Construction of wavelets

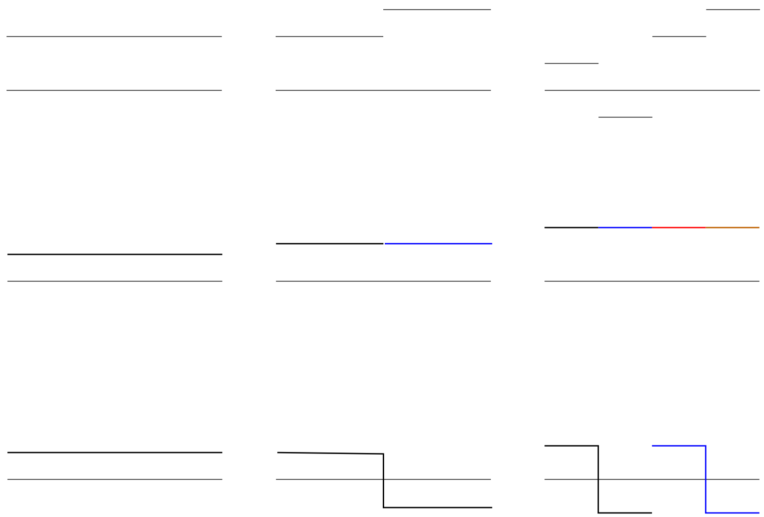
Rob Stevenson
Korteweg-de Vries Institute for Mathematics
University of Amsterdam

Contents

- Stability of biorthogonal wavelets.
- Examples on $\mathbb{R}$, $(0,1)$, and $(0,1)^n$.
- General domains via domain decomposition.
- Wavelets in finite element spaces.

Example: Haar wavelets (1910)

$\Omega = (0,1)$ (sim. $\Omega = I\!R$).



$V_0 \subset V_1 \subset \cdots$. $\overline{\cup_{j \in I\!N_0} V_j} = L_2(\Omega)$

Φ_0, Φ_1, Φ_2. $\Phi_j = \{\phi_{j,k} : k \in I_j\}$, $\phi_{j,k} = 2^{j/2}\phi(2^j \cdot -k)$ (shift & dyadic dilation), $\operatorname{diam}\operatorname{supp}\phi_{j,k} \eqsim 2^{-j}$

Φ_0, Ψ_1, Ψ_2. $\Psi_j = \{\psi_{j,k} : k \in J_j\}$ spans $V_j \cap \tilde{V}_{j-1}^{\perp}$, $\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)$ $\operatorname{diam}\operatorname{supp}\psi_{j,k} \eqsim 2^{-j}$. Vanishing moment

$\Psi := \Phi_0 \cup \cup_{j \in I\!N} \Psi_j = \{\psi_\lambda : \lambda \in \nabla\}$. $\lambda = (j,k)$, $|\lambda| = j$.
Orthonormal basis of $L_2(\Omega)$.

For $|s| < \frac{1}{2}$, $\{2^{-|\lambda|s}\psi_\lambda : \lambda \in \nabla\}$ Riesz basis for $H^s(\Omega)$.

For obtaining a Riesz basis: Orthogonality within and between levels can be relaxed.

Smoothness and so range of stability can be increased.

Riesz bases and biorthogonality

Recall, Ψ Riesz basis for H means

$$\mathcal{F}': \ell_2(\nabla) \to H : \mathbf{c} \mapsto \mathbf{c}^\top \Psi \quad \text{b.i.},$$
$$\mathcal{F}: H' \to \ell_2(\nabla) : g \mapsto g(\Psi) \quad \text{b.i.}.$$

From $\mathcal{F}'^{-1}$ bounded, $u \mapsto$ "c_λ" $\in H'$. Denote it as $\tilde{\psi}_\lambda$, i.e., $c_\lambda = \tilde{\psi}_\lambda(u)$. Then $\tilde{\Psi} := \{\tilde{\psi}_\lambda : \lambda \in \nabla\} \subset H'$ and $\tilde{\Psi}(\Psi) := [\tilde{\psi}_\lambda(\psi_\mu)]_{\lambda,\mu\in\nabla} = \mathrm{Id}$.

Set
$$\tilde{\mathcal{F}}': \ell_2(\nabla) \to H' : \mathbf{c} \mapsto \mathbf{c}^\top \tilde{\Psi}$$
$$\tilde{\mathcal{F}}: H \to \ell_2(\nabla) : u \mapsto \tilde{\Psi}(u)$$

Then $\tilde{\mathcal{F}}\mathcal{F}' = \mathrm{Id} = \mathcal{F}'\tilde{\mathcal{F}}$. So Ψ R.b. $H \iff \tilde{\Psi}$ R.b. H'.

[Via Riesz repr., alternatively $\tilde{\Psi}$ R.b. H and $\langle \tilde{\Psi}, \Psi \rangle_H = \mathrm{Id}$.]

Knowing biorthogonal Ψ and $\tilde{\Psi}$, b.i. of $\mathcal{F}'$ can be *proved* by showing boundedness of $\mathcal{F}'$ and $\tilde{\mathcal{F}}'$, i.e. boundedness of $\langle \Psi, \Psi \rangle_H$ and $\langle \tilde{\Psi}, \tilde{\Psi} \rangle_{H'}$, via "strengthened CS ineq.".

