Evolution problems

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Overview

- With time marching methods, an optimal distribution of grid points over space and time is hard to realize.
- We apply an adaptive method to a simultaneously space-time variational formulation.
- While keeping discrete solutions on all time levels is prohibitive for time marching methods, thanks to the use of tensorized multi-level bases our method produces approximations simultaneously in space and time without penalty in complexity because of the additional time dimension.

Parabolic problems

Let $V \hookrightarrow H \simeq H' \hookrightarrow V'$, $I := (0,T)$. Consider parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,\cdot) + A(t)u(t,\cdot) &= g(t,\cdot) \quad \text{in } V' \quad (t \in I) \\ u(0,\cdot) &= u_0 \quad \text{in } H, \end{cases}$$

where for $t \in I$ a.e.,

$$|(A(t)(\eta))(\zeta)| \leq M_a \|\eta\|_V \|\zeta\|_V \quad (\eta, \zeta \in V) \quad (\textit{boundedness}),$$

$$(A(t)(\eta))(\eta) + \mu\|\eta\|_H^2 \geq \alpha\|\eta\|_V^2 \quad (\eta \in V) \qquad (\textit{Gårding inequality}).$$

We take $H = L_2(\Omega)$, and think of $A(t)$ differential or integrodifferential operator of order $2m \geq 0$, and $V = H^m(\Omega)$ ($H_0^m(\Omega)$).

Ex 1. $m = 1$, $(A(t)(\eta))(\zeta) = \int_\Omega \nabla\eta \cdot \nabla\zeta dx$ (heat eq).

Space-time variational formulation

Multiplying with smooth test functions that vanish at $t = T$, and integration over space and time, and int.-by-parts in time (initial condition becomes a natural b.c.),

$$(Bu)(v) := -\int_I \int_\Omega u(t,x)\tfrac{\partial v}{\partial t}(t,x)dxdt + \int_I (A(t)u(t,\cdot))(v(t,\cdot))dt$$
$$= \int_I \int_\Omega g(t,x)v(t,x)dxdt + \int_\Omega u_0(x)v(0,x)dx =: f(v)$$

With

$$\mathcal{X} := L_2(I;V) = \{u : I \to V : \int_I \|u(t,\cdot)\|_V^2 dt < \infty\} \simeq L_2(I) \otimes V,$$
$$\mathcal{Y} := L_2(I;V) \cap H^1_{0,\{T\}}(I;V')$$

one has $B \in \mathcal{B}(\mathcal{X}, \mathcal{Y}')$. Also
Thm 1. $B^{-1} \in \mathcal{B}(\mathcal{Y}', \mathcal{X})$.

Proof. (for $A(t) \equiv A = A'$ coercive, and $V \hookrightarrow L_2(\Omega)$ compact)

$\exists$ orthon. basis $\{\phi\}$ for $L_2(\Omega)$ of eigenfunctions of A with eigenv. $\lambda_\phi > 0$.

Writing $u(t,x) = \sum_\varphi u_\varphi(t)\varphi(x)$, $f(t,x) = \sum_\varphi f_\varphi(t)\varphi(x)$, $v(t,x) = \sum_\varphi v_\varphi(t)\varphi(x)$,

$$(Bu)(v) = f(v) \Leftrightarrow \int_I -u_\varphi \dot{v}_\varphi + \lambda_\varphi u_\varphi v_\varphi \, dt = \int_I f_\varphi v_\varphi \, dt \quad (\varphi \in \{\varphi\},\, v_\varphi \in L_2(I)).$$

Equipping V by $(A\cdot)(\cdot)^{\frac{1}{2}}$ and $H^1_{0,\{T\}}(I)$ by $|\cdot|_{H^1(I)}$,

$$\|u\|^2_{L_2(I;V)} = \sum_\varphi \lambda_\varphi \|u_\varphi\|^2_{L_2(I)}$$

$$\|v\|^2_{L_2(I;V)\cap H^1_{0,\{T\}}(I;V')} = \sum_\varphi \lambda_\varphi \|v_\varphi\|^2_{L_2(I)} + \lambda_\varphi^{-1} |v_\varphi|^2_{H^1(I)}$$

