Eigenvalue Problems in Electromagnetics
Part I

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Zürich Summer School, August 25-29, 2008

Outline

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  - spectral properties
- Spectrally correct approximations
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- Discontinuous Galerkin (DG) approximations
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  - theory
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- The (indefinite) time-harmonic Maxwell source problem

The Maxwell Eigenproblem

Problem Domain

Let Ω be an open, bounded, simply-connected Lipschitz polyhedron in $\mathbb{R}^3$ with connected boundary $\partial\Omega$.

Let $\mathbf{n}$ be the outward unit normal vector to $\partial\Omega$.

Unknowns and parameters

- ω: frequency
- $\mathbf{E} = \mathbf{E}(\mathbf{x})$: electric field phasor (electric field$=$ $\mathrm{Re}(\mathbf{E}(\mathbf{x})e^{i\omega t})$)
- $\mathbf{H} = \mathbf{H}(\mathbf{x})$: magnetic field phasor
- μ: magnetic permeability
- ε: electric permittivity

The Maxwell Eigenproblem

Find $(\mathbf{0} \neq \mathbf{E}, \mathbf{0} \neq \mathbf{H}, \omega) \in L^2(\Omega)^3 \times L^2(\Omega)^3 \times \mathbb{C}$ s.t.

$$\begin{aligned}
\nabla \times \mathbf{E} &= -i\omega\mu\mathbf{H} && \text{in } \Omega\\
\nabla \cdot (\varepsilon\mathbf{E}) &= 0 && \text{in } \Omega\\[1em]
\nabla \times \mathbf{H} &= i\omega\varepsilon\mathbf{E} && \text{in } \Omega\\
\nabla \cdot (\mu\mathbf{H}) &= 0 && \text{in } \Omega\\[1em]
\mathbf{n} \times \mathbf{E} &= 0 && \text{on } \partial\Omega\\
(\mu\mathbf{H}) \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega
\end{aligned}$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H} \;\Rightarrow \mu^{-1}\nabla \times \mathbf{E} = -i\omega\mathbf{H}$$

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) = -i\omega\nabla \times \mathbf{H} = \omega^2\varepsilon\mathbf{E}$$

The Maxwell Eigenproblem

Assumptions on material coefficients

- μ and ε second order, real, symmetric tensor fields
- $\exists\, \mu_*, \mu^*, \varepsilon_*, \varepsilon^* \in L^\infty(\Omega)$ strictly positive s.t., $\forall\, \xi \in \mathbb{R}^3$, $|\xi|^2 = 1$,

$$\mu_*(\mathbf{x}) \leq \sum_{i,j=1}^{3} \mu_{i,j}(\mathbf{x})\xi_i\xi_j \leq \mu^*(\mathbf{x}) \qquad \text{a.e. in } \Omega$$

$$\varepsilon_*(\mathbf{x}) \leq \sum_{i,j=1}^{3} \varepsilon_{i,j}(\mathbf{x})\xi_i\xi_j \leq \varepsilon^*(\mathbf{x}) \qquad \text{a.e. in } \Omega$$

- there exists a partition $\Omega = \cup_{k=1}^{K} \Omega_k$, such that μ and ε are smooth in each Ω_k

The Maxwell Eigenproblem

Functional Spaces

$$H(\mathrm{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3\}$$
$$H_0(\mathrm{curl};\Omega) = \{\mathbf{v} \in H(\mathrm{curl};\Omega) : \mathbf{n} \times \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$$
$$H_0(\mathrm{curl}^0;\Omega) = \{\mathbf{v} \in H_0(\mathrm{curl};\Omega) : \nabla \times \mathbf{v} = \mathbf{0}\}$$
$$H(\mathrm{div}_\varepsilon^0;\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \cdot (\varepsilon \mathbf{v}) = 0 \text{ in } \Omega\}$$

Denote by $(\cdot,\cdot)$ the standard L^2–inner product and by $(\cdot,\cdot)_\varepsilon$ the ε–weighted L^2–inner product; the L^2–norm and the L^2_ε–norm are clearly equivalent.

Define

$$|\mathbf{v}|^2_{H(\mathrm{curl};\Omega)} = (\mu^{-1}\nabla \times \mathbf{v}, \nabla \times \mathbf{v})$$
$$\|\mathbf{v}\|^2_{H(\mathrm{curl};\Omega)} = |\mathbf{v}|^2_{H(\mathrm{curl};\Omega)} + (\mathbf{v},\mathbf{v})_\varepsilon$$

The Maxwell Eigenproblem

$$\begin{aligned}
\mathbf{V} &= H_0(\mathrm{curl};\Omega) \\
\mathbf{V}^0 &= H_0(\mathrm{curl}^0;\Omega) \\
\mathbf{W} &= \mathbf{V} \cap H(\mathrm{div}_\varepsilon^0;\Omega)
\end{aligned}$$

Hermitian Bilinear Form

$$a(\mathbf{u},\mathbf{v}) = (\mu^{-1}\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) \qquad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{V}$$

Coercivity in Seminorm and Continuity

$$\begin{aligned}
&\mathrm{Re}\,[a(\mathbf{v},\mathbf{v})] = a(\mathbf{v},\mathbf{v}) \geq \alpha|\mathbf{v}|_{\mathbf{V}}^2 \qquad && \forall\,\mathbf{v}\in\mathbf{V} \qquad (\alpha = 1) \\
&|a(\mathbf{u},\mathbf{v})| \leq \gamma\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}} && \forall\,\mathbf{u},\mathbf{v}\in\mathbf{V}
\end{aligned}$$

The Maxwell Eigenproblem

Kernel and its L^2_ε–Orthogonal Complement

$$\mathbf{K} = \{\mathbf{w} \in \mathbf{V} : a(\mathbf{w}, \mathbf{v}) = 0 \ \forall \mathbf{v} \in \mathbf{V}\} = \mathbf{V}^0$$
$$\mathbf{K}^\perp = \mathbf{W}$$