Stability biorthogonal wavelets

Thm 1. *Let*

$$V_0 \subset V_1 \subset \cdots \subset L_2(\Omega), \qquad \tilde{V}_0 \subset \tilde{V}_1 \subset \cdots \subset L_2(\Omega) \quad \text{(MRAs)} \ s.t.$$

$$\inf_{j \in \mathbb{N}_0} \inf_{0 \neq \tilde{v}_j \in \tilde{V}_j} \sup_{0 \neq v_j \in V_j} \frac{|\langle \tilde{v}_j, v_j \rangle_{L_2(\Omega)}|}{\|\tilde{v}_j\|_{L_2(\Omega)} \|v_j\|_{L_2(\Omega)}} > 0 \quad \textit{and sim. with } (V_j)_j \leftrightarrow (\tilde{V}_j)_j$$

For some $0 < \gamma < d$, *let*

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2(\Omega)} \lesssim 2^{-jd} \|v\|_{\mathcal{H}^d(\Omega)} \quad (v \in \mathcal{H}^d(\Omega)) \quad \text{(Jackson estimate)}, \ \textit{and}$$

$$\|v_j\|_{\mathcal{H}^s(\Omega)} \lesssim 2^{js} \|v_j\|_{L_2(\Omega)} \quad (v_j \in V_j,\ s \in [0, \gamma)) \quad \text{(Bernstein estimate)},$$

where for a Hilbert space $\mathcal{H}^d(\Omega) \hookrightarrow L_2(\Omega)$ *with dense embedding,*

$$\mathcal{H}^s(\Omega) := [L_2(\Omega), \mathcal{H}^d(\Omega)]_{s/d} \quad (s \in [0, d]).$$

Let similar estimates be valid at the dual side with $((V_j)_j, d, \gamma, \mathcal{H}^s(\Omega))$ *reading as* $((\tilde{V}_j)_j, \tilde{d}, \tilde{\gamma}, \tilde{\mathcal{H}}^s(\Omega))$.
Assume that for sufficiently small $s > 0$, $\mathcal{H}^s(\Omega) = \tilde{\mathcal{H}}^s(\Omega)$.

Then, with $\Phi_0 = \{\phi_{0,k} \colon k \in I_0\}$ *basis for* V_0 *(scaling functions) and* $\Psi_j = \{\psi_{j,k} \colon k \in J_j\}$ *uniform* $L_2(\Omega)$*-Riesz bases for* $W_j := V_j \cap \tilde{V}_{j-1}^{\perp_{L_2(\Omega)}}$ *(wavelets), for* $s \in (-\tilde{\gamma}, \gamma)$

$$\Phi_0 \cup \cup_{j \in \mathbb{N}} 2^{-sj} \Psi_j$$

is a Riesz basis for $\mathcal{H}^s(\Omega)$, *where* $\mathcal{H}^s(\Omega) := (\tilde{\mathcal{H}}^{-s}(\Omega))'$ *for* $s < 0$.

For the (unique) collections $\tilde{\Phi}_0 = \{\tilde{\phi}_{0,k} \colon k \in I_0\} \subset \tilde{V}_0$ *with* $\langle \Phi_0, \tilde{\Phi}_0 \rangle_{L_2(\Omega)} = \mathrm{Id}$, $\tilde{\Psi}_j = \{\tilde{\psi}_{j,k} \colon k \in J_j\} \subset \tilde{W}_j := \tilde{V}_j \cap V_{j-1}^{\perp_{L_2(\Omega)}}$ *with* $\langle \Psi_j, \tilde{\Psi}_j \rangle_{L_2(\Omega)} = \mathrm{Id}$, *and* $s \in (-\gamma, \tilde{\gamma})$,

$$\tilde{\Phi}_0 \cup \cup_{j \in \mathbb{N}} 2^{-sj} \tilde{\Psi}_j$$

is a Riesz basis for $\tilde{\mathcal{H}}^s(\Omega)$, *where* $\tilde{\mathcal{H}}^s(\Omega) := (\mathcal{H}^{-s}(\Omega))'$ *for* $s < 0$.

inf-inf-sup conditions eq. to

- $\exists$ unif. bounded $Q_j : L_2(\Omega) \to L_2(\Omega)$ with $\Im Q_j = V_j$, $\Im(\mathrm{Id} - Q_j) = \tilde{V}_j^{\perp}$ (biorthogonal projector), or
- $\exists$ unif. $L_2(\Omega)$-Riesz bases Φ_j, $\tilde{\Phi}_j$ for V_j, $\tilde{V}_j$ s.t. $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-1}$ is uniformly bounded.

Two ways of appl. Thm.:

1): You ***have*** unif. $L_2(\Omega)$-Riesz coll. Ψ_j, $\tilde{\Psi}_j$, Φ_j, $\tilde{\Phi}_j$ s.t. $\Psi_j \perp \tilde{\Psi}_k$ when $j \neq k$, $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)} = \mathrm{Id}$, $\langle \Psi_j, \tilde{\Psi}_j \rangle_{L_2(\Omega)} = \mathrm{Id}$, and
$(V_j :=) \operatorname{span} \Phi_j = \operatorname{span} \Phi_0 \cup \Psi_1 \cup \ldots \cup \Psi_j$,
$(\tilde{V}_j :=) \operatorname{span} \tilde{\Phi}_j = \operatorname{span} \tilde{\Phi}_0 \cup \tilde{\Psi}_1 \cup \ldots \cup \tilde{\Psi}_j$.
Standard setting: Shift & dyadic dilation case on $\mathbb{R}$ with 1 (or a few) generating primal and dual "mother" scaling function(s) and wavelet(s).
Then inf-inf-sup is valid, and Ψ_j, $\tilde{\Psi}_j$ unif. $L_2(\Omega)$ Riesz for $V_j \cap \tilde{V}_{j-1}^{\perp_{L_2(\Omega)}}$, $\tilde{V}_j \cap V_{j-1}^{\perp_{L_2(\Omega)}}$. To **verify** J & B estimates.

2): You ***specify*** MRA's $(V_j)_j$, $(\tilde{V}_j)_j$ that satisfy J & B. To **verify** inf-inf-sup and to **construct** unif. $L_2(\Omega)$-Riesz bases Ψ_j for $W_j = V_j \cap \tilde{V}_{j-1}^{\perp_{L_2(\Omega)}}$, preferably **local** ones.
For ***some*** applications, needed that resulting bases $\tilde{\Psi}_j$ for $\tilde{V}_j \cap V_{j-1}^{\perp_{L_2(\Omega)}}$ are also local. Not needed for `awgm`.

Proof of Thm. (sketch)

$W_j = \Im(Q_j - Q_{j-1})$ $(Q_{-1} := 0)$, $\tilde{W}_j = \Im(Q_j^* - Q_{j-1}^*)$.

