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Master Thesis

Capparelli's identity, a refinement and a generalization

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Abstract

The aim of this thesis is to present Andrews' proof of Capparelli's identity on integer partitions and a refinement and a generalization of this result. Both have first been presented in a paper of Krishnaswami Alladi, George E. Andrews and Basil Gordon in 1995. For this purpose, we will give an introduction into the general theory on combinatorics of integer partitions and the method of weighted words due to Alladi and Gordon.

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1. Introduction

"In how many ways can we split 5 into a sum of integers?" The answer to this question is as elementary as the question itself and can be found in a quite natural and straight-forward way: by counting the possibilities. We can write:

$$\begin{aligned}
&5 = 5\\
&5 = 4 + 1\\
&5 = 3 + 2 = 3 + 1 + 1\\
&5 = 2 + 2 + 1 = 2 + 1 + 1 + 1\\
&5 = 1 + 1 + 1 + 1 + 1.
\end{aligned}$$

So, the short and sober answer to this question is: 7. Understanding addition of natural numbers is sufficient to think about this sort of questions (and answer them for small numbers at least). As this operation was presumably known already in the Stone Age, splitting integers has probably been thought about for quite some time, for example for the purpose of distributing discrete objects (think of money or apples) among several people. Note that there are basically two different options for counting ways of splitting natural numbers. On the one hand, we can take care of the order, which would mean in our example that we count $5 = 4 + 1$ and $5 = 1 + 4$ as two different ways. Conventionally, when this approach is chosen, the objects are called *compositions* of integers. On the other hand, we can consider partitions as finite non-increasing sequences, which means that we count $5 = 4 + 1 = 1 + 4$ as one partition and we always write it in the form $5 = 4 + 1$. This is the approach that we will choose for the subjects considered in this thesis. The numbers appearing in such an *integer partition* (see Definition 2.28) are called *parts* of this partition.

1.1. Combinatorics of integer partitions. Once started thinking about splitting integers, it seems natural to ask more refined questions as for example "In how many ways can we split 5 into a sum of odd integers?" or "In how many ways can we split 5 into a sum of distinct integers?". As simple as the concept of partitioning integers is, it is astonishing that these elementary sort of questions leads to a vast number of sophisticated and sometimes surprising results called *partition identities* (see Definition 2.32). These identities describe equalities between the numbers of different (constrained) ways of splitting an integer. The most famous example is due to Leonhard Euler who proved in 1748 the following result (see Corollary 2.1).

Theorem (Euler). [AE04, p. 3] *Every natural number can be split into a sum of odd integers in the same number of ways as it can be split into a sum of distinct integers.*

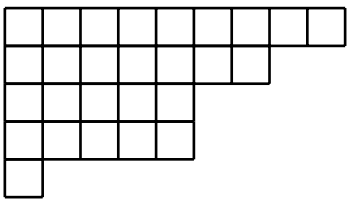
Example. *For our example for the number 5, this can be verified by counting:*

$$5 = 5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$
$$5 = 5 = 4 + 1 = 3 + 2.$$

Mentioned in a scientific context, it was also Euler who "kicked off the study of partitions in the eighteenth century" ([AE04, p. 124]).
To prove such a partition identity we will mainly use the two tools that are most often used for proofs in combinatorics. The first option is using *bijections*. We build pairs between the partitions of a natural number into odd parts and partitions of the same number into distinct parts and use each partition exactly once. Thus it becomes clear that the numbers of the two types of partitions are equal. This can be illustrated using *graphical representations* of integer partitions (presented in Subsection 2.6). We can picture integer partitions by drawing little boxes, each of size one. For example,

$$25 = 9 + 7 + 5 + 5 + 1$$

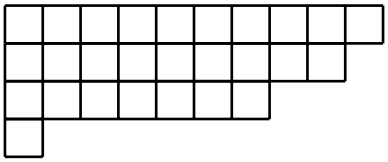
can be drawn in the following way:



For the proof of Euler's partition identity we will pair this partition with the partition obtained by glueing rows of the same length together and reordering the rows in decreasing order. If necessary, we iterate this procedure until we end up with all rows having different length. By splitting rows of even length in the middle and placing them below each other (and reordering if necessary), we can go back to the partition we started with. The partition pairing partner of the one above is

$$25 = 10 + 9 + 7 + 1,$$

and it can be illustrated as:



A more formal description of this proof is given in Subsection 2.5. We will refer to this sort of proofs as *Bijective Proofs*. Note that this method is quite informative as it gives not only the numerical identity between the cardinalities of two sets of such partitions, but also a structural relation between individual partitions of the two types.
The second technique delivers only the numerical identity. However, its advantage is that it can be used much more systematically. In combinatorics (and in mathematics generally), finding bijections is sometimes quite challenging whereas constructing *generating functions* (see Subsection 2.3) is usually much easier. If we denote for every natural number n by a_n the number of integer partitions of n into odd parts and by b_n the number of integer partitions of n into distinct parts, then the formal power series

$$\sum_{j=0}^{+\infty} a_n q^n, \quad \left(\text{resp. } \sum_{j=0}^{+\infty} b_n q^n \right),$$

is the generating function of the first (resp. second) type of partitions. If we manage to show that these two formal power series are equal, using elementary algebraic tools, we can deduce that for all natural numbers n,

$$a_n = b_n.$$

The main idea to translate our questions on integer partitions into this algebraic setting is to consider the size of the parts as exponents of the formal variable and choose each part independently. For each natural number we can choose the number of its appearances as a part in the partition (possibly zero). For example for the number 2 we can choose between

$$q^0, q^2, q^{2+2}, q^{2+2+2}, q^{2+2+2+2}, q^{2+2+2+2+2}, ...$$

This choice can be modelled by considering the product of the geometric series

$$1 + q^j + q^{2j} + q^{3j} + q^{4j} + q^{5j} + ...,$$

for all natural numbers j. Multiplying will then yield the size of the considered partitions in the exponent of the formal variable as the sum of the parts.

1.2. The story behind Capparelli's identity. In 1894 and 1913 ([AE04, p. 3]), Leonard James Rogers and Srinivasa Ramanujan proved independently of each other two famous partition identities that are the origin of a whole collection of partition identities, called *Rogers-Ramanujan type identities*.

Theorem (The first Rogers-Ramanujan identity). [And76, p. 109] *The number of partitions of an integer n in which the difference between any two parts is at least 2 equals the number of partitions of n into parts congruent to 1 or 4 (mod 5).*

Example. *For $n = 9$, the partitions satisfying the difference condition are*

$$9, \quad 8+1, \quad 7+2, \quad 6+3, \quad 5+3+1,$$

and the partitions satisfying the congruence condition are

$$9, \quad 6+1+1+1, \quad 4+4+1, \quad 4+1+1+1+1+1, \quad 1+1+1+1+1+1+1+1+1.$$

Indeed, they are equinumerous.

With those, modular arithmetic became an important tool in the field ([AE04, p. 3]). The characterizing property of these identities is that on one side of the identity the partitions are restricted by using congruence conditions and on the other side the restrictions come from conditions on the difference between consecutive parts ([And76, p. 103 - 120]). Regarding this characteristic, one can also see Euler's mentioned theorem as a Rogers-Ramanujan type partition identity with a congruence condition on one side (odd parts) and a difference condition on the other side (distinct parts).
In the last centuries many famous mathematicians contributed to the field of Combinatorics of Integer Partitions: Leonhard Euler, Adrien-Marie Legendre, Srinivasa Ramanujan, Hans Rademacher and James Joseph Sylvester are only a few among many names ([AE04, p. ix]). Thanks to these contributions and those of others, the field became important for number theoretic, representation theoretic, analytic and even physical questions. At the heart of this thesis lies *Capparelli's identity* which was conjectured by Stefano Capparelli in his PhD dissertation (see [Cap93]) and which is a mathematical statue for the collaboration between combinatorialists and mathematicians investigating in Lie Algebra representation theory ([And94, p. 143]). Capparelli stated it in connection with extending the mathematical merit of James Lepowsky and Robert L. Wilson who started studying "the role of Rogers-Ramanujan-type identities in Lie algebra representation theory" ([And94, p. 143]) in the 1980s (see [LW81], [LW82], [LW84]). In those years, various partition identities have been found by focusing on the interplay between the theories on vertex operators and representations of affine Lie algebras ([AAG95b, p. 636-637]). This work has been continued until today (see [Dou20], [DHK21] and [DK21]). Finding new Rogers-Ramanujan type identities goes together with finding new character formulas that yield more practical ways to compute characters ([DK21]), starting for example from considerations on the Weyl-Kac character formula ([DHK21]).

Theorem (Capparelli's identity). [And94, p. 143] *The number of partitions of an integer n into parts that are congruent to $2, 3, 9$ or 10 modulo 12 equals the number of partitions of n where all parts are greater than 1 and the difference between any two consecutive parts is at least 2 (if the sum of the two parts is a multiple of 3) or 4 (otherwise).*

It is a Rogers-Ramanujan type identity where on one side, we face congruence conditions with respect to division by 6, and on the other side, difference conditions are set with case distinction with respect to division by 3.

Example. *For $n = 10$, the partitions satisfying the congruence condition are*

$$10, \quad 3+3+2+2, \quad 2+2+2+2+2,$$

and the partitions satisfying the congruence condition are

$$10, \quad 8+2, \quad 7+3.$$

Indeed, we can verify the identity for $n = 10$.

The discovery of this partition identity strongly motivates further work on the interactions of combinatorics of integer partitions and representation theory ([AAG95b, p. 637]).
George E. Andrews, probably the most famous mathematician of the last decades working in the field of Combinatorics of Integer Partitions, was the first to prove this identity. He did so by using generating functions.
Before we will present Andrews' first proof of Capparelli's identity, we will focus on another famous Rogers-Ramanujan type identity due to a Russian mathematician. Issai Schur proved in 1926 the following theorem (see Theorem 3.1).

Theorem (Schur). [And94, p. 141] *The number of partitions of an integer n into parts that are congruent to 1 or 5 modulo 6 equals the number of partitions of n such that any two consecutive parts differ by at least 6 (if they are both multiples of 3) or 3 (otherwise).*

Example. *Also this theorem can be verified for small numbers by counting. For $n = 10$, the partitions satisfying the congruence condition are*

$$7+1+1+1, \quad 5+5, \quad 5+1+1+1+1+1, \quad 1+1+1+1+1+1+1+1+1+1,$$

and the partitions satisfying the congruence condition are

$$10, \quad 9+1, \quad 8+2, \quad 7+3.$$

Schur's partition theorem sets the stage for the results and their proofs presented in this thesis. There exists already a vast number of proofs of this theorem (see [AG93], [And67], [And68], [And71], [Bes91], [Bre80] and [And94]). The one presented in Section 3 is due to Andrews ([And94]) and it turns out that a very similar proof strategy is applicable for Andrews' first proof of Capparelli's identity (see Section 4).
It seems legitimate to ask: Why is another proof of Schur's theorem worth studying? In the process of Andrews' work in the field of statistical mechanics concerning the Hard Hexagon Model together with R. J. Baxter, see [AB87], they defined the so-called *q-trinomial coefficients* which is a q-analog of the coefficients of

$$\left(1+x+x^2\right)^n,$$

for n a natural number. These q-trinomial coefficients are polynomials in q and they are called a *q-analog*, since letting $q \to 1$ in their expression yields the coefficients of the mentioned polynomial in x above. These $q-$trinomial coefficients will be explained in detail in Subsection 3.2. They were essential in the exact solution of the above mentioned

model in statistical mechanics ([And94, p. 142], [AB87]) and, as we will see, they will be crucial in the presented proofs of Schur's theorem and Capparelli's identity. Thus these two proofs provide a connection between the fields of statistical mechanics and combinatorics of integer partitions and, more explicitely, they serve for "combinatorial significance of the $q-$trinomial coefficients" ([And94, p. 141]).

1.3. Further work based on Capparelli's identity. In 1995 [AAG95b], together with Krishnaswami Alladi and Basil Gordon, Andrews gave a bijective proof for Capparelli's identity which provides more insight on how the two types of partitions are related. Namely, they prove the following refinement of Capparelli's identity.

Theorem (Alladi, Andrews, Gordon). [AAG95b, p. 637] *The number of partitions of n into distinct parts consisting of i parts that are congruent to 4 modulo 6, j parts that are congruent to 2 modulo 6, and parts that are multiples of 3 of which exactly k are strictly bigger than $3(i+j)$ equals the number of partitions of n satisfying the difference condition in Capparelli's identity and consisting of i parts that are congruent to 1 modulo 3, all greater than 1, j parts congruent to 2 modulo 3 and k parts that are multiples of 3.*

To recognise that this partition identity can be seen, indeed, as a refinement of Capparelli's identity we will prove the following lemma (see Lemma 5.1).

Lemma. [AAG95b, p. 637, eq. (1.1)] *The number of partitions of an integer n into distinct parts that are congruent to $0, 2, 3$ or 4 modulo 6 equals the number of partitions of n satisfying the congruence condition in Capparelli's identity.*

The proof of this theorem underlines the value of the work of David Marius Bressoud in [Bre79]. Ideas presented by Bressoud in the course of giving a bijective proof for an extension of a partition identity due to the German mathematician Heinz Göllnitz reappear. As Alladi, Andrews and Gordon mention themselves, they based their proof of this refinement of Capparelli's identity on Bressoud's work ([AAG95b, p. 637]).
In the process of working on "generalizations and companions" of Schur's mentioned partition identity ([AAG95b, p.638]), Krishnaswami Alladi and Basil Gordon developed the *method of weighted words*. In [AAG95b], they used it (together with George E. Andrews) to prove a generalization of Capparelli's identity.
The main idea of this method is to consider three (coloured) levels of natural numbers, we specify them by the letters a, b and c, and construct symbols with *weights* (i. e. natural numbers that we assign to the colours). Considering finite sequences of these symbols constructs more general objects than integer partitions which we call *words*.
For the result, that we will present based on these more general objects, we will present two different proofs. First we will prove it using generating functions, second we will present a bijective proof that uses the same ideas that have already been used for the bijective proof in Subsection 5.3 for the presented refinement. The two proofs have first been published in [AAG95b].
In the course of studying the proof using generating functions, we will again encounter a $q-$analog, this time referring to the well-known binomial coefficients. More explicitely, we will introduce so-called *Gaussian polynomials* (see Subsection 2.8). Seeing them reappear in playing a crucial role in many proofs throughout the thesis (and beyond, see [And76]) shows how representative the presented work of Andrews, Alladi and Gordon is for getting some insight into important techniques that are used in the world of integer partitions.
Recently, Jehanne Dousse showed that this generalization obtained by the method of weighted words can be seen as a particular case of the "weighted words version" of another partition identity due to Primc ([Dou20]). This is interesting since "from a representation

theoretic point of view, the two identities do not seem to be related a priori" ([Dou20, p. 5]). "Weighted words version" refers to a new variant of the method of weighted words due to Jehanne Dousse which she (together with Jeremy Lovejoy) applied for getting a generalization of the mentioned result due to Primc. Also this work is related to representation theory, as Primc extended the already mentioned work of Lepowsky and Wilson in the course of getting his mentioned partition identity.
The method of weighted words can be used also to prove further results ([AAG95b, p. 657]). For this purpose, we will apply again the same strategy as we will use to show that the before mentioned result is, indeed, a generalization of Capparelli's identity.

Theorem. [AAG95b, p. 657] *The number of partitions of an integer n into distinct parts congruent to $0, 3, 4$ or $5 \pmod 8$ equals the number of partitions of n into distinct parts all greater than 1 which are not congruent to $2 \pmod 4$ such that the difference between consecutive parts is greater than 4 unless they are both multiples of 4 or they add up to a multiple of 8.*

Example. *For $n = 8$, the partitions that satisfy the congruence condition are exactly the same that satisfy the difference condition. Namely,*

$$8, \quad 5 + 3.$$

This theorem gives a glimpse on how useful the method of weighted words is. There are more interesting partition identities of Rogers-Ramanujan type that have been deduced using this method ([AAG95b, p. 638], see [AAG95a]) and theoretically it's possible to deduce infinitely many new partition identities with this tool ([Dou20, p. 2]).
Finally, it's worth mentioning, that Capparelli's identity has also been proved by Capparelli himself ([Cap96]) and by Manvendra Tamba and Chuanfu Xie ([TX95]), using tools form representation theory ([Dou20, p. 2]).

1.4. Personal motivation. The main purpose of this thesis is to explain the main results of the two papers [And94] and [AAG95b] to give formal insight and work out details in order to make the mentioned results around Capparelli's identity understandable for people with a modest combinatorial (or even mathematical) background. By doing this, the author wants to give an introduction to some of the most important tools used in the theory of integer partitions and wants to share his passion about thinking on ways how to prove partition identities.
This thesis is organised in the following way: In the next section (Section 2), we will provide some elementary notation and basic theory on integer partitions. In Sections 3 and 4, we will present Andrews proofs of Schur's theorem and Capparelli's identity. In Section 5, the mentioned refinement is proven bijectively (following the proof of Alladi, Andrews and Gordon) and the last two Sections 6 and 7 are devoted to the method of weighted words.
As a formal remark, I'd like to mention that all statements, ideas and proofs that are not referenced throughout the thesis are written from my mind - if not created during my studies for this thesis, they are based on lecture notes and the mathematical education that I was granted at University of Zurich.

2. Basic theory

We will introduce now some notation and the basic theory of integer partitions with all important results that we will use for later proofs.

2.1. Basic notation. We will now introduce important basic mathematical notations such as sets, maps and sequences. The whole content is based on handwritten lecture notes; other presentations of the content can be found in [Fis14] or [Kö4].

One of the most basic and famous definitions in mathematics is the one of a set.

Definition 2.1. *A **set** is a collection of objects noted between curly brackets.*

Example 2.1. *The set of the three numbers* 1, 2 *and* 3 *is denoted as* $A := \{1, 2, 3\}$ *and* $\{blue, orange\}$ *is the set containing the two colours blue and orange.*

Remark 2.1. Throughout this thesis we will use the common describing notation for sets, namely mentioning the general object of a set before a vertical bar followed by describing the conditions that a general object has to fulfil to belong to this set. The empty set is denoted by $\emptyset$.

Example 2.2. *The set* A *can be described as*

$$A = \{j \mid j \text{ is an integer between 0 and 4.}\}.$$

When writing about elements of sets we use the following symbolic notation.

Definition 2.2. *Let* A *be a set and* a *be an object. If* a *belongs to* A *we write* $a \in A$ *and say that* a *is an **element** of* A*. If* a *does not belong to* A *we write* $a \notin A$*.*

Example 2.3. *Note that* $1 \in \{1, 2\}$ *but* $3 \notin \{1, 2\}$*.*

Often we consider certain smaller collections of elements belonging to a set.

Definition 2.3. *Let* A, B *be sets. We say that* A *is a **subset** of* B *(in symbols* $A \subseteq B$*) if every element of* A *is an element of* B*.*

Example 2.4. *The set of odd numbers is a subset of the set of all numbers.*

The most often used operations on sets are the following.

Definition 2.4. *Let* A, B *be sets. Then the **union** $A \cup B$ is defined to be the set containing all elements contained in* A *or* B*. Moreover, the **intersection** $A \cap B$ is defined to be the set containing all elements that are contained in* A *and* B*. We say that* A *and* B *are **disjoint** if* $A \cap B = \emptyset$*. The **difference** of* A *and* B *is the set* $A \setminus B$ *of elements of* A *that are not contained in* B*.*

Example 2.5. *Note that*

$$\{1, 3, 5\} \cup \{2, 3, 7, 8\} = \{1, 2, 3, 5, 7, 8\},$$

and

$$\{1, 3, 5\} \cap \{2, 3, 7, 8\} = \{3\}.$$

Moreover, $\{1,3,5\}$ *and* $\{2,4\}$ *are disjoint sets and it holds that*

$$\{2,3,4\} \setminus \{1,2\} = \{3,4\}.$$

Remark 2.2. Union and intersection can be inductively generalized for an arbitrary natural number (greater than 1) of sets. For $k \in \mathbb{N}_{\geq 3}$ and sets $A_1, ..., A_k$, define

$$A_1 \cup ... \cup A_k := (A_1 \cup ... \cup A_{k-1}) \cup A_k,$$

and

$$A_1 \cap ... \cap A_k := (A_1 \cap ... \cap A_{k-1}) \cap A_k.$$

In the next definition we already encounter the concept of partitioning a mathematical object. Here we consider sets, later we mainly will consider natural numbers. Note that this definition can be generalized for an infinite number of subsets.

Definition 2.5. *Let* M *be a set. A* ***set partition*** *of* M *is a collection of subsets* $A_1, ..., A_k$ *of* M *such that* $A_1, ..., A_k$ *are pairwise disjoint and* M *is the union of* $A_1, ..., A_k$*. In this case we write*

$$A_1 \sqcup ... \sqcup A_k = M.$$

Example 2.6. *Note that*

$$\{1,2,3,4,5,6\} = \{1\} \sqcup \{2,4\} \sqcup \{3,5,6\}.$$

As in this thesis we are dealing with subjects in the field of discrete mathematics, the most important set is the set of natural numbers.

Definition 2.6. *As conventionally done over all mathematical areas we denote the set of* ***natural numbers*** *by*

$$\mathbb{N} := \{0,1,2,3,...\}.$$

Explicitly note that 0 *is treated as a natural number. Moreover, the set of* ***integers*** *is denoted by*

$$\mathbb{Z} := \{...,-3,-2,-1,0,1,2,3,...\}.$$

The set of ***complex numbers*** *is denoted by* $\mathbb{C}$*.*

Symbols as $<, \leq, >, \geq$ are used with their conventional meaning when standing between elements of $\mathbb{Z}$. For keeping the notation short, consider the following definition.

Definition 2.7. *For all* $n \in \mathbb{N}$*, we define*

$$[n] := \{j \in \mathbb{N} \mid 1 \leq j \leq n\},$$
$$[\infty] := \mathbb{N}^* := \mathbb{N} \setminus \{0\}.$$

Moreover, for all $x \in \mathbb{N}$*,*

$$\mathbb{N}_{\geq x} := \{j \in \mathbb{N} \mid j \geq x\},$$

and for $m \in \mathbb{N}$ *and any subset* $A \subseteq \mathbb{N}$*,*

$$mA := \{m \cdot x \mid x \in A\}.$$

Example 2.7. *Note that*

$$[5] = \{1, 2, 3, 4, 5\},$$

and

$$[0] = \emptyset.$$

Moreover, $2\mathbb{N}$ *is the set of all even natural numbers.*

For defining maps we need the following binary operation on sets.

Definition 2.8. *Let* A, B *be sets. Then the* ***cartesian product*** *of* A *and* B *is defined as*

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

For elements (a, b) *of* $A \times B$*,* a *(resp.* b*) is referred to as the first (resp. second) component of* (a, b)*.*

Example 2.8. *Let* $A := [2]$ *and* $B := [3]$*. Then*

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}.$$

Note that 2 *is the first component of* $(2, 3)$ *and the second component of* $(1, 2)$*.*

Based on the notation of the cartesian product, it seems natural to define powers of sets.

Definition 2.9. *Let* A *be a set,* $k \in \mathbb{N}^*$*. The* k*th power of* A *is defined as*

$$A^k := \{(a_1, ..., a_k) \mid a_j \in A \quad \forall j \in [k]\}.$$

Example 2.9. *Note that*

$$[2]^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}.$$

Maps build the basis of all important concepts in this thesis.

Definition 2.10. *Let* A, B *be sets. A* ***map*** *is a subset* $\varphi \subseteq A \times B$ *such that for all* $a \in A$ *there is exactly one element* $(a, b) \in \varphi$ *for some* $b \in B$*. A map* φ *is denoted by*

$$\begin{aligned} \varphi : \ & A \to B \\ & x \mapsto \varphi(x), \end{aligned}$$

iff

$$\varphi = \{(x, \varphi(x)) \mid x \in A\}.$$

The set B *is called the* ***target set*** *of* φ*. We say that a map is* ***well-defined*** *if, indeed, for every* $a \in A$ *there is exactly one* $b \in B$ *such that* $\varphi(a) = b$*.*

Example 2.10. *Note that*

$$\begin{aligned} \varphi : \mathbb{N} &\to \mathbb{N} \\ x &\mapsto x + 1 \end{aligned}$$

is the well-defined map $\varphi = \{(x, x+1) \mid x \in \mathbb{N}\}$*. However,*

$$\begin{aligned} \psi : \mathbb{N} &\to \mathbb{N} \\ x &\mapsto x - 1 \end{aligned}$$

is not well-defined, since $\psi(0) = -1 \notin \mathbb{N}$*.*

It is possible that not all elements of the target set of a map appear in the second component of an element in the map. This motivates the following definition.

Definition 2.11. *Let $\varphi: A \to B$ be a map. The **image** of φ is defined as*

$$Im(\varphi) := \{b \in B \mid \exists a \in A: \varphi(a) = b\}.$$

Moreover, for $(a, b) \in \varphi$, b is referred to as the image of a under φ.

Example 2.11. *The image of the map*

$$\begin{aligned} \varphi: \mathbb{N} &\to \mathbb{N} \\ x &\mapsto 2x \end{aligned}$$

is the subset of $\mathbb{N}$ consisting of all even natural numbers and 4 is the image of 2 under φ.

Next we introduce basic characterizing properties of maps.

Definition 2.12. *Let A, B be sets. A map*

$$\begin{aligned} \varphi: A &\to B \\ x &\mapsto \varphi(x) \end{aligned}$$

*is called **injective** if for all $x, y \in A$ such that $\varphi(x) = \varphi(y)$ it follows that $x = y$. It is called **surjective** if*

$$Im(\varphi) = B,$$

*and **bijective** if it is injective and surjective. A bijective map is called a **bijection**.*

Example 2.12. *Note that*

$$\begin{aligned} \varphi_1: [3] &\to [6] \\ x &\mapsto x \end{aligned}$$

is injective but not surjective, since

$$4 \notin Im(\varphi) = [3].$$

However,

$$\begin{aligned} \varphi_2: [6] &\to \{-1, 1\} \\ n &\mapsto (-1)^n \end{aligned}$$

is surjective but not injective, since

$$\varphi_2(1) = -1 = \varphi_2(3).$$

Finally,

$$\begin{aligned} \varphi_3: [4] &\to 5[4] \\ y &\mapsto 5y \end{aligned}$$

is bijective.

For combinatorial proofs of partition identities we will need the following crucial lemma.

Lemma 2.1. *Let A, B be sets. A map*

$$\begin{aligned}\varphi :\ & A \to B \\ & x \mapsto \varphi(x)\end{aligned}$$

*is **bijective** if and only if there exists another map*

$$\begin{aligned}\psi :\ & B \to A \\ & x \mapsto \psi(x),\end{aligned}$$

such that for all $a \in A$,

$$\psi\left(\varphi(a)\right) = a,$$

and for all $b \in B$,

$$\varphi\left(\psi(b)\right) = b.$$

*In this case we denote the **inverse** of φ by*

$$\varphi^{-1} := \psi.$$

Example 2.13. *Note that*

$$\begin{aligned}\varphi :\ & \{1,2,3\} \to \{2,4,6\} \\ & x \mapsto 2x\end{aligned}$$

is a bijection. Its inverse is given by

$$\begin{aligned}\psi :\ & \{2,4,6\} \to \{1,2,3\} \\ & y \mapsto \frac{1}{2}y.\end{aligned}$$

Proof. Let A, B be sets and

$$\begin{aligned}\varphi :\ & A \to B \\ & x \mapsto \varphi(x)\end{aligned}$$

a map. Assume first that φ is bijective. Then by definition of injective and surjective, for each $b \in B$ there exists exactly one $a_b \in A$ such that $\varphi(a_b) = b$. Define

$$\begin{aligned}\psi :\ & B \to A \\ & b \mapsto a_b.\end{aligned}$$

Note that for each $a \in A$,

$$\psi\left(\varphi(a)\right) = a_{\varphi(a)} = a,$$

and for all $b \in B$,

$$\varphi\left(\psi(b)\right) = \varphi(a_b) = b.$$

Next assume that there exists an inverse of φ. We show that φ is injective. Let $x, y \in A$ and assume that

$$\varphi(x) = \varphi(y).$$

Thus

$$\varphi^{-1}\left(\varphi(x)\right) = \varphi^{-1}\left(\varphi(y)\right).$$

By definition of the inverse it follows that $x = y$. For proving surjectivity, let $b \in B$. Then

$$\varphi\left(\varphi^{-1}(b)\right) = b,$$

and $\varphi^{-1}(b) \in A$. Thus we can conclude that φ is bijective. ■

The main objects in this thesis are partitions of integers. Since these will be considered as finite sequences we introduce shortly formally what sequences are.

Definition 2.13. *Let M be a set. A **sequence** in M is a map*

$$\begin{aligned} s : \mathbb{N} &\to M \\ j &\mapsto s(j). \end{aligned}$$

For convenience, s is denoted by $\left(s(j)\right)_{j\in\mathbb{N}}$.

Example 2.14. *Note that $(j)_{j\in\mathbb{N}}$ is the sequence that maps every natural number to itself.*

For the purpose of introducing multivariate generating functions we will need a slightly more general version of sequences.

Definition 2.14. *Let M be a set, $k \in \mathbb{N}^*$. A **k-variate sequence** in M is a map*

$$\begin{aligned} s : \mathbb{N}^k &\to M \\ (j_1, ..., j_k) &\mapsto s(j_1, ..., j_k). \end{aligned}$$

For convenience, s is denoted by $\left(s(j_1, ..., j_k)\right)_{j_1,...,j_k\in\mathbb{N}}$.

Example 2.15. *Note that $\left((a, -b)\right)_{a,b\in\mathbb{N}}$ is a bivariate sequence.*

In most of the cases where sequences appear in this thesis they are not defined on $\mathbb{N}$ but on a finite subset of $\mathbb{N}^*$.

Definition 2.15. *Let M be a set, $r \in \mathbb{N}$. A **finite sequence** in M of length r is a map*

$$\begin{aligned} s : [r] &\to M \\ j &\mapsto s(j). \end{aligned}$$

*For convenience, s is denoted by $\left(s(j)\right)_{j\in[r]}$. A finite sequence of length 2 is called a **pair**, finite sequences of length 3 are named **triples**.*

Example 2.16. *Note that $(2j)_{j\in[5]}$ is a finite sequence, $(j)_{j\in[2]}$ is a pair and $(j+2)_{j\in[3]}$ is a triple.*

Remark 2.3. Elements of the image of a (finite) sequence s are referred to as **elements** of s. Sometimes we will denote finite sequences by listing their elements between round brackets, where the jth element denotes the image of j under the sequence. When we write about two **consecutive elements** of a finite sequence s then we mean two elements a and b of s such that b is the image of $j+1$ and a is the image of j under s for some $j \in \mathbb{N}^*$. The first (resp. last) element of a finite sequence

$$s : [r] \to M$$

$$j \mapsto s(j)$$

is the image of 1 (resp. r) under s.

Example 2.17. *Note that* 6 *is an element of* $(2j)_{j\in[5]}$ *and*

$$(2j)_{j\in[5]} = (2,4,6,8,10)\,.$$

6 *and* 8 *are consecutive elements of this finite sequence,* 2 *is its first element,* 10 *its last.*

Listing elements of a finite sequence generates the idea of an order of these elements which justifies the following definition.

Definition 2.16. *A finite sequence* $\left(s_j\right)_{j\in[r]}$ *is called to be* ***non-decreasing*** *if for all* $j \in [r-1]$,

$$s_j \leq s_{j+1},$$

and ***non-increasing*** *if for all* $j \in [r-1]$,

$$s_j \geq s_{j+1}.$$

Example 2.18. *The triple* $(1,3,2)$ *is neither non-increasing nor non-decreasing whereas* $(5,4,4,2)$ *is a non-increasing and* $(2,2,7,18)$ *is a non-decreasing finite sequence.*

For notational reasons sometimes it is convenient to reverse the order of finite sequences.

Definition 2.17. *Let* $s = \left(s_j\right)_{j\in[r]}$ *be a finite sequence. Then*

$$s^{rev} := \left(s_{r-j+1}\right)_{j\in[r]}\,.$$

Example 2.19. *Note that*

$$(2j)^{rev}_{j\in[5]} = (10,8,6,4,2)\,.$$

As we will often deal with congruence conditions, division with remainder plays an important role.

Definition 2.18. *For* $a,b \in \mathbb{Z}, m \in \mathbb{N}^*$, *we write* $a \equiv b \pmod m$ *iff* $b-a$ *is a multiple of* m. *In this case we say that* a *is congruent to* b *modulo* m.

Example 2.20. *Note that* $17 \equiv 2 \pmod 5$.

Division with remainder induces a partition of the set of natural numbers.

Definition 2.19. *The* ***residual classes*** *of* $m \in \mathbb{N}^*$ *are the elements of the finite sequence*

$$\left(\{a \in \mathbb{N} \mid a \equiv j-1\}\right)_{j\in[m]}\,.$$

Example 2.21. *The residual classes of* 3 *are* $\{3j \mid j \in \mathbb{N}\}$, $\{3j+1 \mid j \in \mathbb{N}\}$ *and* $\{3j+2 \mid j \in \mathbb{N}\}$.

For short notation, we introduce a symbol for a natural number being a divisor of another.

Definition 2.20. *For* $a,b \in \mathbb{N}$, *we write that* $a \mid b$ *if* a *divides* b. *If* a *does not divide* b, *we write that* $a \nmid b$.

Example 2.22. *Note that* $2 \mid 6$*, but* $3 \nmid 7$*.*

Remark 2.4. Generally, for summation and product symbols and limits we use the common conventions from analysis. Explicitly note the following convention, for $a, b \in \mathbb{Z}$ such that $b < a$ and $\alpha_j \in \mathbb{C}$ for all $j \in \mathbb{Z}$,

$$\sum_{j=a}^{b} \alpha_j = \sum_{j \in \emptyset} \alpha_j := 0,$$

$$\prod_{j=a}^{b} \alpha_j = \prod_{j \in \emptyset} \alpha_j := 1.$$

For notations that we will use that are not mentioned here we will use the common mathematical conventions.

2.2. Induction principle. As we will consider a few proofs using the principle of mathematical induction, the most important two induction principles are presented below. First we introduce the standard induction principle.

Axiom 2.1. [KÖ4, p. 1] *For all* $n \in \mathbb{N}$*, let* $A(n)$ *be a statement. If* $A(0)$ *is true and for all* $n \geq 0$*,* $A(n)$ *implies* $A(n+1)$*, then* $A(n)$ *is true for all* $n \in \mathbb{N}$*.*

The second used induction principle will be the Strong Induction Principle.

Axiom 2.2. [Ham18, p. 187] *For all* $n \in \mathbb{N}$*, let* $A(n)$ *be a statement. If* $A(0)$ *is true and for all* $n \geq 0$*,* $A(j)$ *for all* $j \in \{0\} \cup [n]$ *implies* $A(n+1)$*, then* $A(n)$ *is true for all* $n \in \mathbb{N}$*.*

As an example we will present the proof of a result due to Carl Friedrich Gauss which is very famous (and will reappear in some proofs throughout the thesis).

Lemma 2.2. [KÖ4, p. 1] *For* $m \in \mathbb{N}$*, the sum of the first* m *positive natural numbers can be computed as*

$$\sum_{j=1}^{m} j = \frac{m(m+1)}{2}.$$

Proof. [KÖ4, p. 1] We prove this by induction. For $m = 0$, note that

$$\sum_{j=1}^{0} j = 0 = \frac{0 \cdot 1}{2}.$$

Let $n \in \mathbb{N}$ and assume that

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

Then

$$\sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+2)(n+1)}{2}.$$

By the induction axiom on natural numbers (Axiom 2.1) we can conclude. ■

2.3. Generating functions. Generating functions will be (besides bijections) one of the two most important tools for proving identities between the cardinalities of sets of integer partitions. The ideas, structure and examples in this subsection are based on personal handwritten notes created while attending the lectures MAT648 Combinatorics of integer partitions (taught in spring 2018 by Dr. Jehanne Dousse) and MAT969 Enumerative Combinatorics (taught in fall 2018 by Prof. Dr. Mathilde Bouvel) at University of Zurich. Additionally a good source for formal basics on formal power series and examples for applications can be found in [Hun21] which is open accessible through the lecture directory of ETH Zurich.

Definition 2.21. *Let $\alpha = \left(a_j\right)_{j\in\mathbb{N}}$ be a sequence. Then we define the **(univariate) generating function** of α to be*

$$G\left(\alpha\right) := \sum_{j=0}^{+\infty} a_j q^j.$$

Remark 2.5. So far, generating functions are not more than another notation for sequences. The convenience of this notation will be clear from proofs in later sections (see for example Corollary 2.1). Despite the same notation (if not specified otherwise) we don't consider them as analytic series but as so-called **formal power series**. This name is motivated by the fact that q plays the role of a formal variable. In the following we will define what it exactly means to sum and multiply formal power series. However, for convenient notation it is possible to use convergence results on the corresponding series as long as the convergence holds for certain replacements of q by complex numbers. What is meant by this is illustrated in the following example. Finally we mention that the formal variable q can be replaced by any other letter.

Example 2.23. *The generating function of $\alpha := \left(\left(-1\right)^j\right)_{j\in\mathbb{N}}$ is*

$$G\left(\alpha\right) = \sum_{j=0}^{+\infty} \left(-q\right)^j = \frac{1}{1+q},$$

where we used the well-known convergence result of the geometric series which holds for $q \in \mathbb{C}$ such that $|q| < 1$.

When considering integer partitions it can be useful to consider multiple parameters of partitions (such as for example the size and the number of parts) which motivates the introduction of the following generalization of the previous definition.

Definition 2.22. *Let $\alpha = \left(a_{j_1,\ldots,j_k}\right)_{j_1,\ldots,j_k\in\mathbb{N}}$ be a k-variate sequence. Then we define the **(multivariate) generating function** of α to be*

$$G\left(\alpha\right) := \sum_{j_1=0}^{+\infty} \cdots \sum_{j_k=0}^{+\infty} a_{j_1,\ldots,j_k} \left(\prod_{s=1}^{k} q_s^{j_s}\right).$$

Remark 2.6. The previous remark applies also for multivariate generating functions. Note that the product appears only for notational reasons but is consistent with the definition of multiplication of generating functions.

Example 2.24. *Consider the following bivariate sequence.*

$$s : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$$
$$(i, j) \mapsto \begin{cases} 1 & \text{if } j = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the bivariate generating function of s is

$$G(s) = \sum_{i=0}^{+\infty} a^i b^{2i} = \frac{1}{1 - ab^2}.$$

Generating functions can be considered for sequences in arbitrary sets.

Definition 2.23. *Let M be a set. Then the set of all (univariate) generating functions of sequences in M with q as their formal variable is denoted by*

$$M[[q]] := \{G(s) \mid s \text{ is a sequence in } M\}.$$

Consistently, for $k \in \mathbb{N}^*$, we define the set of multivariate generating functions of k-variate sequences in M as

$$M[[q_1, ..., q_k]] := \{G(s) \mid s \text{ is a } k\text{-variate sequence in } M\}.$$

Addition of two generating functions is defined in the natural way.

Definition 2.24. *Let M be a set, $k \in \mathbb{N}^*$. We define addition in $M[[q_1, ..., q_k]]$ in the following way. If $\alpha = \left(a_{j_1,...,j_k}\right)_{j_1,...,j_k \in \mathbb{N}}$ and $\beta = \left(b_{j_1,...,j_k}\right)_{j_1,...,j_k \in \mathbb{N}}$ are two univariate or multivariate sequences in M, then*

$$G(\alpha) + G(\beta) := \sum_{j_1=0}^{+\infty} \cdots \sum_{j_k=0}^{+\infty} \left(a_{j_1,...,j_k} + b_{j_1,...,j_k}\right) \left(\prod_{s=1}^{k} q_s^{j_s}\right).$$

Example 2.25. *Let $\alpha = (1)_{j \in \mathbb{N}}$ and $\beta = \left((-1)^j\right)_{j \in \mathbb{N}}$. Then*

$$G(\alpha) + G(\beta) = \sum_{j=0}^{+\infty} q^j + \sum_{j=0}^{+\infty} (-q)^j = \sum_{j=0}^{+\infty} \left(1 + (-1)^j\right) q^j = \sum_{j=0}^{+\infty} 2q^{2j}.$$

Note that this is consistent with the analytic correspondence that holds for $q \in \mathbb{C}$ such that $|q| < 1$:

$$\frac{1}{1-q} + \frac{1}{1+q} = \frac{2}{1-q^2}.$$

Also Multiplication is defined in the natural way as we would do with convergent analytic series.