Now sufficient to prove that

with

$$(B_\lambda u)(v) := \int_I -u\dot{v} + \lambda uv \, dt,$$

$B_\lambda : L_2(I) \to (H^1_{0,\{T\}}(I), |||\cdot|||_\lambda)'$ is b.i., uniformly in $\lambda > 0$, where $|||\cdot|||_\lambda := \sqrt{\lambda^2 \|\cdot\|^2_{L_2(\Omega)} + |\cdot|^2_{H^1(I)}}$.

Invertibility of B_λ, and its unif. boundedness are readily shown.

Given $v \in H^1_{0,\{T\}}(I)$, define $u = -\dot{v} + \lambda v$. Then $\|u\|_{L_2(I)} \leq \sqrt{2}\, |||v|||_\lambda$, and

$$(B_\lambda u)(v) = \int_I \dot{v}(t)^2 - 2\lambda v(t)\dot{v}(t) + \lambda^2 v(t)^2 dt \geq |||v|||^2_\lambda \geq \tfrac{1}{2}\sqrt{2}\, |||v|||_\lambda \|u\|_{L_2(I)},$$

where we used $\int_I -2v(t)\dot{v}(t)dt = -\int_I \frac{d}{dt} v(t)^2 dt = -v(T)^2 + v(0)^2 = v(0)^2 \geq 0$.

This unif. inf-sup condition implies the unif. boundedness of $B_\lambda^{-1} : (H^1_{0,\{T\}}(I), |||\cdot|||_\lambda)' \to L_2(I)$. □

Shift of the smoothness indices in space

Thm 2. *Let* $W \hookrightarrow V \hookrightarrow U$, $A(t)' + \mu I : W \to U'$ *b.i, and* $A(\cdot)' \in C(\bar{I}, \mathcal{B}(W, U'))$. *Then with*

$$\mathcal{X} := L_2(I; U')$$
$$\mathcal{Y} := L_2(I; W) \cap H^1_{0,\{T\}}(I; U)$$

$B \in \mathcal{B}(\mathcal{X}, \mathcal{Y}')$ *is b.i.*

For $W = V = U$, one retrieves original spaces.

Typical application: $m = 1$, $V = H^1_0(\Omega)$, $W = H^2(\Omega) \cap H^1_0(\Omega)$, $U = L_2(\Omega) \simeq U'$, assuming H^2-regularity of the spatial operator.

Tensor product bases

Let $\Theta^{\mathcal{X}}$, $\Theta^{\mathcal{Y}}$, and $\Sigma^{\mathcal{X}}$, $\Sigma^{\mathcal{Y}}$ be collections of temporal or spatial functions such that, normalized in the corresponding norms,

$\Theta^{\mathcal{X}}$	is a Riesz basis for	$L_2(I)$,		
$\Theta^{\mathcal{Y}}$	"	$L_2(I)$	and for	$H^1_{0,\{T\}}(I)$,
$\Sigma^{\mathcal{X}}$	"	U',		
$\Sigma^{\mathcal{Y}}$	"	W	"	U.

Then, with $\mathcal{X} = L_2(I;U')$ and $\mathcal{Y} = L_2(I;W) \cap H^1_{0,\{T\}}(I;U)$, normalized in the corresponding norms,

$\Theta^{\mathcal{X}} \otimes \Sigma^{\mathcal{X}}$	is a Riesz basis for	$\mathcal{X}$,
$\Theta^{\mathcal{Y}} \otimes \Sigma^{\mathcal{Y}}$	"	$L_2(I;W)$, $H^1_{0,\{T\}}(I;U)$, and so for $\mathcal{Y}$,

Best possible rates

Let $U = L_2(\Omega) \simeq U'$, $\Omega \subset \mathbb{R}^n$, so $\mathcal{X} = L_2(I; L_2(\Omega))$

$$s_{\max} = \begin{cases} \min(d_t, \frac{d_x}{n}) & \text{isotropic spatial wavelets} \\ \min(d_t, d_x) & \text{anisotropic spatial wavelets} \end{cases}$$

up to log-factors when $d_t = \frac{d_x}{n}$ or $d_t = d_x$.