$\ell_{2,s}(Q) := \{(w_j)_j : w_j \in W_j : \|(w_j)_j\|_{\ell_{2,s}(Q)} := \sqrt{\sum_j 4^{sj}\|w_j\|^2_{L_2(\Omega)}} < \infty\}$,
$\ell_{2,s}(Q^*) :=\ldots$

To prove

$$G' : \ell_{2,s}(Q) \to \mathcal{H}^s : (w_j)_j \mapsto \sum_j w_j \quad \text{b.i.} \quad (s \in (-\tilde{\gamma}, \gamma))$$

$$\tilde{G}' : \ell_{2,s}(Q^*) \to \tilde{\mathcal{H}}^s : (\tilde{w}_j)_j \mapsto \sum_j \tilde{w}_j \quad \text{b.i.} \quad (s \in (-\gamma, \tilde{\gamma}))$$

Boundedness of G' from

$$|\langle w_j, w_k\rangle_{\mathcal{H}^s}| \lesssim 2^{-\varepsilon|j-k|}(2^{sj}\|w_j\|_{L_2(\Omega)})(2^{sk}\|w_k\|_{L_2(\Omega)})$$

for ε s.t. $s \pm \varepsilon \in (-\tilde{\gamma}, \gamma)$. Sim. for $\tilde{G}'$.
[$j \leq k$: $|\langle w_j, w_k\rangle_{\mathcal{H}^s}| \leq \|w_j\|_{\mathcal{H}^{s+\varepsilon}}\|w_k\|_{\mathcal{H}^{s-\varepsilon}}$ (for $s-\varepsilon < 0 < s+\varepsilon$ use $\mathcal{H}^t = \tilde{\mathcal{H}}^t$ for $t > 0$ small). For $t = s \pm \varepsilon > 0$, use B at primal side. Otherwise, $\|w_i\|_{\mathcal{H}^t} = \sup_v \frac{\langle w_i, (Q_i^* - Q_{i-1}^*)v\rangle}{\|v\|_{\tilde{\mathcal{H}}^{-t}}}$ and J at dual side.]

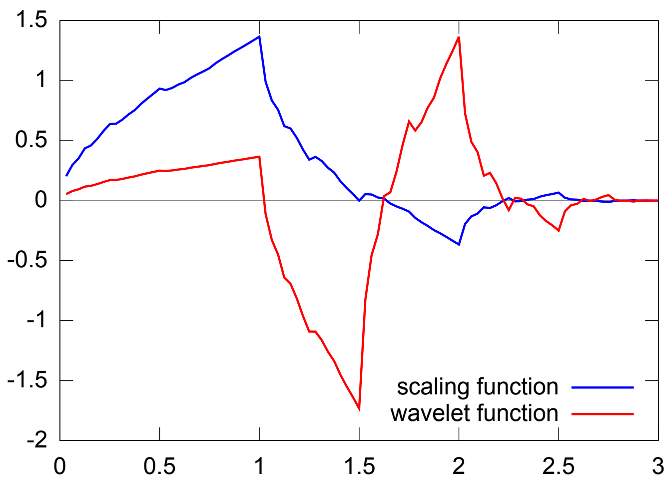
Since $(\mathcal{H}^s)' = \tilde{\mathcal{H}}^{-s}$ and $\ell_{2,s}(Q)' = \ell_{2,-s}(Q^*)$, boundedness of G' or $\tilde{G}'$ gives that of

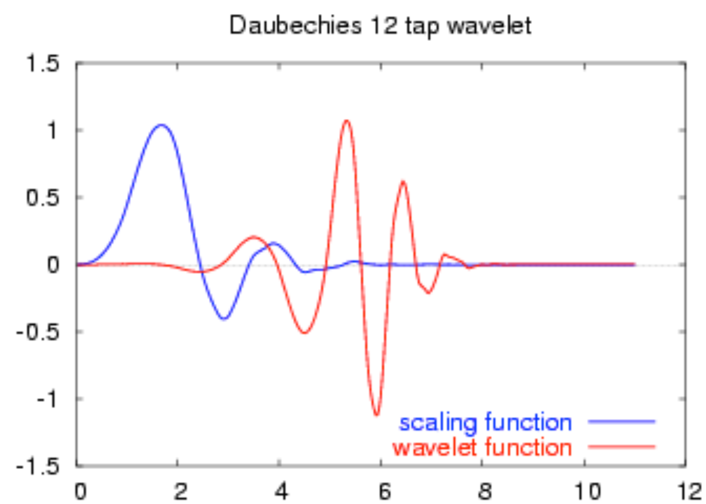
$$G : \tilde{\mathcal{H}}^{-s} \to \ell_{2,-s}(Q^*) : u \mapsto ((Q_j^* - Q_{j-1}^*)u)_j \quad (s \in (-\tilde{\gamma}, \gamma)),$$
$$\tilde{G} : \mathcal{H}^{-s} \to \ell_{2,-s}(Q) : u \mapsto ((Q_j - Q_{j-1})u)_j \quad (s \in (-\gamma, \tilde{\gamma})).$$

Now use $\tilde{G}'G = \mathrm{Id} = G\tilde{G}'$ and $G'\tilde{G} = \mathrm{Id} = \tilde{G}G'$ to conclude boundedness of $(G')^{-1}$ and $(\tilde{G}')^{-1}$.

Examples in shift & dyadic dilation setting

i.e. application of first type. Orthogonal wavelets from the **Daubechies family** of orders $d(=\tilde{d}) = 2$ and 6. $d = 1$ is Haar. Smoothness, and so γ, increases with d. Riesz for $|s| < \gamma$.





Functions implicitly defined as solution of a refinement equation.

Cohen-Daubecies-Feauveau family

Primals are B-splines. Duals are implicitly defined as solution of a ref. eq.