Definition 2.25. *Let M be a set, $k_1, k_2 \in \mathbb{N}$. We define multiplication in $M[[q_1, ..., q_{k_1+k_2}]]$ in the following way. If*

$$\alpha = \left(a_{j_1,\dots,j_{k_1}}\right)_{j_1,\dots,j_{k_1}\in\mathbb{N}} \quad \text{and } \beta = \left(b_{j_{k_1+1},\dots,j_{k_1+k_2}}\right)_{j_{k_1+1},\dots,j_{k_1+k_2}\in\mathbb{N}}$$

are two univariate or multivariate sequences in M, then

$$G(\alpha)\cdot G(\beta) := \sum_{j_1=0}^{+\infty}\cdots\sum_{j_{k_1+k_2}=0}^{+\infty}\left(a_{j_1,\dots,j_{k_1}} b_{j_{k_1+1},\dots,j_{k_1+k_2}}\right)\left(\prod_{s=1}^{k_1+k_2} q_s^{j_s}\right).$$

Note that for $k_1 = k_2 = 1$, if we consider $G(\alpha)$ and $G(\beta)$ as elements of $M[[q]]$, this definition is equivalent to the commonly used Cauchy product

$$G(\alpha)\cdot G(\beta) = \sum_{j=0}^{+\infty}\left(\sum_{l=0}^{j} a_l b_{j-l}\right) q^j.$$

Example 2.26. *Let*

$$\alpha = \beta = \left(\frac{1}{j!}\right)_{j\in\mathbb{N}}.$$

Then (we consider $G(\alpha)$ as an element of $\mathbb{C}[[x]]$ and $G(\beta)$ as an element of $\mathbb{C}[[y]]$)

$$\begin{aligned} G(\alpha)\cdot G(\beta) &= \sum_{j=0}^{+\infty}\sum_{l=0}^{j}\frac{x^l}{l!}\frac{y^{j-l}}{(j-l)!} \\ &= \sum_{j=0}^{+\infty}\frac{1}{j!}\sum_{l=0}^{j}\binom{j}{l}x^l y^{j-l} \\ &= \sum_{j=0}^{+\infty}\frac{1}{j!}(x+y)^j q^j \\ &= \sum_{j=0}^{+\infty}\frac{(x+y)^j}{j!}. \end{aligned}$$

which is consistent with the analytic correspondence

$$e^x \cdot e^y = e^{x+y}.$$

In the third step of the computation above we used the well-known binomial theorem.

Remark 2.7. There are a lot of beautiful algebraic properties of formal power series which are not needed for our purpose. But still it's worth mentioning that for a field K, $K[[q]]$ has the structure of a commutative unitary ring, with respect to $+$ and $\cdot$. More details can be found in [Aig07].

In the following we will often consider generating functions of integer partitions. The formal definition of an integer partition can be found in Definition 2.28.

Definition 2.26. *Let $P(n, i_1, ..., i_k)$ be the number of partitions of size n with parameters $i_1, ..., i_k$. Then we define the generating function of the partitions counted by $P(n, i_1, ..., i_k)$ as*

$$G\left(\left(P(n, i_1, ..., i_k)\right)_{n,i_1,...,i_k \in \mathbb{N}}\right).$$

Examples for this definition can be found in the proof of Euler's partition identity (Corollary 2.1).

Finally we need to be able to refer to single elements of the sequences represented by generating functions.

Definition 2.27. *For all $i_1, ..., i_k \in \mathbb{N}$ and*

$$Q(q_1, ..., q_k) := \sum_{j_1=0}^{+\infty} \cdots \sum_{j_k=0}^{+\infty} a_{j_1,...,j_k} \left(\prod_{s=1}^{k} q_s^{j_s}\right) \in M[[q_1, ..., q_k]],$$

we denote its coefficients by

$$[q_1^{i_1} ... q_k^{i_k}] Q(q_1, ..., q_k) := a_{j_1,...,j_k}.$$

Example 2.27. *Note that*

$$[q^{17}] \frac{1}{1+q} = [q^{17}] \sum_{j=0}^{+\infty} (-q)^j = (-1)^{17} = -1.$$

Remark 2.8. Note that for $l, k \in \mathbb{N}^*$ such that $l < k$ and a set M, by construction

$$M[[q_1, ..., q_l]][[q_{l+1}, ..., q_k]] = M[[q_1, ..., q_k]].$$

Additionally we remark that two generating functions are equal if and only if all their coefficients are equal.

2.4. Integer partitions. The main objects in the following considerations are partitions of natural numbers.

Definition 2.28. [And76, p. 1] *Let $n \in \mathbb{N}$. A **partition** of n is a non-increasing finite sequence $(\lambda_i)_{i\in[r]}$ in $\mathbb{N}^*$ for some $r \in \mathbb{N}$ such that*

$$\sum_{i=1}^{r} \lambda_i = n.$$

*The elements λ_i of this sequence are called **parts** of the partition. The set of partitions of n is denoted by $\mathcal{P}(n)$. If $\lambda \in \mathcal{P}(n)$ we write $\lambda \vdash n$. An additional notation of λ that is convenient is*

$$\lambda = \left(j^{k_j}\right)_{j\in[s]},$$

where k_j is the number of parts equal to j and s is the size of the largest part of λ. Finally we denote

$$\mu_\lambda(j) := k_j.$$

Example 2.28. *$(3,2)$ is a partition of 5 with 2 parts, $(15,8,3,3,1)$ is a partition of 30 with 5 parts and*

$$\lambda := \left(1^3, 2^8, 3^0, 4^2, 5^1\right)$$

is a partition of 32 with 14 parts. Moreover, for example $\mu_\lambda(4) = 2$.

For convenience we introduce a short notation for the size and the largest part of a partition.

Definition 2.29. *Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Then*

$$s(\lambda) := n$$

*is called the **size** of λ. Moreover we denote the largest part of λ by $L(\lambda)$ and the number of parts of λ by $z(\lambda)$.*

Example 2.29. *Note that*

$$s\left((15,8,3,3,1)\right) = 30,$$

$L\left((15,8,3,3,1)\right) = 15$ *and* $z\left((15,8,3,3,1)\right) = 5$.

Remark 2.9. Note that the size of a partition can be computed in the following way. Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Then

$$s(\lambda) = \sum_{j=1}^{L(\lambda)} \mu_\lambda(j) \cdot j.$$

Sometimes we will consider partitions of integers that contain only parts belonging to a subset of $\mathbb{N}$.

Definition 2.30. *Let $A \subseteq \mathbb{N}^*, n \in \mathbb{N}$.For a partition π of n, denote by π_A the partition consisting of the parts of π that are contained in A. Moreover, if π consists of parts that are contained in a set $H \subseteq \mathbb{N}^*$, we write*

$$\pi \to H.$$

Example 2.30. *Note that* $(6,6,4,2) \to 2\mathbb{N}^*$ *and* $(10,8,6,4,3,3,1)_{3\mathbb{N}^*} = (6,3,3)$.

For the sake of introducing generating functions counting integer partitions we introduce the following notation.

Definition 2.31. [And76, p. 3] *Let $n \in \mathbb{N}$. Then we define $p(n)$ to be the number of partitions of n. For convenience note that $p(m) := 0$ for all $m \in \mathbb{Z}_{<0}$. If $H \subseteq \mathbb{N}^*$ we denote by $p^H(n)$ the number of partitions of n whose parts lie in H. Further for $d \in \mathbb{N}^*$ we denote the number of partitions of n with each part appearing at most d times and lying in H by $p^H_{\leq d}(n)$.*

Remark 2.10. Whenever we consider infinite products / series or finite products / sums (if not specified otherwise), we consider them as formal power series and q and t will denote formal variables. In general most transformations and identities used in the proofs hold analytically only under additional restrictions.

The following theorem and its proof grant the most important tool for translating problems about integer partitions into the language of formal power series.

Theorem 2.1. [And76, p. 3] *Let $H \subseteq \mathbb{N}^*$ and*

$$f^H(q) := \sum_{n\geq 0} p^H(n)\, q^n, \tag{1}$$

$$f^H_d(q) := \sum_{n\geq 0} p^H_{\leq d}(n)\, q^n. \tag{2}$$

Then, in $\mathbb{C}[[q]]$,

$$f^H(q) = \prod_{n\in H} (1-q^n)^{-1}, \tag{3}$$

$$f^H_d(q) = \prod_{n\in H} \sum_{j=0}^{d} q^{jn} = \prod_{n\in H} \frac{1-q^{(d+1)n}}{1-q^n}. \tag{4}$$

Proof. [And76, p. 4] Note that the arguments given in this proof only hold in the setting of formal power series. To prove the theorem analytically (for $q \in \mathbb{C}$ such that $|q| < 1$) a consideration of convergence of the involved series is necessary. Let $H = \{h_j \mid j \in [r]\} \subseteq \mathbb{N}^*$. Then

$$\prod_{n\in H} \frac{1}{1-q^n} = \prod_{n\in H} \left(\sum_{j=0}^{\infty} q^{jn} \right) = \sum_{k=0}^{|H|} \sum_{\substack{(a_1,\ldots,a_k)\in\mathbb{N}^k \\ a_k\neq 0}} q^{\sum_{j=1}^k a_j h_j}. \tag{5}$$

Just by notation, there is a bijection between the sums of the form

$$\sum_{j=1}^{k} a_j h_j,$$

and partitions of the form

$$\left(h_j^{a_j}\right)_{j\in[k]}.$$

Thus q^n will occur in the summation of (5) exactly once for each partition of n into parts taken from H. Thus

$$\prod_{n\in H} \frac{1}{1-q^n} = \sum_{n\geq 0} p^H(n)\, q^n.$$

Equality (4) follows similar noting that

$$\prod_{n\in H} \frac{1-q^{(d+1)n}}{1-q^n} = \prod_{n\in H} \left(\sum_{j=0}^{d} q^{jn}\right) = \sum_{k=0}^{|H|} \sum_{\substack{(a_1,\ldots,a_k)\in\mathbb{N}^k \\ a_1,\ldots,a_k\leq d \\ a_k\neq 0}} q^{\sum_{j=1}^{k} a_j h_j}. \tag{6}$$

Again note that $\sum_{j=1}^{k} a_j h_j$ represents the partition

$$\left(h_j^{a_j}\right)_{j\in[k]}.$$

Thus q^n will occur in the summation of (6) exactly once for each partition of n into parts taken from H, each occuring at most d times. Hence

$$\prod_{n\in H} \frac{1-q^{(d+1)n}}{1-q^n} = \sum_{n\geq 0} p^H_{\leq d}(n)\, q^n.$$

■

2.5. Partition identities. The main results in which we are interested in this thesis are all of the same sort, relating integer partitions of different types.

Definition 2.32. [AE04, p. 3] *Let $n \in \mathbb{N}$, a **partition identity** is an equality between the cardinalities of two subsets of $\mathcal{P}(n)$.*

As a first example for a partition identity and as the result of an application of Theorem 2.1 we consider one of the most famous partition identities due to Leonhard Euler.

Corollary 2.1. [And76, p. 5] *(Euler) Let $\mathcal{O} := \{2n+1 \mid n \in \mathbb{N}\}$ be the subset of $\mathbb{N}^*$ consisting of all odd natural numbers. Then for all $n \in \mathbb{N}$,*

$$p^{\mathcal{O}}(n) = p^{\mathbb{N}}_{\leq 1}(n),$$

i.e. the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

Example 2.31. *Note that $p^{\mathcal{O}}(8) = 6$ counts the partitions*

$$(7,1),(5,3),(5,1,1,1),(3,3,1,1),(3,1,1,1,1,1),(1,1,1,1,1,1,1,1).$$

On the other hand, $p^{\mathbb{N}}_{\leq 1}(8) = 6$ counts the partitions

$$(8),(7,1),(6,2),(5,3),(5,2,1),(4,3,1).$$

We will present two different proofs. The first one uses the approach of generating functions.

Proof (using generating functions). [And76, p. 5] By an application of Theorem 2.1 we note that

$$\sum_{n=0}^{+\infty} p^{\mathcal{O}}(n)\, q^n = \prod_{n=1}^{+\infty} \left(1 - q^{2n-1}\right)^{-1},$$

and

$$\sum_{n=0}^{+\infty} p^{\mathbb{N}}_{\leq 1}(n)\, q^n = \prod_{n=1}^{+\infty} \frac{1-q^{2n}}{1-q^n} = \prod_{n=1}^{+\infty} \left(1 - q^{2n-1}\right)^{-1}.$$

Thus we deduce that

$$\sum_{n=0}^{+\infty} p^{\mathcal{O}}(n)\, q^n = \sum_{n=0}^{+\infty} p^{\mathbb{N}}_{\leq 1}(n)\, q^n.$$

From this we can conclude that for all $n \in \mathbb{N}$,

$$p^{\mathcal{O}}(n) = p^{\mathbb{N}}_{\leq 1}(n).$$

■

In the following we present a well-known combinatorial proof of Corollary 2.1.

Proof (Bijective proof). [AE04, p. 8 - 9] Let $n \in \mathbb{N}$, $\mathcal{P}^{\mathcal{O}}(n)$ be the set of all partitions counted by $p^{\mathcal{O}}(n)$ and $\mathcal{P}^{\mathbb{N}}_{\leq 1}(n)$ be the set of all partitions counted by $p^{\mathbb{N}}_{\leq 1}(n)$. Consider the following map:

$$\varphi : \mathcal{P}^{\mathcal{O}}(n) \to \mathcal{P}^{\mathbb{N}}_{\leq 1}(n)$$

$$\lambda \mapsto \varphi(\lambda),$$

where $\varphi(\lambda)$ is constructed in the following way. For all parts a of λ that appear multiple times, replace two of them by the part $2a$. As long as there are still parts with multiple appearances, repeat this step until we end up (after possible reordering) with a partition of n into distinct parts.

Example 2.32. *Note that*

$$\varphi\big((5,5,5,5,5,5,3,1,1,1)\big) = (20,10,3,2,1).$$

By construction, this map is **well-defined** since replacing two equal parts a by $2a$ reduces $\mu_\lambda(a)$ by 1 and preserves the size of the partition. Iterative application of this modification results in a partition where for each part a it holds that $\mu_\lambda(a) = 1$. Reordering insures the non-increasing property.
For proving **bijectivity** we define the following map which turns out to be the inverse of φ:

$$\begin{aligned} \psi:\ \mathcal{P}^{\mathbb{N}}_{\leq 1}(n) &\to \mathcal{P}^{\mathcal{O}}(n) \\ \mu &\mapsto \psi(\mu), \end{aligned}$$

where $\psi(\mu)$ is obtained from μ by replacing all even parts b by two parts equal to $\frac{b}{2}$ and reorder the finite sequence to preserve the non-increasing property. As long as there are still even parts repeat this procedure. By construction this map is well-defined: Note that repeatedly dividing a natural number by 2 results eventually in an odd number. The size is preserved by the multiple insertion of the new parts and so is the non-increasing property by the possibly necessary reordering.

Example 2.33. *Note that*

$$\psi\big((20,10,3,2,1)\big) = (5,5,5,5,5,5,3,1,1,1).$$

To conclude we will use Lemma 2.1. Let $\lambda \in \mathcal{P}^{\mathcal{O}}(n)$ and consider $\psi\big(\varphi(\lambda)\big)$. By construction, this partition is obtained by preserving the parts of λ with single appearance and first doubling and replacing the parts with multiple appearance. But for each doubling we will divide it again by 2 and replace it in the course of applying ψ. Since all parts of λ are odd we will return to the parts with which we started. Hence

$$\psi\big(\varphi(\lambda)\big) = \lambda.$$

Now let $\mu \in \mathcal{P}^{\mathbb{N}}_{\leq 1}$ and consider $\varphi\big(\psi(\mu)\big)$. This partition is obtained from μ by preserving all odd parts and first dividing the even parts by 2 and replacing them. In the course of applying φ we will again double and replace them (because it is exactly those that appear multiple times). Since we started by a partition with distinct parts we will return to the even parts we started with. Reordering assures therefore that

$$\varphi\big(\psi(\mu)\big) = \mu.$$

■

As a second example we will prove a generalization of a partition identity due to Subbarao (see [And76, p. 13]).

Corollary 2.2. *Let $m \in \mathbb{N}_{>2}$ be an odd natural number. Then for all $n \in \mathbb{N}$ the number of partitions of n in which each part appears exactly two, m or $2+m$ times equals the number of partitions of n into parts congruent to $2, 6, 10, ..., 4m-2, m, 3m \pmod{4m}$.*

Proof. Let $m \in \mathbb{N}_{>2}$ be an odd natural number. Using the idea of the proof of Theorem 2.1 the generating function of partitions in which each part appears exactly two, m or $m+2$ times is

$$\begin{aligned}
\prod_{k=1}^{+\infty} \left(1 + q^{2k} + q^{mk} + q^{(2+m)k}\right) &= \prod_{k=1}^{+\infty} \left(1 + q^{2k}\right)\left(1 + q^{mk}\right) \\
&= \prod_{k=1}^{+\infty} \left(1 + q^{2k}\right)\left(1 + q^{mk}\right) \frac{\left(1 - q^{2k}\right)\left(1 - q^{mk}\right)}{\left(1 - q^{2k}\right)\left(1 - q^{mk}\right)} \\
&= \prod_{k=1}^{+\infty} \frac{\left(1 - q^{4k}\right)\left(1 - q^{2mk}\right)}{\left(1 - q^{2k}\right)\left(1 - q^{mk}\right)} \\
&= \prod_{\substack{k=1 \\ k \equiv 2 \pmod 4 \\ \text{or} \\ k \equiv m \pmod{2m}}}^{+\infty} \frac{1}{1 - q^k}.
\end{aligned}$$

Since $lcm\,(4, 2m) = 4m$ we see that

$$\prod_{\substack{k=1 \\ k \equiv 2 \pmod 4 \\ \text{or} \\ k \equiv m \pmod{2m}}}^{+\infty} \frac{1}{1 - q^k} = \prod_{k \equiv 2, 6, 10, ..., 4m-2, m, 3m \pmod{4m}}^{+\infty} \frac{1}{1 - q^k},$$

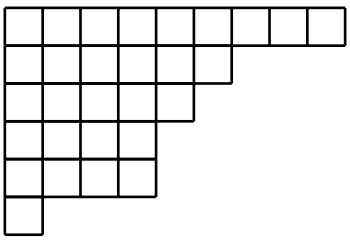
which is the generating function of partitions of n into parts congruent to $2, 6, 10, ..., 4m-2, m, 3m \pmod{4m}$. ■

Further examples for applications of Theorem 2.1 can be found in Lemmas 3.1 and 4.1.

2.6. Graphical representation of integer partitions. Sometimes it's possible to find bijections between certain types of partitions using graphical representations. This illustrates the relation between individual partitions of the two types.

Definition 2.33. [And76, p. 6] *The **graphical representation** (Ferrers graph) of a partition $\lambda = (\lambda_j)_{j\in[n]}$ is a drawing where, for each part λ_i, we draw a row of λ_i boxes, and place the rows left-aligned below each other. We order the rows by the indices of the corresponding parts.*

Example 2.34. *The partition $\lambda = (9, 6, 5, 4, 4, 1)$ has the following graphical representation:*



Definition 2.34. [And76, p. 7] *For every partition $\lambda = (\lambda_j)_{j\in[r]}$, the partition*

$$\lambda' := \left(\lambda'_j\right)_{j\in[\lambda_1]},$$

*where λ'_i is the number of parts of λ that are greater or equal i, is called the **conjugate** of λ. Its graphical representation can be obtained by mirroring the Ferrers graph of λ along its diagonal.*

Remark 2.11. Since mirroring a Ferrers graph along its diagonal preserves the number of boxes, conjugation preserves the size of a partition.

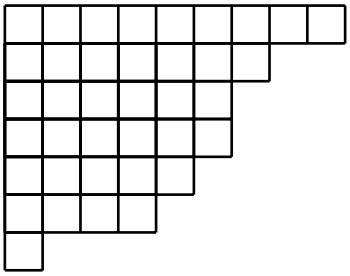
Conjugation provides a bijection, that leads to the following result. It can be deduced by mirroring the graphical representation.

Theorem 2.2. [And76, p. 8] *Let $n, m \in \mathbb{N}$. The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.*

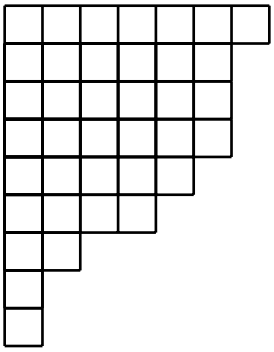
Example 2.35. *Let's look at a graphical example. Consider the partition*

$$\lambda = (9, 7, 6, 6, 5, 4, 1)\,.$$

Its graphical representation is



Note that $\lambda' = (7,6,6,6,5,4,2,1,1)$ *and* $z\left(\lambda'\right) = 9 = L(\lambda)$. *The Ferrers graph of* λ' *is obtained by mirroring the graph above along its diagonal.*



Proof. This theorem follows by noting that the inverse map of conjugation is conjugation itself, the fact that for every partition λ,

$$z(\lambda) = L\left(\lambda'\right),$$

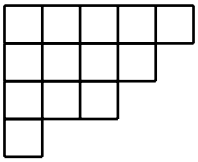
and Remark 2.11 (the size of a partition is preserved under conjugation). ■

We will use the following corollary for a combinatorial proof of a generalization of Capparelli's identity.

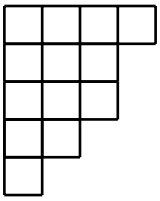
Corollary 2.3. *Let* $n, m \in \mathbb{N}^*$, $\mathcal{D}_m(n)$ *be the set of partitions of* n *into distinct parts that are smaller than or equal to* m *and* $\mathcal{P}_m(n)$ *denote the set of partitions into at most* m *parts such that the smallest part is equal to* 1 *and the difference between consecutive parts is at most* 1. *Then the following map is a bijection:*

$$\begin{aligned} \alpha: \ \mathcal{D}_m(n) &\to \mathcal{P}_m(n) \\ \lambda &\mapsto \lambda'. \end{aligned}$$

Example 2.36. *Also this bijection can best be understood by considering Ferrers graphs. Consider the example* $\lambda = (5,4,3,1)$ *for* $n = 13$ *and* $m = 5$. *Its graphical representation is*



Conjugating this partition yields $\alpha(\lambda) = \lambda' = (4,3,3,2,1)$ *which can be drawn as*



Proof. Let $\lambda \in \mathcal{D}_m(n)$. We show that two consecutive parts of λ' differ by at most 1. We can write

$$\lambda' = (\lambda_i)_{i \in L(\lambda)},$$

where λ_i is the number of parts of λ that are greater than or equal to i. Since for all $i \in [L(\lambda)]$ the number of parts in λ equal to i can be expressed by $\lambda_i - \lambda_{i+1}$ and λ consists of distinct parts the difference condition follows.
Now let $\mu \in \mathcal{P}_m(n)$. We show that the parts of μ' are distinct. Assume that there is a part $j \in \mathbb{N}^*$ that appears at least twice in μ'. Then j is the number of parts of μ that are greater than or equal to k and the number of parts of μ that are greater than or equal to $k+1$ for some $k \in \mathbb{N}^*$ such that $k < L(\mu)$. Hence there is no part equal to k in μ which means that the greatest part of μ smaller than k and the smallest part greater than k differ by at least 2, or there is no part equal to 1. This is a contradiction. Hence the parts of μ' are distinct.
Using Theorem 2.2 and the fact shown in its proof that the inverse map of conjugation is conjugation itself we can conclude that α is a bijection. ■

In the proofs of a refinement of Caparelli's Identity we will use the following operation involving conjugation.

Definition 2.35. [AAG95b, p. 640] *Let* $n \in \mathbb{N}$ *and*

$$\pi = \left(j^{\mu_\pi(j)}\right)_{j \in [s]} \vdash n,$$

such that $\pi \to 3\mathbb{N}^*$. *The* ***3-fold conjugate*** *of* π *is defined to be*

$$\pi^* := \left(\left(\frac{j}{3}\right)^{3\mu_\pi(j)}\right)'_{j \in L(\pi)}.$$

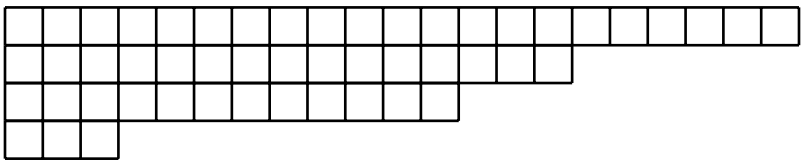
Example 2.37. *Consider the following partition of 51:*

$$\pi = (21, 15, 12, 3).$$

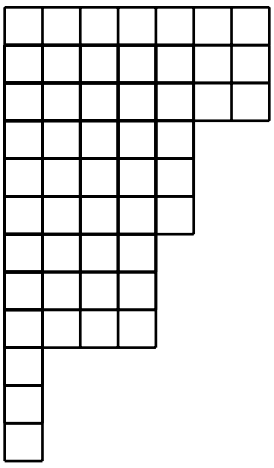
Its 3-fold conjugate is

$$\pi^* = (7,7,7,5,5,5,4,4,4,1,1,1)' = (12,9,9,9,6,3,3)\,.$$

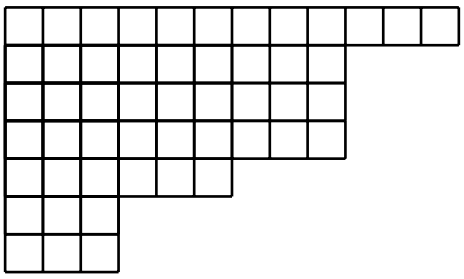
This map can be illustrated by using graphs. Consider the graphical representation of π:



The first step is then dividing each row into three parts of equal length and placing them below each other.



For getting the conjugate of the partition corresponding to this graph we need to mirror it at its diagonal.



Taking the 3-fold conjugate yields a bijection which will be useful in the proof of a refinement of Capparelli's identity.

Proposition 2.1. [AAG95b, p. 640] *Let $n \in \mathbb{N}$ and define*

$$\mathcal{D}_3(n) := \{\lambda \vdash n \mid \lambda \to 3\mathbb{N}^*, \forall j \in \mathbb{N}^* : \ \mu_\lambda(j) \leq 1\},$$

and

$$\mathcal{P}_3(n) := \{\lambda \vdash n \mid \lambda \to 3\mathbb{N}^*, \forall j \in [L(\lambda)] : \ 3 \mid j \Rightarrow \mu_\lambda(j) \geq 1\}.$$

Then

$$\begin{aligned} \varphi : \ \mathcal{D}_3(n) &\to \mathcal{P}_3(n) \\ \pi &\mapsto \pi^* \end{aligned}$$

is a bijection.

Proof. Let $n \in \mathbb{N}$. First we prove that φ is **well-defined**. For this purpose let

$$\pi = \left(j^{\mu_\pi(j)}\right)_{j \in [r]} \in \mathcal{D}_3(n).$$

Using Remark 2.11 we can deduce that the size is preserved under φ:

$$s(\pi^*) = s\left(\left(\left(\frac{j}{3}\right)^{3\mu_\pi(j)}\right)_{j\in[r]}\right) = \sum_{j=1}^{r} 3\mu_\pi(j) \cdot \frac{j}{3} = \sum_{j=1}^{r} \mu_\pi(j) \cdot j = s(\pi) = n.$$

Denote the parts of π^* by π_j^* for $j \in [3L(\pi)]$. Then by definition of π^* all parts are multiples of 3. Indeed, for each $j \in [3L(\pi)]$,

$$\pi_j^* = \sum_{k=j}^{\frac{L(\pi)}{3}} 3\mu_\pi(3k) = 3\sum_{k=j}^{\frac{L(\pi)}{3}} \mu_\pi(3k) \in 3\mathbb{N}^*.$$

Moreover the difference between two consecutive parts is either 0 or 3. This can be deduced by the definition of $\mathcal{D}_3(n)$. Note that for all $j \in [\frac{L(\pi)}{3} - 1]$,

$$\pi_j^* - \pi_{j+1}^* = \sum_{k=j}^{\frac{L(\pi)}{3}} 3\mu_\pi(3k) - \sum_{k=j+1}^{\frac{L(\pi)}{3}} 3\mu_\pi(3k) = 3\mu_\pi(3j) \in \{0, 3\}.$$

Hence we proved that $\pi^* \in \mathcal{P}_3(n)$.
For proving **bijectivity** consider the following map

$$\begin{aligned} \psi : \mathcal{P}_3(n) &\to \mathcal{D}_3(n) \\ \pi &\mapsto \left(3j^{\frac{\mu_{\pi'}(j)}{3}}\right)_{j\in[z(\pi)]}. \end{aligned}$$

Let us prove that ψ is well-defined. For this purpose take $\pi \in \mathcal{P}_3(n)$. First note that for all $k \in \mathbb{N}$,

$$\mu_\pi(3k+1) = \mu_\pi(3k+2) = 0.$$

Thus

$$\sum_{j=3k+1}^{L(\pi)} \mu_\pi(j) = \sum_{j=3k+2}^{L(\pi)} \mu_\pi(j) = \sum_{j=3k+3}^{L(\pi)} \mu_\pi(j),$$

and therefore

$$\pi'_{3k+1} = \pi'_{3k+2} = \pi'_{3k+3},$$

where π'_j denotes the jth part of π'. Hence for all $j \in [z(\pi)]$,

$$\frac{\mu_{\pi'}(j)}{3} \in \mathbb{N}.$$

Moreover, using Lemma 2.11 note that the size is preserved under ψ:

$$s\left(\psi(\pi)\right) = \sum_{j=1}^{z(\pi)} \frac{\mu_{\pi'}(j)}{3} \cdot 3j = \mu_{\pi'}(j) \cdot j = s\left(\pi'\right) = s(\pi) = n.$$

By definition of ψ all parts of $\psi(\pi)$ are multiples of 3. Lastly assume that there is a $j \in [z(\pi)]$ such that

$$\frac{\mu_{\pi'}(j)}{3} \geq 2.$$

This would imply that $\mu_{\pi'}(j) \geq 6$ and thus there is a $k \in [z(\pi)]$ such that k is a multiple of 3 and $\mu_\pi(k) = 0$ which contradicts that $\pi \in \mathcal{P}_3(n)$. Hence

$$\frac{\mu_{\pi'}(j)}{3} \leq 1,$$

for all $j \in [z(\pi)]$. Thus

$$\psi(\pi) \in \mathcal{D}_3(n).$$

To conclude the proof note that by construction for all $\pi \in \mathcal{D}_3(n)$ it holds that

$$\psi\left(\varphi(\pi)\right) = \psi\left(\pi^*\right) = \left(3j^{\frac{\mu_{\pi^{*\prime}}(j)}{3}}\right)_{j\in[z(\pi^*)]} = \pi.$$

Further for all $\lambda \in \mathcal{P}_3(n)$,

$$\varphi\left(\psi(\lambda)\right) = \varphi\left(\left(3j^{\frac{\mu_{\pi'}(j)}{3}}\right)_{j\in[z(\pi)]}\right) = \left(3j^{\frac{\mu_{\pi'}(j)}{3}}\right)^*_{j\in[z(\pi)]} = \left(j^{\mu_{\pi'}(j)}\right)'_{j\in[z(\pi)]} = \left(\pi'\right)' = \pi.$$

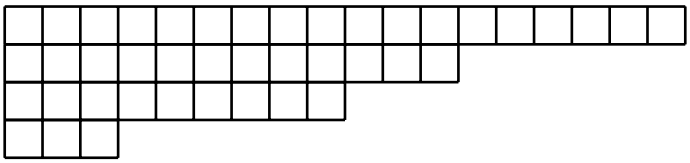
■

The following result is useful for constructing generating functions of partitions into distinct parts.

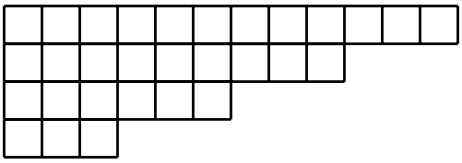
Proposition 2.2. *Let $n, m, j \in \mathbb{N}^*$ and $\mathcal{D}_{m,j}(n)$ be the set of partitions with exactly m distinct parts all congruent to $0 \pmod j$ and $\mathcal{P}_{m,j}(n)$ be the set of partitions into at most m parts all congruent to $0 \pmod j$. Then*

$$|\mathcal{D}_{m,j}(n)| = |\mathcal{P}_{m,j}\left(n - \frac{jm(m+1)}{2}\right)|.$$

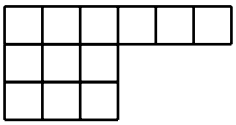
Proof. The idea for the proof is to construct a bijection that divides the Ferrers graph of a partition $\pi \in \mathcal{D}_{m,j}(n)$ into a staircase with m steps and the remaining diagram. This can be drawn as follows (for the example $m = 4, j = 3, \pi = (18, 12, 9, 3), n = 42$).



can be decomposed into



and the remaining diagram.



More precisely we define the map

$$\begin{aligned} \varphi :\ \mathcal{D}_{m,j}(n) &\to \mathcal{P}_{m,j}\left(n - \frac{jm(m+1)}{2}\right) \\ \pi &\mapsto \varphi(\pi), \end{aligned}$$

where $\varphi(\pi)$ is the partition represented by the remaining diagram of the Ferrers graph of π after removing the staircase representing the partition $(js)_{s\in[m]}$.
By construction and the fact that

$$s\left((js)_{s\in[m]}\right) = \frac{jm(m+1)}{2},$$

using Lemma 2.2, φ is **well-defined**.
For seeing **bijectivity**, we note that the inverse φ^{-1} can be constructed as

$$\begin{aligned} \varphi^{-1} :\ \mathcal{P}_{m,j}\left(n - \frac{jm(m+1)}{2}\right) &\to \mathcal{D}_{m,j}(n) \\ \mu &\mapsto \varphi^{-1}(\mu), \end{aligned}$$

where $\varphi^{-1}(\mu)$ is the partition represented by the Ferrers graph obtained by glueing together the corresponding rows of the graphs of μ and $T := (js)_{s\in[m]}$. Note that for the

glueing, we start with the largest rows. This glueing process corresponds to addition of μ and T. Addition of partitions is described more formally in Definition 5.5. Note that since μ is non-increasing, $\varphi^{-1}(\mu)$ is indeed a partition into distinct parts. ■

The following corollary provides a tool for describing generating functions of partitions into distinct parts.

Corollary 2.4. *Let $t \in \mathbb{N}^*$. The generating function for partitions into distinct parts all multiples of t, keeping track of the number of parts, is given by*

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}|D_{m,t}(n)|a^m q^n = \sum_{s=0}^{+\infty}\frac{a^s q^{t\frac{s(s+1)}{2}}}{(q^t;q^t)_s}.$$

Proof. Let $t \in \mathbb{N}^*$. Note that by Proposition 2.2, the partitions counted by

$$\left(D_{m,t}(n)\right)_{m,n\in\mathbb{N}}$$

can be divided in $(ts)_{s\in[m]}$ and an arbitrary second partition with at most m parts. Noting that these two partitions can be chosen independently of each other and using Theorems 2.1 and 2.2 we can deduce that the generating function of $\left(D_{m,t}(n)\right)_{m,n\in\mathbb{N}}$ is

$$\sum_{s=0}^{+\infty} a^s q^{t\frac{s(s+1)}{2}}\frac{1}{(q^t;q^t)_s}.$$

Note that the variable a keeps track of the number of parts. ■

2.7. Some equalities between formal power series. In this rather technical subsection we will provide a few identities of formal power series that will prove very helpful in the proofs of the following sections.
The following products appear often in generating functions that we will consider and allow a clearer notation.

Notation. [And76, p. 17] Denote, for $a \in \mathbb{C}$,

$$\begin{aligned} (a)_n &:= (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right), \\ (a)_\infty &:= (a;q)_\infty := \lim_{n\to\infty} (a;q)_n, \\ (a)_0 &:= 1. \end{aligned} \tag{7}$$

Remark 2.12. [And76, p. 17] Note that for all $a \in \mathbb{C}, n \in \mathbb{N}$

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

This can be obtained directly from the definition of this notation.

To prove Jacobi's triple product identity, which is used as an important tool in Andrews' proofs of Schur's theorem, and Caparelli's identity, we need the following theorem due to Cauchy.

Theorem 2.3. *(Cauchy)* [And76, p. 17] *Let $a \in \mathbb{C}$. In $\mathbb{C}[[t,q]]$ it holds that*

$$\sum_{n=0}^{+\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(at)_\infty}{(t)_\infty}.$$

Proof. [And76, p. 18] Let $a \in \mathbb{C}$. In $\mathbb{C}[[t,q]]$ denote

$$\phi_{a,q}(t) := \frac{(at)_\infty}{(t)_\infty}.$$

Since $\mathbb{C}[[t,q]] = \mathbb{C}[[q]][[t]]$ we can write

$$\phi_{a,q}(t) = \sum_{j=0}^{+\infty} A_j(a,q)\, t^j,$$

with $A_j(a,q) \in \mathbb{C}[[q]]$ for all $j \in \mathbb{N}$. Note that

$$(1-t)\,\phi_{a,q}(t) = (1-t)\,\frac{(at)_\infty}{(t)_\infty} = (1-at)\,\frac{(atq)_\infty}{(tq)_\infty} = (1-at)\,\phi_{a,q}(tq).$$

Now compare the coefficients for $n \in \mathbb{N}^*$ on both extremes of the previous computation:

$$[t^n](1-t)\,\phi_{a,q}(t) = [t^n]\phi_{a,q}(t) - [t^{n-1}]\phi_{a,q}(t) = A_n(a,q) - A_{n-1}(a,q),$$

$$[t^n](1-at)\,\phi_{a,q}(tq) = q^n[t^n]\phi_{a,q}(t) - aq^{n-1}[t^{n-1}]\phi_{a,q}(t) = q^n A_n(a,q) - aq^{n-1}A_{n-1}(a,q).$$

Thus we get

$$A_n(a,q) - A_{n-1}(a,q) = q^n A_n(a,q) - aq^{n-1} A_{n-1}(a,q)$$
$$\Leftrightarrow A_n(a,q) = \frac{1-aq^{n-1}}{1-q^n} A_{n-1}(a,q),$$

and therefore, by iterating,

$$A_n(a,q) = \frac{(a)_n}{(q)_n} A_0(a,q).$$

Further,

$$A_0(a,q) = [t^0]\phi_{a,q}(t) = \phi_{a,q}(0) = 1.$$

Thus

$$\phi_{a,q}(t) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(q)_n} t^n,$$

which proves the theorem. ■

More explicitly we will use the following consequence of Theorem 2.3.

Corollary 2.5. *(Euler)* [And76, p. 19] *In* $\mathbb{C}[[t,q]]$ *it holds that*

$$\sum_{n=0}^{+\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}, \tag{8}$$

$$\sum_{n=0}^{+\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = (-t)_\infty. \tag{9}$$

Proof. [And76, p. 19] By applying Theorem 2.3 for $a = 0$ we obtain (8). To prove (9), let $a \in \mathbb{C}$. Then by Theorem 2.3

$$\sum_{n=0}^{+\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(at)_\infty}{(t)_\infty}.$$

Now let $b \in \mathbb{C} \setminus \{0\}$ and replace a by $\frac{a}{b}$ and t by bz. Then

$$\sum_{n=0}^{+\infty} \frac{\left(\frac{a}{b}\right)_n}{(q)_n} (bz)^n = \frac{(az)_\infty}{(bz)_\infty},$$

which is equivalent to

$$\sum_{n=0}^{+\infty} \frac{\prod_{j=0}^{n-1}\left(1-\frac{a}{b}q^j\right)}{(q)_n} b^n z^n = \frac{(az)_\infty}{(bz)_\infty},$$

or

$$\sum_{n=0}^{+\infty} \frac{\prod_{j=0}^{n-1}\left(b-aq^j\right)}{(q)_n} z^n = \frac{(az)_\infty}{(bz)_\infty}.$$

Now letting $b \to 0$, setting $a = -1$ and using the well-known summation formula (due to Gauss) that for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n-1} j = \frac{n\,(n-1)}{2},$$

we get

$$\sum_{n=0}^{+\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q)_n} z^n = (-z)_\infty\,,$$

which completes the proof of the corollary. ■

Now we can prove an identity that is useful in many proofs of partition identities when using generating functions.

