Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

University of Zurich[UZH]

MASTER'S THESIS

A Posteriori Error Estimation of *hp*-DG Finite Element Methods for Highly Indefinite Helmholtz Problems

written at the
Institut für Mathematik
Mathematisch-naturwissenschaftliche Fakultät
Universität Zürich[1]

Supervisor

Prof. Dr. Stefan Sauter

by

Jakob Zech

March 28, 2014

(revised June 10, 2014)

[1] I thank Prof. Ch. Schwab (SAM, D-MATH, ETH) for administrative sponsoring of this MSc project in my MSc Math. studies at ETH.

Abstract

In this thesis we introduce and analyse a residual-type a posteriori error estimator for a discontinuous Galerkin (DG) method applied to the Helmholtz equation with Robin boundary condition at high wavenumbers k. The estimator can be used for adaptive mesh refinement, and controls the error caused by the nonconformity of the method as well as the approximation error in the norm $k\|\cdot\|_{L^2}+|\cdot|_{H^1}$. As a model problem, we consider the case of convex polygons and hp-finite elements. In the classical theory, a posteriori estimators provide bounds of the error up to constants depending linearly on k. In the present work, we prove a reliability estimate, where the occuring constant becomes k-independent under certain assumptions on the FEM space, which require a minimum of $\mathcal{O}(k^2)$ degrees of freedom in two dimensions. In this setting, efficiency will be shown to hold with a constant depending on a power of $\log(k)$. The proof is based on recent findings in the study of highly indefinite Helmholtz problems, which show that our choice of space allows a pollution-free discretization. This theory is also applicable to the DG formulation, which is moreover known to admit unique solutions under much weaker conditions than conventional Galerkin methods. This leaves us with an adaptive method, that avoids pollution and additionally has favourable stability properties. A series of numerical experiments is conducted to illustrate the performance of the error estimator and the adaptive algorithm.

Contents

List of Figures

Introduction

Consider the Helmholtz equation with Robin boundary condition:

$$\begin{aligned} -\Delta u - k^2 u &= f \qquad \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n} + \mathrm{i}ku &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\boldsymbol{n}$ is the outer normal, $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$, $\Omega \subseteq \mathbb{R}^2$ is a convex polygonal domain, and $k > 0$ is the wavenumber. If k is large, this problem is highly indefinite, and it is well-known that low order finite element methods suffer from the so called pollution effect. This means that they fail to approximate the solution properly, unless a surprisingly stringent resolution condition is imposed on the meshwidth h, typically this is $k^2 h = \mathcal{O}(1)$. The condition implicates, that a minimum of $\mathcal{O}(k^4)$ degrees of freedom is needed in two dimensions. Moreover, a posteriori error estimators are marred by their dependence on the inf-sup condition, which is known to grow linearly in k for certain domains. This may result in an underestimation of the error, thereby challenging the estimators usefulness in adaptive algorithms. Naturally it is of interest to improve such methods in order to avoid these problems.

A new k-explicit a priori analysis was conducted in [26, 27]. There it is shown, that a pollution-free discretization of the Helmholtz equation is possible under certain requirements regarding the FEM space: In the case of convex polygons and hp-finite elements, the mesh needs to be geometrically refined towards the vertices of the polygon, the polynomial degree p must be of size $\mathcal{O}(\log(k))$, and hk/p should be sufficiently small. This theory relies on a splitting of the solution of the Helmholtz equation, and we will shortly recall the essential points in Section 2.2 and further elaborate in Section 2.4.

In [25] a discontinuous Galerkin method, based on the mentioned splitting, was analysed. The advantages of this formulation manifest themselves in the following properties: Only very mild assumptions are necessary to ensure the existence of a discrete solution (see Theorem 2.3.5). Moreover, k-independent quasi optimality of the method is guaranteed under the above resolution condition of the FEM space, which mandates a minimum of $\mathcal{O}(k^2)$ degrees of freedom in two dimensions as compared to $\mathcal{O}(k^4)$ (cf. Theorem 2.4.2 and Remark 2.4.3).

Particularly in view of the necessary mesh refinement in neighbourhoods of the vertices, it is desireable to have an adaptive algorithm for this problem. Therefore, the primary goal of this thesis is to find an a posteriori error estimator for the discontinuous Galerkin formulation of [25], which is efficient and reliable with constants depending very weakly on k. To be more precise, we will restrict ourselves to the case of hp-finite elements on regular triangular meshes. The constant

in the reliability estimate will be independent of the wavenumber, provided that the FEM space meets the mentioned requirements. The constant in the efficiency estimate will likewise not depend on k, but on some power of p. The choice $p = \mathcal{O}(\log(k))$, effectively entails polynomial dependence on $\log(k)$ in this case. The specific problem of the k-dependence will be overcome as in [15] by an Aubin-Nitsche type argument and the approximability of the adjoint problem, which again relies heavily on the theory from [26, 27].

The structure of this thesis is as follows: In Chapter 1, we define function spaces and set up necessary notation. In Chapter 2, a model problem is given, and we briefly summarize the relevant material concerning the DG method from [25]. Furthermore, we recall the findings of [27], and, based on this, extend some results on convergence and stability from [25] to our specific setting. Chapter 3 will be concerned with two approximation theorems. It serves as a preparation for the subsequent chapter on a posteriori estimators, but is of interest in itself: First, we will introduce an H^2-conforming Clément type interpolant for the Sobolev space $W^{n,q}$, with the property that it locally reproduces the optimal convergence rates for derivatives up to order $n-1$ on admissible triangulations. Afterwards, we proceed with the study of the approximation of discontinuous piecewise polynomials by globally continuous piecewise polynomials on regular triangular meshes. In Chapter 4, an a posteriori error estimator is presented, and efficiency and reliability will be proved. Finally, Chapter 5 is dedicated to numerical experiments. We will compare convergence rates for uniform and adaptive refinement, and test the adaptive algorithm with problems that are not covered by our theory.

Chapter 1

Preliminaries

In this chapter, we introduce the notation and function spaces which will be used in the following. Most of it is fairly standard, and we keep close to [9, Chapter 2 § 5], [27, Section 1], [30, Section 4.4.1], and [24, Section 1.1].

Let us start with a few general remarks. Throughout this thesis $C > 0$ will be a constant which does not depend on the meshwidth h, the polynomial degree p, the wavenumber k, and the solution u. Yet, it possibly depends on the shape regularity constant γ, which will be defined in (1.1.4), and occasionally on some other parameters. The constant C is not fixed and may change in each occurence. Moreover, we use the notation

$$f(x) \lesssim g(x)$$

to indicate that there exists a constant $C > 0$, not depending on the aforementioned parameters, such that $f(x) \leq Cg(x)$ for every x in the domain of the two real valued functions f and g. If we have $f \lesssim g$ together with $f \gtrsim g$ we write $f \sim g$.

Further, the (weak) j-th derivative of v in x direction is denoted by $D_x^j v$, and for some multiindex $\alpha \in \mathbb{N}_0^d$, we write $D^\alpha v := D_{x_1}^{\alpha_1} \dots D_{x_d}^{\alpha_d} v$.