With non-adaptive approx, these rates require boundedness of certain mixed derivatives in L_2. Relaxed regularity conditions with best N-term approx.

Realization of rates of best N-term approximation: Application of awgm to

$$\mathbf{B}^*\mathbf{B}\mathbf{u} = \mathbf{B}^*\mathbf{f}.$$

Numerics heat equation

Heat equation

$$\begin{aligned} \tfrac{\partial}{\partial t}u - \Delta_{\mathbf{x}}u = g &\quad \text{on } (0,T) \times \square, \\ u = 0 &\quad \text{on } (0,T) \times \partial\square \\ u(0,\cdot) = u_0. \end{aligned}$$

$\square = (0,1)^n$. Temporal wavelets of order $d_t = 5$, tensor product spatial wavelets of order $d_x = 5$.

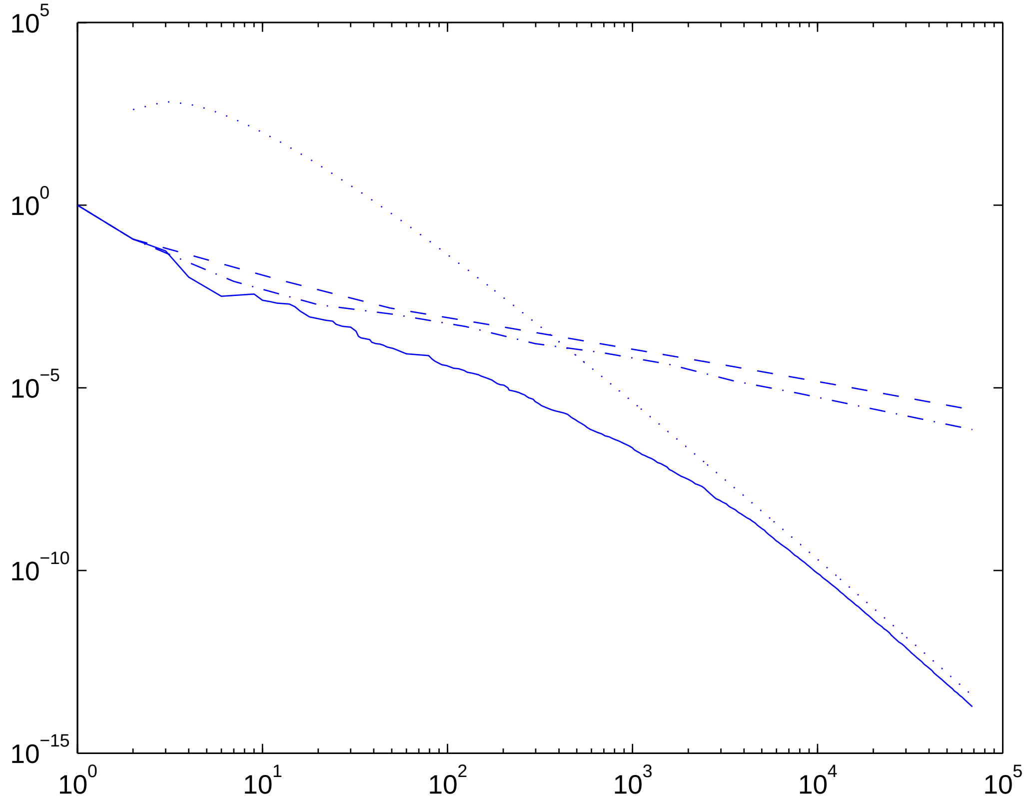

Figure 1: Heat eqn. in $n = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 0$. $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $N = \#\mathrm{supp}\, \mathbf{u}_\varepsilon$ for `awgm` (solid), full-grid (dashed) and sparse-grid method (dashed-dotted). The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$.

