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# Quantitative Multiple Mixing and Equidistribution in the Moduli Space of Translation Surfaces

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# Abstract

In this thesis, we study ergodic properties of the Teichmüller geodesic flow on the moduli space of area one translation surfaces, as well as the $SL_2(\mathbb{R})$ action. In the first part of the thesis, we show that unstable leafs equidistribute under the Teichmüller geodesic flow using ideas from homogeneous dynamics introduced in [EM93]. We then prove multiple equidistribution of unstable leafs under the geodesic flow by borrowing ideas from [Dol04], where a connected component of moduli space is partitioned into subsets satisfying similar properties to those of an almost Markov family. In the second part of the thesis, we prove that the action is exponentially mixing of all orders. We first use the exponential mixing result of the geodesic flow from [AGY06] to establish that the full $SL_2(\mathbb{R})$ action is exponentially mixing using ideas from [Rat87]. For higher order correlations, we follow similar arguments to the ones used to establish multiple equidistribution of the geodesic flow. We moreover give an exposition of notions in ergodic theory, representation theory, and translation surfaces and their parameter spaces.

# Zusammenfassung

In dieser Masterarbeit untersuchen wir die ergodischen Eigenschaften des Teichmüllerschen geodätischen Flusses auf dem Modulraum von Translationsflächen mit Fläche eins, sowie die $SL_2(\mathbb{R})$ Gruppenaktion. Im ersten Teil der Arbeit zeigen wir, dass instabile Blätter unter dem Teichmüllerschen geodätischen Fluss sich exponentiell gleichverteilen, indem wir Ideen aus dem Bereich der homogenen dynamischen Systemen verwenden, die in [EM93] eingeführt wurden. Danach beweisen wir mehrfacher Gleichverteilung von instabilen Blättern unter dem geodätischen Fluss, indem wir Ideen aus [Dol04] entlehnen, wo eine Zusammenhangsomponente des Modulraums in Teilmengen zerlegt wird, die ähnliche Eigenschaften wie die einer fast Markov-Familie erfüllen. Im zweiten Teil der Arbeit beweisen wir, dass die $SL_2(\mathbb{R})$ Aktion exponentiell mehrfach mischend ist. Zunächst verwenden wir das Ergebnis der exponentiellen Mischung des geodätischen Flusses aus [AGY06], um zu beweisen, dass die vollständige $SL_2(\mathbb{R})$ Aktion exponentiell mischend ist, indem wir Ideen aus [Rat87] verwenden. Für Korrelationen höherer Ordnung folgen wir ähnlichen Argumenten, wie sie zur Begründung der mehrfachen Gleichverteilung des geodätischen Flusses verwendet wurden. Darüber hinaus geben wir eine Darstellung von Begriffen aus der Ergodentheorie, der Darstellungstheorie und von Translationsflächen und deren Parameterräumen.

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# Introduction

There are two main goals in this thesis. First, we show that unstable leafs in the moduli space of area one translation surfaces equidistribute exponentially under the Teichmüller geodesic flow. Second, we show that the $SL_2(\mathbb{R})$ action on moduli space is exponentially mixing of all orders. Before stating the results in detail, we give some motivation for the study of translation surfaces and the associated Teichmüller geodesic flow, as well as for the study of quantitative multiple mixing estimates. We do not give detailed explanations, and will include references for more in depth introductions. After stating the results, we give the main ideas in the proofs of our results. We close by explaining the organization of the thesis.

The study of translation surfaces spans various areas in mathematics, from dynamics, to number theory, and mathematical physics. We refer to [Zor06] for a survey of translation surfaces. A translation surface can be viewed as a polygon with parallel sides of the same length identified with each other. To start, they arise from the study of polygonal billiards. This is a dynamical system which comes from following a straight line in a polygon and where it reflects off a wall with the angle of reflection being the same as the angle of incidence. For certain polygons, one can "unfold" the polygon so that the resulting shape corresponds to a translation surface as in figure 1. For an introduction to such systems, see [MT02].

There are a number of open questions regarding billiard flows, such as whether the flow is ergodic in a generic triangle, if there are periodic trajectories for general billiards, and if any, how many of these there are [see Zor06, section 2.1]. There are also relations to mathematical physics, where translation surfaces arise in the study of electron transport in metals. We do not give details here, and refer to section 2.2 of [Zor06].

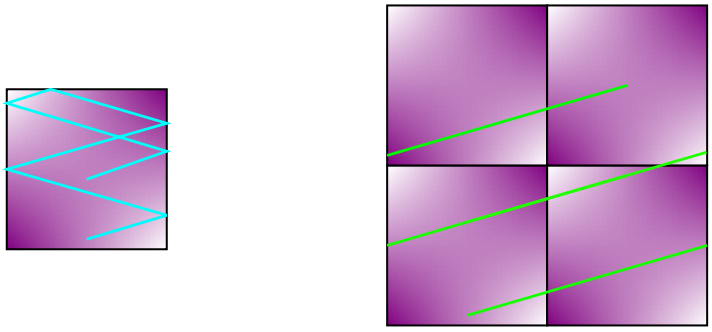

Figure 1: Unfolding billiards

Translation surfaces also arise in number theory. One instance of this is inspired by the study of the distribution of the points $\left\{\sqrt{1}\right\}, \left\{\sqrt{2}\right\}, \ldots, \left\{\sqrt{N}\right\}$ in $\mathbb{S}^1$, where $\{x\} = x \pmod 1$. This problem was considered in [EM04] and was studied using methods in homogeneous dynamics. More recently, a generalization of this was considered in [FSU18], where a connection to translation surfaces is made. Another area of number theory where translation surfaces arise, is in generalizations of Diophantine approximations. This theory deals with how well real numbers can be approximated by rational numbers. In Diophantine approximation, an object of study is the Lagrange spectrum $L$. The elements in the spectrum intuitively correspond to the following. For a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the number $L(\alpha)$ is the best constant such that

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{L(\alpha)q^2}$$

has infinitely many solutions $p, q \in \mathbb{Q}$. This problem is related to how geodesics on the modular surface $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ escape to the cusps. This idea was generalized in [HMU15] to the dynamics of the geodesic flow on the moduli space of translation surfaces, and more recently studied in [AMU20].

Some of the above ideas concern themselves specifically with properties of the geodesic flow on moduli space. Another motivation to study this flow and the whole $SL_2(\mathbb{R})$ action, is the general philosophy presented in [Zor06]. Namely, that studying the $SL_2(\mathbb{R})$ orbits of a particular translation surface, may give information about said surface. An instance of this is the celebrated theorem from Mirzakhani and Eskin [EM18], where they show that orbit closures in moduli space have particular algebro-geometric properties (they are affine submanfiolds). Some of the

consequences of this result are outlined in sections 3.6-3.7 in [Zor06]. It is thus natural to study the ergodic properties of the $SL_2(\mathbb{R})$ action on moduli space.

An ergodic group action may be mixing, but it is in general unknown whether ergodicity or mixing, imply higher order mixing. The question of whether mixing implies higher order mixing was originally posed by Rokhlin in [Rok67], and there is a counterexample by Ledrappier of an action that is mixing of order 2 but not mixing of order 3 [Led78]. However, as $SL_2(\mathbb{R})$ is in particular a simple Lie group with finite centre, and there is a measure on moduli space with respect to which it is ergodic (found independently by Masur [Mas82] and Veech [Vee82]), several mixing results follow immediately. First, the Howe-Moore theorem [HM79] implies that the action is mixing. It then follows from the main theorem in [Mar78], that the horocycle flow, that is, the $SL_2(\mathbb{R})$ action restricted to

$$\left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

is mixing of all orders. More importantly, it follows by the work of Mozes in [Moz92] that the full $SL_2(\mathbb{R})$ action is indeed mixing of all orders.

What we are interested in, is whether there are quantitative estimates for higher order mixing. It was shown by Avila, Gouëzel and Yoccoz in [AGY06] that the Teichmüller flow is exponentially mixing for functions with certain regularity. The goal of this thesis is to obtain such estimates for higher order correlations. Quantitative multiple mixing results were established in [BEG20] for homogeneous dynamics, and for higher rank Lie group actions in [Gor19]. We note that these results do not directly apply to moduli space, as it is not homogeneous, and the action is by $SL_2(\mathbb{R})$, which is not a higher rank Lie group.

In the following, we give some motivations to study quantitative ergodic theorems, and refer to [GN15] for a survey on this topic and their applications to different areas of number theory. We begin with the lattice counting problem. The Gauss circle problem is a long studied subject whose premise is to count how many integer points there are in a ball in $\mathbb{R}^n$ of a certain radius. A similar question can be asked for a locally compact group $G$ and a lattice $\Gamma < G$, i.e., a discrete subgroup for which there exists a finite $G$-invariant measure on $G/\Gamma$. If $G$ can be made into a metric space (which is the case for Lie groups) one is interested in estimating the

quantity

$$|\Gamma \cap B_t|,$$

where $B_t$ is the ball of radius $t > 0$ around the identity (we are assuming that the group $G$ is metrizable, which holds for Lie groups). As explained in [GN15] and in more detail in [Gor19], mixing in homogeneous dynamics helps in obtaining a quantitative estimate for the value $|\Gamma \cap B_t|$. As explained in [Mau07], quantitative mixing estimates help in obtaining an error term for the estimate rather than simply an asymptotic estimate. The counterpart to this problem in the world of translation surfaces is outlined in [ABEM12]. Briefly, if $q$ and $p$ are points in Teichmüller space, and $\Gamma$ is the mapping class group, then one is interested in the quantity $|\Gamma \cdot p \cap B_t(q)|$, where $B_t(q)$ is the ball of radius $t > 0$ in Teichmüller space around $q$. In [ABEM12], asymptotics of this quantity are obtained. Although we do not do this in this thesis, there may be a relation between quantitative mixing estimates and this lattice point counting problem as in the case of homogeneous spaces. Other counting problems in Teichmüller and moduli space which use ergodic theory methods are outlined in [Esk06].

Another application of quantitative mixing estimates is to central limit theorems. This questions was first studied by Sinai in the context of the geodesic flow $g_t$ on a compact Riemannian manifold $X$ with constant negative curvature in [Sin60], and volume measure $\mu$ on the unit tangent bundle $T^1X$. Here, he obtained that for regular enough observables $\varphi\colon X \to \mathbb{R}$, the functions

$$F_T(x) = \frac{1}{\sqrt{T}} \int_0^T \varphi(g_s x)\, ds, \qquad x \in X \tag{1}$$

converge in distribution to the normal law as $T \to \infty$. That is, for $\xi \in \mathbb{R}$ and some $\sigma \in \mathbb{R}_+$,

$$\frac{\mu\left(\{x \in T^1X :\ F_T(x) < \xi\}\right)}{\mu(T^1X)} \xrightarrow{T\to\infty} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-s^2/(2\sigma^2)}\, ds$$

For homogeneous spaces, this is the main subject of [BG17] where instead of an action of $\mathbb{R}$, a more general group action is considered. In this context, we have a locally compact group $G$ acting continuously on a finite measure Borel space $(X, \mu)$ in a measure preserving way. If $G$ is a Lie group, then we may endow it with a left invariant Riemannian metric $\rho$. Then let

$$B_T = \{g \in G :\ \rho(\mathrm{id}, g) < T\},$$

and vol be the volume measure arising from the Riemannian metric (this defines a Haar measure on G). This allows us to define the equivalent of (1) as

$$F_T(x) = \frac{1}{\mathrm{vol}(B_T)^{1/2}} \int_{B_T} \varphi \circ g(x)\, dg, \qquad x \in X$$

for a compactly supported continuous function $\varphi : X \to \mathbb{R}$. Then for this system to satisfy the central limit theorem, we mean that for $\xi \in \mathbb{R}$

$$\frac{\mu(\{x \in X : F_T < \xi\})}{\mu(X)} \xrightarrow{T\to\infty} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\xi} e^{-s^2/(2\sigma^2)}\, ds,$$

for some $\sigma \in \mathbb{R}_+$. Once again, quantitative multiple mixing estimates form part of the argument for establishing central limit theorems. In terms of translation surfaces, one can ask whether a central limit theorem holds for the $SL_2(\mathbb{R})$ action on a connected component of moduli space. Such a question was considered in [Buf06], and a central limit theorem was obtained for a discretization of the geodesic flow, known as the Rauzy-Veech induction algorithm. This algorithm arises from the study of interval exchange transformations, which themselves are suspended to give rise to translation surfaces. Again, we do not explore this application in this thesis, but we merely point out that this is a potential direction for future work.

# Results

Let $M$ be a compact surface of genus $g \geqslant 1$ and fix a finite subset $\Sigma \subset M$. With $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$ being a partition of $2g-2$, this defines a stratum in the moduli space of area one translation surfaces $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$. We will consider a connected component $\mathscr{C} \subset \mathscr{M}^{(1)}(M, \Sigma, \kappa)$. This space admits an ergodic action by the group $SL_2(\mathbb{R})$, with respect to the Masur-Veech measure $\mu$. The action by the subgroup $\{g_t = \mathrm{diag}(e^t, e^{-t}) : t \in \mathbb{R}\} < SL_2(\mathbb{R})$ is known as the Teichmüller geodesic flow, which gives rise to an unstable foliation $W^{uu}$. The leafs of this foliation come with a measure $\mu_u$ (which is compatible with $\mu$ as explained in section 1.8.3).

There are two parts to the thesis. In the first, we show that with $\alpha \in (0, 1]$, unstable leafs equidistribute exponentially for compactly supported $\alpha$-Hölder functions with respect to a Riemannian metric $\rho$ on $SL_2(\mathbb{R})$. Having shown this, we show

quantitative multiple equidistribution of unstable leafs under the geodesic flow; which is the content of the following theorem.

**Theorem.** *Let $n \in \mathbb{N}$ and $r \geqslant 1$. Then there exists a constant $\delta_r > 0$ satisfying the following. Let $\mathscr{F} \in W^{uu}$ be an unstable leaf and $\varphi_0$ be an $\alpha$-Hölder compactly supported function on $\mathscr{C}$ such that $\operatorname{int}(\operatorname{supp}(\varphi_0) \cap \mathscr{F})$ is relatively compact in $\mathscr{F}$ with respect to the subspace topology. Then for every $\alpha$-Hölder compactly supported function $\varphi_1, \dots, \varphi_r$ and $t_1, \dots, t_r \geqslant 0$, we have that*

$$\int_{\mathscr{F}} \varphi_0 \, (\varphi_1 \circ g_{t_1}) \cdots (\varphi_r \circ g_{t_r}) \, d\mu_u = \frac{1}{\mu(\mathscr{C})^r} \int_{\mathscr{F}} \varphi_0 \, d\mu_u \prod_{k=1}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu$$
$$+ \mathcal{O}\left( \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha,\infty}} \right) \operatorname{diam}(F)^{-(2g+\sigma-1)-(1-\alpha)} \, \mathfrak{M}(t_1, \dots, t_r)^{-\delta_r} \right),$$

*where $F = \operatorname{supp}(\varphi_0) \cap \mathscr{F}$ and*

$$\mathfrak{M}(t_1, \dots, t_r) = \exp\left( \min_{i \neq j} |t_i - t_j| \right).$$

The norm $\|\cdot\|_{\mathscr{D}_{n,\alpha,\infty}}$ will be described in section 1.8.2. In the second part of the thesis we show that the full $\mathrm{SL}_2(\mathbb{R})$ action is exponentially mixing of all orders, which is the content of the following theorem.

**Theorem.** *Let $n \in \mathbb{N}$ and $r \geqslant 1$. Then there exists a constant $\delta_r > 0$ such that for every $\alpha$-Hölder compactly supported function $\varphi_0, \varphi_1, \dots, \varphi_r$ on $\mathscr{C}$ and any $\gamma_1, \dots, \gamma_r \in \mathrm{SL}_2(\mathbb{R})$, the following holds*

$$\int_{\mathscr{C}} \varphi_0 \cdot \prod_{k=1}^{r} \varphi_k \circ \gamma_k \, d\mu = \frac{1}{\mu(\mathscr{C})^r} \prod_{k=0}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu$$
$$+ \mathcal{O}\left( \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha,\infty}} \right) \mathfrak{M}(\gamma_1, \dots, \gamma_r)^{-\delta_r} \right),$$

*where*

$$\mathfrak{M}(\gamma_1, \dots, \gamma_r) \coloneqq \exp\left( \min_{i \neq j} \rho(\gamma_i, \gamma_j) \right).$$

We note here that this theorem implies that the Teichmüller geodesic flow is exponentially mixing of all orders. Although we do not use the first theorem to prove the second, we use similar methods for the proofs.

## Outline of Proofs

We now outline the methods we use in this thesis to prove the main theorems. The goal is to borrow ideas from results in homogeneous dynamics as well as in hyperbolic dynamics, and extend them to the case of the $SL_2(\mathbb{R})$ action on moduli space. We first focus on the geodesic flow and equidistribution of unstable leafs, and then move on to the whole group action and multiple mixing. The first idea comes from section 8 in [Dol04], where quantitative equidistribution of unstable leafs of the geodesic flow is used to establish quantitative higher order mixing for hyperbolic maps. In this work, the notion of an almost Markov partition is introduced which, among other properties, is a partition of an unstable leaf into subsets with small enough diameter, but large enough volume. If the diameter is small enough, then one of the observables is almost constant on these subsets. We also need the volume to be large enough, because the smaller the volume, the worse the rate of equidistribution is. The proof then proceeds by induction on the number of observables and the main result in [AGY06] for the base case. One still needs to establish that the unstable leafs equidisitribute quantitatively under the geodesic flow. For this portion, we borrow ideas from [EM93] where they work with homogeneous spaces. We note that similar ideas were used in [KM96] to establish quantitative equidistribution results. Equidistribution is established by using the mixing property of the flow. However, before being able to use the mixing property, one needs to consider a thickening of an unstable leaf, which behaves well with respect to the flow. In [EM93], this intermediate step is named the wavefront lemma. In our case, we adapt this idea to moduli space and establish a quantitative version of this lemma. It is crucial in this step to use the (non-uniform) hyperbolicity of the Teichmüller geodesic flow. With these tools, we establish quantitative multiple equidistribution for the geodesic flow.

Next, we consider on the full action of $SL_2(\mathbb{R})$. The proof of this is also by induction on the number of observables. The first step is thus to establish quantitative mixing. This easily follows from the fact that the geodesic flow is exponentially mixing and by arguments similar to those used by Ratner in [Rat87] where the Cartan decomposition of $SL_2(\mathbb{R})$ into $SO(2)\left\{\mathrm{diag}\left(a, a^{-1}\right) : a \in \mathbb{R}\right\} SO(2)$ is considered. One would hope that this method can also be used for higher order correlations. This is however no longer possible, as the integral of interest does no longer correspond to an inner product on the space of square integrable functions.

To go around this, we follow the same arguments used to show that unstable leafs equidistribute under the geodesic flow and adapt them to the full $\mathrm{SL}_2(\mathbb{R})$ action.

The rest of the thesis is organized as follows. Chapter 1 introduces some notation and all the necessary notions to state and prove the main theorem. Besides basic definitions in topological groups, Lie groups, ergodic theory, representation theory, the mapping class group, Teichmüller space, and moduli space, we also give some proofs of statements that are useful for understanding the above objects, as well as a sketch of the proof of the Howe-Moore theorem for $\mathrm{SL}_2(\mathbb{R})$. In chapter 2, we establish quantitative multiple equidistribution for the Teichmüller geodesic flow. Finally, we prove that the whole $\mathrm{SL}_2(\mathbb{R})$ action is exponentially mixing of all orders in chapter 3.

# Chapter 1

# Background and Notation

In this chapter, we give the background and definitions we need for this thesis. We start out with general notation in 1.1, and then continue on the some basic definitions in functional analysis in 1.2. It is followed by a short section on topological groups and Lie groups in 1.3, as we will be dealing with actions of such groups. The definition of ergodicity, mixing and mixing of all orders is given in section 1.4, together with some consequences of ergodicity for certain group actions, as well as notions hyperbolicity in dynamical systems. Mixing is also studied with representation theoretic methods, which we introduce next in 1.5. There we will also give a sketch of a proof of the Howe-Moore theorem for $SL_2(\mathbb{R})$. In the rest of the chapter, we introduce the main object of interest, namely translation surfaces in 1.6, Teichmüller and moduli spaces in 1.7, and the $SL_2(\mathbb{R})$ action on these in 1.8.

## 1.1 Notation

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$ denote the positive integers, the integers, the rational numbers, the real numbers, the positive real numbers, and the complex numbers respectively. For a number $z = x + iy \in \mathbb{C}$, we denote by $\Re(z) = x$ the real part, and by $\Im(z) = y$ the imaginary part. For a topological space $X$ and a subset $E \subset X$, the interior will be denoted by $\mathrm{int}(E)$. When $V$ is a vector space, and $U, W$ are subspaces, we write $U \oplus W$ for the direct sum in $V$ (when this makes sense). Moreover, we write $\mathrm{span}\{E\}$ for the linear span of a subset $E \subset V$. For a set $X$ and functions $f, g$ on $X$, we write $f = \mathcal{O}(g)$ or $f \ll g$ if there exists a constant $A > 0$ such that $\|f(x)\| \leqslant A\|g(x)\|$ for all $x \in X$. We denote by $\|f\|_\infty = \sup_{x\in X}\|f(x)\|$ the

infinity norm of $f$, and by $\operatorname{supp}(f)$ the support of $f$. When $X$ is a metric space and $E$ is a subset, we will write $\operatorname{diam}(E)$ for its diameter. Now let $X$ and $Y$ be topological spaces, we will denote by $C(X, Y)$ the set of continuous functions between $X$ and $Y$. When the functions are from $X$ to $\mathbb{R}$ or $\mathbb{C}$, we write $C(X)$. The set of continuous functions with compact support is denoted by $C_c(X)$. Given a smooth manifold $M$, we write $TM_p$ for the tangent space of $M$ at $p \in M$, and $TM_p^*$ for the cotangent space. If $N$ is another smooth manifold, and $f\colon M \to N$ is a differentiable map, we will write $d_p f\colon TM_p \to TN_{f(p)}$ for the differential map. The Lebesgue measure on $\mathbb{R}$ will be denoted by $\mathrm{Leb}$. For an arbitrary measure $\mu$ on a measurable space $X$, if $Y$ is another measurable space and $f\colon X \to Y$ is a measurable function, the push-forward measure on $Y$ is denoted by $f_*\mu$. In addition, we let $\chi_E$ be the characteristic function of a subset $E$. Given a matrix $A$, we will write ${}^{\intercal}A$ for its transpose, and $\operatorname{tr}(A)$ for its trace. For $d \in \mathbb{N}$, we write $\mathrm{GL}_d(\mathbb{R})$ for the real invertible $d \times d$ matrices, and work with the following subgroups $\mathrm{SL}_d(\mathbb{R}) = \{A \in \mathrm{GL}_d(\mathbb{R}) : \det(A) = 1\}$ of determinant 1 matrices, and $\mathrm{SO}(d) = \{A \in \mathrm{SL}_d(\mathbb{R}) : A\,{}^{\intercal}A = \mathrm{id}\}$. Here $\mathrm{id}$ denotes the identity matrix in $\mathrm{GL}_d(\mathbb{R})$. When $d = 2$,

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\},$$

in which case we write

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Finally, for a group $G$, we denote the identity element by $e_G$.

## 1.2 Functional Analysis

We collect some basic facts and definitions from functional analysis that will be used in the later sections.

We will need the following notion of a topology on the space of continuous functions [for details, see Mun00, section 46].

**Definition 1.1.** *Let $X$ and $Y$ be topological spaces. For a compact subset $K \subset X$ and an open subset $U \subset Y$, let*

$$S(K, U) \coloneqq \{f \in C(X, Y) : f\{K\} \subset U\}.$$

*These sets form a subbasis for a topology on $C(X, Y)$, called the* ***compact-open topology****.*

In this definition, a subbasis is a collection of subsets whose union is the whole space, and the topology they define consists of arbitrary unions and finite intersections of the elements in the subbasis.