Theorem 2.4. *(Jacobi's triple product identity)* [And76, p. 21] *Let $z, q \in \mathbb{C}$. If $z \neq 0, |q| < 1$,*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \left(q^2;q^2\right)_\infty \left(-zq;q^2\right)_\infty \left(-\frac{q}{z};q^2\right)_\infty.$$

Proof. [And76, p. 21] Let $z, q \in \mathbb{C}$ such that $z \neq 0$ and $|q| < 1$. Note that using (9) (for $t = zq$ and $q = q^2$) and Remark 2.12,

$$\begin{aligned}
\left(-zq;q^2\right)_\infty &= \sum_{n=0}^{+\infty} \frac{z^n q^n q^{n(n-1)}}{\left(q^2;q^2\right)_n} \\
&= \sum_{n=0}^{+\infty} \frac{z^n q^{n^2}}{\left(q^2;q^2\right)_n} \\
&= \frac{1}{\left(q^2;q^2\right)_\infty} \sum_{n=0}^{+\infty} z^n q^{n^2} \left(q^{2n+2};q^2\right)_\infty.
\end{aligned}$$

Note that

$$\left(q^{2n+2};q^2\right)_\infty = 0,$$

for all $n < 0$ (since the product contains a factor equal to $\left(1 - q^0\right)$). Thus, using again (9) (for $t = -q^{2n+2}$ and $q = q^2$) we get

$$\begin{aligned}
\left(-zq;q^2\right)_\infty &= \frac{1}{\left(q^2;q^2\right)_\infty} \sum_{n=-\infty}^{+\infty} z^n q^{n^2} \left(q^{2n+2};q^2\right)_\infty \\
&= \frac{1}{\left(q^2;q^2\right)_\infty} \sum_{n=-\infty}^{+\infty} z^n q^{n^2} \left(\sum_{m=0}^{+\infty} \frac{\left(-q^{2n+2}\right)^m q^{m(m-1)}}{\left(q^2;q^2\right)_m} \right) \\
&= \frac{1}{\left(q^2;q^2\right)_\infty} \sum_{n=-\infty}^{+\infty} z^n q^{n^2} \left(\sum_{m=0}^{+\infty} \frac{(-1)^m q^{m^2+2nm+m}}{\left(q^2;q^2\right)_m} \right)
\end{aligned}$$

$$= \frac{1}{\left(q^2;q^2\right)_\infty} \sum_{m=0}^{+\infty} \frac{(-1)^m q^m}{\left(q^2;q^2\right)_m} \left(\sum_{n=-\infty}^{+\infty} q^{(m+n)^2} z^n \right).$$

Now shifting the index in the second sum by $-m$ and using the first equality in Corollary 1 (for $t = -\frac{q}{z}$ and $q = q^2$) we get that

$$\begin{aligned}
\left(-zq;q^2\right)_\infty &= \frac{1}{\left(q^2;q^2\right)_\infty} \sum_{m=0}^{+\infty} \frac{(-1)^m q^m z^{-m}}{\left(q^2;q^2\right)_m} \left(\sum_{n=-\infty}^{+\infty} q^{(m+n)^2} z^{m+n} \right) \\
&= \frac{1}{\left(q^2;q^2\right)_\infty} \left(\sum_{m=0}^{+\infty} \frac{(-1)^m q^m z^{-m}}{\left(q^2;q^2\right)_m} \right) \left(\sum_{n=-\infty}^{+\infty} q^{n^2} z^n \right) \\
&= \frac{1}{\left(q^2;q^2\right)_\infty} \frac{1}{\left(-\frac{q}{z};q^2\right)_\infty} \left(\sum_{n=-\infty}^{+\infty} q^{n^2} z^n \right).
\end{aligned}$$

Thus we get the desired result by multiplying both sides of this equality by

$$\left(q^2;q^2\right)_\infty \left(-\frac{q}{z};q^2\right)_\infty.$$

Note that this proof is not complete as soon as one wants analytic identities of the involved series. ■

The following identity is based on a special case of the fundamental lemma (see [And66a, p. 65]) and will be used to prove the next result.

Lemma 2.3. [And66b, p. 576, eq. (I1)] *For $\alpha, \beta, \gamma, \tau \in \mathbb{C}$ such that $\beta \neq 0$ it holds that*

$$\sum_{n=0}^{+\infty} \frac{(\alpha)_n (\beta)_n}{(q)_n (\gamma)_n} \tau^n = \frac{(\beta)_\infty (\alpha\tau)_\infty}{(\gamma)_\infty (\tau)_\infty} \sum_{m=0}^{+\infty} \frac{\left(\frac{\gamma}{\beta}\right)_m (\tau)_m}{(q)_m (\alpha\tau)_m} \beta^m.$$

Proof. [And66b, p. 577 - 578], [And66a, p. 65 - 67] Let $\alpha, \beta, \gamma, \tau \in \mathbb{C}$ such that $\beta \neq 0$. Then using Remark 2.12 we see that

$$\sum_{n=0}^{+\infty} \frac{(\alpha)_n (\beta)_n}{(q)_n (\gamma)_n} \tau^n = \frac{(\beta)_\infty}{(\gamma)_\infty} \sum_{n=0}^{+\infty} \frac{(\alpha)_n (\gamma q^n)_\infty}{(q)_n (\beta q^n)_\infty} \tau^n.$$

Now we apply twice Theorem 2.3, first for $a = \frac{\gamma}{\beta}$ and $z = \beta q^n$ and then for $z = q^m \tau$:

$$\begin{aligned}
\frac{(\beta)_\infty}{(\gamma)_\infty} \sum_{n=0}^{+\infty} \frac{(\alpha)_n (\gamma q^n)_\infty}{(q)_n (\beta q^n)_\infty} \tau^n &= \frac{(\beta)_\infty}{(\gamma)_\infty} \sum_{n=0}^{+\infty} \frac{(\alpha)_n}{(q)_n} \tau^n \left(\sum_{m=0}^{+\infty} \frac{\left(\frac{\gamma}{\beta}\right)_m}{(q)_m} (\beta q^n)^m \right) \\
&= \frac{(\beta)_\infty}{(\gamma)_\infty} \sum_{m=0}^{+\infty} \frac{\left(\frac{\gamma}{\beta}\right)_m}{(q)_m} \beta^m \left(\sum_{n=0}^{+\infty} \frac{(\alpha)_n}{(q)_n} (\tau q^m)^n \right)
\end{aligned}$$

$$= \frac{(\beta)_\infty}{(\gamma)_\infty} \sum_{m=0}^{+\infty} \frac{\left(\frac{\gamma}{\beta}\right)_m (\alpha\tau q^m)_\infty}{(q)_m (\tau q^m)_\infty} \beta^m.$$

Using again the quite useful Remark 2.12 we can conclude that

$$\sum_{n=0}^{+\infty} \frac{(\alpha)_n (\beta)_n}{(q)_n (\gamma)_n} \tau^n = \frac{(\beta)_\infty (\alpha\tau)_\infty}{(\gamma)_\infty (\tau)_\infty} \sum_{m=0}^{+\infty} \frac{\left(\frac{\gamma}{\beta}\right)_m (\tau)_m}{(q)_m (\alpha\tau)_m} \beta^m.$$

■

The following lemma serves well for proving a crucial result concerning the limit of $q-$trinomial coefficients (see Definition 3.4 and Lemma 3.4).

Lemma 2.4. [And66b, p. 576, eq. (I2)] *For $\alpha, \beta, \gamma, \tau \in \mathbb{C}$ such that $\beta, \gamma \neq 0$ it holds that*

$$\sum_{n=0}^{+\infty} \frac{(\alpha)_n (\beta)_n}{(q)_n (\gamma)_n} \tau^n = \frac{(\beta\tau)_\infty \left(\frac{\gamma}{\beta}\right)_\infty}{(\tau)_\infty (\gamma)_\infty} \sum_{n=0}^{+\infty} \frac{(\beta)_n \left(\frac{\alpha\beta\tau}{\gamma}\right)_n}{(q)_n (\beta\tau)_n} \left(\frac{\gamma}{\beta}\right)^n.$$

Proof. [And66b, p. 578] Let $\alpha, \beta, \gamma, \tau \in \mathbb{C}$ such that $\beta, \gamma \neq 0$. Then by lemma 2.3,

$$\sum_{n=0}^{+\infty} \frac{(\alpha)_n (\beta)_n}{(q)_n (\gamma)_n} \tau^n = \frac{(\beta)_\infty (\alpha\tau)_\infty}{(\gamma)_\infty (\tau)_\infty} \sum_{m=0}^{+\infty} \frac{\left(\frac{\gamma}{\beta}\right)_m (\tau)_m}{(q)_m (\alpha\tau)_m} \beta^m. \tag{10}$$

Additionally, by a second application of Lemma 2.3 for $\alpha = \tau, \beta = \frac{\gamma}{\beta}, \gamma = \alpha\tau, \tau = \beta$ it holds that

$$\sum_{m=0}^{+\infty} \frac{(\tau)_m \left(\frac{\gamma}{\beta}\right)_m}{(q)_m (\alpha\tau)_m} \beta^m = \frac{\left(\frac{\gamma}{\beta}\right)_\infty (\tau\beta)_\infty}{(\alpha\tau)_\infty (\beta)_\infty} \sum_{n=0}^{+\infty} \frac{\left(\frac{\alpha\beta\tau}{\gamma}\right)_n (\beta)_n}{(q)_n (\beta\tau)_n} \left(\frac{\gamma}{\beta}\right)^n. \tag{11}$$

Combining (10) and (11) we can conclude that

$$\sum_{n=0}^{+\infty} \frac{(\alpha)_n (\beta)_n}{(q)_n (\gamma)_n} \tau^n = \frac{(\beta\tau)_\infty \left(\frac{\gamma}{\beta}\right)_\infty}{(\tau)_\infty (\gamma)_\infty} \sum_{n=0}^{+\infty} \frac{(\beta)_n \left(\frac{\alpha\beta\tau}{\gamma}\right)_n}{(q)_n (\beta\tau)_n} \left(\frac{\gamma}{\beta}\right)^n.$$

■

2.8. Gaussian polynomials. We will introduce another technical notation belonging to the world of q-analogues that we will use often in the following sections. Generally q-analogues play an important role in combinatorics (see [GR04]).

Definition 2.36. *A formal power series $A(q) \in \mathbb{C}[[q]]$ is called a* ***q-analogue*** *of $B \in \mathbb{C}$, if*

$$\lim_{q \to 1} A(q) = B.$$

As an example and since it is an important notation in Andrews' proofs of Schur's theorem and Caparelli's identity, we introduce a q-analogue of the binomial coefficient. A second example of a q-analog will be presented at the end of Subsection 3.2.

Definition 2.37. [And76, p. 35] *For $n, m \in \mathbb{Z}$, the* ***Gaussian polynomial*** *is defined as*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} (q)_n (q)_m^{-1} (q)_{n-m}^{-1} & \text{if } 0 \leq m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.13. [And76, p. 35 - 36] Indeed, Gaussian polynomials are $q-$analogues of the corresponding binomial coefficients. Note that for $k \in \mathbb{N}$,

$$\lim_{q \to 1} \frac{1 - q^k}{1 - q} = \lim_{q \to 1} \sum_{j=0}^{k-1} q^j = k, \tag{12}$$

where we used the well-known partial summation formula for the geometric series. Let $m, n \in \mathbb{N}$ such that $0 \leq m \leq n$. By definition of Gaussian polynomials (Definition 2.37) it holds that

$$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \lim_{q \to 1} (q)_n (q)_m^{-1} (q)_{n-m}^{-1}.$$

Dividing and multiplying by $(1-q)^n$ yields

$$\lim_{q \to 1} (q)_n (q)_m^{-1} (q)_{n-m}^{-1} = \lim_{q \to 1} \frac{(q)_n}{(1-q)^n} \frac{(1-q)^m}{(q)_m} \frac{(1-q)^{n-m}}{(q)_{n-m}}.$$

Finally, using Equation (12), we can deduce that

$$\lim_{q \to 1} \frac{(q)_n}{(1-q)^n} \frac{(1-q)^m}{(q)_m} \frac{(1-q)^{n-m}}{(q)_{n-m}} = n! \frac{1}{m!} \frac{1}{(n-m)!} = \binom{n}{m}.$$

In the following theorem a few important properties of Gaussian polynomials are presented.

Theorem 2.5. [And76, p. 35] *Let $m, n \in \mathbb{N}$ such that $0 \leq m \leq n$. The Gaussian polynomial*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q$$

is a polynomial of degree $m(n-m)$ in q that satisfies:

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1, \tag{13}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q. \tag{14}$$

Moreover, if $n > 0$ it holds that

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q, \tag{15}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q. \tag{16}$$

Remark 2.14. In [And76, p. 35] it is claimed that Equations (15) and (16) hold also for $n = m = 0$. Apparently this is not true, since by Equation (13)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1,$$

but by Definition 2.37

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}_q + q^0 \begin{bmatrix} -1 \\ -1 \end{bmatrix}_q = 0,$$

and

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}_q + q^0 \begin{bmatrix} -1 \\ 0 \end{bmatrix}_q = 0.$$

Proof. [And76, p. 35 - 36] Let $m, n \in \mathbb{N}$ such that $0 \leq m \leq n$. First we show Equation (13). By Definition 2.37 it holds that

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = (q)_n \, (q)_0^{-1} \, (q)_n^{-1} = 1,$$

and

$$\begin{bmatrix} n \\ n \end{bmatrix}_q = (q)_n \, (q)_n^{-1} \, (q)_0^{-1} = 1.$$

Also Equation (14) follows directly from Definition 2.37. Note that

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = (q)_n \, (q)_m^{-1} \, (q)_{n-m}^{-1} = \begin{bmatrix} n \\ n-m \end{bmatrix}_q.$$

To prove Equation (15), assume that $n > 0$ and $0 < m < n$ and note that by using the definition of Gaussian polynomials,

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix}_q - \begin{bmatrix} n-1 \\ m \end{bmatrix}_q &= \frac{(q)_n}{(q)_m \, (q)_{n-m}} - \frac{(q)_{n-1}}{(q)_m \, (q)_{n-m-1}} \\ &= \frac{(q)_{n-1}}{(q)_m \, (q)_{n-m}} \left((1-q^n) - \left(1 - q^{n-m}\right) \right) \\ &= \frac{(q)_{n-1}}{(q)_m \, (q)_{n-m}} \left(q^{n-m} - q^n \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{(q)_{n-1}}{(q)_m (q)_{n-m}} q^{n-m} (1 - q^m) \\
&= q^{n-m} \frac{(q)_{n-1}}{(q)_{m-1} (q)_{n-m}} \\
&= q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q .
\end{aligned}
$$

If $m = n$, by Equation (13),

$$
\begin{bmatrix} n \\ n \end{bmatrix}_q - \begin{bmatrix} n-1 \\ n \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1 = q^0 \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_q .
$$

Finally, if $m = 0$, again by Equation (13),

$$
\begin{bmatrix} n \\ 0 \end{bmatrix}_q - \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_q = 1 - 1 = 0 = q^n \begin{bmatrix} n-1 \\ -1 \end{bmatrix}_q .
$$

Now, let again $0 \leq m \leq n$ and apply Equation (15), replacing m by $n - m$. Thus

$$
\begin{bmatrix} n \\ n-m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ n-m \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ n-m-1 \end{bmatrix}_q .
$$

Applying Equation (14) we obtain

$$
\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q .
$$

Finally, we prove by induction that

$$
\begin{bmatrix} n \\ m \end{bmatrix}_q
$$

is a polynomial in q of degree $m(n-m)$. For this purpose, note that

$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1
$$

is a polynomial in q of degree 0 with positive coefficient. Now let $n \in \mathbb{N}$ and assume that for all $0 \leq m \leq n$,

$$
\begin{bmatrix} n \\ m \end{bmatrix}_q
$$

is a polynomial of degree $m(n-m)$ in q with positive coefficients. Then, for any $\tilde{m} \in \mathbb{N}$ such that $0 \leq \tilde{m} < n + 1$, by Equation (16),

$$
\begin{bmatrix} n+1 \\ \tilde{m} \end{bmatrix}_q = \begin{bmatrix} n \\ \tilde{m}-1 \end{bmatrix}_q + q^{\tilde{m}} \begin{bmatrix} n \\ \tilde{m} \end{bmatrix}_q
$$

is a polynomial in q such that

$$
\deg \begin{bmatrix} n+1 \\ \tilde{m} \end{bmatrix}_q = \max\{\deg \begin{bmatrix} n \\ \tilde{m}-1 \end{bmatrix}_q, \deg q^{\tilde{m}} \begin{bmatrix} n \\ \tilde{m} \end{bmatrix}_q\}.
$$

Using the assumption, we get that

$$\begin{aligned}\max\{\deg \begin{bmatrix} n \\ \tilde{m}-1 \end{bmatrix}_q, \deg q^{\tilde{m}} \begin{bmatrix} n \\ \tilde{m} \end{bmatrix}_q\} &= \max\{(\tilde{m}-1)(n-\tilde{m}+1), \tilde{m}+\tilde{m}(n-\tilde{m})\} \\ &= \max\{(\tilde{m}-1)(n+1-\tilde{m}), \tilde{m}(n+1-\tilde{m})\} \\ &= \tilde{m}(n+1-\tilde{m}).\end{aligned}$$

To conclude using the axiom of induction (Axiom 2.1), note that by Equation (13),

$$\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_q = 1,$$

is a polynomial in q of degree 0. ■

The following result is crucial for the proof of the generalization of Caparelli's identity in Section 6.

Corollary 2.6. *For $i, j, k \in \mathbb{N}$ such that $i+j>0$, it holds that*

$$\begin{aligned}\begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} &= \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{k+2j} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \\ &+ q^{i+j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{2j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}.\end{aligned}$$

Proof. Let $i, j, k \in \mathbb{N}$ such that $i+j>0$. Note that by Theorem 2.5, Equations (15) and (16), the following four equations hold:

$$\begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q = \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q + q^k \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q, \tag{17}$$

$$\begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q = \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q + q^{i+j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q, \tag{18}$$

$$\begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} = \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}, \tag{19}$$

$$\begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} = \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2}. \tag{20}$$

By Equations (18) and (19) it follows that

$$\begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} = \left(\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q + q^{i+j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \right)$$

$$\cdot \left(\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \right).$$

Multiplying and another application of Equation (18) yields that this is equal to

$$\begin{aligned} &\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{i+j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{2j} \begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}. \end{aligned}$$

Apply Equation (17) to see that this is equal to

$$\begin{aligned} &\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{i+j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{2j} \left(\begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q + q^k \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \right) \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} = &\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{k+2j} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \\ &+ q^{i+j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &+ q^{2j} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}. \end{aligned}$$

■

3. Schur's theorem

3.1. Statement. The content of this subsection is based on [And94, p. 141]. In Schur's theorem two types of partitions are considered. On the one hand we consider partitions that satisfy certain congruence conditions.

Definition 3.1. *For all $n \in \mathbb{N}$, let $A(n)$ denote the number of partitions of n into parts that are congruent to 1 or 5 modulo 6.*

Example 3.1. *For $n = 9$, $A(9) = 3$ counts the partitions*

$$(7,1,1), (5,1,1,1,1), (1,1,1,1,1,1,1,1,1).$$

As a third example to Theorem 2.1 and tool to conclude Andrews' proof of Schur's theorem, we compute the generating function of the partitions counted by $\left(A(n)\right)_{n\in\mathbb{N}}$.

Lemma 3.1. *The generating function of $\left(A(n)\right)_{n\in\mathbb{N}}$ is*

$$\sum_{n=0}^{+\infty} A(n)\, q^n = \prod_{n=0}^{+\infty} \left(1 - q^{6n+5}\right)^{-1} \left(1 - q^{6n+1}\right)^{-1}.$$

Proof. This lemma can be obtained by a direct application of Theorem 2.1 for $H = \{n \in \mathbb{N}^* \mid n \equiv 1, 5 \pmod 6\}$ using the definition of $\left(A(n)\right)_{n\in\mathbb{N}}$. ■

On the other hand we consider partitions satisfying difference conditions.

Definition 3.2. *For all $n \in \mathbb{Z}$, let $B(n)$ denote the number of partitions $(\lambda_i)_{i\in[r_2]}$ of n such that any two consecutive parts satisfy*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 6 & \text{if } 3 \mid \lambda_i \text{ and } 3 \mid \lambda_{i+1}, \\ 3 & \text{otherwise.} \end{cases}$$

Example 3.2. *For $n = 9$, $B(9) = 3$ counts the partitions*

$$(9), (8,1), (7,2).$$

Issai Schur, a Russian mathematician, proved in 1926 in [Sch26] the following partition identity.

Theorem 3.1. *For all $n \in \mathbb{N}$ it holds that*

$$A(n) = B(n).$$

3.2. *q*-trinomial coefficients. At the heart of Andrews' proof of Schur's theorem lie the so-called q-trinomial coefficients which are a q-analog of trinomial coefficients.

Definition 3.3. [And94, p. 142] *Let $n \in \mathbb{N}$. Consider the product $\left(1+x+x^{-1}\right)^n$ and write it in the form*

$$\left(1+x+x^{-1}\right)^n = \sum_{j=-n}^{n} \alpha_n(j)\, x^j, \tag{21}$$

*for $\alpha_n(j) \in \mathbb{N}$ for all $j \in \{-n, ..., n\}$. The $\alpha_n(j)$ are called the **trinomial coefficients** and we denote them by*

$$\binom{n}{j}_2 := \alpha_n(j).$$

For convenience, for $j \in \mathbb{Z}$ such that $j < -n$ or $j > n$, we set

$$\binom{n}{j}_2 := 0. \tag{22}$$

Remark 3.1. [AB87, p. 298] In [And94], Andrews refers to another definition of trinomial coefficients which are obtained considering the polynomial $\left(1+x+x^2\right)^n$, written in the form

$$\left(1+x+x^2\right)^n = \sum_{j=0}^{2n} \gamma_n(j)\, x^j.$$

The relation between the two notions can be seen by shifting the index in (21) by n, resulting in

$$\left(1+x+x^{-1}\right)^n = \sum_{j=0}^{2n} \alpha_n(j-n)\, x^{j-n},$$

and then multiplying on both sides by x^n. Thus for all $j \in \{0\} \cup [2n]$,

$$\gamma_n(j) = \binom{n}{j-n}_2.$$

Trinomial coefficients satisfy the following basic property which reminds of the well-known Chu-Vandermonde's identity for binomial coefficients, which is that for all $m, n, k \in \mathbb{N}$ it holds that

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}.$$

Proposition 3.1. [And94, p. 142] *For all $n \in \mathbb{N}, j \in \{-n, ..., n\}$ it holds that*

$$\binom{n}{j}_2 = \sum_{h=0}^{n} \binom{n}{h}\binom{n-h}{h+j}.$$

There are two simple ways to prove this identity. First we consider a very short combinatorial proof.

Proof (Combinatorial proof). Let $n \in \mathbb{N}$. For all $j \in \{-n, ..., n\}$, to obtain the term x^j when we multiply $\left(1 + x + x^{-1}\right)^n$ we can choose x^{-1} in h out of the n factors and then choose the term x in $h + j$ out of the remaining $n - h$ factors for every arbitrary $h \in \mathbb{N}_{\leq n}$. Thus considering basic combinatorial rules we get that

$$\binom{n}{j}_2 = \sum_{h=0}^{n} \binom{n}{h}\binom{n-h}{h+j}.$$

■

However there is also a straight-forward proof using the binomial theorem.

Proof (using the binomial theorem). Let $n \in \mathbb{N}$. Applying the binomial theorem twice we get that

$$\begin{aligned}\left(1 + t + t^{-1}\right)^n &= \sum_{h=0}^{n} \binom{n}{h} (1+t)^h t^{h-n} \\ &= \sum_{h=0}^{n} \binom{n}{h} \left(\sum_{j=0}^{h} \binom{h}{j} t^j\right) t^{h-n} \\ &= \sum_{h=0}^{n} \sum_{j=0}^{h} \binom{n}{h}\binom{h}{j} t^{j+h-n}.\end{aligned}$$

Then replacing h by $n - h$ and j by $j + h$, it follows that

$$\begin{aligned}\sum_{h=0}^{n} \sum_{j=0}^{h} \binom{n}{h}\binom{h}{j} t^{j+h-n} &= \sum_{h=0}^{n} \sum_{j=0}^{n-h} \binom{n}{h}\binom{n-h}{j} t^{j-h} \\ &= \sum_{h=0}^{n} \sum_{j=-h}^{n-2h} \binom{n}{h}\binom{n-h}{h+j} t^{j}.\end{aligned}$$

Noting that

$$\binom{n-h}{h+j} = 0,$$

if $j < -h$ or $j > n - 2h$, we can deduce that

$$\left(1 + t + t^{-1}\right)^n = \sum_{j=-n}^{n} \sum_{h=0}^{n} \binom{n}{h}\binom{n-h}{h+j} t^j,$$

and therefore by comparing coefficients,

$$\binom{n}{j}_2 = \sum_{h=0}^{n} \binom{n}{h}\binom{n-h}{h+j},$$

for all $j \in \{-n, ..., n\}$. ■

Moreover we have the following symmetry.

Proposition 3.2. [And94, p. 142] *For all* $n \in \mathbb{N}, j \in \mathbb{Z}$,

$$\binom{n}{j}_2 = \binom{n}{-j}_2.$$

Moreover,

$$\binom{n}{n}_2 = \binom{n}{-n}_2 = 1.$$

Proof. Let $n \in \mathbb{N}, j \in \mathbb{Z}$. If $j \notin \{-n, ..., n\}$ then the proposition follows directly from (22). Using (21) and exchanging first t by t^{-1} and then j by $-j$, we get that

$$\sum_{j=-n}^{n} \binom{n}{j}_2 t^j = \left(1 + t + t^{-1}\right) = \sum_{j=-n}^{n} \binom{n}{j}_2 t^{-j} = \sum_{j=-n}^{n} \binom{n}{-j}_2 t^j.$$

Then comparing coefficients yields for all $j \in \{-n, ..., n\}$,

$$\binom{n}{j}_2 = \binom{n}{-j}_2.$$

We can conclude the proof using Proposition 3.1. Note that

$$\binom{n}{n}_2 = \sum_{h=0}^{n} \binom{n}{h}\binom{n-h}{h+n} = \binom{n}{0}\binom{n}{n} = 1.$$

■

Remark 3.2. Recall that for the binomial coefficient there is a recursion formula: Note that

$$\binom{0}{0} = 1,$$

and for $n \in \mathbb{N}^*, j \in \mathbb{N}_{\leq n}$ it holds that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This formula is usually illustrated by the so-called Pascal's triangle:

$$\begin{array}{lccccccccc}
n = 0: & & & & & 1 & & & & \\
n = 1: & & & & 1 & & 1 & & & \\
n = 2: & & & 1 & & 2 & & 1 & & \\
n = 3: & & 1 & & 3 & & 3 & & 1 & \\
n = 4: & 1 & & 4 & & 6 & & 4 & & 1.
\end{array}$$

For the trinomial coefficients there is also a recursion formula which can be illustrated similarly.

Proposition 3.3. *It holds that*

$$\binom{0}{0}_2 = 1,$$

and for $n \in \mathbb{N}_{\geq 1}$ and $j \in \{-n, ..., n\}$ the trinomial coefficients satisfy the recursion

$$\binom{n}{j}_2 = \binom{n-1}{j-1}_2 + \binom{n-1}{j}_2 + \binom{n-1}{j+1}_2.$$

Remark 3.3. This recursion can be illustrated in a similar triangle to Pascal's triangle:

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & 1 & 1 & & & \\
n=2: & & & 1 & 2 & 3 & 2 & 1 & & \\
n=3: & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
n=4: & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1.
\end{array}$$

Proof. By Proposition 3.1 it holds that

$$\binom{0}{0}_2 = \binom{0}{0}\binom{0}{0} = 1.$$

To prove the recursion formula, let $n \in \mathbb{N}_{\geq 1}, j \in \{-n, ..., n\}$. Then, using the definition of the trinomial coefficients,

$$\begin{aligned}
\binom{n}{j}_2 &= [t^j]\left(1 + t + t^{-1}\right)^n \\
&= [t^j]\left(1 + t + t^{-1}\right)\left(1 + t + t^{-1}\right)^{n-1} \\
&= [t^j]\left(1 + t + t^{-1}\right)^{n-1} + [t^j]t\left(1 + t + t^{-1}\right)^{n-1} + [t^j]t^{-1}\left(1 + t + t^{-1}\right)^{n-1} \\
&= [t^j]\left(1 + t + t^{-1}\right)^{n-1} + [t^{j-1}]\left(1 + t + t^{-1}\right)^{n-1} + [t^{j+1}]\left(1 + t + t^{-1}\right)^{n-1} \\
&= \binom{n-1}{j}_2 + \binom{n-1}{j-1}_2 + \binom{n-1}{j+1}_2.
\end{aligned}$$

■

In Andrews' proofs of Schur's theorem and Caparelli's identity we will use a $q-$analog of these trinomial coefficients, the so-called $q-$trinomial coefficients.

Definition 3.4. [And94, p. 142] *For all $n \in \mathbb{N}, A, B \in \mathbb{Z}$, we define*

$$\binom{n; B; q}{A}_2 = \sum_{j=0}^{n} q^{j(j+B)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ j+A \end{bmatrix}.$$

Example 3.3. *The following expressions that will be used in the next subsections are trivial.*

$$\binom{2;-2;q}{-2}_2 = \binom{2;2;q}{2}_2 = \binom{1;-1;q}{-1}_2 = \binom{0;0;q}{0}_2 = \binom{1;0;q}{0}_2 = \binom{1;1;q}{1}_2 = 1.$$

A non-trivial example is

$$\binom{2;0;q}{0}_2 = 1 + q\frac{(q)_2}{(q)_1^2} = 1 + q\frac{1-q^2}{1-q} = 1 + q + q^2.$$

Remark 3.4. Using Remark 2.13, note that for all $n \in \mathbb{N}, A, B \in \mathbb{Z}$,

$$\begin{aligned}\lim_{q\to 1}\binom{n;B;q}{A}_2 &= \lim_{q\to 1}\sum_{j=0}^{n} q^{j(j+B)}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n-j \\ j+A \end{bmatrix} \\ &= \sum_{j=0}^{n}\left(\lim_{q\to 1}\begin{bmatrix} n \\ j \end{bmatrix}\right)\left(\lim_{q\to 1}\begin{bmatrix} n-j \\ j+A \end{bmatrix}\right) \\ &= \sum_{j=0}^{n}\binom{n}{j}\binom{n-j}{j+A}.\end{aligned}$$

Keeping an eye on Proposition 3.1, it becomes clear that indeed, $q-$trinomial coefficients are $q-$analogues of the corresponding trinomial coefficients.

In fact there exists also an analogue to the recursion formula in Proposition 3.3. It turns out that this formula plays a crucial role in Andrews' proof of Schur's theorem.

Lemma 3.2. [And94, p. 144] *For all $n \in \mathbb{N}^*, A \in \mathbb{Z}$, it holds that*

$$\begin{aligned}\binom{n;A;q}{A}_2 = \binom{n-1;A;q}{A}_2 &+ q^{A+n}\binom{n-1;A+1;q}{A+1}_2 \\ &+ q^{n-A}\binom{n-1;A-1;q}{A-1}_2 \\ &+ q^{n-1}\left(1-q^{n-1}\right)\binom{n-2;A;q}{A}_2.\end{aligned} \tag{23}$$

Remark 3.5. The corresponding triangle starts as the following:

$$\begin{array}{lccccc} n=0: & & & 1 & & \\ n=1: & & 1 & & 1 & \\ n=2: & 1 & 1+q & 1+q+q^2 & 1+q & 1. \end{array}$$

Note that letting $q \to 1$ on both sides of (23) and using Remark 3.4 yields that for all $n \in \mathbb{N}^*, A \in \mathbb{Z}$,

$$\binom{n}{A}_2 = \binom{n-1}{A}_2 + \binom{n-1}{A+1}_2 + \binom{n-1}{A-1}_2,$$

which is exactly the recursion in Proposition 3.3.

Proof. [And94, p. 144] We use the following notation. For all $h, n \in \mathbb{N}, A \in \mathbb{Z}$, set

$$T(n, A, h) := q^{h(h+A)} \begin{bmatrix} n \\ h \end{bmatrix} \begin{bmatrix} n-h \\ h+A \end{bmatrix}.$$

With this notation, we have

$$\binom{n; A; q}{A}_2 = \sum_{h=0}^{n} T(n, A, h).$$

Let us prove now that $T(n, A, h)$ can be written as

$$\begin{aligned} T(n-1, A, h) &+ q^{A+n} T(n-1, A+1, h-1) \\ &+ q^{n-A} T(n-1, A-1, h) \\ &+ q^{n-1}\left(1-q^{n-1}\right) T(n-2, A, h-1). \end{aligned} \tag{24}$$

Using the definition of $T(n, A, h)$ this expression rewrites

$$\begin{aligned} q^{h(h+A)} \begin{bmatrix} n-1 \\ h \end{bmatrix} \begin{bmatrix} n-h-1 \\ h+A \end{bmatrix} &+ q^{A+n} q^{(h-1)(h+A)} \begin{bmatrix} n-1 \\ h-1 \end{bmatrix} \begin{bmatrix} n-h \\ h+A \end{bmatrix} \\ &+ q^{n-A} q^{h(h+A-1)} \begin{bmatrix} n-1 \\ h \end{bmatrix} \begin{bmatrix} n-h-1 \\ h+A-1 \end{bmatrix} \\ &+ q^{n-1}\left(1-q^{n-1}\right) q^{(h-1)(h+A-1)} \\ &\quad \cdot \begin{bmatrix} n-2 \\ h-1 \end{bmatrix} \begin{bmatrix} n-h-1 \\ h+A-1 \end{bmatrix}. \end{aligned}$$

To proceed we transform this into the following product in which $T(n, A, h)$ appears as a factor.

$$\begin{aligned} q^{h(h+A)} \Bigg(&\begin{bmatrix} n-1 \\ h \end{bmatrix} \begin{bmatrix} n-h-1 \\ h+A \end{bmatrix} + q^{n-h} \begin{bmatrix} n-1 \\ h-1 \end{bmatrix} \begin{bmatrix} n-h \\ h+A \end{bmatrix} \\ &+ q^{n-A-h} \begin{bmatrix} n-1 \\ h \end{bmatrix} \begin{bmatrix} n-h-1 \\ h+A-1 \end{bmatrix} \\ &+ q^{n-1}\left(1-q^{n-1}\right) q^{-2h-A+1} \\ &\quad \cdot \begin{bmatrix} n-2 \\ h-1 \end{bmatrix} \begin{bmatrix} n-h-1 \\ h+A-1 \end{bmatrix} \Bigg) \end{aligned}$$

$$\begin{aligned} &= q^{h(h+A)} \begin{bmatrix} n \\ h \end{bmatrix} \begin{bmatrix} n-h \\ h+A \end{bmatrix} \\ &\cdot \left(\frac{1-q^{n-2h-A}}{1-q^n} + q^{n-h} \frac{1-q^h}{1-q^n} + q^{n-A-h} \frac{1-q^{h+A}}{1-q^n} + q^{n-2h-A} \frac{\left(1-q^h\right)\left(1-q^{h+A}\right)}{1-q^n} \right). \end{aligned}$$

Extracting $(1-q^n)^{-1}$ in the bracket and realizing that the remaining terms in the bracket sum up to $1-q^n$, we get

$$\begin{aligned}
&\frac{q^{h(h+A)}}{1-q^n}\begin{bmatrix} n \\ h \end{bmatrix}\begin{bmatrix} n-h \\ h+A \end{bmatrix} \\
&\cdot\left(1-q^{n-2h-A}+q^{n-h}-q^n+q^{n-A-h}-q^n+q^{n-2h-A}-q^{n-h-A}-q^{n-h}+q^n\right) \\
= \quad & q^{h(h+A)}\begin{bmatrix} n \\ h \end{bmatrix}\begin{bmatrix} n-h \\ h+A \end{bmatrix} \\
= \quad & T(n,A,h)\,. \qquad (25)
\end{aligned}$$

Therefore, summing (24) and (25) over h,

$$\begin{aligned}
\sum_{h=0}^{n} T(n-1,A,h) &+ q^{A+n}\sum_{h=0}^{n} T(n-1,A+1,h-1) \\
&+ q^{n-A}\sum_{h=0}^{n} T(n-1,A-1,h) \\
&+ q^{n-1}\left(1-q^{n-1}\right)\sum_{h=0}^{n} T(n-2,A,h-1) \\
&= \sum_{h=0}^{n} T(n,A,h) \\
&= \binom{n;A;q}{A}_2 .
\end{aligned}$$

Shifting indices, this is equivalent to

$$\begin{aligned}
\binom{n-1;A;q}{A}_2 + T(n-1,A,h) &+ q^{A+n}\sum_{h=-1}^{n-1} T(n-1,A+1,h) \\
&+ q^{n-A}\binom{n-1;A-1;q}{A-1}_2 \\
&+ q^{n-A}T(n-1,A-1,h) \\
&+ q^{n-1}\left(1-q^{n-1}\right)\sum_{h=-1}^{n-1} T(n-2,A,h) \\
&= \binom{n;A;q}{A}_2 .
\end{aligned}$$

And, using again the definition of (T,n,A,h), we can conclude that

$$\binom{n-1;A;q}{A}_2 + q^{A+n}\binom{n-1;A+1;q}{A+1}_2$$

$$
\begin{aligned}
&+ q^{n-A} \begin{pmatrix} n-1; A-1; q \\ A-1 \end{pmatrix}_2 \\
&+ q^{n-1} \left(1 - q^{n-1}\right) \begin{pmatrix} n-2; A; q \\ A \end{pmatrix}_2 \\
&= \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 .
\end{aligned}
$$

■

Another important variant of the Pascal-type recursion for $q-$trinomial coefficients is the following which is used in Andrews' proof of Caparelli's identity.

Lemma 3.3. [And94, p. 145] *For all $n \in \mathbb{N}^*, A \in \mathbb{Z}$, it holds that*

$$
\begin{aligned}
\begin{pmatrix} n+1; A; q \\ A \end{pmatrix}_2 = (1+q^n) \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 &+ q^{n+A+1} \begin{pmatrix} n-1; A+2; q \\ A+2 \end{pmatrix}_2 \\
&+ q^{n-A+1} \begin{pmatrix} n-1; A-2; q \\ A-2 \end{pmatrix}_2 \\
&+ q^{2n} \begin{pmatrix} n-1; A; q \\ A \end{pmatrix}_2 \\
&+ q^{2n-1} \left(1 - q^{n-1}\right) \begin{pmatrix} n-2, A; q \\ A \end{pmatrix}_2 .
\end{aligned} \tag{26}
$$

In Andrews' proof of Schur's theorem we will use the following statement on the limit of $q-$trinomial coefficients.

Lemma 3.4. [AB87, p. 309, eq. (2.48)] *For all $A \in \mathbb{Z}$ it holds that*

$$
\lim_{n \to +\infty} \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 = (q)_\infty^{-1} .
$$

Proof. [AB87, p. 310] Let $A \in \mathbb{N}$. Then by Definitions 3.4 and 2.37, we see that

$$
\begin{aligned}
\lim_{n \to +\infty} \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 &= \lim_{n \to +\infty} \sum_{j=0}^{n} q^{j(j+A)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ j+A \end{bmatrix} \\
&= \lim_{n \to +\infty} \sum_{j=0}^{n} q^{j(j+A)} \frac{(q)_n (q)_{n-j}}{(q)_j (q)_{n-j} (q)_{j+A} (q)_{n-2j-A}} \\
&= \sum_{j=0}^{+\infty} q^{j(j+A)} \frac{(q)_\infty^2}{(q)_j (q)_{j+A} (q)_\infty^2} \\
&= \sum_{j=0}^{+\infty} \frac{q^{j(j+A)}}{(q)_j (q)_{j+A}} .
\end{aligned} \tag{27}
$$

Let $t \in \mathbb{R} \setminus \{0\}$. Then by Lemma 2.4, setting $\alpha = \beta = t^{-1}, \gamma = q^{A+1}, \tau = t^2 q^{1+A}$,

$$\sum_{n=0}^{+\infty} \frac{\left(t^{-1}\right)_n \left(t^{-1}\right)_n t^{2n} q^{n+An}}{(q)_n \left(q^{A+1}\right)_n} = \frac{\left(tq^{1+A}\right)_\infty \left(tq^{A+1}\right)_\infty}{\left(t^2 q^{1+A}\right)_\infty \left(q^{A+1}\right)_\infty} \sum_{n=0}^{+\infty} \frac{\left(t^{-1}\right)_n \left(q^0\right)_n t^n q^{An+n}}{(q)_n \left(tq^{1+A}\right)_n}. \tag{28}$$

Note that for all $n \in \mathbb{N}^*$, $(1)_n = 0$ and

$$\lim_{t\to 0} t^n \left(t^{-1}\right)_n = \lim_{t\to 0} t^n \prod_{j=0}^{n-1} \left(1 - t^{-1} q^j\right) = \lim_{t\to 0} \prod_{j=0}^{n-1} \left(t - q^j\right) = (-1)^n q^{\frac{n(n-1)}{2}},$$

using Gauss' well-known summation formula, i.e. for all $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} j = \frac{n\,(n+1)}{2}.$$

Hence, letting $t \to 0$ in (28), we get

$$\sum_{n=0}^{+\infty} \frac{q^{n^2+An}}{(q)_n \left(q^{A+1}\right)_n} = \frac{1}{\left(q^{A+1}\right)_\infty}.$$

Multiplying on both sides by $(q)_A^{-1}$ yields

$$\sum_{n=0}^{+\infty} \frac{q^{n^2+An}}{(q)_n (q)_{A+n}} = \frac{1}{(q)_\infty},$$

and thus

$$\lim_{n\to+\infty} \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 = (q)_\infty^{-1}. \tag{29}$$

Let $A \in \mathbb{Z}_{<0}$. Then note that for all $n \in \mathbb{N}$

$$\begin{bmatrix} n-j \\ j+A \end{bmatrix} = 0,$$

if $j \in \mathbb{N}$ such that $j < -A$. Thus using the index shift $j \leftarrow j - A$ and Definition 2.37,

$$\begin{aligned}
\lim_{n\to+\infty} \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 &= \lim_{n\to+\infty} \sum_{j=-A}^{n} q^{j(j+A)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ j+A \end{bmatrix} \\
&= \lim_{n\to+\infty} \sum_{j=0}^{n+A} q^{j(j-A)} \begin{bmatrix} n \\ j-A \end{bmatrix} \begin{bmatrix} n-j+A \\ j \end{bmatrix} \\
&= \lim_{n\to+\infty} \sum_{j=0}^{n+A} q^{j(j-A)} \frac{(q)_n (q)_{n-j+A}}{(q)_{j-A} (q)_{n-j+A} (q)_j (q)_{n-2j+A}} \\
&= \sum_{j=0}^{+\infty} q^{j(j-A)} \frac{(q)_\infty^2}{(q)_{j-A} (q)_j (q)_\infty^2}
\end{aligned}$$

$$= \sum_{j=0}^{+\infty} \frac{q^{j(j-A)}}{(q)_{j-A}\,(q)_j}.$$

Using (27) and (29) we can conclude that

$$\lim_{n\to+\infty} \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 = \lim_{n\to+\infty} \begin{pmatrix} n; -A; q \\ -A \end{pmatrix}_2 = (q)_\infty^{-1}.$$

■

As a further example for a q-analog and application of $q-$trinomial coefficients, we will present a q-analog of

$$\left(\sum_{j=0}^{k} x^j\right)^n, \tag{30}$$

for $x \in \mathbb{C}$, $k, n \in \mathbb{N}$.

