On the completion of locally refined partitions

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Variational problem

Let $a(\cdot,\cdot)$ bounded, coercive, bil. form on H, $f \in H'$. Seek $u \in H$, s.t.

$$a(u,v) = f(v) \quad (v \in H).$$

$H = H^m(\Omega)$ or $H = H_0^m(\Omega)$, $\Omega \subset \mathbb{R}^n$ polytope.

Galerkin

Let $V \subset H$ closed subspace. Seek $\tilde{u} \in V$, s.t.

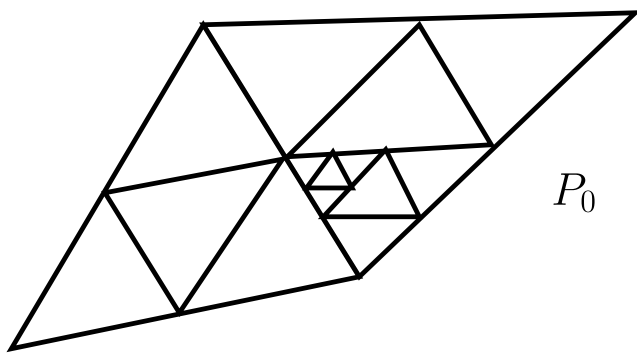
$$a(\tilde{u},v) = f(v) \quad (v \in V).$$

Cea's lemma: $\|u - \tilde{u}\|_H \leq \frac{C}{\alpha} \inf_{v \in V} \|u - v\|_H$.

(A)FEM

FEM: $V = V_P$ fin. dim subspace "associated to" a partition P of $\bar{\Omega}$ into essentially disjoint "elements", e.g., n-simplices ($V_P = H \cap \prod_{T \in P} P_{d-1}(T)$), or n-rectangles ($V_P = H \cap \prod_{T \in P} Q_{d-1}(T)$ ($Q_{d-1} = P_{d-1} \otimes \cdots \otimes P_{d-1}$)).

We consider partitions that, starting from some **fixed initial partition P_0**, are created by a recurrent application of some 'deterministic' **refinement rule**, that applied to T, replaces it by its K children, e.g.,

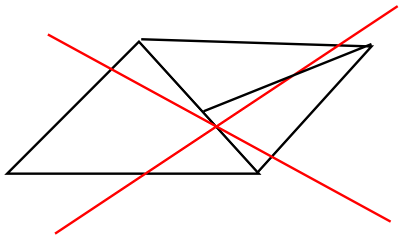


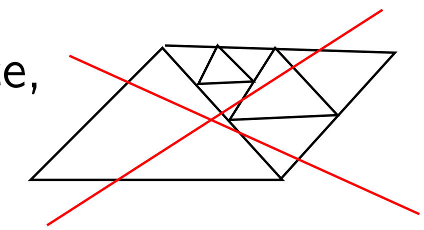
Level $\ell(T)$ is depth of refinement. $\boldsymbol{\tau}$ is collection of all T that can be created.

AFEM: create seq. of partitions by performing following loop:

$$\textbf{GALSOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{REFINE}$$

For the a posteriori error estimation it is needed to impose some **admissibility condition** on the partitions. E.g.

- **conformity**  , or
- if $T \cap T'$ is $(n-1)$-dimensional, then
  - either $\ell(T) = \ell(T')$ and $T \cap T'$ is a common face,
  - or $|\ell(T) - \ell(T')| = 1$,



So in **REFINE**, besides elements that were marked for refinement, generally surrounding elements have to be refined ("completion").

Complexity of completion

Let $\mathbf{refine}[P,T]$ output the smallest admissible refinement P' of an admissible P in which $T \in P$ has been refined.
(uniquely determined by **ref. rule**, the **admissibility condition** and P_0)

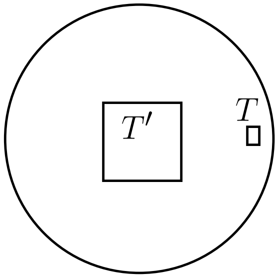
Th 1 ([Binev, Dahmen, DeVore '04] for newest vertex bisection in $n = 2$).