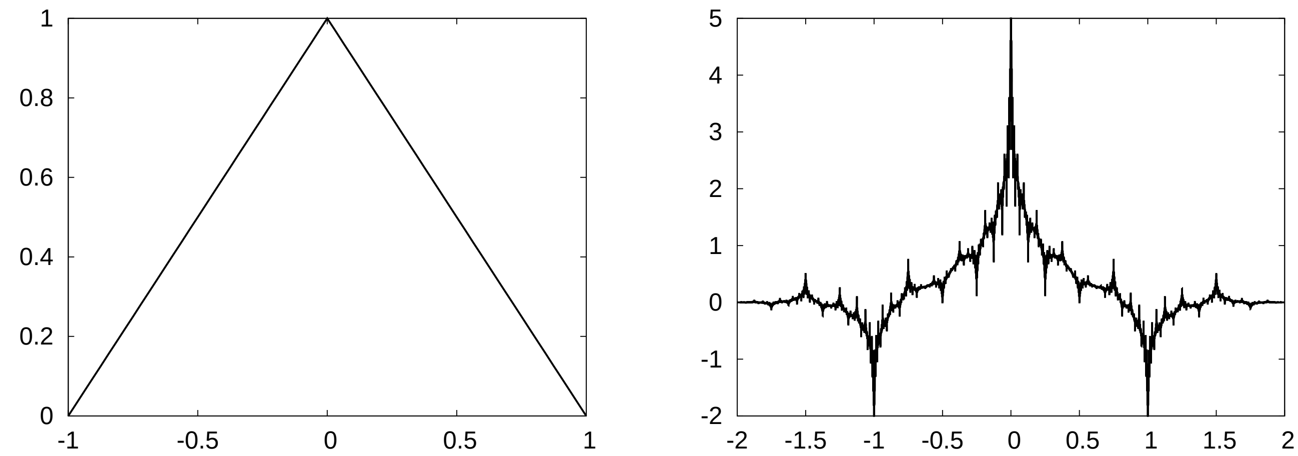


Figure 1: Primal and dual scaling function of orders $d = \tilde{d} = 2$

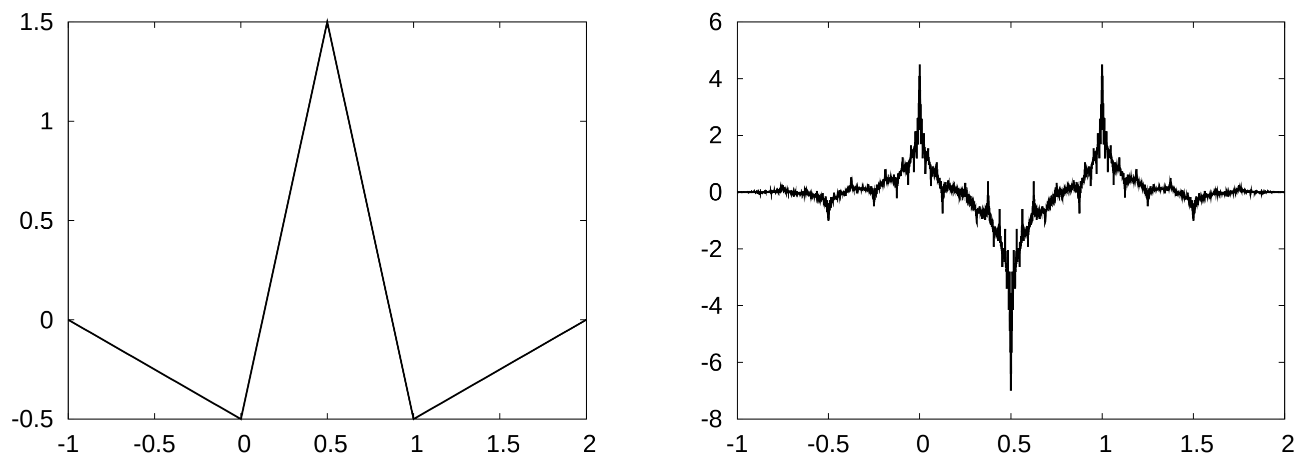


Figure 2: Corresponding primal and dual wavelet

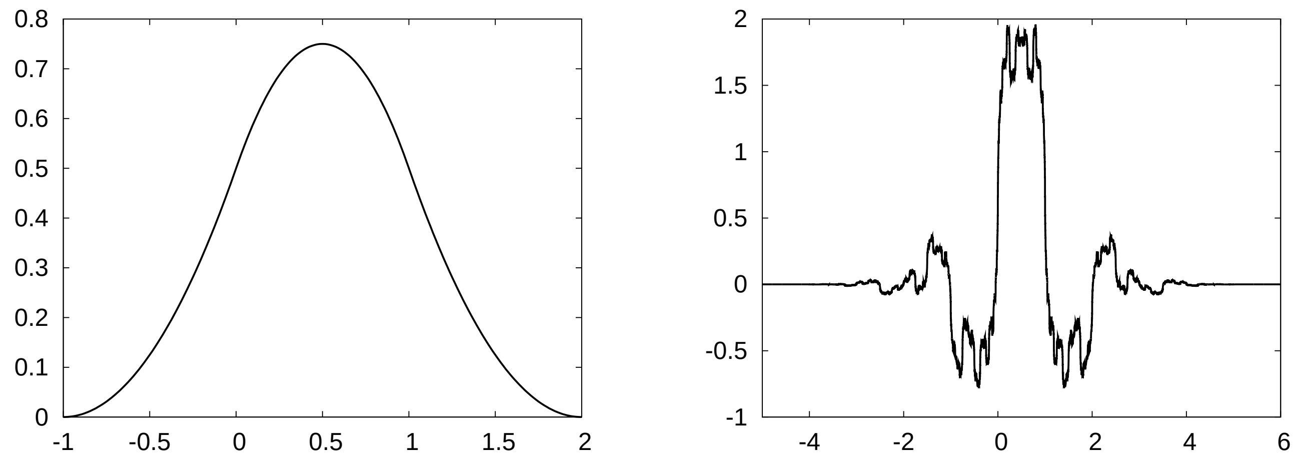


Figure 3: Primal and dual scaling function of orders $d = 3$, $\tilde{d} = 5$

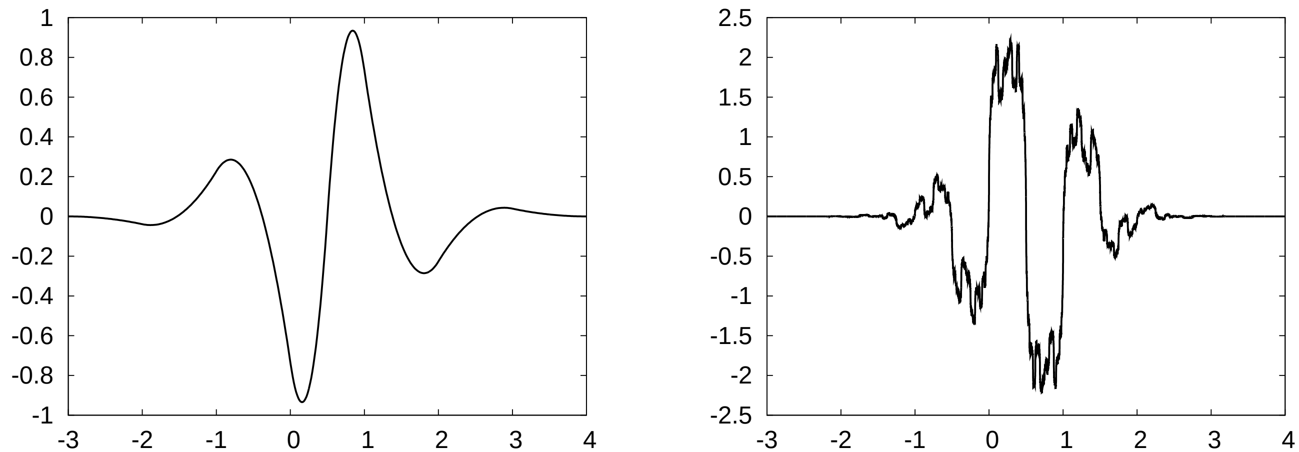


Figure 4: Corresponding primal and dual wavelet

$\gamma = d - \frac{1}{2}$, and $\tilde{\gamma}$ increases linearly with $\tilde{d} \geq d$, $d + \tilde{d}$ even.