L^2_ε–Orthogonal Helmholtz Decompositions

$$L^2(\Omega)^3 = \mathbf{V}^0 \oplus H(\mathrm{div}^0_\varepsilon; \Omega)$$
$$\mathbf{V} = \mathbf{V}^0 \oplus \mathbf{W} = \mathbf{K} \oplus \mathbf{K}^\perp \qquad (\mathbf{V}\text{–orthogonality})$$

The Maxwell Eigenproblem

Problem 1

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{W} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon$$

Remark: If $(\mathbf{E}, \mathbf{H}, \omega)$ is a Maxwell eigenpair, then $(\mathbf{E}, \omega)$ is an eigenpair of Problem 1; if $(\mathbf{u}, \omega)$ is an eigenpair of Problem 1, then $(\mathbf{u}, -(1/i\omega)\mu^{-1}\nabla \times \mathbf{u}, \omega)$ is a Maxwell eigenpair.

The Maxwell Eigenproblem

Problem 1

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{W} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall \mathbf{v} \in \mathbf{W}$$

Problem 2

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall \mathbf{v} \in \mathbf{V}$$

We will study approximations to Problem 2

The Maxwell Eigenproblem

Problem 2

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall\, \mathbf{v} \in \mathbf{V}$$

- $\omega^2 = 0$ is an eigenvalue with *infinite dimensional* associated eigenspace $\mathbf{V}^0$ ($\{0\}$ = essential spectrum)
- $\omega^2 = 0$ is an isolated eigenvalue and all the other eigenvalues are real and strictly positive and form a sequence accumulating only at $+\infty$
- all the eigenspaces associated with eigenvalues $\neq 0$ are finite dimensional
- eigenfunctions associated with different eigenvalues are L^2_ε–orthogonal and $\mathbf{V}$–orthogonal

The Maxwell Eigenproblem

Problem 2

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall\, \mathbf{v} \in \mathbf{V}$$

Problem 3

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v})_\varepsilon =: b(\mathbf{u}, \mathbf{v}) = \widetilde{\omega}^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall\, \mathbf{v} \in \mathbf{V}$$

Remark: All the eigenvalues of Problem 2 and Problem 3 are such that

$$\widetilde{\omega}^2 = \omega^2 + 1$$

thus $\widetilde{\omega}^2 = 1$ is an eigenvalue of Problem 3 with infinite multiplicity and associate eigenspace $\mathbf{V}^0$.

Spectrally Correct Approximation

Following [Descloux, Nassif & Rappaz, 1978]:

i) isolation of the discrete essential spectrum, i.e., all the discrete eigenvalues approaching $\omega^2 = 0$ (or $\widetilde{\omega}^2 = 1$) are separated from the other ones

ii) non-pollution of the spectrum, i.e., there are no discrete spurious eigenvalues

iii) completeness of the spectrum, i.e., all continuous eigenvalues smaller than an arbitrarily large fixed value are approximated when the mesh is sufficiently fine

iv) non-pollution and completeness of the eigenspaces, i.e., there are no spurious eigenfunctions and the eigenspace approximations associated with eigenvalues which are not approaching $\omega^2 = 0$ have the right dimension

Lack of spectral correctness: we expect *spurious* solutions for the associated parabolic or hyperbolic evolution problems

Some References

Conforming Discretizations

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Discontinuous Galerkin Discretizations

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Finite Element Approximation

Finite element spaces

$\{\mathbf{V}_h\}_h$ finite element spaces on meshes $\{\mathcal{T}_h\}_h$

- $\mathbf{V}_h \subset \mathbf{V} \rightarrow$ *conforming* approximation
- $\mathbf{V}_h \not\subset \mathbf{V} \rightarrow$ *non conforming* approximation

Discrete Bilinear Forms

$$a_h(\mathbf{u}_h, \mathbf{v}_h) \qquad \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$$

Discrete Problem

Find $(\mathbf{0} \neq \mathbf{u}_h, \omega_h) \in \mathbf{V}_h \times \mathbb{C}$ s.t.

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \omega_h^2 (\mathbf{u}_h, \mathbf{v}_h)_\varepsilon \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h$$

Conforming Finite Elements

Example I: Nodal Elements

$$\mathbf{V}_h = \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v}|_K \in \mathcal{S}^\ell(K)^3 \ \forall K \in \mathcal{T}_h \ + \ \text{b.c.}\}$$

with $\mathcal{S}^\ell(K) = \mathcal{P}^\ell(K)$ if K is a tetrahedron
or $\mathcal{S}^\ell(K) = \mathcal{Q}^\ell(K)$ K is a parallelepiped

(standard *continuous* vector-valued finite elements)

$$\mathbf{V}_h \subset H^1(\Omega)^3 \subset H(\mathrm{curl}; \Omega)$$

Approximation Properties

- order ℓ in H^1–norm (and therefore in $\mathbf{V}$–norm)
- order $\ell + 1$ in L^2–norm

Conforming Finite Elements

Example II: Edge Elements ([Nédélec, 1980, 1982])

- D.o.f.: edge, face and volume moments
- Continuity of the *tangential component only* imposed at interelement boundaries (contained in $H(\mathrm{curl};\Omega)$ but not in $H^1(\Omega)^3$)
- First family on tetrahedra: the local space is

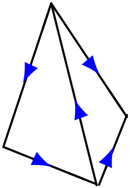
$$\mathbf{R}^\ell(K) = \mathcal{P}^{\ell-1}(K)^3 \oplus \mathcal{S}^\ell(K) \subset \mathcal{P}^\ell(K)^3$$

  where $\mathcal{S}^\ell(K) = \{\mathbf{p} \text{ homog. pol. degree } \ell \text{ on } K : \mathbf{p}\cdot\mathbf{x} = 0\}$
- Second family on tetrahedra: the local space is $\mathcal{P}^\ell(K)^3$
- Extension to parallelepipeds
- Approximation properties: order ℓ in $H(\mathrm{curl};\Omega)$–norm