Let

$$\mathrm{diam}(T)^n \eqsim \mathrm{meas}(T) \eqsim K^{-\ell(T)} \quad (T \in \tau)$$

Let any T' newly created by a call $\mathbf{refine}[P,T]$ satisfy

- $\ell(T') \leq \ell(T) + 1$
- $d(T', T) \lesssim K^{-\ell(T')/n}$



Then at any time in AFEM algorithm

$$\#(P \backslash P_0) \lesssim \#M.$$

Theorem 1 is one of the keys to obtain

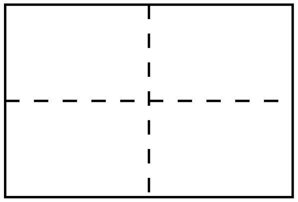
Th 2 (quasi-optimality of AFEM [St '07]). *Suppose* $\exists (P_i)_i$ *of admissible partitions with* $\lim_{i\to\infty} \# P_i = \infty$ *and for some* $s > 0$,

$$\lim_{i\to\infty} (\# P_i)^s \|u - u_{P_i}\|_H < \infty.$$

Then the seq of partitions produced by AFEM has the same property, with limit at most a constant factor larger.
With inexact Galsolves also optimal comput. compl.

Refinement of n-rectangles

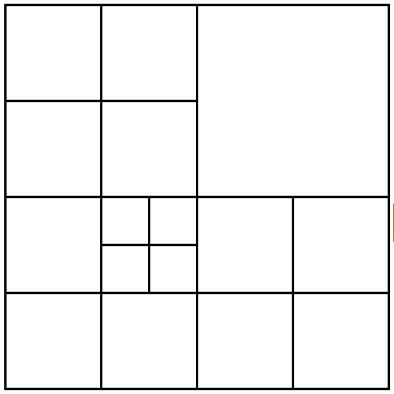
Refinement rule



$K = 2^n$

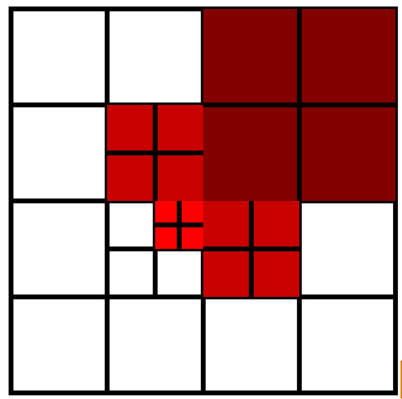
$$\operatorname{diam}(T)^n \eqsim \operatorname{meas}(T) \eqsim K^{-\ell(T)}.$$

Admissibility condition



Generalizes to n-dimensions, also with n-**simplices** ("red-refinement") (some care for $n \geq 3$ to ensure shape regularity, and for $n > 3$ to ensure conformity of uniform refinements. Freudenthal algorithm, see [Bey '00])

$\mathbf{refine}[P,T] \to P'$
% P is admissible and $T \in P$
forall neighbours T'' of T in P do
    if $\ell(T'') < \ell(T)$ then $P = \mathbf{refine}[P,T'']$ endif
enddo
refine T into its 2^n children



Th 3. $P' := \mathbf{refine}[P, T]$ *terminates, and* P' *is smallest admissible refinement of* P *in which* T *has been refined.*

If $T' \in P'$ *is newly created by the call, then* $\boxed{\ell(T') \leq \ell(T) + 1}$.

Proof. Induction to $\ell(T)$. $\square$

Th 4. *With* $\zeta := \sup_{T \in \mathcal{T}} 2^{\ell(T)} \text{diam}(T)$, *any newly created* T' *by the call* $\mathbf{refine}[P, T]$ *satisfies*

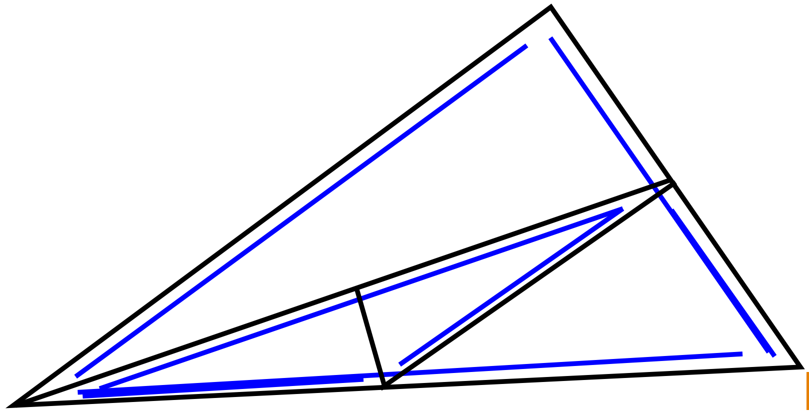
$$\boxed{d(T', T)} \leq 2\zeta \sum_{k=\ell(T')}^{\ell(T)} 2^{-k} \qquad \boxed{\left(\lesssim 2^{-\ell(T')}\right)}$$