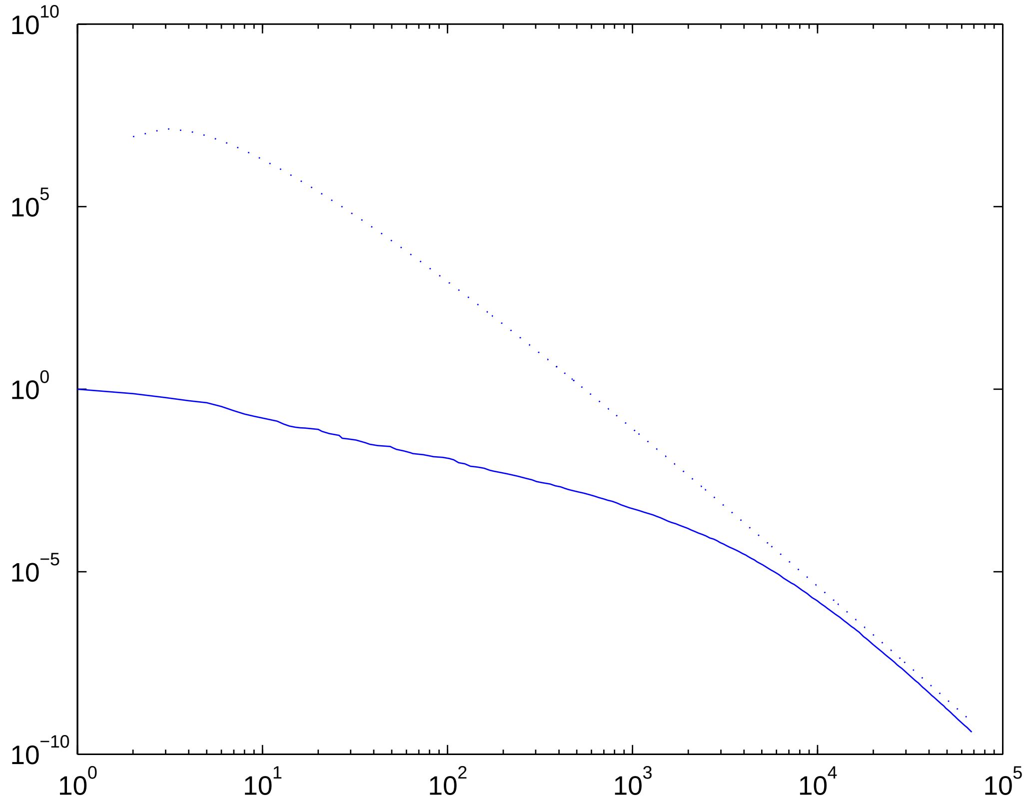

Figure 2: awgm applied to heat eqn. in $n = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 1$. $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_\varepsilon$. The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$.