Conforming Finite Elements

D.o.f. for the Nédélec elements, first family, lowest degree

$$\begin{bmatrix} \alpha_1 + \beta_2 z - \beta_3 y \\ \alpha_2 + \beta_3 x - \beta_1 z \\ \alpha_3 + \beta_1 y - \beta_2 x \end{bmatrix}$$



6 d.o.f.: $\int_{e_i} \mathbf{n} \times \mathbf{v}$

Conforming Finite Elements

$$\mathbf{V}_h \subset \mathbf{V} \qquad \text{and} \qquad a_h(\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}) \qquad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{V}_h$$

Remark: The condition $a_h(\mathbf{u}_h,\mathbf{v}_h) = a(\mathbf{u}_h,\mathbf{v}_h)$ rules out the *ellipticized variational formulation*.

Discrete Problem

Find $(\mathbf{0} \neq \mathbf{u}_h, \omega_h) \in \mathbf{V}_h \times \mathbb{C}$ s.t.

$$a(\mathbf{u}_h,\mathbf{v}_h) = \omega_h^2 (\mathbf{u}_h,\mathbf{v}_h)_\varepsilon \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h$$

Conforming Finite Elements

Define

$$\mathbf{V}_h^0 = \mathbf{V}_h \cap \mathbf{V}^0 \subset \mathbf{V}^0$$
$$\mathbf{W}_h = \{\mathbf{v} \in \mathbf{V}_h :\ (\mathbf{v}, \mathbf{w})_{\mathbf{V}} = (\mathbf{v}, \mathbf{w})_\varepsilon = 0\ \forall\, \mathbf{w} \in \mathbf{V}_h^0\} \not\subset \mathbf{W}$$

L^2_ε and **V**–Orthogonal Decomposition

$$\mathbf{V}_h = \mathbf{V}_h^0 \oplus \mathbf{W}_h$$

Discrete Kernel and its Orthogonal Complement

$$\mathbf{K}_h = \{\mathbf{v} \in \mathbf{V}_h :\ a(\mathbf{v}, \mathbf{w}) = 0\ \forall\, \mathbf{w} \in \mathbf{V}_h\} = \mathbf{V}_h \cap \mathbf{V}^0$$
$$\mathbf{K}_h^\perp = \mathbf{W}_h$$

$\mathbf{K}_h \subset \mathbf{K}$ but $\mathbf{K}_h^\perp \not\subset \mathbf{K}^\perp$

Conforming Finite Elements

Discrete Problem

Find $(\mathbf{0} \neq \mathbf{u}_h, \omega_h) \in \mathbf{V}_h \times \mathbb{C}$ s.t.

$$a(\mathbf{u}_h, \mathbf{v}_h) = \omega_h^2 (\mathbf{u}_h, \mathbf{v}_h)_\varepsilon \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h$$

Properties of the Discrete Eigenproblem

- All the discrete eigenvalues are real and non negative
- $\omega_h = 0$ is an eigenvalue with associated eigenspace $\mathbf{K}_h = \mathbf{V}_h^0$

Assumption (CAS)

Completeness of the Approximating Subspace (CAS)

$$\lim_{h\to 0} \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}} = 0 \qquad \forall\, \mathbf{v} \in \mathbf{V}$$

Coercivity in Seminorm and Continuity + (CAS)

$\Rightarrow$

Convergence for the Positive Definite Source Problem

$$b(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \text{with } \mathbf{f} \in H(\mathrm{div}^0_\varepsilon; \Omega)$$

(recall: $b(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v})_\varepsilon$)

Assumptions (DFI) and (DCP)

Discrete Friedrichs Inequality (DFI)

$$\|\mu^{-1/2}\nabla \times \mathbf{v}_h\|_{L^2(\Omega)^3} = a(\mathbf{v}_h, \mathbf{v}_h) \geq \widetilde{\alpha}\|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^3} \qquad \forall\, \mathbf{v}_h \in \mathbf{K}_h^\perp$$

Discrete Compactness Property (DCP)

Any $\{\mathbf{v}_h\}_h \subset \mathbf{K}_h^\perp$ s.t. $\|\mathbf{v}_h\|_{\mathbf{V}} \leq 1$ for all h contains a subsequence (still denoted by $\{\mathbf{v}_h\}_h$) s.t.

$$\exists\, \mathbf{v} \in L^2(\Omega)^3 : \quad \lim_{h\to 0} \|\mathbf{v} - \mathbf{v}_h\|_{L^2_\varepsilon(\Omega)^3} = 0$$

Remarks on (DCP)

- (DCP) is the discrete counterpart of the compact embedding of $\mathbf{W} = H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0_\varepsilon;\Omega)$ in $L^2_\varepsilon(\Omega)^3$
- In (DCP), the limit $\mathbf{v}$ actually belongs to $\mathbf{V}$;
  in fact: $\{\mathbf{v}_h\}_h$ bounded in $\mathbf{V} \Rightarrow \mathbf{v}_h \rightharpoonup \widetilde{\mathbf{v}}$ in $\mathbf{V}$; if, by (DCP), $\{\mathbf{v}_h\}_h \to \mathbf{v}$ in $L^2_\varepsilon(\Omega)$, then $\mathbf{v} = \widetilde{\mathbf{v}} \in \mathbf{V}$
- (DCP), as well as (DFI), is a condition on $\mathbf{K}_h^\perp$, hence on $\mathbf{V}_h$

Main Theorem

Recall:

Problem 2

Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall \, \mathbf{v} \in \mathbf{V}$$

Main Theorem (in words)

(CAS), (DFI) and (DCP) are necessary and sufficient conditions for a *spectrally correct* conforming approximation to Problem 2

Remark: Unlike for problems with compact inverse operators, the choice of a complete approximating subspace does not necessarily provide a correct approximation for an eigenproblem.