Proof. $\ell(T) = 0$: T' child of T so $d(T, T') = 0$.
$\ell(T) = \ell$: If T' child of T, then as above. Otherwise, T' is created by recursive call $\mathbf{refine}[P, T'']$ with $\ell(T'') = \ell(T) - 1$. $T \cap T'' \neq \emptyset$, so

$$d(T', T) \leq d(T', T'') + \text{diam}(T'') \leq 2\zeta \sum_{k=\ell(T')}^{\ell(T'')} 2^{-k} + \zeta 2^{-\ell(T'')} = 2\zeta \sum_{k=\ell(T')}^{\ell(T)} 2^{-k}. \quad \square$$

Newest vertex bisection [Sewell '72]

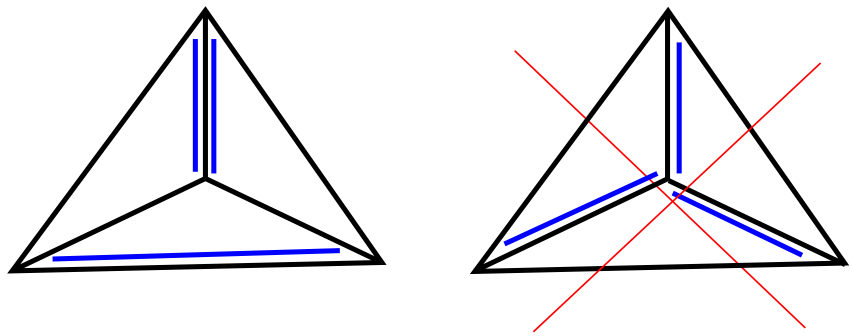
Refinement rule: $K = 2$. Vertex opposite to **refinement edge** is called **newest vertex**.



$$\text{diam}(T)^n \eqsim \text{meas}(T) \eqsim 2^{-\ell(T)}$$ ([Mitchell '91]).

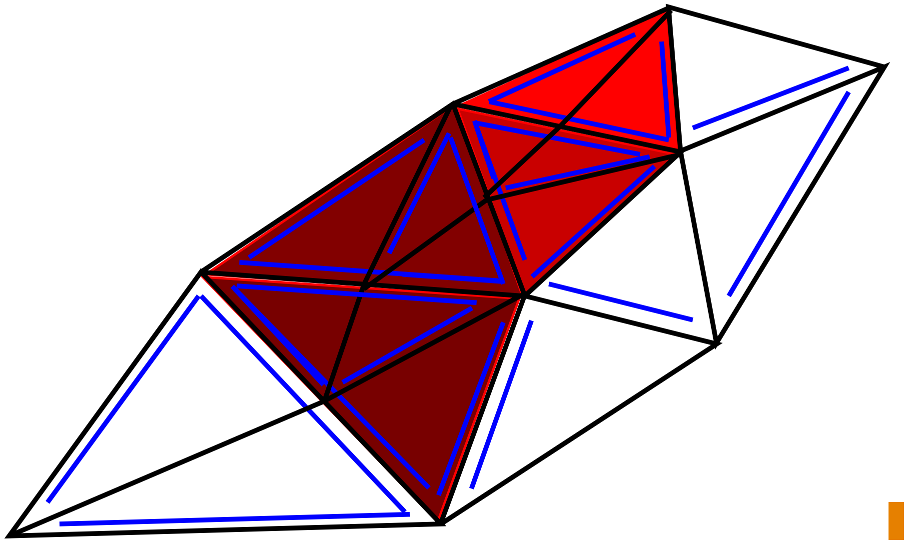
Admissibility condition: conformity.

To satisfy conditions Th. 1 start with conf. init. part. with **matching refinement edges**. Such an assignment of ref. edges exists in any init. conf. part. ([Binev, Dahmen, DeVore '04]).



Matching cond. on P_0 is equiv. to conformity of any uniform refinement of P_0.