We will be working with measure spaces $(X, \mu)$, and will denote by $L^p(X) = L^p(X, \mu)$ the space of $p$ integrable measurable functions on $X$, for $p \geqslant 1$. When $p = 2$, this is a Hilbert space with an inner product, denoted by $\langle \cdot, \cdot \rangle$, and defined by

$$\langle \varphi, \psi \rangle = \int_X \varphi \overline{\psi} \, d\mu.$$

For a subspace $V \subset L^2(X)$, the **orthogonal complement** of $V$, denoted by $V^\perp$, is the subspace

$$V^\perp \coloneqq \left\{ w \in L^2(X) \, : \, \langle w, \vartheta \rangle = 0 \text{ for all } \vartheta \in V \right\}.$$

Given a Hilbert space $\mathscr{H}$, we say that a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathscr{H}$ **weakly converges** to $x \in \mathscr{H}$ if

$$\langle x_n, y \rangle \to \langle x, y \rangle, \quad \text{for all } y \in \mathscr{H}.$$

## 1.3 Topological Groups and Lie Groups

As we deal with the action of $SL_2(\mathbb{R})$ on Teichmüller space, we introduce some notions of topological and Lie groups. We refer to [Fol95] for more on topological groups, and to [Kna02] for Lie groups.

**Definition 1.2.** *A **topological group** is a group together with a topology such that the group operation and inversion are continuous maps.*

When a topological group $G$ is locally compact (and Hausdorff), i.e. every point has a compact neighbourhood, the group admits a unique, up to positive scaling, **left Haar measure** on $G$. This is a nonzero Radon measure $\lambda$ such that for every $g \in G$ and every Borel subset $E \subset G$,

$$\lambda(gE) = \lambda(E).$$

For a proof of this, see chapter 2.2 in [Fol95].

As we deal with asymptotic properties of a group action, we say that a sequence $(g_n)_{n \in \mathbb{N}} \subset G$ **goes to infinity**, and write $g_n \to \infty$, if for every compact set $K \subset G$,

there exists an $N \in \mathbb{N}$ such that for each $n \geqslant N$, $g_n \notin K$. That is, the sequence escapes every compact subset.

A **Lie group** $G$ is defined to be a topological group together with a smooth structure such that multiplication and inversion are smooth maps. The **Lie algebra** of $G$ is the tangent space at the identity, and is denoted by $\mathfrak{g}$. This is a vector space which comes with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, such that for every $X, Y, Z \in \mathfrak{g}$,

(i) $[X, Y] = -[Y, X]$, and

(ii) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

A Lie algebra is called **abelian** if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Moreover, $\mathfrak{g}$ is called **simple** if it is not abelian and the only ideals (with respect to the Lie bracket) are $\{0\}$ and $\mathfrak{g}$. The algebra is called **semisimple** if it is a direct sum of simple Lie algebras. A Lie group is then called **simple** (respectively **semisimple**) if its Lie algebra is simple (respectively semisimple).

There is a map connecting the Lie algebra to the Lie group, namely the **exponential map**, $\exp\colon \mathfrak{g} \to G$ defined as follows. Let $X \in \mathfrak{g}$ and consider the curve $c\colon [0, T] \to G$ (for some $T > 0$) with $c(0) = e_G$, and $c'(0) = X$. Then $\exp(X) \coloneqq c(1)$. Although this definition is useful for general Lie groups, we will only be dealing with matrix Lie groups, for which the exponential map is the matrix exponential.

Finally, we define the **adjoint representation** of $G$ as the map

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto d_{e_G} c_g, \end{aligned}$$

where $c_g\colon G \to G$ is conjugation by $g$, i.e. $c_g(h) = ghg^{-1}$. Accordingly, the **adjoint representation** of $\mathfrak{g}$ is the map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad} \coloneqq d_{e_G}\mathrm{Ad}$. In our case, where we are dealing with closed subgroups of $\mathrm{GL}_n(\mathbb{R})$, the Lie bracket is given by $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{g}$. Moreover, we have that for any $g \in G$,

$$\mathrm{Ad}(g)(X) = gXg^{-1}, \quad \text{and}, \quad \mathrm{ad}(X)(Y) = [X, Y].$$

We note that the second equality is one for general Lie groups, while the first one is for matrix groups. We will only use these maps in Appendix B.

### 1.3.1 The Case of $SL_2(\mathbb{R})$

The group $SL_2(\mathbb{R})$ is a Lie group, since it inherits the subspace topology when viewed as a subset of $\mathbb{R}^4$. The Lie algebra of $SL_2(\mathbb{R})$ is

$$\mathfrak{sl}(2,\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \operatorname{tr}(A) = 0\},$$

and the Lie bracket is given by $[A, B] = AB - BA$. The group $SL_2(\mathbb{R})$ is also simple, as we see below. The Lie algebra is not abelian, as we can take

$$A = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad B = \begin{pmatrix} & 2 \\ -1 & \end{pmatrix},$$

and with that

$$AB - BA = \begin{pmatrix} & 4 \\ 2 & \end{pmatrix}.$$

To see that it has no non-trivial ideals, note that a basis for $\mathfrak{sl}(2,\mathbb{R})$ is

$$E = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad U^+ = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad U^- = \begin{pmatrix} & \\ 1 & \end{pmatrix},$$

and that

$$[E, U^+] = 2U^+, \quad [E, U^-] = -2U^-, \quad [U^+, U^-] = E.$$

Assume now that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Then $\mathfrak{h}$ cannot be $\mathbb{R}E$ by the first two equalities above. Neither can it be $\mathbb{R}U^+$ nor $\mathbb{R}U^-$ since $[U^+, U^-] = E$. Moreover, it cannot be $\mathbb{R}E \oplus \mathbb{R}U^+$ nor $\mathbb{R}E \oplus \mathbb{R}U^-$ again by the first two equalities. The last possibility of $\mathbb{R}U^+ \oplus \mathbb{R}U^-$ is ruled out by the last equality above. This means in particular that no linear combination of the above subspaces can be an ideal. Hence $\mathfrak{h}$ is either $\{0\}$ or $\mathfrak{g}$.

We are interested in quantitative estimates of the $SL_2(\mathbb{R})$ action on Teichmüller space. This is done in terms of a metric on the group. One can define a Riemannian metric on the group, by first fixing an inner product on the Lie algebra. The inner product is then defined for all other tangent spaces by pushing the one on the Lie algebra with the differential of left translation giving rise to Riemannian metric on the Lie group. See Appendix B for the specific metric we use.

## 1.4 Ergodic Theory and Dynamical Systems

Here we introduce the dynamical and ergodic notions that we deal with in this thesis. We follow [BM00] for basic ergodic theory, [EW11] for definitions of higher order mixing, and Margulis' PhD thesis [Mar04] and [BP07] for stable and unstable manifold definitions. We begin with general group actions.

**Definition 1.3.** *Let* $G$ *be a topological group, and* $(X, \mu)$ *a Borel space with a finite* $\sigma$*-finite measure* $\mu$*. A (continuous)* ***group action*** *of* $G$ *on* $X$ *is a continuous map*

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

*such that the following hold:*

*(i) For every* $g_1, g_2 \in G$ *and* $x \in X$*,*

$$g_1 (g_2 x) = (g_1 g_2) x \quad \text{and} \quad e_G x = x,$$

*and*

*(ii)* $G$ *preserves the measure* $\mu$*, i.e.*

$$\mu (gE) = \mu (E),$$

*for every* $g \in G$*, and every Borel subset* $E \subset X$*.*

*The group action is said to be* ***ergodic*** *if for every* $G$*-invariant Borel subset* $E \subset X$*, either* $\mu (E) = 0$*, or* $\mu (X \smallsetminus E) = 0$*.*

In general, one can deal with group actions that only preserve the measure class of $\mu$ (that is, the action preserves the measure zero sets of $\mu$). Moreover, one could also define the group action on simply a measurable space (the action is then asked to be a measurable map). However, we will only be dealing with measure preserving systems on topological spaces. We will also write $G \curvearrowright X$ for a group action, and may call a group action a **dynamical system**.

The above definition of ergodicity is in terms of the group action and the space itself. We will be interested in more functional analytic definitions of such notions.

**Definition 1.4.** *A group action $G \curvearrowright X$ is*

*(i)* ***ergodic*** *if every measurable function $\varphi\colon X \to \mathbb{C}$ for which $\varphi(gx) = \varphi(x)$ $\mu$-almost everywhere, is constant almost everywhere,*

*(ii)* ***mixing*** *if for every $\varphi, \psi \in L^2(X)$,*

$$\int_X \psi \cdot \varphi \circ \gamma \, d\mu \xrightarrow{\gamma \to \infty \text{ in } G} \frac{1}{\mu(X)} \int_X \varphi \, d\mu \int_X \psi \, d\mu,$$

*(iii)* ***mixing of order*** $r \geqslant 2$ *if for every $\varphi_1, \ldots, \varphi_r \in L^\infty(X)$,*

$$\int_X (\varphi_1 \circ \gamma_1) \cdots (\varphi_r \circ \gamma_r) \, d\mu \longrightarrow \frac{1}{\mu(X)^{r-1}} \int_X \varphi_1 \, d\mu \cdots \int_X \varphi_r \, d\mu,$$

*as $\gamma_i \to \infty$, and $\gamma_i \gamma_j^{-1} \to \infty$ for $i \neq j$ in $G$,*

*(iv)* ***mixing of all orders*** *if it is mixing of order $r$ for all $r \geqslant 2$.*

For an equivalence of the ergodic definitions, see proposition 2.14 in [EW11] or 2.1.11 in [Zim84] for group actions. We will say that an action is **quantitativley mixing** (respectively **quantitatively mixing of order** $r$, **quantitatively mixing of all orders**) if the convergence to the right hand side depends on $\rho(e_G, \gamma)$ (respectively on $\rho(\gamma_i, \gamma_j)$) for some metric $\rho$ on $G$.

For completeness, we state the classical Birkhoff ergodic theorem, along with some comments on the generalization to group actions.

**Theorem 1.5** (Birkhoff Ergodic Theorem)**.** *Let $(X, \mu)$ be a probability measure space, and $T\colon X \to X$ be a measure preserving measurable map. If $T$ is ergodic, then for any $\varphi \in L^1(X)$ and $\mu$-almost every $x \in X$*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi \circ T^k(x) = \int_X \varphi \, d\mu.$$

This was proved by Birkhoff in [Bir31], and we refer to theorem 2.30 in [EW11] for various proofs. In the above, ergodicity of a map means that for any $T$-invariant measurable subset $E \subset X$, one has that $\mu(E) \in \{0, 1\}$.

The above theorem can be interpreted as the result of an action of $\mathbb{Z}$ on X by $(n, x) \mapsto T^n(x)$. The theorem itself is a type of equidistribution of points as T acts on them. It is then natural to ask whether such a result holds for more general group actions.

One class of groups for which such a result holds is the following. A topological group G is said to be **amenable** if for every continuous action of G on a compact metrizable space X, there exists a G-invariant probability measure on X. There are multiple equivalent definitions of amenability, but we work with this one, from [Zim84]. As proven in 4.12 in [Zim84], every abelian group is amenable. In particular, $\mathbb{Z}$ is amenable.

It turns out that results similar to theorem 1.5 hold for actions of amenable groups [see EW11, chapter 8]. Since $\mathbb{Z}$ is amenable, then these results are indeed generalizations of theorem 1.5.

**Remark 1.6.** Our main interest in this thesis is the action of $SL_2(\mathbb{R})$. However, $SL_2(\mathbb{R})$ is not amenable and hence the above cannot be directly applied to actions of this group. To see that it is not amenable, consider its action on the projective line $\mathbb{R}P^1$ by matrix multiplication on homogeneous coordinates. The measures invariant under

$$\left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{R} \right\} < SL_2(\mathbb{R})$$

are the measures of the form $s\,dx + t\delta_\infty$ (where $\delta_\infty$ is the Dirac measure supported at $\infty \in \mathbb{R}P^1$ and $dx$ is the Lebesgue measure on $\mathbb{R}$). Out of these measures, the ones invariant under $\left\{ \mathrm{diag}\left(a, a^{-1}\right) : a \neq 0 \right\} < SL_2(\mathbb{R})$ must be supported at $\infty$. However, none of these are invariant under

$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

See figure 1.1 for a schematic of how these matrices act.

Nevertheless, there is an ergodic theorem for $SL_2(\mathbb{R})$ actions, presented in [GN09], and even a quantitative version for the $SL_2(\mathbb{R})$ action on moduli space. As we require further definitions in representation theory, and the action, we leave this portion to section 1.8.4.

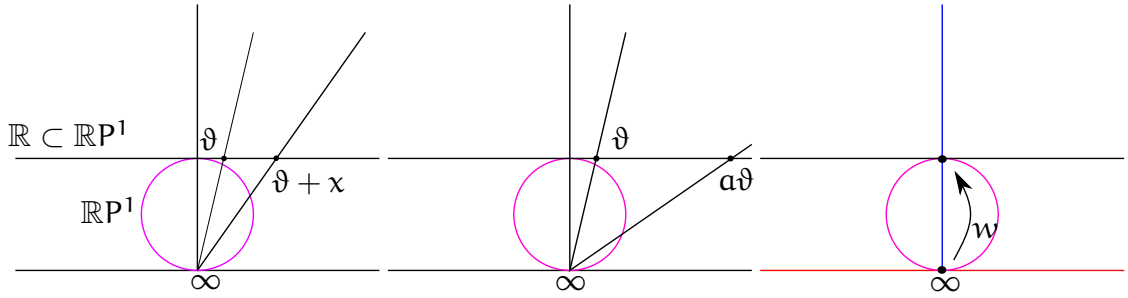


Figure 1.1: $SL_2(\mathbb{R})$ is not amenable: Left shows action by upper triangular matrix, middle shows action by diagonal matrix, and right shows action of $w$ on $\infty$

Now we move on to describe how a group action gives rise to foliations on $X$. For this section, we also assume that $X$ is a smooth manifold, and follow [Mar04]. We deal with **flows** on $X$, that is, dynamical systems coming from an action of $\mathbb{R}$ on $X$. We begin with the notion of a foliation.

**Definition 1.7.** *A **foliation** $\mathcal{G} = \mathcal{G}_k$ of $X$ is a decomposition of $X$ into smooth $k$-dimensional submanifolds, called **leafs**, with the following properties:*

*(i) For $p \in X$, denoting by $\mathcal{G}(p) \in \mathcal{G}$ a leaf in the foliation, we ask that the tangent spaces $T\mathcal{G}(p)_p$ vary continuously with $p$.*

*(ii) Every smooth curve tangent to $\mathcal{G}(p)$ at some point $p \in X$ is completely contained in $\mathcal{G}(p)$.*

*Moreover, we say that two foliations $\mathcal{G}_1$ and $\mathcal{G}_2$ on $X$ are **complementary** (in the sense of Margulis) if at each $p \in X$,*

$$TX_p = T\mathcal{G}_1(p)_p \oplus T\mathcal{G}_2(p)_p .$$

The last part means that, in particular, for a point in the intersection of leafs from different foliations is transversal. A flow is said to be **hyperbolic** if there are complementary foliations $\mathcal{G}^u$ and $\mathcal{G}^s$ such that the flow is contracting on one, and expanding on the other with respect to the differential of the flow, and a norm on the tangent spaces. See [KH95] for a more detailed description in terms of single maps, and [FH19] for an account of hyperbolic flows. The foliations are called the **unstable** and **stable** foliations respectively. For a hyperbolic flow $\Phi^t : X \to X$ on a metric space $(X, d)$ (the action is to be understood as $(t, x) \mapsto \Phi^t(x)$), the strong

stable manifold at $x \in X$ (respectively strong unstable manifold) denoted by $W^s(x)$ (respectively $W^u(x)$) is given by

$$W^s(x) = \left\{ y \in X : \ d\left(\Phi^t(x), \Phi^t(y)\right) \xrightarrow{t\to\infty} 0 \right\}, \text{ and respectively}$$
$$W^u(x) = \left\{ y \in X : \ d\left(\Phi^{-t}(x), \Phi^{-t}(y)\right) \xrightarrow{t\to\infty} 0 \right\}.$$

The system we deal with in this thesis is not hyperbolic. It is however non-uniformly hyperbolic, and there will be a similar description of the foliations. We do not give the full definition here, and refer to [BP07] (in particular definition 2.2.1) for details. Let $\mathbb{R} \curvearrowright X$ be a flow on a smooth manifold X, which we denote by $(t, x) \mapsto \Phi^t(x)$. For a fixed time $t \in \mathbb{R}$, this is a map from X to X. For $x \in X$, we write $d_x\Phi^t \colon TX_x \to TX_{\Phi^t(x)}$ for its differential, and let $\|\cdot\|$ be a norm on the tangent spaces (we omit the index in the norm indicating the particular tangent space).

**Definition 1.8.** *Assume there exist measurable functions* $C, a, b \colon X \to (0, \infty)$, $\varepsilon \colon X \to [0, \varepsilon_0]$, *with* $\varepsilon_0 > 0$ *such that for every* $x \in X$, *and* $t \in \mathbb{R}$,

$$a\left(\Phi^t(x)\right) = a(x), \quad b\left(\Phi^t(x)\right) = b(x), \quad \varepsilon\left(\Phi^t(x)\right) = \varepsilon(x),$$

*and*

$$a(x)\, e^{\varepsilon(x)} < b(x)\, e^{-\varepsilon(x)}.$$

*Then the above action is* ***non-uniformly hyperbolic*** *if the following hold for each* $x \in X$.

*(i) The tangent space at each point decomposes as*

$$TX_x = E^1(x) \oplus E^2(x)$$

*depending measurably on* x *and such that* $d_x\Phi^t\left(E^i(x)\right) = E^i\left(\Phi^t(x)\right)$, *for* $i = 1, 2$, *i.e. the decomposition is invariant under the flow.*

*(ii) For any* $v \in E^1(x)$ *and* $t > 0$

$$\| d_x\Phi^t(v)\| \leqslant C(x)\, a(x)^t\, e^{\varepsilon(x)t} \|v\|.$$

*(iii) For any* $v \in E^2(x)$ *and* $t < 0$

$$\| d_x\Phi^t(v)\| \leqslant C(x)\, b(x)^t\, e^{\varepsilon(x)t} \|v\|.$$

In this case, there are still complementary foliations into stable and unstable manifolds [see BP07, chapter 7]. These foliations are given by the integral manifolds of the decomposition of the particular tangent spaces. The reason for the name non-uniformly hyperbolic is that the constants $C, a$ and $b$ are not uniform in $X$. Moreover, $a$ could grow similarly to $e^{\varepsilon(x)}$ and hence there does not necessarily have to be any strict contraction. This would be closer to having non-expansion or partially hyperbolic systems [BW10, see]. We will describe the particular foliations that arise in the system we are interested in in section 1.8.

## 1.5 Representation Theory

We begin with general notions in representation theory, for which we follow the presentation in chapter 3 of [Fol95]. Let $\mathscr{H}$ be a Hilbert space over $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{U}(\mathscr{H})$ the space of unitary operators on $\mathscr{H}$, i.e. operators on $\mathscr{H}$ for which $A^* = A^{-1}$, where $A^*$ is the adjoint of $A$ with respect to the inner product on $\mathscr{H}$. A **unitary representation** of $G$ is a group homomorphism $\pi\colon G \to \mathcal{U}(\mathscr{H})$ which is continuous in the strong operator topology, i.e. for every $\xi \in \mathscr{H}$, the map $g \mapsto \pi(g)\xi$ is continuous. A consequence of the definition, is that for every $g \in G$,

$$(\pi(g))^* = \pi\left(g^{-1}\right).$$

**Remark 1.9.** Given a Borel space $(X, \mu)$ and a continuous group action $G \curvearrowright X$, there is a natural unitary representation on the Hilbert space $\mathscr{H} = L^2(X)$. We define $\varrho\colon G \to \mathcal{U}(\mathscr{H})$ as follows. For $g \in G$ and $\vartheta \in \mathscr{H}$,

$$(\varrho(g)\vartheta)(x) \coloneqq \vartheta\left(g^{-1}x\right), \quad x \in X. \tag{1.1}$$

Using $g^{-1}$ is what makes this a homomorphism. Indeed, let $\vartheta \in \mathscr{H}$, and $g, h \in G$. Then for $x \in X$

$$\begin{aligned}\varrho(gh)\vartheta(x) &= \vartheta\left(h^{-1}g^{-1}x\right) = \vartheta \circ h^{-1} \circ g^{-1}(x)\\ &= \varrho(g)\left(\vartheta \circ h^{-1}\right)(x) = (\varrho(g)\varrho(h))\vartheta(x).\end{aligned}$$

To see that it is unitary, let $\varphi, \psi \in \mathscr{H}$, and $g \in G$, then

$$\langle\varrho(g)\varphi, \psi\rangle = \int_X \left(\varphi \circ g^{-1}\right)\psi \, d\mu = \int_X \varphi\,(\psi \circ g)\; d\mu = \left\langle\varphi, \varrho\left(g^{-1}\right)\psi\right\rangle \; d\mu,$$

where the second equality follows from the fact that the group action is measure preserving. The continuity of the representation comes from the continuity of the group action. This is the main representation we will be interested in. However, we continue with general representations for now.

TO continue, we introduce the notion of a spectral gap, as in [EW17].

**Definition 1.10.** *Let* $\pi\colon G \to \mathscr{H}$ *be a unitary representation. Then* $\pi$ *is said to have* ***almost-invariant vectors*** *if for every* $\varepsilon > 0$ *and compact subset* $Q \subseteq G$ *there exists a unit vector* $\vartheta \in \mathscr{H}$ *such that*

$$\sup_{g \in Q} \|\pi(g)\,\vartheta - \vartheta\| \leqslant \varepsilon.$$

*A unitary representation is then said to have* ***spectral gap*** *if* $\pi$ *restricted to*

$$\mathscr{H}^G \coloneqq \{\vartheta \in \mathscr{H} : \pi(g)\,\vartheta = \vartheta \text{ for all } g \in G\}$$

*does not have almost-invariant vectors.*

This definition is relevant in section 1.8.4.

To continue, if we have two unitary representations $\pi\colon G \to \mathcal{U}(\mathscr{H})$ and $\pi'\colon G \to \mathscr{H}'$ we call a bounded linear map

$$T\colon \mathscr{H} \to \mathscr{H}'$$

an **intertwining operator** if

$$T \circ \pi(g) = \pi'(g) \circ T,$$

for all $g \in G$. The set of all such operators is denoted by $\mathcal{C}(\pi, \pi')$. Two such representations are then called (unitarily) **equivalent** if there exists a bijection $U \in \mathcal{C}(\pi, \pi')$ such that

$$\pi'(g) = U \circ \pi(g) \circ U^{-1},$$

for all $g \in G$.

Given a unitary representation $\pi\colon G \to \mathcal{U}(\mathscr{H})$, a subspace $\mathcal{M}$ of $\mathscr{H}$ is said to be an **invariant subspace** for $\pi$ if $\pi(g)\mathcal{M} \subseteq \mathcal{M}$ for every $g \in G$, where $\pi(g)\mathcal{M} = \{\pi(g)\,\xi \in \mathscr{H} : \xi \in \mathcal{M}\}$. If $\mathcal{M} \neq \{0\}$, then $\pi|_{\mathcal{M}}$ is said to be a **subrepresentation** of

$\pi$. We say that $\pi$ is **reducible** if it admits a nontrivial invariant subspace. The representation is said to be **irreducible** otherwise. A vector $\xi \in \mathscr{H}$ is said to be $G$**-invariant** if $\pi(g)\xi = \xi$ for all $g \in G$. Notice that the set of all invariant vectors is also an invariant subspace. We further define the **direct sum** of a countable family of representations $(\pi_i \colon G \to \mathscr{H}_i)_{i\in I}$, denoted by $\bigoplus_{i\in I} \pi_i$, as the representation $\pi$ on $\mathscr{H} = \bigoplus_{i\in I} \mathscr{H}_i$ given by

$$\pi(g)\left(\sum_{i\in I} \vartheta_i\right) = \sum_{i\in I} \pi_i(g)\,\vartheta_i, \quad \vartheta_i \in \mathscr{H}_i.$$

With the above notions, if $\pi$ is a reducible unitary representation on $\mathscr{H}$, one could try to decompose the space into irreducible subspaces. If it is possible to decompose the space into countably many subspaces, then one can decompose the representation itself by constructing a direct sum as above. For the general case, we need the notion of a direct integral.