In the following let $\Omega \subseteq \mathbb{R}^2$ be a connected open polygonal domain with

$$\partial\Omega = \partial\overline{\Omega}. \tag{1.0.1}$$

1.1 Triangulations

1.1.1 Elements, Edges, Vertices

We set up the terminology for elements, edges, and vertices of a triangulation. Let $\mathcal{T}$ be a finite set of open triangles $K \subseteq \Omega$ such that

$$\bigcup_{K \in \mathcal{T}} \overline{K} = \overline{\Omega} \quad \text{and} \quad K \cap K' = \emptyset,$$

for all $K, K' \in \mathcal{T}$ with $K \neq K'$. Let $K \in \mathcal{T}$ be an **element** of $\mathcal{T}$. We call the three corners V_1, V_2, V_3 of K its **vertices**, and each side of K is an **edge** of K. The set of all vertices belonging

to an element $K \in \mathcal{T}$ is denoted by $\mathcal{N}(K)$. Let e be an edge of K. Then e is always considered to be closed in $\mathbb{R}^2$. We say e is an **interior edge** if there exists an element $K' \neq K$ in $\mathcal{T}$, such that e is an edge of both elements and $e \not\subseteq \partial\Omega$. Otherwise e is called a **boundary edge** of the triangulation $\mathcal{T}$. The set of all interior edges belonging to some $K \in \mathcal{T}$ is denoted by $\mathcal{E}^I(K)$. Similarly, for the boundary edges we write $\mathcal{E}^B(K)$, and for all edges of the element K we write $\mathcal{E}(K) := \mathcal{E}^I(K) \cup \mathcal{E}^B(K)$. A subset $\omega \subseteq \Omega$ is called a **patch** of $\mathcal{T}$, if for some $n \in \mathbb{N}$ it can be written as

$$\omega = \bigcup_{j=1}^{n} \overline{K_j}, \tag{1.1.1}$$

where $K_j \in \mathcal{T}$ for all $j \in \{1, \dots, n\}$ and for every $j \in \{2, \dots, n\}$ it holds that K_j shares an interior edge with an element from the set $\{K_1, \dots, K_{j-1}\}$. For the triangulation of this patch we write $\mathcal{T}_\omega$, and sometimes consider ω to be a set of elements rather than a polygon. The sets of elements belonging to a patch ω, an edge e, or a vertex V are denoted by

$$\begin{aligned}
\mathcal{K}(\omega) &:= \left\{K \in \mathcal{T} : \overline{K} \subseteq \omega\right\}, \\
\mathcal{K}(e) &:= \left\{K \in \mathcal{T} : \overline{K} \cap e \neq \emptyset\right\}, \\
\mathcal{K}(V) &:= \left\{K \in \mathcal{T} : \overline{K} \cap V \neq \emptyset\right\}.
\end{aligned}$$

We use the following notation for sets of vertices and edges belonging to a patch ω:

$$\mathcal{N}(\omega) := \bigcup_{\{K \in \mathcal{T} : K \subseteq \omega\}} \mathcal{N}(K), \qquad \text{and} \qquad \mathcal{E}(\omega) := \bigcup_{\{K \in \mathcal{T} : K \subseteq \omega\}} \mathcal{E}(K)$$

with similar definitions for interior and boundary edges. In the special case where the patch is the whole domain we write for the fixed triangulation $\mathcal{T}$

$$\mathcal{N} := \mathcal{N}(\mathcal{T}) := \mathcal{N}(\Omega), \qquad \text{and} \qquad \mathcal{E} := \mathcal{E}(\mathcal{T}) := \mathcal{E}(\Omega),$$

with analogue definitions for interior and boundary edges. The set $\mathcal{E}$ is called the **skeleton** of $\mathcal{T}$. Finally, the notation $\mathcal{N}(e)$ and $\mathcal{E}(V)$ stands for the endpoints of the edge e respectively for the set of edges with endpoint V.

Patches that will be of importance to us are those associated with an element K, an edge e, or a vertex V (without further assumptions on $\mathcal{T}$ the following sets generally are no patches in the sense of (1.1.1)):

$$\omega_K := \omega_K^1 := \bigcup_{\left\{K' \in \mathcal{T} : \overline{K'} \cap \overline{K} \neq \emptyset\right\}} \overline{K'},$$

and further for $j \in \mathbb{N}$, $j > 1$

$$\omega_K^j := \bigcup_{K' \in \mathcal{K}(\omega_K^{j-1})} \omega_{K'}^1.$$

Replacing K with e or V we obtain analogue definitions for ω_e^j and ω_V^j.

1.1.2 Admissible Triangulations

Definition 1.1.1 (Admissible triangulations). *Let $\mathcal{T}$ be a finite set of open triangles $K \subseteq \Omega$. We say that $\mathcal{T}$ is an* **admissible triangulation** *of Ω provided that*

(i) $\bigcup_{K\in\mathcal{T}} \overline{K} = \overline{\Omega}$.

(ii) Let $K \neq K'$ be two elements of $\mathcal{T}$. Then $\overline{K} \cap \overline{K'}$ is either empty, it consists of a common vertex V, or it consists of an entire common edge e of K and K'.

(iii) Let $K \in \mathcal{T}$ and $|\mathcal{T}| > 1$. Then there exists an element $K' \in \mathcal{T}$, $K' \neq K$ and $\overline{K} \cap \overline{K'}$ is an entire common edge of K and K'.

(iv) For every $V \in \mathcal{N}(\mathcal{T})$ it holds that ω_V is a patch in the sense of (1.1.1).

In this case we also write $(\Omega, \mathcal{T})$ for the tupel. Moreover, a property of a function defined on Ω is said to hold **piecewise** *if it holds for the restriction of the function to each element $K \in \mathcal{T}$.*

As a consequence of (1.0.1) and due to the requirements of an admissible triangulation, hanging nodes and slit domains are excluded. Let us make some further comments on the notation.

- Even though $\mathcal{E}$ is a set of edges, we write short

$$\int_{\mathcal{E}} h \, \mathrm{d}S := \sum_{e\in\mathcal{E}} \int_e h \, \mathrm{d}S,$$

  for functions h with $h \in L^1(e)$ for every $e \in \mathcal{E}$.