Definition 3.5. [And94, p. 152] *For $n \in \mathbb{N}, x \in \mathbb{C}$ we define*

$$p_{0,n}(x) := 1,$$

and for $k \in \mathbb{N}^*$,

$$p_{k,n}(x) := \sum_{j=0}^{n} x^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q p_{k-1,j}\left(xq^n\right).$$

Here, we use the convention that for all $j \in \mathbb{N}$,

$$\binom{j}{2} := \frac{j\,(j-1)}{2}.$$

Remark 3.6. By letting $q \to 1$, we can prove that $p_{k,n}(x)$ is indeed a q-analog for (30), for all $k, n \in \mathbb{N}, x \in \mathbb{C}$ ([And94, p. 152]). We do a proof by induction on k.
Note that for all $n \in \mathbb{N}$,

$$\lim_{q\to1} p_{0,n}(x) = 1 = \left(\sum_{j=0}^{0} x^j\right)^n.$$

Assume that for some fixed $k \in \mathbb{N}$, for all $n \in \mathbb{N}$ it holds that

$$\lim_{q\to1} p_{k,n}(x) = \left(\sum_{j=0}^{k} x^j\right)^n. \tag{31}$$

Note that for all $n \in \mathbb{N}$,

$$\lim_{q\to1} p_{k+1,n}(x) = \sum_{j=0}^{n} x^j 1^{\binom{j}{2}} \left(\lim_{q\to1} \begin{bmatrix} n \\ j \end{bmatrix}\right) \left(\lim_{q\to1} p_{k,j}\left(x1^n\right)\right).$$

Using Remark 2.13 and Equation (31), we see that

$$\sum_{j=0}^{n} x^j 1^{\binom{j}{2}} \left(\lim_{q\to 1} \begin{bmatrix} n \\ j \end{bmatrix}\right) \left(\lim_{q\to 1} p_{k,j}\left(x1^n\right)\right) = \sum_{j=0}^{n} x^j \binom{n}{j} \left(\lim_{q\to 1} p_{k,j}\left(x\right)\right)$$
$$= \sum_{j=0}^{n} x^j \binom{n}{j} \left(\sum_{h=0}^{k} x^h\right)^j.$$

By multiplying and shifting the index h by 1, we can deduce that

$$\begin{aligned}\sum_{j=0}^{n} x^j \binom{n}{j} \left(\sum_{h=0}^{k} x^h\right)^j &= \sum_{j=0}^{n} \binom{n}{j} \left(\sum_{h=0}^{k} x^{h+1}\right)^j \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(\sum_{h=1}^{k+1} x^h\right)^j \\ &= \left(1 + \sum_{h=1}^{k+1} x^h\right)^n \\ &= \left(\sum_{h=0}^{k+1} x^h\right)^n,\end{aligned}$$

where we used the well-known binomial theorem, i.e. for all $a, b \in \mathbb{C}, n \in \mathbb{N}$,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

By mathematical induction (Axiom 2.1) the claim follows. ■

We have the following form for the case $k = 1$.

Proposition 3.4. [And94, eq. (6.3)] *For all $n \in \mathbb{N}, x \in \mathbb{C}$,*

$$p_{1,n}\left(x\right) = \sum_{j=0}^{n} x^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}.$$

Proof. This follows directly by Definition 3.5. For all $n \in \mathbb{N}, x \in \mathbb{C}$,

$$\begin{aligned}p_{1,n}\left(x\right) &= \sum_{j=0}^{n} x^j q^{\binom{n}{j}} \begin{bmatrix} n \\ j \end{bmatrix} p_{0,j}\left(xq^n\right) \\ &= \sum_{j=0}^{n} x^j q^{\binom{n}{j}} \begin{bmatrix} n \\ j \end{bmatrix}.\end{aligned}$$

■

Furthermore, in [And94, eq. (6.4)], Andrews claims that for all $n \in \mathbb{N}$,

$$p_{2,n}(x) = \sum_{j=-n}^{n} x^{j+n} q^{\binom{j}{2}} \binom{n; j; q}{j}_2. \tag{32}$$

But this turns out to be wrong. Applying Definition 3.5 and Proposition 3.4, it holds that

$$\begin{aligned} p_{2,1}(x) &= \sum_{j=0}^{1} x^j q^{\binom{j}{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} p_{1,j}(xq) \\ &= p_{1,0}(xq) + x p_{1,1}(xq) \\ &= \sum_{j=0}^{0} x^j q^{j+\binom{j}{2}} \begin{bmatrix} 0 \\ j \end{bmatrix} + \sum_{j=0}^{1} x^{j+1} q^{j+\binom{j}{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} \\ &= 1 + x + x^2 q. \end{aligned}$$

But assuming (32) was correct, we get a contradiction since

$$\begin{aligned} p_{2,1}(x) &= \sum_{j=-1}^{1} x^{j+1} q^{\binom{j}{2}} \binom{1; j; q}{j}_2 \\ &= q \binom{1; -1; q}{-1}_2 + x \binom{1; 0; q}{0}_2 + x^2 \binom{1; 1; q}{1}_2 \\ &= q + x + x^2, \end{aligned}$$

using Example 3.3. So (32) is not consistent with Definition 3.5.

Instead, the following holds

Proposition 3.5. *For all $n \in \mathbb{N}, x \in \mathbb{C}$,*

$$p_{2,n}(x) = \sum_{j=-n}^{n} x^{j+n} q^{nj+\binom{n}{2}+\binom{j}{2}} \binom{n; j; q}{j}_2.$$

Proof. Using Definition 3.5, for all $n \in \mathbb{N}, x \in \mathbb{C}$,

$$p_{2,n}(x) = \sum_{j=0}^{n} x^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} p_{1,j}(xq^n).$$

With Proposition 3.4,

$$\begin{aligned} p_{2,n}(x) &= \sum_{j=0}^{n} x^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \left(\sum_{h=0}^{j} x^h q^{nh+\binom{h}{2}} \begin{bmatrix} j \\ h \end{bmatrix} \right) \\ &= \sum_{j=0}^{n} \sum_{h=0}^{j} x^{j+h} q^{nh+\binom{j}{2}+\binom{h}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} j \\ h \end{bmatrix}. \end{aligned}$$

Replacing h by $h-j+n$ and j by $n-j$,

$$\begin{aligned} p_{2,n}(x) &= \sum_{j=0}^{n} \sum_{h=j-n}^{2j-n} x^{h+n} q^{n(h-j+n)+\binom{j}{2}+\binom{h-j+n}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} j \\ h-j+n \end{bmatrix} \\ &= \sum_{j=0}^{n} \sum_{h=-j}^{n-2j} x^{h+n} q^{n(h+j)+\binom{n-j}{2}+\binom{h+j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ h+j \end{bmatrix}. \end{aligned}$$

Noting that $h > n-2j$ implies that $h+j > n-j$, $h < -j$ implies that $h+j < 0$ and

$$\begin{aligned} & n(h+j) + \binom{n-j}{2} + \binom{h+j}{2} \\ &= nh + nj + \frac{n^2 - 2nj - n + j^2 + j + h^2 + 2hj - h + j^2 - j}{2} \\ &= nh + \binom{n}{2} + \binom{h}{2} + j(j+h), \end{aligned}$$

it follows that

$$\begin{aligned} p_{2,n}(x) &= \sum_{h=-n}^{n} \sum_{j=0}^{n} x^{h+n} q^{nh+\binom{n}{2}+\binom{h}{2}+j(j+h)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ h+j \end{bmatrix} \\ &= \sum_{h=-n}^{n} x^{h+n} q^{nh+\binom{n}{2}+\binom{h}{2}} \left(\sum_{j=0}^{n} q^{j(j+h)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ h+j \end{bmatrix} \right) \\ &= \sum_{h=-n}^{n} x^{h+n} q^{nh+\binom{n}{2}+\binom{h}{2}} \binom{n; h; q}{h}_2. \end{aligned}$$

■

3.3. Preparation for Andrews' proof of Schur's theorem. As mentioned before, in Andrews' proof of Schur's theorem generating functions are considered. More precisely we will consider bivariate generating functions that keep track of the size of a partition but also the difference between the numbers of parts of different types. For this purpose we refine the notion of $B(n)$.

Definition 3.6. [And94, p. 143] *For all $n \in \mathbb{Z}, h, j \in \mathbb{Z}$, let $B_n(h,j)$ denote the number of partitions $(\lambda_i)_{i\in[r]}$ that are counted by $B(j)$ such that the largest part is bounded by n and*

$$|\{i \in [r] \mid \lambda_i \equiv 1 \pmod 3\}| - |\{i \in [r] \mid \lambda_i \equiv 2 \pmod 3\}| = h.$$

Example 3.4. *For $n = 7$, $B_7(0,9) = 1$ counts the partition*

$$(7,2).$$

In the following we introduce the before mentioned bivariate generating function.

Definition 3.7. [And94, p. 143] *For $n \in \mathbb{Z}$ we denote*

$$S_n(t,q) := \sum_{h=-\infty}^{\infty} \sum_{j=0}^{\infty} B_n(h,j)\, t^h q^j.$$

Example 3.5. *For $n = 2$,*

$$S_2(t,q) = 1 + tq + t^{-1}q^2,$$

referring to $\emptyset$, (1) and (2).

Next we consider a theorem about a rearrangement of this generating function for $n \equiv 2 \pmod 3$. From this Schur's theorem can be deduced as a corollary.

Theorem 3.2. [And94, p. 143] *For all $n \in \mathbb{N}$,*

$$S_{3n-1}(t,q) = \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n; j; q^3 \\ j \end{pmatrix}_2. \tag{33}$$

Example 3.6. *For $n = 1$, using Example 3.3,*

$$\begin{aligned} \sum_{j=-1}^{1} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} 1; j; q^3 \\ j \end{pmatrix}_2 &= q^2 t^{-1} \begin{pmatrix} 1; -1; q^3 \\ -1 \end{pmatrix}_2 + \begin{pmatrix} 1; 0; q^3 \\ 0 \end{pmatrix}_2 + qt \begin{pmatrix} 1; 1; q^3 \\ 1 \end{pmatrix}_2 \\ &= q^2 t^{-1} + 1 + qt \\ &= S_2(t,q). \end{aligned} \tag{34}$$

In order to find a recurrence for $(S_{3n-1})_{n\in\mathbb{N}}$ to prove Theorem 3.2 we need the following lemma.

Lemma 3.5. [And94, p. 145] *For all $n \in \mathbb{N}$ the following equations hold*

$$S_{3n-2}(t,q) = S_{3n-3}(t,q) + tq^{3n-2} S_{3n-5}(t,q), \tag{35}$$

$$S_{3n-1}(t,q) = S_{3n-2}(t,q) + t^{-1}q^{3n-1} S_{3n-4}(t,q), \tag{36}$$

$$S_{3n}(t,q) = S_{3n-1}(t,q) + q^{3n} S_{3n-4}(t,q). \tag{37}$$

Proof. [And94, p. 145 - 146] Let $h \in \mathbb{Z}, n, j \in \mathbb{N}$. We show Equation (35). Equations (36) and (37) can be shown in the exact same way. Denote by $P_{3n-2}(h,j)$ the set of partitions counted by $B_{3n-2}(h,j)$. Moreover, define the two following subset of $P_{3n-2}(h,j)$ considering the largest part of the partitions contained in this set:

$$P^{\mathcal{L}}_{3n-2}(h,j) := \{(\lambda_i)_{i\in[r]} \in P_{3n-2}(h,j) \mid \lambda_1 = 3n-2\}.$$

Note that

$$P_{3n-2}(h,j) = P_{3n-3}(h,j) \sqcup P^{\mathcal{L}}_{3n-2}(h,j).$$

First, we show that

$$|P^{\mathcal{L}}_{3n-2}(h,j)| = |P_{3n-5}(h-1, j-3n+2)|.$$

For this purpose define

$$\begin{aligned} \varphi: \ P^{\mathcal{L}}_{3n-2}(h,j) &\to P_{3n-5}(h-1, j-3n+2) \\ \lambda &\mapsto \varphi(\lambda), \end{aligned}$$

where $\varphi(\lambda)$ is obtained from λ by removing the largest part. We show that φ is **well-defined**. To see this let $\lambda = (\lambda_i)_{i\in[r]} \in P^{\mathcal{L}}_{3n-2}(h,j)$. Then $\varphi(\lambda)$ is a (possibly empty) partition of size

$$\sum_{i=2}^{r} \lambda_i = \left(\sum_{i=1}^{r} \lambda_i\right) - \lambda_1 = j - (3n-2) = j - 3n + 2,$$

satisfying the same difference condition as λ. By the difference condition,

$$\lambda_2 \leq \lambda_1 - 3 = (3n-2) - 3.$$

Since $3n - 2 \equiv 1 \pmod 3$ the difference h changes and it follows that

$$\varphi(\lambda) \in P_{3n-5}(h-1, j-3n+2).$$

We prove now that φ is a **bijection**. Injectivity follows directly by definition of φ and by noting that all partitions in $P^{\mathcal{L}}_{3n-2}(h,j)$ have the same largest part. For seeing surjectivity consider a partition

$$\lambda = (\lambda_i)_{i\in[s]} \in P_{3n-5}(h-1, j-3n+2).$$

Then define $\mu = (\mu_i)_{i\in[s+1]} \in P^{\mathcal{L}}_{3n-2}(h,j)$ by $\mu_1 := 3n-2$ and $\mu_j := \lambda_{j-1}$ for all $j \in [s+1] \setminus \{1\}$ and note that $\varphi(\mu) = \lambda$.
Therefore it holds that

$$\begin{aligned} B_{3n-2}(h,j) &= |P_{3n-2}(h,j)| \\ &= |P_{3n-3}(h,j) \sqcup P^{\mathcal{L}}_{3n-2}(h,j)| \\ &= |P_{3n-3}(h,j)| + |P^{\mathcal{L}}_{3n-2}(h,j)| \\ &= |P_{3n-3}(h,j)| + |P_{3n-5}(h-1, j-3n+2)| \\ &= B_{3n-3}(h,j) + B_{3n-5}(h-1, j-3n+2). \end{aligned}$$

By summing on both extremes, we can deduce that

$$\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} B_{3n-2}(h,j)\,t^h q^j = \sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} B_{3n-3}(h,j)\,t^h q^j + \sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} B_{3n-5}(h-1,j-3n+2)\,t^h q^j.$$

Using Definition 3.7 this is equivalent to

$$S_{3n-2}(t,q) = S_{3n-3}(t,q) + \sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} B_{3n-5}(h,j)\,t^{h+1}q^{j+3n-2}.$$

Therefore we reach

$$S_{3n-2}(t,q) = S_{3n-3}(t,q) + tq^{3n-2}S_{3n-5}(t,q).$$

■

In the next lemma, a recurrence for $(S_{3n-1})_{n\in\mathbb{N}}$ is derived from the three equations in Lemma 3.5.

Lemma 3.6. [And94, p. 146] *The sequence* $\left(S_{3n-1}(t,q)\right)_{n\in\mathbb{N}}$ *satisfies the following recurrence. For all* $n \in \mathbb{N}_{\geq 2}$,

$$S_{3n-1}(t,q) = \left(1 + t^{-1}q^{3n-1} + tq^{3n-2}\right) S_{3n-4}(t,q) + q^{3n-3}\left(1 - q^{3n-3}\right) S_{3n-7}(t,q),$$

and the initial values are

$$\begin{aligned} S_{-1}(t,q) &= 1, \\ S_2(t,q) &= 1 + tq + t^{-1}q^2. \end{aligned} \tag{38}$$

Proof. [And94, p. 146] Let $n \in \mathbb{N}^*$.

Step 1: First we want to get rid of the term $S_{3n-3}(t,q)$ in Equation (35). By (37),

$$S_{3n-3}(t,q) = S_{3n-4}(t,q) + q^{3n-3}S_{3n-7}(t,q). \tag{39}$$

Substitute (39) in (35) to get

$$S_{3n-2}(t,q) = S_{3n-4}(t,q) + q^{3n-3}S_{3n-7}(t,q) + tq^{3n-2}S_{3n-5}(t,q). \tag{40}$$

Step 2: Next we prepare for substituting $S_{3n-2}(t,q)$. Rewriting (36) yields

$$S_{3n-2}(t,q) = S_{3n-1}(t,q) - t^{-1}q^{3n-1}S_{3n-4}(t,q). \tag{41}$$

Step 3: Use (41) to derive a recurrence for $\left(S_{3n-1}(t,q)\right)_{n\in\mathbb{N}}$ from (40). Substituting the expressions for $S_{3n-2}(t,q)$ and $S_{3n-5}(t,q)$ from (41) in (40) yields

$$S_{3n-1}(t,q) - t^{-1}q^{3n-1}S_{3n-4}(t,q) = S_{3n-4}(t,q) + q^{3n-3}S_{3n-7}(t,q) + tq^{3n-2}\left(S_{3n-4}(t,q) - t^{-1}q^{3n-4}S_{3n-7}(t,q)\right).$$

This is equivalent to

$$S_{3n-1}(t,q) = \left(1 + t^{-1}q^{3n-1} + tq^{3n-2}\right) S_{3n-4}(t,q) + q^{3n-3}\left(1 - q^{3n-3}\right) S_{3n-7}(t,q).$$

Step 4: Note that the two initial values are

$$S_{-1}(t,q) = \sum_{h=-\infty}^{\infty} \sum_{j=0}^{\infty} B_{-1}(h,j)\, t^h q^j = B_{-1}(0,0) = 1,$$

and

$$S_2(t,q) = 1 + tq + t^{-1}q^2,$$

by Example 3.6. ■

To prove Theorem 3.2 we show that the right hand side in (33) satisfies the same recurrence. For this purpose we consider the following definition.

Definition 3.8. [And94, p. 146] *For all $n \in \mathbb{N}$ define*

$$T_n(t,q) := \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n; j; q^3 \\ j \end{pmatrix}_2.$$

Example 3.7. *(i) For $n = 0$,*

$$T_0(t,q) = q^0 t^0 \begin{pmatrix} 0; 0; q^3 \\ 0 \end{pmatrix}_2 = 1. \tag{42}$$

(ii) For $n = 1$, as seen in (34),

$$T_1(t,q) = 1 + qt + q^2 t^{-1}. \tag{43}$$

To proceed with the proof of Theorem 3.2, note that the recurrence in Lemma 3.6 determines $\left(S_{3n-1}(t,q)\right)_{n\in\mathbb{N}}$ uniquely.

Proof (Theorem 3.2). [And94, p. 146] From (42) and (43) it is clear that $\left(T_n(t,q)\right)_{n\in\mathbb{N}}$ satisfies the initial conditions in (38). Let $n \in \mathbb{N}^*, j \in \mathbb{Z}$. Then, note that by Lemma 3.2,

$$\begin{aligned}\begin{pmatrix} n; j; q^3 \\ j \end{pmatrix}_2 = \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 &+ q^{3(n+j)} \begin{pmatrix} n-1; j+1; q^3 \\ j+1 \end{pmatrix}_2 \\ &+ q^{3(n-j)} \begin{pmatrix} n-1; j-1; q^3 \\ j-1 \end{pmatrix}_2 \\ &+ q^{3(n-1)}\left(1 - q^{3(n-1)}\right) \begin{pmatrix} n-2; j; q^3 \\ j \end{pmatrix}_2.\end{aligned}$$

Therefore

$$\begin{aligned}T_n(t,q) &= \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n; j; q^3 \\ j \end{pmatrix}_2 \\ &= \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}+3n+3j} t^j \begin{pmatrix} n-1; j+1; q^3 \\ j+1 \end{pmatrix}_2 \\
&+ \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}+3n-3j} t^j \begin{pmatrix} n-1; j-1; q^3 \\ j-1 \end{pmatrix}_2 \\
&+ \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}+3n-3} \left(1-q^{3n-3}\right) t^j \begin{pmatrix} n-2; j; q^3 \\ j \end{pmatrix}_2 .
\end{aligned}
$$

Note that for all $A, B \in \mathbb{Z}$ such that $A < -n$ or $A > n$, by Definition 3.4,

$$
\begin{pmatrix} n; B; q \\ A \end{pmatrix}_2 = 0.
$$

Thus, replacing j by $j-1$ in the second summand and j by $j+1$ in the third summand and using Definition 3.8, we get that

$$
\begin{aligned}
T_n(t,q) = &\sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ \sum_{j=-n+1}^{n+1} q^{\frac{(j-1)(3j-4)}{2}+3n+3j-3} t^{j-1} \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ \sum_{j=-n+1}^{n-1} q^{\frac{(j+1)(3j+2)}{2}+3n-3j-3} t^{j+1} \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ \sum_{j=-(n-2)}^{n-2} q^{\frac{j(3j-1)}{2}+3n-3} \left(1-q^{3n-3}\right) t^j \begin{pmatrix} n-2; j; q^3 \\ j \end{pmatrix}_2 \\
= &\, T_{n-1}(t,q) \\
&+ \sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j-4)}{2}+3n-1+\frac{6j-4-3j+4}{2}} t^{j-1} \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ \sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j+2)}{2}+3n-2-\frac{6j+2-3j-2}{2}} t^{j+1} \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ q^{3n-3} \left(1-q^{3n-3}\right) \sum_{j=-(n-2)}^{n-2} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n-2; j; q^3 \\ j \end{pmatrix}_2 .
\end{aligned}
$$

Then by using again Definition 3.8, we get that

$$
T_n(t,q) = T_{n-1}(t,q)
$$

$$
\begin{aligned}
&+ \sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j-1)}{2}+3n-1} t^{j-1} \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ \sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j-1)}{2}+3n-2} t^{j+1} \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ q^{3n-3} \left(1 - q^{3n-3}\right) T_{n-2}(t,q) \\
= T_{n-1}(t,q) \\
&+ q^{3n-1} t^{-1} \sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ q^{3n-2} t \sum_{j=-(n-1)}^{n-1} q^{\frac{j(3j-1)}{2}} t^j \begin{pmatrix} n-1; j; q^3 \\ j \end{pmatrix}_2 \\
&+ q^{3n-3} \left(1 - q^{3n-3}\right) T_{n-2}(t,q) \\
= \left(1 + q^{3n-1} t^{-1} + q^{3n-2} t\right) T_{n-1}(t,q) + q^{3n-3} \left(1 - q^{3n-3}\right) T_{n-2}(t,q).
\end{aligned}
$$

Therefore it follows that for all $n \in \mathbb{N}$,

$$S_{3n-1}(t,q) = T_n(t,q),$$

which proves Theorem 3.2. ■

3.4. Andrews' Proof of Schur's theorem. Using the material in the last subsection and Jacobi's triple product identity (Theorem 2.4), Andrews proved Schur's theorem by showing that the generating function of the partitions counted by $\left(A\left(n\right)\right)_{n\in\mathbb{N}}$ equals the generating function of the partitions counted by $\left(B\left(n\right)\right)_{n\in\mathbb{N}}$.

Proof (Theorem 3.1). [And94, p. 147] Using the definitions of $B_n\left(h,m\right)$ and $S_n\left(t,q\right)$, Theorem 3.2 and Lemma 3.4 we see that

$$\begin{aligned}
\sum_{m=0}^{+\infty} B\left(m\right) q^m &= \sum_{m=0}^{+\infty} \sum_{h=-\infty}^{+\infty} \lim_{n\to+\infty} B_n\left(h,m\right) 1^h q^m \\
&= \lim_{n\to+\infty} \sum_{m=0}^{+\infty} \sum_{h=-\infty}^{+\infty} B_n\left(h,m\right) 1^h q^m \\
&= \lim_{n\to+\infty} S_n\left(1,q\right) \\
&= \lim_{n\to+\infty} S_{3n-1}\left(1,q\right) \\
&= \lim_{n\to+\infty} \sum_{j=-n}^{n} q^{\frac{j(3j-1)}{2}} \begin{pmatrix} n;j;q^3 \\ j \end{pmatrix}_2 \\
&= \sum_{j=-\infty}^{+\infty} q^{\frac{j(3j-1)}{2}} \lim_{n\to+\infty} \begin{pmatrix} n;j;q^3 \\ j \end{pmatrix}_2 \\
&= \frac{\sum_{j=-\infty}^{+\infty} q^{\frac{j(3j-1)}{2}}}{\left(q^3,q^3\right)_\infty}.
\end{aligned}$$

Then by Jacobi's triple product identity (Theorem 2.4) applied for z replaced by $q^{-\frac{1}{2}}$ and q replaced by $q^{\frac{3}{2}}$,

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} \left(q^{\frac{3}{2}}\right)^{j^2} \left(q^{-\frac{1}{2}}\right)^j &= \prod_{n=0}^{+\infty} \left(1-\left(q^{\frac{3}{2}}\right)^{2n+2}\right)\left(1+\left(q^{-\frac{1}{2}}\right)\left(q^{\frac{3}{2}}\right)^{2n+1}\right)\left(1+q^{\frac{1}{2}}\left(q^{\frac{3}{2}}\right)^{2n+1}\right) \\
&= \prod_{n=0}^{+\infty} \left(1-q^{3n+3}\right)\left(1+q^{3n+1}\right)\left(1+q^{3n+2}\right).
\end{aligned}$$

Note that

$$\prod_{n=0}^{+\infty} \left(1-q^{3n+3}\right) = \prod_{n=1}^{+\infty} \left(1-q^{3n}\right) = \left(q^3,q^3\right)_\infty.$$

Thus we get, using Lemma 3.1,

$$\begin{aligned}
\sum_{m=0}^{+\infty} B\left(m\right) q^m &= \prod_{n=0}^{+\infty} \left(1+q^{3n+1}\right)\left(1+q^{3n+2}\right) \\
&= \prod_{n=0}^{+\infty} \left(1+q^{3n+1}\right)\frac{1-q^{3n+1}}{1-q^{3n+1}}\left(1+q^{3n+2}\right)\frac{1-q^{3n+2}}{1-q^{3n+2}}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{n=0}^{+\infty} \frac{\left(1-q^{6n+4}\right)\left(1-q^{6n+2}\right)}{\left(1-q^{3n+1}\right)\left(1-q^{3n+2}\right)} \\
&= \prod_{n=0}^{+\infty} \left(1-q^{6n+5}\right)^{-1} \left(1-q^{6n+1}\right)^{-1} \\
&= \sum_{m=0}^{+\infty} A\left(m\right) q^{m}.
\end{aligned}
$$

Comparing the coefficients on both extremes of the last computation, we get Schur's theorem. ■

4. Capparelli's identity

With the method Andrews used to prove Schur's theorem, he also proved Capparelli's conjecture. The statement of this conjecture is slightly more involved since it has the additional restriction for the second considered type of partitions that all parts should be greater than 1. However, the proof uses the exact same ideas as presented in the last section; as a crucial tool for the conclusion we will use again Lemma 3.4 which states that

$$\lim_{n\to+\infty} \begin{pmatrix} n; A; q \\ A \end{pmatrix}_2 = (q)_\infty^{-1},$$

for all $A \in \mathbb{Z}$.

4.1. Statement. The content of this subsection is based on [And94, p. 143]. Capparelli's identity is another Rogers-Ramanujan type identity where two different types of partitions are related. On the one hand we consider partitions that satisfy certain congruence conditions, this time with respect to division by 12. On the other hand we will face a difference condition that is similar to the one we set in Schur's theorem, again involving a case distinction with respect to division by 3.

Definition 4.1. *For all $n \in \mathbb{N}$, let $C(n)$ denote the number of partitions $(\lambda_i)_{i\in[r]}$ of n into parts that are congruent to $2, 3, 9$ or 10 modulo 12.*

Example 4.1. *For $n = 9$, $C(9) = 3$ counts the partitions*

$$(9), (3,3,3), (3,2,2,2).$$

As a fourth example to Theorem 2.1 and tool to conclude Andrews' proof of Capparelli's identity we compute the generating function of the partitions counted by $\big(C(n)\big)_{n\in\mathbb{N}}$.

Lemma 4.1. *The generating function of $\big(C(n)\big)_{n\in\mathbb{N}}$ is*

$$\sum_{n=0}^{+\infty} C(n)\, q^n = \prod_{n=0}^{+\infty} \left(1-q^{6n+3}\right)^{-1} \left(1-q^{12n+2}\right)^{-1} \left(1-q^{12n+10}\right)^{-1}.$$

Proof. This lemma can be obtained by another direct application of Theorem 2.1 for $H = \{n \in \mathbb{N}^* \mid n \equiv 2, 3, 9, 10 \pmod{12}\}$ using the definition of $\big(C(n)\big)_{n\in\mathbb{N}}$. ■

We introduce the difference conditions for the second type of partitions in the same way as we did it for Schur's theorem.

Definition 4.2. *For all $n \in \mathbb{Z}$, let $D(n)$ denote the number of partitions $(\lambda_i)_{i\in[r]}$ of n such that $\lambda_i > 1$ for all $i \in [r]$ and any two consecutive parts satisfy*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 & \text{if } 3 \mid \lambda_i + \lambda_{i+1}, \\ 4 & \text{otherwise.} \end{cases}$$

Example 4.2. *For $n = 9$, $D(9) = 3$ counts the partitions*

$$(9), (7,2), (6,3).$$

The Italian mathematician Stefano Capparelli who is currently a professor in Rome conjectured the following identity in his dissertation [Cap88].

Theorem 4.1. *For all $n \in \mathbb{N}$ it holds that*

$$C(n) = D(n).$$

4.2. Preparation for Andrews' proof of Capparelli's identity. As already mentioned, Andrews' proof of Caparelli's identity follows the same idea, method and structure as his proof of Schur's theorem. Therefore we will proceed with the same preparation steps and thus again, for the purpose of considering recursive formulas for specific generating functions, we refine the notion of $D(n)$. Besides the size of the partitions, it takes an upper bound on the parts (which we will need for finding the recursions) and a measure for the balance between numbers of parts of different residual classes into account.

Definition 4.3. [And94, p. 147] *For all $n \in \mathbb{Z}, h, j \in \mathbb{Z}$, let $D_n(h,j)$ denote the number of partitions $(\lambda_i)_{i\in[r]}$ that are counted by $D(j)$ such that the largest part is bounded by n and*

$$|\{i \in [r] \mid \lambda_i \equiv 1 \pmod 3\}| - |\{i \in [r] \mid \lambda_i \equiv 2 \pmod 3\}| = h.$$

Example 4.3. *$D_6(0,9) = 1$ counts the partition*

$$(6,3).$$

Again we introduce the corresponding bivariate generating function involving this new notation.

Definition 4.4. [And94, p. 147] *For $n \in \mathbb{Z}$ we denote*

$$U_n(t,q) := \sum_{h=-\infty}^{\infty} \sum_{j=0}^{\infty} D_n(h,j)\, t^h q^j.$$

Example 4.4. *For $n = 4$,*

$$U_4(t,q) = 1 + t^{-1}q^2 + q^3 + tq^4 + q^6,$$

referring to $\emptyset$, (2), (3), (4) and $(4,2)$.

Again we consider a theorem about a rearrangement of this generating function, this time for $n \equiv 1 \pmod 3$. From this, we will be able to deduce Capparelli's identity as a corollary.

Theorem 4.2. [And94, p. 149] *For all $n \in \mathbb{N}$,*

$$U_{3n-2}(t,q) = \sum_{j=-n}^{n} t^j q^{j(3j+1)} \begin{pmatrix} n; 2j; q^3 \\ 2j \end{pmatrix}_2. \tag{44}$$

Example 4.5. *For $n = 2$, noting that*

$$\begin{pmatrix} 2; -4; q^3 \\ -4 \end{pmatrix}_2 = \begin{pmatrix} 2; 4; q^3 \\ 4 \end{pmatrix}_2 = 0,$$

and using Examples 3.3 and 4.4 it holds that

$$\begin{aligned} \sum_{j=-2}^{2} t^j q^{j(3j+1)} \begin{pmatrix} 2; 2j; q^3 \\ 2j \end{pmatrix}_2 &= t^{-1}q^2 \begin{pmatrix} 2; -2; q^3 \\ -2 \end{pmatrix}_2 + \begin{pmatrix} 2; 0; q^3 \\ 0 \end{pmatrix}_2 + tq^4 \begin{pmatrix} 2; 2; q^3 \\ 2 \end{pmatrix}_2 \\ &= t^{-1}q^2 + 1 + q^3 + q^6 + tq^4 \\ &= U_4(t,q). \end{aligned} \tag{45}$$

In order to find a recurrence for $(U_{3n-2})_{n\in\mathbb{N}}$ we need the following lemma, which provides a recursion for each residual class with respect to division by 3.

Lemma 4.2. [And94, p. 148] *For all $n \in \mathbb{N}$ the following equations hold:*

$$U_{3n}(t,q) = U_{3n-1}(t,q) + q^{3n}U_{3n-3}(t,q), \tag{46}$$

$$U_{3n+1}(t,q) = U_{3n}(t,q) + tq^{3n+1}U_{3n-3}(t,q) + q^{6n}U_{3n-5}, \tag{47}$$

$$U_{3n+2}(t,q) = U_{3n+1}(t,q) + t^{-1}q^{3n+2}U_{3n-2}(t,q). \tag{48}$$

Proof. [And94, p. 148] Equations (46) and (48) are proved in the exact same way as in the proof of Lemma 3.5. We prove Equation (47).
Let $h \in \mathbb{Z}, n, j \in \mathbb{N}$. Denote by $Q_{3n+1}(h,j)$ the set of partitions counted by $D_{3n+1}(h,j)$. Moreover, define the following subset of $Q_{3n+1}(h,j)$, considering the largest part of the partitions contained in this set,

$$Q^{\mathcal{L}}_{3n+1}(h,j) := \{(\lambda_i)_{i\in[r]} \in Q_{3n+1}(h,j) \mid \lambda_1 = 3n+1\}.$$

Note that

$$Q_{3n+1}(h,j) = Q_{3n}(h,j) \sqcup Q^{\mathcal{L}}_{3n+1}(h,j).$$

A partition counted by $D(j)$ with largest part $3n+1$ can have $3n-1$ or $3n-3$ as the second largest part, but not $3n-2$. Thus we have to slightly adapt our proof strategy. Define the following subsets of $Q^{\mathcal{L}}_{3n+1}(h,j)$:

$$M_1(h,j) := \{(\lambda_i)_{i\in[r]} \in Q^{\mathcal{L}}_{3n+1}(h,j) \mid r = 1 \text{ or } \lambda_2 < 3n-1\},$$
$$M_2(h,j) := \{(\lambda_i)_{i\in[r]} \in Q^{\mathcal{L}}_{3n+1}(h,j) \mid \lambda_2 = 3n-1\}.$$

First we show that

$$|M_1(h,j)| = |Q_{3n-3}(h-1, j-3n-1)|.$$

For this purpose define

$$\varphi_1 : M_1(h,j) \to Q_{3n-3}(h-1, j-3n-1)$$
$$\lambda \mapsto \varphi_1(\lambda),$$

where $\varphi_1(\lambda)$ is obtained from λ by removing the largest part.
We show that φ_1 is **well-defined**. To see this let $\lambda = (\lambda_i)_{i\in[r]} \in M_1(h,j)$. Then $\varphi_1(\lambda)$ is a (possibly empty) partition of size

$$\sum_{i=2}^{r}\lambda_i = \left(\sum_{i=1}^{r}\lambda_i\right) - \lambda_1 = j - (3n+1) = j-3n-1,$$

satisfying the same difference condition as λ. By the difference condition, since $\lambda_2 \neq 3n-1$,

$$\lambda_2 \leq \lambda_1 - 4 = (3n+1) - 4 = 3n-3.$$

Since $3n+1 \equiv 1 \pmod 3$ the difference h changes and it follows that

$$\varphi_1(\lambda) \in Q_{3n-4}(h-1, j-3n-1).$$

We show that φ_1 is a **bijection**: Injectivity follows directly by definition of φ_1 and by noting that all partitions in $M_1(h,j)$ have the same largest part. For seeing surjectivity, consider a partition

$$\lambda = (\lambda_i)_{i\in[s]} \in Q_{3n-3}(h-1, j-3n-1).$$

Then define $\mu = (\mu_i)_{i\in[s+1]} \in M_1(h,j)$ by $\mu_1 := 3n+1$ and $\mu_j := \lambda_{j-1}$ for all $j \in [s+1]\backslash\{1\}$. Note that

$$\varphi_1(\mu) = \lambda.$$

Second, we show that

$$|M_2(h,j)| = |Q_{3n-5}(h, j-6n)|.$$

For this purpose define

$$\begin{aligned} \varphi_2 :\ M_2(h,j) &\to Q_{3n-5}(h, j-6n) \\ \lambda &\mapsto \varphi_2(\lambda), \end{aligned}$$

where $\varphi_2(\lambda)$ is obtained from λ by removing the largest two parts. We show that φ_2 is **well-defined**. To see this, let $\lambda = (\lambda_i)_{i\in[r]} \in M_2(h,j)$. Then $\varphi_2(\lambda)$ is a (possibly empty) partition of size

$$\sum_{i=3}^{r} \lambda_i = \left(\sum_{i=1}^{r} \lambda_i\right) - \lambda_1 - \lambda_2 = j - (3n+1) - (3n-1) = j - 6n,$$

satisfying the same difference condition as λ. By the difference condition,

$$\lambda_3 \leq \lambda_2 - 4 = (3n-1) - 4 = 3n-5.$$

Since $3n+1 \equiv 1 \pmod 3$ and $3n-1 \equiv 2 \pmod 3$, the difference h doesn't change and it follows that

$$\varphi_2(\lambda) \in Q_{3n-5}(h, j-6n).$$

We show that φ_2 is a **bijection**: Injectivity follows again directly by the definition of φ_2 and by noting that all partitions in $M_2(h,j)$ have the same largest two parts. For seeing surjectivity, consider a partition

$$\lambda = (\lambda_i)_{i\in[s]} \in Q_{3n-5}(h, j-6n).$$

Then define $\mu = (\mu_i)_{i\in[s+2]} \in M_2(h,j)$ by $\mu_1 := 3n+1, \mu_2 := 3n-1$ and $\mu_j := \lambda_{j-2}$ for all $j \in [s+2] \setminus \{1,2\}$ and note that

$$\varphi_2(\mu) = \lambda.$$

Therefore it holds that

$$\begin{aligned} D_{3n+1}(h,j) &= |Q_{3n+1}(h,j)| \\ &= |Q_{3n}(h,j) \sqcup Q^{\mathcal{L}}_{3n+1}(h,j)| \\ &= |Q_{3n}(h,j) \sqcup M_1(h,j) \sqcup M_2(h,j)| \\ &= |Q_{3n}(h,j)| + |M_1(h,j)| + |M_2(h,j)| \\ &= |Q_{3n}(h,j)| + |Q_{3n-3}(h-1, j-3n-1)| + |Q_{3n-5}(h, j-6n)| \\ &= D_{3n}(h,j) + D_{3n-3}(h-1, j-3n-1) + D_{3n-5}(h, j-6n). \end{aligned}$$

Summing on both extremes of this computation we can deduce that

$$\begin{aligned}\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} D_{3n+1}(h,j)\,t^h q^j &= \sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} D_{3n}(h,j)\,t^h q^j \\ &\quad+\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} D_{3n-3}(h-1,j-3n-1)\,t^h q^j \\ &\quad+\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} D_{3n-5}(h,j-6n)\,t^h q^j.\end{aligned}$$

Shifting indices and using Definition 4.4 we see that this is equivalent to

$$\begin{aligned}U_{3n+1}(t,q) = U_{3n}(t,q) &+\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} D_{3n-3}(h,j)\,t^{h+1}q^{j+3n+1} \\ &+\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty} D_{3n-5}(h,j)\,t^h q^{j+6n}.\end{aligned}$$

Thus

$$U_{3n+1}(t,q) = U_{3n}(t,q) + tq^{3n+1}U_{3n-3}(t,q) + q^{6n}U_{3n-5}(t,q).$$

■

With these three equations we are now ready to deduce a recurrence for $(U_{3n-2})_{n\in\mathbb{N}}$.

Lemma 4.3. [And94, p. 148] *The sequence* $\left(U_{3n-2}(t,q)\right)_{n\in\mathbb{N}}$ *satisfies the following recurrence. For all* $n\in\mathbb{N}_{\geq 2}$,

$$\begin{aligned}U_{3n+1}(t,q) &= \left(1+q^{3n}\right)U_{3n-2}(t,q) \\ &\quad+\left(t^{-1}q^{3n-1}+tq^{3n+1}+q^{6n}\right)U_{3n-5}(t,q) \\ &\quad+q^{6n-3}\left(1-q^{3n-3}\right)U_{3n-8}(t,q),\end{aligned}$$

and the initial values are

$$\begin{aligned}U_{-2}(t,q) = U_1(t,q) &= 1 \\ U_4(t,q) &= 1+t^{-1}q^2+q^3+tq^4+q^6.\end{aligned} \tag{49}$$

Proof. [And94, p. 148] Let $n\in\mathbb{N}^*$.