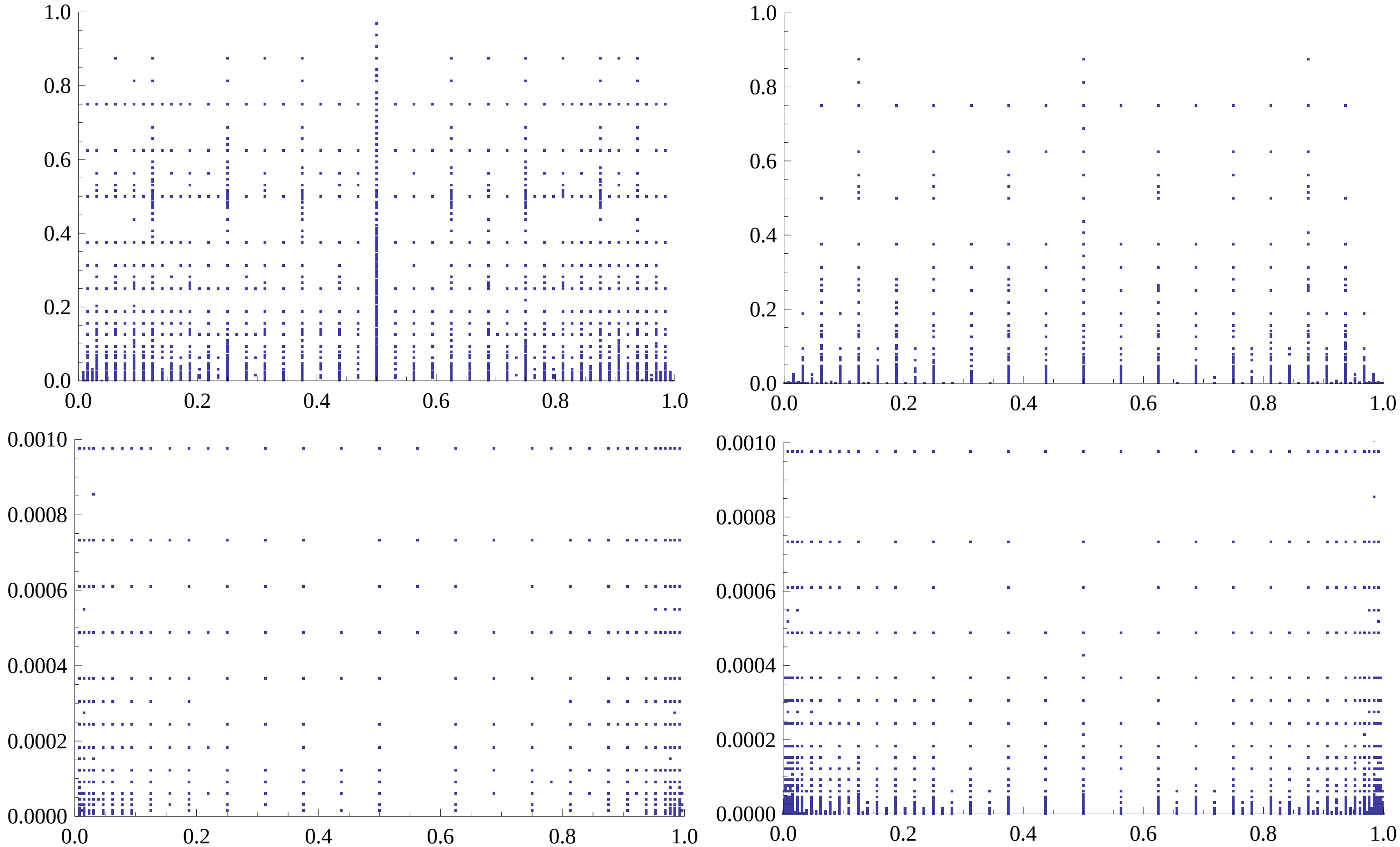

Figure 3: Heat eqn. in $n = 1$ spatial dimension and right-hand side $g = 1$. Centers of the supports of the wavelets selected by `awgm` . Left $u_0 = 0$ and $\#\mathbf{u}_\varepsilon = 13420$. Right $u_0 = 1$ and $\#\mathbf{u}_\varepsilon = 13917$. A zoom in near $t = 0$ is given at the bottom row.