Some Details of the Analysis

Recall: $b(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v})_\varepsilon$

Define the solution operators:

$$T : L^2(\Omega)^3 \to \mathbf{V} \qquad b(T\mathbf{f}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\varepsilon$$
$$T_h : L^2(\Omega)^3 \to \mathbf{V}_h \qquad b(T_h\mathbf{f}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_\varepsilon$$

$(\mathbf{u}, \omega)$ Maxwell eigenpair $\Leftrightarrow$ $(\mathbf{u}, \lambda = \frac{1}{\omega^2+1})$ eigenpair of T

If λ_h is an eigenvalue of T_h, then λ_h is real and $0 < \lambda_h \leq 1$
1 is an eigenvalue of T_h with associated eigenspace $\mathbf{K}_h$

Recall: Spectrally Correct Approximation

i) isolation of the discrete essential spectrum (IDES), i.e., all the discrete eigenvalues approaching $\lambda = 1$ are separated from the other ones

ii) non-pollution of the spectrum, i.e., there are no discrete spurious eigenvalues

iii) completeness of the spectrum, i.e., all continuous eigenvalues smaller than an arbitrarily large fixed value are approximated when the mesh is sufficiently fine

iv) non-pollution and completeness of the eigenspaces, i.e., there are no spurious eigenfunctions and the eigenspace approximations associated with eigenvalues which are not approaching $\omega^2 = 0$ have the right dimension

Scheme of the Analysis

- (DFI) $\Leftrightarrow$ isolation of the discrete essential spectrum (IDES) (i.e., condition i))
- (CAS) + (DCP) $\Rightarrow$ convergence in a mesh dependent norm (CMDN) of $\{T_h\}_h$ to T
- (CMDN) $\Rightarrow$ non-pollution of the spectrum (i.e., condition ii))
- Completeness of the spectrum, non-pollution and completeness of the eigenspaces are also satisfied (i.e., conditions iii) and iv))
- Convergence rates for eigenvalue and eigenfunction approximations

First Key Result

Discrete Friedrichs Inequality (DFI)

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \widetilde{\alpha} \|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^3} \qquad \forall\, \mathbf{v}_h \in \mathbf{K}_h^\perp$$

is equivalent to

Isolation of the Discrete Essential Spectrum (IDES)

If λ_h is an eigenvalue of T_h with $\lambda_h \neq 1$, then

$$\lambda_h \leq \beta < 1$$

Proof of (DFI) $\Rightarrow$ (IDES)

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \widetilde{\alpha} \|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^3} \qquad \forall\, \mathbf{v}_h \in \mathbf{K}_h^\perp$$

If λ_h is an eigenvalue of T_h with $\lambda_h \neq 1$ and $\mathbf{v}_h$ is an associated eigenfunction, then $\mathbf{v}_h \in \mathbf{K}_h^\perp$

$$\begin{aligned}
\widetilde{\alpha}\lambda_h^2 \|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^3} &\leq a(\lambda_h \mathbf{v}_h, \lambda_h \mathbf{v}_h) = a(T_h \mathbf{v}_h, \lambda_h \mathbf{v}_h) \\
&= b(T_h \mathbf{v}_h, \lambda_h \mathbf{v}_h) - (T_h \mathbf{v}_h, \lambda_h \mathbf{v}_h)_\varepsilon \\
&= (\mathbf{v}_h, \lambda_h \mathbf{v}_h)_\varepsilon - (\lambda_h \mathbf{v}_h, \lambda_h \mathbf{v}_h)_\varepsilon \\
&= (\lambda_h - \lambda_h^2) \|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^3}
\end{aligned}$$

Then, $\widetilde{\alpha}\lambda_h \leq 1 - \lambda_h$ which gives

$$\lambda_h \leq 1/(1 + \widetilde{\alpha}) =: \beta < 1$$

Proof of (IDES) $\Rightarrow$ (DFI)

If λ_h is an eigenvalue of T_h with $\lambda_h \neq 1$, then

$$\lambda_h \leq \beta < 1$$

If $\mathbf{v}_h \in \mathbf{K}_h^{\perp}$, then

$$\mathbf{v}_h = \sum_{1 \neq \lambda_h \text{ eigenv}} \mathbf{v}_{\lambda_h}$$

Thus

$$a(\mathbf{v}_h, \mathbf{v_h}) = \sum_{\lambda_h} \sum_{\nu_h} a(\mathbf{v}_{\lambda_h}, \mathbf{v}_{\nu_h}) = \sum_{\lambda_h} \sum_{\nu_h} (\lambda_h^{-1} - 1)(\mathbf{v}_{\lambda_h}, \mathbf{v}_{\nu_h})_\varepsilon$$

$$\geq (\beta^{-1} - 1) \sum_{\lambda_h} \sum_{\nu_h} (\lambda_h^{-1} - 1)(\mathbf{v}_{\lambda_h}, \mathbf{v}_{\nu_h})_\varepsilon = (\beta^{-1} - 1) \|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^2}$$

Therefore

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \widetilde{\alpha} \|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^3} \qquad \text{with } \widetilde{\alpha} := (\beta^{-1} - 1)$$

Second Key Result ((CAS) is assumed)

Discrete Compactness Property (DCP)

Any $\{\mathbf{v}_h\}_h \subset \mathbf{K}_h^\perp$ s.t. $\|\mathbf{v}_h\|_{\mathbf{V}} \leq 1$ for all h contains a subsequence (still denoted by $\{\mathbf{v}_h\}_h$) s.t.