$P' := \mathbf{refine}[P, T]$:



Under matching condition on P_0, remaining two conditions of Th 1 can be shown.

(without matching cond., the recursive implementation of **refine** does not necessarily terminate)

Bisection ($K = 2$) of n-simplices

n-simplex is convex hull of $x_0, \ldots, x_n$ such that $\{x_1 - x_0, \cdots, x_n - x_0\}$ is independent.

To obtain uniform shape regularity, a proper cyclic choice of refinement edges should be made. Given $\{x_0, \ldots, x_n\}$ we distinguish between $n(n+1)!$ **tagged** simplices given by all possible *ordered* sequences $(x_0, x_1, \ldots, x_n)_\gamma$ and *types* $\gamma \in \{0, \ldots, n-1\}$.

Refinement rule: Given a tagged simplex

$$T = (x_0, x_1, \ldots, x_n)_\gamma,$$

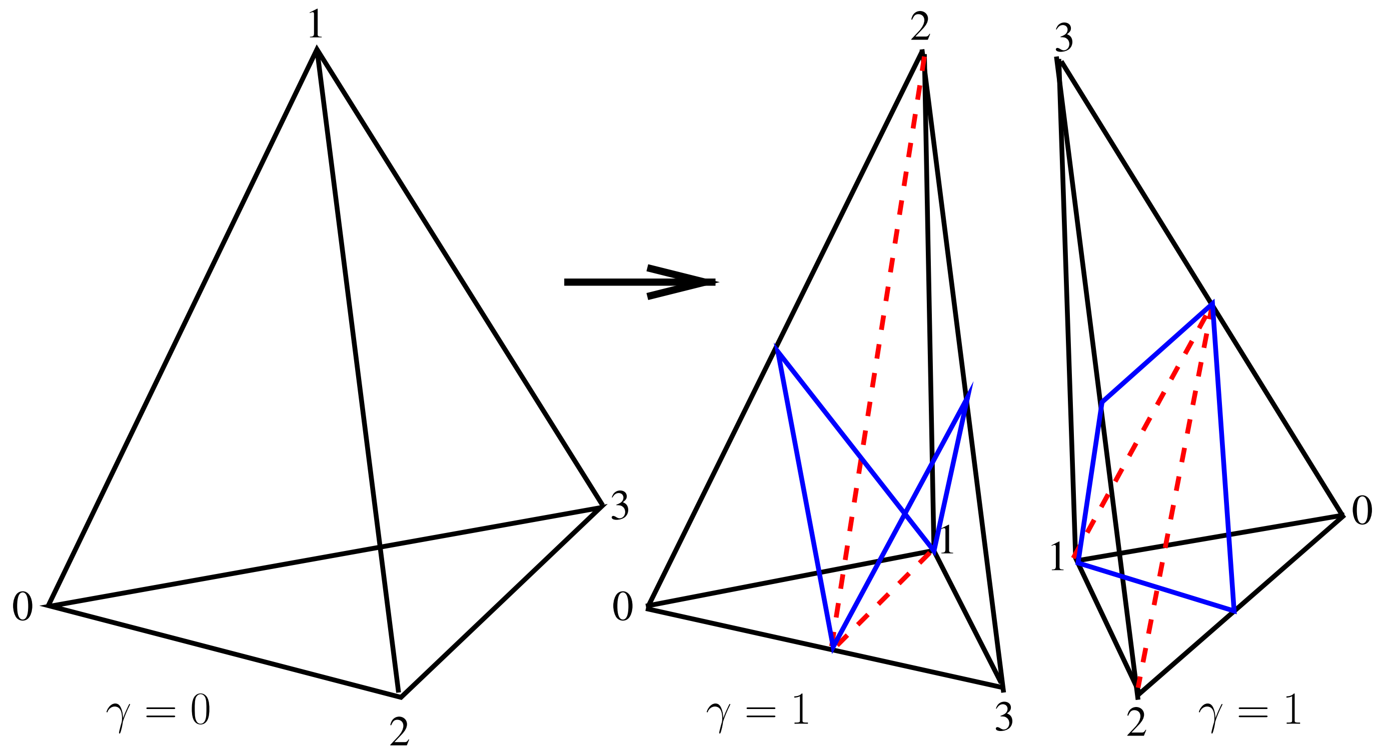
its children are the tagged simplices

$$\left(x_0, \tfrac{x_0+x_n}{2}, x_1, \ldots, x_\gamma, x_{\gamma+1}, \ldots, x_{n-1}\right)_{(\gamma+1)\mathrm{mod}n}$$
$$\left(x_n, \tfrac{x_0+x_n}{2}, x_1, \ldots, x_\gamma, x_{n-1}, \ldots, x_{\gamma+1}\right)_{(\gamma+1)\mathrm{mod}n}.$$

$\overline{x_0 x_n}$ is the **refinement edge** of T.