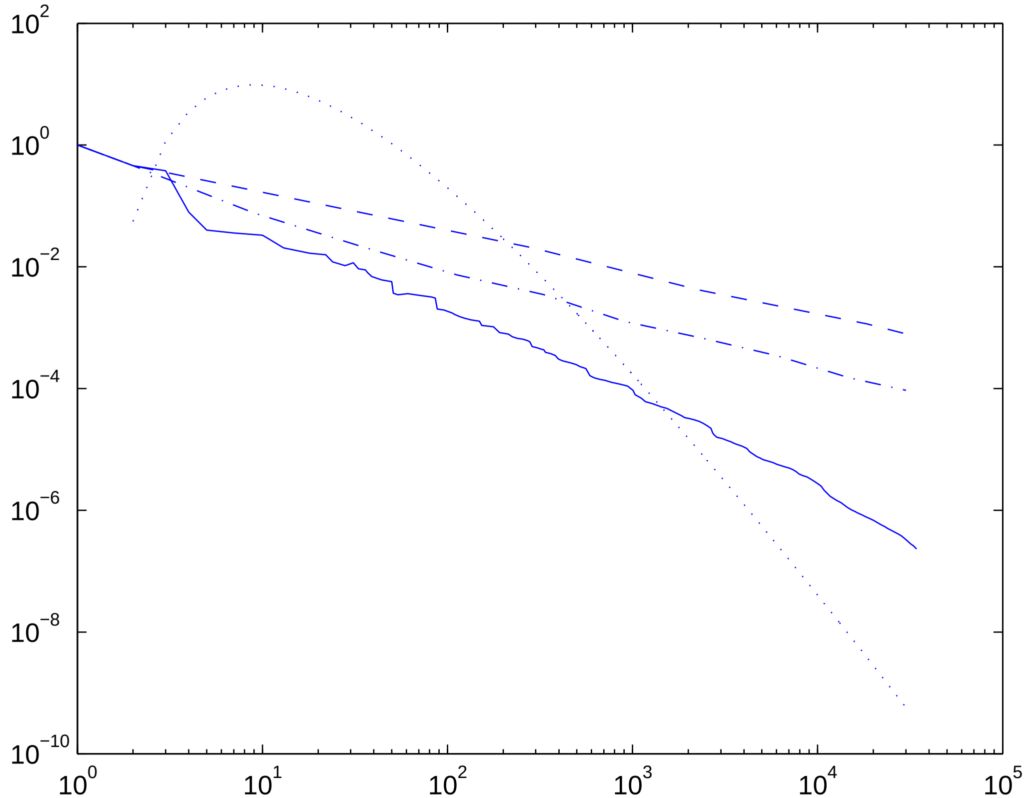

Figure 4: Heat eqn. in $n = 2$ spatial dimensions, right-hand side $g = 1$ and initial condition $u_0 = 0$. $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_\varepsilon$ for `awgm` (solid), full-grid (dashed) and sparse-grid method (dashed-dotted). The dotted line is a multiple of $N^{-5}(\log N)^{11}$.