Step 1: First we want to get rid of U_{3n-1} in the first recurrence equation (46). Consider (48) for $n-1$,

$$U_{3n-1}(t,q) = U_{3n-2}(t,q) + t^{-1}q^{3n-1}U_{3n-5}(t,q).$$

Use it to substitute $U_{3n-1}(t,q)$ in (46). We get

$$U_{3n}(t,q) - q^{3n}U_{3n-3}(t,q) = U_{3n-2}(t,q) + t^{-1}q^{3n-1}U_{3n-5}(t,q). \tag{50}$$

Step 2: Next we want to express $U_{3n-3}(t,q)$ as a combination of

$$U_{3n-2}(t,q), U_{3n-5}(t,q) \text{ and } U_{3n-8}(t,q).$$

Multiplying (50) on both sides by tq yields

$$tqU_{3n}(t,q) - tq^{3n+1}U_{3n-3}(t,q) = tqU_{3n-2}(t,q) + q^{3n}U_{3n-5}(t,q), \tag{51}$$

and by (47) we have that

$$U_{3n}(t,q) + tq^{3n+1}U_{3n-3}(t,q) = U_{3n+1}(t,q) - q^{6n}U_{3n-5}(t,q). \tag{52}$$

Add this equation to (51) to get

$$(1+tq)U_{3n}(t,q) = U_{3n+1}(t,q) + tqU_{3n-2}(t,q) + q^{3n}\left(1-q^{3n}\right)U_{3n-5}(t,q).$$

Replace n by $n-1$ and then multiply the equation on both sides by q^{3n}. Thus we reach

$$\begin{aligned}&(1+tq)q^{3n}U_{3n-3}(t,q) = \\ &q^{3n}U_{3n-2}(t,q) + tq^{3n+1}U_{3n-5}(t,q) + q^{6n-3}\left(1-q^{3n-3}\right)U_{3n-8}(t,q).\end{aligned} \tag{53}$$

Step 3: To get rid of the term $U_{3n-3}(t,q)$ we construct a second equation with the same left hand side as Equation (53).
Subtract (50) from (52) to get

$$(1+tq)q^{3n}U_{3n-3}(t,q) = U_{3n+1}(t,q) - U_{3n-2}(t,q) - \left(t^{-1}q^{3n-1}+q^{6n}\right)U_{3n-5}(t,q). \tag{54}$$

Combining the right hand sides of (53) and (54) we get the desired recursion formula.

$$\begin{aligned}U_{3n+1}(t,q) = &\left(1+q^{3n}\right)U_{3n-2}(t,q) \\ &+\left(t^{-1}q^{3n-1}+q^{6n}+tq^{3n+1}\right)U_{3n-5}(t,q) + q^{6n-3}\left(1-q^{3n-3}\right)U_{3n-8}(t,q).\end{aligned}$$

Step 4: Note that the three initial values are

$$U_{-2}(t,q) = \sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty}D_{-2}(h,j)t^hq^j = \sum_{h=-\infty}^{\infty}D_{-2}(h,0)t^h = D_{-2}(0,0) = 1,$$

$$U_1(t,q) = \sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty}D_1(h,j)t^hq^j = \sum_{h=-\infty}^{\infty}D_1(h,0)t^h + D_1(h,1)t^hq = D_1(0,0) = 1,$$

and

$$U_4(t,q) = 1 + t^{-1}q^2 + q^3 + tq^4 + q^6.$$

by Example 4.5. ■

To prove Theorem 4.2 again we show that the right hand side in (44) satisfies the same recurrence. For this purpose we consider the following definition.

Definition 4.5. [And94, p. 148] *For all $n \in \{-1\} \cup \mathbb{N}$, define*

$$Z_n(t,q) := \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)} \begin{pmatrix} n+1; 2j; q^3 \\ 2j \end{pmatrix}_2.$$

Example 4.6. *(i) For $n = -1$, using Example 3.3,*

$$Z_{-1}(t,q) = t^0 q^0 \begin{pmatrix} 0; 0; q^3 \\ 0 \end{pmatrix}_2 = 1. \tag{55}$$

(ii) For $n = 0$, noting that

$$\begin{pmatrix} 1; -2; q^3 \\ -2 \end{pmatrix}_2 = \begin{pmatrix} 1; 2; q^3 \\ 2 \end{pmatrix}_2 = 0,$$

and using Example 3.3,

$$Z_0(t,q) = t^{-1} q^2 \begin{pmatrix} 1; -2; q^3 \\ -2 \end{pmatrix}_2 + \begin{pmatrix} 1; 0; q^3 \\ 0 \end{pmatrix}_2 + tq^4 \begin{pmatrix} 1; 2; q^3 \\ 2 \end{pmatrix}_2 = 1. \tag{56}$$

(iii) For $n = 1$, using Example 4.5,

$$Z_1(t,q) = 1 + t^{-1} q^2 + q^3 + tq^4 + q^6. \tag{57}$$

To proceed with the proof of Theorem 4.2, note that the recurrence in Lemma 4.3 determines $\left(U_{3n-2}(t,q)\right)_{n \in \mathbb{N}}$ uniquely.

Proof (Theorem 4.2). [And94, p. 149] From Example 4.6 it is clear that

$$\left(Z_n(t,q)\right)_{n \in \mathbb{N} \cup \{-1\}}$$

satisfies the initial conditions in (49). Let $n \in \mathbb{N}^*, j \in \mathbb{Z}$. Then, note that by Lemma 3.3, for A replaced by $2j$ and q replaced by q^3,

$$\begin{aligned} \begin{pmatrix} n+1; 2j; q^3 \\ 2j \end{pmatrix}_2 = \left(1 + q^{3n}\right) \begin{pmatrix} n; 2j; q^3 \\ 2j \end{pmatrix}_2 &+ q^{3n+6j+3} \begin{pmatrix} n-1; 2j+2; q^3 \\ 2j+2 \end{pmatrix}_2 \\ &+ q^{3n-6j+3} \begin{pmatrix} n-1; 2j-2; q^3 \\ 2j-2 \end{pmatrix}_2 \\ &+ q^{6n} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\ &+ q^{6n-3} \left(1 - q^{3n-3}\right) \begin{pmatrix} n-2; 2j; q^3 \\ 2j \end{pmatrix}_2. \end{aligned}$$

Therefore, using this decomposition of the q-trinomial coefficients,

$$\begin{aligned} Z_n(t,q) &= \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)} \begin{pmatrix} n+1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\ &= \left(1 + q^{3n}\right) \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)} \begin{pmatrix} n; 2j; q^3 \\ 2j \end{pmatrix}_2 \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)+3n+6j+3} \begin{pmatrix} n-1; 2j+2; q^3 \\ 2j+2 \end{pmatrix}_2 \\
&+ \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)+3n-6j+3} \begin{pmatrix} n-1; 2j-2; q^3 \\ 2j-2 \end{pmatrix}_2 \\
&+ \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)+6n} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&+ \left(1-q^{3n-3}\right) \sum_{j=-(n+1)}^{n+1} t^j q^{j(3j+1)+6n-3} \begin{pmatrix} n-2; 2j; q^3 \\ 2j \end{pmatrix}_2.
\end{aligned}
$$

Recall that for all $A, B \in \mathbb{Z}$ such that $A < -n$ or $A > n$ by Definition 3.4

$$
\begin{pmatrix} n; B; q \\ A \end{pmatrix}_2 = 0.
$$

Thus, replacing j by $j+1$ in the second summand and j by $j-1$ in the third summand, and using Definition 4.5 we get that

$$
\begin{aligned}
Z_n(t,q) &= \left(1+q^{3n}\right) \sum_{j=-n}^{n} t^j q^{j(3j+1)} \begin{pmatrix} n; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&\quad + \sum_{j=-n}^{n+2} t^{j-1} q^{3(j-1)^2+7j-4+3n} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&\quad + \sum_{j=-(n+2)}^{n} t^{j+1} q^{3(j+1)^2-5j-2+3n} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&\quad + q^{6n} \sum_{j=-(n-1)}^{n-1} t^j q^{j(3j+1)} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&\quad + q^{6n-3} \left(1-q^{3n-3}\right) \sum_{j=-(n-2)}^{n-2} t^j q^{j(3j+1)} \begin{pmatrix} n-2; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&= \left(1+q^{3n}\right) Z_{n-1}(t,q) \\
&\quad + \sum_{j=-n-1}^{n-1} t^{j-1} q^{3j^2+j+3n-1} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&\quad + \sum_{j=-(n-1)}^{n-1} t^{j+1} q^{3j^2+j+3n+1} \begin{pmatrix} n-1; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&\quad + q^{6n} Z_{n-2}(t,q)
\end{aligned}
$$

$$+ q^{6n-3}\left(1-q^{3n-3}\right) Z_{n-3}(t,q).$$

Then by using again Definition 4.5, we see that

$$\begin{aligned} Z_n(t,q) = &\left(1+q^{3n}\right) Z_{n-1}(t,q) \\ &+ \left(t^{-1}q^{3n-1} + tq^{3n+1} + q^{6n}\right) Z_{n-2}(t,q) \\ &+ q^{6n-3}\left(1-q^{3n-3}\right) Z_{n-3}(t,q). \end{aligned}$$

Therefore it follows that for all $n \in \{-1\} \cup \mathbb{N}$,

$$Z_n(t,q) = U_{3n+1}(t,q),$$

which proves Theorem 4.2. ■

4.3. Andrews' Proof of Caparelli's identity. Using the material in the last section and Jacobi's triple product identity (Theorem 2.4), Andrews proved Caparelli's identity by showing that the generating function of the partitions counted by

$$\left(C\left(n\right)\right)_{n\in\mathbb{N}}$$

equals the generating function of the partitions counted by

$$\left(D\left(n\right)\right)_{n\in\mathbb{N}}.$$

Proof (Theorem 4.1). [And94, p. 150] Using the definitions of $D_n\left(h,m\right)$ (Definition 4.3) and $U_n\left(t,q\right)$ (Definition 4.4), Theorem 4.2 and Lemma 3.4 we see that

$$\begin{aligned}
\sum_{m=0}^{+\infty} D\left(m\right) q^m &= \sum_{m=0}^{+\infty} \sum_{h=-\infty}^{+\infty} \lim_{n\to+\infty} D_n\left(h,m\right) 1^h q^m \\
&= \lim_{n\to+\infty} \sum_{m=0}^{+\infty} \sum_{h=-\infty}^{+\infty} D_n\left(h,m\right) 1^h q^m \\
&= \lim_{n\to+\infty} U_n\left(1,q\right) \\
&= \lim_{n\to+\infty} U_{3n-2}\left(1,q\right) \\
&= \lim_{n\to+\infty} \sum_{j=-n}^{n} q^{j(3j+1)} \begin{pmatrix} n; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&= \sum_{j=-\infty}^{+\infty} q^{j(3j+1)} \lim_{n\to+\infty} \begin{pmatrix} n; 2j; q^3 \\ 2j \end{pmatrix}_2 \\
&= \frac{\sum_{j=-\infty}^{+\infty} q^{j(3j+1)}}{\left(q^3,q^3\right)_\infty}.
\end{aligned}$$

Note that by Jacobi's triple product identity (Theorem 2.4) applied for z replaced by q and q replaced by q^3,

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} q^{j+3j^2} &= \left(q^6,q^6\right)_\infty \left(-q^4,q^6\right)_\infty \left(-q^2,q^6\right)_\infty \\
&= \prod_{n=0}^{+\infty} \left(1-q^{6n+6}\right)\left(1+q^{6n+4}\right)\left(1+q^{6n+2}\right).
\end{aligned}$$

Note that

$$\prod_{n=0}^{+\infty} \left(1-q^{3n+3}\right) = \prod_{n=1}^{+\infty} \left(1-q^{3n}\right) = \left(q^3,q^3\right)_\infty.$$

Thus we get, using Lemma 4.1,

$$\sum_{m=0}^{+\infty} D\left(m\right) q^m = \frac{\sum_{j=-\infty}^{+\infty} q^{j(3j-1)}}{\left(q^3,q^3\right)_\infty}$$

$$
\begin{aligned}
&= \prod_{n=0}^{+\infty} \frac{\left(1-q^{6n+6}\right)\left(1+q^{6n+4}\right)\left(1+q^{6n+2}\right)}{\left(1-q^{3n+3}\right)} \\
&= \prod_{n=0}^{+\infty} \frac{\left(1-q^{6n+6}\right)\left(1-q^{12n+8}\right)\left(1-q^{12n+4}\right)}{\left(1-q^{3n+3}\right)\left(1-q^{6n+4}\right)\left(1-q^{6n+2}\right)} \\
&= \prod_{n=0}^{+\infty} \left(1-q^{6n+3}\right)^{-1}\left(1-q^{12n+2}\right)^{-1}\left(1-q^{12n+10}\right)^{-1} \\
&= \sum_{m=0}^{+\infty} C\left(m\right) q^{m}.
\end{aligned}
$$

Comparing the coefficients on both extremes of the last computation we get Caparelli's identity. ■

5. A refinement of Capparelli's identity

It turns out that Capparelli's Identity can be refined in the sense that we can specify on both sides of the identity how many parts belong to which residual class with respect to division by 6 or 3. After rewriting Caparelli's identity and introducing some more notation, the mentioned refinement is stated and a combinatorial proof is given which has first been presented in 1995 by Andrews, Alladi and Gordon in [AAG95b].

5.1. Statement. In view of the two refinements of Caparelli's identity that we will consider it seems a good choice to rewrite Caparelli's identity by replacing the partitions counted by

$$\left(C\left(n\right)\right)_{n\in\mathbb{N}}.$$

For this purpose consider the following type of partitions.

Definition 5.1. [AAG95b, p. 637] *For all $n \in \mathbb{N}$, let $E\left(n\right)$ denote the number of partitions of n into distinct parts that are congruent to $0, 2, 3$ or 4 modulo 6.*

Example 5.1. *For $n = 9$, $E\left(9\right) = 3$ counts the partitions*

$$\left(9\right), \left(6,3\right), \left(4,3,2\right).$$

Comparing this example with Example 4.1 it seems that partitions counted by $E\left(9\right)$ can be obtained from the partitions counted by $C\left(9\right)$ by replacing two equal parts by the sum of them. It turns out that this intuition is true as it is shown in the second proof of the following partition identity.

Lemma 5.1. [AAG95b, p. 637, eq. (1.1)] *For all $n \in \mathbb{N}$ it holds that*

$$E\left(n\right) = C\left(n\right).$$

Proof (using generating functions). [AAG95b, p. 637] Using Theorem 2.1 for $H = \{n \in \mathbb{N} \mid n \equiv 0, 2, 3, 4 \pmod 6\}$, and $d = 1$, note that

$$\begin{aligned}
\sum_{n=0}^{+\infty} E\left(n\right) q^n &= \prod_{j=0}^{+\infty} \left(1 + q^{6j+2}\right)\left(1 + q^{6j+3}\right)\left(1 + q^{6j+4}\right)\left(1 + q^{6j+6}\right) \\
&= \prod_{j=0}^{+\infty} \left(1 + q^{6j+2}\right)\left(1 + q^{6j+4}\right)\left(1 + q^{3j+3}\right) \\
&= \prod_{j=0}^{+\infty} \frac{\left(1 - q^{12j+4}\right)\left(1 - q^{12j+8}\right)\left(1 - q^{6j+6}\right)}{\left(1 - q^{6j+2}\right)\left(1 - q^{6j+4}\right)\left(1 - q^{3j+3}\right)}.
\end{aligned}$$

By cancelling out the terms in the product that appear in the numerator and in the denominator it turns out that

$$\sum_{n=0}^{+\infty} E\left(n\right) q^n = \prod_{j=0}^{+\infty} \left(1 - q^{12j+2}\right)^{-1} \left(1 - q^{12j+10}\right)^{-1} \left(1 - q^{6j+3}\right)^{-1}.$$

By Lemma 4.1 we can deduce that

$$\sum_{n=0}^{\infty} E(n)\, q^n = \sum_{n=0}^{\infty} C(n)\, q^n.$$

Comparing coefficients, we can conclude that for all $n \in \mathbb{N}$

$$E(n) = C(n).$$

■

We can also construct a combinatorial proof of this lemma.

Proof (Bijective Proof). Let $n \in \mathbb{N}$ and $\mathcal{E}(n)$ (resp. $\mathcal{C}(n)$) be the set of partitions counted by $E(n)$ (resp. $C(n)$). Then we define the following map:

$$\begin{aligned} \varphi :\ \mathcal{C}(n) &\to \mathcal{E}(n) \\ \lambda &\mapsto \varphi(\lambda), \end{aligned}$$

where $\varphi(\lambda)$ is obtained from λ by changing it in the following way: If there is a $j \in \mathbb{N}^*$ such that $\mu_\lambda(j) \geq 2$, replace two parts equal to j by one part equal to $2j$. Iterate this procedure and reorder the parts if necessary, until we reach a partition consisting of distinct parts. This is **well-defined** since doubling a part that is

(i) congruent to 2 (mod 6) yields a natural number congruent to 4 (mod 6, 12).
(ii) congruent to 3 (mod 6) yields a multiple of 6 that is congruent to 6 (mod 12).
(iii) congruent to 4 (mod 6) yields a natural number congruent to 2 (mod 6) and 8 (mod 12).
(iv) a multiple of 6 yields another multiple of 6 and 12.

Therefore we get that $\varphi(\lambda) \in \mathcal{E}(n)$ for all $\lambda \in \mathcal{C}(n)$.

Example 5.2. *Note that*

$$\begin{aligned} \varphi\big((22,14,10,9,9,3,3,3,3,3,2)\big) &= \varphi\big((22,18,14,10,6,6,3,2)\big) \\ &= (22,18,14,12,10,3,2). \end{aligned}$$

To prove that φ is **bijective** we construct the inverse. Define the map

$$\begin{aligned} \psi :\ \mathcal{E}(n) &\to \mathcal{C}(n) \\ \pi &\mapsto \psi(\pi), \end{aligned}$$

where $\psi(\pi)$ is obtained from π by changing it in the following way: If there is a part in π that is congruent to $0, 4, 6$ or $8 \pmod{12}$ replace this part by inserting twice the part which is exactly this part divided by two. Iterate this procedure and reorder the parts if necessary, until we reach a partition consisting of parts congruent to $2, 3, 9$ or $10 \pmod{12}$. This is well-defined since dividing a part by 2 that is

(i) a multiple of 12 yields another multiple of 12 or a natural number congruent to 6 (mod 12).
(ii) congruent to 4 (mod 12) yields a natural number that is congruent to 2 or 8 (mod 12).

(iii) congruent to 6 (mod 12) yields a natural number that is congruent to 3 or 9 (mod 12).
(iv) congruent to 8 (mod 12) yields a natural number that is congruent to 4 or 10 (mod 12).

As iterated division by 2 of natural numbers ends eventually always in an odd number, this process is deterministic and therefore we get that $\psi(\pi) \in \mathcal{C}(n)$ for all $\pi \in \mathcal{E}(n)$.

Example 5.3. *Note that*

$$\begin{aligned}\psi\left((22,18,14,12,10,3,2)\right) &= \psi\left((22,14,10,9,9,6,6,3,2)\right)\\ &= (22,14,10,9,9,3,3,3,3,3,2)\,.\end{aligned}$$

To conclude, we note that by construction, indeed,

$$\psi = \varphi^{-1}.$$

■

With this lemma we can rewrite Capparelli's identity in the following way.

Theorem 5.1. [AAG95b, p. 636 - 637] *For all $n \in \mathbb{N}$ it holds that*

$$E(n) = D(n)\,.$$

For the mentioned refinement we consider the number of parts in each residual class. For this purpose we need the following two definitions.

Definition 5.2. [AAG95b, p. 637] *For all $n, i, j, k \in \mathbb{N}$, let $E(n; i, j, k)$ denote the number of partitions counted by $E(n)$ with exactly i parts that are congruent to $4 \pmod 6$, exactly j parts that are congruent to $2 \pmod 6$ and exactly k parts that are multiples of 3 and strictly bigger than $3(i+j)$.*

Example 5.4. *For $n = 9$, $E(9; 1, 1, 0) = 1$ counts the partition*

$$(4,3,2)\,.$$

Definition 5.3. [AAG95b, p. 637] *For all $n, i, j, k \in \mathbb{N}$, let $D(n; i, j, k)$ denote the number of partitions counted by $D(n)$ consisting of exactly i parts that are congruent to $1 \pmod 3$, exactly j parts that are congruent to $2 \pmod 3$ and exactly k parts that are multiples of 3.*

Example 5.5. *For $n = 9$, $D(9; 1, 1, 0) = 1$ counts the partition*

$$(7,2)\,.$$

Now we are ready to state the following three-parameter refinement of Capparelli's identity.

Theorem 5.2. [AAG95b, p. 637] *For all $n, i, j, k \in \mathbb{N}$ it holds that*

$$E(n; i, j, k) = D(n; i, j, k)\,.$$

5.2. Preparation for the proof. The proof of this statement is constructed as an algorithmically described bijection. Here we shall present it in the same steps but written more in a formal way than describing the procedure. The reason for this is that we tried to find another refinement and in the proof of it the formal description of the inverse of the map would have been helpful. In the end, we must admit that it didn't work: the reason can be understood by the description of the inverse (see Subsection 5.3). In this subsection we will present the most important tools that Alladi, Andrews and Gordon use in their paper [AAG95b]. We start with an elementary binary operation on partitions.

Definition 5.4. *Let $k_1, k_2 \in \mathbb{N}$ and $\pi_1 \vdash k_1, \pi_2 \vdash k_2$. Then we define $\pi_1 \cup \pi_2 \vdash k_1 + k_2$ to be the partition consisting of all parts of π_1 and π_2 in which each part j occurs exactly $\mu_{\pi_1}(j) + \mu_{\pi_2}(j)$ times.*

Example 5.6. *Note that*

$$(5,3,3,2) \cup (7,3,2,1) = (7,5,3,3,3,2,2,1).$$

Moreover, we will use a bijection which describes a certain decomposition of partitions.

Lemma 5.2. *Let $A \sqcup B \sqcup C$ be a set partition of $\mathbb{N}^*$. Moreover, let $k \in \mathbb{N}$ and define*

$$\mathcal{P}_{A,B,C}(k) := \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \to A, \pi_2 \to B, \pi_3 \to C, s(\pi_1) + s(\pi_2) + s(\pi_3) = k\}.$$

Then

$$\begin{aligned} \varphi : \ \mathcal{P}(k) &\to \mathcal{P}_{A,B,C}(k) \\ \pi &\mapsto (\pi_A, \pi_B, \pi_C) \end{aligned}$$

is a bijection.

Proof. Note that by Definition 2.30, φ is well-defined. Consider the following map

$$\begin{aligned} \psi : \mathcal{P}_{A,B,C}(k) &\to \mathcal{P}(k) \\ (\pi_1, \pi_2, \pi_3) &\mapsto (\pi_1 \cup \pi_2) \cup \pi_3. \end{aligned}$$

By Definition 5.4 this map is well-defined. To conclude, note that for all $\pi \in \mathcal{P}(k)$,

$$\psi(\varphi(\pi)) = \psi(\pi_A, \pi_B, \pi_C) = (\pi_A \cup \pi_B) \cup \pi_C = \pi,$$

using that $A \sqcup B \sqcup C = \mathbb{N}^*$. Further, note that for all $(\pi_1, \pi_2, \pi_3) \in \mathcal{P}_{A,B,C}(k)$ it holds that

$$\begin{aligned} \varphi(\psi(\pi_1, \pi_2, \pi_3)) &= \varphi((\pi_1 \cup \pi_2) \cup \pi_3) \\ &= \left(((\pi_1 \cup \pi_2) \cup \pi_3)_A, ((\pi_1 \cup \pi_2) \cup \pi_3)_B, ((\pi_1 \cup \pi_2) \cup \pi_3)_C\right) \\ &= (\pi_1, \pi_2, \pi_3). \end{aligned}$$

■

In a natural way we can also define an addition on integer partitions.

Definition 5.5. [AAG95b, p. 640] *Let $n_1, n_2 \in \mathbb{N}$. Define*

$$\begin{aligned} + : \mathcal{P}(n_1) \times \mathcal{P}(n_2) &\to \mathcal{P}(n_1 + n_2) \\ \left((\lambda_l)_{l \in [r_1]}, (\mu_s)_{s \in [r_2]}\right) &\mapsto (\lambda_i + \mu_i)_{i \in \max\{r_1, r_2\}}, \end{aligned}$$

where $\lambda_i = 0$ *for* $i > r_1$ *and* $\mu_j = 0$ *for* $j > r_2$. *For* $(\lambda, \mu) \in \mathcal{P}(n_1) \times \mathcal{P}(n_2)$ *we denote*

$$\lambda + \mu := +(\lambda, \mu).$$

Example 5.7. *For* $n_1 = 5, n_2 = 4$, *note that*

$$(9, 7, 6, 5, 2) + (12, 5, 3, 1) = (21, 12, 9, 6, 2).$$

Remark 5.1. Indeed, $+$ is well-defined. Let $n_1, n_2 \in \mathbb{N}$ and $\left((\lambda_l)_{l\in[r_1]}, (\mu_s)_{s\in[r_2]}\right) \in \mathcal{P}(n_1) \times \mathcal{P}(n_2)$. Then

$$\sum_{j=1}^{\max\{r_1,r_2\}} \lambda_j + \mu_j = \sum_{j=1}^{r_1} \lambda_j + \sum_{j=1}^{r_2} \mu_j = s\left((\lambda_l)_{l\in[r_1]}\right) + s\left((\mu_s)_{s\in[r_2]}\right) = n_1 + n_2,$$

and for all $j \in [\max\{r_1, r_2\} - 1]$ it holds that

$$\lambda_j + \mu_j \geq \lambda_{j+1} + \mu_{j+1}.$$

Thus

$$(\lambda_l)_{l\in[r_1]} + (\mu_s)_{s\in[r_2]} \in \mathcal{P}(n_1 + n_2).$$

To specify objects at an intermediate step of the proof of the presented refinement (Theorem 5.2), we need a further specific term to describe the order of finite sequences.

Definition 5.6. *Let* $s = (s_t)_{t\in[r]}$ *be a finite sequence in* $\mathbb{N}^*$. *Then* s *is said to be* ***almost non-decreasing*** *if it is increasing up to elements* a *that are congruent to* 2 *or* $5 \pmod 6$ *that may be followed by* $a - 1$.

Example 5.8. *Note that* $(1, 2, 4, 8, 7, 10)$ *is an almost non-decreasing finite sequence in* $\mathbb{N}^*$.

As a final part for the preparation we will need a corollary of Proposition 2.1. More concretely it describes a restriction of the 3-fold conjugation.

Corollary 5.1. *Let* $n, m \in \mathbb{N}$ *and define*

$$\mathcal{D}_3^m(n) := \{\lambda \vdash n \mid \lambda \to 3\mathbb{N}^*, \forall j \in \mathbb{N}^* : \ \mu_\lambda(j) \leq 1, L(\lambda) \leq 3m\},$$

and

$$\mathcal{P}_3^m(n) := \{\lambda \vdash n \mid \lambda \to 3\mathbb{N}^*, \forall j \in [L(\lambda)] : \ 3 \mid j \Rightarrow \mu_\lambda(j) \geq 1, z(\lambda) \leq m\}.$$

Then

$$\tilde{\varphi} : \ \mathcal{D}_3^m(n) \to \mathcal{P}_3^m(n)$$
$$\pi \mapsto \pi^*$$

is a bijection.

Proof. Let $n, m \in \mathbb{N}$. By Proposition 2.1, the following map is a bijection:

$$\varphi : \ \mathcal{D}_3(n) \to \mathcal{P}_3(n)$$
$$\pi \mapsto \pi^*.$$

It remains to show that

$$z\left(\varphi\left(\pi\right)\right) \leq m,$$

for all $\pi \in \mathcal{D}_3\left(n\right)$ such that $L\left(\pi\right) \leq 3m$, and

$$L\left(\varphi^{-1}\left(\lambda\right)\right) \leq 3m,$$

for all $\lambda \in \mathcal{P}_3\left(n\right)$ such that $z\left(\lambda\right) \leq m$. Let $\pi \in \mathcal{D}_3^m\left(n\right)$. Then by the definition of the 3-fold-conjugate (Definition 2.35) $\varphi\left(\pi\right)$ has at most

$$\frac{L\left(\pi\right)}{3} \leq m$$

parts. Recall that φ^{-1} can be described by the following map (see the proof of Proposition 2.1):

$$\begin{aligned} \psi: \ \mathcal{P}_3\left(n\right) &\to \mathcal{D}_3\left(n\right) \\ \pi &\mapsto \left(3j^{\frac{\mu_{\pi'}(j)}{3}}\right)_{j\in[z(\pi)]}. \end{aligned}$$

Thus, if $\lambda \in \mathcal{P}_3^m\left(n\right)$, the largest part of $\psi\left(\lambda\right)$ is at most

$$3z\left(\lambda\right) \leq 3m.$$

Hence $\tilde{\varphi}$ is a bijection. ■

5.3. Bijective proof. As mentioned before, Alladi, Andrews and Gordon presented in their paper a combinatorial proof by constructing a bijection between the two types of partitions. It shall be described here in the same step structure.

Proof. [AAG95b, p. 639 - 641] Let $n, i, j, k \in \mathbb{N}$.

Step 1: First we decompose the partitions counted by $E(n; i, j, k)$ into three partitions. One contains the $i + j$ parts that are not multiples of 3, one contains the multiples of 3 which are greater than $3(i + j)$ (i.e. those that are counted by k) and one contains the smaller multiples of 3. The motivation behind this decomposition is an application of 3-fold conjugation to these smaller multiples of 3 in Step 2 and reordering the partition for the further steps.
Denote by $\mathcal{E}(n; i, j, k)$ the set of partitions counted by $E(n; i, j, k)$. Further let $\mathcal{T}_1(n; i, j, k)$ be the set of finite sequences (π_1, π_2, π_3) of partitions such that π_1 consists of i distinct parts congruent to 4 (mod 6) and j distinct parts congruent to 2 (mod 6), π_2 consists of distinct parts that are multiples of 3 and smaller than or equal to $3(i + j)$ and π_3 consists of k distinct parts that are multiples of 3 and greater than $3(i + j)$. Further we demand that

$$s(\pi_1) + s(\pi_2) + s(\pi_3) = n.$$

Let $C := 3\left(\mathbb{N}^* \setminus [i + j]\right)$, $B := 3[i + j]$ and $A := \mathbb{N}^* \setminus (B \cup C)$. We can formalize this decomposition process using the following map:

$$\begin{aligned} \varphi_1 : \ \mathcal{E}(n; i, j, k) &\to \mathcal{T}_1(n; i, j, k) \\ \pi &\mapsto (\pi_A, \pi_B, \pi_C). \end{aligned}$$

Indeed, by definition of $\mathcal{E}(n; i, j, k)$ and $\mathcal{T}_1(n; i, j, k)$ and Lemma 5.2, φ_1 is a **well-defined bijection**. The inverse can be described by the union $(\pi_1 \cup \pi_2) \cup \pi_3$ for $(\pi_1, \pi_2, \pi_3) \in \mathcal{T}_1(n; i, j, k)$.

Example 5.9. *Let* $n = 211, i = 3, j = 5, k = 2$ *and consider*

$$\lambda := (32, 30, 27, 26, 24, 22, 14, 10, 9, 8, 4, 3, 2) \in \mathcal{E}(n; i, j, k).$$

Then

$$\varphi_1(\lambda) = \big((32, 26, 22, 14, 10, 8, 4, 2), (9, 3), (30, 27)\big).$$

Step 2: As a second step we will apply 3-fold-conjugation to the partition consisting of the smaller parts that are multiples of 3.
Let $\mathcal{T}_2(n; i, j, k)$ be the set of triples (π_1, π_2, π_3) of partitions, where π_1 consists precisely of i distinct parts congruent to 4 (mod 6) and j distinct parts congruent to 2 (mod 6), π_2 consists of at most $(i + j)$ parts that are multiples of 3, with the additional condition that every multiple of 3 smaller than or equal to $L(\pi_2)$ appears at least once as a part of π_2, π_3 consists of precisely k parts that are multiples of 3 and greater than $3(i + j)$ and

$$s(\pi_1) + s(\pi_2) + s(\pi_3) = n.$$

This step can be described using the following map:

$$\varphi_2 : \ \mathcal{T}_1(n; i, j, k) \to \mathcal{T}_2(n; i, j, k)$$

$$(\pi_1, \pi_2, \pi_3) \mapsto \left(\pi_1, \pi_2^*, \pi_3\right).$$

Indeed, by an application of Corollary 5.1 for $m = i + j$, φ_2 is a **well-defined bijection**.

Example 5.10. *Continue Example 5.9. Note that*

$$\begin{aligned}\varphi_2\left(\varphi_1\left(\lambda\right)\right) &= \varphi_2\left(\left(32, 26, 22, 14, 8, 4, 2\right), \left(9, 3\right), \left(30, 27\right)\right)\\ &= \left(\left(32, 26, 22, 14, 8, 4, 2\right), \left(3, 3, 3, 1, 1, 1\right)', \left(30, 27\right)\right)\\ &= \left(\left(32, 26, 22, 14, 8, 4, 2\right), \left(6, 3, 3\right), \left(30, 27\right)\right).\end{aligned}$$

Step 3: Next we sum the first two partitions of the obtained triple. Note that since we add parts that are multiples of 3 to the first partition we obtain parts of more different residual classes.
Let $\mathcal{T}_3\left(n; i, j, k\right)$ be the set of pairs (π_1, π_2) of partitions where π_1 consists of i distinct parts all greater than 1 congruent to 1 or 4 (mod 6) and j distinct parts congruent to 2 or 5 (mod 6), and consecutive parts with different parity differ by at least 5. Moreover, π_2 consists of k parts that are multiples of 3 and greater than $3\left(i + j\right)$, and

$$s\left(\pi_1\right) + s\left(\pi_2\right) = n.$$

This summation is processed by the following map:

$$\begin{aligned}\varphi_3 :\ \mathcal{T}_2\left(n; i, j, k\right) &\to \mathcal{T}_3\left(n; i, j, k\right)\\ \left(\pi_1, \pi_2, \pi_3\right) &\mapsto \left(\pi_1 + \pi_2, \pi_3\right).\end{aligned}$$

First we will prove that this map is **well-defined**. Let $(\pi_1, \pi_2, \pi_3) \in \mathcal{T}_2\left(n; i, j, k\right)$. Since π_1 consists of i parts congruent to 4 (mod 6) and j parts congruent to 2 (mod 6) and π_2 consists of at most $i + j$ parts that are all multiples of 3, $\pi_1 + \pi_2$ satisfies the congruence conditions in $\mathcal{T}_3\left(n; i, j, k\right)$.
To show the difference conditions, let p_t and p_{t+1} be consecutive parts of $\pi_1 + \pi_2$ with distinct parities. Since π_1 consists of even parts and π_2 consists of multiples of 3 the corresponding parts $\pi_{2,t}$ and $\pi_{2,t+1}$ (possibly equal to 0) differ by 3. As π_1 consists of distinct parts and partitions are non-increasing, this implies that p_t and p_{t+1} differ by at least $3 + 1 = 4$. The minimal difference must therefore be 5 because they have distinct parities. If p_t and p_{t+1} share the same parity they differ by at least 2 and the corresponding parts $\pi_{2,t}$ and $\pi_{2,t+1}$ are equal. Thus

$$\left|p_t - p_{t+1}\right| \geq 2 + 0 = 2.$$

For the size note that by the definition of addition of partitions (Definition 5.5),

$$s\left(\pi_1 + \pi_2\right) + s\left(\pi_3\right) = s\left(\pi_1\right) + s\left(\pi_2\right) + s\left(\pi_3\right) = n.$$

Hence $\varphi_3\left(\pi_1, \pi_2, \pi_3\right) \in \mathcal{T}_3\left(n; i, j, k\right)$.
To show **bijectivity** we construct the inverse of φ_3. The idea is to use the two facts that in the first partition of pairs in $\mathcal{T}_3\left(n; i, j, k\right)$ a change in parity means that there was at least difference 3 in the originally second partition and consecutive parts in that partition differ by at most 3 in any case. Note that the first summand consists only of even parts. Thus starting with the smallest odd part of the sum we can reconstruct the two summands, by

increasing the parts in the second summand by 3 whenever we reach a change in parity. This can be formalized using the following definition of a map. Define

$$\begin{aligned}\psi_3 :\ \mathcal{T}_3(n;i,j,k) &\to \mathcal{T}_2(n;i,j,k)\\ (\pi_1,\pi_2) &\mapsto (\mu_1,\mu_2,\pi_2),\end{aligned}$$

where μ_1 and μ_2 are constructed in the following way. For $(\pi_1,\pi_2)\in\mathcal{T}_3(n;i,j,k)$, denote $\pi_1=\left(\pi_{1,t}\right)_{t\in[r]}$ and let

$$\left(\pi_{1,a_l}\right)_{l\in[k_1]}$$

be the finite sequence consisting of the odd parts of π_1 that are followed by an even part or that are $\pi_{1,r}$. We assume the elements to be ordered reversely, i.e. for all $m\in[k_1-1]$,

$$\pi_{1,a_m}<\pi_{1,a_{m+1}}.$$

Analogously let

$$\left(\pi_{1,b_l}\right)_{l\in[k_2]}$$

be the finite sequence of even parts of π_1 that are followed by an odd part. We again assume that the elements are ordered reversely, i.e. for all $m\in[k_2-1]$,

$$\pi_{1,b_m}<\pi_{1,b_{m+1}}.$$

Example 5.11. *For* $\pi_1=(41,34,26,19,14,10,5)$*,*

$$\left(\pi_{1,a_l}\right)_{l\in[3]}=(5,19,41),$$

and

$$\left(\pi_{1,b_l}\right)_{l\in[2]}=(10,26).$$

Note that in this example $a_1=7,a_2=4,a_3=1$ *and* $b_1=6,b_2=3$.

Then define $\mu_1=\left(\mu_{1,t}\right)_{t\in[r]}$ by

$$\mu_{1,t}:=\begin{cases}\pi_{1,t}-3(2k_1-1) & \text{if } t\le a_{k_1} \text{ and } \pi_{1,1} \text{ is odd},\\ \pi_{1,t}-3\cdot 2k_2 & \text{if } t\le b_{k_2} \text{ and } \pi_{1,1} \text{ is even},\\ \pi_{1,t}-3(2l-1) & \text{if } t\in(b_l,a_l],\\ \pi_{1,t}-3\cdot 2l & \text{if } t\in(a_{l+1},b_l],\\ \pi_{1,t} & \text{if } t>a_1,\end{cases}$$

and $\mu_2=\left(\mu_{2,t}\right)_{t\in[a_1]}$ by

$$\mu_{2,t}:=\begin{cases}3(2k_1-1) & \text{if } t\le a_{k_1} \text{ and } \pi_{1,1} \text{ is odd},\\ 3\cdot 2k_2 & \text{if } t\le b_{k_2} \text{ and } \pi_{1,1} \text{ is even},\\ 3(2l-1) & \text{if } t\in(b_l,a_l],\\ 3\cdot 2l & \text{if } t\in(a_{l+1},b_l].\end{cases}$$

Example 5.12. *Continuing the previous example, note that*

$$\begin{aligned}\mu_1 &= (41-5\cdot 3, 34-4\cdot 3, 26-4\cdot 3, 19-3\cdot 3, 14-2\cdot 3, 10-2\cdot 3, 5-3)\\ &= (26,22,14,10,8,4,2)\,,\end{aligned}$$

and

$$\mu_2 = (5\cdot 3, 4\cdot 3, 4\cdot 3, 3\cdot 3, 2\cdot 3, 2\cdot 3, 3) = (15,12,12,9,6,6,3)\,.$$

Next we show that ψ_3 is well-defined. Let $(\pi_1,\pi_2)\in\mathcal{T}_3(n;i,j,k)$. Note that by definition of $\mathcal{T}_3(n;i,j,k)$ every two consecutive parts of π_1 with different parities differ by at least 5 and π_{1,a_1} is at least 5 (by the congruence conditions and since it is odd). Hence the subtraction in the definition of μ_1 is well-defined and μ_1 is non-increasing. Therefore it forms, indeed, a partition. Since from each odd (resp. even) part of π_1 an odd (resp. even) multiple of 3 is subtracted, μ_1 consists of even parts, more precisely i parts that are congruent to 4 (mod 6) and j parts that are congruent to 2 (mod 6).
Moreover, μ_2 forms a non-increasing finite sequence. It consists of $a_1\le i+j$ parts that are consecutive multiples of 3 (with possible multiple appearances) and the smallest part equals 3.
Finally, noting that

$$s(\mu_1)+s(\mu_2)=s(\pi_1)\,,$$

we can deduce that

$$\psi_3(\pi_1,\pi_2)=(\mu_1,\mu_2,\pi_2)\in\mathcal{T}_2(n;i,j,k)\,.$$

We show that $\psi_3=\varphi_3^{-1}$. Let $(\pi_1,\pi_2,\pi_3)\in\mathcal{T}_2(n;i,j,k)$. Then by construction it holds that

$$\psi_3\left(\varphi_3(\pi_1,\pi_2,\pi_3)\right)=\psi_3(\pi_1+\pi_2,\pi_3)=(\pi_1,\pi_2,\pi_3)\,.$$

For $(\pi_1,\pi_2)\in\mathcal{T}_3(n;i,j,k)$,

$$\varphi_3\left(\psi_3(\pi_1,\pi_2)\right)=\varphi_3(\mu_1,\mu_2,\pi_2)=(\mu_1+\mu_2,\pi_2)=(\pi_1,\pi_2)\,.$$

Example 5.13. *Continue Example 5.10. Note that*

$$\begin{aligned}\varphi_3\left(\varphi_2\left(\varphi_1(\lambda)\right)\right)&=\varphi_3\left((32,26,22,14,8,4,2),(6,3,3),(30,27)\right)\\ &=\left((38,29,25,14,8,4,2),(30,27)\right).\end{aligned}$$

Step 4: We combine the two partitions to a single (not necessarily non-increasing) finite sequence.
Let $\mathcal{T}_4(n;i,j,k)$ be the set of finite sequences in $\mathbb{N}^*$ with $i+j+k$ elements greater than 1 such that the first $i+j$ elements form a partition in reversed order into distinct parts of which i are congruent to 1 or 4 (mod 6) and j are congruent to 2 or 5 (mod 6) and consecutive parts with different parity differ by at least 5. The last k elements form a partition in reversed order into distinct parts that are multiples of 3 and all greater than $3(i+j)$. The sum of the elements of the sequences should be n.
We are ready to define the fourth bijection:

$$\begin{aligned}\varphi_4:\ \mathcal{T}_3(n;i,j,k)&\to\mathcal{T}_4(n;i,j,k)\\ (\pi_1,\pi_2)&\mapsto(u_t)_{t\in[i+j+k]}\,,\end{aligned}$$

where

$$(u_t)_{t\in[i+j]} := \pi_1^{rev},$$

and

$$(u_t)_{t\in[i+j+k]\setminus[i+j]} := \pi_2^{rev}.$$

By definition of $\mathcal{T}_3(n;i,j,k)$ and $\mathcal{T}_4(n;i,j,k)$, φ_4 is a **well-defined bijection**. The inverse can be described by separating the finite sequences into two finite sequences π_1 and π_2 and reversing their order where π_1 contains the first $i+j$ elements and π_2 contains the last k elements.