Generalizes newest vertex bisec.

We identify T with $T_R = (x_n, x_1, \ldots, x_{n-1}, x_0)_\gamma$ since they have same descendants.

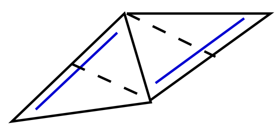


Th 5 ([Maubach '95], [Traxler '97]). $\operatorname{diam}(T)^n \eqsim \operatorname{meas}(T) \eqsim 2^{-\ell(T)}$

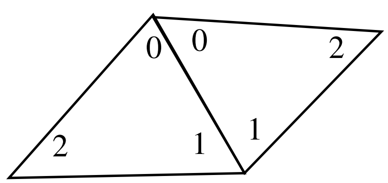
Admissibility condition: conformity.

Th 6. *All unif. refs of init part P_0 with fixed type γ are conforming iff*

- *P_0 is conforming*
- *Any two neighbouring tagged simplices $T = (x_0, \ldots, x_n)_\gamma$, $T' = (x'_0, \ldots, x'_n)_\gamma$ from P_0* **match** *in the sense that*
  - *if $\overline{x_0 x_n}$ or $\overline{x'_0 x'_n}$ is on $T \cap T'$, then T and T' are reflected neighbours,*
  - *otherwise, the pair of neighbouring children of T and T' are reflected neighbours.*



$T = (x_0, \ldots, x_n)_\gamma$, $T' = (x'_0, \ldots, x'_n)_\gamma$ **reflected neighbours** when the ordered sequence of vertices of either T or T_{R} coincides with that of T' on all but one position (necessarily corresponding to $x_i, x'_i \notin T \cap T'$)



Proof. For all ℓ, level ℓ descendants of any T form conforming part.

For all ℓ, level ℓ descendants of neighbours T and T' with either $\overline{x_0 x_n}$ or $\overline{x'_0 x'_n}$ on $T \cap T'$ form conforming part iff T and T' are refl. neighb.

If for neighbours T and T', both $\overline{x_0 x_n}, \overline{x'_0 x'_n} \notin T \cap T'$, then they have children S and S' with their refinement edges on $S \cap S'$ $(= T \cap T')$. □

Matching cond. generalizes the one for $n = 2$. Much milder than in [Maubach '95] or [Traxler '97].

For $n > 2$, we don't know whether a suitable numbering exists for any conforming initial partition. It does exist after a suitable refinement of each simplex into $\frac{1}{2}(n+1)!$ subsimplices (generalization of [Kossaczký '94] in 3D).

Let P_0 be conforming and satisfy matching condition:

```
refine[P,T] → P':
% P is a conforming descendant of P_0 and T ∈ P.
K := ∅; F = {T}
do Fnew := ∅
   forall T' ∈ F do
      forall neighbours T'' ∉ F ∪ K of T' that contain its ref. edge do
         if T'', T' have the same refinement edge
         then Fnew := Fnew ∪ {T''}
         else P := refine[P,T'']
              add to Fnew the child of T'' that is a neighbour of T'
         endif
      endfor
   endfor
   K := K ∪ F
   F := Fnew
until F = ∅
create P' from P by simultaneously bisecting all T' ∈ K ▮
```

Remaining two conds of Th. 1 are satisfied.

Proof of Th. 1 [Binev, Dahmen, DeVore '04]

Th 1. *Let ref. rule, admis. cond. and P_0 be s.t.*

$$\mathrm{diam}(T)^n \eqsim \mathrm{meas}(T) \eqsim K^{-\ell(T)} \quad (T \in \tau).$$

and s.t. any T' newly created by a call **refine**$[P, T]$ *satisfies*

- $\ell(T') \leq \ell(T) + 1$
- $d(T', T) \lesssim K^{-\ell(T')/n}$.