### 1.5.1 Direct Integrals

We do not give a detailed introduction of this notion here, we only explain the basics. This section and the next are not used in the the rest of thesis, but we include them to be able to make sense of the work of Ratner in [Rat87] and the last portion of [AGY06] where it is shown that the $SL_2(\mathbb{R})$ action we are interested in has a spectral gap, as both of these are related to the notions we work with.

The construction of direct integrals is due to von Neumann in [von49], and for a careful treatment, see [Mau50a] for an overview of direct integrals, [Mau50b] for integral decompositions for locally compact groups, and see chapter 8 of [Dix64] (in particular theorem 8.3.2) for the integral decomposition of unitary representations into irreducible components. In this subsection we borrow from these three sources.

Let $(Y, \nu)$ be a standard Borel space (i.e. a Borel space measurably isomorphic to $[0,1] \subset \mathbb{R}$ with the Lebesgue measure and Borel $\sigma$-algebra) of finite measure and assume that for each $\eta \in Y$, there is a Hilbert space $\mathscr{H}_\eta$ of dimension $k(\eta) \in \mathbb{N}\cup\{\infty\}$. Now let

$$\vartheta \colon Y \to \prod_{\eta\in Y} \mathscr{H}_\eta$$

be a $\nu$-measurable function (with respect to the product topology) such that for each $\eta \in Y$, $\vartheta(\eta) \in \mathscr{H}_\eta$. We say that two such functions are equivalent if they differ on a set of $\nu$-measure zero. For each $\eta \in Y$, we can then write out the components of $\vartheta(\eta)$ in $\mathscr{H}_\eta$ as

$$\vartheta(\eta) = \sum_{m=1}^{k(\eta)} \vartheta_m(\eta).$$

Then let $\mathscr{H}$ be the Hilbert space of equivalence classes of such complex valued $\nu$-measurable functions such that

$$\int_Y \sum_{m=1}^{k(\xi)} |\vartheta_m(\eta)|^2 \, d\nu < \infty.$$

The functions $\vartheta \in \mathscr{H}$ evaluated at some $\eta \in Y$ are to be thought of as components of a vector in $\mathscr{H}$. With this definition, we call $\mathscr{H}$ the **direct integral** of the spaces $\mathscr{H}_\eta$, and write

$$\mathscr{H} = \int_Y^\oplus \mathscr{H}_\eta.$$

Now, given a unitary representation $\pi$ of $G$ on $\mathscr{H}$, we have from theorem 1.1 in [Mau50b] that we can decompose $\mathscr{H}$ into invariant subspaces. Moreover, we may write

$$\pi = \int_Y^\oplus \pi_\eta \, d\nu(\eta),$$

where $\pi_\eta$ is a representation of $G$ on $\mathscr{H}_\eta$ defined as follows. For $\vartheta \in \mathscr{H}$ and $g \in G$,

$$\pi_\eta(g)(\vartheta(\eta)) = (\pi(g)\vartheta)(\eta).$$

Finally, it follows from theorem 8.3.2 in [Dix64] that the $\mathscr{H}_\eta$ can be chosen such that for $\nu$-almost every $\eta \in Y$, $\pi_\eta$ is irreducible.

### 1.5.2 Compact Abelian Groups

In this thesis, we will be working with the group $SL_2(\mathbb{R})$, and its Cartan decomposition into $KA^+K$, with $K = SO(2)$, and $A^+ = \{\mathrm{diag}\{e^t, e^{-t}\} : t \geqslant 0\}$ (see Appendix A). Since $SL_2(\mathbb{R})$ is a locally compact group, the above discussion applies to it. We will be interested in further decomposing restrictions of representations of $SL_2(\mathbb{R})$ to $K$. Since $K$ is a compact abelian group, we will restrict to the theory of such

groups below. Here we follow chapters 4 and 5 of [Fol95].

In the following, $G$ will denote a compact abelian topological group. Although we do not need the full strength of the following definition, we give it to more easily state some results below.

**Definition 1.11.** *The **dual** of $G$, denoted by $\widehat{G}$ is the set of all unitary equivalence classes of irreducible unitary representations of $G$. The equivalence class of a representation $\pi$ will be denoted by $[\pi]$.*

In our case, where $G$ is also abelian, all irreducible representations are one dimensional. In particular, any unitary irreducible representation $\pi$ of $G$ is (up to isomorphism of Hilbert spaces) a representation on $\mathbb{C}$. This allows us to define a **unitary character** of $G$ to be a continuous homomorphism from $G$ to the group $\mathbb{S}^1 \subset \mathbb{C}$, and identify the set of such unitary characters with $\widehat{G}$.

Moreover, for $u, w \in \mathscr{H}$, we call the functions

$$\begin{aligned} \psi_{u,w} \colon G &\to \mathbb{C} \\ g &\mapsto \langle \pi(g)u, w \rangle \end{aligned}$$

the **matrix coefficients** of $\pi$. Since the representation is continuous and $G$ is compact, the Haar measure on it is finite, and thus the matrix coefficients are elements of $L^2(G, \lambda)$. In fact, as the representation is continuous, the matrix coefficients are in $C(G)$. We can now consider the linear span of the matrix coefficients of the representation, namely

$$\mathcal{E}_\pi \coloneqq \operatorname{span}\{\psi_{u,w} : u, w \in \mathscr{H}\} \subset L^2(G, \lambda).$$

The above definition is for a particular unitary representation $\pi$. However, this subspace depends only on the equivalence class of $\pi$. We can therefore define

$$\mathcal{E} \coloneqq \bigcup_{[\pi] \in \widehat{G}} \mathcal{E}_\pi.$$

We may now state the Peter-Weyl Theorem.

**Theorem 1.12** (The Peter-Weyl Theorem)**.** *Let $G$ be a compact group. Then*

*(i) The subset $\mathcal{E}$ is dense in $C(G)$,*

*(ii) the space of square integrable functions on* $G$ *decomposes as*

$$L^2(G, \lambda) = \bigoplus_{[\pi] \in \widehat{G}} \mathcal{E}_\pi,$$

*and,*

*(iii) any unitary representation of* $G$ *on a Hilbert space decomposes into a direct sum of irreducible finite dimensional representations.*

For the first two parts of the theorem, see chapter 5.2 in [Fol95] and for the last part, see theorem 1.12 in [Kna86]. For us, the important part of the theorem is part (iii), as it allows us to further decompose unitary representations of $SL_2(\mathbb{R})$ by restricting them to the compact group $SO(2)$.

In our particular case, as $SO(2)$ is also abelian, any unitary representation $\pi$ of $SO(2)$ on a Hilbert space $\mathscr{H}$ decomposes the space into

$$\mathscr{H} = \bigoplus_{n \in \mathbb{Z}} \mathscr{H}_n,$$

where each $\mathscr{H}_n$ is finite dimensional. In what follows, we focus on a particular unitary representation of $SL_2(\mathbb{R})$.

### 1.5.3 The Case of $SL_2(\mathbb{R})$

We denote by $G = SL_2(\mathbb{R})$ and $K = SO(2)$. Consider a continuous action of $G$ on a finite measure space $(X, \mathscr{B}, \mu)$ and the unitary representation $\varrho$ it induces on $L^2(X)$ from (1.1). Given an arbitrary action of $G$ on $X$, it is highly unlikely that the representation it induces is irreducible. However, we may decompose the representation into a direct integral of irreducible representations as in section 1.5.1. For a standard Borel space $(Y, \nu)$, we write

$$\varrho = \int_Y^{\oplus} \varrho_\eta \, d\nu(\eta),$$

and the corresponding decomposition of the Hilbert space as

$$\mathscr{H} = \int_Y^{\oplus} \mathscr{H}_\eta.$$

Now consider the representations $\varrho_\eta$ restricted to the subgroup $K$. Since $K$ is compact, we may apply part (iii) of the Peter-Weyl theorem to get that

$$\mathscr{H}_\eta = \bigoplus_{m\in\mathbb{Z}} \mathscr{H}_{\eta,m},$$

where each $\mathscr{H}_{\eta,m}$ is finite dimensional. Up to relabelling, these have the following form:

$$\mathscr{H}_{\eta,m} = \left\{\vartheta \in \mathscr{H}_\eta : \ \varrho_\eta\left(k(\theta)\right)\vartheta = e^{im\theta}\vartheta, \ \theta \in [0, 2\pi)\right\}.$$

This is because the unitary characters of $K$ all have the form $e^{im\theta}$ [see EW17, section 3.5 and in particular 3.5.2]. With this decomposition, it follows from lemma 1.1 in [Rat87] that each $\mathscr{H}_\eta$ has an orthonormal basis consisting of $K$-eigenfunctions.

We do not explicitly use these properties of the $SL_2(\mathbb{R})$ action in this thesis, but mention them as they are essential in deducing that if the action of $A^+$ is quantitatively mixing, then so is the action of the whole group [Rat87].

### 1.5.4 The Howe-Moore Theorem

In this section we state the Howe-Moore theorem and explain its relation to the mixing property when dealing with unitary representations arising from a group action. We will only explain some ideas of the proof in the $SL_2(\mathbb{R})$ case. We formulate the theorem as in chapter III of [BM00], and it is due to Howe and Moore [HM79].

Let $G$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Since $G$ is in particular a manifold, it is locally compact, and hence possesses a Haar measure. By semisimplicity of $G$, its Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,$$

for some $n \in \mathbb{N}$, and where $\mathfrak{g}_i$ is a simple Lie algebra for $i = 1, \ldots, n$. Letting $S_i \coloneqq \exp(\mathfrak{g}_i)$, we have that

$$G = S_1 \cdots S_n.$$

The subgroups $S_i$ are called the **simple factors** of $G$. We may now state the theorem.

**Theorem 1.13** (Howe-Moore Vanishing Theorem)**.** *Let* $G$ *be a connected semisimple Lie group with finite centre, and* $\pi\colon G \to \mathscr{H}$ *be a unitary representation. Assume that the restriction of* $\pi$ *to any non-compact simple factor* $S_i$ *of* $G$ *has no non-trivial invariant vectors. Then all the matrix coefficients vanish at* $\infty$*, i.e.*

$$\psi_{\xi,\eta} = \langle \pi(\gamma)\,\xi, \eta\rangle \xrightarrow{\gamma\to\infty \text{ in } G} 0$$

*for all* $\xi, \eta \in \mathscr{H}$*.*

Before going through the ideas of the proof, we show the following corollary, which is the reason this theorem is important in ergodic theory.

**Corollary 1.14.** *Let* $G$ *be as in theorem 1.13, and assume it acts ergodically on a finite measure Borel space* $(X, \mu)$*. Then, the action is mixing.*

**Proof.** Let

$$L^2_0(X) \coloneqq (\mathbb{C}\cdot 1)^\perp = \left\{ \varphi \in L^2(X) \,:\, \int_X \varphi \, d\mu = 0 \right\}.$$

Then let $\pi\colon G \to L^2_0(X)$ be the induced unitary representation of the group action. By ergodicity, the only non-trivial invariant vectors in $L^2(X)$ are the constant functions. Hence, $\pi$ has no non-trivial invariant functions, and we may apply the Howe-Moore theorem to get that all matrix coefficients vanish at $\infty$. Now let $\varphi, \psi \in L^2(X)$, and write

$$\varphi = \varphi_0 + \varphi', \quad \psi = \psi_0 + \psi',$$

for $\varphi_0, \psi_0 \in (\mathbb{C}\cdot 1)$, and $\varphi', \psi' \in L^2_0(X)$. For $\gamma \in G$, we have that

$$\begin{aligned}
\int_X \psi\cdot\varphi\circ\gamma\, d\mu &= \left\langle \psi, \pi\left(\gamma^{-1}\right)\varphi\right\rangle = \langle \pi(\gamma)\,\psi, \varphi\rangle \\
&= \underbrace{\langle \pi(\gamma)\,\psi_0, \varphi_0\rangle}_{=\langle\psi_0,\varphi_0\rangle} + \cancelto{0}{\langle \pi(\gamma)\,\psi', \varphi_0\rangle} + \cancelto{0}{\langle \pi(\gamma)\,\psi_0, \varphi'\rangle} + \underbrace{\langle \pi(\gamma)\,\psi', \varphi'\rangle}_{\to 0} \\
&\xrightarrow{\gamma\to\infty} \langle \psi_0, \varphi_0\rangle = \int_X \psi_0\varphi_0\, d\mu \\
&= \psi_0\varphi_0\mu(X) = \frac{1}{\mu(X)}\int_X \psi_0\, d\mu \int_X \varphi_0\, d\mu \\
&= \frac{1}{\mu(X)}\int_X \psi_0\, d\mu \int_X \varphi_0\, d\mu + \frac{1}{\mu(X)}\cancelto{0}{\int_X \psi'\, d\mu}\int_X \varphi_0\, d\mu
\end{aligned}$$

$$+\frac{1}{\mu(X)}\int_X \psi_0\,d\mu\cancelto{0}{\int_X \varphi'\,d\mu}+\frac{1}{\mu(X)}\cancelto{0}{\int_X \psi'\,d\mu\int_X \varphi'\,d\mu}$$
$$=\frac{1}{\mu(X)}\int_X \psi\,d\mu\int_X \varphi\,d\mu.$$

where the cancellations happen by orthogonality, and we apply the Howe-Moore theorem in the third line. ■

**Remark 1.15.** To see that the assumption on the Lie group is important, consider the circle rotations by some angle $\alpha \in \mathbb{R}$. This corresponds to an action of $\mathbb{Z}$. Although the action is ergodic when $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, it is not mixing [see EW11, example 2.33]. Clearly the Howe-Moore theorem does not apply in this case as $\mathbb{Z}$ is abelian (it is also not connected, and so none of the required properties of the theorem apply).

**Remark 1.16.** Nevertheless, if one wants to know whether an ergodic dynamical system is mixing, one can ask whether the group action extends to an action of a Lie group with the properties required by theorem 1.13. For example, geodesic flows on hyperbolic surfaces are described by actions of $\mathbb{R}$, which is an abelian group and hence one cannot apply the Howe-Moore theorem. However, these flows are modelled on

$$\mathrm{PSL}_2(\mathbb{R})/\Gamma,$$

where $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ is a lattice (this is isomorphic to the unit tangent bundle of the surface). There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on this space, and the restriction to the subgroup

$$A^+ = \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} : t \geqslant 0 \right\} < \mathrm{SL}_2(\mathbb{R})$$

induces the geodesic flow. Then (after proving that the action is ergodic), the Howe-Moore theorem implies that the action is also mixing, and in particular that the flow is mixing. See [BM00] for details.

We now go through some of the ideas of the proof of the Howe-Moore theorem for $\mathrm{SL}_2(\mathbb{R})$, where we follow [BM00]. We note that the Howe-Moore theorem applies to $\mathrm{SL}_2(\mathbb{R})$ as it is a simple Lie group (and so as it is not compact, it has no non-compact non-trivial simple factors), and its centre is $\{\pm \mathrm{id}\}$. To start out, define

$$A^+ \coloneqq \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} : a \geqslant 1 \right\}, \quad N \coloneqq \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

$$N^- := \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad K := SO(2). \tag{1.2}$$

The proof is based on two ideas. First, by the Cartan decomposition of $SL_2(\mathbb{R})$ into $KA^+K$ and compactness of $K$, it is enough to show the theorem for a sequence $(a_n)_n \in \mathbb{N} \subset A^+$ going to $\infty$. The second part is based on the Mautner phenomenon. The general phenomenon is explained in detail in [Moo80]. Morally, it asserts that if a vector $\vartheta \in \mathscr{H}$ is invariant under some one-parameter subgroup of $G$, then it is invariant under a much larger subgroup.

As for the first step, we have the following.

**Reduction 1.17.** *With the above notations, assume that for all sequences $(a_n)_{n\in\mathbb{N}} \subset A^+$ going to $\infty$ and all $\xi, \eta \in \mathscr{H}$*

$$\lim_{n\to\infty} \psi_{\xi,\eta}(a_n) = 0.$$

*Then all matrix coefficients vanish at $\infty$.*

**Proof.** The proof goes by contradiction. Assume that there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset G$ going to $\infty$, but there is $\xi, \eta \in \mathscr{H}$ with

$$\lim_{n\to\infty} \psi_{\xi,\eta}(g_n) \neq 0.$$

Now, using the Cartan decomposition of $SL_2(\mathbb{R})$, write

$$g_n = k(\theta'_n)\, a_n k(\theta_n)$$

for $k(\theta_n), k(\theta'_n) \in K$ and $a_n \in A^+$. Since $g_n \to \infty$ and $K$ is compact, we have that $a_n \to \infty$. Then, by compactness of $K$ and continuity of the representation, up to passing to a subsequence, we may assume that

$$\lim_{n\to\infty} \pi\left(k(\theta'_n)^{-1}\right)\eta = \overline{\eta}, \quad \text{and} \quad \lim_{n\to\infty} \pi(k(\theta_n))\,\xi = \overline{\xi},$$

for some $\overline{\eta}, \overline{\xi} \in \mathscr{H}$. Then we have that

$$\begin{aligned} \psi_{\xi,\eta}(g_n) = &\left(\psi_{\xi,\pi\left(k(\theta'_n)^{-1}\right)\eta}(a_n k(\theta_n)) - \psi_{\xi,\overline{\eta}}(a_n k(\theta_n))\right) \\ &+ \left(\psi_{\pi(k(\theta_n)\xi),\overline{\eta}}(a_n) - \psi_{\overline{\xi},\overline{\eta}}(a_n)\right) + \psi_{\overline{\xi},\overline{\eta}}(a_n). \end{aligned}$$

The first two summands go to zero, and hence

$$0 \neq \lim_{n\to\infty} \psi_{\xi,\eta}\left(k(\theta'_n)\, a_n k(\theta_n)\right) = \lim_{n\to\infty} \psi_{\overline{\xi},\overline{\eta}}(a_n).$$

This is a contradiction since we assume that all matrix coefficients go to zero with $A^+$. ■

We now move on the the Mautner phenomenon. Before stating the version of the phenomenon that we need, we state the following lemma.

**Lemma 1.18.** *Let*

$$a = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} \in A^+,$$

$h \in N$, *and* $h^- \in N^-$. *Then*

$$\lim_{n\to\infty} a^n h a^{-n} = \mathrm{id}, \quad \text{if } \alpha < 1, \text{ and}$$
$$\lim_{n\to\infty} a^n h^- a^{-n} = \mathrm{id}, \quad \text{if } \alpha > 1.$$

**Proof.** The proof is a simple computation:

$$a^n h a^{-n} = \begin{pmatrix} 1 & \alpha^{2n}x \\ & 1 \end{pmatrix}, \quad \text{and}$$
$$a^n h^- a^{-n} = \begin{pmatrix} 1 & \\ \alpha^{2n}x & 1 \end{pmatrix}.$$

The lemma follows. ■

We now state the Mautner phenomenon, where we see that the non-commutativity of the group is essential.

**Lemma 1.19** (Mautner Phenomenon)**.** *Let* $G = SL_2(\mathbb{R})$ *and* $\pi\colon G \to \mathscr{H}$ *a unitary representation. Assume that for* $g, h \in G$,

$$\lim_{n\to\infty} g^n h g^{-n} = \mathrm{id}.$$

*Then any vector* $\xi \in \mathscr{H}$ *that is fixed by* $g$ *is also fixed by* $h$.

**Proof.** Let $\xi \in \mathscr{H}$ and assume it is fixed by $g$. Then

$$\begin{aligned}\|\pi(h)\,\xi - \xi\|^2 &= \langle \pi(h)\,\xi - \xi, \pi(h)\,\xi - \xi\rangle \\ &= \langle \pi(h)\,\pi(g^{-n})\,\xi - \xi, \pi(h)\,\pi(g^{-n})\,\xi - \xi\rangle \\ &= \langle \pi(g^n)\,\pi(h)\,\pi(g^{-n}) - \pi(g^n)\,\xi, \pi(g^n)\,\pi(h)\,\pi(g^{-n}) - \pi(g^n)\,\xi\rangle \\ &= \|\pi(g^n h g^{-n})\,\xi - \xi\|^2.\end{aligned}$$

Finally, since we are assuming that $g^n h g^{-n} \to \mathrm{id}$, we have that $\|\pi(h)\,\xi - \xi\| = 0$. Thus $\xi$ is also fixed by $h$. ■

With these auxiliary results, one can prove the Howe-Moore theorem as follows. By 1.17, it is enough to prove the theorem for $A^+$. Now let $\xi \in \mathscr{H}$. Since $\pi$ is a unitary representation, we have that

$$\|\pi(a_n)\,\xi\| = \|\xi\|.$$

Hence, the sequence $(\pi(a_n)\,\xi)_{n\in\mathbb{N}} \subset \mathscr{H}$ is bounded. Up to passing to a subsequence, there is a weak limit point $\xi_0 \in \mathscr{H}$. Noting that $N \cup N^-$ generate all of $SL_2(\mathbb{R})$, it follows from lemmas 1.18 and 1.19 that $\xi_0$ is invariant under all of $SL_2(\mathbb{R})$. Now, since $SL_2(\mathbb{R})$ is simple, the assumption of the theorem asserts that there are no non-trivial $SL_2(\mathbb{R})$ invariant vectors. Thus, $\xi_0 = 0$, which is what we wanted to have.

## 1.6 Abelian Differentials and Translations Surfaces

We now introduce translation surfaces mainly trough Abelian differentials. We then move on to define the Teichmüller space and moduli space of translation surfaces. In the presentation, we follow the lecture notes by Forni and Matheus [FM14], and the notes by Yoccoz [Yoc10]. For smaller details we also refer to Wright's introduction [Wri15], and Viana's article [Via06].

We begin with a compact **Riemann surface** $M$ of genus $g \geqslant 1$, that is a smooth manifold with an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ with charts $\phi_\alpha \colon U_\alpha \to \mathbb{C}$ such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are holomorphic whenever $U_\alpha \cap U_\beta \neq \varnothing$ [see Jos06, for an introduction].

### 1.6.1 Translation Surfaces as Abelian Differentials

We begin with with a description of translation surfaces in the language of differential geometry.

**Definition 1.20.** *A global section of the cotangent bundle of $M$ is called an **Abelian differential**. For a nonzero Abelian differential $\omega$, the pair $(M, \omega)$ is called a **translation surface**.*

In other words, an Abelian differential is a holomorphic 1-form on $M$, $\omega\colon M \to TM^*$ which in a chart $(U, z)$ can be written as $\omega_z = \alpha(z)\,dz$, for a nonzero holomorphic function $\alpha\colon M \to \mathbb{C}$. The finitely many zeros of $\omega$ are called **singularities**. We denote them by $\Sigma(\omega)$, or simply $\Sigma$ when it is understood which Abelian differential we are working with.

The Abelian differential naturally gives rise to an area form on the corresponding translation surface as follows.

**Definition 1.21.** *Let $(M, \omega)$ be a translation surface. Define the **area** of the translation surface as*

$$A(\omega) \coloneqq \frac{i}{2}\int_M \omega \wedge \overline{\omega}.$$

The reason for the name translation surface, is that the Abelian differential also defines an atlas on $M \setminus \Sigma$ where the transition maps are translations. To define the charts, pick a point $p \in M \setminus \Sigma$, and a neighbourhood $U_p \subset M \setminus \Sigma$ around this point. Then define

$$\begin{aligned} \phi_p\colon U_p &\to \mathbb{C} \\ x &\mapsto \int_p^x \omega, \end{aligned} \tag{1.3}$$

where the integral is over an arbitrary path within $U_p$ from $p$ to $x$ (note that we can do this because the charts themselves are simply connected). This definition does not depend on the path taken, since $\omega$ is a global section. These charts then define an atlas $\{(U_p, \phi_p)\}_{p\in M}$ on $M \setminus \Sigma$. We now see that the transition maps are indeed translations.

**Proposition 1.22.** *With notation as above, let* $p, q \in M \setminus \Sigma$ *such that* $U_p \cap U_q \neq \varnothing$. *Then the transition map*

$$\phi_p \circ \phi_q^{-1} \colon \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$$

*is a translation in* $\mathbb{C}$, *i.e.* $\phi_p \circ \phi_q^{-1}(z) = z + c$, *for some* $c \in \mathbb{C}$.