$$\exists\, \mathbf{v} \in L^2(\Omega)^3 : \quad \lim_{h\to 0} \|\mathbf{v} - \mathbf{v}_h\|_{L^2_\varepsilon(\Omega)^3} = 0$$

implies

Convergence in Mesh-Dependent Norm (CMDN)

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(\mathbf{V},\mathbf{V})} = 0$$

Equivalently: for h small enough,

$$\|(T - T_h)\mathbf{f}_h\|_{\mathbf{V}} \leq \xi_h \|\mathbf{f}_h\|_{\mathbf{V}} \qquad \forall\, \mathbf{f}_h \in \mathbf{V}_h$$

with $\xi_h \to 0$ as $h \to 0$

Proof of (CAS)+(DCP) $\Rightarrow$ (CMDN)

Convergence in Mesh-Dependent Norm (CMDN)

$$\|(T - T_h)\mathbf{f}_h\|_{\mathbf{V}} \leq \xi_h \|\mathbf{f}_h\|_{\mathbf{V}} \qquad \forall\, \mathbf{f}_h \in \mathbf{V}_h$$

- $\mathbf{f}_h = \mathbf{f}_h^0 + \mathbf{f}_h^\perp \in \mathbf{K}_h \oplus \mathbf{K}_h^\perp$ (**V**–orthogonal decomp.); $\|\mathbf{f}_h^\perp\|_{\mathbf{V}} \leq \|\mathbf{f}_h\|_{\mathbf{V}}$
- uniqueness of sol. of PD source pbl. $\Rightarrow (T - T_h)\mathbf{f}_h^0 = 0$
  $\rightarrow$ prove $\|(T - T_h)\mathbf{f}_h^\perp\|_{\mathbf{V}} \leq \xi_h \|\mathbf{f}_h^\perp\|_{\mathbf{V}}$
- (DCP) $\Rightarrow$ for h small enough, for any $\mathbf{f}_h^\perp \in \mathbf{K}_h^\perp$ there exists $\mathbf{f} \in H(\mathrm{div}_\varepsilon^0; \Omega)$ s.t.

$$\|\mathbf{f} - \mathbf{f}_h^\perp\|_{L^2_\varepsilon(\Omega)^3} \leq \eta_h \|\mathbf{f}_h^\perp\|_{\mathbf{V}}$$

  with $\eta_h \rightarrow 0$ as $h \rightarrow 0$ (by contradiction)

Proof of (CAS)+(DCP) $\Rightarrow$ (CMDN)

- For h small enough,

$$\begin{aligned}\|(T - T_h)\mathbf{f}_h^{\perp}\|_{\mathbf{V}} &\leq \|(T - T_h)(\mathbf{f} - \mathbf{f}_h^{\perp})\|_{\mathbf{V}} + \|(T - T_h)\mathbf{f}\|_{\mathbf{V}} \\ &=: (I) + (II)\end{aligned}$$

- $(I) \leq C\|\mathbf{f} - \mathbf{f}_h^{\perp}\|_{L^2_\varepsilon(\Omega)^3} \leq C\eta_h\|\mathbf{f}_h^{\perp}\|_{\mathbf{V}}$
- (CAS) $\Rightarrow$ convergence of the PD source pbl. $\Rightarrow$

$$\begin{aligned}(II) &\leq C\chi_h\|\mathbf{f}\|_{L^2_\varepsilon(\Omega)^3} \\ &\leq C\chi_h(\|\mathbf{f} - \mathbf{f}_h^{\perp}\|_{L^2_\varepsilon(\Omega)^3} + \|\mathbf{f}_h^{\perp}\|_{L^2_\varepsilon(\Omega)^3}) \\ &\leq C\xi_h(\eta_h + 1)\|\mathbf{f}_h^{\perp}\|_{L^2_\varepsilon(\Omega)^3}\end{aligned}$$

Non-Pollution of the Spectrum

Convergence in mesh-dependent norm implies

Non-Pollution of the Spectrum

If z is not an eigenvalue of T then, for h small enough,

$$\|(z - T_h)\mathbf{f}_h\|_{\mathbf{V}} \geq C\|\mathbf{f}_h\|_{\mathbf{V}} \qquad \forall\, \mathbf{f}_h \in \mathbf{V}_h$$

In words: if z is not an eigenvalue of T, then, for h small enough, z is not an eigenvalue of T_h

$\Rightarrow$ if $G \subset \mathbb{C}$ is an open set containing all the eigenvalue of T, then, for h small enough, G also contains all the eigenvalues of T_h
$\rightarrow$ no spurious eigenvalues

Proof of (CMDN) $\Rightarrow$ Non-pollution of Spectrum

Non-Pollution of the Spectrum

$$\|(z - T_h)\mathbf{f}_h\|_{\mathbf{V}} \geq C\|\mathbf{f}_h\|_{\mathbf{V}} \qquad \forall\, \mathbf{f}_h \in \mathbf{V}_h$$

- Triangle inequality:

$$\|(z - T_h)\mathbf{f}_h\|_{\mathbf{V}} \geq \|(z - T)\mathbf{f}_h\|_{\mathbf{V}} - \|(T - T_h)\mathbf{f}_h\|_{\mathbf{V}}$$

- Since $\mathbf{f}_h \in \mathbf{V}$ and z is not an eigenvalue of T, then

$$\|(z - T)\mathbf{f}_h\|_{\mathbf{V}} \geq C\|\mathbf{f}_h\|_{\mathbf{V}}$$

- (CMDN): for h small enough,

$$\|(T - T_h)\mathbf{f}_h\|_{\mathbf{V}} \leq \xi_h\|\mathbf{f}_h\|_{\mathbf{V}}$$

- Conclusion:

$$\|(z - T)\mathbf{f}_h\|_{\mathbf{V}} \geq (C - \xi_h)\|\mathbf{f}_h\|_{\mathbf{V}}$$