Then at any time in AFEM algorithm

$$\#(P \backslash P_0) \lesssim \#M.$$

Let P, $\bar{P}$ and M denote final part, any intermediate part, and set of marked elem., e.g. set of $T \in \bar{P}$ for which a call $\mathbf{refine}[\bar{P}, T]$ has been made.

Going to constr. λ on $P \times M$ s.t.

$$\sum_{T' \in P} \lambda(T', T) \lesssim 1 \quad (T \in M), \quad \sum_{T \in M} \lambda(T', T) \gtrsim 1 \quad (T' \in P \backslash P_0).$$

Then

$$\#(P \backslash P_0) \lesssim \sum_{T' \in P \backslash P_0} \sum_{T \in M} \lambda(T', T) \leq \sum_{T \in M} \sum_{T' \in P} \lambda(T', T) \lesssim \#M,$$

($\lambda(T', T)$ is amount of money spent by T on T').

Let

$$\sum_{p=-1}^{\infty} a(p) < \infty, \quad \sum_{p=0}^{\infty} b(p) K^{-p/n} < \infty, \quad \inf_{p \geq 0} b(p) a(p) > 0,$$

$$A := C_1 \sum_{p=0}^{\infty} b(p) K^{-p/n}$$

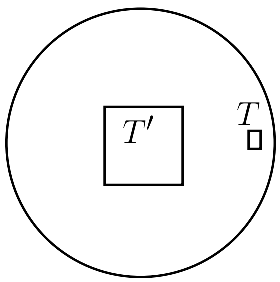
$$\lambda(T', T) := \begin{cases} a(\ell(T) - \ell(T')) \text{ if } d(T', T) < AK^{-\ell(T')/n} \text{ and } \ell(T') \leq \ell(T) + 1, \\ \text{zero otherwise.} \end{cases}$$

(T spends only money on T' which might be created by $\textbf{refine}[\bar{P}, T]$)

For any $\ell' \leq \ell(T) + 1$, $\# T' \in P$ with $\ell(T') = \ell'$ and $d(T', T) \leq AK^{-\ell'/n}$ is unif. bounded. So

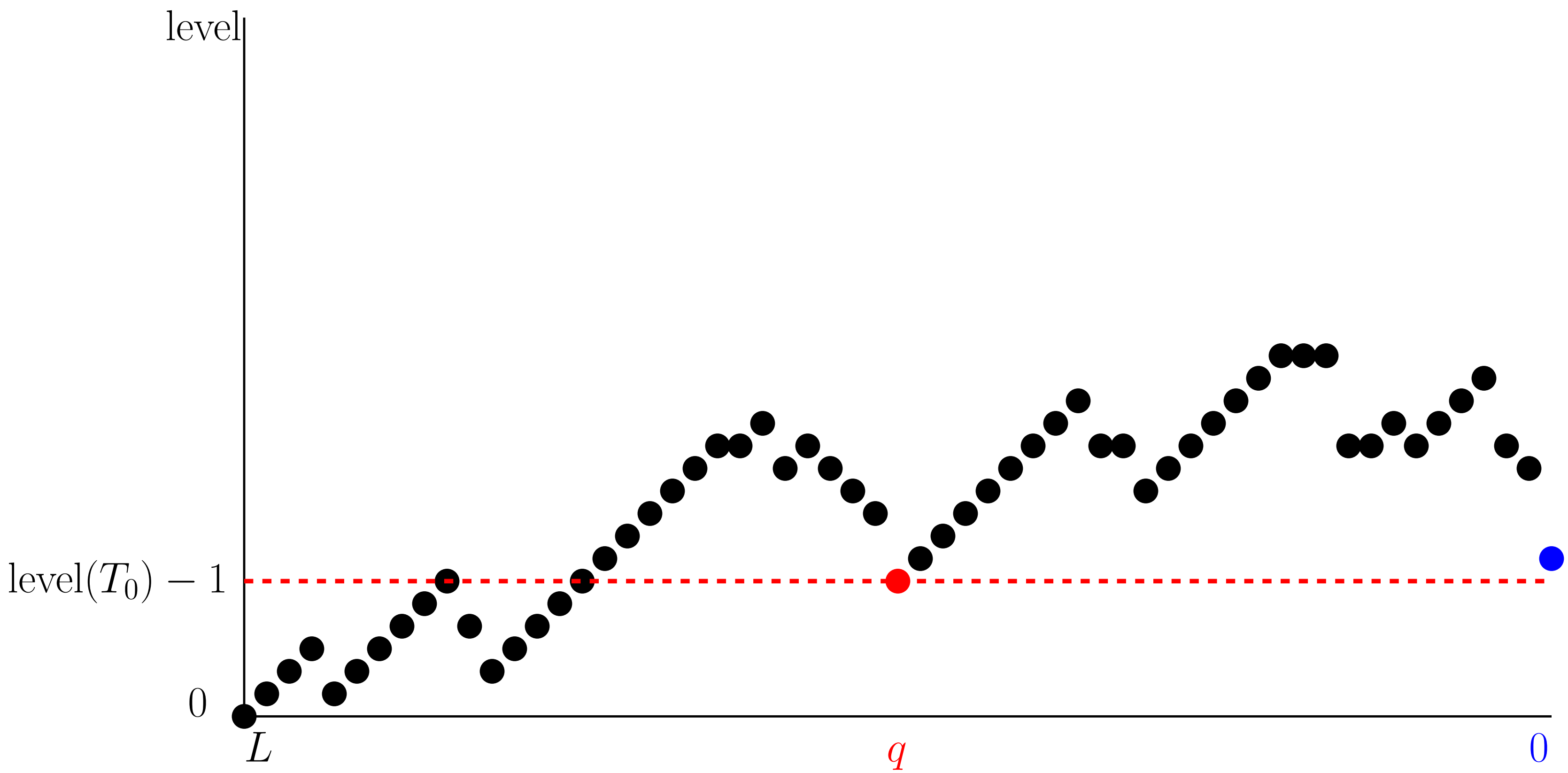
$$\sum_{T' \in P} \lambda(T', T) \lesssim \sum_{\ell'=0}^{\ell(T)+1} a(\ell(T) - \ell') < \infty.$$