Primals Riesz for $s \in (-\tilde{\gamma}, \gamma)$, duals for $s \in (-\gamma, \tilde{\gamma})$.

Donovan-Geronimo-Hardin piecewise *orthogonal* multi-wavelets

Piecewise polynomial *orthogonal* scaling functions (and) wavelets. Multiple generators. Here continuous ones (i.e. $\gamma = \frac{3}{2}$) of order $d = 2$. Riesz for $|s| < \gamma$

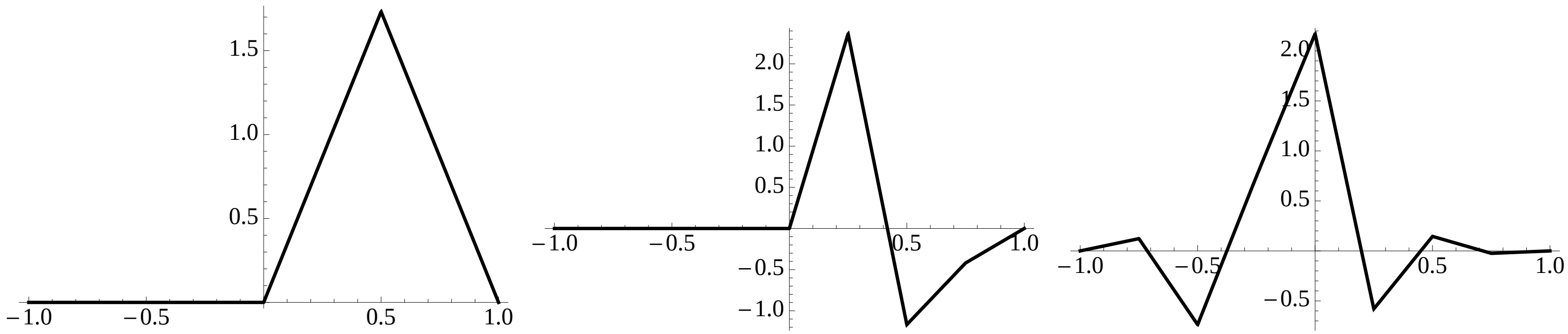


Figure 5: Scaling functions

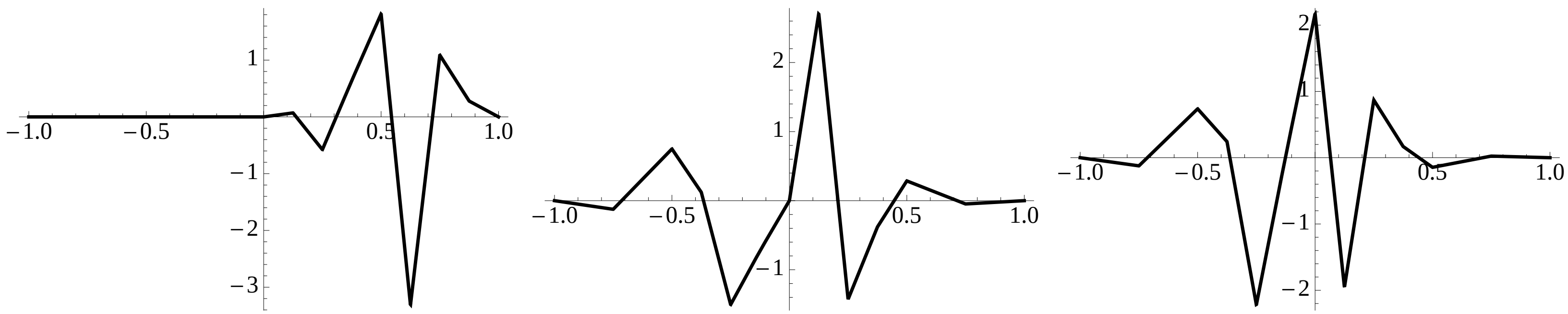


Figure 6: Corresponding wavelets

(Figures CDF and DGH wavelets by Sebastian Kestler, Uni Ulm)

Towards more interesting domains for num. appl.