- To capture the size of elements we use the three quantities

$$h_K := \operatorname{diam}(K), \qquad h_e := |e|, \qquad h_V := \min_{K\in\omega_V} h_K. \tag{1.1.2}$$

  The **meshwidth** $h_{\mathcal{T}}$ of $\mathcal{T}$ is the maximal size of an element:

$$h_{\mathcal{T}} := \max_{K\in\mathcal{T}} h_K. \tag{1.1.3}$$

- The **reference element** $\hat{K} \subseteq \mathbb{R}^2$ and the **reference edge** $\hat{e} \subseteq \mathbb{R}$ are defined as

$$\hat{K} := \left\{(x,y) \in \mathbb{R}^2 : x, y > 0 \text{ and } x + y < 1\right\}, \qquad \hat{e} := [0,1],$$

  where sometimes $\hat{e}$ is considered to be naturally embedded in $\mathbb{R}^2$. For every $K \in \mathcal{T}$ there is an affine isomorphism $F_K : \mathbb{R}^2 \to \mathbb{R}^2$ called the **element map** with $F_K(\hat{K}) = K$. In case we want $\hat{e}$ or $\mathbf{0} \in \mathbb{R}^2$ to be mapped onto a specific edge $e \in \mathcal{E}(K)$ or a vertex $V \in \mathcal{N}(K)$, we use the notation $F_{K,V}$ and $F_{K,e}$ instead of F_K. To shorten notation we define the **edge map** $F_e := F_{K,e}|_{\hat{e}} : \hat{e} \to e$, and for $V \in \mathcal{N}(e)$ the map $F_{e,V}$ as the edge map with the additional property $F_{e,V}(\mathbf{0}) = V$. The element maps allow us to define shape regularity.

Definition 1.1.2 (Shape regularity). *Let $\mathcal{T}$ be an admissible triangulation of Ω and let $\gamma > 0$. Then we say $\mathcal{T}$ is* **γ-shape regular** *if we have*

$$h_K^{-1}\|F_K'\|_{L^\infty(\hat{K})} + h_K\|(F_K')^{-1}\|_{L^\infty(\hat{K})} \leq \gamma, \tag{1.1.4}$$

for every $K \in \mathcal{T}$.

Shape regularity implies two important characteristics of the mesh that are stated in the following lemma [24, Lemma 1.3].

Lemma 1.1.3. *Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of Ω. Then there exists an integer $M = M(\gamma) > 0$ such that the following is true.*

(i) For every $V \in \mathcal{N}(\mathcal{T})$ it holds that no more than M elements of $\mathcal{T}$ share the vertex V.

(ii) Let $K, K' \in \mathcal{T}$ with $\overline{K} \cap \overline{K'} \neq \emptyset$. Then

$$M^{-1}h_K \leq h_{K'} \leq Mh_K. \tag{1.1.5}$$

If (1.1.5) holds, we also say that K and K' are **comparable in size**. The last definitions we need to make in this section are about reference patches.

Definition 1.1.4 (Reference patches of the first type). *Let $\gamma > 0$ and let $M = M(\gamma) > 0$ be the constant from Lemma 1.1.3. For every $j \in \{3, \dots, M\}$ we consider a fixed closed regular polygon $\hat{\omega}_j \subseteq \mathbb{R}^2$ with j sides, center $\mathbf{0} \in \mathbb{R}^2$ and such that the farthest distance from a point in $\hat{\omega}_j$ to its center is one. By defining interior edges on $\hat{\omega}_j$ as the lines connecting $\mathbf{0}$ with a corner of $\hat{\omega}_j$, we then obtain a triangulation $\mathcal{T}_j$ of $\hat{\omega}_j$ in a natural way. The elements in this triangulation are denoted by $\{\hat{K}_1^j, \dots, \hat{K}_j^j\}$. The finite set $\hat{B}(\gamma)$ of reference patches is given as the set of all patches (in the sense of (1.1.1)) $\hat{\omega}$ belonging to one of the polygons $\hat{\omega}_j$ with the triangulation $\mathcal{T}_j$, $j \in \{3, \dots, M\}$.*

With the above notation let now $V \in \mathcal{N}(\mathcal{T})$. Additional to element maps we also consider **patch maps** $F_V : \mathbb{R}2 \to \mathbb{R}^2$ such that $F_V(\hat{\omega}_V) = \omega_V$ for a suitable $\hat{\omega}_V \in \hat{B}(\gamma)$. They are defined to be the continuous piecewise affine extension of a fixed Lipschitz continuous piecewise affine (with regards to the triangulations $\mathcal{T}|_{\omega_V}$ and $\mathcal{T}_j$, where $\hat{\omega}_V = \hat{\omega}_j$) homeomorphism between $\mathring{\omega}_V$ and $\mathring{\hat{\omega}}_V$.

1.1.3 Geometric Meshes

For $\sigma \in (0,1)$ and $L \in \mathbb{N}$ we construct meshes $\hat{\mathcal{T}}(\sigma, L)$ on the reference element $\hat{K}$ as follows: $\hat{\mathcal{T}}(\sigma, 0)$ is a fixed admissible triangulation such that the only vertices on the boundary are $(0,0)$, $(\sigma, 0)$, $(0, \sigma)$, $(1, 0)$, $(0, 1)$, and the only element abutting at the origin is determined by the three vertices $(0,0)$, $(\sigma, 0)$ and $(0, \sigma)$. Starting with $\hat{\mathcal{T}}(\sigma, L)$, the triangulation $\hat{\mathcal{T}}(\sigma, L+1)$ is obtained by additionally furnishing the single element of $\hat{\mathcal{T}}(\sigma, L)$ that abuts at the vertex $\mathbf{0}$ with a scaled version of the triangulation $\hat{\mathcal{T}}(\sigma, 1)$, such that this is still an admissible triangulation (i.e. the scaling maps $\mathbf{0}$ to $\mathbf{0}$).

Definition 1.1.5. *Let $\Omega \subseteq \mathbb{R}^2$ be a polygon with $J \in \mathbb{N}$ corners $A_1, \dots, A_J$, which we call its* **apices**. *Let $h > 0$, $L \in \mathbb{N}$, $\sigma \in (0,1)$, and let $\mathcal{T}$ be a γ-shape regular admissible triangulation of Ω. Then we call $\mathcal{T}$ a* **geometric mesh** *with* **grading factor** *σ and L* **layers** *provided that the following is satisfied: Assume that no element of $\mathcal{T}$ touches more than one apex. Then, starting from a quasi-uniform triangulation $\hat{\mathcal{T}}$ on Ω with mesh size h, that means for each $K \in \hat{\mathcal{T}}$ we have*

$$c_2^{-1} h_K \leq h \leq c_2 h_K,$$

for some fixed $c_2 > 0$, $\mathcal{T}$ is obtained by furnishing elements abutting at an apex with a scaled version of the triangulation $\hat{\mathcal{T}}(\sigma, L)$, such that the resulting mesh is regular and refined towards the apices (i.e. the scaling maps $\mathbf{0}$ to the apex). The triangulation restricted to the uniform part of the mesh is denoted by $\mathcal{T}_{unif}$, and the rest of the triangulation is denoted by $\mathcal{T}_{geo}$. In this case we also write $\mathcal{T} = \mathcal{T}(h, \sigma, L)$, respectively $\mathcal{T}_{unif}(h, \sigma, L)$ and $\mathcal{T}_{geo}(h, \sigma, L)$ for the uniform and the geometric part of the triangulation.