(amount of money spent by T is unif. bounded)



Let $T_0 \in P \backslash P_0$. Define $T_1, \ldots, T_L \in M$ by

$$P_0 \ni T_L \xrightarrow{\mathbf{refine}[\bar{P}, T_L]} T_{L-1} \xrightarrow{\mathbf{refine}[\bar{P}, T_{L-1}]} \ldots \xrightarrow{\mathbf{refine}[\bar{P}, T_2]} T_1 \xrightarrow{\mathbf{refine}[\bar{P}, T_1]} T_0$$



Let $q \leq L$ smallest with $\ell(T_q) = \ell(T_0) - 1$.

For $1 \le j \le q$,

$$d(T_0, T_j) \le d(T_0, T_1) + \mathrm{diam}(T_1) + d(T_1, T_j) \le \sum_{k=1}^{j} d(T_{k-1}, T_k) + \sum_{k=1}^{j-1} \mathrm{diam}(T_k)$$

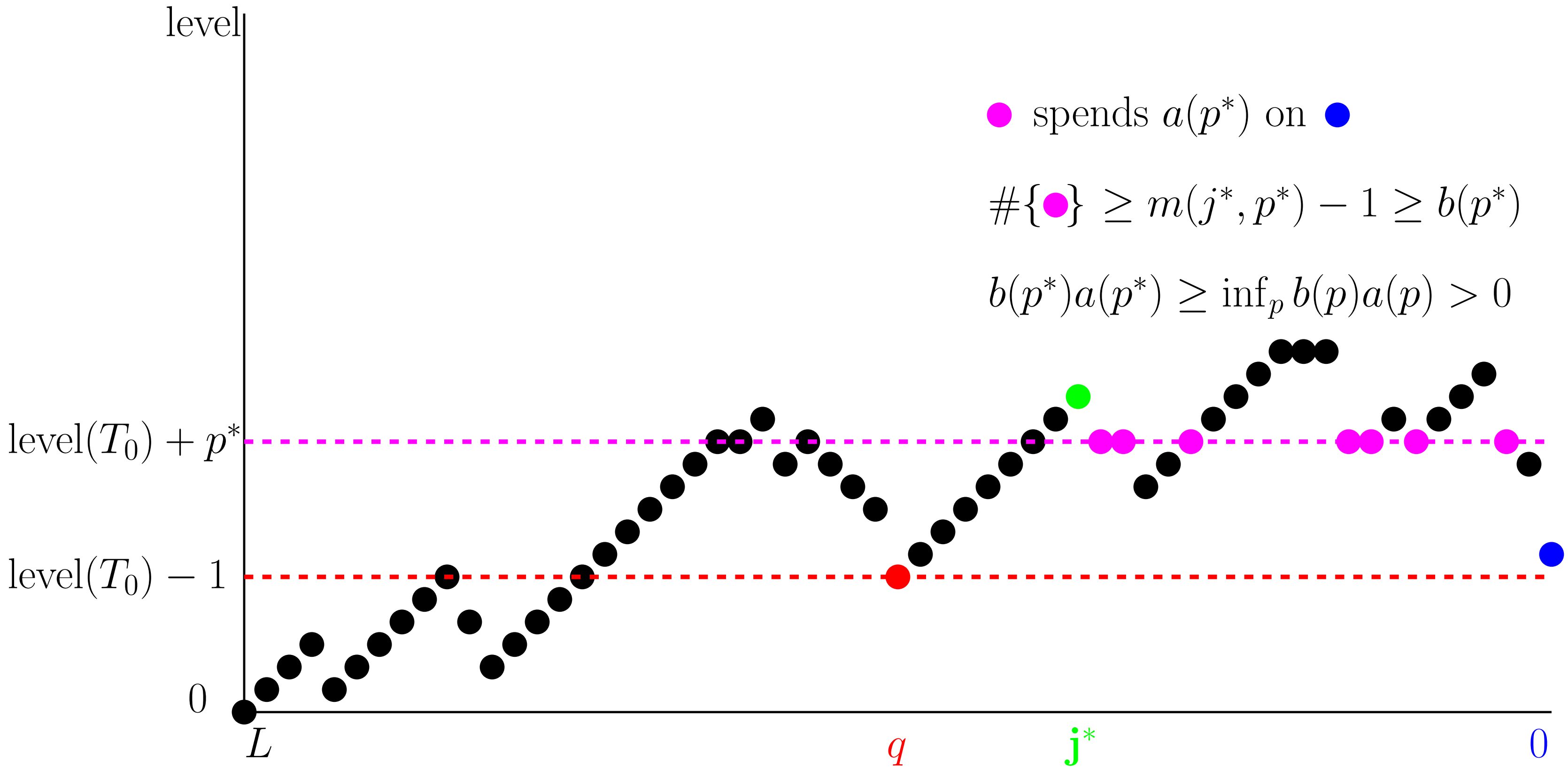
$$\lesssim \sum_{k=1}^{j} K^{-\frac{\ell(T_{k-1})}{n}} + \sum_{k=1}^{j-1} K^{-\frac{\ell(T_k)}{n}} \le C_1 \sum_{k=0}^{j-1} K^{-\frac{\ell(T_k)}{n}} = K^{-\frac{\ell(T_0)}{n}} C_1 \sum_{p=0}^{\infty} m(p,j) K^{-\frac{p}{n}},$$

with $m(p,j) = \#\{k \le j-1 : \ell(T_k) = \ell(T_0) + p\}$.