Final Results

Non-Pollution of the Spectrum

If z is not an eigenvalue of T then, for h small enough,

$$\|(z - T_h)\mathbf{f}_h\|_{\mathbf{V}} \geq C\|\mathbf{f}_h\|_{\mathbf{V}} \qquad \forall\, \mathbf{f} \in \mathbf{V}_h$$

- Completeness of the spectrum
- Non-pollution and completeness of the eigenspaces
- Eigenvalue and eigenfunction convergence rates

Convergence Rates

Eigenvalue Approximation

Let $\lambda \neq 1$ be an eigenvalue of T with multiplicity m; for h small enough, there exist m discrete eigenvalues $\lambda_{i,h}$ s.t.

$$\sup_{1\leq i\leq m} |\lambda - \lambda_{i,h}| \leq Ch^{2t}$$

$t = \min\{\ell, \sigma_\lambda\}$, with σ_λ s.t. $\mathbf{v} \in H^{\sigma_\lambda}(\mathrm{curl};\Omega)$ for all $\mathbf{v} \in E_\lambda$

Distance between closed subspaces of $\mathbf{V} + \mathbf{V}_h$:

$$\delta(Y, Z) := \sup_{y\in Y,\ \|y\|_{\mathbf{V}(h)}=1} \inf_{z\in Z} \|y - z\|_{\mathbf{V}(h)}$$

$$\widehat{\delta}(Y, Z) := \max\{\delta(Y, Z), \delta(Z, Y)\}$$

Eigenfunction Approximation

For h small enough, $\quad \widehat{\delta}(E_\lambda, E_{\{\lambda_{i,h}\}}) \leq Ch^t$

Analysis of Edge Element Approximations

Simplicial mesh $\mathcal{T}_h$;

$$V_h = \{\mathbf{v}_h \in \mathbf{V} : \, v_h|_K \in \mathbf{R}^\ell(K) \; \forall \, K \in \mathcal{T}_h\}$$

(Nédélec's elements, first family)

Scheme of the Analysis

- (CAS) is satisfied
- Prove (DFI) and (DCP) for $\mu = \varepsilon = I$
- Extend the results to generic μ and ε

Remark: Everything ok for Nédélec I and II on tetrahedra, and for Nédélec I on parallelepipeds, but *not* for Nédélec II on parallelepipeds (see [Boffi, Costabel, Dauge & Demkowicz, 2006])

Preliminary Results

Recall $\mathbf{V}_h = \mathbf{V}_h^0 \oplus \mathbf{W}_h$, where $\mathbf{V}_h^0 = \mathbf{V}_h \cap \mathbf{V}^0$ and $\mathbf{W}_h$ its L^2_ε–orthogonal complement; $\mathbf{K}_h = \mathbf{V}_h^0$ and $\mathbf{K}_h^\perp = \mathbf{W}_h$.
Set

$$S_h = \{p_h \in H_0^1(\Omega) : \, p_h|_K \in \mathcal{P}^\ell \; \forall K \in \mathcal{T}_h\}$$

Discrete Helmholtz Decomposition

$$\mathbf{V}_h = \nabla S_h \oplus \mathbf{W}_h$$

Let Π_h the Nédélec's interpolation operator onto $\mathbf{V}_h$.
Let π_h be the standard nodal element interpolation operator onto S_h.

Commuting Diagram Property

Provided that p is smooth enough, it holds

$$\Pi_h(\nabla p) = \nabla(\pi_h p)$$

Proof of Discrete Helmholtz Decomposition

Discrete Helmholtz Decomposition

$$\mathbf{V}_h = \nabla S_h \oplus \mathbf{W}_h$$

- $\mathbf{v}_h \in \mathbf{V}_h^0 \subset \mathbf{V}^0 \Rightarrow \mathbf{v}_h = \nabla p$, with $p \in H_0^1(\Omega)$
- $\mathbf{v}_h|_K \in \mathbf{R}^\ell(K) \subset \mathcal{P}^\ell(K)^3 \Rightarrow p|_K \in \mathcal{P}^{\ell+1}(K)$
- $p|_K = p_1 + p_2$, with $p_1 \in \mathcal{P}^\ell$ and p_2 homog. polyn. of degree $\ell + 1$; we have to prove that $p_2 = 0$, from which $\mathbf{V}_h^0 = \nabla S_h$
- $\mathbf{x} \cdot \nabla p_2 = (\ell + 1) p_2$
  $\mathbf{v}_h|_K \in \mathbf{R}^\ell(K) \Rightarrow \mathbf{x} \cdot \nabla p_2 = 0$
  $\Rightarrow p_2 = 0$

Proof of (DCP) for $\mu = \varepsilon = I$

Discrete Compactness Property (DCP)

Any $\{\mathbf{v}_h\}_h \subset \mathbf{K}_h^{\perp}$ s.t. $\|\mathbf{v}_h\|_{\mathbf{V}} \leq 1$ for all h contains a subsequence (still denoted by $\{\mathbf{v}_h\}_h$) s.t.

$$\exists\, \mathbf{v} \in L^2(\Omega)^3 : \quad \lim_{h \to 0} \|\mathbf{v} - \mathbf{v}_h\|_{L^2_\varepsilon(\Omega)^3} = 0$$

- $\{h_n\}_n$ sequence, $h_n \to 0$ as $n \to \infty$, $\{\mathbf{v}_{h_n}\}_n \subset \mathbf{K}_h^{\perp}$
- Let $p^n \in H_0^1(\Omega)$ be s.t. $(\nabla p^n, \nabla \xi) = (\mathbf{v}_{h_n}, \nabla \xi)\ \forall\, \xi \in H_0^1(\Omega)$
- Set $\mathbf{v}^n = \mathbf{v}_{h_n} - \nabla p^n$; we have:

$$\nabla \times \mathbf{v}^n = \nabla \times \mathbf{v}_{h_n},\ \nabla \cdot \mathbf{v}^n = 0 \text{ in } \Omega, \quad \mathbf{n} \times \mathbf{v}^n = 0 \text{ on } \partial\Omega$$