We mention that this definition implies that elements touching the apex are of size $\mathcal{O}(h\sigma^L)$. Moreover, there exists a constant $c_{geo} > 0$ such that for every element $K \in \mathcal{T}_{geo}$ with $A_j \notin \overline{K}$ for every $j \in \{1, \dots, J\}$ and $\operatorname{dist}(K, A_j) \leq c_1 h$ we have

$$c_{geo}^{-1} h_K \leq \operatorname{dist}(K, A_j) \leq c_{geo} h_K. \tag{1.1.6}$$

1.2 Spaces

1.2.1 Finite Element Space

Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of Ω. In our setting we work with polynomials of degree p_K on each element K. The polynomial degrees are collected in the vector $\boldsymbol{p} := (p_K)_{K \in \mathcal{T}}$, which is called a **polynomial degree distribution**. For $d, p \in \mathbb{N}$ we define the following spaces of polynomials

$$\mathcal{P}_p(\mathbb{R}^d) := \operatorname{span}\left\{ x_1^{j_1} \dots x_d^{j_d} : j_1, \dots, j_d \in \mathbb{N}_0 \ \wedge \ \sum_{j=1}^{d} j_d \leq p \right\},$$

$$\mathcal{Q}_p(\mathbb{R}^d) := \bigotimes_{j=1}^{d} \mathcal{P}_p(\mathbb{R}),$$

and further for every $e \in \mathcal{E}(\mathcal{T})$ and $K \in \mathcal{T}$

$$\mathcal{P}_p(e) := \left\{ q \circ F_e^{-1}|_e : q \in \mathcal{P}_p(\mathbb{R}) \right\},$$
$$\mathcal{P}_p(K) := \left\{ r \circ F_K^{-1}|_K : r \in \mathcal{P}_p(\mathbb{R}^2) \right\}.$$

The space in which we approximate our solution is

$$S^{\boldsymbol{p}}(\mathcal{T}) := \left\{ v \in L^2(\Omega) : v|_K \in \mathcal{P}_{p_K}(K) \ \forall K \in \mathcal{T} \right\}.$$

If $p_K = p$ for all $K \in \mathcal{T}$, we also write $S^p(\mathcal{T}) := S^{\boldsymbol{p}}(\mathcal{T})$. Similar to (1.1.5) we want to bound the change of polynomial degree for neighbouring elements. To ensure this, it will generally be assumed that if $K' \subseteq \omega_K$, then

$$\gamma^{-1} p_K \leq p_{K'} \leq \gamma p_K, \tag{1.2.1}$$

where $\gamma > 0$ is a constant. For simplicity of notation we use the same letter as for γ-shape regularity, and consider γ to be a constant associated with the finite element space $S^{\boldsymbol{p}}(\mathcal{T})$. Furthermore we employ the notation

$$p_e := \min_{K \in \omega_e} p_K, \qquad p_V := \min_{K \in \omega_V} p_K,$$

to get local polynomial degrees, and

$$p_{\mathcal{T}} := \min_{K \in \mathcal{T}} p_K, \tag{1.2.2}$$

to get the lowest polynomial degree of $\boldsymbol{p}$.

1.2.2 Broken Sobolev Spaces

The usual notation $H^s(\omega)$ and $W^{s,p}(\omega)$ is used to denote Sobolev spaces for $s > 0$, $p \in [1, \infty]$, and some open nonempty set $\omega \subseteq \mathbb{R}^2$ (see [2, Paragraphs 3.2 and 7.57]). The norms on these spaces are $\|\cdot\|_{H^s(\omega)}$, $\|\cdot\|_{W^{s,p}(\omega)}$, and $|\cdot|_{H^s(\omega)}$, with the last one being the standard seminorm on $H^s(\omega)$ (defined via Fourier transformation). On the domain Ω with the triangulation $\mathcal{T}$ we define the **broken Sobolev space** $H^s_{\mathcal{T}}(\Omega)$ for $s > 0$ as

$$H^s_{\mathcal{T}}(\Omega) := \prod_{K \in \mathcal{T}} H^s(K). \tag{1.2.3}$$

For convenience, functions in this product space are identified with functions defined on Ω, and by the notation $v|_K \in H^s(K)$ we mean the component of $v \in H^s_{\mathcal{T}}(\Omega)$ that belongs to K. If $v|_K \in C^0(K)$, we define for $\boldsymbol{x} \in \overline{K}$

$$v|_K(\boldsymbol{x}) := \lim_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ \boldsymbol{y} \in K}} v(\boldsymbol{y}).$$

On the spaces $H^s_{\mathcal{T}}(\Omega)$ we introduce the piecewise gradient $\nabla_{\mathcal{T}} : H^1_{\mathcal{T}}(\Omega) \to L^2(\Omega)^3$ and the piecewise Laplace operator $\Delta_{\mathcal{T}} : H^2_{\mathcal{T}}(\Omega) \to L^2(\Omega)$. They are simply the elementwise application of ∇ and Δ, and are therefore well-defined on the broken Sobolev spaces.

For $s > 1/2$ and $v|_K \in H^s(K)$ there exists a trace $(v|_K)|_{\partial K} \in H^{s-1/2}(\partial K)$ [2, Chapter 7]. Hence the trace of $v \in H^s_{\mathcal{T}}(\Omega)$ belongs to

$$\prod_{K \in \mathcal{T}} H^{s-1/2}(\partial K).$$

In this case we write e.g., $(v|_K)\,|_e$ for some $e \in \mathcal{E}(K)$ to denote the restriction of the boundary trace of $v|_K \in H^s(K)$ to the edge $e \subseteq \partial K$. Moreover we will need jumps and mean values on interior edges. Let $e \in \mathcal{E}^I$ be shared by $K, K' \in \mathcal{T}$. Then

$$\begin{aligned}[\![v]\!]_N|_e &:= (v|_K)|_e \cdot \boldsymbol{n}_K + (v|_{K'})|_e \cdot \boldsymbol{n}_{K'},\\ \{v\}|_e &:= \frac{1}{2}\Big((v|_K)|_e + (v|_{K'})|_e\Big),\end{aligned}$$

where $\boldsymbol{n}_K, \boldsymbol{n}_{K'}$ are the respective outer normal vectors on the boundary of K and K' and v can also be vector valued. Additionally, we define the jump

$$[\![v]\!]|_e := (v|_K)|_e - (v|_{K'})|_e,$$

where the sign of this term is arbitrary, unless explicitly stated otherwise. Finally, we write $\partial_{\boldsymbol{n}} u := \nabla u \cdot \boldsymbol{n}$ for the normal derivative on the boundary.