**Proof.** Let $z \in \phi_q(U_p \cap U_q)$, and set $P \coloneqq \phi_q^{-1}(z) \in U_p \cap U_q$. Since $U_p \cap U_q \neq \varnothing$, there exists a path $\gamma_1 \colon [0,1] \to M$ such that $\gamma_1([0,1]) \subset U_p \cup U_q$, $\gamma_1(0) = p$, and $\gamma_1(1) = q$. Now choose two other paths $\gamma_2 \colon [0,1] \to M$ contained in $U_q$ starting at $q$ and ending at $P$, and $\gamma_3 \colon [0,1] \to M$ contained in $U_p$ starting at $p$ and ending at $P$. Note that the concatenation $\gamma_1\gamma_2$ is also a path from $p$ to $P$, as in figure 1.2. Therefore, we have that

$$\begin{aligned}\phi_p \circ \phi_q^{-1}(z) &= \int_p^{\phi_q(z)^{-1}} \omega = \int_p^P \omega = \int_{\gamma_3} \omega = \int_{\gamma_1\gamma_2} \omega \\ &= \int_{\gamma_1} \omega + \int_{\gamma_2} \omega = \int_p^q \omega + \int_q^P \omega.\end{aligned}$$

Now note that $\int_p^q \omega$ does not depend on $z$, and hence it is constant. Thus, for $c \coloneqq \int_p^q \omega$, we have that

$$\phi_p \circ \phi_q^{-1}(z) = z + c,$$

which is what we wanted to prove. ∎

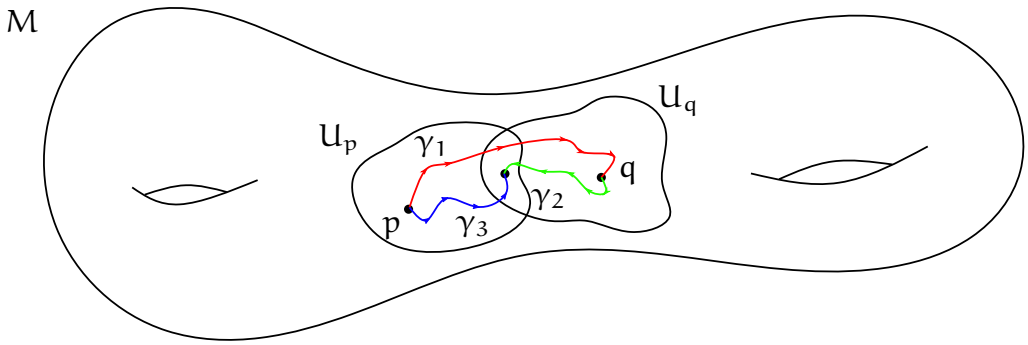


Figure 1.2: Proof that transition maps are translations

This motivates calling an atlas where all the changes of coordinates are translations, a **translation structure** (we will also call this a **translation atlas**).

We now look more carefully at the form of the Abelian differential.

**Proposition 1.23.** *Let* $(M, \omega)$ *be a translation surface. Let* $p \in M$ *such that* $\omega_p \neq 0$*. Then there is a chart* $(U, z)$ *around* $p$ *such that in coordinates* $\omega$ *takes the form* $dz$*. If* $q \in M$ *is a zero of order* $k \in \mathbb{N}$ *of* $\omega$*, there is a chart* $(U, z)$ *of around* $q$ *such that* $\omega$ *takes the form* $z^k\, dz$*.*

**Proof.** Since $\omega$ is an Abelian differential, there exists a chart $(U, w)$ on which $\omega_p = \alpha(p)\, dw$ (i.e. $\omega_p(v) = \alpha(p)\, d_p\phi(v)$, for $v \in TM_p$). Then, since the zeros of $\omega$ are isolated, up to shrinking $U$, we may assume that for all $p \in U$, $\omega_p \neq 0$. In particular $\alpha(p) \neq 0$ for any $p \in U$. We then define a new chart $(U, z)$ by setting $z(p) = \frac{w(p)}{\alpha(p)}$, for $p \in U$. Hence, we have that in these charts

$$\omega = dz.$$

On the other hand, let $q \in M$ be a zero of order $k$ of $\omega$. Let $(U, w)$ be a chart around $q$, with $w(q) = 0$. We may then write $\omega = w^k f(w)$ for some holomorphic function $f$ on $\mathbb{C}$, with $f(0) \neq 0$. Then

$$\int_0^w f(t)t^k dt$$

vanishes with order $k+1$ at $0$, and thus one can take a $(k+1)$st root. We may therefore define a coordinate chart

$$z \coloneqq \left((k+1)\int_0^w f(t)t^k dt\right)^{\frac{1}{k+1}}.$$

That is

$$z^{k+1} = (k+1)\int_0^w f(t)t^k dt.$$

Finally, taking the differential of both sides, we see that

$$(k+1)z^k\, dz = (k+1)\left(f(w)w^k - 0\right).$$

Therefore, $\omega = w^k f(w) = z^k\, dz$. ∎

This gives us that if $p \in \Sigma$ is a zero of order $k$ of $\omega$, the chart from proposition 1.23 gives a $(k+1)$-fold covering of $\mathbb{C}$. Thus, we call the zeros of the Abelian differential **conical singularities**. This is because for a short enough loop around

the singularity, the index of the loop is $k+1$, and the singularity is said to have **angle** $2\pi(k+1)$, and denoted by $\measuredangle(p_0)$ for a singularity $p_0 \in \Sigma$.This will be clearer when we give an equivalent definition of translation surfaces by identifying parallel sides of polygons.

We now work with the curvature of translation surfaces. Given a translation surface $(M, \omega)$, we introduce a Riemannian metric on $M \setminus \Sigma$ as the pullback by the coordinate map (1.3) of the inner product on $\mathbb{C}$. That is, for $p \in M \setminus \Sigma$, choose a chart $(U_p, \phi_p)$ around $p$ from the translation atlas of the surface determined by $\omega$. Then, for $q \in U_p$, and for $v, w \in TM_q$, we define

$$\langle v, w \rangle_q \coloneqq \langle \, d_q\phi_p(v), \, d_q\phi_p(w) \rangle_{\mathbb{C}} \, ,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the Euclidean inner product in $\mathbb{C}$. We need to verify that this is well defined, and that it is indeed a Riemannian metric.

**Proposition 1.24.** *The metric $\langle \cdot, \cdot \rangle$ on $M \setminus \Sigma$ is a Riemannian metric.*

**Proof.** We begin by showing that the metric is well defined. Let $p, q \in M \setminus \Sigma$ be such that for their corresponding charts, $U_p \cap U_q \neq \varnothing$. Let $x \in U_p \cap U_q$, and $v, w \in TM_x$. Note that since $\phi_p \circ \phi_q^{-1}(z) = z + c$ for some $c \in \mathbb{C}$, we have that for $z = \phi_q(x)$,

$$d_z \left( \phi_p \circ \phi_q^{-1} \right) = \mathrm{id}_{T\mathbb{C}_z}.$$

We therefore have that

$$d_x\phi_q(v) = \, d_z \left( \phi_p \circ \phi_q^{-1} \right) ( \, d_x\phi_q(v)) = \, d_x\phi_p(v),$$

where the last line follows by the chain rule. In particular, this means that

$$\langle \, d_x\phi_p(v), \, d_x\phi_p(w) \rangle_{\mathbb{C}} = \langle \, d_x\phi_q(v), \, d_x\phi_q(w) \rangle_{\mathbb{C}} \, ,$$

and hence the metric is well defined. The fact that the metric varies smoothly with respect to the base point follows directly from the fact that $\langle \cdot, \cdot \rangle$ is defined as the pullback of a chart. Thus, $\langle \cdot, \cdot \rangle$ is a Riemannian metric on $M \setminus \Sigma$. ∎

In particular, this means that $M \setminus \Sigma$ is **flat**, that is, it has zero sectional curvature. Since the Euler characteristic of the surface is $\chi(M) = 2g - 2$, the Gauss-Bonnet theorem [do 76] implies that if $g \geqslant 2$, the sectional curvature cannot be everywhere zero. It therefore makes sense that the flat metric cannot be extended to all of $M$. Moreover, having a translation atlas defined everywhere except at the singularities allows us to make sense of direction on the surface.

**Definition 1.25.** *Let* $(M, \omega)$ *be a translation surface. Let* $p \in M \setminus \Sigma$ *and* $v \in TM_p$. *Then the* ***direction*** *of* $v$ *is the angle* $\theta \in [0, 2\pi)$ *that* $d_p\phi_p(v) \in \mathbb{C}$ *makes with the real axis.*

The direction of a vector does not depend on the choice of chart, since all the changes of coordinates are translations. This allows us to define the **vector field in direction** $\theta$ as the vector field $X^\theta$ on $M \setminus \Sigma$ such that $X^\theta_p$ is the unit vector in direction $\theta$, and the corresponding flow is denoted by $\varphi^\theta$. The **vertical vector field** is the vector field in direction $\pi/2$, denoted by $X^v$. With this, we can give the following definition of a cone angle.

**Definition 1.26.** *Let* $(M, \omega)$ *be a translation surface, and* $\Sigma$ *the set of singularities. The* ***cone angle*** *of a singularity* $p$ *is the index of the vector field* $q \mapsto \omega_q\left(X^v_q\right) X^v_q$ *at* $p$.

We now calculate the cone angles of the singularities. Given a translation surface $(M, \omega)$, we denote by $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$ the orders of the zeros of $\omega$. Since there are only finitely many singularities, they are isolated and hence for each one, there is a chart on which $\omega = z^{\kappa_j}\, dz$, for $j = 1, \ldots, \sigma$. The index of the vertical vector field around the singularities is easily seen to be $\kappa_j + 1$ [see do 76, Chapter 4-5] for $j = 1, \ldots, \sigma$. This gives us two things. First, the above definition of the cone angle as $2\pi(\kappa_j + 1)$ is justified. Secondly, we may use the Poincaré-Hopf theorem [see do 76, Chapter 4-5] to infer that

$$\sum_{j=1}^{\sigma} \kappa_j = 2g - 2 + \sigma. \tag{1.4}$$

Thus, the orders of the zeros of the Abelian differential are restricted by the topology of the surface.

### 1.6.2 Translation Surfaces as Translation Structures

Translation surfaces may also be defined as follows.

**Definition 1.27.** *A* ***translation surface*** *is a triple* $(M, \Sigma, \zeta)$ *consisting of a compact topological surface* $M$ *and a finite set of points* $\Sigma \subset M$ *together with an atlas* $\zeta = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ *where all the transition maps are translations, i.e. for* $\alpha, \beta \in A$ *for which* $U_\alpha \cap U_\beta \neq \varnothing$, *we have that*

$$\phi_\alpha \circ \phi_\beta^{-1}(z) = z + c,$$

*for some* $c \in \mathbb{C}$. *For each singularity* $p_0 \in \Sigma$ *there exists a number* $k \in \mathbb{N}$ *and a* $k+1$ *covering of a neighbourhood in* $\mathbb{C}$ *around* $0$. *Moreover, two translation atlases are said to be* ***equivalent*** *if their union is also a translation atlas.*

We already saw that an Abelian differential gives rise to a translation atlas in proposition 1.22. What is left to show is that the singularities have cone angle $2\pi c$, and that a translation atlas determines an Abelian differential.

The first assertion follows from the calculation of the constant in the proof of proposition 1.22, and the form of the chart around singularities from the proof of proposition 1.23.

For the second part, we need to construct an Abelian differential from a translation atlas. Let $(M, \Sigma, \zeta)$ be a translation surface in the sense of definition 1.27. To define an Abelian differential, let $p \in M \setminus \Sigma$, and a chart $(U_p, \phi_p)$ around $p$ not containing any singularities. Then set

$$\omega_p \coloneqq d_p \phi_p.$$

This is well defined since the transition maps are translations (see the calculation in the proof of proposition 1.24). This differential is nonzero since otherwise there would exist a chart that is not a homeomorphism. Now we need to define the Abelian differential close to the singularities. Let $p_0 \in \Sigma$. Then by the definition, there exists a chart $(U, \phi)$ around $p_0$ with $\phi(p_0) = 0$, and for which $\frac{\phi^{k+1}}{k+1}$ is a covering map of a neighbourhood in $\mathbb{C}$. We again define the Abelian differential close to $p_0$ as the differential of the chart, meaning that for $q \in U \setminus \{p_0\}$, we have $\omega_q = \phi^k \, d_q \phi$. Finally, we define the Abelian differential at the singularity as $\omega_{p_0} = \phi^k \, d_{p_0} \phi$. This completes the definition of the Abelian differential and hence the two definitions of translation surfaces are equivalent.

### 1.6.3 Translation Surfaces as Polygons

For the final equivalent definition of a translation surface, we follow Masur [Mas06].

**Definition 1.28.** *A* ***translation surface*** *is an equivalence of a union of a finite amount of polygons* $\{\Delta_1, \ldots, \Delta_n\} \subset \mathbb{C}$ *such that*

*(i) the boundary of every polygon is oriented such that the interior lies to the left,*

*(ii) and the edges of the polygons come in pairs, being parallel and of equal length.*

*The equivalence is through translations gluing parallel pairs of edges to one another.*

The singularities in this case are given by the equivalence classes of the vertices of the polygons. This definition gives us the classical picture for a translation surface as in figure 1.3. Here, sides with the same colour are identified.

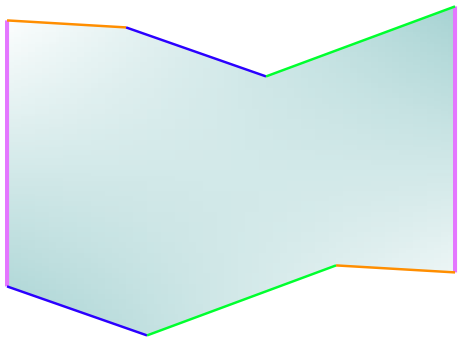

Figure 1.3: Example of a translation surface

We will only explain how this definition is equivalent to definition 1.27, and refer to [Wri15] for further details. We first need the following definition.

**Definition 1.29.** *Let* $(M, \Sigma, \zeta)$ *be a translation surface. A geodesic in* $M$ *(in the flat metric this is the lift of a straight line in* $\mathbb{C}$*) where its endpoints are singularities, and without singularities in its interior, is called a* ***saddle connection****. The shortest saddle connection on the translation surface is called its* ***systole****, which we will denote by* $\mathrm{sys}\,(M, \Sigma, \zeta)$*.*

In order to get from a translation surface in the sense of definition 1.27 to the above definition with polygons, one can use all the saddle connections on the surface to give polygons in $\mathbb{C}$. For the other direction, assume that we have a single polygon with edges identified, and where the interiors of the edges are pairwise disjoint. For any point in the interior of the polygon, the chart can simply be taken to be the identity as the polygon lies in $\mathbb{C}$. For points in the interior of an edge, since the edges are glued via translations, this gives a local chart for these points. For the singularities, we see that since the edges come in pairs, the total angle around the singularity is an integer multiple of $2\pi$.

Using this definition, we can see the definition of the cone angle around a singularity agrees with the angle formed by going around the singularity in the polygon.

Hence, for a translation surface $(M, \Sigma, \zeta)$ given as the gluing of a 2d-gon, and singularities $p_1, \ldots, p_\sigma$ and orders $\kappa_1, \ldots, \kappa_\sigma$,

$$\sum_{j=1}^{\sigma} \measuredangle(p_j) = 2\pi(d-1).$$

But since the angles are given by $2\pi(\kappa_j + 1)$, we have that

$$\sum_{j=1}^{\sigma} \kappa_j = d - \sigma - 1.$$

Then combining this with equation (1.4), we have that

$$\sigma + 1 - d = 2g - 2. \tag{1.5}$$

This finishes our discussion of translation surfaces, and we move on to their parameter spaces. We will use the definitions of translation surfaces interchangeably.

## 1.7 Teichmüller Space and Moduli Space

We begin by fixing a compact topological surface $M$ of genus $g \geqslant 1$, and fix a finite set of singularities $\Sigma \subset M$ with orders $\kappa = \kappa_1, \ldots, \kappa_\sigma$ (where $|\Sigma| = \sigma$). We will denote the space of all translation surface structures on the triple $(M, \Sigma, \kappa)$ by $\mathcal{L}(M, \Sigma, \kappa)$. We begin with a description of the mapping class group, then define the Teichmüller and moduli spaces.

### 1.7.1 The Mapping Class Group

For this section, we follows some of the presentation in [FM11] adapted to our setting. We denote by $\mathrm{Homeo}(M, \Sigma)$ the group of homeomorphisms of $M$ for which the points in $\Sigma$ are fixed points. This set carries a group structure given by composition of maps. Moreover, the space carries the compact-open topology, with respect to which it also becomes a topological group (since $M$ is compact). We further let $\mathrm{Homeo}^+(M, \Sigma)$ be the subgroup of orientation preserving homeomorphisms, and $\mathrm{Homeo}_0(M, \Sigma)$ the connected component of $\mathrm{Homeo}(M, \Sigma)$ containing the identity map. These two sets are equipped with the subspace topology.

To continue, we define the notion of an isotopy. Two homeomorphisms $f_1, f_2 \in \mathrm{Homeo}(M, \Sigma)$ are called **isotopic** if there exists a continuous map $H\colon M \times [0,1] \to M$ such that

(i) $H(\cdot, 0) = f_1$,

(ii) $H(\cdot, 1) = f_2$, and

(iii) for every $t \in [0,1]$, the map $H(\cdot, t) \in \mathrm{Homeo}(M, \Sigma)$.

Since the connected component $\mathrm{Homeo}_0(M, \Sigma)$ is also path-connected, we may define it as the set of homeomorphisms isotopic to the identity. The isotopies in this case would be the paths between the homeomorphism and the identity. We may now define the **mapping class group** of $M$ and $\Sigma$ as

$$\Gamma(M, \Sigma) \coloneqq \mathrm{Homeo}(M, \Sigma) \Big/ \mathrm{Homeo}_0(M, \Sigma).$$

That is, two homeomorphisms $f_1, f_2 \in \mathrm{Homeo}(M, \Sigma)$ are equivalent if there exists a homeomorphism $f \in \mathrm{Homeo}_0(M, \Sigma)$ such that $f_1 = f \circ f_2$. The mapping class group therefore consists of isotopy classes.

To see this more explicitly, assume that $f_1$ and $f_2$ are in the same equivalence class in $\Gamma(M, \Sigma)$, and are identified through $f \in \mathrm{Homeo}_0(M, \Sigma)$. Then there exists an isotopy $\overline{H}\colon M \times [0,1] \to M$ between $f$ and $\mathrm{id}_M$, the identity map on $M$. By following this isotopy backwards and pushing it with $f_2$, we get an isotopy between $f_1$ and $f_2$ as in figure 1.4 below.

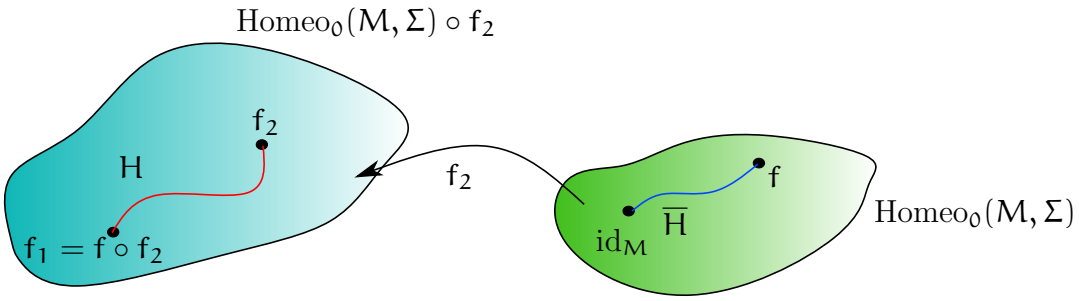


Figure 1.4: Isotopy in the Mapping Class Group

That is, defining

$$H\colon M \times [0,1] \to M$$

$$(x,t) \mapsto \overline{H}(x,1-t) \circ f_2,$$

we have that $H(\cdot,0) = f \circ f_2 = f_1$, that $H(\cdot,1) = f_2$, and that for every $t \in [0,1]$, $H(\cdot,t)$ is a homeomorphism since it is a composition of two homeomorphisms. Therefore $f_1$ and $f_2$ are isotopic and it is justified to call elements in the mapping class group isotopy classes.

The mapping class group is discrete as it consists of cosets of a connected component. Also note that $\mathrm{Homeo}_0(M,\Sigma)$ is itself is a normal subgroup. To see that it is a subgroup, let $f_1, f_2 \in \mathrm{Homeo}_0(M,\Sigma)$, and $H_1$ and $H_2$ the respective isotopies to the identity. Then the map

$$H(x,t) \coloneqq \begin{cases} H_2(x,2t), & t \in [0,1/2], \\ H_1(f_2(x),2t-1), & t \in (1/2,1] \end{cases}$$

is an isotopy between the identity and $f_1 \circ f_2$. To see that it is normal, take $f_1 \in \mathrm{Homeo}(M,\Sigma)$ and $f \in \mathrm{Homeo}_0(M,\Sigma)$ with isotopy with the identity $\overline{H}$. Then

$$H(x,t) \coloneqq f_1 \circ \overline{H}(x,t) \circ f_1^{-1}$$

is an isotopy between the identity and $f_1 \circ f \circ f_1^{-1}$. This means that $\Gamma(M,\Sigma)$ is indeed a group, which we endow with the quotient topology. Similarly, we can restrict to only the orientation preserving homeomorphisms and define

$$\Gamma^+(M,\Sigma) \coloneqq \mathrm{Homeo}^+(M,\Sigma) \Big/ \mathrm{Homeo}_0(M,\Sigma).$$

Finally, in order to define the Teichmüller space for the the surface $M$ together with its singularities $\Sigma$, we need to define the action of $\mathrm{Homeo}(M,\Sigma)$ on translation surface structures on $M$ with singularities $\Sigma$. However, translation surfaces are themselves Riemann surfaces, meaning that they come with a smooth structure. Thus when acting on these surfaces, this structure should still be respected. To take this into account, we consider the group of diffeomorphisms $\mathrm{Diff}(M,\Sigma)$ and its corresponding subgroups in place of $\mathrm{Homeo}(M,\Sigma)$. The mapping class group can equivalently be defined using diffeomorphisms as explained in 2.1 in [FM11], which is what we do for what follows.

Now to continue with the definition of the action of the mapping class group on translation surfaces, we use definition 1.27. Let $(M,\Sigma,\zeta)$ be a translation surface

and $f \in \mathrm{Diff}(M, \Sigma)$. Then if $\zeta = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, we define

$$f_* \zeta \coloneqq \left\{(U_\alpha), \phi_\alpha \circ f^{-1}\right\}_{\alpha \in A}.$$

Note that since $f \in \mathrm{Diff}(M, \Sigma)$, the triple $(M, \Sigma, f_* \zeta)$ is still a translation surface (that is, the set of singularities does not change). Moreover, if the Abelian differential corresponding to $(M, \Sigma, \zeta)$ had zeros with orders $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$, then the action of $\mathrm{Diff}(M, \Sigma)$ does not change the order of the zeros. Hence, $(M, \Sigma, f_* \zeta)$ is still in $\mathcal{L}(M, \Sigma, \kappa)$. This action also defines an action of the mapping class group. We can now move on to define Teichmüller space.

### 1.7.2 Teichmüller Space

For this section, we follow [Yoc10]. Here we will focus on the sets of translation surface structures on $M$, with singularities $\Sigma$ and with prescribed orders of the zeros of the corresponding Abelian differential $\kappa$. By the above discussion, the set of these structures is invariant under the $\mathrm{Diff}(M, \Sigma)$ action.

**Definition 1.30.** *With the notation as above, the* ***Teichmüller space*** *of $\mathcal{L}(M, \Sigma, \kappa)$, denoted by $\mathcal{Q}(M, \Sigma, \kappa)$ is the quotient*

$$\mathcal{Q}(M, \Sigma, \kappa) \coloneqq \mathcal{L}(M, \Sigma, \kappa) \Big/ \mathrm{Diff}_0(M, \Sigma).$$

We will say that two translation surface structures $\zeta_1, \zeta_2 \in \mathcal{Q}(M, \Sigma, \kappa)$ are **isotopic** if there exists a diffeomorphism $f \in \mathrm{Diff}_0(M, \Sigma)$ such that $f_* \zeta_1 = \zeta_2$. Teichmüller space hence consists of the isotopy classes of translation surface structures. This space is also a complex manifold, and we need to define a topology and a smooth structure on it to see this.