$\Rightarrow \mathbf{v}^n \in \mathbf{W}$ and $\|\mathbf{v}^n\|_{\mathbf{V}} \leq C$

Proof of (DCP) for $\mu = \varepsilon = I$

- $\mathbf{W}$ comp. in $L^2(\Omega)^3$: up to subseq., $\mathbf{v}^n \to \mathbf{v} \in \mathbf{W}$ in $L^2(\Omega)^3$
- $\mathbf{v}^n$ is smooth enough for $\Pi_{h_n}\mathbf{v}^n$ to be defined; since $\Pi_{h_n}\mathbf{v}_{h_n} = \mathbf{v}_{h_n}$, also $\Pi_{h_n}(\nabla p^n)$ is well-defined and, by the commuting diagram property, $\Pi_{h_h}\mathbf{v}^n = \mathbf{v}_{h_n} - \nabla(\pi_{h_n} p^n)$
- Adding and subtracting $\Pi_{h_h}\mathbf{v}^n$, we have

$$\begin{aligned}
\|\mathbf{v} - \mathbf{v}_{h_n}\|^2_{L^2(\Omega)^3} &= (\mathbf{v} - \mathbf{v}_{h_n}, \mathbf{v} - \Pi_{h_h}\mathbf{v}^n) + (\mathbf{v} - \mathbf{v}_{h_n}, \Pi_{h_h}\mathbf{v}^n - \mathbf{v}_{h_n}) \\
&= (\mathbf{v} - \mathbf{v}_{h_n}, \mathbf{v} - \Pi_{h_h}\mathbf{v}^n) + (\mathbf{v} - \mathbf{v}_{h_n}, -\nabla(\pi_{h_n} p^n)) \; (\ldots + 0) \\
&\leq \|\mathbf{v} - \mathbf{v}_{h_n}\|_{L^2(\Omega)^3} \|\mathbf{v} - \Pi_{h_h}\mathbf{v}^n\|_{L^2(\Omega)^3}
\end{aligned}$$

thus,

$$\|\mathbf{v} - \mathbf{v}_{h_n}\|_{L^2(\Omega)^3} \leq \|\mathbf{v} - \Pi_{h_h}\mathbf{v}^n\|_{L^2(\Omega)^3}$$

Proof of (DCP) for $\mu = \varepsilon = I$

- By triangle inequality,

$$\|\mathbf{v}-\mathbf{v}_{h_n}\|_{L^2(\Omega)^3} \leq \|\mathbf{v}-\Pi_{h_h}\mathbf{v}^n\|_{L^2(\Omega)^3} \leq \|\mathbf{v}-\mathbf{v}^n\|_{L^2(\Omega)^3}+\|\mathbf{v}^n-\Pi_{h_h}\mathbf{v}^n\|_{L^2(\Omega)^3}$$

- By construction, $\|\mathbf{v}-\mathbf{v}^n\|_{L^2(\Omega)^3} \to 0$; by approximation properties and $\|\mathbf{v}^n\|_{\mathbf{V}} \leq C$, $\|\mathbf{v}^n - \Pi_{h_h}\mathbf{v}^n\|_{L^2(\Omega)^3} \to 0$ and we conclude

(DCP) actually holds true with strong limit in **W** (SDCP)

Proof of (DFI) for $\mu = \varepsilon = I$

Discrete Friedrichs Inequality (DFI)

$$\|\nabla \times \mathbf{v}_h\|^2_{L^2(\Omega)^3} = a(\mathbf{v}_h, \mathbf{v}_h) \geq \widetilde{\alpha} \|\mathbf{v}_h\|^2_{L^2(\Omega)^3} \qquad \forall \, \mathbf{v}_h \in \mathbf{K}_h^\perp$$

- $\|\nabla \times \mathbf{v}_h\|^2_{L^2(\Omega)^3} = 0 \Rightarrow \mathbf{v}_h = \nabla p_h$ with $p_h \in S_h$
  $\Rightarrow \mathbf{v}_h = 0 \Rightarrow \|\nabla \times \cdot \, \|_{L^2(\Omega)^3}$ is a norm in $\mathbf{K}_h^\perp$
- Norm equivalence in finite dimensional spaces $\Rightarrow$ inequality ok but with $\widetilde{\alpha} = \widetilde{\alpha}(h)$; we need to prove that $\widetilde{\alpha}$ is indep. of h
- By contradiction: there exists a sequence $\{\mathbf{v}_{h_n}\}_n \subset \mathbf{K}_h^\perp$ s.t. $\|\mathbf{v}_{h_n}\|_{L^2(\Omega)^3} = 1$ and $\|\nabla \times \mathbf{v}_{h_n}\|_{L^2(\Omega)^3} < 1/n$
  - (SDCP) $\Rightarrow$ up to subseq., $\mathbf{v}_{h_n} \to \mathbf{v} \in \mathbf{W}$ in $L^2(\Omega)^3$; obviously, $\|\mathbf{v}\|_{L^2(\Omega)^3} = 1$
  - $\|\nabla \times \mathbf{v}\|_{L^2(\Omega)^3} = 0$ implies $\mathbf{v} \in \mathbf{V}^0$, therefore $\mathbf{v} = 0$

(DCP) for Generic μ and ε

$$\mathbf{V} = \nabla H_0^1(\Omega) \oplus_1 \mathbf{W}^1 \qquad \mathbf{V} = \nabla H_0^1(\Omega) \oplus_\varepsilon \mathbf{W}^\varepsilon$$

$$\mathbf{V}_h = \nabla S_h \oplus_1 \mathbf{W}_h^1 \qquad \mathbf{V}_h = \nabla S_h \oplus_\varepsilon \mathbf{W}_h^\varepsilon$$