1.2.3 Weighted Sobolev Spaces

We introduce a set of weighted Sobolev spaces. Let $\mu \in [0,1)$, $k > 0$, and $n \in \mathbb{N}_0$, then we define

$$\Phi_{n,\mu,k+1}(\boldsymbol{x}) := \min\left\{1, \frac{|\boldsymbol{x}|}{\min\left\{1, \frac{n+1}{k+1}\right\}}\right\}^{n+\mu}. \tag{1.2.4}$$

For a polygonal domain Ω with apices $A_1, \dots, A_J$ and some $\boldsymbol{\mu} \in [0,1)^J$ we further set

$$\Phi_{n,\boldsymbol{\mu},k+1}(\boldsymbol{x}) := \prod_{j=1}^{J} \Phi_{n,\mu_j,k+1}(\boldsymbol{x} - A_j). \tag{1.2.5}$$

Moreover, if $n \in \mathbb{N}_0$ then

$$|\nabla^n u(\boldsymbol{x})|^2 := \sum_{\{\alpha \in \mathbb{N}_0^2 : |\alpha| = n\}} \frac{n!}{\alpha!} |D^\alpha u(\boldsymbol{x})|^2.$$

Definition 1.2.1. *Let $C_u, k, \nu > 0$ and $\boldsymbol{\mu} \in [0,1)^J$. The space $\mathcal{B}_{\boldsymbol{\mu},k}(C_u, \nu)$ is given by*

$$\begin{aligned}\mathcal{B}_{\boldsymbol{\mu},k}(C_u,\nu) := \Big\{u \in H^1(\Omega) : & k\|u\|_{L^2(K)} + |u|_{H^1(\Omega)} \le C_u k \quad \wedge \\ & \|\Phi_{n,\boldsymbol{\mu},k+1} \nabla^{n+2} u\|_{L^2(\Omega)} \le C_u (\nu \max\{n,k\})^{n+2} \ \forall n \in \mathbb{N}_0\Big\},\end{aligned} \tag{1.2.6}$$

where $\Phi_{n,\boldsymbol{\mu},k+1}$ is as in (1.2.5).

Chapter 2

Discontinuous Galerkin (DG) Formulation and Splitting of the Solution

The purpose of this chapter is to introduce a DG method for the Helmholtz equation and gather a few important results on stability and existence of a solution of this formulation. In [26] a splitting of the solution of the Helmholtz equation was derived for the special case where the domain is a ball centered at zero, and it was shown that this allows a pollution free discretization of the Helmholtz equation. In [27] the theory was extended to domains with analytic boundary and to convex polygonal domains, and in [25] a DG method based on this splitting was analysed again on domains with analytic boundary. Here, we consider this method on convex polygonal domains and give a k-explicit convergence result in Section 2.4. Moreover, in Section 2.2 we shortly touch on the subject of pollution, which explains the reasoning behind the whole theory, and subsequently recall the decomposition of the solution.

2.1 Model Problem

Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal domain satisfying (1.0.1). Let $f \in L^2(\Omega)$ and let $g \in H^{1/2}(\partial\Omega)$. Then we consider the **Helmholtz equation** on Ω with **Robin boundary condition**

$$-\Delta u - k^2 u = f \qquad \text{in } \Omega, \tag{2.1.1a}$$

$$\nabla u \cdot \boldsymbol{n} + \mathrm{i}ku = g \qquad \text{on } \partial\Omega, \tag{2.1.1b}$$

where $\boldsymbol{n}$ is the outer normal vector and $k > 0$ is a constant. The variational formulation of this problem is: Find $u \in H^1(\Omega)$ such that

$$a(u, v) = F(v) \qquad \forall v \in H^1(\Omega), \tag{2.1.2}$$

with the sesquilinear form

$$a(u, v) := \int_\Omega \nabla u \overline{\nabla v} - k^2 u \overline{v} \, \mathrm{d}\boldsymbol{x} + \mathrm{i}k \int_{\partial\Omega} u \overline{v} \, \mathrm{d}S,$$

and the linear functional

$$F(v) := \int_\Omega f\overline{v}\ \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} g\overline{v}\ \mathrm{d}S.$$

This problem has a unique solution that depends continuously on the data [22, Proposition 8.1.3]:

Theorem 2.1.1. *Let Ω be a bounded Lipschitz domain. Then there exists a constant $C(\Omega, k) > 0$ such that for every $f \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of* (2.1.2)*. Moreover,*

$$k\|u\|_{L^2(\Omega)} + |u|_{H^1(\Omega)} \leq C(\Omega, k)\left(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}\right).$$

On convex domains the constant can be chosen independent of k [22, Proposition 8.1.4]:

Theorem 2.1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex domain, and let $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Then there exists a constant $C(\Omega) > 0$ independent of k such that the solution u of* (2.1.2) *satisfies*

$$\begin{aligned} k\|u\|_{L^2(\Omega)} + |u|_{H^1(\Omega)} &\leq C(\Omega)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right), \\ |u|_{H^2(\Omega)} &\leq C(\Omega)\left((1+k)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right) + \|g\|_{H^{1/2}(\Omega)}\right). \end{aligned}$$

2.2 Splitting of the Solution

2.2.1 Pollution

Before we discuss the splitting of the solution, we shortly explain why, when applying h-FEM to the Helmholtz equation, one needs to choose the meshwidth h in dependence of the wavenumber k. We further describe the problem of pollution, which is intended to motivate our preceedings henceforth. In order to do so, let us take a look at two examples given in [7, Section 2.2] and [21, Section 4.5.3]:

Consider the problem

$$-u'' - k^2 u = f \qquad \text{in } (0,1), \tag{2.2.1a}$$

$$u(0) = 1, \tag{2.2.1b}$$

$$u'(1) - \mathrm{i}ku(1) = 0, \tag{2.2.1c}$$

for some $k > 0$ and $f \in L^2(0,1)$. A variational formulation for this problem is: Find $u \in V$ such that

$$b(u,v) = (f,v)_{L^2(0,1)} \qquad \forall v \in V, \tag{2.2.2}$$

where

$$V := \left\{ v \in H^1(0,1) : v(0) = 0 \right\},$$
$$b(u,v) := \int_{(0,1)} u'\overline{v}' \, \mathrm{d}\boldsymbol{x} - k^2 \int_{(0,1)} u\overline{v} \, \mathrm{d}\boldsymbol{x} - \mathrm{i}ku(1)\overline{v}(1).$$

A mesh on the domain $(0,1)$ is obtained by dividing it into uniform intervalls of the size $h > 0$. We then denote the space of piecewise linear functions v_h satisfying $v_h(0) = 0$ on this mesh by V_h. Furthermore, we write u_h for the associated FEM solution in V_h, and u_I for the linear interpolant of u in the nodes of our mesh. As it is mentioned in [21, Section 4.5.3], u_I is the best approximation of u in V_h w.r.t. to $|\cdot|_{H^1(0,1)}$. Also, it is well-known [21, (4.4.2)] that in this situation $|u - u_I|_{H^1} \leq Ch|u|_{H^2}$, where $C > 0$ does not depend on either k or h. Now let $f \equiv 0$. Then the exact solution is given by $u(x) = \exp(\mathrm{i}kx)$, and it fulfills $k\|u\|_{L^2} = |u|_{H^1(0,1)}$. Hence we have for the relative error of the best approximation in the H^1-seminorm

$$\frac{|u - u_I|_{H^1}}{|u|_{H^1}} \leq C\frac{h|u|_{H^2}}{|u|_{H^1}} \leq Chk.$$

As a first observation, it is therefore sensible to impose $hk \leq 1$ as a minimal condition on the meshwidth h. Otherwise one cannot expect the FEM solution u_h to be a proper approximation of u, which is also intuitively clear, since $u(x) = \exp(\mathrm{i}kx)$ oscillates with frequency k. If $h \gg 1/k$ this oscillation cannot be reproduced by u_h.