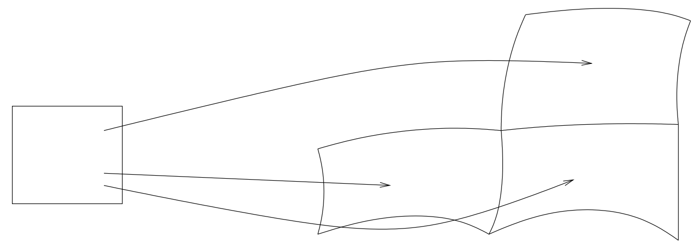
Step 1: Adaptation to $(0,1)$: Several constructions that yield on each level a few boundary adapted scaling functions and wavelets. Needed to maintain simultaneously biorthogonality and polynomial reproduction of orders d and $\tilde{d}$.
Boundary conditions can be incorporated at primal and/or dual side, so $\mathcal{H}^d$ can read as $H^d(0,1)$, $H^d_0(0,1)$ or, say $H^d(0,1) \cap H^1_0(0,1)$. Sim. for $\tilde{\mathcal{H}}^d$.
Recent constructions yield condition numbers on $(0,1)$ that are only slightly worse than those on $\mathbb{R}$.

Step 2: Wavelets on $(0,1)^n$.

Let $(V_j)_j, (\tilde{V}_j)_j$ MRA's on $(0,1)$, orders d, $\tilde{d}$, regularity γ, $\tilde{\gamma}$. $W_j = V_j \cap \tilde{V}^{\perp}_{j-1}$, $\tilde{W}_j = \tilde{V}_j \cap V^{\perp}_{j-1}$.

Then $(V_j \otimes V_j)_j, (\tilde{V}_j \otimes \tilde{V}_j)_j$ MRA's on $(0,1)$, orders d, $\tilde{d}$, regularity γ, $\tilde{\gamma}$.
$V_j \otimes V_j \cap (\tilde{V}_{j-1} \otimes \tilde{V}_{j-1})^{\perp} = W_j \otimes V_{j-1} \oplus V_{j-1} \otimes W_j \oplus W_j \otimes W_j$, so primal wavs $\psi_{j,k} \otimes \phi_{j-1,k'}$, $\phi_{j-1,k'} \otimes \psi_{j,k}$, $\phi_{j-1,k} \otimes \phi_{j-1,k'}$.
Sim. at dual side.

Step 3: DD



First approach: Composite wavelets. Connect primal and dual MRAs by identifying DOFs on the interfaces. Construct bases (wavelets) for the biorthogonal complements. New types of wavelets near the interfaces.

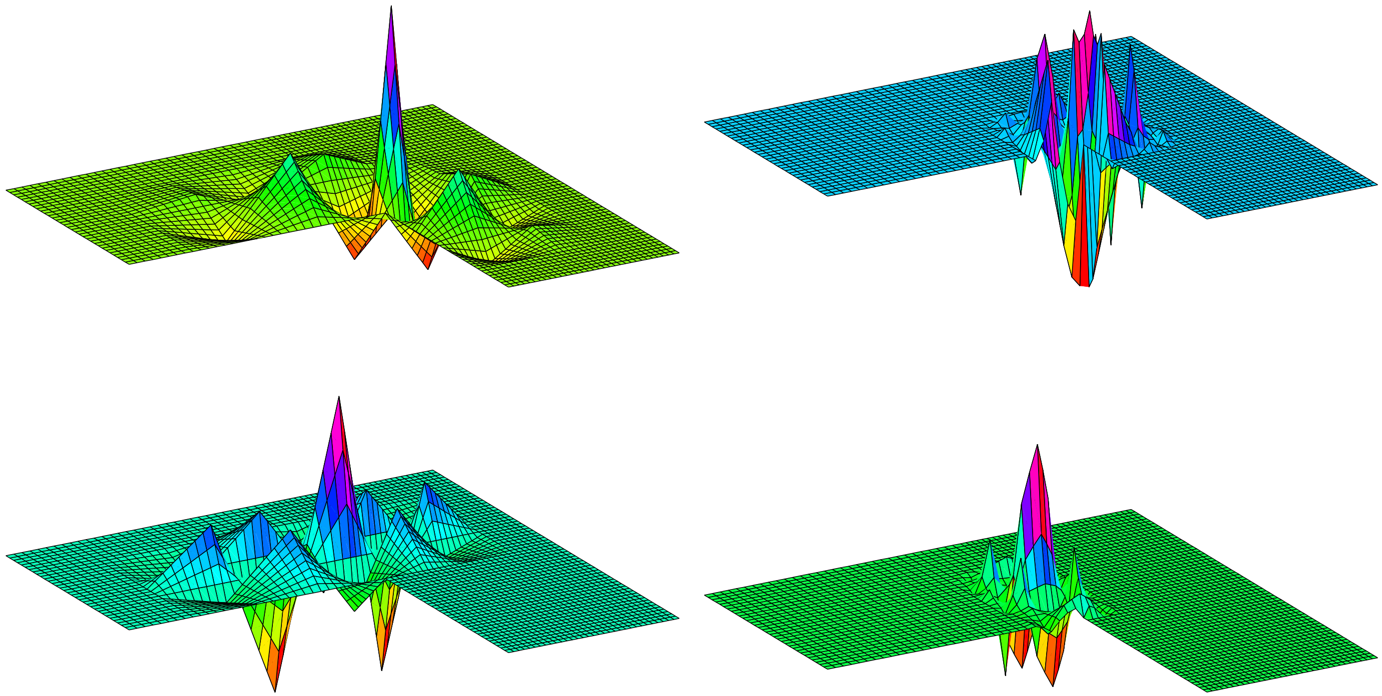


Figure 7: Primal (left) and corresponding dual (right) wavelets built from CDF $(2, 2)$.

Second approach (DD): Using extension operators.