Example 5.14. *Continue Example 5.13. Note that*

$$\begin{aligned}\varphi_4\Big(\varphi_3\Big(\varphi_2\left(\varphi_1(\lambda)\right)\Big)\Big) &= \varphi_4\left((38,29,25,14,8,4,2),(30,27)\right)\\ &= (2,4,8,14,25,29,38,27,30).\end{aligned}$$

Step 5: In the obtained finite sequence we subtract consecutive multiples of 3 starting with 0.
Let $\mathcal{T}_5(n;i,j,k)$ be the set of finite sequences $(u_t)_{t\in[i+j+k]}$ in $\mathbb{N}^*$ such that $u_1>1$, the last k elements are non-decreasing multiples of 3 and the first $i+j$ elements form an almost non-decreasing sequence consisting of i elements that are congruent to 1 or 4 (mod 6) and j elements that are congruent to 2 or 5 (mod 6). Every two consecutive elements among these $i+j$ are distinct. Moreover,

$$\sum_{t=1}^{i+j+k} u_t = n - 3\frac{(i+j+k)(i+j+k-1)}{2}.$$

We define the fifth bijection by

$$\begin{aligned}\varphi_5:\ \mathcal{T}_4(n;i,j,k) &\to \mathcal{T}_5(n;i,j,k)\\ (u_t)_{t\in[i+j+k]} &\mapsto \left(u_t-(t-1)\cdot 3\right)_{t\in[i+j+k]}.\end{aligned}$$

Let us show that φ_5 is **well-defined**. Let

$$u=(u_t)_{t\in[i+j+k]}\in\mathcal{T}_4(n;i,j,k).$$

We show first that the elements of $\varphi_5(u)$ lie in $\mathbb{N}^*$. Note that for $t\in[i+j+k]\setminus[i+j]$, $u_t\geq t\cdot 3$. Thus,

$$u_t-(t-1)\cdot 3\geq 3t-3(t-1)=3>0.$$

Further, $u_1\geq 2$ and by the difference and congruence conditions in $\mathcal{T}_4(n;i,j,k)$ two consecutive elements differ by at least 2 and

$$u_{t+2}-u_t\geq 6,$$

for all $t\in[i+j-2]$. Thus if $t\in[i+j]$ is odd, then

$$u_t\geq 2+\frac{t-1}{2}\cdot 6=3t-1.$$

If $t \in [i+j]$ is even, then

$$u_t \geq 2 + \frac{t-2}{2} \cdot 6 + 2 = 3t - 2.$$

Hence for all $t \in [i+j]$,

$$u_t - (t-1) \cdot 3 \geq 3t - 2 - (3t-3) > 0,$$

which proves that the subtraction is well-defined.
Since we subtract multiples of 3, the congruence conditions in $\mathcal{T}_5(n; i, j, k)$ are satisfied.
For $t \in [i+j+k]$ such that $i+j < t < i+j+k$, it holds that

$$(u_{t+1} - 3t) - \big(u_t - (t-1) \cdot 3\big) = u_{t+1} - u_t - 3 \geq 3 - 3 = 0,$$

which shows that the last k elements of $\varphi_5(u)$ form a non-decreasing sequence.
Note that for $t \in [i+j-1]$, by the difference conditions in $\mathcal{T}_4(n; i, j, k)$,

$$u_{t+1} - u_t \geq 2.$$

Assume that

$$u_{t+1} - u_t = 2.$$

Then by the congruence conditions in $\mathcal{T}_4(n; i, j, k)$ it follows that u_t is congruent to 2 or 5 (mod 6). In this case,

$$(u_{t+1} - 3t) - \big(u_t - 3(t-1)\big) = (u_{t+1} - u_t) - 3 = -1. \tag{58}$$

If

$$u_{t+1} - u_t > 2,$$

by the distinct parity difference condition, we have that

$$u_{t+1} - u_t > 3.$$

By (58) we see that

$$(u_{t+1} - 3t) - \big(u_t - 3(t-1)\big) > 0.$$

Thus the first $i+j$ elements of $\varphi_5(u)$ form an almost non-decreasing sequence.
Moreover, by Lemma 2.2,

$$\sum_{t=1}^{i+j+k} u_t - 3(t-1) = \sum_{t=1}^{i+j+k} u_t - 3 \sum_{t=0}^{i+j+k-1} t = n - 3\frac{(i+j+k-1)(i+j+k)}{2}.$$

Hence

$$\varphi_5(u) \in \mathcal{T}_5(n; i, j, k).$$

For proving **bijectivity** we construct the inverse of φ_5. Define

$$\begin{aligned} \psi_5 : \ \mathcal{T}_5(n; i, j, k) &\to \mathcal{T}_4(n; i, j, k) \\ (v_t)_{t \in [i+j+k]} &\mapsto \big(v_t + (t-1) \cdot 3\big)_{t \in [i+j+k]}. \end{aligned}$$

We show that this map is well-defined. Let

$$v = (v_t)_{t \in [i+j+k]} \in \mathcal{T}_5(n; i, j, k).$$

Since we add multiples of 3 the elements of $\varphi(v)$ satisfy the congruence conditions. For $t \in [i+j-1]$, note that

$$(v_{t+1}+3t)-\big(v_t+3(t-1)\big)=(v_{t+1}-v_t)+3\geq -1+3=2,$$

and by definition of $\mathcal{T}_5(n;i,j,k)$, $u_1>1$. Assume $v_t+3(t-1)$ and $v_{t+1}+3t$ have distinct parities. Then v_t and v_{t+1} have the same parity and thus

$$v_{t+1}-v_t\geq 2.$$

Hence

$$(v_{t+1}+3t)-\big(v_t+3(t-1)\big)=(v_{t+1}-v_t)+3\geq 2+3=5.$$

Next we show that the last k elements form a partition in reversed order into distinct parts that are multiples of 3 and all greater than $3(i+j)$. Note that for all $t \in [i+j+k-1]\backslash[i+j]$,

$$(v_{t+1}+3t)-\big(v_t+3(t-1)\big)=(v_{t+1}-v_t)+3\geq 3.$$

Moreover,

$$v_t+3(t-1)\geq v_t+3(i+j)>3(i+j).$$

Finally, note that by Lemma 2.2,

$$\begin{aligned}\sum_{t=1}^{i+j+k} v_t+3(t-1)&=\sum_{t=1}^{i+j+k} v_t+3\sum_{t=0}^{i+j+k-1} t\\&=n-3\frac{(i+j+k)(i+j+k-1)}{2}+3\frac{(i+j+k)(i+j+k-1)}{2}\\&=n.\end{aligned}$$

Hence

$$\psi_5(v)\in\mathcal{T}_4(n;i,j,k).$$

To conclude the proof on bijectivity of φ_5 note that for all

$$(u_t)_{t\in[i+j+k]}\in\mathcal{T}_4(n;i,j,k),$$

it holds that

$$\psi_5\left(\varphi_5\left((u_t)_{t\in[i+j+k]}\right)\right)=\psi_5\left(\big(u_t-3(t-1)\big)_{t\in[i+j+k]}\right)=(u_t)_{t\in[i+j+k]},$$

and for all $(v_t)_{i\in[i+j+k]}\in\mathcal{T}_5(n;i,j,k)$ it holds that

$$\varphi_5\left(\psi_5\left((v_t)_{t\in[i+j+k]}\right)\right)=\varphi_5\left(\big(v_t+3(t-1)\big)_{i\in[i+j+k]}\right)=(v_t)_{t\in[i+j+k]}.$$

Example 5.15. *Continue Example 5.14. Note that*

$$\begin{aligned}\varphi_5\left(\varphi_4\left(\varphi_3\left(\varphi_2\left(\varphi_1(\lambda)\right)\right)\right)\right)&=\varphi_5\left((2,4,8,14,25,29,38,27,30)\right)\\&=(2,1,2,5,13,14,20,6,6).\end{aligned}$$

Step 6: We rearrange the obtained finite sequence in such a way that all multiples of 3 are placed where it naturally belongs (with respect to the order on the natural numbers). Let $\mathcal{T}_6(n;i,j,k)$ be the set of almost non-decreasing finite sequences $(u_t)_{t\in[i+j+k]}$ in $\mathbb{N}^*$ such that $u_t > 1$ for all $t \in [i+j+k]$, precisely k elements are multiples of 3, i elements are congruent to 1 or 4 (mod 6) and j elements are congruent to 2 or 5 (mod 6) and consecutive elements are distinct, unless they are both multiples of 3. Moreover,

$$\sum_{s=1}^{i+j+k} u_s = n - 3\frac{(i+j+k)(i+j+k-1)}{2}.$$

The rearranging procedure is described by the following map.

$$\begin{aligned}\varphi_6 :\ \mathcal{T}_5(n;i,j,k) &\to \mathcal{T}_6(n;i,j,k)\\ u &\mapsto \tilde{u},\end{aligned}$$

where $\tilde{u}$ is the unique almost non-decreasing sequence with the same elements (with same multiplicities) as u such that the order of all elements that are not multiples of 3 is preserved.
By definition of $\mathcal{T}_5(n;i,j,k)$ and $\mathcal{T}_6(n;i,j,k)$, this map is a **well-defined bijection**. The inverse can be described by taking all elements that are multiples of 3 in increasing order to the end of the finite sequence.

Example 5.16. *Continue Example 5.15. Note that*

$$\begin{aligned}\varphi_6\left(\varphi_5\left(\varphi_4\left(\varphi_3\left(\varphi_2\left(\varphi_1(\lambda)\right)\right)\right)\right)\right) &= \varphi_6\left((2,1,2,5,13,14,20,6,6)\right)\\ &= (2,1,2,5,6,6,13,14,20).\end{aligned}$$

Step 7: The final step consists in adding again the subtracted multiples of 3 to the rearranged sequence, reversing the order and obtaining a partition of the type counted by $D(n;i,j,k)$.
Denote by $\mathcal{D}(n;i,j,k)$ the set of partitions counted by $D(n;i,j,k)$. The last piece in the considered row of bijections is the following map:

$$\begin{aligned}\varphi_7 :\ \mathcal{T}_6(n;i,j,k) &\to \mathcal{D}(n;i,j,k)\\ (u_t)_{t\in[i+j+k]} &\mapsto \left(u_t + 3(t-1)\right)_{t\in[i+j+k]}^{rev}.\end{aligned}$$

We show that this map is **well-defined**. Let $u = (u_t)_{t\in[i+j+k]} \in \mathcal{T}_6(n;i,j,k)$. Since adding multiples of 3 doesn't change the residual classes with respect to division by 3, $\varphi_7(u)$ satisfies the congruence conditions in $\mathcal{D}(n;i,j,k)$. Note that again by Lemma 2.2,

$$\begin{aligned}\sum_{t=1}^{i+j+k} u_t + 3(t-1) &= \sum_{t=1}^{i+j+k} u_t + 3\sum_{t=0}^{i+j+k-1} t\\ &= n - 3\frac{(i+j+k)(i+j+k-1)}{2} + 3\frac{(i+j+k)(i+j+k-1)}{2}\\ &= n.\end{aligned}$$

By definition of $\mathcal{T}_6(n;i,j,k)$, $u_1 > 1$. Let $t \in [i+j+k-1]$ and assume that

$$3 \nmid \big(u_t + 3(s-1)\big) + (u_{t+1} + 3t).$$

Then $3 \nmid (u_t + u_{t+1})$ and therefore

$$u_{t+1} - u_t \geq 1.$$

Since u is almost non-decreasing it follows that

$$(u_{t+1} + 3t) - \big(u_t + 3(t-1)\big) = (u_{t+1} - u_t) + 3 \geq 1 + 3 = 4.$$

Generally it holds that

$$(u_{t+1} + 3t) - \big(u_t + 3(t-1)\big) = (u_{t+1} - u_t) + 3 \geq -1 + 3 = 2.$$

Hence

$$\varphi_7(u) \in \mathcal{D}(n;i,j,k).$$

To prove **bijectivity**, consider the following map which turns out to be the inverse of φ_7:

$$\begin{aligned} \psi_7 :\ & \mathcal{D}(n;i,j,k) \to \mathcal{T}_6(n;i,j,k) \\ & (\pi_t)_{t\in[i+j+k]}^{rev} \mapsto \big(\pi_t - 3(t-1)\big)_{t\in[i+j+k]}. \end{aligned}$$

We show that this map is well-defined. Let

$$\pi = (\pi_t)_{t\in[i+j+k]}^{rev} \in \mathcal{D}(n;i,j,k).$$

Since addition of multiples of 3 preserves the residual classes with respect to division by 3, $\psi_7(\pi)$ satisfies the congruence conditions in $\mathcal{T}_6(n;i,j,k)$. Moreover, by definition of $\mathcal{D}(n;i,j,k)$, $\pi_t > 1$ for all $t \in [i+j+k]$.
To check the difference conditions, let $t \in [i+j+k]$ such that

$$3 \mid \pi_t + \pi_{t+1}. \tag{59}$$

Then

$$(\pi_{t+1} - 3t) - \big(\pi_t - 3(t-1)\big) = \pi_{t+1} - \pi_t - 3 \geq 2 - 3 = -1.$$

Assume that

$$(\pi_{t+1} - 3t) - \big(\pi_t - 3(t-1)\big) = -1.$$

Then

$$\pi_{t+1} - \pi_t = 2,$$

and thus

$$\pi_t \equiv \pi_t - 3(t-1) \equiv 2 \pmod 3 \quad \text{and} \quad \pi_{t+1} \equiv \pi_{t+1} - 3t \equiv 1 \pmod 3,$$

by the difference conditions in $\mathcal{D}(n;i,j,k)$ and (59).
Moreover, if

$$(\pi_{t+1} - 3t) - \big(\pi_t - 3(t-1)\big) = 0,$$

it follows that $\pi_{t+1} - \pi_t = 3$ and thus

$$\pi_t \equiv \pi_t - 3(t-1) \equiv \pi_{t+1} \equiv \pi_{t+1} - 3t \equiv 0 \pmod 3.$$

Otherwise,

$$\big(\pi_t - 3(t-1)\big) - (\pi_{t+1} - 3t) < 0.$$

For $t \in [i+j+k]$ such that $3 \nmid (\pi_t + \pi_{t+1})$ it holds that

$$(\pi_{t+1} - 3t) - \big(\pi_t - 3\,(t-1)\big) = \pi_{t+1} - \pi_t - 3 \geq 4 - 3 = 1.$$

Finally, by Lemma 2.2, note that $\psi_7\,(\pi)$ has size

$$\sum_{t=1}^{i+j+k} \pi_t - 3\,(t-1) = \sum_{t=1}^{i+j+k} \pi_t - 3 \sum_{t=0}^{i+j+k-1} t = n - 3\frac{(i+j+k)\,(i+j+k-1)}{2}.$$

Thus, $\psi_7\,(\pi) \in \mathcal{T}_7\,(n;i,j,k)$.
To conclude the proof on bijectivity of φ_7 note that for all $(u_t)_{t\in[i+j+k]} \in \mathcal{T}_6\,(n;i,j,k)$ it holds that

$$\psi_7\left(\varphi_7\left((u_t)_{t\in[i+j+k]}\right)\right) = \psi_7\left(\big(u_t + 3\,(t-1)\big)^{rev}_{t\in[i+j+k]}\right) = (u_t)_{t\in[i+j+k]}\,,$$

and for all $(\pi_t)^{rev}_{t\in[i+j+k]} \in \mathcal{D}\,(n;i,j,k)$ it holds that

$$\varphi_7\left(\psi_7\left((\pi_t)^{rev}_{t\in[i+j+k]}\right)\right) = \varphi_7\left(\big(\pi_t - 3\,(t-1)\big)_{t\in[i+j+k]}\right) = (\pi_t)^{rev}_{t\in[i+j+k]}\,.$$

Example 5.17. *Continue Example 5.16. Note that*

$$\begin{aligned}\varphi_7\left(\varphi_6\left(\varphi_5\left(\varphi_4\left(\varphi_3\left(\varphi_2\left(\varphi_1\left(\lambda\right)\right)\right)\right)\right)\right)\right) &= \varphi_7\left((2,1,2,5,6,6,13,14,20)\right)\\ &= (2,4,8,14,18,21,31,35,44)^{rev}\\ &= (44,35,31,21,18,14,8,4,2)\,.\end{aligned}$$

Since φ_t is a bijection for all $t \in [7]$, we constructed a bijection

$$\varphi_7 \circ \varphi_6 \circ \varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1 :\ \mathcal{E}\,(n;i,j,k) \to \mathcal{D}\,(n;i,j,k)\,,$$

which proves the presented refinement of Capparelli's identity (Theorem 5.2). ■

5.4. Further possible refinements. In their paper "Refinements and Generalizations of Capparelli's Conjecture on Partitions" Alladi, Andrews and Gordon conjectured a further refinement of Capparelli's identity. To state it we need a bit more notation.

Definition 5.7. [AAG95b, p. 638] *For $n, i, j, k, N \in \mathbb{N}$, let $\tilde{E}(n; i, j, k, N)$ denote the number of partitions π of n counted by $E(n; i, j, k)$ such that $\mathcal{L} \leq 3N - 3k - 2$, where $\mathcal{L}$ is the largest part $\not\equiv 0 \pmod 3$ of π.*

Example 5.18. *For $n = 9$, $\tilde{E}(9; 1, 1, 0, 3) = 1$ counts the partition*

$$(4, 3, 2).$$

Definition 5.8. [AAG95b, p. 638] *For $n, i, j, k, N \in \mathbb{N}$, let $\tilde{D}(n; i, j, k, N)$ denote the number of partitions of n counted by $D(n; i, j, k)$ such that the largest part is $\leq 3N - 2$.*

Example 5.19. *For $n = 9$, $\tilde{D}(9; 1, 1, 0, 3) = 1$ counts the partition*

$$(7, 2).$$

Based on the proof of their first refinement, Alladi, Andrews and Gordon presented the following identity.

Theorem 5.3. [AAG95b, p. 638] *For all $n, i, j, k, N \in \mathbb{N}$ it holds that*

$$\tilde{E}(n; i, j, k, N) = \tilde{D}(n; i, j, k, N).$$

As it turns out, there are counter examples that disprove this conjecture.

Disproof. For $n = 16$, note that the partitions counted by $E(16)$ are

$$(16), (14, 2), (12, 4), (10, 6), (10, 4, 2), (9, 4, 3) \text{ and } (8, 6, 2).$$

The partitions counted by $D(16)$ are

$$(16), (14, 2), (13, 3), (12, 4), (11, 5), (10, 6) \text{ and } (10, 4, 2).$$

Set $i = 1$, $j = 0$, $k = 1$ and $N = 4$. Then

$$\tilde{E}(16; 1, 0, 1, 4) = 2,$$

since the two partitions counted by $\tilde{E}(16; 1, 0, 1, 4)$ are

$$(12, 4) \text{ and } (9, 4, 3).$$

Meanwhile,

$$\tilde{D}(16; 1, 0, 1, 4) = 1,$$

since the only partition counted by $\tilde{D}(16; 1, 0, 1, 4)$ is

$$(10, 6).$$

Thus

$$\tilde{E}(16; 1, 0, 1, 4) \neq \tilde{D}(16; 1, 0, 1, 4),$$

which disproves the theorem.

To find this counter example we used MATLAB programs whose code can be found in the appendix.

A difficulty in this conjecture is that we compare a restriction on the largest part restricted on two residual classes with respect to division by 3 with the generally largest part among all residual classes. The partitions counted by $\tilde{E}(n;i,j,k,N)$ can have arbitrary large parts that are multiples of 3, so bounding all parts of the corresponding partitions counted by $D(n;i,j,k)$ misses the necessary foundation.
One could guess that this conjecture can be fixed by restricting the condition on the largest part of the partitions counted by $D(n;i,j,k)$ to the corresponding two residual classes. But as it turns out, the "almost decreasing" condition in Steps 5 to 7 of the proof of the first refinement complicates identifying the largest part and so far, from the combinatorial proof of the presented first refinement (Theorem 5.2), we could not deduce another refinement involving restrictions on the size of the parts.

6. The method of weighted words

The method of weighted words has first been presented in a paper of Alladi and Gordon for the purpose of finding generalizations of Schur's theorem (Theorem 3.1, see [AG93] and [AG95]). In this thesis we will use it to provide an algebraic and a combinatorial proof of a generalization of Capparelli's identity (Theorem 4.1) which is stated in this section as Theorem 6.1. This generalization and its two mentioned proofs have first been presented in [AAG95b, p. 647 - 657].

6.1. Introduction. The main idea in this method is to use letters to be able to consider different types of parts in partitions. More explicitely for our purpose we introduce the letters a, b and c (call them **colors**, see [AAG95b, p.647]) and consider them as symbols with numerical indices. More explicitly we consider the elements of the sequences

$$\left(a_j\right)_{j\in\mathbb{N}^*}, \left(b_j\right)_{j\in\mathbb{N}_{\geq 2}} \text{ and } \left(c_j\right)_{j\in\mathbb{N}^*}.$$

We denote the set of these symbols as

$$\mathcal{S} := \{a_j \mid j \in \mathbb{N}^*\} \cup \{b_j \mid j \in \mathbb{N}_{\geq 2}\} \cup \{c_j \mid j \in \mathbb{N}^*\}.$$

For the purpose of considering finite sequences in $\mathcal{S}$ as generalizations of integer partitions we need to be able to determine if such a sequence is non-increasing. Therefore we define the following **lexicographic ordering** (see [AAG95b, p. 647, eq. (5.1)]). For all $j \in \mathbb{N}^*$,

$$a_j < b_{j+1} < c_j < a_{j+1}. \tag{60}$$

Example 6.1. *The smallest symbols according to this ordering are*

$$a_1 < b_2 < c_1 < a_2 < b_3 < c_2.$$

The concrete choice of this ordering - especially that $b_{j+1} < c_j$ for all $j \in \mathbb{N}^*$ - can be a bit confusing at first sight. However, the choice receives a natural legitimation as it is related to the ordering of natural numbers via the residual classes of division by 3. This reminds of congruence conditions in Capparelli's identity (Theorem 4.1).

Remark 6.1. [AAG95b, p. 647] Substituting a_j by $3j-2$, b_j by $3j-4$ and c_j by $3j$ we get that for all $j \in \mathbb{N}^*$,

$$3j-2 < 3j-1 < 3j < 3j+1.$$

Example 6.2. *For $j = 1$ and $j = 2$, as expected we get*

$$1 < 2 < 3 < 4 < 5 < 6.$$

We introduce the following definition concerning the numerical value of a symbol which is crucial when writing about generalizations of integer partitions. With this definition it is possible to consider the size of parts (elements of the mentioned finite sequences) and difference conditions.

Definition 6.1. [AAG95b, p. 648] *For all $x \in \{a, b, c\}$ and $j \in \mathbb{N}^*$ we define the **weight** of the symbol x_j to be*

$$\omega\left(x_j\right) := j.$$

*Further, for $x, y \in \{a, b, c\}$ and $i, j \in \mathbb{N}^*$, the **gap** between the two symbols x_i and y_j is defined to be*

$$g\left(x_i, y_j\right) := \left|\omega\left(x_i\right) - \omega\left(y_j\right)\right| = |i - j|.$$

*For all $j \in \mathbb{N}^*$, a_j is called an **a-part** and c_j is called a **c-part**. Consistently, for all $j \in \mathbb{N}_{\geq 2}$ we call b_j a **b-part**.*

Example 6.3. *With this definition, a_1 has weight*

$$\omega\left(a_1\right) = 1,$$

and the gap between the a-part a_1 and the c-part c_3 is

$$g\left(a_1, c_3\right) = |1 - 3| = 2.$$

Finally we introduce the before mentioned generalized integer partitions.

Definition 6.2. [AAG95b, p. 648] *A **word** is a finite sequence in $\mathcal{S}$ such that its elements form a non-increasing finite sequence. The **size** of a word α, denoted by $s\left(\alpha\right)$, is the sum of the weights of the elements of α. For $n \in \mathbb{N}$ we denote the set of words of size n by $\Omega\left(n\right)$.*

Example 6.4. *The word $\alpha := \left(a_5, c_2, b_3, b_3, c_1, b_2\right)$ is of size*

$$s\left(\alpha\right) = 5 + 2 + 3 + 3 + 1 + 2 = 16,$$

with 1 a-part, 3 b-parts and 2 c-parts.

Indeed, words can be seen as generalized integer partitions as there is a distinction of colours in addition to the distinction of the size of parts (or, in the language of words, weights).

Remark 6.2. Using Remark 6.1 there is a natural correspondence between words and integer partitions. Note that the size of a word is not the same as the size of its corresponding integer partition.

Example 6.5. *The corresponding partition of the considered word α in Example 6.4 is*

$$\overline{\alpha} = \left(3 \cdot 5 - 2, 3 \cdot 2, 3 \cdot 3 - 4, 3 \cdot 3 - 4, 3 \cdot 1, 3 \cdot 2 - 4\right) = \left(13, 6, 5, 5, 3, 2\right).$$

Note that $\overline{\alpha}$ is a partition of

$$13 + 6 + 5 + 5 + 3 + 2 = 34 \neq 16 = s\left(\alpha\right).$$

6.2. Generalization of Capparelli's identity. As mentioned before we will present a generalization of Caparelli's identity. To keep the statement short we will first introduce a bit more notation.

Definition 6.3. [AAG95b, p. 648] *Let $n, i, j, k \in \mathbb{N}$. Define $K(n; i, j, k)$ to be the number of finite sequences of words (π_1, π_2, π_3) where the elements of π_1 are i distinct a-parts with even weights, the ones of π_2 are j distinct b-parts with even weights and those of π_3 distinct c-parts such that precisely k of them have a weight that is greater than $i+j$. Additionally, the weights of all elements of the elements of (π_1, π_2, π_3) sum up to n.*

Example 6.6. *$K(10; 2, 1, 0) = 3$ counts the finite sequences*

$$\left((a_6, a_2), (b_2), \emptyset\right), \left((a_4, a_2), (b_4), \emptyset\right), \left((a_4, a_2), (b_2), (c_2)\right).$$

Definition 6.4. [AAG95b, p. 648] *Let $n, i, j, k \in \mathbb{N}$. Define $\mathcal{G}(n; i, j, k)$ to be the set of words of size n where each element is greater than a_1 and the minimal gap between consecutive elements is given by the following table.*

	a	b	c
a	2	2	1
b	0	2	0
c	2	3	1

This table should be read in the following way: Consider two consecutive parts of such a word, say $x_{k_1} \geq y_{k_2}$. Then y corresponds to the row and x corresponds to the column. Further, we denote

$$G(n; i, j, k) := \mathcal{G}(n; i, j, k).$$

Example 6.7. *For $n = 10$, $G(10; 2, 1, 0) = 3$ counts the words*

$$(a_6, a_2, b_2), (a_5, a_3, b_2), (a_4, b_4, a_2).$$

Now we can state a generalization of Capparelli's identity.

Theorem 6.1. [AAG95b, p. 648] *For all $n, i, j, k \in \mathbb{N}$ it holds that*

$$K(n; i, j, k) = G(n; i, j, k).$$

Remark 6.3. Indeed, this is a generalization of Capparelli's identity. Note that Theorem 5.1 can be deduced by replacing q by q^3, a by aq^{-2} and b by bq^{-4} in the generating functions in Lemma 6.2, which we will use to prove Theorem 6.1 ([AAG95b, p. 647]). Using the ideas how generating functions of partitions are constructed (see the proof of Theorem 2.1) and the generalization Theorem 6.1, we get the following correspondences. Note that we change the weights of the a-parts by multiplying them by 3 and subtracting 2, the weights of the b-parts by multiplying them by 3 and subtracting 4 and the weights of the c-parts by multiplying them by 3.

(i) Words consisting of distinct a-parts with even weights correspond to partitions into distinct parts congruent to $2 \cdot 3 - 2 \equiv 4 \pmod 6$.

(ii) Words consisting of distinct b-parts with even weights correspond to partitions into distinct parts congruent to $2 \cdot 3 - 4 \equiv 2 \pmod 6$.

(iii) Words consisting of distinct c-parts correspond to partitions into distinct parts congruent to $3 \equiv 0 \pmod 3$, i.e. multiples of 3.

If we still use letters a (for i)), b (for ii)) and c (for iii)) in the correspondences we get the following table for the minimal differences.

	a	b	c
a	6	4	5
b	2	6	4
c	4	5	3

This can be justified by looking at the following transformations of pairs of possible consecutive symbols (recall the minimal gap conditions in Definition 6.4). For $k \in \mathbb{N}^*$,

$$\begin{aligned}
(a_{k+2}, a_k) &\longrightarrow (3k+4, 3k-2)\,,\\
(a_k, b_k) &\longrightarrow (3k-2, 3k-4)\,,\\
(a_{k+2}, c_k) &\longrightarrow (3k+4, 3k)\,,\\
(b_{k+2}, a_k) &\longrightarrow (3k+2, 3k-2)\,,\\
(b_{k+2}, b_k) &\longrightarrow (3k+2, 3k-4)\,,\\
(b_{k+3}, c_k) &\longrightarrow (3k+5, 3k)\,,\\
(c_{k+1}, a_k) &\longrightarrow (3k+3, 3k-2)\,,\\
(c_k, b_k) &\longrightarrow (3k, 3k-4)\,,\\
(c_{k+1}, c_k) &\longrightarrow (3k+3, 3k)\,.
\end{aligned}$$

Therefore by Theorem 6.1, the number of partitions of $n \in \mathbb{N}$ into distinct parts congruent to $0, 2, 3$ or $4 \pmod 6$ is equal to the number of partitions of n into distinct parts congruent to $3, 3-2, 6-4 \equiv 0, 1, 2 \pmod 3$ all greater than $3 \cdot 1 - 2$ such that the difference between two consecutive parts is given by the table above. Note that a represents parts congruent to $1 \pmod 3$, b represents parts congruent to $2 \pmod 3$ and c represents multiples of 3. This is exactly the number of partitions of n into distinct parts all greater than 1 such that the difference between consecutive parts is greater than 3 unless they are both multiples of 3 or they add up to a multiple of 6 (which happens when a part represented by b is followed by a part represented by a with difference 2).
Thus we can deduce Theorem 5.1.

6.3. Preparation for the proof using generating functions. At the heart of the proof of Theorem 6.1 lies determining the generating functions of $\left(G\left(n;i,j,k\right)\right)_{n\in\mathbb{N}}$ which involves a good amount of computations. We will use the concept of minimal words and recursive formulas for Gaussian polynomials (see Theorem 2.5).

Definition 6.5. *Let $n,i,j,k\in\mathbb{N}$. A **minimal word** in $\mathcal{G}\left(n;i,j,k\right)$ is a word $\omega\in\mathcal{G}\left(n;i,j,k\right)$ in which the gap between every two consecutive elements is exactly the allowed minimal gap in Definition 6.4. The set of minimal words in $\mathcal{G}\left(n;i,j,k\right)$ is denoted by $\mathcal{H}\left(n;i,j,k\right)$. For convenience, we extend the definition to $i,j,k\in\mathbb{Z}$ by defining $\mathcal{H}\left(n;i,j,k\right)$ to be the empty set if $i<0,j<0$ or $k<0$.*

Example 6.8. *For $n=10,i=2,j=1$ and $k=0$ there is exactly one minimal word in $\mathcal{G}\left(10;2,1,0\right)$, namely (a_4,b_4,a_2). Thus*

$$\mathcal{H}\left(10;2,1,0\right)=\{(a_4,b_4,a_2)\}.$$

Since we consider words with elements with distinct weights (see Definition 6.3) we will often use the following term in the corresponding generating functions (recall the idea behind Corollary 2.4).

Definition 6.6. [AAG95b, p. 648] *For all $i\in\mathbb{N}$, define the ith triangular number as*

$$T_i:=\frac{i\left(i+1\right)}{2}.$$

Example 6.9. *Note that*

$$T_4=\frac{4\cdot 5}{2}=10=4+3+2+1.$$

Remark 6.4. Indeed, for all $i\in\mathbb{N}$,

$$T_i=\sum_{j=1}^{i}j.$$

This well-known result is due to Gauss and proven in Lemma 2.2. Thus it holds for all $i\in\mathbb{N}^*$ that

$$T_{i-1}+i=T_i.$$

For convenience we introduce a bit of notation used in the proof of the following lemma.

Definition 6.7. [AAG95b, p. 650] *For all $i,j,k\in\mathbb{N}$, let $H_m\left(i,j,k\right)$ be the generating function of minimal words in*

$$\bigcup_{n\in\mathbb{N}}\mathcal{G}\left(n;i,j,k\right).$$

Moreover $H_a\left(i,j,k\right)$, $H_b\left(i,j,k\right)$ and $H_c\left(i,j,k\right)$ are the generating functions of those minimal words in

$$\bigcup_{n\in\mathbb{N}}\mathcal{G}\left(n;i,j,k\right)$$

with last element a_1,b_2 or c_1 respectively.

Remark 6.5. Let $i,j,k \in \mathbb{N}$. Note that by Definition 6.7, for $i+j+k>0$,

$$H_a(i,j,k) + H_b(i,j,k) + H_c(i,j,k) = H_m(i,j,k).$$

Definition 6.8. *Let $n,i,j,k \in \mathbb{N}$ and μ be a word of length $s \leq i+j+k$. Then*

$$\mathcal{H}_\mu(n;i,j,k) := \{(a_i)_{i\in[i+j+k]} \in \mathcal{H}(n;i,j,k) \mid (a_i)_{i\in[i+j+k]\setminus[i+j+k-s]} = \mu\}.$$

Example 6.10. *Let $n,i,j,k \in \mathbb{N}$. Then $\mathcal{H}_{(a_4,a_2)}(n;i,j,k)$ is the set of minimal words in $\mathcal{G}(n;i,j,k)$ ending with (a_4,a_2).*

The following lemma provides the most important technical tool to prove Lemma 6.2 to compute the generating function in (71).

Lemma 6.1. [AAG95b, p. 652 - 654] *Let $n,i,j,k \in \mathbb{N}$. Then the following equalities between cardinalities of sets hold:*

$$|\mathcal{H}_{(a_4,a_2)}(n;i,j,k)| = |\mathcal{H}_{(a_2)}(n-2(i+j+k);i-1,j,k)|, \tag{61}$$
$$|\mathcal{H}_{(b_4,a_2)}(n;i,j,k)| = |\mathcal{H}_{(b_2)}(n-2(i+j+k);i-1,j,k)|, \tag{62}$$
$$|\mathcal{H}_{(c_3,a_2)}(n;i,j,k)| = |\mathcal{H}_{(c_1)}(n-2(i+j+k);i-1,j,k)|, \tag{63}$$
$$|\mathcal{H}_{(a_2,b_2)}(n;i,j,k)| = |\mathcal{H}_{(a_2)}(n-2;i,j-1,k)|, \tag{64}$$
$$|\mathcal{H}_{(b_4,b_2)}(n;i,j,k)| = |\mathcal{H}_{(b_2)}(n-2(i+j+k);i,j-1,k)|, \tag{65}$$
$$|\mathcal{H}_{(c_2,b_2)}(n;i,j,k)| = |\mathcal{H}_{(c_1)}(n-(i+j+k)-1;i,j-1,k)|, \tag{66}$$
$$|\mathcal{H}_{(a_3,c_1)}(n;i,j,k)| = |\mathcal{H}_{(a_2)}(n-(i+j+k);i,j,k-1)|, \tag{67}$$
$$|\mathcal{H}_{(b_4,c_1)}(n;i,j,k)| = |\mathcal{H}_{(b_2)}(n-2(i+j+k)+1;i,j,k-1)|, \tag{68}$$
$$|\mathcal{H}_{(c_2,c_1)}(n;i,j,k)| = |\mathcal{H}_{(c_1)}(n-(i+j+k);i,j,k-1)|. \tag{69}$$

Proof. [AAG95b, p. 652 - 654] Let $n,i,j,k \in \mathbb{N}$. We start with proving Equation (61) by constructing a bijection between the two mentioned sets. Define

$$\varphi : \mathcal{H}_{(a_4,a_2)}(n;i,j,k) \to \mathcal{H}_{(a_2)}(n-2(i+j+k);i-1,j,k)$$
$$(\mu_s)_{s\in[i+j+k]} \mapsto (\lambda_s)_{s\in[i+j+k-1]},$$

where λ_s is obtained from μ_s by reducing its weight by 2.

Example 6.11. *For $n=50, i=3, j=2, k=2$, note that*

$$\varphi((b_{13},b_{11},c_8,a_7,c_5,a_4,a_2)) = (b_{11},b_9,c_6,a_5,c_3,a_2).$$

First we prove that φ is well-defined. For this purpose let

$$\mu = (\mu_s)_{s\in[i+j+k]} \in \mathcal{H}_{(a_4,a_2)}(n;i,j,k).$$

Note that by Definition 6.2 the elements of a word form a non-increasing finite sequence. Since $\mu_{i+j+k-1} = a_4$ it follows that $\omega(\mu_s) \geq 4$ for all $s \in [i+j+k-1]$. Therefore subtracting 2 from each weight is admissible. By this subtraction step it is also clear that indeed the size of $\varphi(\mu)$ is $n-2(i+j+k)$ and since the last element of μ is cancelled $\varphi(\mu)$

has only $i-1$ a-parts. Moreover, by construction the last element of $\varphi(\mu)$ is $a_{4-2}=a_2$. Hence

$$\varphi(\mu) \in \mathcal{H}_{(a_2)}\left(n-2(i+j+k);i-1,j,k\right).$$

To show bijectivity of φ we construct its inverse. Define

$$\begin{aligned}\psi : \mathcal{H}_{(a_2)}\left(n-2(i+j+k);i-1,j,k\right) &\to \mathcal{H}_{(a_4,a_2)}(n;i,j,k)\\ (\lambda_s)_{s\in[i+j+k-1]} &\mapsto (\mu_s)_{s\in[i+j+k-1]},\end{aligned}$$

where $\mu_{i+j+k}=a_2$ and for $s<i+j+k$, μ_s is obtained from λ_s by increasing its weight by 2.

Example 6.12. *For $n=50, i=3, j=2, k=2$, note that*

$$\psi\left((b_{11},b_9,c_6,a_5,c_3,a_2)\right)=(b_{13},b_{11},c_8,a_7,c_5,a_4,a_2).$$

Indeed this map is well-defined: Take $\lambda \in \mathcal{H}_{(a_2)}\left(n-2(i+j+k);i-1,j,k\right)$. Since we add 2 to each weight and attach a_2 as the new last element, the size of $\psi(\lambda)$ is n and its number of a-parts is $(i-1)+1=i$. Moreover by construction $\varphi(\mu)$ ends with (a_4,a_2). Hence

$$\psi(\lambda) \in \mathcal{H}_{(a_4,a_2)}(n;i,j,k).$$

To conclude we note that $\varphi\circ\psi$ and $\psi\circ\varphi$ are both identity maps. Therefore φ is indeed bijective and thus

$$|\mathcal{H}_{(a_4,a_2)}(n;i,j,k)| = |\mathcal{H}_{(a_2)}\left(n-2(i+j+k);i-1,j,k\right)|.$$

For Equations (62) to (69) the proof works in the exact same way up to adjusting the number subtracted from the weights and consequently the size of the words in the set on the right hand side of the equation. In equations (66), (67) and (69) each weight is only reduced by 1 and in Equation (64) the map from the set on the left hand side to the set on the right hand side consists only in cancelling the last element and preserves the weights of the remaining elements. ■

Using the techniques of generating functions, Theorem 6.1 can be deduced using the following lemma.