Now let us return to the general situation where f in (2.2.1) is not necessarily zero. By subtracting $b(u - u_h, v_h)$ from the left-hand side, integrating by parts, and using that u_I interpolates u (which e.g. gives $u_I(1) = u(1)$), it can be checked that

$$b(u_h - u_I, v_h) = k^2(u - u_I, v_h) \qquad \forall v_h \in V_h. \tag{2.2.3}$$

Moreover, if $kh \leq 1$, then according to [21, Lemma 4.12] we have for the finite element solution u_h of (2.2.2) (with V_h instead of V) that $|u_h|_{H^1} \leq C\|f\|_{L^2(0,1)}$, where $C > 0$ does not depend on h and k. Thus with (2.2.3)

$$|u_h - u_I|_{H^1} \leq Ck^2\|u - u_I\|_{L^2(0,1)}$$

and further with $\|u - u_I\|_{L^2} \leq Ch|u - u_I|_{H^1}$ [21, (4.4.2)]

$$\begin{aligned} |u - u_h|_{H^1} &\leq |u - u_I|_{H^1} + |u_I - u_h|_{H^1} \lesssim |u - u_I|_{H^1} + k^2\|u - u_I\|_{L^2} \\ &\lesssim (1 + k^2h)|u - u_I|_{H^1} \sim (1 + k^2h) \inf_{v_h \in V_h} |u - v_h|_{H^1}. \end{aligned}$$

This estimate is not too pesimistic and the factor $1 + k^2h$ can indeed be seen in numerical experiments (cf. Figure 5.1).

Let us sum up these deliberations. Whereas $kh \leq 1$ seems to be a necessary resolution condition to approximate the function u in the space V_h, the FEM additionally requires $k^2h \leq 1$, in order for u_h to be close to the best approximation of u in V_h. This is because even though the FEM is

quasi optimal, the constant indicating the difference between the FEM solution u_h and the best approximation u_I depends on the wavenumber k. In dimension $d \in \mathbb{N}$ the condition $k^2 h \leq 1$ entails that we need at least $\mathcal{O}(k^{2d})$ degrees of freedom. Therefore, large values of k can make the FEM impractical. This strong dependence of the quasi optimality constant on the wavenumber is known as **pollution**. A precise definition can be found in [7, Def. 2.1] or [21, Section 4.6.1].

2.2.2 Decomposition Theorem

The solution u of (2.1.2) allows the following splitting [27, Theorem 4.9]:

Theorem 2.2.1 (Decomposition). *Let $\Omega \subseteq \mathbb{R}^2$ be a convex polygonal domain. Assume that the solution u of* (2.1.2) *satisfies an a priori estimate of the type*

$$k\|u\|_{L^2(\Omega)} + |u|_{H^1(\Omega)} \lesssim k^{\vartheta} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right),$$

for some $\vartheta \geq 0$. Then there exist constants C, ν, and $\boldsymbol{\mu} \in [0,1)^J$ independent of k such that for every $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, the solution u of (2.1.2) *can be written as $u = u_{\mathcal{A}} + u_{H^2}$, where for all $n \in \mathbb{N}_0$ there holds*

$$k\|u_{\mathcal{A}}\|_{L^2(\Omega)} + |u_{\mathcal{A}}|_{H^1(\Omega)} \leq Ck^{\vartheta} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right), \tag{2.2.4a}$$

$$\|\Phi_{n,\boldsymbol{\mu},k+1} \nabla^{n+2} u_{\mathcal{A}}\|_{L^2(\Omega)} \leq C\nu^n k^{\vartheta-1} \max(n,k)^{n+2} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right), \tag{2.2.4b}$$

$$k^2\|u_{H^2}\|_{L^2(\Omega)} + k|u_{H^2}|_{H^1(\Omega)} + |u_{H^2}|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right). \tag{2.2.4c}$$

Remark 2.2.2. Recall that if Ω is a convex polygon, then, according to Theorem 2.1.2, the constant ϑ can be chosen as 0. Moreover, as it is mentioned in [27, Remark 4.10], in this case any (small) value $\mu \in (0,1)$ is permitted for the entries of $\boldsymbol{\mu}$.

Let us roughly describe the idea of the proof of Theorem 2.2.1. At first the data f, g is decomposed by Fourier transformation into a low frequency part containing frequencies up to $\mathcal{O}(k)$ and a high frequency part containing the remaining frequencies. Then the linearity of the equation is exploited, and one obtains a smooth but oscillating function $u_{\mathcal{A}}$, belonging to the low frequency part, and the function u_{H^2}, which belongs to the high frequence part and is less smooth, but whose H^2-norm can be controlled independent of k. For the details, see [27].

With this splitting at hand, the strategy is as follows: The function u_{H^2} is approximated with a Clément type interpolant, and the constant stemming from this approximation is independent of k since $\|u_{H^2}\|_{H^2(\Omega)} \sim \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}$ independent of k. The smooth part $u_{\mathcal{A}}$ on the other hand, can be approximated with an exponential convergence rate in the polynomial degree p. The choice $p \sim \log(k)$ will then absorb unfavorable polynomial k dependencies in this case. All in all, we can thus approximate u optimally such that the approximation constants do not depend on k, and the minimal number of degrees of freedom is $\mathcal{O}(k^d)$. We will demonstrate this in detail in Section 2.4.

2.3 DG Method

We recall the DG method described in [25].

2.3.1 DG Formulation

The following formulation was originally proposed in [16] (cf. Remark 2.3.2). A DG formulation for (2.1.1) is: For a test space $S \subseteq H^2_{\mathcal{T}}(\Omega)$ and real valued weight functions $\alpha, \beta, \delta \in L^\infty(\mathcal{E})$, find $u_{\mathcal{T}} \in S$ such that

$$a_{\mathcal{T}}(u_{\mathcal{T}}, v) = F_{\mathcal{T}}(v) \qquad \forall v \in S, \tag{2.3.1}$$

with the sesquilinear form $a_{\mathcal{T}} : H^{3/2}_{\mathcal{T}}(\Omega) \times H^{3/2}_{\mathcal{T}}(\Omega) \to \mathbb{C}$

$$\begin{aligned} a_{\mathcal{T}}(u,v) :=& (\nabla_{\mathcal{T}} u, \nabla_{\mathcal{T}} v)_{L^2(\Omega)} - \int_{\mathcal{E}^I} [\![u]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}} v}\} \,\mathrm{d}S - \int_{\mathcal{E}^I} \{\nabla_{\mathcal{T}} u\} \cdot [\![\overline{v}]\!]_N \,\mathrm{d}S \\ &- \int_{\mathcal{E}^B} \delta u \overline{\nabla_{\mathcal{T}} v \cdot \boldsymbol{n}} \,\mathrm{d}S - \int_{\mathcal{E}^B} \delta \nabla_{\mathcal{T}} u \cdot \boldsymbol{n} \overline{v} \,\mathrm{d}S \\ &- \frac{1}{\mathrm{i}k} \int_{\mathcal{E}^I} \beta [\![\nabla_{\mathcal{T}} u]\!]_N [\![\overline{\nabla_{\mathcal{T}} v}]\!]_N \,\mathrm{d}S - \frac{1}{\mathrm{i}k} \int_{\mathcal{E}^B} \delta \nabla_{\mathcal{T}} u \cdot \boldsymbol{n} \overline{\nabla_{\mathcal{T}} v \cdot \boldsymbol{n}} \,\mathrm{d}S \\ &+ \mathrm{i}k \int_{\mathcal{E}^I} \alpha [\![u]\!]_N [\![\overline{v}]\!]_N \,\mathrm{d}S + \mathrm{i}k \int_{\mathcal{E}^B} (1-\delta) u \overline{v} \,\mathrm{d}S - k^2 (u,v)_{L^2(\Omega)} \end{aligned} \tag{2.3.2}$$