Let $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. Let

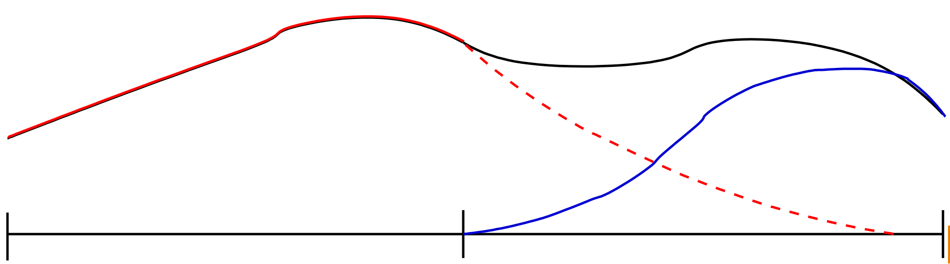
R_i	restriction	of functions on Ω	to Ω_i
E_i	extension	of functions on Ω_i	to Ω
η_i	extension by zero	of functions on Ω_i	to Ω

From $R_i E_i = \mathrm{Id}$ $(= R_i \eta_i)$, and $R_i \eta_j = 0$,

$$\begin{bmatrix} R_1 \\ R_2(I - E_1 R_1) \end{bmatrix} : L_0(\Omega) \to L_0(\Omega_1) \times L_0(\Omega_2)$$

$$[E_1 \quad \eta_2] : L_0(\Omega_1) \times L_0(\Omega_2) \to L_0(\Omega)$$

are each others inverse.



Note if **black** is C^k, and E is C^k, then **blue** extended with zero is C^k.

For $m \in \mathbb{N}$, if $E_1 \in B(H^m(\Omega_1), H^m(\Omega))$, then

$$[E_1 \quad \eta_2] \in \mathcal{B}(H^m(\Omega_1) \times H^m_{0,\partial\Omega_1\cap\partial\Omega_2}(\Omega_2), H^m(\Omega))$$

$$\begin{bmatrix} R_1 \\ R_2(I - E_1 R_1) \end{bmatrix} \in \mathcal{B}(H^m(\Omega), H^m(\Omega_1) \times H^m_{0,\partial\Omega_1\cap\partial\Omega_2}(\Omega_2)).$$

So for Ψ_1 and Ψ_2 Riesz bases of $H^m(\Omega_1)$ and $H^m_{0,\partial\Omega_1\cap\partial\Omega_2}(\Omega_2)$,

$$E_1\Psi_1 \cup \eta_2\Psi_2 \quad \text{is Riesz basis for } H^m(\Omega).$$

Construction can be applied recursively in the multi-subdomain case.

If Ω_1, Ω_2 are hypercubes that share a face, E_1 can be univariate extension. Hestenes extensions can be applied, e.g., reflection. Latter suitable for construction of Riesz bases for $H^s(\Omega)$ for $s \in (-\frac{1}{2}, \frac{3}{2})$.

Only basis functions that don't vanish at interface have to be extended.

Kind of basis functions irrelevant.

Construction of a basis for the biorthogonal complement

(second appl. of Thm.) Let $(V_j)_j$, $(\tilde{V}_j)_j$ MRAs that satisfy J & B and inf-inf-sup. [(e.g., $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-1}$ unif. bounded for unif. $L_2(\Omega)$-Riesz bases Φ_j, $\tilde{\Phi}_j$ for V_j, $\tilde{V}_j$).]

Thm 2. *Let* $\begin{Bmatrix} \tilde{\Phi}_{j-1} \\ \Theta_j \cup \Xi_j \end{Bmatrix}$*, prop. sc., be unif.* $L_2(\Omega)$*-Rsz bases for* $\begin{Bmatrix} \tilde{V}_{j-1} \\ V_j \end{Bmatrix}$*, with* $\langle \Theta_j, \tilde{\Phi}_{j-1} \rangle_{L_2(\Omega)}$ *an invertible diagonal matrix. Then, properly scaled,*

$$\Psi_j := \Xi_j - \langle \Xi_j, \tilde{\Phi}_{j-1} \rangle_{L_2(\Omega)} \langle \Theta_j, \tilde{\Phi}_{j-1} \rangle_{L_2(\Omega)}^{-1} \Theta_j$$

is unif. $L_2(\Omega)$*-Riesz basis for* $V_j \cap \tilde{V}_{j-1}^{\perp_{L_2(\Omega)}}$. [*Indeed,* $\langle \Psi_j, \tilde{\Phi}_{j-1} \rangle_{L_2(\Omega)} = 0$]

Note Ψ_j unif. local when Ξ_j, Θ_j, $\tilde{\Phi}_{j-1}$ unif. local. ▮

[If additionally, $\operatorname{span} \Theta_j = V_{j-1}$, i.e. $\Theta_j =$ "Φ_{j-1}" (so "classical" wavelet setting) ("severe"), and both transformations $\Phi_j \leftrightarrow \Phi_{j-1} \cup \Xi_j$ unif. local ("mild"), then resulting $\tilde{\Psi}_j$ also unif. local.]

Application: (Pre)wavelets on polytopes

Let $(V_j)_j$ seq. of cont. piecewise f.e. spaces on Ω corr. to dyadic refs. ("red-refinements") of initial conforming subdivision $\mathcal{T}_0$ into n-simplices. Then J & B with $d = 2$, $\gamma = \frac{3}{2}$.

Let $\Phi_{j-1}(= \tilde{\Phi}_{j-1}) = \{\phi_{j-1,x} : x \in I_{j-1}\}$ nodal basis V_{j-1} (bdr. vertices may or may not be included depending on bdr. conds.), being prop. sc. a unif. $L_2(\Omega)$-Riesz basis.

Set $\Xi_j = \{\phi_{j,x} : x \in I_j \backslash I_{j-1}\}$ and $\Theta_j = \{\phi_{j,x} - 2^{-(n+1)}\phi_{j-1,x} : x \in I_{j-1}\}$. Then, $\Theta_j \cup \Xi_j$ is, properly scaled, unif. $L_2(\Omega)$-Riesz bases for V_j, and $\langle \Theta_j, \Phi_{j-1}\rangle_{L_2(\Omega)}$ is an invertible diagonal matrix.

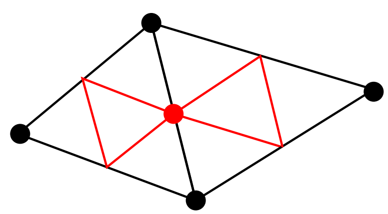


Figure 8: Nonzero coefficients $\langle \Xi_j, \Phi_{j-1}\rangle_{L_2(\Omega)}$.