For the topology, we fix a universal cover of $M$, say $\widetilde{M}$, a point $* \in \widetilde{M}$, and denote by $\bullet$ a point in $\Sigma$. Let

$$P\colon \left(\widetilde{M}, *\right) \to (M, \bullet)$$

be the covering map, where $P(*) = \bullet$. Then for a translation surface $(M, \Sigma, \zeta) \in \mathcal{L}(M, \Sigma, \kappa)$, we define the **development map** of this surface as

$$\mathrm{dev}_\zeta\colon \left(\widetilde{M}, *\right) \to (\mathbb{C}, 0)$$

$$q \mapsto \int_*^q P^*\omega,$$

where $P^*$ is the pullback of the 1-form $\omega$ defined by the translation surface. Note also that the map $(M, \Sigma, \zeta) \mapsto \mathrm{dev}_\zeta$ is injective (the integration over different 1-forms will differ on an open set). Hence, we identify $\mathcal{L}(M, \Sigma, \kappa)$ with a subset of $C\left(\widetilde{M}\right)$, which itself carries the compact-open topology. The space $\mathcal{L}(M, \Sigma, \kappa)$ then carries the subspace topology, and the topology on Teichmüller space is the quotient topology.

For the smooth structure, we need to define charts on $\mathscr{Q}(M, \Sigma, \kappa)$. Denote by $H_1(M, \Sigma; \mathbb{Z})$ the relative homology of $M$ with respect to $\Sigma$ , and by

$$H^1(M, \Sigma; \mathbb{C}) = \mathrm{Hom}\left(H_1(M, \Sigma; \mathbb{Z}), \mathbb{C}\right)$$

the relative cohomology (see [Hat02]). We define the **period map** as

$$\begin{aligned} \Theta\colon \mathscr{Q}(M, \Sigma, \kappa) &\to H^1(M, \Sigma; \mathbb{C}) \\ [(M, \Sigma, \zeta)] &\mapsto \left(\tau \mapsto \int_\tau \omega\right), \end{aligned}$$

where $\omega$ is the Abelian differential defined by a representative atlas $\zeta$ in an equivalence class in $\mathscr{Q}(M, \Sigma, \kappa)$. Since the definition is in terms of a representative, we need to argue that the map is well defined. By the homotopy invariance property of the first homology [see Hat02, section 2.1], $\mathrm{Diff}_0(M, \Sigma)$ acts trivially on homology. Thus, the period map only depends on the isotopy class of a translation surface.

The period map defines coordinates for Teichmüller space. We do not prove this here, and refer to section 6.5 in [Yoc10]. Since $H^1(M, \Sigma; \mathbb{C})$ is a vector space over $\mathbb{C}$, Teichmüller space is a complex manifold. The (complex) dimension is the same as that of the relative homology, which is $2g + \sigma - 1$. Intuitively, the $2g$ is for the fundamental group of the surface without considering the singularities, and the $\sigma - 1$ is for connecting a starting singularity to the rest. In the example in figure 1.5 below, we have a translation surface represented by an octagon, with $\Sigma = \{\bullet, \bullet, \bullet\}$. By equation (1.5), the surface has genus 2. The red lines in the figure are representatives of the generators in relative homology, of which there are $6 = 2(2) + 3 - 1$ elements.

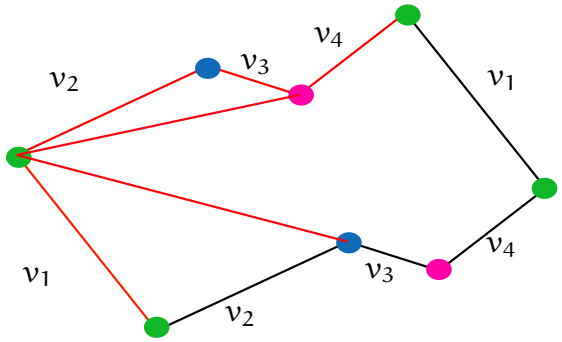


Figure 1.5: Translation surface example for homology dimension

Similarly, we define

$$\mathscr{Q}^{+}(M,\Sigma,\kappa) \coloneqq \mathcal{L}(M,\Sigma,\kappa) \Big/ \mathrm{Diff}_0^{+}(M,\Sigma),.$$

which is the space we will be more interested in. Finally, we will want to restrict further to surfaces of area one. Since we are quotienting by the diffeomorphisms isotopic to the identity, and we are dealing with Abelian differentials (which themselves are closed 1-forms), it follows that the area of a surface is constant on equivalence classes in Teichmüller space. The area function will still be denoted by $A$. We may therefore define

$$\mathscr{Q}^{(1)}(M,\Sigma,\kappa) \coloneqq \{(M,\Sigma,\zeta) \in \mathscr{Q}^{+}(M,\Sigma,\kappa) \,:\, A(\zeta) = 1\}.$$

For the same reasons as $A$ is well defined on Teichmüller space, the group $\mathrm{Diff}(M,\Sigma)$ preserves the area and hence $\mathscr{Q}^{(1)}(M,\Sigma,\kappa)$.

### 1.7.3 Moduli Space

We now consider the action of the mapping class group on Teichmüller space. Since Teichmüller space is the quotient by the diffeomorphisms isotopic to the identity, the action of the mapping class group on Teichmüller space is well defined. More explicitly, assume $\zeta_1$ and $\zeta_2$ are two translation surface structures in $\mathcal{L}(M,\Sigma,\kappa)$ representing the same point in Teichmüller, being isotopic through some $f \in \mathrm{Diff}_0(M,\Sigma)$, and let $\gamma \in \Gamma(M,\Sigma)$. Then $\gamma_* \zeta_1$ and $\gamma_* \zeta_2$ are isotopic through the same map $f$. We may therefore define the following.

**Definition 1.31.** *The* ***moduli space*** *of* $\mathcal{L}(M, \Sigma, \kappa)$*, denoted by* $\mathscr{M}(M, \Sigma, \kappa)$*, is the quotient*

$$\mathscr{M}(M, \Sigma, \kappa) \coloneqq \mathscr{Q}(M, \Sigma, \kappa) \Big/ \Gamma(M, \Sigma).$$

*Similarly as above, the* ***moduli space of area one translation surfaces*** *is the quotient*

$$\mathscr{M}^{(1)}(M, \Sigma, \kappa) \coloneqq \mathscr{Q}^{(1)}(M, \Sigma, \kappa) \Big/ \Gamma^{+}(M, \Sigma).$$

By choosing a different set of orders $\kappa'$, one would get a different space. For a fixed $\kappa$, we will call $\mathscr{Q}(M, \Sigma, \kappa)$ a **stratum**.Quotienting by this larger class of (equivalence classes of) diffeomorphisms gives us more freedom when identifying surfaces. In particular, as elements in the mapping class group are not necessarily isotopic to the identity, the mapping class group can permute the representatives of the relative homology. Hence, one can "cut and paste" a surface as in figure 1.4 below. In the example, we cut the surface along one of the representatives in the homology (the red line), and paste along curves that are identified by translations.

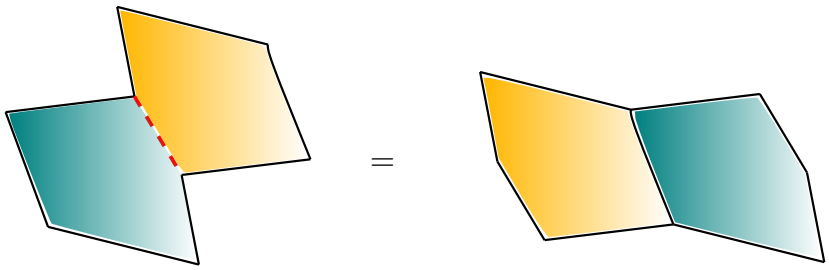

Figure 1.6: Translation surfaces in the same equivalence class in moduli space

The mapping class group acts properly discontinuously on Teichmüller space. However, the action is not free. This means that moduli space is not a manifold, but rather an orbifold. At points in Teichmüller space which are not fixed by $\Gamma(M, \Sigma)$, we may use the coordinates from the period map.

### 1.7.4 A Metric on Teichmüller Space

We now introduce a metric on $\mathscr{Q}(M, \Sigma, \kappa)$ that will allow us to define the stable and unstable manifolds, as well as Hölder functions on it. The metric we use is the one introduced in [AGY06]. There are other metrics one can define on Teichmüller space (see [FM14]), but we do not deal with those here.

The metric we will work with is a Finsler metric. Given a smooth manifold $X$, a **Finsler metric** assigns to each $p \in X$ a norm $\|\cdot\|_p$ on $TX_p$ which depends smoothly on $p$. The length of a curve $c\colon [a,b] \to X$, denoted by $\ell(c)$, is then defined as

$$\ell(c) = \int_a^b \|c'(t)\|_{c(t)}\, dt.$$

The metric $d$ on $X$ is then defined as follows. For $p, q \in X$,

$$d(p,q) \coloneqq \inf_c \ell(c),$$

where the infimum is taken over all $C^1$ paths from $p$ to $q$. Note that the norm need not come from an inner product, and hence the manifold is not necessarily Riemannian.

To continue, we define a norm on each tangent space of $\mathscr{Q}(M,\Sigma,\kappa)$. Let $q \in \mathscr{Q}(M,\Sigma,\kappa)$, and let $\{c_n\}$ be the set of saddle connections of $q$. Then, as $T\mathscr{Q}(M,\Sigma,\kappa)_q$ is identified with $H^1(M,\Sigma;\mathbb{C})$, we define

$$\|v\|_q \coloneqq \sup_{n\in\mathbb{N}} \left| \frac{v([c_n])}{(\Theta(q))([c_n])} \right|,$$

where $[c]$ is the equivalence class of $c$ in $H_1(M,\Sigma;\mathbb{Z})$. For a proof of why this is a Finsler metric, see section 2.2.2 in [AGY06]. We will denote the resulting metric by $d$. Since the mapping class group acts on homology by permuting the generators, the norm is invariant under its action. Therefore, the distance descends to $\mathscr{M}(M,\Sigma,\kappa)$. We will also denote the metric there by $d$.

## 1.8 The $\mathrm{SL}_2(\mathbb{R})$ Action and Dynamics on Moduli Space

We are finally ready to introduce the main object in this thesis, the $\mathrm{SL}_2(\mathbb{R})$ action on $\mathscr{M}^{(1)}(M,\Sigma,\kappa)$. We will first describe the action on $\mathcal{L}(M,\Sigma,\kappa)$, and then argue that it induces an action on the moduli space. Before we begin, we note that $\mathscr{M}^{(1)}(M,\Sigma,\kappa)$ is in general not connected. However, the connected components have been classified in [KZ03], and they are invariant under the action we will describe. We will now focus on some connected component $\mathscr{C} \subset \mathscr{M}^{(1)}(M,\Sigma,\kappa)$.

### 1.8.1 The $SL_2(\mathbb{R})$ Action

Let $\gamma \in SL_2(\mathbb{R})$ and consider its action on $\mathbb{C}$ through the identification $\mathbb{R}^2 \simeq \mathbb{C}$. More explicitly, for $z = x + iy$,

$$\gamma.z \coloneqq \gamma \cdot \begin{pmatrix} x \\ iy \end{pmatrix}.$$

Now consider a translation surface structure $(M, \Sigma, \kappa) \in \mathcal{L}(M, \Sigma, \kappa)$ with translation atlas $\zeta = \{\phi_\alpha\}_{\alpha \in A}$. The $SL_2(\mathbb{R})$ action is defined by

$$\gamma_* \zeta \coloneqq \{\gamma \circ \phi_\alpha\}_{\alpha \in A}.$$

To see that this indeed defines a new translation surface structure, let $\phi_\alpha, \phi_\beta \in \zeta$. Then

$$\begin{aligned}(\gamma \circ \phi_\alpha) \circ (\gamma \circ \phi_\beta)^{-1}(z) &= \gamma \circ \phi_\beta \circ \phi_\alpha^{-1} \circ \gamma^{-1}(z) \\ &= \gamma\left(\gamma^{-1}(z) + c\right) = z + \gamma(c),\end{aligned}$$

where the constant $c \in \mathbb{C}$ comes from the fact that $\zeta$ is a translation surface structure. There is an easier way to view this action, namely through polygons. Since one can define the atlas directly through the polygon representation of a translation surface structure, we get that $SL_2(\mathbb{R})$ acts on the polygon simply by multiplication with the matrix. Below in figures 1.7, 1.8, and 1.9 are some examples of this.

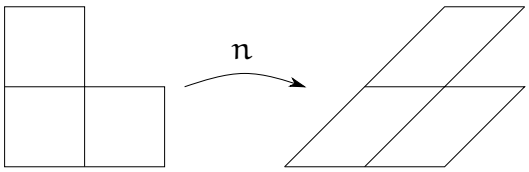


Figure 1.7: Action of $n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ on a translation surface with $x > 0$

This action descends to an action on Teichmüller space because the action is on the left in terms of the atlas, whereas the action of $\mathrm{Diff}_0(M, \Sigma)$ is on the right. The same argument works for moduli space, and we therefore have a group action on $\mathscr{M}(M, \Sigma, \kappa)$. Finally, since the determinant of matrices in $SL_2(\mathbb{R})$ is one, $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$ is invariant under this action. The restriction of this action to the subgroup

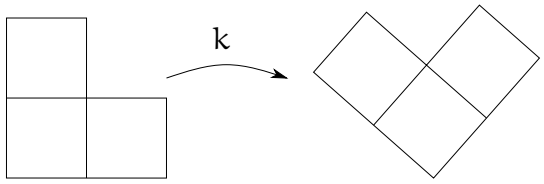


Figure 1.8: Action of $k \in SO(2)$ on a translation surface

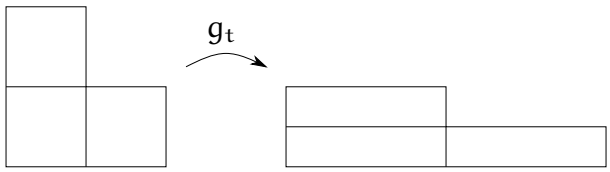


Figure 1.9: Action of $g_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ on a translation surface for $t > 0$

$$A = \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} < SL_2(\mathbb{R})$$

is called the **Teichmüller geodesic flow**, and we will write

$$g_t \coloneqq \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}.$$

The flow is indeed geodesic (in terms of metric spaces) for the Finsler metric described in section 1.7.4 since lemma 5.1 in [AG13] states that

$$d(p, g_t p) \leqslant |t|, \quad t \in \mathbb{R},\ p \in \mathscr{C}. \tag{1.6}$$

This flow is one of the main areas of study for translation surfaces.

We still require a measure on the moduli space that is invariant in order to have a group action as in definition 1.3. To define a measure, first make the identification

$$H^1(M, \Sigma; \mathbb{C}) \simeq \mathbb{C}^{2g+\sigma-1}$$

as follows. Choose a basis $(c_1, \ldots, c_{2g+\sigma-1})$ of the relative homology $H_1(M, \Sigma; \mathbb{Z})$, and send

$$H^1(M, \Sigma; \mathbb{C}) \ni v \mapsto (v(c_1), \ldots, v(c_{2g+\sigma-1})) \in \mathbb{C}^{2g+\sigma-1}. \tag{1.7}$$

Then let $m$ be the Lebesgue measure on $\mathbb{C}^{2g+\sigma-1}$. The pullback of this measure through the period map gives a measure on $\mathscr{Q}^+(M,\Sigma,\kappa)$ and also on $\mathscr{M}(M,\Sigma,\kappa)$. To get a measure on the area one translation surfaces, we consider the natural Lebesgue measure on the level set with $A = 1$. This measure is denoted by $\mu$. It was shown by Masur in [Mas82] and Veech in [Vee82] independently that this measure is finite. For this reason, we call $\mu$ the Masur-Veech measure.

Given an Abelian differential $\omega$, it has coordinates

$$\left(\int_{c_1}\omega,\ldots,\int_{c_{2g+\sigma-1}}\omega\right)\in\mathbb{C}^{2g+\sigma-1}$$

through the identification (1.7). These are called **period coordinates**, and one can then see that $SL_2(\mathbb{R})$ acts linearly in these coordinates. For an abelian differential $\omega\in\mathscr{M}^{(1)}(M,\Sigma,\kappa)$, we will denote by $\Re(\omega)$ the real components, and by $\Im(\omega)$ the imaginary components in period coordinates. Since the measure on $\mathbb{C}^{2g+\sigma-1}$ is the Lebesgue measure, the Masur-Veech measure is invariant under the $SL_2(\mathbb{R})$ action. Moreover, it was shown by Veech in [Vee86] that the action is ergodic.

It immediately follows by the Howe-Moore theorem (in particular corollary 1.14) that the whole action is mixing. Then, by the main theorem in [Moz92] that the action is actually mixing of all orders (see theorem 11.26 in [EW11] for a proof for the $SL_2(\mathbb{R})$ case). In terms of quantitative estimates however, there is only the exponential mixing result in [AGY06], described in the following section.

### 1.8.2 Exponential Mixing of the Teichmüller Geodesic Flow

The result holds for functions with certain regularity. These functions are morally functions that are $\alpha$-Hölder and behave well a the cusps of a connected component $\mathscr{C}\subset\mathscr{M}^{(1)}(M,\Sigma,\kappa)$. To describe this class of functions, let $\alpha\in(0,1]$. For a function $\varphi\colon\mathscr{C}\to\mathbb{R}$ we define for $p\in\mathscr{C}$

$$\omega_\alpha(f,p)\coloneqq\sup_{\substack{d(p,q)\leqslant 1\\ q\neq p}}\frac{|\varphi(p)-\varphi(q)|}{d(p,q)^\alpha},$$

which is the local Hölder constant of $\varphi$ at $p$. Then for $n\in\mathbb{N}$, define

$$\|\varphi\|_{\mathscr{D}_{n,\alpha}}\coloneqq\sup_{p\in\mathscr{C}}|\varphi(p)|\operatorname{sys}(p)^n+\sup_{p\in\mathscr{C}}\omega_\alpha(\varphi,p)\operatorname{sys}(p)^n,$$

and with that $\mathscr{D}_{n,\alpha}$ is the space of functions for which $\|\cdot\|_{\mathscr{D}_{n,\alpha}}$ is finite. The main theorem by Avila, Gouëzel and Yoccoz in [AGY06] is as follows.

**Theorem 1.32** (Exponential Mixing of the Teichmüller Geodesic Flow)**.** *Let* $n \in \mathbb{N}$, $\alpha \in (0,1]$*, and* $p, q \in \mathbb{R}_+$ *such that* $1/p + 1/q < 1$*. Then there exists a constant* $\delta > 0$ *(depending on* $k, \alpha, p$ *and* $q$*) such that for all functions* $\varphi \in \mathscr{D}_{n,\alpha} \cap L^p(\mathscr{C})$ *and* $\psi \in \mathscr{D}_{n,\alpha} \cap L^q(\mathscr{C})$ *and all* $t \geqslant 0$

$$\int_{\mathscr{C}} \varphi \cdot \psi \circ g_t \, d\mu = \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \, d\mu \int_{\mathscr{C}} \psi \, d\mu + \mathcal{O}\left( \left( \|\varphi\|_{\mathscr{D}_{n,\alpha}} + \|\varphi\|_{L^p} \right) \left( \|\psi\|_{\mathscr{D}_{n,\alpha}} + \|\psi\|_{L^q} \right) e^{-\delta t} \right).$$

In this thesis, we will be dealing with an even smaller class of functions, namely compactly supported $\alpha$-Hölder functions. For a function $\varphi \colon \mathscr{C} \to \mathbb{R}$, we define

$$\|\varphi\|_{C^{0,\alpha}} \coloneqq \|\varphi\|_\infty + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\alpha}.$$

With this, we define $C^{0,\alpha}(\mathscr{C})$ as the space of those functions for which $\|\cdot\|_{C^{0,\alpha}}$ is finite. As functions in this space are bounded, and $\mu$ is a finite measure on $\mathscr{C}$, we have that $C^{0,\alpha}(\mathscr{C}) \subset L^q(\mathscr{C})$ for every $q \geqslant 1$. Then, if $\varphi \in C^{0,\alpha}(\mathscr{C})$,

$$\|\varphi\|_{L^q} \leqslant \mu(\mathscr{C}) \, \|\varphi\|_\infty \tag{1.8}$$

for every $q \geqslant 1$. Moreover, if $\varphi \in C^{0,\alpha}(\mathscr{C})$ and it has compact support, then for any $n \in \mathbb{N}$,

$$\begin{aligned} \|\varphi\|_{\mathscr{D}_{n,\alpha}} &\leqslant \|\varphi\|_\infty \sup_{x \in \operatorname{supp}(\varphi)} \operatorname{sys}(x)^n + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\alpha} \sup_{x \in \operatorname{supp}(\varphi)} \operatorname{sys}(x)^n \\ &= \|\varphi\|_{C^{0,\alpha}} \sup_{x \in \operatorname{supp}(\varphi)} \operatorname{sys}(x)^n < \infty, \end{aligned}$$

where the systole is bounded since it is bounded on compact sets [see KMS86, section 3]. Therefore, $C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C}) \subset \mathscr{D}_{n,\alpha}$ for every $n \in \mathbb{N}$. Now, by considering functions in $C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$, we essentially get rid of the $L^p$ and $L^q$ requirement in theorem 1.32. Hence, we define

$$\|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \coloneqq \|\varphi\|_{\mathscr{D}_{n,\alpha}} + \|\varphi\|_{C^{0,\alpha}}.$$

This is still a norm since both summands are norms. The following version of theorem 1.32 restricted to our space of interest, $C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ follows immediately.

**Theorem 1.33** (Exponential Mixing for compactly supported $\alpha$-Hölder functions)**.** *Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Then there exists a constant $\delta > 0$ (depending on $\alpha$ and $n$) such that for all functions $\varphi, \psi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ and all $t \geqslant 0$,*

$$\int_{\mathscr{C}} \varphi \cdot \psi \circ g_t \, d\mu = \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \, d\mu \int_{\mathscr{C}} \psi \, d\mu + \mathcal{O}\left( \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} e^{-\delta t} \right).$$

We note that since we are dealing with compactly supported functions, there is a global Hölder constant encoded in $\|\cdot\|_{C^{0,\alpha}}$.

### 1.8.3 Stable and Unstable Manifolds for the Teichmüller Geodesic Flow

Here we describe the (strong) stable and unstable manifolds alluded to in section 1.4. Since the geodesic flow acts linearly on period coordinates, we have explicitly that for $t \geqslant 0$ and $\omega \in \mathscr{M}^{(1)}(M, \Sigma, \kappa)$,

$$g_t \omega = e^t \mathfrak{R}(\omega) + e^{-t} \mathfrak{I}(\omega).$$

The imaginary part of these coordinates goes to zero as $t \to \infty$. If now $\omega' \in \mathscr{M}^{(1)}(M, \Sigma, \kappa)$ is another point in moduli space, we expect that $d(g_t\omega, g_t\omega') \to 0$ if their real parts are equal. This is indeed the case for $\mu$-almost every $\omega \in \mathscr{M}^{(1)}(M, \Sigma, \kappa)$ [see Wri15, theorem 3.11]. Following time backwards, one would get that two point approach each other if they have the same imaginary part. In the spirit of the definitions in section 1.4, this motivates the following definition.

