \\
&\quad = \mu(q(K_d(F)a_0)) \int_{K_d(F)/\mathrm{stab}_{K_d(F)}(q(a_0)} f(q(ka_0))\, dk \\
&\quad = \frac{\mu(q(K_d(F)a_0))}{|q(K_d(F)a_0)|} = 1.
\end{aligned}$$

By [Sil15, Lem. 4.1.1], there exist constants $c_1, c_2 > 0$ such that

$$c_1\Delta^{-1}_{\widetilde{P}_d^+(F)}(a) \leq \mu(q(K_d(F)a_0)) \leq c_2\Delta^{-1}_{\widetilde{P}_d^+(F)}(a),$$

which completes the proof. □

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