- Decompose $\mathbf{v}_{h_n}$ using $\mathbf{V}_h = \nabla S_h \oplus_1 \mathbf{W}_h^1$:

$$\mathbf{v}_{h_n} = \nabla p_n^1 + \mathbf{w}_n^1$$

- Apply (DCP) for $\mu = \varepsilon = I$ to $\{\mathbf{w}_n^1\}_n$ and call $\mathbf{v}$ the strong limit (subsequence still denoted by $\{\mathbf{w}_n^1\}_n$)
- Decompose $\mathbf{v}$ using $\mathbf{V} = \nabla H_0^1(\Omega) \oplus_\varepsilon \mathbf{W}^\varepsilon$:

$$\mathbf{v} = \nabla p^\varepsilon + \mathbf{w}^\varepsilon$$

- Prove that $\{\mathbf{v}_{h_n}\}_n$ converges to $\mathbf{w}^\varepsilon$ strongly in $L^2_\varepsilon(\Omega)^3$.

(DFI) for Generic μ and ε

$$\mathbf{V}_h = \nabla S_h \oplus_1 \mathbf{W}_h^1 \qquad \mathbf{V}_h = \nabla S_h \oplus_\varepsilon \mathbf{W}_h^\varepsilon$$

- Decompose $\mathbf{v}_h \in \mathbf{K}_h^\perp$ using $\mathbf{V}_h = \nabla S_h \oplus_1 \mathbf{W}_h^1$:

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h^1$$

  $(\nabla \times \mathbf{w}_h^1 = \nabla \times \mathbf{v}_h)$
- Since $\mathbf{v}_h$ is L^2_ε–orthogonal to ∇S_h:

$$\|\mathbf{v}_h\|^2_{L^2_\varepsilon(\Omega)^2} = (\varepsilon \mathbf{v}_h, \mathbf{v}_h - \nabla p_h) = (\varepsilon \mathbf{v}_h, \mathbf{w}_h^1) \le \|\mathbf{v}_h\|_{L^2_\varepsilon(\Omega)^3} \|\mathbf{w}_h^1\|_{L^2_\varepsilon(\Omega)^3}$$

- From the equivalence between $L^2_{\varepsilon,\mu}$ and L^2 norms and (DFI) for $\mu = \varepsilon = I$:

$$\|\mathbf{v}_h\|_{L^2_\varepsilon(\Omega)^2} \le C \|\mathbf{w}_h^1\|_{L^2(\Omega)^3} \le \frac{C}{\widetilde{\alpha}} \|\nabla \times \mathbf{w}_h^1\|_{L^2(\Omega)^3} = \frac{C}{\widetilde{\alpha}} \|\mu^{-1/2} \nabla \times \mathbf{v}_h\|_{L^2(\Omega)^3}$$

Nodal Elements

Simplicial mesh $\mathcal{T}_h$;

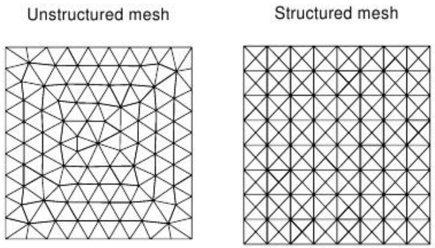
$$V_h = \{\mathbf{v}_h \in H^1(\Omega)^3 : \, v_h|_K \in \mathcal{P}^\ell(K)^3 \;\forall\, K \in \mathcal{T}_h, \; \mathbf{n} \times \mathbf{v}_h = \mathbf{0} \text{ on } \partial\Omega\}$$

- (CAS) is satisfied
- On general meshes, (DFI) is not satisfied
  $\Rightarrow$ pollution of the whole spectrum
- On particular meshes, (DFI) might be satisfied
  $\Rightarrow$ zero frequencies exactly individuated and the physical eigenvalues are correctly approximated
  on the other hand, there are sequences of discrete eigenvalues converging to values which are not in the spectrum
  $\rightarrow$ (DCP) is not satisfied

Nodal Elements

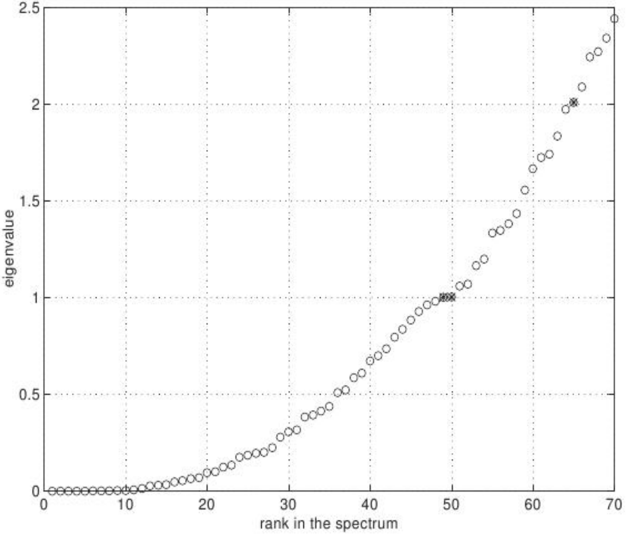
- $\Omega = (0, \pi) \times (0, \pi)$, $\mu = 1$, $\varepsilon = I$
- eigenvalues: $0 \neq \omega^2 = m^2 + n^2$, $m, n = 0, 1, 2, \dots$
  $\{1, 1, 2, 4, 4, 5, 5, 8, 9, 9, \dots\}$

Meshes



Spurious Eigenvalues

Unstructured Meshes



Spurious Eigenvalues

Structured Meshes