Lemma 6.2. [AAG95b, p. 648 - 649] *The following two identities hold:*

$$\sum_{n=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} K(n;i,j,k)\,a^ib^jc^kq^n = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{a^ib^jq^{2T_i+2T_j}(-q)_{i+j}\left(-cq^{i+j+1}\right)_\infty}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j}, \tag{70}$$

$$\begin{aligned}\sum_{n=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} G(n;i,j,k)\,a^ib^jc^kq^n &= \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{a^ib^jc^kq^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}}\\ &\quad\cdot\begin{bmatrix} i+j+k \\ i+j\end{bmatrix}_q\begin{bmatrix} i+j \\ i\end{bmatrix}_{q^2}.\end{aligned} \tag{71}$$

Remark 6.6. [AAG95b, p. 649] For $c = 1$, note that (70) becomes

$$\sum_{n=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} K(n;i,j,k)\, a^i b^j q^n = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{a^i b^j q^{2T_i+2T_j}\,(-q)_{i+j}\left(-q^{i+j+1}\right)_\infty}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j}.$$

Noting that

$$(-q)_{i+j}\left(-q^{i+j+1}\right)_\infty = \left(\prod_{s=1}^{i+j}(1+q^s)\right)\left(\prod_{s=i+j+1}^{+\infty}(1+q^s)\right) = (-q)_\infty$$

yields

$$\begin{aligned}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{a^i b^j q^{2T_i+2T_j}\,(-q)_{i+j}\left(-q^{i+j+1}\right)_\infty}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j} &= \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{a^i b^j q^{2T_i+2T_j}\,(-q)_\infty}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j} \\ &= (-q)_\infty\left(\sum_{i=0}^{+\infty}\frac{a^i q^{2T_i}}{\left(q^2;q^2\right)_i}\right)\left(\sum_{j=0}^{+\infty}\frac{b^j q^{2T_j}}{\left(q^2;q^2\right)_j}\right).\end{aligned}$$

By applying twice Equation (9) in Corollary 2.5 for $q = q^2$, once for $t = aq^2$ and once for $t = bq^2$, we get that

$$(-q)_\infty\left(\sum_{i=0}^{+\infty}\frac{a^i q^{2T_i}}{\left(q^2;q^2\right)_i}\right)\left(\sum_{j=0}^{+\infty}\frac{b^j q^{2T_j}}{\left(q^2;q^2\right)_j}\right) = (-q)_\infty\left(-aq^2;q^2\right)_\infty\left(-bq^2;q^2\right)_\infty.$$

Replacing q by q^3, a by q^{-2} and b by q^{-4} this expression becomes

$$\begin{aligned}\left(-q^3;q^3\right)_\infty\left(-q^4;q^6\right)_\infty\left(-q^2;q^6\right)_\infty &= \prod_{j=0}^{+\infty}\left(1+q^{3+3j}\right)\left(1+q^{4+6j}\right)\left(1+q^{2+6j}\right) \\ &= \sum_{n=0}^{+\infty} E(n)\, q^n,\end{aligned}$$

where we used again Theorem 2.1 for $H = \{n \in \mathbb{N} \mid n \equiv 0,2,3,4 \pmod 6\}$, and $d = 1$.

Proof. [AAG95b, p. 649 - 654] Equation (70) follows straight from the definition of $K(n;i,j,k)$ using Theorem 2.1 and Corollary 2.4. Note that the choices of the parts in π_1, π_2 and π_3 for finite sequences (π_1,π_2,π_3) counted by $K(n;i,j,k)$ are independent of each other. The words π_1 (consisting of i distinct a-parts with even weights) can be considered as partitions into i distinct even parts. By Corollary 2.4 for $t = 2$ the generating function for such partitions is

$$\sum_{i=0}^{+\infty}\frac{a^i q^{2T_i}}{\left(q^2;q^2\right)_i}.$$

Similarly, the generating function for the words π_2 (consisting of j distinct b-parts with even weights) is

$$\sum_{j=0}^{+\infty} \frac{b^j q^{2T_j}}{\left(q^2;q^2\right)_j}.$$

Moreover, for all $i,j \in \mathbb{N}$, using Theorem 2.1 for $H = [i+j]$ and $d = 1$, $(-q)_{i+j}$ is the generating function of partitions into distinct parts all less or equal than $i+j$ and by another application of Theorem 2.1 for $H = \mathbb{N}_{\geq i+j+1}$ and $d = 1$, $\left(-cq^{i+j+1}\right)_\infty$ is the generating function of partitions into distinct parts greater than $i+j$ with c keeping track of the number of parts. Note that the words π_3 (consisting of c-parts with distinct weights where exactly k parts have a weight greater than $i+j$) can be considered as pairs (λ_1, λ_2) where λ_1 is a partition into distinct parts smaller or equal $i+j$ and λ_2 is a partitions into k distinct parts greater than $i+j$. Therefore

$$\begin{aligned}\sum_{n=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} K\left(n;i,j,k\right) a^i b^j c^k q^n &= \sum_{i=0}^{+\infty} \frac{a^i q^{2T_i}}{\left(q^2;q^2\right)_i} \sum_{j=0}^{+\infty} \frac{b^j q^{2T_j}}{\left(q^2;q^2\right)_j} \left(-q\right)_{i+j} \left(-cq^{i+j+1}\right)_\infty \\ &= \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{a^i b^j q^{2T_i+2T_j} \left(-q\right)_{i+j} \left(-cq^{i+j+1}\right)_\infty}{\left(q^2;q^2\right)_i \left(q^2;q^2\right)_j}.\end{aligned}$$

The proof of Equation (71) involves the essential ideas of the method of weighted words of Alladi and Gordon. The goal is to show that for all $i,j,k \in \mathbb{N}$,

$$\sum_{n=0}^{+\infty} G\left(n;i,j,k\right) q^n = \frac{q^{2T_i+2T_j+T_k+(i+j)k}}{\left(q\right)_{i+j+k}} \begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}. \tag{72}$$

This is equivalent to the statement that for all $i,j,k \in \mathbb{N}$,

$$\begin{aligned}&[a^i b^j c^k] \sum_{n=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} G\left(n;i,j,k\right) a^i b^j c^k q^n \\ &= [a^i b^j c^k] \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k}}{\left(q\right)_{i+j+k}} \begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2},\end{aligned}$$

which is equivalent to Equation (71). To prove (72), define for all $i,j,k \in \mathbb{N}$,

$$H\left(i,j,k\right) := \sum_{n=0}^{+\infty} G\left(n;i,j,k\right) q^n.$$

Then using the exact same techniques as in the proofs of Proposition 2.2 and Corollary 2.4, we can deduce that

$$\frac{H_m\left(i,j,k\right)}{\left(q\right)_{i+j+k}} = H\left(i,j,k\right). \tag{73}$$

Substituting $H\left(i,j,k\right)$ in (72) yields the equivalent equation

$$H_m\left(i,j,k\right) = q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}. \tag{74}$$

To prove it, we will compute $H_a(i,j,k)$, $H_b(i,j,k)$ and $H_c(i,j,k)$ and use Remark 6.5 to conclude.
More precisely, we prove that for all $i,j,k \in \mathbb{N}$ the following equations hold:

$$H_a(i,j,k) = q^{Z(i,j,k)+k+2j} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}, \tag{75}$$

$$H_b(i,j,k) = q^{Z(i,j,k)} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2}. \tag{76}$$

And despite the claim in [AAG95b, p. 652, eq. (5.10c)], if $\mathbf{k} > \mathbf{0}$,

$$H_c(0,0,k) = q^{T_k}, \tag{77}$$

and if $\mathbf{i}+\mathbf{j} > \mathbf{0}$ or $\mathbf{i}+\mathbf{j}+\mathbf{k} = \mathbf{0}$ (consistent with the source),

$$\begin{aligned} H_c(i,j,k) = q^{Z(i,j,k)} &\begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \\ &\cdot \left(q^{i+j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \right), \end{aligned} \tag{78}$$

where

$$Z(i,j,k) := 2T_i + 2T_j + T_k + (i+j)\,k.$$

We prove them by induction on $i+j+k$. If $i+j+k = 0$, $i = j = k = 0$ and hence by definition,

$$H_a(i,j,k) = H_b(i,j,k) = H_c(i,j,k) = 0.$$

Moreover, note that by the definition of Gaussian polynomials (Definition 2.37),

$$q^{2T_0+2T_0+T_0} \begin{bmatrix} -1 \\ -1 \end{bmatrix}_q \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{q^2} = 0,$$

$$q^{2T_0+2T_0+T_0} \begin{bmatrix} -1 \\ -1 \end{bmatrix}_q \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{q^2} = 0,$$

$$q^{2T_0+2T_0+T_0} \begin{bmatrix} -1 \\ 0 \end{bmatrix}_q \left(q^0 \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{q^2} + q^0 \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{q^2} \right) = 0.$$

Now let $m \in \mathbb{N}$ and assume that Equations (75), (76) and (78) hold for all $i,j,k \in \mathbb{N}$ such that $i+j+k = m$.
Let $i,j,k \in \mathbb{N}$ such that $i+j+k = m+1$. If $\mathbf{i} = \mathbf{0}$, by Definition 6.8 for all $n \in \mathbb{N}$,

$$\mathcal{H}_{(a_2)}(n;i,j,k) = \emptyset,$$

and thus

$$H_a(0,j,k) = 0 = q^{Z(0,j,k)+k+2j} \begin{bmatrix} j+k-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} j-1 \\ -1 \end{bmatrix}_{q^2}.$$

Assume now that $\mathbf{i} > \mathbf{0}$. If $\mathbf{i}+\mathbf{j}+\mathbf{k} = \mathbf{1}$ it follows that $(i,j,k) = (1,0,0)$. Note that the only minimal word with these parameters ending with an a-part is (a_2). Thus

$$H_a(1,0,0) = q^2 = q^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q^2}.$$

Assume that $\mathbf{i}+\mathbf{j}+\mathbf{k} > \mathbf{1}$. Then, by Definition 6.4, we can partition the set of minimal words ending with an a-part. For all $n \in \mathbb{N}$,

$$\mathcal{H}_{(a_2)}(n;i,j,k) = \mathcal{H}_{(a_4,a_2)}(n;i,j,k) \sqcup \mathcal{H}_{(b_4,a_2)}(n;i,j,k) \sqcup \mathcal{H}_{(c_3,a_2)}(n;i,j,k). \tag{79}$$

Using Lemma 6.1 and the induction hypothesis (Equation (75) for i replaced by $i-1$) we see that the generating function of

$$\left(|\mathcal{H}_{(a_4,a_2)}(n;i,j,k)|\right)_{n\in\mathbb{N}}$$

is

$$q^{2(i+j+k)} q^{2T_{i-1}+2T_j+T_k+(i+j)k+2j} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2},$$

where the term $q^{2(i+j+k)}$ is justified by the change of the size of the considered words in Equation (61). The exponent of q can be simplified using Remark 6.4. So the generating function equals

$$q^{2T_i+2T_j+T_k+(i+j)k+4j+2k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2}.$$

In the same way Equation (62) and the induction hypothesis (Equation (76) for i replaced by $i-1$) yield that the generating function of

$$\left(|\mathcal{H}_{(b_4,a_2)}(n;i,j,k)|\right)_{n\in\mathbb{N}}$$

is

$$\begin{aligned} &q^{2(i+j+k)} q^{2T_{i-1}+2T_j+T_k+(i+j-1)k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} \\ &= q^{2T_i+2T_j+T_k+(i+j)k+2j+k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2}, \end{aligned}$$

using again Remark 6.4.
Finally, also in the third case we can use a bijection given in Lemma 6.1. Equation (63) and the induction hypothesis (Equation (77) for i replaced by $i-1$) provide that if $\mathbf{i}+\mathbf{j} = \mathbf{1}$ (i.e. $(i,j) = (1,0)$, in this case $\mathbf{k} > \mathbf{0}$),

$$q^{2(1+k)} q^{T_k} = q^{T_k+2k+2}$$

is the generating function of

$$\left(|\mathcal{H}_{(c_3,a_2)}(n;i,j,k)|\right)_{n\in\mathbb{N}}.$$

Then using (79) we get

$$H_a(1,0,k) = q^{T_k+2k+2} = q^{2+T_k+k+k} \begin{bmatrix} k \\ 0 \end{bmatrix}_q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q^2}.$$

However, if $\mathbf{i+j>1}$ Equation (63) and the induction hypothesis (Equation (78) for i replaced by $i-1$) provide that

$$q^{2(i+j+k)} q^{2T_{i-1}+2T_j+T_k+(i+j-1)k} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \cdot \left(q^{i+j-1} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2} \right)$$

is the generating function of

$$\left(\left| \mathcal{H}_{(c_3,a_2)}(n;i,j,k) \right| \right)_{n\in\mathbb{N}}.$$

It equals (again by Remark 6.4)

$$q^{2T_i+2T_j+T_k+(i+j)k+2j+k} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \cdot \left(q^{i+j-1} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2} \right).$$

Denote

$$Z_a(i,j,k) := Z(i,j,k) + 2j + k.$$

Then using (79) we get

$$\begin{aligned} H_a(i,j,k) &= q^{Z_a(i,j,k)+2j+k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2} \\ &\quad + q^{Z_a(i,j,k)} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} \\ &\quad + q^{Z_a(i,j,k)} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \left(q^{i+j-1} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2} \right) \\ &= q^{Z_a(i,j,k)} \left(\left(\begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q + q^{i+j-1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \right) \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} \right. \\ &\quad \left. + \left(q^k \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q + \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \right) q^{2j} \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2} \right). \end{aligned}$$

Using the recursion formulas (18), (17) and (19) (in this order), replacing i by $i-1$, we can conclude that

$$H_a(i,j,k) = q^{Z_a(i,j,k)} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-2 \\ i-2 \end{bmatrix}_{q^2} \right)$$
$$= q^{Z_a(i,j,k)} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2},$$

which is exactly (75) (recall that $i+j-1>0$).
For proving Equation (76) we follow the same procedure. Let again $i,j,k \in \mathbb{N}$ be arbitrary such that $i+j+k = m+1$. If $\mathbf{j} = \mathbf{0}$, by Definition 6.8 for all $n \in \mathbb{N}$,

$$\mathcal{H}_{(b_2)}(n;i,j,k) = \emptyset,$$

and thus

$$H_b(i,0,k) = 0 = q^{Z(i,0,k)} \begin{bmatrix} i+k-1 \\ i-1 \end{bmatrix}_q \begin{bmatrix} i-1 \\ i \end{bmatrix}_{q^2}.$$

Assume now that $\mathbf{j} > \mathbf{0}$. If $\mathbf{i}+\mathbf{j}+\mathbf{k} = \mathbf{1}$ it follows that $(i,j,k) = (0,1,0)$. Note that the only minimal word with these parameters ending with a b-part is (b_2). Thus

$$H_b(0,1,0) = q^2 = q^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q^2}.$$

Assume that $\mathbf{i}+\mathbf{j}+\mathbf{k} > \mathbf{1}$. Then, by Definition 6.4, we can partition the set of minimal words ending with a b-part. For all $n \in \mathbb{N}$,

$$\mathcal{H}_{(b_2)}(n;i,j,k) = \mathcal{H}_{(a_2,b_2)}(n;i,j,k) \sqcup \mathcal{H}_{(b_4,b_2)}(n;i,j,k) \sqcup \mathcal{H}_{(c_2,b_2)}(n;i,j,k). \tag{80}$$

Using Lemma 6.1 and the induction hypothesis (Equation (75) for j replaced by $j-1$) we see that the generating function of

$$\left(|\mathcal{H}_{(a_2,b_2)}(n;i,j,k)| \right)_{n\in\mathbb{N}}$$

is

$$q^2 q^{2T_i+2T_{j-1}+T_k+(i+j)k+2(j-1)} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2}, \tag{81}$$

where the term q^2 is justified by the change of the size of the considered words in Equation (64). It can be written as

$$q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2}.$$

In the same way Equation (65) and the induction hypothesis (Equation (76) for j replaced by $j-1$) yield that the generating function of

$$\left(|\mathcal{H}_{(b_4,b_2)}(n;i,j,k)| \right)_{n\in\mathbb{N}}$$

is

$$q^{2(i+j+k)} q^{2T_i+2T_{j-1}+T_k+(i+j-1)k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2}$$

$$= q^{2T_i+2T_j+T_k+(i+j)k+2i+k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2},$$

using Remark 6.4.
Also again in the third case we can use a bijection given in Lemma 6.1. Equation (66) and the induction hypothesis (Equation (77) for j replaced by $j-1$) provide that if $\mathbf{i}+\mathbf{j}=\mathbf{1}$ (i.e. $(i,j)=(0,1)$, in this case $\mathbf{k}>\mathbf{0}$),

$$q^{1+k+1}q^{T_k} = q^{T_k+k+2}$$

is the generating function of

$$\left(\left|\mathcal{H}_{(c_2,b_2)}\left(n;i,j,k\right)\right|\right)_{n\in\mathbb{N}}.$$

Then using (79) we get

$$H_b\left(0,1,k\right) = q^{T_k+k+2} = q^{2+T_k+k} \begin{bmatrix} k \\ 0 \end{bmatrix}_q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q^2}.$$

However, if $\mathbf{i}+\mathbf{j}>\mathbf{1}$ Equation (66) and the induction hypothesis (Equation (78) for j replaced by $j-1$) provide that

$$q^{i+j+k+1}q^{2T_i+2T_{j-1}+T_k+(i+j-1)k} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q$$
$$\cdot \left(q^{i+j-1} \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2} + q^{2(j-1)} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2}\right)$$

is the generating function of

$$\left(\left|\mathcal{H}_{(c_2,b_2)}\left(n;i,j,k\right)\right|\right)_{n\in\mathbb{N}}.$$

It equals (using again Remark 6.4)

$$q^{2T_i+2T_j+T_k+(i+j)k+i-j+1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q$$
$$\cdot \left(q^{i+j-1} \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2} + q^{2j-2} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2}\right).$$

Then using (80) we get

$$\begin{aligned} H_b\left(i,j,k\right) = {} & q^{Z(i,j,k)} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} \\ & + q^{Z(i,j,k)+2i+k} \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2} \\ & + q^{Z(i,j,k)+i-j+1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \end{aligned}$$

$$\cdot \left(q^{i+j-1} \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2} + q^{2j-2} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} \right)$$
$$= q^{Z(i,j,k)} \left(\left(\begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q + q^{i+j-1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \right) \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} \right.$$
$$\left. + \left(q^k \begin{bmatrix} i+j+k-2 \\ i+j-2 \end{bmatrix}_q + \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \right) q^{2i} \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2} \right).$$

Using the recursion formulas (18), (17) and (20) (in this order), replacing j by $j-1$, we can conclude that

$$H_b(i,j,k) = q^{Z(i,j,k)} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^2} + q^{2i} \begin{bmatrix} i+j-2 \\ i \end{bmatrix}_{q^2} \right)$$
$$= q^{Z(i,j,k)} \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2},$$

which is exactly (76) (recall that $i+j-1>0$).
Finally, for proving Equations (77) and (78) we follow once more the same procedure. Let again $i,j,k \in \mathbb{N}$ be arbitrary such that $i+j+k = m+1$. If $\mathbf{k} = \mathbf{0}$, by Definition 6.8 for all $n \in \mathbb{N}$,

$$\mathcal{H}_{(c_1)}(n;i,j,k) = \emptyset,$$

and thus

$$H_c(i,j,0) = 0 = q^{Z(i,0,k)} \begin{bmatrix} i+k-1 \\ i-1 \end{bmatrix}_q \begin{bmatrix} i-1 \\ i \end{bmatrix}_{q^2}.$$

Assume now that $\mathbf{k} > \mathbf{0}$. If $\mathbf{i}+\mathbf{j}+\mathbf{k} = \mathbf{1}$ it follows that $(i,j,k) = (0,0,1)$. Note that the only minimal word with these parameters ending with a c-part is (c_1). Thus

$$H_c(0,0,1) = q = q^{T_1}.$$

Assume that $\mathbf{i}+\mathbf{j}+\mathbf{k} > \mathbf{1}$. Then, by Definition 6.4, we can partition the set of minimal words ending with a c-part. For all $n \in \mathbb{N}$,

$$\mathcal{H}_{(c_1)}(n;i,j,k) = \mathcal{H}_{(a_3,c_1)}(n;i,j,k) \sqcup \mathcal{H}_{(b_4,c_1)}(n;i,j,k) \sqcup \mathcal{H}_{(c_2,c_1)}(n;i,j,k). \tag{82}$$

Using Lemma 6.1 and the induction hypothesis (Equation (75) for k replaced by $k-1$) we see that the generating function of

$$\left(|\mathcal{H}_{(a_3,c_1)}(n;i,j,k)| \right)_{n \in \mathbb{N}}$$

is

$$q^{i+j+k} q^{2T_i+2T_j+T_{k-1}+(i+j+1)(k-1)+2j} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}. \tag{83}$$

Using Remark 6.4, it can be written as

$$q^{2T_i+2T_j+T_k+(i+j)k+k-1+2j} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}.$$

In the same way Equation (68) and the induction hypothesis (Equation (76) for k replaced by $k-1$) yields that the generating function of

$$\left(\left|\mathcal{H}_{(b_4,c_1)}\left(n;i,j,k\right)\right|\right)_{n\in\mathbb{N}}$$

is

$$\begin{aligned} &q^{2(i+j+k)-1} q^{2T_i+2T_j+T_{k-1}+(i+j)(k-1)} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\ &= q^{2T_i+2T_j+T_k+(i+j)k+i+j+k-1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2}, \end{aligned}$$

using again Remark 6.4.
In the third case we can use a bijection given in Lemma 6.1. Equation (69) and the induction hypothesis (Equation (77) for k replaced by $k-1$) provide that if $\mathbf{i}+\mathbf{j}=\mathbf{0}$ (i.e. $i=j=0$, in this case $k>1$), using Remark 6.4,

$$q^k q^{T_{k-1}} = q^{T_k}$$

is the generating function of

$$\left(\left|\mathcal{H}_{(c_2,c_1)}\left(n;i,j,k\right)\right|\right)_{n\in\mathbb{N}}.$$

Then using (79) we get

$$H_b\left(0,1,k\right) = q^{T_k}.$$

However, if $\mathbf{i}+\mathbf{j}>\mathbf{0}$, Equation (66) and the induction hypothesis (Equation (78) for k replaced by $k-1$) provide that

$$\begin{aligned} &q^{i+j+k} q^{2T_i+2T_j+T_{k-1}+(i+j)(k-1)} \begin{bmatrix} i+j+k-2 \\ i+j \end{bmatrix}_q \\ &\cdot \left(q^{i+j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \right) \end{aligned}$$

is the generating function of

$$\left(\left|\mathcal{H}_{(c_2,c_1)}\left(n;i,j,k\right)\right|\right)_{n\in\mathbb{N}}.$$

It equals (using again Remark 6.4),

$$\begin{aligned} &q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} i+j+k-2 \\ i+j \end{bmatrix}_q \\ &\cdot \left(q^{i+j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \right). \end{aligned}$$

Then using (82) we get

$$\begin{aligned}
H_c(i,j,k) &= q^{Z(i,j,k)+k-1+2j} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \\
&\quad + q^{Z(i,j,k)+i+j+k-1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\
&\quad + q^{Z(i,j,k)} \begin{bmatrix} i+j+k-2 \\ i+j \end{bmatrix}_q \left(q^{i+j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \right) \\
&= q^{Z(i,j,k)} \Bigg(\left(q^{k-1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q + \begin{bmatrix} i+j+k-2 \\ i+j \end{bmatrix}_q \right) \\
&\quad \cdot q^{i+j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} \\
&\quad + \left(q^{k-1} \begin{bmatrix} i+j+k-2 \\ i+j-1 \end{bmatrix}_q + \begin{bmatrix} i+j+k-2 \\ i+j \end{bmatrix}_q \right) q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \Bigg).
\end{aligned}$$

Using the recursion formula (17), replacing k by $k-1$, we can conclude that

$$H_c(i,j,k) = q^{Z(i,j,k)} \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \left(q^{i+j} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2} \right),$$

which is exactly (78) (recall that $i+j+k-1 > 0$).
By the induction principle we can deduce Equations (75) to (78). Recall now that the goal is to prove Equation (74), i.e.

$$H_m(i,j,k) = q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}.$$

For this purpose let $i,j,k \in \mathbb{N}$ and first assume that $\mathbf{i}+\mathbf{j} = \mathbf{0}$ (i.e. $i = j = 0$). Then if $\mathbf{k} = \mathbf{0}$, by Definition 6.7,

$$H_m(0,0,0) = 1 = q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q^2},$$

counting the only minimal word of size 0 which is $\emptyset$. If $\mathbf{k} > \mathbf{0}$, note that by Remark 6.5,

$$\begin{aligned}
H_m(0,0,k) &= H_a(0,0,k) + H_b(0,0,k) + H_c(0,0,k) \\
&= q^{Z(0,0,k)} \begin{bmatrix} k-1 \\ -1 \end{bmatrix}_q \left(q^k \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{q^2} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{q^2} \right) + q^{T_k} \\
&= q^{T_k} \\
&= q^{T_k} \begin{bmatrix} k \\ 0 \end{bmatrix}_q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q^2}.
\end{aligned}$$

Finally if $\mathbf{i}+\mathbf{j}>\mathbf{0}$, using again Remark 6.5,

$$\begin{aligned}
H_m(i,j,k) &= H_a(0,0,k)+H_b(0,0,k)+H_c(0,0,k)\\
&= q^{Z(i,j,k)}\Bigg(q^{k+2j}\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}\\
&\quad + \begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2}\\
&\quad + \begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \left(q^{i+j}\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2} + q^{2j}\begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}\right)\Bigg).
\end{aligned}$$

Rearranging the terms and using Lemma 2.6,

$$\begin{aligned}
H_m(i,j,k) &= q^{Z(i,j,k)}\Bigg(\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2}\\
&\quad + q^{k+2j}\begin{bmatrix} i+j+k-1 \\ i+j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}\\
&\quad + q^{i+j}\begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_{q^2}\\
&\quad + q^{2j}\begin{bmatrix} i+j+k-1 \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_{q^2}\Bigg)\\
&= q^{Z(i,j,k)}\begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2},
\end{aligned}$$

which is what we wanted to prove.

6.4. Proof using generating functions. To deduce Theorem 6.1 from Lemma 6.2, we need to show that the right hand side of (70) equals the right hand side of (71).

Proof (Theorem 6.1). Note that by Definition 2.37 (Gaussian polynomials),

$$\begin{aligned}
&\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{a^ib^jc^kq^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}}\cdot\begin{bmatrix} i+j+k \\ i+j \end{bmatrix}_q\begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}\\
=&\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{a^ib^jc^kq^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}}\cdot\frac{(q)_{i+j+k}}{(q)_{i+j}(q)_k}\cdot\frac{\left(q^2;q^2\right)_{i+j}}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j}\\
=&\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{a^ib^jq^{2T_i+2T_j}\left(q^2;q^2\right)_{i+j}}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j(q)_{i+j}}\sum_{k=0}^{+\infty}\frac{c^kq^{T_k+(i+j)k}}{(q)_k}.
\end{aligned}\tag{84}$$

Note that for all $i,j\in\mathbb{N}$, by the notation defined in (7),

$$\begin{aligned}
(q)_{i+j}(-q)_{i+j}&=\prod_{t=1}^{i+j}\left(1-q^t\right)\prod_{r=1}^{i+j}\left(1+q^t\right)\\
&=\prod_{t=1}^{i+j}\left(1-q^t\right)\left(1+q^t\right)\\
&=\prod_{t=1}^{i+j}\left(1-q^{2t}\right)\\
&=\left(q^2;q^2\right)_{i+j}.
\end{aligned}$$

Therefore by substituting $\left(q^2;q^2\right)_{i+j}$ in (84),

$$\begin{aligned}
&\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{a^ib^jq^{2T_i+2T_j}\left(q^2;q^2\right)_{i+j}}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j(q)_{i+j}}\sum_{k=0}^{+\infty}\frac{c^kq^{T_k+(i+j)k}}{(q)_k}\\
=&\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{a^ib^jq^{2T_i+2T_j}(-q)_{i+j}}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j}\sum_{k=0}^{+\infty}\frac{c^kq^{T_k+(i+j)k}}{(q)_k}.
\end{aligned}$$

Note that by equation (9) in Corollary 2.5, for $t=cq^{i+j+1}$, it holds that

$$\sum_{k=0}^{+\infty}\frac{c^kq^{k(i+j+1)+\frac{k(k-1)}{2}}}{(q)_k}=\left(-cq^{i+j+1}\right)_\infty,$$

or equivalently

$$\sum_{k=0}^{+\infty}\frac{c^kq^{k(i+j)+\frac{k(k+1)}{2}}}{(q)_k}=\left(-cq^{i+j+1}\right)_\infty.$$

Hence,

$$\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{a^i b^j q^{2T_i+2T_j}\left(-q\right)_{i+j}}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j}\sum_{k=0}^{+\infty}\frac{c^k q^{T_k+(i+j)k}}{\left(q\right)_k}=\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{a^i b^j q^{2T_i+2T_j}\left(-q\right)_{i+j}\left(-cq^{i+j+1}\right)_\infty}{\left(q^2;q^2\right)_i\left(q^2;q^2\right)_j},$$

which is the right hand side of (70). Thus using Lemma 6.2 and comparing coefficients we get that for all $n,i,j,k\in\mathbb{N}$,

$$K\left(n;i,j,k\right)=G\left(n;i,j,k\right),$$

which proves Theorem 6.1. ■

6.5. Preparation for the bijective proof. The most technical ideas that are needed for the combinatorial proof of Alladi, Andrews and Gordon of the presented generalization can be taken from generalizations of definitions we stated in Subsection 5.2. In that subsection we prepared technical notations for the combinatorial proof of the refinement presented in Subsection 5.1.

First we will generalize the definition of the union of partitions to be applicable to words.

Definition 6.9. *Let $k_1, k_2 \in \mathbb{N}$ and π_1, π_2 be words of size k_1 (resp. k_2). Then we define $\pi_1 \cup \pi_2$ to be the word of size $k_1 + k_2$ consisting of all elements of π_1 and π_2 in which each symbol occurs exactly the number of appearances in π_1 plus the number of appearances in π_2 times.*

Example 6.13. *Note that*

$$(c_{10}, a_8, b_7, a_6, c_5) \cup (c_8, a_7, b_7, c_4) = (c_{10}, c_8, a_8, a_7, b_7, b_7, a_6, c_5, c_4)\,.$$

For being able to decompose words we will also generalize the notion of decomposing partitions.

Definition 6.10. *Let $A \subseteq \mathbb{N}^*$. For a word π denote by π_A the word consisting of the elements of π whose weight is contained in A. Moreover, for $x \in \{a, b, c\}$, π_x is the word consisting of all x-parts of π.*

Example 6.14. *Note that*

$$(c_{10}, c_8, a_8, a_7, b_7, b_7, a_6, c_5, c_4)_{2\mathbb{N}^*} = (c_{10}, c_8, a_8, a_6, c_4)\,,$$

and

$$(c_{10}, c_8, a_8, a_7, b_7, b_7, a_6, c_5, c_4)_a = (a_8, a_7, a_6)\,.$$

For convenience we will also need an adapted version of our term "almost non-decreasing".

Definition 6.11. *Let $s = (s_t)_{t\in[r]}$ be a finite sequence in $\mathcal{S}$. Then s is said to be **$\mathcal{S}$-almost non-decreasing** if it is increasing up to b-parts that may be followed by a smaller a-part such that the gap between the two elements is 1.*

Example 6.15. *Note that $(a_1, c_1, b_3, a_2, b_3, c_5)$ is an $\mathcal{S}$-almost non-decreasing finite sequence in $\mathcal{S}$.*

6.6. Bijective Proof. The combinatorial proof of Theorem 6.1 presented by Alladi, Andrews and Gordon is very similar to the one of the presented refinement (Theorem 5.2) and has the same step structure.

Proof (Theorem 6.1). [AAG95b, p. 655 - 657] Let $n, i, j, k \in \mathbb{N}$.

Step 1: Denote by $\mathcal{K}(n; i, j, k)$ the set of triples of words counted by $K(n; i, j, k)$. First we rearrange the triples in $\mathcal{K}(n; i, j, k)$ similar to Step 1 in the combinatorial proof of Theorem 5.2. We build a union out of the first two elements (containing the a-parts and b-parts) and separate the c-parts that are greater than $i + j$ i.e. those that are counted by k. Also the smaller c-parts are written in a separate word as we will apply conjugation to the partition formed by their weights in the next step.
Further let $\mathcal{L}_1(n; i, j, k)$ be the set of triples (π_1, π_2, π_3) of words such that π_1 consists of i distinct a-parts with even weights and j distinct b-parts with even weights, π_2 consists of distinct c-parts that have weights smaller than or equal to $i + j$ and π_3 consists of k distinct c-parts whose weights are greater than $i + j$. Further we demand that

$$s(\pi_1) + s(\pi_2) + s(\pi_3) = n.$$

Let $B := \mathbb{N}^* \setminus [i + j]$ and $A := [i + j]$. We can formalize this rearranging process using the following map:

$$\begin{aligned} \alpha_1 : \ & \mathcal{K}(n; i, j, k) \to \mathcal{L}_1(n; i, j, k) \\ & (\pi_1, \pi_2, \pi_3) \mapsto \left(\pi_1 \cup \pi_2, \pi_{3,A}, \pi_{3,B}\right). \end{aligned}$$

Indeed, by definition of $\mathcal{K}(n; i, j, k)$, $\mathcal{L}_1(n; i, j, k)$ and the used operations on words (Definitions 6.9 and 6.10), α_1 is a **well-defined bijection**. The inverse can be described by the map

$$\begin{aligned} \beta_1 : \ & \mathcal{L}_1(n; i, j, k) \to \mathcal{K}(n; i, j, k) \\ & (\pi_1, \pi_2, \pi_3) \mapsto \left(\pi_{1,a}, \pi_{1,b}, \pi_2 \cup \pi_3\right). \end{aligned}$$

Example 6.16. *Let* $n = 89, i = 3, j = 4, k = 2$ *and consider*

$$(\pi_1, \pi_2, \pi_3) := \left((a_8, a_6, a_2), (b_{12}, b_{10}, b_8, b_2), (c_{17}, c_{13}, c_6, c_3, c_2)\right) \in \mathcal{K}(89; 3, 4, 2).$$

Then

$$\alpha_1(\pi_1, \pi_2, \pi_3) = \left((b_{12}, b_{10}, a_8, b_8, a_6, a_2, b_2), (c_6, c_3, c_2), (c_{17}, c_{13})\right).$$

Step 2: As already mentioned we will apply a conjugation to the partition formed by the weights of the smaller c-parts.
Let $\mathcal{L}_2(n; i, j, k)$ be the set of triples (π_1, π_2, π_3) of finite sequences, where π_1 is a word consisting of i distinct a-parts with even weights and j distinct b-parts with even weights, π_2 is a partition consists of at most $(i + j)$ parts such that 1 appears at least once as a part of π_2 and the difference between consecutive parts is at most 1, π_3 is a word consisting of k c-parts that are greater than $i + j$ and

$$s(\pi_1) + s(\pi_2) + s(\pi_3) = n.$$

This step can be described using the following map:

$$\begin{aligned}\alpha_2 :\ & \mathcal{L}_1(n;i,j,k) \to \mathcal{L}_2(n;i,j,k)\\ & \left(\pi_1,(\mu_t)_{t\in[r]},\pi_3\right) \mapsto \left(\pi_1,\left(\omega(\mu_t)\right)'_{t\in[r]},\pi_3\right).\end{aligned}$$

Indeed, by an application of Corollary 2.3 for $m = i+j$, α_2 is a **well-defined bijection**.

Example 6.17. *Continue Example 6.16. Note that*

$$\begin{aligned}\alpha_2\left(\alpha_1(\pi_1,\pi_2,\pi_3)\right) &= \alpha_2\left((b_{12},b_{10},a_8,b_8,a_6,a_2,b_2),(c_6,c_3,c_2),(c_{17},c_{13})\right)\\ &= \left((b_{12},b_{10},a_8,b_8,a_6,a_2,b_2),(6,3,2)',(c_{17},c_{13})\right)\\ &= \left((b_{12},b_{10},a_8,b_8,a_6,a_2,b_2),(3,3,2,1,1,1),(c_{17},c_{13})\right).\end{aligned}$$

Step 3: Next we add the parts of the second element of the obtained triple to the weights of the first word.
Let $\mathcal{L}_3(n;i,j,k)$ be the set of pairs (π_1,π_2) of words where π_1 consists of i distinct a-parts and j distinct b-parts satisfying the additional conditions that the gap between consecutive elements is at least 2 (with the exception that if an a-part is followed by a b-part the minimal gap is 0) and the last element is greater than a_1. The word π_2 consists of k distinct c-parts that are greater than $i+j$, and

$$s(\pi_1)+s(\pi_2)=n.$$

This summation is processed by the following map:

$$\begin{aligned}\alpha_3 :\ & \mathcal{L}_2(n;i,j,k) \to \mathcal{L}_3(n;i,j,k)\\ & (\pi_1,\lambda,\pi_3) \mapsto \left(\nu(\pi_1,\lambda),\pi_3\right),\end{aligned}$$

where $\nu(\pi_1,\lambda)$ is obtained from π_1 by adding λ_t to the weight of $\pi_{1,t}$ for all $t\in[i+j]$. We use the notation

$$\lambda = (\lambda_t)_{t\in[r]} \text{ and } \pi_1 = \left(\pi_{1,t}\right)_{t\in[i+j]},$$

and $\lambda_t := 0$ for all $t > r$.
First we prove that this map is **well-defined**. Let

$$(\pi_1,\lambda,\pi_3)\in\mathcal{L}_2(n;i,j,k).$$

Since π_1 has $i+j$ elements and λ has at most $i+j$ parts the increasing of the weights is well-defined. Note that since π_1 consists of i distinct a-parts and j distinct b-parts and λ is non-increasing, $\nu(\pi_1,\lambda)$ consists still of i distinct a-parts and j distinct b-parts.
For checking the minimal gap between consecutive elements, consider two consecutive elements of $\nu(\pi_1,\lambda)$. If both are a-parts or both are b-parts then the minimal gap between the corresponding elements of π_1 is 2. Since $b_l < a_l$ for all $l\in\mathbb{N}_{\geq 2}$, whenever a b-part in π_1 is followed by an a-part the gap is at least 2. By the lexicographic ordering (see (60)), the weight of a b-part that is preceded by an a-part can't be greater than the weight of this a-part. Thus as λ is non-increasing the minimal gap condition for $\alpha_3(\pi_1,\lambda,\pi_3)$ follows. Moreover note that the last element of π_1 is greater than a_1 which implies that the same holds for $\nu(\pi_1,\lambda)$.

For the size note that by the definition of the addition step,

$$s\left(\nu\left(\pi_1,\lambda\right)\right)+s\left(\pi_3\right)=s\left(\pi_1\right)+s\left(\lambda\right)+s\left(\pi_3\right)=n.$$

Hence $\alpha_3\left(\pi_1,\lambda,\pi_3\right)\in\mathcal{L}_3\left(n;i,j,k\right)$.
To show **bijectivity** we construct the inverse of α_3. Recall that all a-parts and b-parts involved in $\mathcal{L}_2\left(n;i,j,k\right)$ have even weights. The idea is to consider the changes in parity in the weights of consecutive elements in the first word of pairs in $\mathcal{L}_3\left(n;i,j,k\right)$. If we consider this word as a first word of an element in $Im\left(\alpha_3\right)$, then a change of parity means that there was a change of parity between the corresponding consecutive parts in the added partition. Further, note that consecutive parts in that partition differ by at most 1. Thus starting with the smallest element with an odd weight of the sum we can reconstruct the original two objects, by starting with 1 and increasing the parts in the partition by 1 whenever we reach a change in parity in the weights.
This can be formalized using the following definition of a map. Define

$$\begin{aligned}\beta_3:\ \mathcal{L}_3\left(n;i,j,k\right)&\to\mathcal{L}_2\left(n;i,j,k\right)\\ \left(\pi_1,\pi_2\right)&\mapsto\left(\mu,\lambda,\pi_2\right),\end{aligned}$$

where μ and λ are constructed in the following way. For $\left(\pi_1,\pi_2\right)\in\mathcal{L}_3\left(n;i,j,k\right)$, denote $\pi_1=\left(\pi_{1,t}\right)_{t\in[r]}$ and let

$$\left(\pi_{1,p_l}\right)_{l\in[k_1]}$$

be the finite sequence consisting of the elements with odd weight of π_1 that are followed by an element with even weight or that are $\pi_{1,r}$. We assume the elements to be ordered reversely, i.e. for all $m\in[k_1-1]$,

$$\pi_{1,p_m}<\pi_{1,p_{m+1}}.$$

Analogously let

$$\left(\pi_{1,q_l}\right)_{l\in[k_2]}$$

be the finite sequence of elements with even weights of π_1 that are followed by an element with odd weight. We again assume that the elements are ordered reversely, i.e. for all $m\in[k_2-1]$,

$$\pi_{1,q_m}<\pi_{1,q_{m+1}}.$$

Example 6.18. *For* $\pi_1=(a_{15},b_{15},a_{12},a_9,b_9,a_6,a_3)$,

$$\left(\pi_{1,p_l}\right)_{l\in[3]}=\left(a_3,b_9,b_{15}\right),$$

and

$$\left(\pi_{1,q_l}\right)_{l\in[2]}=(a_6,a_{12}).$$

Note that in this example $p_1=7,p_2=5,p_3=2$ *and* $q_1=6,q_2=3$.