and the linear functional $F_{\mathcal{T}} : H^{3/2}_{\mathcal{T}}(\Omega) \to \mathbb{C}$

$$F_{\mathcal{T}}(v) := (f,v)_{L^2(\Omega)} - \int_{\mathcal{E}^B} \delta \frac{1}{\mathrm{i}k} g \overline{\nabla_{\mathcal{T}} v \cdot \boldsymbol{n}} \,\mathrm{d}S + \int_{\mathcal{E}^B} (1-\delta) g \overline{v} \,\mathrm{d}S.$$

Moreover, we introduce the mesh-dependent norms

$$\begin{aligned} \|v\|^2_{DG} :=& \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)} + k^{-1} \|\beta^{1/2} [\![\nabla_{\mathcal{T}} v]\!]_N\|^2_{L^2(\mathcal{E}^I)} + k \|\alpha^{1/2} [\![v]\!]_N\|^2_{L^2(\mathcal{E}^I)} \\ &+ k^{-1} \|\delta^{1/2} \nabla_{\mathcal{T}} v \cdot \boldsymbol{n}\|^2_{L^2(\mathcal{E}^B)} + k \|(1-\delta)^{1/2} v\|^2_{L^2(\mathcal{E}^B)} + k^2 \|v\|^2_{L^2(\Omega)}, \end{aligned} \tag{2.3.3}$$

$$\|v\|^2_{DG^+} := \|v\|^2_{DG} + k^{-1} \|\alpha^{-1/2} \{\nabla_{\mathcal{T}} v\}\|^2_{L^2(\mathcal{E}^I)}. \tag{2.3.4}$$

The mesh-dependent weights α, β and δ have to be chosen in consideration of a) balancing the different error terms of the DG-norm in a priori error estimates b) ensuring that there exists a unique solution of (2.3.1) c) ensuring favorable stability properties of the formulation. Here we consider *hp*-finite elements (i.e. $S = S^{\boldsymbol{p}}(\mathcal{T})$). Then these weights are defined on each edge e of the skeleton as

$$\alpha(\boldsymbol{x}) := \mathfrak{a} \frac{p_e^2}{k h_e}, \qquad \beta(\boldsymbol{x}) := \mathfrak{b} \frac{k h_e}{p_e}, \qquad \delta(\boldsymbol{x}) := \mathfrak{d} \frac{k h_e}{p_e} \qquad \forall \boldsymbol{x} \in e, \tag{2.3.5}$$

where $\mathfrak{a} > 0$, $\mathfrak{b} \geq 0$, and $\mathfrak{d} > 0$ are at our disposal and do not depend on h_e, p_e or k. The parameter $\mathfrak{a}$ still needs be chosen with care, in particular it has to be sufficiently large (see [25, Remark 2.2]).

From now on we will assume that this is the case. The choice of $\mathfrak{b}$ and $\mathfrak{d}$ is less critical, and we consider them to be of size $\mathcal{O}(1)$ (cf. Theorem 2.3.5).

Remark 2.3.1. In [25, (2.8)] β and δ are defined to be constant on the whole skeleton:

$$\beta = \mathfrak{b}\frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}, \qquad \delta = \mathfrak{d}\frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}. \tag{2.3.6}$$

In (2.3.5) we defined them to depend on the local meshwidth and local polynomial degree. Therefore they are smaller, which in turn weakens the DG-norm, but it has no affect on the analysis or proofs given in [25]. That is to say, all statements of [25] remain to be true with the use of β, δ from (2.3.5) instead of (2.3.6), but the DG-norm and consequently the convergence results are weaker in this case.

Remark 2.3.2. A derivation of the formulation in the case of local plane waves instead of polynomials can be found in [16, 17]. It is related to the so called ultra weak formulation. The idea is to use integration by parts and replace derivatives on the boundary with appropriate numerical fluxes. In [25] the method was adapted for *hp*-finite elements. For a general discussion on such DG methods see [4, 11].

2.3.2 Adjoint Helmholtz Problem

The adjoint Helmholtz problem and its approximabilty play a major in the analysis.

Definition 2.3.3. *Let $w \in L^2(\Omega)$. The* **adjoint Helmholtz problem** *is given by: Find $z \in H^1(\Omega)$ such that*

$$a(v, z) = (v, w)_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega). \tag{2.3.7}$$

With this notation we introduce the adjoint solution operator N_k^*, which maps the right-hand side $w \in L^2(\Omega)$ to its solution z:

$$N_k^*(w) := z.$$

Moreover, we call

$$\sigma_k^*(S) := \sup_{w \in L^2(\Omega)} \inf_{\Psi_S \in S} \frac{k\|N_k^*(w) - \Psi_S\|_{DG^+}}{\|w\|_{L^2(\Omega)}} \tag{2.3.8}$$

the **adjoint approximation property**.

We will need that the discrete adjoint formulation is consistent:

Lemma 2.3.4 (Adjoint consistency). *Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal domain and let $w \in L^2(\Omega)$. Then (2.3.7) is a well-posed problem. Denote its solution by z. Then z satisfies*

$$a_{\mathcal{T}}(v, z) = (v, w)_{L^2(\Omega)} \qquad \forall v \in H_{\mathcal{T}}^{3/2+\varepsilon}(\Omega).$$

PROOF. This follows from Remark 2.6 and Lemma 2.7 in [25]. □

With Remark 2.3.1 in mind, let us now quote the following result of the a priori analysis conducted in [25, Section 3]. The theorem states that a solution to the discrete problem exists, and that the method is in some sense quasi-optimal [25, Theorem 3.8]:

Theorem 2.3.5. *Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal Lipschitz domain, and let $S = S^{\boldsymbol{p}}(\mathcal{T})$. Let the assumptions on α, β, and δ from Section 2.3.1 be satisfied, and assume moreover that $\|\delta\|_{L^\infty(\Omega)} < 1/3$. Then (2.3.1) has a unique solution $u_{\mathcal{T}} \in S^{\boldsymbol{p}}(\mathcal{T})$. Additionally, there exist constants $C, C^* > 0$ independent of $h_{\mathcal{T}}$, $p_{\mathcal{T}}$, and k such that if $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T})) < C^*$ then*

$$\|u - u_{\mathcal{T}}\|_{DG} \leq C \inf_{v \in S^{\boldsymbol{p}}(\mathcal{T})} \|u - v\|_{DG^+}.$$