**Definition 1.34.** *The **strong stable foliation** of the Teichmüller geodesic flow, denoted by $W^{ss}$ is the foliation on $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$ consisting of leafs of the following form. For $p \in \mathscr{M}^{(1)}(M, \Sigma, \kappa)$,*

$$W^{ss}(p) = \left\{ q \in \mathscr{M}^{(1)}(M, \Sigma, \kappa) \,:\, \mathfrak{R}(q) = \mathfrak{R}(p) \right\}.$$

*Similarly, the **strong unstable foliation**, denoted by $W^{uu}$, consists of the leafs of the form*

$$W^{ss}(p) = \left\{ q \in \mathscr{M}^{(1)}(M, \Sigma, \kappa) \,:\, \mathfrak{I}(q) = \mathfrak{I}(p) \right\}.$$

*The tangent space at $p$ to $W^{ss}(p)$ will be denoted by $E^s(p)$, and the tangent space at $p$ to $W^{uu}(p)$ by $E^u(p)$.*

We will further denote by $d^-$ the metric $d$, coming from the Finsler metric restricted to stable leafs as in section 3 of [EM18]. The above defined foliations are not complementary however, as $E^s(p) + E^u(p) \neq T\mathscr{C}_p$. The missing part is the flow direction. Define now the **stable foliation**, denoted by $W^s$, as the foliation whose leafs are of the form

$$W^s(p) \coloneqq \bigcup_{t\in\mathbb{R}} g_t W^{ss}(p).$$

We will also write $\mathscr{F} \in W^{uu}$ for a leaf in the foliation. Similarly, the **unstable foliation**, denoted by $W^u$, as the foliation whose leafs are of the form

$$W^u(p) \coloneqq \bigcup_{t\in\mathbb{R}} g_t W^{uu}(p).$$

Then we have that $W^{uu}$ and $W^s$ are complementary foliations [ABEM12] (similarly $W^u$ and $W^{ss}$ are complementary, but we do not deal with these). The tangent space at a point $p$ then decomposes as

$$T\mathscr{C}_p = E^u(p) \oplus E^s(p) \oplus \mathcal{F}(p), \tag{1.9}$$

where $\mathcal{F}(p)$ is the tangent space to the flow, i.e. it is the tangent to the curve $t \mapsto g_t p$. By proposition 4.1 in [AG13], for every $p \in \mathscr{C}$, there are volume forms $d\mu_u(p)$ on $E^u(p)$ and $d\mu_s(p)$ on $E^s(p)$ that depend smoothly on $p$. Letting $d\mathrm{Leb}$ be the Lebesgue measure on $\mathcal{F}(p)$, we have the Masur-Veech locally decomposes as

$$d\mu(p) = d\mu_u(p) \otimes d\mu_s(p) \otimes d\mathrm{Leb}.$$

As we are interested in the complementary foliations $W^{uu}$ and $W^s$, we will write $d\mu_{\mathfrak{S}}(p) = d\mu_s(p) \otimes d\mathrm{Leb}$ for the volume form on the tangent space to leafs of $W^s$. For the rest of the thesis, we drop the dependence on $p$ and assume it is understood. These volume forms themselves give rise to measures on the leafs of the foliations. They will respectively be denoted by $\mu_u$ and $\mu_{\mathfrak{S}}$.

To see how the geodesic flow is a non-uniformly hyperbolic system as in definition 1.8, we see from lemma 5.2 in [AG13] that if $t \geqslant 0$, $v \in E^s(p)$, and $w \in E^u(p)$ for some $p \in \mathscr{M}^{(1)}(M, \Sigma, \kappa)$, then

$$\begin{aligned} \| d_p g_t v\|_{g_t p} &\leqslant \|v\|_p \\ \| d_p g_t w\|_{g_t p} &\geqslant \|w\|_p. \end{aligned} \tag{1.10}$$

Although there are no explicit constants as required in definition 1.8 one can see that these conditions only give that the system is non-contracting in the strong unstable foliation, and non-expanding in the strong stable foliation.

### 1.8.4 An Ergodic Theorem for the $SL_2(\mathbb{R})$ Action on Moduli Space

We finish this chapter by stating an ergodic theorem for the $SL_2(\mathbb{R})$ action, where we follow [GN09]. By the exponential mixing property of $SL_2(\mathbb{R})$ action, we have that from the discussion at the end of [AGY06], the induced representation on $L^2(\mathscr{C})$ has a spectral gap. Now denote by $\lambda$ the Haar measure on $SL_2(\mathbb{R})$ and set

$$G_t \coloneqq \{g \in SL_2(\mathbb{R}) : \ \log\|g\| < t\} \subset SL_2(\mathbb{R}),$$

for $t > 0$ and some norm $\|\cdot\|$ on $SL_2(\mathbb{R})$. The following result is from theorem A in [GN09].

**Theorem 1.35.** *Let $\varphi \in L^2(\mathscr{C})$. Then*

$$\frac{1}{\lambda(G_t)} \int_{G_t} \varphi\left(\gamma^{-1}x\right) \, d\lambda(\gamma) \xrightarrow{t\to\infty} \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \, d\mu$$

*for $\mu$-almost every $x \in \mathscr{C}$. Moreover, as the action has a spectral gap,*

$$\left| \frac{1}{\lambda(G_t)} \int_{G_t} \varphi\left(\gamma^{-1}x\right) \, d\lambda(\gamma) - \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \, d\mu \right| \leqslant Ce^{-\theta t},$$

*for a constant $C > 0$ depending on $x$ and $\varphi$, and a function $\theta > 0$ depending on the group action.*

# Chapter 2

# Quantitative Multiple Equidistribution of Unstable Leafs under the Teichmüller Geodesic Flow

The main goal of this chapter is to prove a quantitative estimate for the distribution of unstable leafs with the Teichmüller flow on a connected component $\mathscr{C}$ of the moduli space of area 1 translation structures $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$.

**Theorem 2.1.** *Let $n \in \mathbb{N}$ and $r \geqslant 1$. Then there exists a constant $\delta_r > 0$ depending on $r$ satisfying the following. Let $\mathscr{F} \in W^{uu}$ be an unstable leaf and $\varphi_0 \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ such that $\operatorname{int}(\operatorname{supp}(\varphi_0) \cap \mathscr{F})$ is relatively compact in $\mathscr{F}$ with respect to the subspace topology. Then for every $\varphi_1, \ldots, \varphi_r \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ and $t_1, \ldots, t_r \geqslant 0$, we have that*

$$\int_{\mathscr{F}} \varphi_0 \, (\varphi_1 \circ g_{t_1}) \cdots (\varphi_r \circ g_{t_r}) \, d\mu_u = \frac{1}{\mu(\mathscr{C})^r} \int_{\mathscr{F}} \varphi_0 \, d\mu_u \prod_{k=1}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu$$
$$+ \mathcal{O}\left( \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha,\infty}} \right) \operatorname{diam}(F)^{-(2g+\sigma-1)-(1-\alpha)} \, \mathfrak{M}(t_1, \ldots, t_r)^{-\delta_r} \right),$$

*where $F = \operatorname{supp}(\varphi_0) \cap \mathscr{F}$ and*

$$\mathfrak{M}(t_1, \ldots, t_r) = \exp\left( \min_{i \neq j} |t_i - t_j| \right).$$

The theorem can be interpreted as follows. We may think of $\varphi_0$ as an indicator

function $\chi_F$ on some subset of $F \subset \mathscr{F}$. Then in the case $r = 1$, we may write

$$\frac{1}{\mu_u(F)} \int_{\mathscr{F}} \chi_F \varphi \circ g_t \, d\mu_u = \frac{1}{\mu_u(g_t)} \int_{g_t F} \varphi \, d\mu_u.$$

Hence, the result is about how $g_t F$ equidistributes in $\mathscr{C}$. However, this is not exactly the case of the theorem, as it considers Hölder functions, and hence an approximation argument is needed to apply the theorem to indicator functions, where regularity conditions on the set $F$ would be required.

The proof is done by induction on $r$, the number of functions. For the base case, we follow the idea from [EM93]. We start with a type of quantitative "wavefront lemma", a phenomenon in homogeneous spaces seen in the above paper, which we briefly explain in the case of $G = \mathrm{PSL}_2(\mathbb{R})$. We may decompose $G$ as $NA^+K$, where the subgroups are as in (1.5.4) (this is also known as the Iwasawa decomposition) modulo $\{\pm \mathrm{id}\}$. The wavefront lemma in [EM93] states that for any open neighbourhood $U$ of the identity in $G$, there is an open set $V \subset G$ such that

$$HVg \subset HgU \tag{2.1}$$

for every $g \in A^+K$. This may be interpreted as follows. One may identify the hyperbolic upper half plane with $G/K$, and then taking a lattice $\Gamma < G$, the quotient $M = \Gamma\backslash G/K$ is a hyperbolic surface (or orbifold if there are cusps). The geodesic flow with respect to the hyperbolic metric on the unit tangent bundle $T^1M \backsimeq \Gamma\backslash G$ of $M$ arises from the action of $A^+$ on $\Gamma\backslash G$. This flow gives rise stable and unstable manifolds on $M$. In this specific case, given a point $p \in M$, the unstable manifold at $p$ is given by the $H$ orbit of $p$. Hence, the term $Hg$ in (2.1) may be thought of as an unstable leaf. Then, since $U$ is an open neighbourhood of the identity, $HgU$ may be thought of as a thickening of the unstable leaf $Hg$. On the other hand, $HVg$ may be thought of as the translate by $g$ of $HV = HV\mathrm{id}$, a thickening of the unstable leaf at the identity. Now, if $g$ varies in $A$, on the left hand side of (2.1) we are flowing a thickening of the unstable leaf at the identity. Meanwhile, the right hand side is a thickening of the unstable leaf to which $g$ has already been applied to. In other words, $V$ is a small enough thickening of the unstable leaf $H$ that behaves well with flowing it with the geodesic flow.

In our case, we will thicken the subset $\mathrm{supp}(\varphi_0) \cap \mathscr{F}$ of the unstable leaf $\mathscr{F}$ and will use the non-uniform hyperbolicity of the Teichmüller geodesic flow to get an

estimate on how the neighbourhood grows after applying the geodesic flow to it. Then, we use the exponential mixing property of the geodesic flow proved in [AGY06] to get equidistribution along unstable leafs.

For the inductive step, we use an idea from Dolgopyat in [Dol04]. There, the notion of an almost Markov family is defined. This consists of a partition of a space into subsets which, among other properties, have small enough diameters, but large enough volume. If the diameter is small enough, the function $\varphi_1 \circ g_{t_1}$ will be close to constant on these sets. Hence, we may take it out on the integral and use the inductive hypothesis. As in the theorem however, the estimate will depend inversely on the diameter of the subset. Hence, if we are restricting ourselves to small neighbourhoods, then we need to be able to bound the diameter of the subsets in the partition from below.

In our arguments, compactness of the functions is essential as it implies boundedness and hence allows us to easily estimate integrals. Moreover, compactly supported functions in $\mathscr{D}_{n,\alpha}$ are the $\alpha$-Hölder functions. Thus, we have a global Hölder constant and we do not need to take into account the non-compact part of $\mathscr{C}$. Taking $\alpha$-Hölder functions is important for two reasons. First, it allows us to have explicit bounds for the wavefront lemma 2.3 with respect to the sizes of the approximating sets. That is, we are able to have a quantitative version of the wavefront lemma since the estimate will depend on the size of the neighbourhood around the subset of the unstable leaf. Secondly, we need $\alpha$-Hölder functions since it is for this class of functions (having already decided that we will be working with compactly supported functions) that the exponential mixing property was proven in [AGY06].

The two main properties of the Teichmüller flow that we use in the proof are the following. First is the exponential mixing property from [AGY06]. Secondly, as explained in [ABEM12], the strong unstable and stable foliations are complementary in the sense of Margulis [Mar04]. Moreover, as shown in [AG13], the geodesic flow is only non contracting in the strong unstable direction, and non expanding in the unstable direction. This second property is one of non-uniformly hyperbolic systems, as explained in section 12.5 of [BP07], and section 1.8. Thus, similar methods should work for systems with the above properties.

Before we begin with the four above steps, we setup some preliminaries.

## 2.1 Preliminaries

For the rest of the thesis, we fix $\alpha \in (0,1]$, and a connected component $\mathscr{C}$ of $\mathscr{M}^{(1)}(M,\Sigma,\kappa)$ with $|\Sigma| = \sigma$, for a compact surface of genus $g \geqslant 1$. We may identify the direct sum in (1.9) with the Cartesian product of the subspaces and write a vector in $T\mathscr{C}_p$ as $(\xi,\eta,r)$. Through the exponential map at $p$ [see BCS00, Chapter 5], there is a coordinate chart at $p$ for a small enough ball around $0 \in T\mathscr{C}_p$. We call the largest radius for which the exponential map is a diffeomorphism at $p$ the radius of injectivity, and denote it by $R_p > 0$. Moreover, for $p \in \mathscr{C}$ and $a > 0$, we define

$$B^{\mathfrak{S}}(p,a) \coloneqq \{q \in W^s(p) \,:\, d^-(p,q) < a\},$$

In words, this is the ball of radius $a$ around $p$ in the stable direction. We will denote by $\mathscr{F}$ a full leaf of a foliation.

A fact we will use regarding the unstable measure is the form of the pushforwards by the geodesic flow. Namely

$$(g_t)_* \mu_u = e^{-ct}\mu_u, \tag{2.2}$$

for some constant $c \in \mathbb{N}$ ([see AG13, proposition 4.1]).

We will make the following abuse of notation. Since the measures come from volume forms on the tangent spaces, when integrating a function $\varphi$ over some subset of $A \subset \mathscr{C}$, we will write

$$\int_A \varphi(p)\, d\mu = \int_A \varphi(\xi,\eta,r)\, d\mu(\xi,\eta,r) = \iint_A \varphi(\xi,\eta,r)\, d\mu_u(\xi)\, d\mu_{\mathfrak{S}}(\eta,r).$$

This can be done explicitly by partitioning $A$ into charts for which the exponential map is a diffeomorphism, and continue as in chapter 16 of [Lee13].

We are now ready to start with the four steps of the multiple mixing proof.

## 2.2 Quantitative Equidistribution of Unstable Leafs

We begin with the base case of theorem 2.1.

**Proposition 2.2.** *Let $n \in \mathbb{N}$. Then there exists a constant $\delta > 0$ satisfying the following. Let $\mathscr{F} \in W^{uu}$ be an unstable leaf, and $\varphi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ such that $\operatorname{int}(\operatorname{supp}(\varphi) \cap \mathscr{F})$ is relatively compact in the subspace topology. Then for every $\psi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ and $t \geqslant 0$, we have that*

$$\int_{\mathscr{F}} \varphi \cdot \psi \circ g_t \, d\mu_u = \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{F}} \varphi \, d\mu_u \int_{\mathscr{C}} \psi \, d\mu$$
$$+\mathcal{O}\left( \|\psi\|_{\mathscr{D}_{n,\alpha,\infty}} \|\varphi\|_{\mathscr{D}_{n,\alpha,\infty}} \operatorname{diam}(F)^{-(2g+\sigma-1)-(1-\alpha)} e^{-\delta t} \right),$$

*where $F = \operatorname{supp}(\varphi) \cap \mathscr{F}$.*

Before the proof of this proposition, we require a type of wavefront lemma.

**Lemma 2.3.** *Let $n \in \mathbb{N}$, $\varepsilon > 0$, and consider an unstable leaf $\mathscr{F} \in W^{uu}$. Moreover, let $\varphi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$, be such that $\operatorname{int}(\operatorname{supp}(\varphi) \cap \mathscr{F})$ is relatively compact in the subspace topology. Then there exists a function $\varphi_\varepsilon \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ such that for every $\psi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ and every $t \geqslant 0$*

$$\int_{\mathscr{C}} \varphi_\varepsilon \, d\mu \int_{\mathscr{F}} \varphi \cdot \psi \circ g_t \, d\mu_u = \int_{\mathscr{F}} \varphi \, d\mu_u \int_{\mathscr{C}} \varphi_\varepsilon \cdot \psi \circ g_t \, d\mu$$
$$+\mathcal{O}\left( \|\varphi\|_{\mathscr{D}_{n,\alpha,\infty}} \|\psi\|_{\mathscr{D}_{n,\alpha,\infty}} \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha,\infty}} \varepsilon^\alpha \right).$$

**Proof.** Denote by $F$ the set $\operatorname{int}(\operatorname{supp}(\varphi) \cap \mathscr{F})$ in the subspace topology. We first define for every $p \in F$ the value

$$\delta_p \coloneqq \frac{1}{4} \min\{\varepsilon, R_p\}.$$

We will construct a subset $V_\varepsilon$ of $F$ which can be though of as a thickening of $F$, and then take $\varphi_\varepsilon$ supported on $V_\varepsilon$. First we construct an intermediate subset $V'_\varepsilon$ defined as

$$V'_\varepsilon \coloneqq \bigcup_{p \in F} B^{\mathfrak{S}}(p, \delta_p).$$

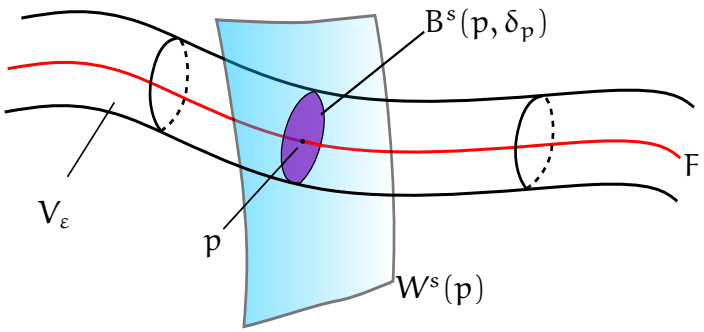


Figure 2.1: Tubular Neighbourhood around F

Now, shrinking $V'_\varepsilon$ if necessary to a subset $V_\varepsilon$, we may assume that it is a tubular neighbourhood of F as in figure 2.1 below.

Such a tubular neighbourhood exists by theorem 3.5 in [AJ19], which we can apply since the unstable leafs are affine submanifolds, F is assumed to be relatively compact, and Teichmüller space is Finsler as in section 1.7.4. We will also define for $p \in F$

$$V_\varepsilon^{\mathfrak{S}}(p) \coloneqq \{q \in V_\varepsilon \cap W^s(p) \,:\, d^-(p,q) < \delta_p\} = V_\varepsilon \cap B^{\mathfrak{S}}(p,\delta_p)\,.$$

To have an estimate on the integral from the lemma, we need to know how $V_\varepsilon$ grows after pushing it with the geodesic flow. Let $p \in F$ and let $q \in V_\varepsilon^{\mathfrak{S}}(p)$. Set $v \coloneqq \exp_p^{-1}(q)$, a vector in $T\mathscr{C}_p$. By the decomposition in (1.9) and since the radius of the balls is smaller than the radius of injectivity, we may write

$$v = \eta + r,$$

where $\eta \in E^s(p)$ and $r \in \mathcal{F}(p)$. Since the geodesic flow is non-expanding in the strong stable direction, we have that by (1.10), for all $t \geqslant 0$,

$$\begin{aligned}\|d_p g_t(v)\|_{g_t p} &\leqslant \|d_p g_t(\eta)\|_{g_t p} + \|d_p g_t(r)\|_{g_t p}\\ &\leqslant \|\eta\|_p + \|r\|_p \leqslant \delta_p + \delta_p.\end{aligned}$$

Here the inequality $\|d_p g_t(r)\|_{g_t p} \leqslant \|r\|_p$ follows from the fact that the flow is geodesic for the metric arising from the Finsler norm $\|\cdot\|$ (see (1.6)). This means in particular, that for every $p \in F$ and every $t \geqslant 0$

$$\operatorname{diam}\left(g_t V_\varepsilon^{\mathfrak{S}}(p)\right) \leqslant \operatorname{diam}\left(g_t B^{\mathfrak{S}}(p,\delta_p)\right) \leqslant 4\delta_p \leqslant \varepsilon. \tag{2.3}$$

Thus, as we flow $V_\varepsilon$ forward with the geodesic flow, the diameter of its stable direction remains bounded by $\varepsilon$.

We now begin with finding a function $\varphi_\varepsilon$ that satisfies the conditions of the lemma. Note that we may assume that $\int_{\mathscr{F}} \varphi \, d\mu_u \neq 0$, since otherwise we could just choose $\varphi_\varepsilon \equiv 0$. We now choose a function $\overline{\varphi}_\varepsilon \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ taking values in $[0,1]$ with support in $V_\varepsilon$, and such that

$$\overline{\varphi}_\varepsilon(p) = \begin{cases} 1, & p \in F \\ 0, & p \notin V_\varepsilon \end{cases}.$$

This acts like a bump function on a neighbourhood of $F$. Then define

$$\varphi_\varepsilon \coloneqq \overline{\varphi}_\varepsilon \cdot \varphi.$$

Note that we may then choose $\varphi_\varepsilon$ to also satisfy

$$\|\varphi_\varepsilon - \varphi\|_{L^2} \ll \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha,\infty}} \varepsilon^\alpha.$$

Since we are assuming $\int_{\mathscr{F}} \varphi \, d\mu_u \neq 0$, we may also assume that $\int_{\mathscr{C}} \varphi_\varepsilon \, d\mu \neq 0$. We thus rewrite

$$\frac{1}{\int_{\mathscr{C}} \varphi_\varepsilon \, d\mu} \int_{\mathscr{C}} \varphi_\varepsilon \cdot \psi \circ g_t \, d\mu = \frac{1}{\int_{\mathscr{C}} \varphi_\varepsilon \, d\mu} \int_{g_t\mathscr{C}} (\varphi_\varepsilon \circ g_{-t}) \cdot \psi \, d\mu,$$

where we used that the geodesic flow is measure preserving. Similarly, using (2.2) we have that

$$\frac{1}{\int_{\mathscr{F}} \varphi \, d\mu_u} \int_{\mathscr{F}} \varphi \cdot \psi \circ g_t \, d\mu_u = \frac{1}{\int_{g_t\mathscr{F}} \varphi \circ g_{-t} \, d\mu_u} \int_{g_t\mathscr{F}} (\varphi \circ g_{-t}) \cdot \psi \, d\mu_u.$$

We further write $\xi_p \in T\mathscr{C}_p$ for the (unique) point in $T\mathscr{C}_p$ that gets mapped to $p$ under the exponential map (here we implicitly assume that $\|\xi_p\| \leqslant R_p$). Now let

$$m_{\xi_p} \coloneqq \mu_{\mathfrak{S}}\left(g_t V_\varepsilon^{\mathfrak{S}}(p)\right).$$

Note that $m_{\xi_p}$ is never zero since the sets $V_\varepsilon^{\mathfrak{S}}(p)$ are open as subsets of $W^s(p)$. We then further rewrite the integral over the leaf above as

$$\int_{g_t\mathscr{F}} (\varphi \circ g_{-t}) \cdot \psi \, d\mu_u =$$

$$= \int_{g_t\mathscr{F}} \frac{((\varphi \circ g_{-t}) \cdot \psi)(\xi_p, 0, 0)}{m_{\xi_p}} \int_{g_t V_\varepsilon^{\mathfrak{S}}(p)} d\mu_{\mathfrak{S}}\, d\mu_u(\xi_p).$$

Moreover, abbreviate

$$B = \int_{g_t\mathscr{F}} \varphi \circ g_{-t}\, d\mu_u, \quad C = \int_{g_t\mathscr{C}} \varphi_\varepsilon \circ g_{-t}\, d\mu,$$

and with that $A = |BC|$. We now begin with the main estimate:

$$\begin{aligned}
&\left| \frac{1}{\int_{\mathscr{F}} \varphi\, d\mu_u} \int_{\mathscr{F}} \varphi \cdot \psi \circ g_t\, d\mu_u - \frac{1}{\int_{\mathscr{C}} \varphi_\varepsilon\, d\mu} \int_{\mathscr{C}} \varphi_\varepsilon \cdot \psi \circ g_t\, d\mu \right| \leqslant \\
&\leqslant \frac{1}{A} \left| \int_{g_t\mathscr{C}} B\left((\psi \cdot \varphi \circ g_{-t})(\xi_p, 0, 0) - (\psi \cdot \varphi_\varepsilon \circ g_{-t})(\xi_p, \eta, r)\right) d\mu(\xi_p, \eta, r) \right| \quad (2.4) \\
&+ \frac{1}{A} \left| \int_{g_t\mathscr{C}} \frac{|(\psi \cdot \varphi \circ g_{-t})(\xi_p, 0, 0)| \left|C - m_{\xi_p} B\right|}{m_{\xi_p}} d\mu(\xi_p, \eta, r) \right|. \quad (2.5)
\end{aligned}$$

The first term is estimated as

$$\begin{aligned}
(2.4) &\leqslant \frac{1}{C} \left| \int_{g_t\mathscr{C}} (\psi \cdot \varphi_\varepsilon \circ g_{-t})(\xi_p, 0, 0) - (\psi \cdot \varphi_\varepsilon \circ g_{-t})(\xi_p, \eta, r)\, d\mu(\xi_p, \eta, r) \right| \\
&\quad + \frac{1}{C} \left| \int_{g_t\mathscr{C}} \psi(\xi_p, 0, 0) \cdot (\varphi_\varepsilon \circ g_{-t}(\xi_p, 0, 0) - \varphi \circ g_{-t}(\xi_p, 0, 0))\, d\mu(\xi_p, \eta, r) \right| \\
&\ll \frac{1}{C} \left( \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon^\alpha + \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \|\varphi_\varepsilon - \varphi\|_{L^2} \right) \\
&\ll \frac{1}{C} \left( \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon^\alpha \right),
\end{aligned}$$

where we used (2.3) in the second to last line. For the second term, we simply get

$$\begin{aligned}
(2.5) &\ll \frac{1}{A} \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \|\varphi_\varepsilon - \varphi\|_{L^2} \\
&\ll \frac{1}{A} \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon^\alpha.
\end{aligned}$$

The result follows. ■

We may now begin with the proof of proposition 2.2.