Then obtain $\mu=\left(\mu_t\right)_{t\in[r]}$ from π_1 by changing the weights in the following way:

$$\omega(\mu_t) := \begin{cases} \omega(\pi_{1,t}) - (2k_1 - 1) & \text{if } t \leq p_{k_1} \text{ and } \omega(\pi_{1,1}) \text{ is odd,} \\ \omega(\pi_{1,t}) - 2k_2 & \text{if } t \leq q_{k_2} \text{ and } \omega(\pi_{1,1}) \text{ is even,} \\ \omega(\pi_{1,t}) - (2l-1) & \text{if } t \in (q_l, p_l], \\ \omega(\pi_{1,t}) - 2l & \text{if } t \in (p_{l+1}, q_l], \\ \omega(\pi_{1,t}) & \text{if } t > p_1. \end{cases}$$

The partition $\lambda = (\lambda_t)_{t \in [p_1]}$ is correspondingly obtained by

$$\lambda_t := \begin{cases} 2k_1 - 1 & \text{if } t \leq p_{k_1} \text{ and } \omega(\pi_{1,1}) \text{ is odd,} \\ 2k_2 & \text{if } t \leq q_{k_2} \text{ and } \omega(\pi_{1,1}) \text{ is even,} \\ 2l-1 & \text{if } t \in (q_l, p_l], \\ 2l & \text{if } t \in (p_{l+1}, q_l]. \end{cases}$$

Example 6.19. *Continuing Example 6.18, note that*

$$\begin{aligned} \mu &= (a_{15-5}, b_{15-5}, a_{12-4}, a_{9-3}, b_{9-3}, a_{6-2}, a_{3-1}) \\ &= (a_{10}, b_{10}, a_8, a_6, b_6, a_4, a_2), \end{aligned}$$

and

$$\lambda = (5, 5, 4, 3, 3, 2, 1).$$

Next we show that β_3 is well-defined. Let $(\pi_1, \pi_2) \in \mathcal{L}_3(n; i, j, k)$. Note that by definition of $\mathcal{L}_3(n; i, j, k)$, π_{1,p_1} is at least b_3 (since it is odd). With every change of parity in the weights there comes a gap of at least 1 (by the nature of changing parity). Hence the subtraction in the definition of μ is well-defined. Since from each odd (resp. even) weight of an element of π_1 an odd (resp. even) number is subtracted, μ consists of elements with even weights, more precisely i a-parts with even weights and j b-parts with even weights. We show that μ is non-increasing and that its elements are distinct. Let $t \in [r-1]$. First assume that μ_t and μ_{t+1} are such that μ_t is a b-part or both are a-parts. Then by definition of $\mathcal{L}_3(n; i, j, k)$,

$$g(\pi_t, \pi_{t+1}) \geq 2.$$

Since, by construction, we subtract from the weight of π_t at most 1 more than from the weight of π_{t+1},

$$g(\mu_t, \mu_{t+1}) \geq 1.$$

Thus $\mu_t > \mu_{t+1}$. If μ_t is an a-part and μ_{t+1} is a b-part and $\omega(\pi_t) = \omega(\pi_{t+1})$ we subtract the same number from both weights. Hence

$$\omega(\mu_t) = \omega(\mu_{t+1}),$$

and thus $\mu_t > \mu_{t+1}$. However, if $\omega(\pi_t) > \omega(\pi_{t+1})$, as we subtract from $\omega(\pi_t)$ at most 1 more than from $\omega(\pi_{t+1})$, we still have $\mu_t > \mu_{t+1}$.
Moreover, λ forms a non-increasing finite sequence. By construction, λ consists of $p_1 \leq i+j$ parts such that consecutive parts differ by at most 1 and the last part λ_{p_1} equals 1.
Finally, noting that

$$s(\mu) + s(\lambda) = s(\pi_1),$$

we can deduce that

$$\beta_3(\pi_1, \pi_2) = (\mu, \lambda, \pi_2) \in \mathcal{L}_2(n; i, j, k).$$

We show that $\beta_3 = \alpha_3^{-1}$. Let $(\pi_1, \lambda, \pi_3) \in \mathcal{L}_2(n; i, j, k)$. Then by construction it holds that

$$\beta_3\left(\alpha_3(\pi_1, \lambda, \pi_3)\right) = \beta_3\left(\nu(\pi_1, \lambda), \pi_3\right) = (\pi_1, \lambda, \pi_3).$$

For $(\pi_1, \pi_2) \in \mathcal{L}_3(n; i, j, k)$,

$$\alpha_3\left(\beta_3(\pi_1, \pi_2)\right) = \alpha_3(\mu, \lambda, \pi_2) = \left(\mu(\nu, \lambda), \pi_2\right) = (\pi_1, \pi_2).$$

Example 6.20. *Continue Example 6.17. Note that*

$$\begin{aligned}\alpha_3\left(\alpha_2\left(\alpha_1(\pi_1, \pi_2, \pi_3)\right)\right) &= \alpha_3\left((b_{12}, b_{10}, a_8, b_8, a_6, a_2, b_2), (3, 3, 2, 1, 1, 1), (c_{17}, c_{13})\right) \\ &= \left((b_{15}, b_{13}, a_{10}, b_9, a_7, a_3, b_2), (c_{17}, c_{13})\right).\end{aligned}$$

Step 4: We combine the two words to a single (not necessarily non-increasing) finite sequence.
Let $\mathcal{L}_4(n; i, j, k)$ be the set of finite sequences in $\mathcal{S}$ with $i + j + k$ elements greater than a_1 such that the first $i + j$ elements form in reversed order a word into i distinct a-parts and j distinct b-parts satisfying the additional condition that the gap between consecutive elements is at least 2 (with the exception that if an a-part is followed by a b-part the minimal gap is 0). The last k elements form a word in reversed order into distinct c-parts whose weights are greater than $i + j$. The sum of the weights of the elements of the sequences should be n.
We are ready to define the fourth bijection:

$$\begin{aligned}\alpha_4 : \ \mathcal{L}_3(n; i, j, k) &\to \mathcal{L}_4(n; i, j, k) \\ (\pi_1, \pi_2) &\mapsto (u_t)_{t \in [i+j+k]},\end{aligned}$$

where

$$(u_t)_{t \in [i+j]} := \pi_1^{rev},$$

and

$$(u_t)_{t \in [i+j+k] \setminus [i+j]} := \pi_2^{rev}.$$

By definition of $\mathcal{L}_3(n; i, j, k)$ and $\mathcal{L}_4(n; i, j, k)$, α_4 is a **well-defined bijection**. The inverse can be described by separating the finite sequences into two finite sequences π_1 and π_2 and reversing their order where π_1 contains the first $i + j$ elements and π_2 contains the last k elements.

Example 6.21. *Continue Example 6.20. Note that*

$$\begin{aligned}\alpha_4\left(\alpha_3\left(\alpha_2\left(\alpha_1(\pi_1, \pi_2, \pi_3)\right)\right)\right) &= \alpha_4\left((b_{15}, b_{13}, a_{10}, b_9, a_7, a_3, b_2), (c_{17}, c_{13})\right) \\ &= (b_2, a_3, a_7, b_9, a_{10}, b_{13}, b_{15}, c_{13}, c_{17}).\end{aligned}$$

Step 5: In the obtained finite sequence we subtract increasing natural numbers from each element starting with 0.
Let $\mathcal{L}_5(n;i,j,k)$ be the set of finite sequences $(u_t)_{t\in[i+j+k]}$ in $\mathbb{N}^*$ such that $u_1 > a_1$, the last k elements are non-decreasing c-parts and the first $i+j$ elements form an $\mathcal{S}$-almost non-decreasing sequence consisting of i a-parts and j b-parts. Every two consecutive elements among these $i+j$ are distinct. Moreover,

$$\sum_{t=1}^{i+j+k} \omega(u_t) = n - \frac{(i+j+k)(i+j+k-1)}{2}.$$

We define the fifth bijection by

$$\begin{aligned} \alpha_5 :\ \mathcal{L}_4(n;i,j,k) &\to \mathcal{L}_5(n;i,j,k) \\ u &\mapsto v, \end{aligned}$$

where $v = (v_t)_{t\in[i+j+k]}$ is obtained from $u = (u_t)_{t\in[i+j+k]}$ by changing the weights in the following way. For all $t \in [i+j+k]$, let

$$\omega(v_t) = \omega(u_t) - (t-1).$$

Let us show that α_5 is **well-defined**. Let

$$u = (u_t)_{t\in[i+j+k]} \in \mathcal{L}_4(n;i,j,k).$$

We show first that the weights of the elements of $\alpha_5(u)$ lie in $\mathbb{N}^*$. Note that by the lower bound for the weights of the distinct c-parts in $\mathcal{L}_4(n;i,j,k)$ for all $t \in [i+j+k] \setminus [i+j]$, $u_t \geq t$. Thus,

$$\omega(u_t) - (t-1) \geq t - t + 1 = 1 > 0.$$

Further, $\omega(u_1) \geq 2$ and by the minimal gap condition in $L_4(n;i,j,k)$ for all $t \in [i+j-2]$ it holds that

$$g(u_{t+2}, u_t) \geq 2,$$

for all $t \in [i+j-2]$. Thus if $t \in [i+j]$ is odd, then

$$\omega(u_t) \geq 2 + \frac{t-1}{2} \cdot 2 = t+1.$$

If $t \in [i+j]$ is even, then

$$\omega(u_t) \geq 2 + \frac{t-2}{2} \cdot 2 = t.$$

Hence for all $t \in [i+j]$,

$$\omega(u_t) - (t-1) \geq t - t + 1 = 1 > 0,$$

which proves that reducing the weights is well-defined.
Since we preserve the letters in u the required numbers of a-parts, b-parts and c-parts are still satisfied. For $t \in [i+j+k]$ such that $i+j < t < i+j+k$, it holds that

$$\big(\omega(u_{t+1}) - t\big) - \big(\omega(u_t) - (t-1)\big) = g(u_{t+1}, u_t) - 1 \geq 0,$$

which shows that the last k elements of $\alpha_5(u)$ form a non-decreasing sequence.

Note that for $t \in [i+j-1]$, by the minimal gap condition in $\mathcal{L}_4(n;i,j,k)$, for consecutive parts u_t and u_{t+1} such that u_t is an a-part or u_t and u_{t+1} are both b-parts,

$$g(u_{t+1}, u_t) \geq 2.$$

Then

$$\big(\omega(u_{t+1}) - t\big) - \big(\omega(u_t) - (t-1)\big) = g(u_{t+1}, u_t) - 1 \geq 1.$$

If u_t is a b-part and u_{t+1} is an a-part by the minimal gap condition we have that

$$\omega(u_{t+1}) - \omega(u_t) \geq 0.$$

Thus

$$\big(\omega(u_{t+1}) - t\big) - \big(\omega(u_t) - (t-1)\big) = \big(\omega(u_{t+1}) - \omega(u_t)\big) - 1 \geq -1.$$

Thus the first $i+j$ elements of $\alpha_5(u)$ form an $\mathcal{S}$-almost non-decreasing sequence whose pairs of consecutive elements are distinct.
Moreover, by Lemma 2.2,

$$\sum_{t=1}^{i+j+k} \omega(u_t) - (t-1) = \sum_{t=1}^{i+j+k} u_t - \sum_{t=0}^{i+j+k-1} 1 = n - \frac{(i+j+k-1)(i+j+k)}{2}.$$

Hence

$$\alpha_5(u) \in \mathcal{L}_5(n;i,j,k).$$

For proving **bijectivity** we construct the inverse of α_5. Define

$$\begin{aligned} \beta_5 : \ \mathcal{L}_5(n;i,j,k) &\to \mathcal{L}_4(n;i,j,k) \\ v &\mapsto u, \end{aligned}$$

where $u = (u_t)_{t\in[i+j+k]}$ is obtained from $v = (v_t)_{t\in[i+j+k]}$ by changing the weights in the following way. For all $t \in [i+j+k]$, let

$$\omega(u_t) = \omega(v_t) + (t-1).$$

We show that this map is well-defined. Let

$$v = (v_t)_{t\in[i+j+k]} \in \mathcal{L}_5(n;i,j,k).$$

Since we preserve the letters in u the required numbers of a-parts, b-parts and c-parts are still satisfied. Let $t \in [i+j-1]$ and v_t and v_{t-1} be two consecutive parts of v. Assume that v_t is an a-part or v_t and v_{t+1} are both b-parts. Then by the minimal gap condition in $\mathcal{L}_5(n;i,j,k)$, note that

$$\big(\omega(v_{t+1}) + t\big) - \big(\omega(v_t) + (t-1)\big) = \big(\omega(v_{t+1}) - \omega(v_t)\big) + 1 \geq 1 + 1 = 2.$$

Further if v_t is a b-part and v_{t+1} is an a-part, by the $\mathcal{S}$-almost non-decreasing property,

$$\big(\omega(v_{t+1}) + t\big) - \big(\omega(v_t) + (t-1)\big) = \big(\omega(v_{t+1}) - \omega(v_t)\big) + 1 \geq -1 + 1 = 0,$$

and by definition of $\mathcal{L}_5(n;i,j,k)$, $v_1 > a_1$. Thus the elements of $\beta_5(v)$ are distinct.
Next we show that the last k elements form a word in reversed order consisting of distinct c-parts whose weights are greater than $(i+j)$. Note that for all $t \in [i+j+k-1] \setminus [i+j]$,

$$\big(\omega(v_{t+1}) + t\big) - \big(\omega(v_t) + (t-1)\big) = g(v_{t+1}, v_t) + 1 \geq 1.$$

Moreover,

$$\omega(v_t) + (t-1) \geq \omega(v_t) + (i+j) > (i+j).$$

Finally, note that by Lemma 2.2,

$$\begin{aligned}\sum_{t=1}^{i+j+k} \omega(v_t) + (t-1) &= \sum_{t=1}^{i+j+k} \omega(v_t) + \sum_{t=0}^{i+j+k-1} t \\ &= n - \frac{(i+j+k)(i+j+k-1)}{2} + \frac{(i+j+k)(i+j+k-1)}{2} \\ &= n.\end{aligned}$$

Hence

$$\beta_5(v) \in \mathcal{L}_4(n; i, j, k).$$

To conclude the proof on bijectivity of φ_5, note that for all

$$(u_t)_{t\in[i+j+k]} \in \mathcal{L}_4(n; i, j, k),$$

it holds that

$$\beta_5\left(\alpha_5\left((u_t)_{t\in[i+j+k]}\right)\right) = \beta_5\left(\left(u_t - 3(t-1)\right)_{t\in[i+j+k]}\right) = (u_t)_{t\in[i+j+k]},$$

and for all $(v_t)_{i\in[i+j+k]} \in \mathcal{L}_5(n; i, j, k)$ it holds that

$$\alpha_5\left(\beta_5\left((v_t)_{t\in[i+j+k]}\right)\right) = \alpha_5\left(\left(v_t + 3(t-1)\right)_{i\in[i+j+k]}\right) = (v_t)_{t\in[i+j+k]}.$$

Example 6.22. *Continue Example 6.21. Note that*

$$\begin{aligned}\alpha_5\left(\alpha_4\left(\alpha_3\left(\alpha_2\left(\alpha_1(\pi_1, \pi_2, \pi_3)\right)\right)\right)\right) &= \varphi_5\left((b_2, a_3, a_7, b_9, a_{10}, b_{13}, b_{15}, c_{13}, c_{17})\right) \\ &= (b_2, a_2, a_5, b_6, a_6, b_8, b_9, c_6, c_9).\end{aligned}$$

Step 6: We rearrange the obtained finite sequence in such a way that all c-parts are placed where they belong to reach a non-decreasing finite sequence (with respect to the lexicographic order in (60)).
Let $\mathcal{L}_6(n; i, j, k)$ be the set of almost non-decreasing finite sequences in $\mathcal{S}$

$$(u_t)_{t\in[i+j+k]},$$

such that $u_t > a_1$ for all $t \in [i+j+k]$. The finite sequences consist of k c-parts, i a-parts and j b-parts and consecutive a-parts and b-parts are distinct. Moreover,

$$\sum_{t=1}^{i+j+k} \omega(u_t) = n - \frac{(i+j+k)(i+j+k-1)}{2}.$$

The rearranging procedure is described by the following map:

$$\begin{aligned}\alpha_6 :\ \mathcal{L}_5(n; i, j, k) &\to \mathcal{L}_6(n; i, j, k) \\ u &\mapsto \tilde{u},\end{aligned}$$

where $\tilde{u}$ is the unique almost non-decreasing sequence with the same elements (with same multiplicities) as u such that the order of all a-parts and b-parts is preserved.
By definition of $\mathcal{L}_5(n;i,j,k)$ and $\mathcal{L}_6(n;i,j,k)$, this map is a **well-defined bijection**. Note that the reordering is uniquely defined since the only a-parts and b-parts that are possibly out of the lexicographic order are of the form $(b_{j+1}a_j)$ for some $j \in \mathbb{N}^*$. But since

$$c_{j-1} < a_j < b_{j+1} < c_j,$$

the places for the c-parts are still uniquely defined. The inverse can be described by taking all c-parts in increasing order to the end of the finite sequence.

Example 6.23. *Continue Example 6.22. Note that*

$$\begin{aligned}\alpha_6\left(\alpha_5\left(\alpha_4\left(\alpha_3\left(\alpha_2\left(\alpha_1\left(\pi_1,\pi_2,\pi_3\right)\right)\right)\right)\right)\right) &= \alpha_6\left((b_2,a_2,a_5,b_6,a_6,b_8,b_9,c_6,c_9)\right)\\ &= (b_2,a_2,a_5,b_6,a_6,c_6,b_8,b_9,c_9).\end{aligned}$$

Step 7: The final step consists in adding again the subtracted numbers to the weights of the elements of the rearranged sequence, reversing the order and obtaining a word of the type counted by $G(n;i,j,k)$.
The last piece in the considered row of bijections is the following map:

$$\begin{aligned}\alpha_7 :\ \mathcal{L}_6(n;i,j,k) &\to \mathcal{G}(n;i,j,k)\\ u &\mapsto v^{rev},\end{aligned}$$

where $v = (v_t)_{t\in[i+j+k]}$ is obtained from $u = (u_t)_{t\in[i+j+k]}$ by changing the weights in the following way. For all $t \in [i+j+k]$, let

$$\omega(v_t) = \omega(u_t) + (t-1).$$

We show that this map is **well-defined**. Let $u = (u_t)_{t\in[i+j+k]} \in \mathcal{L}_6(n;i,j,k)$. Since we preserve the letters in u the required numbers of a-parts, b-parts and c-parts are still satisfied. Note that again by Lemma 2.2,

$$\begin{aligned}\sum_{t=1}^{i+j+k}\omega(u_t)+(t-1) &= \sum_{t=1}^{i+j+k}\omega(u_t) + \sum_{t=0}^{i+j+k-1} t\\ &= n - \frac{(i+j+k)(i+j+k-1)}{2} + \frac{(i+j+k)(i+j+k-1)}{2}\\ &= n.\end{aligned}$$

By definition of $\mathcal{L}_6(n;i,j,k)$, $u_1 > a_1$. To check the minimal gap conditions (see Definition 6.4), we consider two consecutive parts of u, i.e. u_t and u_{t+1} for some $t \in [i+j+k-1]$.

Case 1: Assume u_t and u_{t+1} are both a-parts, both b-parts, u_t is an a-part and u_{t+1} is a b-part or u_t is a c-part and u_{t+1} is an a-part. By the $\mathcal{S}$-almost non-decreasing property (see Definition 6.11) and the lexicographic ordering (60) we know that $u_{t+1} > u_t$ and

$$\left(\omega(u_{t+1})+t\right) - \left(\omega(u_t)+(t-1)\right) = g(u_{t+1},u_t)+1 \geq 1+1 = 2.$$

Case 2: Assume u_t is an a-part or a c-part and u_{t+1} is a c-part. Then again by the ordering, $u_{t+1} > u_t$ and

$$\big(\omega(u_{t+1}) + t\big) - \big(\omega(u_t) + (t-1)\big) = g(u_{t+1}, u_t) + 1 \geq 0 + 1 = 1.$$

Case 3: Assume u_t is a b-part and u_{t+1} is an a-part or a c-part. Then again by the lexicographic ordering (60) and the $\mathcal{S}$-almost non-decreasing property, $u_{t+1} > u_t$ and

$$\big(\omega(u_{t+1}) + t\big) - \big(\omega(u_t) + (t-1)\big) = g(u_{t+1}, u_t) + 1 \geq -1 + 1 = 0.$$

Case 4: Assume u_t is a c-part and u_{t+1} is a b-part. Note that for a c-part c_k for some $k \in \mathbb{N}^*$ by the lexicographic ordering (60) the next greater b-part is b_{k+2}. Thus, by the $\mathcal{S}$-almost non-decreasing property, $u_{t+1} > u_t$ and

$$\big(\omega(u_{t+1}) + t\big) - \big(\omega(u_t) + (t-1)\big) = g(u_{t+1}, u_t) + 1 \geq 2 + 1 = 3.$$

Hence

$$\alpha_7(u) \in \mathcal{G}(n; i, j, k).$$

To prove **bijectivity**, consider the following map which turns out to be the inverse of α_7:

$$\begin{aligned} \beta_7 : \ \mathcal{G}(n; i, j, k) &\to \mathcal{L}_6(n; i, j, k) \\ v^{rev} &\mapsto u, \end{aligned}$$

where $u = (u_t)_{t \in [i+j+k]}$ is obtained from $v = (v_t)_{t \in [i+j+k]}$ by changing the weights in the following way. For all $t \in [i+j+k]$, let

$$\omega(u_t) = \omega(v_t) - (t-1).$$

We show that this map is well-defined. Let

$$v^{rev} = (v_t)^{rev}_{t \in [i+j+k]} \in \mathcal{G}(n; i, j, k).$$

Since we preserve the letters in v the required numbers of a-parts, b-parts and c-parts are still satisfied. Moreover, by definition of $\mathcal{G}(n; i, j, k)$, $v_t > a_1$ for all $t \in [i+j+k]$. Thus $v_t + (t-1) > a_1$.
To check the $\mathcal{S}$-almost non-decreasing condition in $\mathcal{L}_6(n; i, j, k)$, let $t \in [i+j+k-1]$ and consider two consecutive parts v_t and v_{t+1}.

Case 1: Assume v_t and v_{t+1} are both a-parts, both b-parts, v_t is an a-part and v_{t+1} is a b-part or v_t is a c-part and v_{t+1} is an a-part. Then by the minimal gaps stated in Definition 6.4,

$$\big(\omega(v_{t+1}) - t\big) - \big(\omega(v_t) - (t-1)\big) = g(v_{t+1}, v_t) - 1 \geq 2 - 1 = 1.$$

Case 2: Assume v_t is an a-part or a c-part and v_{t+1} is a c-part. Then again by the minimal gaps stated in Definition 6.4,

$$\big(\omega(v_{t+1}) - t\big) - \big(\omega(v_t) - (t-1)\big) = g(v_{t+1}, v_t) - 1 \geq 1 - 1 = 0.$$

Case 3: Assume v_t is a b-part and v_{t+1} is an a-part or a c-part. Then by Definition 6.4,

$$\big(\omega(v_{t+1}) - t\big) - \big(\omega(v_t) - (t-1)\big) = g(v_{t+1}, v_t) - 1 \geq 0 - 1 = -1.$$

Case 4: Assume v_t is a c-part and v_{t+1} is a b-part. Also here by Definition 6.4,

$$\left(\omega\left(v_{t+1}\right)-t\right)-\left(\omega\left(v_t\right)-(t-1)\right)=g\left(v_{t+1},v_t\right)-1\geq 3-1=2.$$

By the definition of $\mathcal{S}$-almost non-decreasing (Definition 6.11) and the lexicographic ordering (60) we need the following minimal gaps in u.

	a	b	c
a	0	1	0
b	−1	0	−1
c	1	2	0

This table is to be read in the same way as the table in Definition 6.4.
Thus by the four inequalities above, $\beta_7\left(v^{rev}\right)$ is $\mathcal{S}$-almost non-decreasing. Moreover, it follows that consecutive a-parts and b-parts are distinct.
Finally, by Lemma 2.2, note that $\beta_7\left(v^{rev}\right)$ has size

$$\sum_{t=1}^{i+j+k} v_t-(t-1)=\sum_{t=1}^{i+j+k} v_t-\sum_{t=0}^{i+j+k-1} t=n-\frac{(i+j+k)(i+j+k-1)}{2}.$$

Thus, $\beta_7\left(v^{rev}\right)\in\mathcal{L}_6\left(n;i,j,k\right)$.
To conclude the proof on bijectivity of α_7 note that for all $\left(u_t\right)_{t\in[i+j+k]}\in\mathcal{L}_6\left(n;i,j,k\right)$ it holds that

$$\beta_7\left(\alpha_7\left(\left(u_t\right)_{t\in[i+j+k]}\right)\right)=\beta_7\left(\left(u_t+(t-1)\right)^{rev}_{t\in[i+j+k]}\right)=\left(u_t\right)_{t\in[i+j+k]},$$

and for all $\left(v_t\right)^{rev}_{t\in[i+j+k]}\in\mathcal{G}\left(n;i,j,k\right)$ it holds that

$$\alpha_7\left(\beta_7\left(\left(v_t\right)^{rev}_{t\in[i+j+k]}\right)\right)=\alpha_7\left(\left(v_t-(t-1)\right)_{t\in[i+j+k]}\right)=\left(v_t\right)^{rev}_{t\in[i+j+k]}.$$

Example 6.24. *Continue Example 6.23. Note that*

$$\begin{aligned}\alpha_7\left(\alpha_6\left(\alpha_5\left(\alpha_4\left(\alpha_3\left(\alpha_2\left(\alpha_1\left(\pi_1,\pi_2,\pi_3\right)\right)\right)\right)\right)\right)\right)&=\alpha_7\left(\left(b_2,a_2,a_5,b_6,a_6,c_6,b_8,b_9,c_9\right)\right)\\&=\left(b_2,a_3,a_7,b_9,a_{10},c_{11},b_{14},b_{16},c_{17}\right)^{rev}\\&=\left(c_{17},b_{16},b_{14},c_{11},a_{10},b_9,a_7,a_3,b_2\right).\end{aligned}$$

Since α_t is a bijection for all $t\in[7]$, we constructed a bijection

$$\alpha_7\circ\alpha_6\circ\alpha_5\circ\alpha_4\circ\alpha_3\circ\alpha_2\circ\alpha_1:\ \mathcal{K}\left(n;i,j,k\right)\to\mathcal{G}\left(n;i,j,k\right),$$

which proves presented generalization of Capparelli's identity (Theorem 6.1). ■

7. Further applications of the method of weighted words

The generality of the objects considered with the method of words allows to find several different consequences by replacing q, a, b and c by suitable alternative terms. As two examples we present two theorems mentioned in [AAG95b, p. 657]. They follow in the same way from Theorem 6.1 as we deduced Capparelli's identity from it in Remark 6.3. The difference is that we replace here q by q^4 instead of q^3.

Theorem 7.1. [AAG95b, p. 657] *The number of partitions of n into distinct parts congruent to $0, 3, 4$ or $5 \pmod 8$ is equal to the number of partitions of n into distinct parts all greater than 1 which are not congruent to $2 \pmod 4$ such that the difference between consecutive parts is greater than 4 unless they are both multiples of 4 or they add up to a multiple of 8.*

Example 7.1. *For $n = 12$, the partitions satisfying the congruence condition are*

$$(12)\,, (8,4) \quad \textit{and} \quad (5,4,3)\,.$$

The corresponding partitions satisfying the difference condition are

$$(12)\,, (9,3) \quad \textit{and} \quad (8,4)\,.$$

Proof. By replacing q by q^4, a by aq^{-3} and b by bq^{-5} in the generating functions in Lemma 6.2 using the ideas how generating functions of partitions are constructed (see the proof of Theorem 2.1) and the generalization Theorem 6.1 we get the following correspondences. Note that we change the weights of the a-parts by multiplying them by 4 and subtracting 3, the weights of the b-parts by multiplying them by 4 and subtracting 5 and the weights of the c-parts by multiplying them by 4.

(i) Words consisting of distinct a-parts with even weights correspond to partitions into distinct parts congruent to $2 \cdot 4 - 3 \equiv 5 \pmod 8$.

(ii) Words consisting of distinct b-parts with even weights correspond to partitions into distinct parts congruent to $2 \cdot 4 - 5 \equiv 3 \pmod 8$.

(iii) Words consisting of distinct c-parts correspond to partitions into distinct parts congruent to $4 \equiv 0 \pmod 4$, i.e. multiples of 4.

If we still use letters a (for i)), b (for ii)) and c (for iii)) in the correspondences we get the following table for the minimal differences.

	a	b	c
a	8	6	7
b	2	8	5
c	5	7	4

This can be justified by looking at the following transformations of pairs of possible consecutive symbols (recall the minimal gap conditions in Definition 6.4). For $k \in \mathbb{N}^*$,

$$\begin{aligned}(a_{k+2}, a_k) &\longrightarrow (4k+5, 4k-3)\,,\\ (a_k, b_k) &\longrightarrow (4k-3, 4k-5)\,,\end{aligned}$$

$$
\begin{aligned}
(a_{k+2}, c_k) &\longrightarrow (4k+5, 4k)\,,\\
(b_{k+2}, a_k) &\longrightarrow (4k+3, 4k-3)\,,\\
(b_{k+2}, b_k) &\longrightarrow (4k+3, 4k-5)\,,\\
(b_{k+3}, c_k) &\longrightarrow (4k+7, 4k)\,,\\
(c_{k+1}, a_k) &\longrightarrow (4k+4, 4k-3)\,,\\
(c_k, b_k) &\longrightarrow (4k, 4k-5)\,,\\
(c_{k+1}, c_k) &\longrightarrow (4k+4, 4k)\,.
\end{aligned}
$$

Therefore by Theorem 6.1, the number of partitions of $n \in \mathbb{N}$ into distinct parts congruent to $0, 3, 4$ or $5 \pmod 8$ is equal to the number of partitions of n into distinct parts congruent to $4, 4-3, 8-5 \pmod 4$ all greater than $4 \cdot 1 - 3$ such that the difference between two consecutive parts is given by the table above. Note that a represents parts congruent to $1 \pmod 4$, b represents parts congruent to $3 \pmod 4$ and c represents multiples of 4. This is exactly the number of partitions of n into distinct parts all greater than 1 which are congruent to $0, 1$ or $3 \pmod 4$ such that the difference between consecutive parts is greater than 4 unless they are both multiples of 4 or they add up to a multiple of 8 (which happens when a part represented by a is followed by a part represented by b with difference 2). ■

With very similar arguments we can deduce also the following theorem which states a result concerning parts belonging to the residual classes containing numbers congruent to 1 or $7 \pmod 8$. The previous theorem used congruences to 5 and $3 \pmod 8$. Note that according to [AAG95b, p. 657] this is eye-catching since

$$
\begin{aligned}
\big(1 \pmod 8\big) + \big(4 \pmod 8\big) &= \big(5 \pmod 8\big)\,,\\
\big(7 \pmod 8\big) + \big(4 \pmod 8\big) &= \big(3 \pmod 8\big)\,.
\end{aligned}
$$

This is way in [AAG95b], they call numbers congruent to 1 or $7 \pmod 8$ to be of *opposite parity* to numbers congruent to 5 or $3 \pmod 8$.

Theorem 7.2. [AAG95b, p. 657] *The number of partitions of n into distinct parts congruent to $0, 1, 4$ or $7 \pmod 8$ is equal to the number of partitions of n into distinct parts all unequal to 3 which are not congruent to $2 \pmod 4$ such that the difference between consecutive parts is greater than 4 unless they are both multiples of 4 or they add up to a multiple of 8.*

Example 7.2. *For $n = 12$, the partitions satisfying the congruence condition are*

$$(12)\,, (8, 4) \text{ and } (7, 4, 1)\,.$$

The corresponding partitions satisfying the difference condition are

$$(12)\,, (11, 1) \text{ and } (8, 4)\,.$$

Proof. By replacing q by q^4, a by aq^{-1} and b by bq^{-7} in the generating functions in Lemma 6.2 using again the ideas how generating functions of partitions are constructed (see the proof of Theorem 2.1) and the generalization Theorem 6.1 we get the following

correspondences. Note that we change the weights of the a-parts by multiplying them by 4 and subtracting 1, the weights of the b-parts by multiplying them by 4 and subtracting 7 and the weights of the c-parts by multiplying them by 4.

(i) Words consisting of distinct a-parts with even weights correspond to partitions into distinct parts congruent to $2 \cdot 4 - 1 \equiv 7 \pmod 8$.
(ii) Words consisting of distinct b-parts with even weights correspond to partitions into distinct parts congruent to $2 \cdot 4 - 7 \equiv 1 \pmod 8$.
(iii) Words consisting of distinct c-parts correspond to partitions into distinct parts congruent to $4 \equiv 0 \pmod 4$, i.e. multiples of 4.

If we still use letters a (for i)), b (for ii)) and c (for iii)) in the correspondences we get the following table for the minimal differences.

	a	b	c
a	8	2	5
b	6	8	7
c	7	5	4

This can be justified by looking at the following transformations of pairs of possible consecutive symbols (recall the minimal gap conditions in Definition 6.4). For $k \in \mathbb{N}^*$,

$$\begin{aligned}
(a_{k+2}, a_k) &\longrightarrow (4k+7, 4k-1)\,, \\
(a_k, b_k) &\longrightarrow (4k-1, 4k-7)\,, \\
(a_{k+2}, c_k) &\longrightarrow (4k+7, 4k)\,, \\
(b_{k+2}, a_k) &\longrightarrow (4k+1, 4k-1)\,, \\
(b_{k+2}, b_k) &\longrightarrow (4k+1, 4k-7)\,, \\
(b_{k+3}, c_k) &\longrightarrow (4k+5, 4k)\,, \\
(c_{k+1}, a_k) &\longrightarrow (4k+4, 4k-1)\,, \\
(c_k, b_k) &\longrightarrow (4k, 4k-7)\,, \\
(c_{k+1}, c_k) &\longrightarrow (4k+4, 4k)\,.
\end{aligned}$$

Therefore by Theorem 6.1, the number of partitions of $n \in \mathbb{N}$ into distinct parts congruent to $0, 1, 4$ or $7 \pmod 8$ is equal to the number of partitions of n into distinct parts congruent to $4, 4-1, 8-7 \pmod 4$ all unequal to $4 \cdot 1 - 1$ such that the difference between two consecutive parts is given by the table above. Note that a represents parts congruent to $3 \pmod 4$, b represents parts congruent to $1 \pmod 4$ and c represents multiples of 4.
This is exactly the number of partitions of n into distinct parts all unequal to 3 which are congruent to $0, 1$ or $3 \pmod 4$ such that the difference between consecutive parts is greater than 4 unless they are both multiples of 4 or they add up to a multiple of 8 (which happens when a part represented by b is followed by a part represented by a with difference 2). ■

As a final remark, note that it is possible finding different Rogers-Ramanujan type partition identities by changing the lexicographic ordering (see (60)). Examples (related to Schur's theorem) can be found in [AG95].

Appendix: MATLAB Codes

To find the counter example to disprove the conjecture on a second refinement in Subsection 5.4 we used the following (self coded) MATLAB programs.

Listing 1. MATLAB Script for finding the counter example

```
%size
s = 50;

%initialize vectors for saving the counts
counts_c = zeros(1, s);
counts_d = zeros (1, s);

%parameter
i = 1;
j = 0;
l = 1;
m = 3;
q = 4;

%counting
for n = 1:s
    counts_c(n) = capCijk_ref_part(n, n, i, j, l, m, q, 7);
    counts_d(n) = capDijk_part(n, 3*q - 2, i, j, l);
end

%printing the result
fprintf('The partitions counted by C(n; i, j, k, N):')
counts_c

fprintf('The partitions counted by D(n; i, j, k, N):')
counts_d
```

Listing 2. Function for counting partitions counted by $\tilde{E}(n; i, j, k, N)$.

```
function counting = capCijk_ref_part(n, lp, i, j, l, m, q, ub)
%Counts the number of partitions of n into distinct parts congruent 2, 3,
%4 or 6 (mod 6) with largest part at most lp, exactly i parts congruent
%4 modulo 6, exactly j parts congruent 2 mod 6, among of those congruent
%0 modulo 3, exactly l are > 3(i + j) and the largest part not congruent 0
%(mod 3) is less or equal 3N - 2 - 3l
%   INPUT:
%       n:          natural number, size of the considered partitions.
%       lp:         natural number, upper bound on the largest part of the
%                   considered partitions.
%       i, j, l:    natural numbers, according to the description above.
%       m:          natural number, lower bound for the "l-condition",
%                   equals 3*(i + j) for initial values i and j
%                   (attention: recursion!)
%       q:          natural number, equals N, parameter for extra condition
%                   on the largest part not congruent to 0 mod 3.
```

```
%       ub:         natural number, upper bound for extra condition, equals
%                   3q - 2 - 3l for initial value of l (attention:
%                   recursion!)
%   OUTPUT:
%       counting:   natural number, the counted number of the described
%                   partitions above.
%

%initialize counter
counting = 0;

%base case of the recursion
if (n == 0) & (i == 0) & (j == 0) & (l == 0)
    counting = 1;

%recursion
elseif n > 0

    %k corresponds to a largest part equal lp - k
    for k = 0:(lp - 1)

        %modulo conditions
        if ~((mod((lp - k), 6) == 1) | (mod((lp - k), 6) ==5))

            %parts that are multiples of 3
            if ~((mod((lp - k), 6) == 4) | (mod((lp - k), 6) == 2))

                %parts that are bigger than the lower bound
                if (l > 0) & ((lp - k) > m)
                    counting = counting + ...
                        capCijk_ref_part(n - lp + k, lp - k - 1, i, j, ...
                        l - 1, m, q, ub);

                %parts that are smaller than the lower bound
                elseif (lp - k) <= m
                    counting = counting + ...
                        capCijk_ref_part(n - lp + k, lp - k - 1, i, j, ...
                        l, m, q, ub);
                end

            %parts congruent to 4 modulo 6
            elseif (i > 0) & (mod((lp - k), 6) == 4) & ((lp - k) <= ub)
                counting = counting + ...
                    capCijk_ref_part(n - lp + k, lp - k - 1, i - 1, j, ...
                    l, m, q, ub);

            %parts congruent to 2 modulo 6
            elseif (j > 0) & (mod((lp - k), 6) == 2) & ((lp - k) <= ub)
                counting = counting + ...
                    capCijk_ref_part(n - lp + k, lp - k - 1, i, j - 1, ...
                    l , m, q, ub);
```

```
            end
        end
    end
end

end
```

LISTING 3. Function for counting partitions counted by $\tilde{D}(n;i,j,k)$

```
function counting = capDijk_part(n, lp, i, j, p)
%Counts the number of partitions of n into distinct parts > 1 with
%minimal difference 2 where the difference is at least 4 unless
%consecutive parts are both multiples of 3 or add up to a multiple of 6
%with largest part at most lp, exactly i parts congruent 1 (mod 3), exactly
%j parts congruent 2 (mod 3) and exactly p parts congruent 0 (mod 3).
%   INPUT:
%       n:          natural number, size of the considered partitions.
%       lp:         natural number, upper bound on the largest part of the
%                   considered partitions.
%       i, j, p:    natural numbers, according to the description above.
%   OUTPUT:
%       counting:   natural number, the counted number of the described
%                   partitions above.

%initialize counter
counting = 0;

%base case of the recursion
if (n == 0) & (i == 0) & (j == 0) & (p == 0)
    counting = 1;

%recursion
elseif n > 0

        %the parts have to be > 1; k corresponds to a largest part = lp - k
        for k = 0:(lp - 2)

            %parts that are multiples of 3
            if (mod(lp - k, 3) == 0) & (p > 0)
                counting = counting + ...
                    capDijk_part(n - lp + k, lp - k - 3, i, j, p - 1);

            %parts x that are congruent 1 mod 3 and are followed by x - 2
            %or at most x - 4 (x - 3 is not allowed)
            elseif (mod(lp - k, 3) == 1) & (i > 0) & (j > 0)
                counting = counting + ...
                    capDijk_part(n - lp + k, lp - k - 4, i - 1, j, p) + ...
                        capDijk_part(n - 2*(lp - k - 1), ...
                        lp - k - 6, i - 1, j - 1, p);

            %parts that are congruent 1 mod 3 and can't be followed by x -
```

```
43                  %2
44                  elseif (mod(lp - k, 3) == 1) & (i > 0) & (j == 0)
45                      counting = counting + ...
46                          capDijk_part(n - lp + k, lp - k - 4, i - 1, j, p);
47
48                  %parts that are congruent 2 mod 3
49                  elseif (mod(lp - k, 3) == 2) & (j > 0)
50                      counting = counting + ...
51                          capDijk_part(n - lp + k, lp - k - 4, i, j - 1, p);
52              end
53      end
54  end
```

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