If $m(p,j) \le b(p)$ ($\forall p$), then $d(T_0, T_j) \le AK^{-\ell(T_0)/n}$, and so from $\ell(T_0) \le \ell(T_j) + 1$ thus $\lambda(T_0, T_j) = a(\ell(T_j) - \ell(T_0))$.

If this holds for $j = q$, then $\sum_{T \in M} \lambda(T_0, T) \ge \lambda(T_0, T_q) = a(-1) > 0$.
(T_0 receives enough money from T_q)

Otherwise let $1 \le j^* \le q$ smallest s.t. $m(p, j^*) > b(p)$ for some p, say p^*.
By def., for $1 \le j < j^*$, $\lambda(T_0, T_j) = a(\ell(T_j) - \ell(T_0))$.
By def., $\#\{0 \le j < j^* : \ell(T_j) = \ell(T_0) + p^*\} = m(p^*, j^*) > b(p^*)$.
So $\sum_{T \in M} \lambda(T_0, T) \ge (m(p^*, j^*) - 1)a(p^*) = b(p^*)a(p^*) \ge \inf_p b(p)a(p) > 0$.


level
level$(T_0) + p^*$
level$(T_0) - 1$
0
L
q
$\mathbf{j}^*$
0
spends $a(p^*)$ on
$\#\{\,\} \geq m(j^*, p^*) - 1 \geq b(p^*)$
$b(p^*)a(p^*) \geq \inf_p b(p)a(p) > 0$

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