**Proof of Proposition 2.2.** Let $\varepsilon > 0$, to be chosen later, and let $\varphi_\varepsilon$ be the function from lemma 2.3. Note that we may further require that

$$\|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \ll \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \operatorname{diam}(\operatorname{supp}(\varphi_\varepsilon))^{-(1-\alpha)}, \tag{2.6}$$

in a similar fashion to the estimates in [Pau81] and in appendix 1 of [Shu92]. Now, by lemma 2.3, we have that

$$\int_{\mathscr{C}} \varphi_\varepsilon \, d\mu \int_{\mathscr{F}} \varphi \cdot \psi \circ g_t \, d\mu_u = \int_{\mathscr{F}} \varphi \, d\mu_u \int_{\mathscr{C}} \varphi_\varepsilon \cdot \psi \circ g_t \, d\mu + \mathcal{O}\left(\|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon^\alpha\right).$$

We may then apply the exponential mixing property of the geodesic flow from theorem 1.33. Letting $A = \int_{\mathscr{C}} \varphi_\varepsilon \, d\mu$, and $\delta > 0$ be the constant from the theorem, we have that

$$A \int_{\mathscr{F}} \varphi \cdot \psi \circ g_t \, d\mu_u = \int_{\mathscr{F}} \varphi \, d\mu_u \int_{\mathscr{C}} \varphi_\varepsilon \, d\mu \int_{\mathscr{C}} \psi \, d\mu + \mathcal{O}\left(\|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} e^{-\delta t}\right) + \mathcal{O}\left(\|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \|\varphi_\varepsilon\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon^\alpha\right).$$

Before continuing, we may assume that $\int_{\mathscr{F}} \varphi \, d\mu_u \neq 0$ since otherwise we can choose $\varphi_\varepsilon \equiv 0$, and then there is nothing to prove. This means we can also assume that $A = \int_{\mathscr{C}} \varphi_\varepsilon \, d\mu \neq 0$. Moreover, as $\mu$ is locally the Lebesgue measure on the tangent space, we may estimate $A$ as

$$A \ll \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \operatorname{diam}(\operatorname{supp}(\varphi_\varepsilon))^{2g+\sigma-1}. \tag{2.7}$$

Then dividing both sides of the last equality by $A$, and using (2.7) and (2.6), we get that

$$\int_{\mathscr{F}} \varphi \cdot \psi \circ g_t \, d\mu_u = \int_{\mathscr{F}} \varphi \, d\mu_u \int_{\mathscr{C}} \psi \, d\mu + \mathcal{O}\left(\operatorname{diam}(\operatorname{supp}(\varphi_\varepsilon))^{-(1-\alpha)-(2g+\sigma-1)} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} e^{-\delta t}\right) + \mathcal{O}\left(\operatorname{diam}(\operatorname{supp}(\varphi_\varepsilon))^{-(1-\alpha)-(2g+\sigma-1)} \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon^\alpha\right).$$

Then choosing

$$\varepsilon \leqslant e^{-\delta t/\alpha},$$

and also small enough such that $\operatorname{diam}(\operatorname{supp}(\varphi_\varepsilon)) = \operatorname{diam}(F)$, we get the result. ∎

## 2.3 Quantitative Multiple Mixing Along Unstable Leafs

In this section, we prove theorem 2.1.

**Proof of Theorem 2.1.** The case $r = 1$ is the content of proposition 2.2. Now let $r \geqslant 2$ and assume the theorem holds for $r-1$. Fix $t_1, \dots, t_r \geqslant 0$ and $0 < \tilde{\varepsilon}_{t_1} < \varepsilon_{t_1}$ to be chosen later, and denote by $E$ the subset $\operatorname{supp}(\varphi_1 \circ g_{t_1})$. As this set is compact, we may choose a partition of unity $(f_i)_{i=1}^m$ of functions in $C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ of $E$. More precisely, these functions take values in $[0,1]$ and are such that

$$\sum_{i=1}^{m} f_i(p) = 1, \quad p \in E,$$

and zero outside a neighbourhood of $E$. Writing $F_i = \operatorname{supp}(f_i)$, we will ask that

$$\tilde{\varepsilon}_{t_1} \leqslant \operatorname{diam}(F_i) \leqslant \varepsilon_{t_1}. \tag{2.8}$$

This condition is the one borrowed from [Dol04] of an almost Markov family. The condition will also give that the number of functions needed in the partition of unity depends on $t_1$ as

$$m \ll \tilde{\varepsilon}_{t_1}^{-(2g+\sigma-1)}. \tag{2.9}$$

We will also need control over the Hölder norms of the $f_i$. Since we are asking that the diameter of their supports be within a certain range, we may require that

$$\|f_i\|_{\mathscr{D}_{n,\alpha},\infty} \ll \tilde{\varepsilon}_{t_1}^{-(1-\alpha)}, \tag{2.10}$$

which is a similar to the requirement in proposition 5.8 in [AG13]. Now arbitrarily pick points $x_{t_1}^i \in F_i$ for each $i = 1, \dots, m$. Before we begin with the main estimate, we make some abbreviations to simplify the notation. Let

$$\Phi_i = \varphi_1 \circ g_{t_1}\left(x_{t_1}^i\right), \quad \psi = \prod_{k=2}^{r} \varphi_k \circ g_{t_k}.$$

Now note that

$$\int_{\mathscr{F}} \varphi_0 (\varphi_1 \circ g_{t_1}) \cdots (\varphi_r \circ g_{t_r}) \, d\mu_u = \sum_{i=1}^{m} \int_{F_i} f_i \cdot \varphi_0 (\varphi_1 \circ g_{t_1}) \cdots (\varphi_r \circ g_{t_r}) \, d\mu_u.$$

We can now begin with the main estimate:

$$\left|\int_{\mathscr{F}} \varphi_0\,(\varphi_1\circ g_{t_1})\cdots(\varphi_r\circ g_{t_r})\,\mathrm{d}\mu_u - \frac{1}{\mu(\mathscr{C})^r}\int_{\mathscr{F}}\varphi_0\,\mathrm{d}\mu_u\prod_{k=1}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu\right|$$

$$= \left|\sum_{i=1}^{m}\int_{F_i} f_i\cdot\varphi_0\,(\varphi_1\circ g_{t_1})\cdot\psi\,\mathrm{d}\mu_u - \frac{1}{\mu(\mathscr{C})^r}\int_{\mathscr{F}}\varphi_0\,\mathrm{d}\mu_u\prod_{k=1}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu\right|$$

$$\leqslant \left|\sum_{i=1}^{m}\int_{F_i} f_i\cdot\varphi_0\,(\varphi_1\circ g_{t_1})\cdot\psi\,\mathrm{d}\mu_u - \sum_{i=1}^{m}\Phi_i\int_{\mathscr{F}} f_i\cdot\varphi_0\cdot\psi\,\mathrm{d}\mu_u\right| \tag{2.11}$$

$$+\left|\sum_{i=1}^{m}\Phi_i\int_{\mathscr{F}} f_i\cdot\varphi_0\cdot\psi\,\mathrm{d}\mu_u - \sum_{i=1}^{m}\Phi_i\frac{\int_{\mathscr{F}} f_i\varphi_0\,\mathrm{d}\mu_u}{\mu(\mathscr{C})^{r-1}}\prod_{k=2}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu\right| \tag{2.12}$$

$$+\left|\sum_{i=1}^{m}\Phi_i\frac{\int_{\mathscr{F}} f_i\varphi_0\,\mathrm{d}\mu_u}{\mu(\mathscr{C})^{r-1}}\prod_{k=2}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu - \frac{1}{\mu(\mathscr{C})^{r-1}}\int_{\mathscr{F}}\varphi_0\cdot\varphi_1\circ g_{t_1}\,\mathrm{d}\mu_u\prod_{k=2}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu\right| \tag{2.13}$$

$$+\left|\frac{1}{\mu(\mathscr{C})^{r-1}}\int_{\mathscr{F}}\varphi_0\cdot\varphi_1\circ g_{t_1}\,\mathrm{d}\mu_u\prod_{k=2}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu - \frac{1}{\mu(\mathscr{C})^{r}}\int_{\mathscr{F}}\varphi_0\,\mathrm{d}\mu_u\prod_{k=1}^{r}\int_{\mathscr{C}}\varphi_k\,\mathrm{d}\mu\right| \tag{2.14}$$

The first term is estimated using the fact that the diameters of the sets $F_i$ are bounded as in (2.8):

$$\begin{aligned}(2.11) &\leqslant \sum_{i=1}^{m}\int_{F_i}|\varphi_1\circ g_{t_1}-\Phi_i|\,f_i\varphi_0\psi\,\mathrm{d}\mu_u\\ &\leqslant \varepsilon_{t_1}^{\alpha}\|\varphi_1\|_{\mathscr{D}_{n,\alpha},\infty}\|\varphi_0\|_{\mathscr{D}_{n,\alpha},\infty}\prod_{k=2}^{r}\|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty}\\ &\ll \left(\prod_{k=0}^{r}\|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty}\right)\varepsilon_{t_1}^{\alpha},\end{aligned}$$

where the second line follows from the Hölder condition. For the second term, we use the inductive hypothesis and consider $f_i \cdot \varphi_0$ as a single function. We get that

$$\begin{aligned}(2.12) &\leqslant \|\varphi_1\|_{\mathscr{D}_{n,\alpha},\infty} \sum_{i=1}^{m} \left| \int_{\mathscr{F}} (f_i \varphi_0)\, \psi \, d\mu_u - \frac{1}{\mu(\mathscr{C})^{r-1}} \int_{\mathscr{F}} f_i \varphi_0 \, d\mu_u \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right| \\ &\leqslant \|\varphi_1\|_{\mathscr{D}_{n,\alpha},\infty} \sum_{i=1}^{m} \left( \prod_{k=2}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \operatorname{diam}(F_i)^{-(2g+\sigma-1)-(1-\alpha)} \\ &\qquad \times \|f_i \varphi_0\|_{\mathscr{D}_{n,\alpha},\infty} \mathfrak{M}(t_1, \dots, t_r)^{-\delta_{r-1}} \\ &\ll \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \mathfrak{M}(t_1, \dots, t_r)^{-\delta_{r-1}} \\ &\qquad \times \sum_{i=1}^{m} \|f_i\|_{\mathscr{D}_{n,\alpha},\infty} \operatorname{diam}(F_i)^{-(2g+\sigma-1)-(1-\alpha)} \\ &\ll \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \mathfrak{M}(t_1, \dots, t_r)^{-\delta_{r-1}} \, \tilde{\varepsilon}_{t_1}^{-2(2g+\sigma-1)-2(1-\alpha)},\end{aligned}$$

where the second line follows by the inductive hypothesis and the last line follows from (2.9) and (2.10). The third term is again estimated with the size of the sets $F_i$. We have

$$\begin{aligned}(2.13) &\leqslant \left( \prod_{k=2}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \left| \sum_{i=1}^{m} \Phi_i \int_{F_i} f_i \varphi_0 \, d\mu_u - \sum_{i=1}^{m} \int_{F_i} f_i \varphi_0 \varphi_1 \circ g_{t_1} \, d\mu_u \right| \\ &\leqslant \left( \prod_{k=2}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \sum_{i=1}^{m} \int_{F_i} |\Phi_i - \varphi_1 \circ g_{t_1}| \, f_i \varphi_0 \, d\mu_u \\ &\ll \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \varepsilon_{t_1}^{\alpha},\end{aligned}$$

where the last line follows from the Hölder condition. For the last term, we simply apply proposition 2.2 to get that

$$(2.14) \leqslant \left( \prod_{k=2}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \left| \int_{\mathscr{F}} \varphi_0 \cdot \varphi_1 \circ g_{t_1} \, d\mu_u - \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{F}} \varphi_0 \, d\mu_u \int_{\mathscr{C}} \varphi_1 \, d\mu \right|$$

$$\ll \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \operatorname{diam}(F)^{-(2g+\sigma-1)} e^{-\delta_1 t_1}.$$

Now we need to choose $\varepsilon_{t_1}$ and $\tilde{\varepsilon}_{t_1}$ to get the result. Let $\delta_r < \delta_{r-1}$ be such that the following estimates hold:

$$\tilde{\varepsilon}_{t_1} \gg \exp\left( \frac{(\delta_{r-1} - \delta_r) \min_{i \neq j} |t_i - t_j|}{-2(2g + \sigma - 1) - 2(1 - \alpha)} \right),$$

and

$$\varepsilon_{t_1} \ll \mathfrak{M}(t_2, \ldots, t_r)^{-\delta_r/\alpha}.$$

With these choices one gets a result for considering only $g_{t_1}$. We then replace $\varphi_1$ with $\varphi_j$, and $g_{t_1}$ with $g_{t_j}$ for $j = 2, \ldots, r$ and we get the result. ■

# Chapter 3

# Quantitative Mixing of the $SL_2(\mathbb{R})$ Action on Moduli Space

The goal of this chapter is to prove a quantitative estimate for the $SL_2(\mathbb{R})$ action on a connected component of the moduli space. The estimate is in terms of a Riemannian metric on $SL_2(\mathbb{R})$. The particular metric we used is described in Appendix B, and will be denoted by $\rho$. The main result of this chapter takes the following form.

**Theorem 3.1.** *Let $n \in \mathbb{N}$ and $r \geqslant 1$. Then there exists a constant $\delta_r > 0$ such that for every $\varphi_0, \varphi_1, \ldots, \varphi_r \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ and any $\gamma_1, \ldots, \gamma_r \in SL_2(\mathbb{R})$, the following holds*

$$\int_{\mathscr{C}} \varphi_0 \cdot \prod_{k=1}^{r} \varphi_k \circ \gamma_k \, d\mu = \frac{1}{\mu(\mathscr{C})^r} \prod_{k=0}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu$$
$$+ \mathcal{O}\left( \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \mathfrak{M}(\gamma_1, \ldots, \gamma_r)^{-\delta_r} \right),$$

*where*

$$\mathfrak{M}(\gamma_1, \ldots, \gamma_r) \coloneqq \exp\left( \min_{i \neq j} \rho(\gamma_i, \gamma_j) \right).$$

At first glance, one may try to use the results in chapter 2 and extend them to the whole group action by using the Cartan decomposition of $SL_2(\mathbb{R})$ into $KA^+K$, where $A^+ = \{\mathrm{diag}(e^t, e^{-t}) : t \geqslant 0\}$, and $K = SO(2)$ (see Appendix A for details). This is the classical approach for mixing of order two. The approach is taken, for

example, by Ratner in [Rat87] to deduce that the horocycle flow on homogeneous spaces of $SL_2(\mathbb{R})$ is polynomially mixing by knowing that the geodesic flow is exponentially mixing. The key tool in this method is that the integral of interest can be written as an inner product. Then, properties of the unitary representation of $SL_2(\mathbb{R})$ on $L^2(\mathscr{C})$ can be used to manipulate the inner product and separate the $SO(2)$ components from the geodesic flow. This is no longer possible for higher order correlations as the integrals can no longer be represented by inner products.

To go around this, we first establish that the $SL_2(\mathbb{R})$ mixes exponentially. This is done by using a similar argument to the one Ratner uses in [Rat87] to show a quantitative mixing estimate for the horocycle flow. This result is already stated at the end of [AGY06], and we spell it out in section 3.1. Then for the case when $r \geqslant 2$, we follow the same arguments as in the proof of theorem 2.1.

## 3.1 Quantitative Mixing of the $SL_2(\mathbb{R})$ Action

Here we follow a similar argument as that in [Rat87] to prove the following.

**Proposition 3.2.** *Let $n \in \mathbb{N}$. Then there exists a constant $\delta > 0$ such that for any $\varphi, \psi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$, and any $\gamma \in SL_2(\mathbb{R})$,*

$$\int_{\mathscr{C}} \varphi \cdot \psi \circ \gamma \, d\mu = \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \, d\mu \int_{\mathscr{C}} \psi \, d\mu$$
$$+ \mathcal{O}\left( \|\varphi\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi\|_{\mathscr{D}_{n,\alpha},\infty} \exp(\rho(\mathrm{id}, \gamma))^{-\delta} \right).$$

Before we prove this proposition, we need a lemma that will allow us to apply theorem 1.33.

**Lemma 3.3.** *The group $K$ acts isometrically on Teichmüller space.*

**Proof.** As in section 1.7.4, the Finsler metric on $\mathscr{Q}(M, \Sigma, \kappa)$ is given by

$$\|v\|_p = \sup_{n \in \mathbb{N}} \left| \frac{v(c)}{\Theta(p)(c)} \right|, \quad p \in \mathscr{Q}(M, \Sigma, \kappa),$$

where $c$ runs over the saddle connections of the surface $p$, and $T\mathscr{C}_p$ is identified with $H^1(M, \Sigma; \mathbb{C})$ (here $M$ is the topological surface, and $\Sigma$ is the set of singularities). Let now $p, q \in \mathscr{C}$ and $k(\theta) \in K$. Moreover, let $\beta \colon [0, T] \to \mathscr{C}$ be a unit

speed continuously differentiable path from $p$ to $q$ (the assumption on the curve being unit speed is not restrictive, as the Finsler length of curves is not affected by reparametrizations). Then, the curve $\tilde{\beta} \coloneqq k(\theta)\,\beta$ is a curve from $k(\theta)\,p$ to $k(\theta)\,q$. We now have that for $s \in [0, T]$, if $\Theta(\beta(s)) = a(s) + ib(s)$, then

$$\Theta\left(\tilde{\beta}(s)\right) = \cos(\theta)\,a(s) - \sin(\theta)\,b(s) + i\left(\sin(\theta)\,a(s) + \cos(\theta)\,b(s)\right).$$

Moreover,

$$\tilde{\beta}'(s) = d_{\beta(s)}k(\theta)\left(\beta'(s)\right) = k(\theta)\left(\beta'(s)\right),$$

since $k(\theta)$ acts linearly in period coordinates. Thus,

$$\|\beta'(s)\|_{\beta(s)} = \|\tilde{\beta}'(s)\|_{\tilde{\beta}(s)}$$

for each $s \in [0, T]$. This means that the Finsler length of both $\beta$ and $\tilde{\beta}$ is the same. Since $\beta$ was arbitrary, this holds for any $C^1$ path from $p$ to $q$. Finally, since the distance induced by the Finsler metric is computed by taking the infimum over all continuously differentiable paths between two points, we have that

$$d(p, q) = d(k(\theta)\,p, k(\theta)\,q).$$

This finishes the proof. ■

Since the distance on Teichmüller space descends to the moduli space, this lemma gives that the action of $K$ is also isometric on moduli space. In particular, it says that for $\gamma \in SL_2(\mathbb{R})$ and $\varphi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$, the function $\varphi \circ \gamma$ is still in $C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ with the same Hölder constant. Indeed, the function is still compactly supported since $\varphi$ is compactly supported. As for the Hölder condition, we have that for $p, q \in \mathscr{C}$

$$\begin{aligned} |\varphi(k(\theta')\,p) - \varphi(k(\theta)\,q)| &\leqslant \|\varphi\|_{C^{0,\alpha}} d(k(\theta)\,p, k(\theta)\,q)^{\alpha} \\ &= \|\varphi\|_{C^{0,\alpha}} d(p, q)^{\alpha}, \end{aligned} \tag{3.1}$$

where the equality follows from lemma 3.3. We will now prove proposition 3.2.

**Proof of Proposition 3.2.** Consider the unitary representation $\varrho$ defined in (1.1) arising from the action of $SL_2(\mathbb{R})$ on $\mathscr{C}$. Note that the subset $C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ is contained $L^2(\mathscr{C})$ since $\mu$ is a finite measure, and hence $\varrho$ also acts on $C^{0,\alpha}(\mathscr{C}) \cap$

$C_c(\mathscr{C})$. Let $\varphi, \psi \in C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$, and $\gamma = k(\theta') g_t k(\theta) \in K$. By the definition of the unitary representation induced by the $SL_2(\mathbb{R})$ action,

$$\int_{\mathscr{C}} \varphi \cdot \psi \circ \gamma \, d\mu = \int_{\mathscr{C}} \varphi \cdot \varrho\left(\gamma^{-1}\right) \psi \, d\mu.$$

Now we may continue by representing the integral as an inner product and use the properties of the unitary representation to get that

$$\begin{aligned}
\int_{\mathscr{C}} \varphi \cdot \varrho\left(\gamma^{-1}\right) \psi \, d\mu &= \left\langle \varphi, \varrho\left(\gamma^{-1}\right) \psi \right\rangle \\
&= \langle \varphi, \varrho(k(-\theta)\, g_{-t} k(-\theta'))\, \psi \rangle \\
&= \langle \varphi, \varrho(k(-\theta))\, \varrho(g_{-t} k(-\theta'))\, \psi \rangle \\
&= \langle \varrho(k(\theta))\, \varphi, \varrho(g_{-t} k(-\theta'))\, \psi \rangle \\
&= \langle \varphi \circ k(-\theta), \psi \circ k(\theta') \circ g_t \rangle \\
&= \int_{\mathscr{C}} \varphi \circ k(-\theta) \cdot \psi \circ k(\theta') \circ g_t \, d\mu.
\end{aligned}$$

By the discussion after the proof of lemma 3.3, we may apply the exponential mixing result in theorem 1.33 to get that for $\delta_1 > 0$ from the theorem,

$$\begin{aligned}
\int_{\mathscr{C}} \varphi \circ k(-\theta) \cdot \varrho\left(\gamma^{-1}\right) \psi \, d\mu &= \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \circ k(-\theta) \, d\mu \int_{\mathscr{C}} \psi \circ k(\theta') \, d\mu + \\
&\quad + \mathcal{O}\left(\|\varphi\|_{C^{0,\alpha}} \|\psi\|_{C^{0,\alpha}} e^{-\delta_1 t}\right). \\
&= \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi \int_{\mathscr{C}} \psi + \mathcal{O}\left(\|\varphi\|_{C^{0,\alpha}} \|\psi\|_{C^{0,\alpha}} e^{-\delta_1 t}\right),
\end{aligned}$$

where the equality follows from the fact that $SL_2(\mathbb{R})$ acts in a measure preserving way on $\mathscr{C}$, and hence in particular

$$\int_{\mathscr{C}} \varphi \circ k(-\theta) \, d\mu = \int_{\mathscr{C}} \varphi \, d\mu, \quad \text{and} \quad \int_{\mathscr{C}} \psi \circ k(\theta') \, d\mu = \int_{\mathscr{C}} \psi \, d\mu.$$

To finish, we estimate $e^{-t}$ with $\rho(\mathrm{id}, \gamma)$. By (B.1) and (B.2), we have that

$$e^{-t} = \|g_t\|_{op}^{-1/2} = \|\gamma\|_{op}^{-1/2} \leqslant \left(e^{c_2} e^{\rho(\gamma, \mathrm{id}) c_1}\right)^{-1/2}.$$

Setting $\delta := \frac{\delta_1 c_1}{2} > 0$, we have that for a constant $C > 0$,

$$e^{-\delta t} \leqslant C \exp(\rho(\gamma, \mathrm{id}))^{-\delta}.$$

The result follows. ∎

## 3.2 Quantitative Multiple Mixing for the $\mathrm{SL}_2(\mathbb{R})$ Action

In this section, we prove theorem 3.1 with similar ideas as those in the proof of 2.1.

