ZSS'08: Multigrid for Eigenproblems

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1 Setting

$$u \in H_0^1(\Omega)\colon \quad \underbrace{\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}\boldsymbol{x}}_{=:a(u,v)} = \lambda \underbrace{\int_\Omega uv \,\mathrm{d}\boldsymbol{x}}_{=:d(u,v)} \quad \forall v \in H_0^1(\Omega)\ . \tag{1.1}$$

$$u \in S\colon \quad \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}\boldsymbol{x} = \lambda \int_\Omega uv \,\mathrm{d}\boldsymbol{x} \quad \forall v \in S\ . \tag{1.2}$$

$$\vec{u}_h \in \mathbb{R}^{N,N}\colon \quad \mathbf{A}\vec{u}_h = \lambda \mathbf{D}\vec{u}_h\ . \tag{1.3}$$

☞ ZSS'08 lecture by S. Sauter

2 Multigrid iterative solvers

$$u_h \in S\colon \quad a(u_h, v_h) = l(v_h) \quad \forall v_h \in S \quad \Leftrightarrow \quad \mathbf{A}\vec{u}_h = \lambda \mathbf{D}\vec{u}_h\ . \tag{2.1}$$

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2.1 Successive subspace corrections (SSC)

$$S = \sum_{i=1}^{J} H_i \;, \quad \text{closed subspaces } H_i \subset S \;. \tag{2.2}$$

$$\begin{aligned} &\texttt{function}\quad u_h = \texttt{sscstep} \\ &\texttt{for}\quad i = 1:J \\ &\quad u_h \leftarrow \underset{v_h \in u_h + H_i}{\operatorname{argmin}}\; J(v_h) \\ &\texttt{end} \end{aligned} \tag{2.3}$$

$$u_h \leftarrow u_h + w_i \;, \quad w_i \in H_i\colon \quad a(w_i, v_i) = l(v_i) - a(u_h, v_i) \quad \forall v_i \in H_i \;. \tag{2.4}$$

$$\exists C_s > 0\colon \quad \inf\{\sum_{i=1}^{J} \|v_i\|_A^2 \,,\, v_i \in H_i,\; \sum_{i=1}^{J} v_i = v_h\} \leq C_s \|v_h\|_A^2 \quad \forall v_h \in S \;, \tag{STAB}$$

$$\exists 0 \leq \epsilon_{ij} \leq 1\colon \quad a(v_i, v_j) \leq \epsilon_{ij} \|v_i\|_A \|v_j\|_A \quad \forall v_i \in H_i,\; v_j \in H_j \;. \tag{SCSI}$$

☞ see [23, 24]

2.2 Multigrid SSC

$$S_0 \subset S_1 \subset \cdots \subset S_{L-1} \subset S = S_L \;. \tag{2.5}$$

$\mathbf{A}_l$, $\mathbf{D}_l$ $\hat{=}$ stiffness/mass matrix on level l

$$S = \sum_{l=0}^{L} (S_0 + \sum_{j=1}^{N_l} \mathrm{Span}\,\{b_j^l\}) \;. \tag{2.6}$$

$\mathbf{I}_l^{l-1} \in \mathbb{R}^{N_l, N_{l-1}}$ $\hat{=}$ prolongation matrix level $l-1 \rightarrow$ level l.

☞ see [14, Ch. 4] for implementation, [4, 6, 22, 24] for theory, [14, Ch. 5] for nested iteration

3 Rayleigh quotient multigrid (RQMG)

$$\rho(v) = \frac{a(v,v)}{d(v,v)} \quad \Leftrightarrow \quad \rho(\vec{v}_h) = \frac{\vec{v}_h^T \mathbf{A} \vec{v}_h}{\vec{v}_h^T \mathbf{D} \vec{v}_h} \tag{3.1}$$

3.1 SSC for elliptic eigenvalue problems

$$
\begin{array}{l}
\texttt{function} \quad \mathrm{V} = \texttt{evpsscstep}(V) \\
\texttt{for} \quad i = 1 : J \\
\quad V \leftarrow \underset{\substack{Y \subset V + H_i \\ \dim Y = p}}{\operatorname{argmin}} \quad \max_{y \in Y} \rho(y) \\
\texttt{end}
\end{array}
\tag{3.2}
$$

$$
(3.2) \quad \Leftrightarrow \quad \mathbf{A}_i \vec{v} = \lambda \mathbf{D}_i \vec{v} \; .
$$

☞ see [9] for discussion for elliptic EVP & theory of sorts

3.2 Implementation of RQMG

$$
\begin{pmatrix} a(v^l_{m-1}, v^l_{m-1}) & a(v^l_{m-1}, b^l_m) \\ a(v^l_{m-1}, b^l_m) & a(b^l_m, b^l_m) \end{pmatrix} \begin{pmatrix} \xi \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} d(v^l_{m-1}, v^l_{m-1}) & d(v^l_{m-1}, b^l_m) \\ d(v^l_{m-1}, b^l_m) & d(b^l_m, b^l_m) \end{pmatrix} \; . \tag{3.3}
$$

$$
v^l_m = \xi_m v^l_{m-1} + \beta_m b^l_m = \alpha_m \left(v_0 + \sum_{k=1}^{N_l} g_k b^l_k \right) \; . \tag{3.4}
$$

☞ see [2] for discussion of minimal resolution requirement on coarsest mesh, [19] for numerical experiments.

3.3 Linearized RQMG

$$
\xi^* = \underset{\xi \in \mathbb{R}}{\operatorname{argmin}} \, \rho(v_0 + \xi b) \tag{3.5}
$$

$$
\Rightarrow \qquad a(v_0 + \xi^* b, b) - \rho(v_0 + \xi^* b) d(v_0 + \xi^* b, b) = 0 \tag{3.6}
$$

☞ [14, Ch. 12], [2, 17] for variant with projections, [8, 13] for theory, [5, 11] for extensions

4 Multigrid and PINVIT

→ Part 3 of ZSS'08 lecture by D. Kressner

☞ [21]

5 Multigrid for Maxwell EVP

☞ [16] and [15, Ch. 6]

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