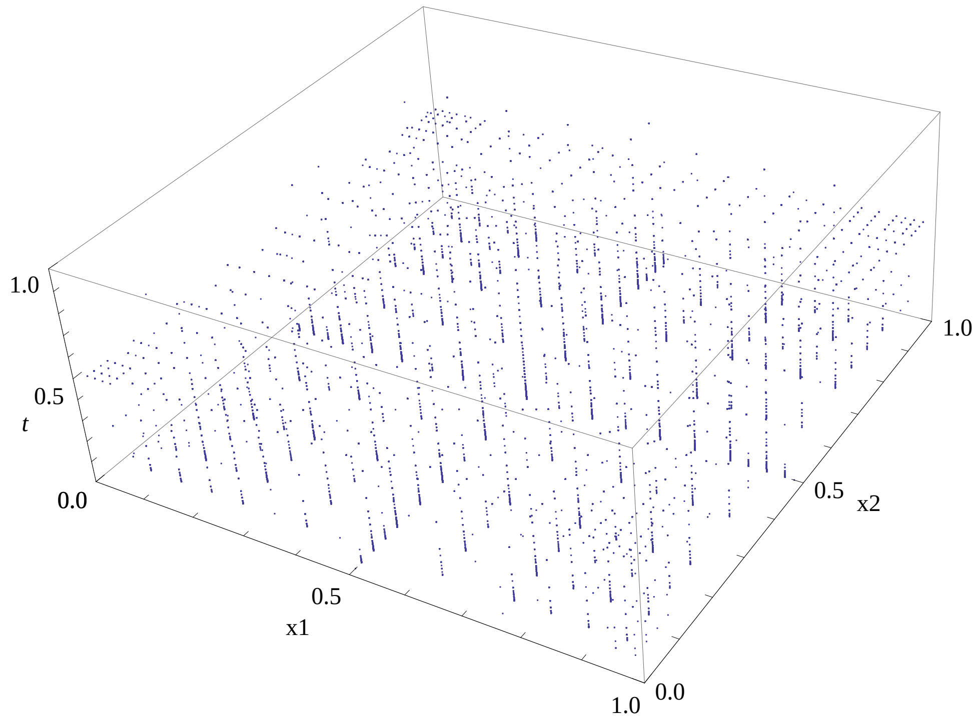

Figure 5: Heat eqn. in $n = 2$ spatial dimensions, right-hand side $g = 1$, and initial condition $u_0 = 0$. Centers of the supports of the wavelets selected by `awgm`. $\#\mathbf{u}_\varepsilon = 34316$.