**Proof of Theorem 3.1.** The proof goes by induction. The case $r = 1$ is the content of proposition 3.2. Now let $r \geqslant 2$, assuming that the theorem holds for $r-1$, and fix $\gamma_1, \ldots, \gamma_r \in \mathrm{SL}_2(\mathbb{R})$, as well as $0 < \tilde{\varepsilon}_{\gamma_1} < \varepsilon_{\gamma_1}$ depending on $\gamma_1$ (this will be chosen later). Since $\varphi_1$ is compactly supported, so is $\varphi_1 \circ \gamma_1$. This allows us to choose a partition of unity of $\operatorname{supp}(\varphi_1 \circ \gamma_1)$ with functions $(f_i)_i \subset C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$ taking values in $[0,1]$ satisfying

$$\sum_i f_i(p) = 1 \quad \text{for all } p \in \operatorname{supp}(\phi_1 \circ \gamma_1)$$

as well as the following properties. The value $\varepsilon_{\gamma_1}$ is used when applying the Hölder condition of the functions, and $\tilde{\varepsilon}_{\gamma_1}$ is used to upper bound the number of functions needed for the partition as well as their the Hölder norm. Letting $F_i \coloneqq \operatorname{supp}(f_i)$, we require that

$$\tilde{\varepsilon}_{\gamma_1} \leqslant \operatorname{diam}(F_i) \leqslant \varepsilon_{\gamma_1}. \tag{3.2}$$

Since $\varphi_1 \circ \gamma_1$ is compactly supported, this can be done with a finite number of functions $(f_i)_{i=1}^m$ with $m = m(\gamma_1)$. This constant will behave like

$$m \ll \tilde{\varepsilon}_{\gamma_1}^{-(2g+\sigma-1)}, \tag{3.3}$$

where $2g + \sigma - 1$ is the dimension of $\mathscr{C}$ (see section 1.7.2). Similarly as in [Pau81] with smooth functions and in appendix 1 in [Shu92], one can ask that

$$\|f_i\|_{\mathscr{D}_{n,\alpha},\infty} \ll \tilde{\varepsilon}_{\gamma_1}^{-(1-\alpha)}, \tag{3.4}$$

for all $i = 1, \ldots, m$. Now, as we want to estimate the integral over $\varphi_1 \circ \gamma_1$ with a single point in the supports $F_i$, we pick points $x_{\gamma_1}^i \in F_i$ for $i = 1, \ldots, m$. These points can be chosen arbitrarily.

Before we begin with our main estimate, we make the following abbreviations. First let

$$\Phi_i = \varphi_1 \circ \gamma_1\left(x_{\gamma_1}^i\right), \quad \psi = \prod_{k=2}^{r} \varphi_k \circ \gamma_k, \quad I_0 = \int_{\mathscr{C}} \varphi_0 \, d\mu,$$

as well as

$$I = \left| \int_{\mathscr{C}} \varphi_0 \cdot \prod_{k=1}^{r} \varphi_k \circ \gamma_k \, d\mu - \frac{1}{\mu(\mathscr{C})^r} \left( \int_{\mathscr{C}} \varphi_0 \, d\mu \right) \prod_{k=1}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right|.$$

Note now that

$$\int_{\mathscr{C}} \varphi_0 \cdot \varphi_1 \circ \gamma_1 \cdot \psi \, d\mu = \sum_{i=1}^{m} \int_{\mathscr{C}} f_i \cdot \varphi_0 \cdot \varphi_1 \circ \gamma_1 \cdot \psi \, d\mu.$$

We now proceed with the main estimate:

$$I \leqslant \left| \sum_{i=1}^{m} \int_{\mathscr{C}} f_i \cdot \varphi_0 \cdot \varphi_1 \circ \gamma_1 \cdot \psi \, d\mu - \sum_{i=1}^{m} \Phi_i \int_{\mathscr{C}} f_i \cdot \varphi_0 \cdot \psi \, d\mu \right| \tag{3.5}$$

$$+ \left| \sum_{i=1}^{m} \Phi_i \int_{\mathscr{C}} f_i \cdot \varphi_0 \cdot \psi \, d\mu - \sum_{i=1}^{m} \Phi_i \frac{\int_{\mathscr{C}} f_i \cdot \varphi_0 \, d\mu}{\mu(\mathscr{C})^{r-1}} \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right| \tag{3.6}$$

$$+ \left| \sum_{i=1}^{m} \Phi_i \frac{\int_{\mathscr{C}} f_i \cdot \varphi_0 \, d\mu}{\mu(\mathscr{C})^{r-1}} \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu - \frac{1}{\mu(\mathscr{C})^{r-1}} \int_{\mathscr{C}} \varphi_0 \cdot \varphi_1 \circ \gamma_1 \, d\mu \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right| \tag{3.7}$$

$$+ \left| \frac{1}{\mu(\mathscr{C})^{r-1}} \int_{\mathscr{C}} \varphi_0 \cdot \varphi_1 \circ \gamma_1 \, d\mu \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu - \frac{1}{\mu(\mathscr{C})^r} \prod_{k=0}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right|. \tag{3.8}$$

The first term is small because the supports $F_i$ are small. Indeed

$$\begin{aligned}
(3.5) &\leqslant \left| \sum_{i=1}^{m} \int_{F_i} |\varphi_1 \circ \gamma_1 - \Phi_i| \, f_i \cdot \varphi_0 \cdot \psi \, d\mu \right| \\
&\leqslant \|\varphi_1\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon_{\gamma_1}^{\alpha} \sum_{i=1}^{m} \int_{F_i} \varphi_0 \cdot \psi \, d\mu \\
&\ll \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \varepsilon_{\gamma_1}^{\alpha}
\end{aligned}$$

Here we only used the Hölder properties of $\varphi_1$ and that the other functions are bounded. The specific properties of the partitions of unity have only come in with the fact that $\operatorname{diam}(F_i) \leqslant \varepsilon_{\gamma_1}$. For the second estimate, we use the inductive hypothesis. We have

$$(3.6) \leqslant \|\varphi_1\|_{\mathscr{D}_{n,\alpha},\infty} \left| \sum_{i=1}^{m} \int_{\mathscr{C}} (f_i \cdot \varphi_0) \cdot \psi \, d\mu - \frac{\sum_{i=1}^{m} \int_{\mathscr{C}} f_i \cdot \varphi_0 \, d\mu}{\mu(\mathscr{C})^{r-1}} \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right|$$

$$\ll \left( \prod_{k=1}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \mathfrak{M}(\gamma_2, \ldots, \gamma_r)^{-\delta_{r-1}} \sum_{i=1}^{m} \|f_i \cdot \varphi_0\|_{\mathscr{D}_{n,\alpha},\infty}$$

$$\ll \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \mathfrak{M}(\gamma_2, \ldots, \gamma_r)^{-\delta_{r-1}} \, \tilde{\varepsilon}_{\gamma_1}^{-(2g+\sigma-1)} \tilde{\varepsilon}_{\gamma_1}^{-(1-\alpha)}$$

where the second line follows from the inductive hypothesis, and the last line follows from assumptions (3.3) and (3.4). For the next term, we have

$$(3.7) \leqslant \frac{1}{\mu(\mathscr{C})^{r-1}} \left| \prod_{k=2}^{r} \int_{\mathscr{C}} \varphi_k \, d\mu \right| \left| \sum_{i=1}^{m} \int_{\mathscr{C}} f_i \cdot \varphi_0 \cdot \Phi_i \, d\mu - \sum_{i=1}^{m} \int_{\mathscr{C}} f_i \cdot \varphi_0 \cdot \varphi_1 \circ \gamma_1 \, d\mu \right|$$

$$\ll \left( \prod_{k=2}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \left| \sum_{i=1}^{m} \int_{F_i} f_i \cdot \varphi_0 \, |\Phi_i - \varphi_1 \circ \gamma_1| \, d\mu \right|$$

$$\ll \left( \prod_{k=1}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \varepsilon_{\gamma_1}^{\alpha} \|\varphi_0\|_{\mathscr{D}_{n,\alpha},\infty},$$

where the third line follows from (3.2) (the size of the supports $F_i$), and the last line follows from the fact that the $f_i$ form a partition of unity. Finally, the third term is estimated with the base case of the induction. We have

$$(3.8) \leqslant \left( \prod_{k=2}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \left| \int_{\mathscr{C}} \varphi_0 \cdot \varphi_1 \gamma_1 \, d\mu - \frac{1}{\mu(\mathscr{C})} \left( \int_{\mathscr{C}} \varphi_0 \, d\mu \right) \left( \int_{\mathscr{C}} \varphi_1 \, d\mu \right) \right|$$

$$\ll \left( \prod_{k=0}^{r} \|\varphi_k\|_{\mathscr{D}_{n,\alpha},\infty} \right) \mathfrak{M}(\mathrm{id}, \gamma_1)^{-\delta_1},$$

where the second line follows from proposition 3.2.

Now we need them make a choice of $\varepsilon_{\gamma_1}$ and $\tilde{\varepsilon}_{\gamma_1}$. We need it to decay to zero exponentially given the estimate for (3.5) and (3.7). However, this cannot be too quickly because we also need term (3.6) to go to zero exponentially. Let $0 < \delta_r < \delta_{r-1}$ be such that the following estimates hold:

$$\tilde{\varepsilon}_{\gamma_1} \gg \exp\left(\frac{(\delta_{r-1} - \delta_r)\min_{i\neq j} \rho(\gamma_i, \gamma_j)}{-2g - \sigma + \alpha}\right),$$

and

$$\varepsilon_{\gamma_1} \ll \mathfrak{M}(\gamma_2, \ldots, \gamma_r)^{-\delta_r/\alpha}.$$

Repeating this process and replacing $\gamma_1$ with $\gamma_2, \ldots, \gamma_r$, we get the result. ■

We finish the chapter with some comments on the proofs. First of all, one can see from the proof the constants $\delta_r$ decrease as $r$ increases. That is, the rate deteriorates as we take higher correlations. Moreover, the reader will notice that chapter 2 and chapter 3 are quite unrelated at first glance, yet they are related through how the proof of quantitative multiple mixing was originally approached. As spelled out in chapter 2, one of the main steps in showing equidistribution under the geodesic flow, is to use the exponential mixing property in [AGY06]. The idea was to use theorem 2.1 to then prove quantitative multiple mixing, i.e. one could try to go the other way around with the proof. Although such an approach is reasonable for the geodesic flow, it does no longer work for the $\mathrm{SL}_2(\mathbb{R})$ action. Indeed, one would need that unstable leafs equidistribute quantitativley under the $\mathrm{SL}_2(\mathbb{R})$ action. However, one could simply conjugate the matrix

$$g_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$$

to

$$g_{-t} = \begin{pmatrix} e^{-t} & \\ & e^t \end{pmatrix}$$

with elements in $K$. Thus, the unstable leaf for the geodesic flow is exactly the stable leaf for the flow $t \mapsto g_{-t}$. One can then see that proof in chapter 2 would not work for the flow $g_{-t}$, and one should not expect the unstable leafs of the geodesic flow to equidistribute under the $\mathrm{SL}_2(\mathbb{R})$ action. We therefore take the approach in

this chapter and start simply with the exponential mixing result from [AGY06].

Our final comment is on the proof of theorem 3.1. The main ingredient in the proof is the exponential mixing property of the $SL_2(\mathbb{R})$ action, and the use of an appropriate partition of unity. However, the partition of unity does not depend on the action of $SL_2(\mathbb{R})$. Hence, similar methods of proof may be used for systems for which a quantitative mixing estimate has already been established. Nevertheless, such a result would hold exclusively for compactly supported $\alpha$-Hölder functions, as using this class of functions was essential in every step of our proofs. If the result were to be extended to a larger class of functions, these would have to be approximated with functions with the above regularity. It is here where our results lose generality. Indeed, considering functions $\varphi$ and $\psi \in C^\infty(\mathscr{C})$ and approximating them with functions $(\varphi_i), (\psi_j) \subset C^{0,\alpha}(\mathscr{C}) \cap C_c(\mathscr{C})$, we have from theorem 3.1 that for $\gamma \in SL_2(\mathbb{R})$,

$$\int_{\mathscr{C}} \varphi_i \cdot \psi_j \circ \gamma \, d\mu = \frac{1}{\mu(\mathscr{C})} \int_{\mathscr{C}} \varphi_i \, d\mu \int_{\mathscr{C}} \psi_j \, d\mu$$
$$+\mathcal{O}\left( \|\varphi_i\|_{\mathscr{D}_{n,\alpha},\infty} \|\psi_j\|_{\mathscr{D}_{n,\alpha},\infty} \exp(\rho(\mathrm{id}, \gamma))^{-\delta} \right).$$

Depending on the functions $\varphi$ and $\psi$, the terms $\|\varphi_i\|_{\mathscr{D}_{n,\alpha},\infty}$ and $\|\psi_j\|_{\mathscr{D}_{n,\alpha},\infty}$ (i.e. the Hölder constants) may blow up as $i, j \to \infty$ with a rate that cannot be controlled by $\exp(\rho(\mathrm{id}, \gamma))^{-\delta}$. This means that different approaches would be necessary to establish such quantitative estimates for other classes of functions, such as infinitely differentiable functions.

# Appendix

## A Cartan Decomposition of $SL_d(\mathbb{R})$

We give a proof of the Cartan decomposition of $SL_d(\mathbb{R})$ by following the proof of the $SL_2(\mathbb{R})$ case given in [DE09]. Although one can prove a Cartan decomposition for semisimple Lie groups as shown in [Kna02], we only give the $SL_d(\mathbb{R})$ case, which is the only one we need.

**Lemma A.1** (Cartan Decomposition of $SL_d(\mathbb{R})$)**.** *Let* $G = SL_d(\mathbb{R})$, $K = SO(d)$, *and*

$$A^+ \coloneqq \left\{ \operatorname{diag}(a_1, \dots, a_d) : \prod_{k=1}^{d} a_k = 1, a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d > 0 \right\}.$$

*Then,*

$$G = KA^+K,$$

*i.e., for every* $g \in G$, *there exist* $k, l \in K$, *and* $a \in A^+$ *such that*

$$g = kal.$$

**Proof.** Let $g \in G$. Then ${}^{\intercal}g\,g$ is a symmetric and positive definite matrix with respect to the Euclidean inner product. Thus, by the spectral theorem for real symmetric matrices [see Axl15, Chapter 7], ${}^{\intercal}g\,g$ is diagonalizable and has real eigenvalues $a_1, \dots, a_d$. Since ${}^{\intercal}g\,g$ is positive definite, all the eigenvalues are positive. Hence, up to reordering, we may assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d > 0$. Once in this order, $a$ is uniquely defined as $\operatorname{diag}(a_1, \dots, a_d)$. Moreover, since $\det {}^{\intercal}g\,g = 1$, and the matrix is symmetric, it is diagonalizable by some $h \in SO(d)$, that is

$${}^{\intercal}h\,{}^{\intercal}g\,g\,h = \operatorname{diag}(a_1, \dots, a_d). \tag{A.1}$$

We now make the following observation. If $h_1\,{}^\intercal h_1 = h_2\,{}^\intercal h_2$, for $h_1, h_2 \in G$, then $h_1^{-1}h_2 \in K$. Indeed, the condition on $h_1$ and $h_2$ implies that

$$\mathrm{id} = h_1^{-1}h_2\,{}^\intercal h_2\,{}^\intercal h_1^{-1} = \left(h_1^{-1}h_2\right){}^\intercal\left(h_1^{-1}h_2\right).$$

Hence $h_1^{-1}h_2 \in K$. Now, rewriting equation (A.1) as

$${}^\intercal h g\,{}^\intercal(\,{}^\intercal h g) = \mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_d})\,{}^\intercal(\mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_d})),$$

and using the claim, we get that $({}^\intercal h g)^{-1}\mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_d}) \in K$, and hence so is its transpose. We now write

$$g = {}^\intercal h \mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_d})\,{}^\intercal((\,{}^\intercal h g)^{-1}\mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_d})).$$

Thus, by setting

$$k = {}^\intercal h,\ a = \mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_2}),$$

and

$$l = {}^\intercal((\,{}^\intercal h g)^{-1}\mathrm{diag}(\sqrt{a_1},\dots,\sqrt{a_d})),$$

we have that $g = kal$, with $k, l \in K$, and $a \in A^+$. ∎

# B Estimate for Riemannian metric on $SL_2(\mathbb{R})$ with a Norm

In this section we fix a Riemannian metric on $SL_2(\mathbb{R})$ as well as a norm $\|\cdot\|_{\mathrm{op}}$ on it such that it can be estimated with the Riemannian metric. This is done following lemma 2.1 in [BEG20]. It is useful to have this estimate as we can explicitly compute the norm, while it is hard to do so for the metric.

To define a Riemannian metric on $SL_2(\mathbb{R})$, we first define an inner product on $\mathfrak{sl}(2,\mathbb{R})$ and then use the differential of left multiplication to define the inner product at each tangent space. For $X, Y \in \mathfrak{sl}(2,\mathbb{R})$, we define

$$\langle X, Y\rangle \coloneqq \frac{1}{2}\mathrm{tr}\left(X\,{}^\intercal Y\right),$$

where $\mathrm{tr}$ denotes the trace of a matrix. This inner product is symmetric and bilinear since the trace is. We also have that

$$\langle X, X\rangle = \sum_{i=1}^{2} (X\,{}^{\intercal}X)_{ii} = \sum_{i=1}^{2}\sum_{j=1}^{2} X_{ij}\,({}^{\intercal}X)_{ji} = \sum_{i,j=1}^{2} X_{ij}^2 \geqslant 0.$$

Thus the inner product is positive definite and it therefore defines a left invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$ [see do 92, chapter 2.1]. Denote the induced distance by $\rho$. To use lemma 2.1 in [BEG20], we also need that the inner product be $\mathrm{Ad}\,(K)$ invariant. It turns out that the above defined inner product has this property. To see this, let $k \in K$, and $X, Y \in \mathfrak{sl}(2,\mathbb{R})$. Then

$$\begin{aligned}\langle \mathrm{Ad}\,(k)\,X, \mathrm{Ad}\,(k)\,Y\rangle &= \left\langle kXk^{-1}, kYk^{-1}\right\rangle = \langle kX\,{}^{\intercal}k, kY\,{}^{\intercal}y\rangle \\ &= \frac{1}{2}\mathrm{tr}\,(kX\,{}^{\intercal}k\,{}^{\intercal}(kY\,{}^{\intercal}k)) = \frac{1}{2}\mathrm{tr}\,(kX\,{}^{\intercal}kk\,{}^{\intercal}Y\,{}^{\intercal}k) \\ &= \frac{1}{2}\mathrm{tr}\,(kX\,{}^{\intercal}Y\,{}^{\intercal}k) = \frac{1}{2}\mathrm{tr}\left(kX\,{}^{\intercal}Yk^{-1}\right) = \frac{1}{2}\mathrm{tr}\,(X\,{}^{\intercal}Y) = \langle X, Y\rangle\,,\end{aligned}$$

where the second to last equality follows from the fact that the trace is invariant under changes of bases. Now following [BEG20], we define for $\gamma \in \mathrm{SL}_2(\mathbb{R})$

$$\|\gamma\|_{\mathrm{op}} \coloneqq \max\{\|\mathrm{Ad}\,(\gamma)\,Z\| :\ Z \in \mathfrak{sl}(2,\mathbb{R}),\ \|Z\| = 1\},$$

where the norm $\|\cdot\|$ comes from the inner product on $\mathfrak{sl}(2,\mathbb{R})$ defined above. We now want to have a more explicit form of this norm. We need the root space decomposition of $\mathfrak{sl}(2,\mathbb{R})$ for this, and will follow chapter II.1 of [Kna02] for what follows.

Begin by consider the Cartan decomposition of $\mathrm{SL}_2(\mathbb{R})$ into $KA^+K$. The Lie algebra of $A^+$ is then

$$\mathfrak{a}_+ = \left\{ a\begin{pmatrix} 1 & \\ & -1\end{pmatrix} :\ a \in \mathbb{R}\right\}.$$

Denote by $E_{ij}$ the $2\times 2$ matrix with $1$ in the $(ij)$-th entry, and $0$ everywhere else. With this, define for $i = 1, 2$ the function $e_i$ in the dual $\mathfrak{a}_+^*$ as

$$e_i\begin{pmatrix} h_1 & \\ & h_2\end{pmatrix} \coloneqq h_i, \quad \begin{pmatrix} h_1 & \\ & h_2\end{pmatrix} \in \mathfrak{a}_+.$$

Then we have that for $H = \mathrm{diag}\,(h, -h) \in \mathfrak{a}_+$, (where $h \in \mathbb{R}$),

$$\mathrm{ad}\,(H)\,E_{ij} = [H, E_{ij}] = HE_{ij} - E_{ij}H = \begin{pmatrix} h & \\ & -h\end{pmatrix}E_{ij} - E_{ij}\begin{pmatrix} h & \\ & -h\end{pmatrix}$$

$$= (e_i(H) - e_j(H))\, E_{ij}.$$

The functions $e_i - e_j$ are called the **roots**. We see from the above computation that the matrices $E_{ij}$ are the eigenvectors of $\mathrm{ad}(H)$ with eigenvalues $e_i(H) - e_j(H)$. Note now that if $i \neq j$, the trace of $E_{ij}$ is zero and the matrix is thus in $\mathfrak{sl}(2,\mathbb{R})$. Observing that a basis for $\mathfrak{sl}(2,\mathbb{R})$ consists of $\mathrm{diag}(1,-1)$, $E_{12}$, and $E_{21}$, we write

$$\begin{aligned}\mathfrak{sl}(2,\mathbb{R}) &= \mathfrak{a}_+ \oplus \mathbb{R}E_{12} \oplus \mathbb{R}E_{21} \\ &= \mathbb{R}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} & 1 \\ & \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} & \\ 1 & \end{pmatrix},\end{aligned}$$

which is called the **root space decomposition** of $\mathfrak{sl}(2,\mathbb{R})$. Moreover, we call a root $e_i - e_j$ **positive** if $i < j$. In our case, as there only two roots, there is only one positive root, namely $e_1 - e_2$ with corresponding root space

$$\mathbb{R}\begin{pmatrix} & 1 \\ & \end{pmatrix}.$$

We make the remark here that when lifting this to the group through the exponential map, it induces the horocycle flow.

Now, lemma 2.1 in [BEG20] states that for every $\gamma \in G$,

$$\|\gamma\|_{\mathrm{op}} = \max_{\alpha \in \Sigma^+} \alpha(a_\gamma),$$

where $\Sigma^+ \subset \mathrm{Hom}\left(A^+, \mathbb{R}_+^\times\right)$ is the positive root system, and $a_\gamma$ denotes the (unique) element in $A^+$ corresponding to the Cartan decomposition of $\gamma$. In our case, $\Sigma^+$ consists of exactly one element, the one corresponding to the root $e_1 - e_2$. To compute this explicitly, let $a \in A^+$. We may write this as

$$a = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix},$$

for some $t \geqslant 0$. Writing this as the exponential of the corresponding element in the Lie algebra, we have that

$$a = \exp\begin{pmatrix} t & \\ & -t \end{pmatrix}.$$

Calling $h \coloneqq \mathrm{diag}\,(t, -t)$, it follows that

$$\alpha_{12}\,(a) = \exp\,(e_1\,(h) - e_2\,(h)) = e^{2t}.$$

Therefore, for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$, and writing $\gamma = k\,(\theta')\,g_t k\,(\theta)$ for $\theta', \theta \in [0, 2\pi)$ and $t \geqslant 0$, we have that

$$\|\gamma\|_{\mathrm{op}} = e^{2t}. \tag{B.1}$$

Finally, we have that from the third part of lemma 2.1 in [BEG20], there exist constants $c_1 \geqslant 1$ and $c_2 > 0$ such that

$$c_1^{-1} \log\|\gamma\|_{\mathrm{op}} - c_2 \leqslant \rho\,(\gamma, e_G) \leqslant c_1 \log\|\gamma\|_{\mathrm{op}} + c_2.$$

It follows that

$$\|\gamma\|_{\mathrm{op}} \leqslant \left(e^{c_2} e^{\rho(\gamma, e_G)}\right)^{c_1}. \tag{B.2}$$

We note here that this norm does not depend on the $K$ components of $\gamma$.

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