Riesz for $|s| < \frac{3}{2}$.

[Orthogonality between levels, but not inside levels.]

Generalizations possible. E.g. take $\tilde{V}_j$ continuous piecewise *quadratics* w.r.t $\mathcal{T}_j$, V_j continuous piecewise linears w.r.t. $\mathcal{T}_{j+1}$. inf-inf-sup satisfied. J & B with $d = 2$, $\gamma = \frac{3}{2}$, $\tilde{d} = 3$, $\tilde{\gamma} = \frac{3}{2}$.

(Primal) wavelets Riesz for $|s| < \frac{3}{2}$. *Three* vanishing moments.

Three-point hierarchical basis

For $(\tilde{V}_j)_j = (V_j)_j$, recall construction unif. $L_2(\Omega)$-stable basis for *orth.* complement $V_j \cap V_{j-1}^{\perp_{L_2(\Omega)}}$. (wavconstr)

This two-level result equally applies with $\langle\cdot,\cdot\rangle_{L_2(\Omega)}$ reading as *some* scalar product $\langle\cdot,\cdot\rangle_j$ on V_j.

Again, let $(V_j)_j$ seq. of cont. piecewise f.e. spaces on Ω corr. to dyadic ref. of initial conforming subd. $\mathcal{T}_0$ into n-simplices.

Take $\langle u,v\rangle_j = \sum_{x\in I_j} w_{j,x} u(x)v(x)$, where $w_{j,x} = \sum_{\{T\in\mathcal{T}_j : T\ni x\}} \frac{\mathrm{vol}(T)}{n+1}$.

Take $\Xi_j = \{\phi_{j,x} : x \in I_j \setminus I_{j-1}\}$, $\Theta_j = \{\phi_{j,x} : x \in I_{j-1}\}$. Then $\Theta_j \cup \Xi_j$ is, properly scaled, unif. $\|\cdot\|_j$ Riesz basis for V_j, and $\langle\Theta_j, \Phi_{j-1}\rangle_j$ is invertible diagonal matrix. So

$$\Psi_j := \Xi_j - \langle\Xi_j, \Phi_{j-1}\rangle_j \langle\Theta_j, \Phi_{j-1}\rangle_j^{-1} \Theta_j$$

is unif. $\|\cdot\|_j$ Riesz basis for $V_j \cap V_{j-1}^{\perp_j}$.

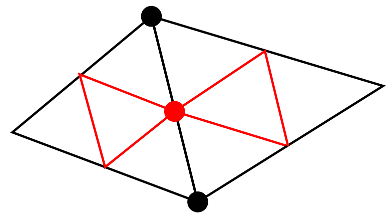


Figure 9: Nonzero coefficients $\langle \Xi_j, \Phi_{j-1} \rangle_j$.

So, by definition of Ξ_j and Θ_j, each wavelet is a linear combi. of only *three* nodal basis functions (any n).

Multi-level stability? Inner product changes from level to level. So space decomposition isn't orthogonal. No information about dual side.

Stability $L_2(\Omega)$-orthogonal decomposition + perturbation arguments gives that, properly scaled, 3-pt hb is Riesz for $s > \begin{cases} 0.29 & n=1 \\ 0.42 & n=2 \\ 0.76 & n=3 \end{cases}$

Likely not sharp. In shift-invariant case, Riesz for $H^s(\mathbb{R}^n)$ for $s \in (-0.99023, \frac{3}{2})$ for $n \in \{1,2,3\}$ (sharp).

The (one-point) **hierarchical basis** (Schauder basis) is Riesz for $H^s(\Omega)$ for $s > \frac{n}{2}$ (sharp).

Conclusion

Wavelet bases suitable for the application of `awgm` can be constructed on general domains.

References

[1] N.G. Chegini, S. Dahlke, U. Friedrich, and R.P. Stevenson. Piecewise tensor product wavelet bases by extensions and approximation rates. Technical report, KdV Institute for Mathematics, University of Amsterdam, 2011. To appear in *Math. Comp.*

[2] A. Cohen, I. Daubechies, and J.C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. Pur. Appl. Math.*, 45:485–560, 1992.

[3] A. Cohen, I. Daubechies, and P. Vial. Wavelets on the interval and fast wavelet transforms. *Appl. Comput. Harmon. Anal.*, 1(1):54–81, 1993.

[4] W. Dahmen. Stability of multiscale transformations. *J. Fourier Anal. Appl.*, 2(4):341–362, 1996.

[5] W. Dahmen, A. Kunoth, and K. Urban. Biorthogonal spline-wavelets on the interval - Stability and moment conditions. *Appl. Comp. Harm. Anal.*, 6:132–196, 1999.

[6] W. Dahmen and R. Schneider. Composite wavelet bases for operator equations. *Math. Comp.*, 68:1533–1567, 1999.

[7] W. Dahmen and R. Schneider. Wavelets on manifolds I: Construction and domain decomposition. *SIAM J. Math. Anal.*, 31:184–230, 1999.

[8] W. Dahmen and R.P. Stevenson. Element-by-element construction of wavelets satisfying stability and moment conditions. *SIAM J. Numer. Anal.*, 37(1):319–352, 1999.

[9] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41:909–996, 1988.

[10] T.J. Dijkema. *Adaptive tensor product wavelet methods for solving PDEs*. PhD thesis, Utrecht University, 2009.

[11] G.C. Donovan, J.S. Geronimo, and D.P. Hardin. Orthogonal polynomials and the construction of piecewise polynomial smooth wavelets. *SIAM J. Math. Anal.*,

[14] M. Primbs. New stable biorthogonal spline-wavelets on the interval. *Results Math.*, 57(1-2):121–162, 2010.

[15] R.P. Stevenson. Piecewise linear (pre-)wavelets on non-uniform meshes. In *Multigrid methods V (Stuttgart, 1996)*, volume 3 of *Lect. Notes Comput. Sci. Eng.*, pages 306–319. Springer, Berlin, 1998.

[16] R.P. Stevenson. Stable three-point wavelet bases on general meshes. *Numer. Math.*, 80:131–158, 1998.