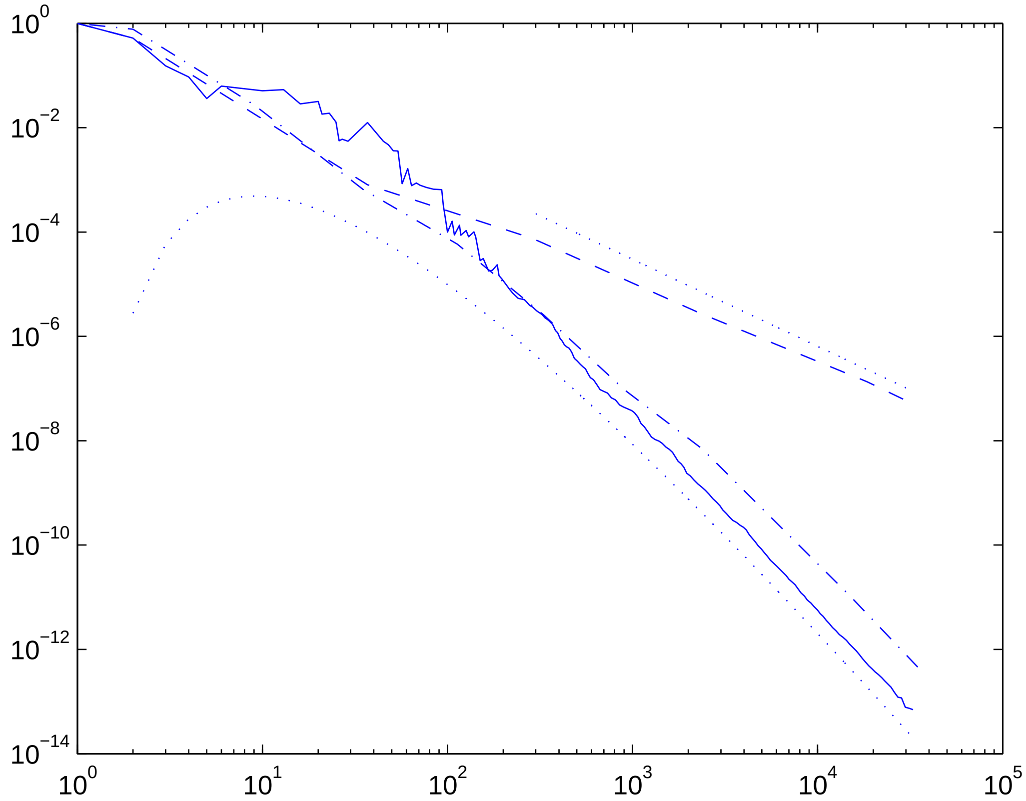

Figure 6: Heat eqn. in $n = 2$ spatial dimensions, right-hand side $g(t, \mathbf{x}) = t^4 x_1(x_1 - 1)(x_1^2 - x_1 - 1)x_2(x_2 - 1)(x_2^2 - x_2 - 1)$ and initial condition $u_0 = 0$. $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_\varepsilon$ for `awgm` (solid), full-grid (dashed) and sparse-grid method (dashed-dotted). The dotted lines are multiples of $N^{-5}(\log N)^{11}$ and $N^{-5/3}$, respectively.

Instationary Stokes equations

$$\begin{cases} \frac{\partial \vec{u}}{\partial t} - \nu\Delta\vec{u} - \nabla p &= \vec{g} & \text{on } I \times \Omega \\ \operatorname{div} \vec{u} &= 0 & \text{on } I \times \Omega \\ \vec{u} &= 0 & \text{on } I \times \partial\Omega \\ \vec{u}(0,\cdot) &= u_0 & \text{on } \Omega \end{cases}$$

Multiplying with smooth, *divergence-free* test functions v, integrating over $I \times \Omega$, int-by-parts over space and time, yields

$$(B\vec{u})(\vec{v}) := -\int_I \int_\Omega \vec{u} \cdot \tfrac{\partial \vec{v}}{\partial t} dxdt + \int_I \int_\Omega \nu \nabla \vec{u} : \nabla \vec{v}\, dxdt$$
$$= \int_I \int_\Omega \vec{g} \cdot \vec{v} dxdt + \int_\Omega \vec{u}_0 \cdot \vec{v}(0,\cdot)dx =: \vec{f}(\vec{v})$$

Set $H := H_0(\operatorname{div} 0; \Omega) = \{\vec{u} \in L_2(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ on } \Omega,\ \vec{u} \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$,

$V := H_0^1(\Omega)^n \cap H.$

$(A\vec{\eta})(\vec{\zeta}) = \int_\Omega \nu \nabla \vec{\eta} : \nabla \vec{\zeta}\, dx$ satisfies the boundedness assump. and the Gårding inequality.
So with $\mathcal{X} = L_2(I;V)$ and $\mathcal{Y} = L_2(I;V) \cap H^1_{0,\{T\}}(I;V')$, $B : \mathcal{X} \to \mathcal{Y}$ b.i.
Problem: The construction of a wavelet Riesz basis for V. Moreover, method won't yield p.

Space-time variational saddle point formulation

Drop divergence-free condition on the test functions, and impose divergence-free condition on $\vec{u}$ weakly.

$$(B(\vec{u},p))(\vec{v},q) := -\int_I \int_\Omega \vec{u} \cdot \tfrac{\partial \vec{v}}{\partial t}\, dx dt + \int_I \int_\Omega \nu \nabla \vec{u} : \nabla \vec{v}\, dx dt$$
$$+ \int_I \int_\Omega \vec{v} \cdot \nabla p\, dx dt - \int_I \int_\Omega \vec{u} \cdot \nabla q\, dx dt$$

Thm 3. *Let* $\Omega \subset \mathbb{R}$ *for* $n \in \{2,3\}$ *either* C^2, *or convex with a piecewise smooth bdr. Then with*

$$\mathcal{X} := L_2(I; L_2(\Omega)) \times \Big(L_2(I; H^1(\Omega)/\mathbb{R}) \cap H^1_{0,\{T\}}\big(I; (H^1(\Omega)/\mathbb{R})'\big)\Big)'$$
$$\mathcal{Y} := L_2(I; (H^1_0(\Omega) \cap H^2(\Omega))^n) \cap H^1_{0,\{T\}}(I; L_2(\Omega)^n) \times L_2(I; H^1(\Omega)/\mathbb{R})$$

$B : \mathcal{X} \to \mathcal{Y}'$ *is b.i.*

Can be extended to NSE.

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