2.4 Stability and Convergence

In [25] it was announced without proofs that their a priori analyis, performed for smooth domains, can be extended to the case of convex polygons by means of the techniques proposed in [27], where convex polygons were considered in a conforming FEM setting. It is the purpose of this section to carry this out in detail. In order to do so we will adapt proofs from [27] and [23, Chapter 3] to fit our situation. The assumption that the domain has smooth boundary, has the advantage that the solution is smooth enough such that it can be approximated well on uniform meshes. In the case of polygonal domains on the other hand, one observes singularities at the apices, which is why we need to work with the geometric meshes defined in Section 1.1.3, and the weighted Sobolev spaces introduced in Section 1.2.3. Our goal is to prove the following theorem:

Theorem 2.4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a convex polygonal domain. Let $\mathcal{T}(h, \sigma, L)$ be an admissible γ-shape regular geometric triangulation of Ω as in Definition 1.1.5. Let $k \geq k_0 > 1$, $p \in \mathbb{N}$, $p \geq 3$, and assume that there exist constants $C_1, C_2 > 0$ such that*

$$\frac{kh}{p} \leq C_1 \tag{2.4.1a}$$

$$p \leq C_2 L. \tag{2.4.1b}$$

Further assume that δ in (2.3.5) satisfies $\|\delta\|_{L^\infty(\Omega)} < 1/3$. Then, for every $\mu \in (0,1)$ there exist constants $b, c, C, \lambda > 0$, independent of k, h, L, and p such that for every $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ it holds for the solution u of (2.1.1) that

$$\begin{aligned}\inf_{v \in S^p(\mathcal{T})} \|u - v\|_{DG^+} &\leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right) \\ &\times \left(\frac{kh}{p} + (h^{1-\mu} + h^2) \left(h^\mu p k \sigma^L \left(1 + \sqrt{k\sigma^L h} + k\sigma^L h \right) + p k e^{chk - bp} + \frac{kh^\mu}{\lambda p} \left(\frac{kh}{\lambda p} \right)^{p-1} \right) \right). \end{aligned} \tag{2.4.2}$$

Before we prove this theorem, let us discuss its implications.

Theorem 2.4.2. *Let the assumptions of the previous theorem be satisfied. Let the grading factor $\sigma \in (0,1)$, let $\mu \in (0,1)$, and let $C_3 > 0$. Then there exist constants $C_4, C_5, C_6 > 0$ independent of k, h, L, and p such that if*

$$\frac{kh}{p} \leq C_4, \qquad \log(k) + 1 \leq C_5 p \leq C_6 L, \tag{2.4.3}$$

then (see (2.3.8)*)*

$$\sigma_k^* \left(S^p(\mathcal{T}(h, \sigma, L)) \right) \leq C_3. \tag{2.4.4}$$

Denote the unique solution of (2.3.1) *by $u_{\mathcal{T}}$. If additionally $C_3 \leq C^*$, where C^* is the constant from Theorem 2.3.5, then there exists $C > 0$ independent of k, h, and p such that*

$$\|u - u_{\mathcal{T}}\|_{DG} \leq C \frac{h^{1-\mu} + h}{p} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right). \tag{2.4.5}$$

Proof. According to the proof of [25, Corollary 4.9], Theorem 2.2.1 is applicable to the solution of the adjoint problem as well (with $g \equiv 0$ in this case). Hence Theorem 2.4.1 gives the existence of the constants C_4, C_5, C_6: First we assume $h < 1$ and consider k times the right-hand side of (2.4.2) without the term $\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}$, as $g \equiv 0$ and $\|f\|_{L^2(\Omega)}$ cancels (cf. (2.3.8)). Polynomial growth in k can then be absorbed by the exponential terms $\exp(chk - bp)$ and $(kh/(\lambda p))^{p-1}$ upon adjusting C_4, C_5 in (2.4.3) (also use $kh \leq C_4 p$). Moreover, polynomial growth in p and k is absorbed by the factor σ^L if C_6 is small enough. We are then left with an estimate of the type

$$\sigma_k^*(S^p(\mathcal{T}(h, \sigma, L))) \leq C(k_0^{-n} + C_4),$$

where n depends on C_4, C_5, C_6 and can be chosen arbitrarily large by decreasing those constants. This gives (2.4.4) if $h < 1$. We turn to the case $h \geq 1$. According to (2.4.3) we have $h \leq C_4 p / k_0$. Thus, polynomial terms in h amount to polynomial growth in p, and can again be absorbed as in the first case. The second part of the statement is then an immediate consequence of Theorem 2.3.5 and Theorem 2.4.1, as we may assume

$$(h^{1-\mu} + h^2) \left(h^\mu p k \sigma^L \left(1 + \sqrt{k \sigma^L h} + k \sigma^L h \right) + p k e^{chk - bp} + \frac{k h^\mu}{\lambda p} \left(\frac{kh}{\lambda p} \right)^{p-1} \right) \lesssim \frac{h^{1-\mu}}{p}.$$

This is shown as above and by adjusting constants once more. □

Remark 2.4.3. For a fixed number $J \in \mathbb{N}$ of apices, the resolution condition in Theorem 2.4.2 implies a minimum of

$$\mathcal{O}(k^2) + \mathcal{O}(J \log(k)^3) \qquad \text{for } k \to \infty$$

degrees of freedom: From the condition $kh/p \leq C_4$ we obtain in two dimensions $\mathcal{O}(k^2)$ degrees of freedom belonging to the uniform mesh, viz $(p+2)(p+1)/2$ per element of which there are in $\mathcal{T}_{unif}$ at least $\mathcal{O}(h^{-2}) \lesssim \mathcal{O}(k^2/p^2)$ (if we choose h as large as possible). The number of elements in $\mathcal{T}_{geo}$ is bounded by $\mathcal{O}(JL) \sim \mathcal{O}(J\log(k))$, and with the polynomial degree $p \sim \log(k)$ we therefore get at most $\mathcal{O}(J\log(k)^3)$ degrees of freedom in this case. This is a significant improvement over the $\mathcal{O}(k^4)$ degrees of freedom one needs in the case of h refinement, as we heuristically illustrated in Section 2.2.1. In terms of DOF vs. error, the pollution effect is thus avoided with this choice of finite element space, at least up to the factor $\mu > 0$ which can be chosen arbitrarily small.

Remark 2.4.4. Theorem 2.4.2 was stated with the assumptions that Ω is convex, $f \in L^2(\Omega)$, and $g \in H^{1/2}(\partial\Omega)$. This implies $u \in H^2(\Omega)$. With stronger requirements, respectively for smoother solutions, one probably obtains better approximation rates. However, this does not follow directly from the present analysis.

Let us come to the proof of Theorem 2.4.1. We distinguish between the approximation on $\mathcal{T}_{unif}(h,\sigma,L)$, the approximation on elements abutting at apices, and the approximation on $\mathcal{T}_{geo}(h,\sigma,L)$ without those elements. The next lemma treats the latter.

Lemma 2.4.5. *Let $u \in \mathcal{B}_{\boldsymbol{\mu},k}(C_u,\nu)$, where this space is as in Definition 1.2.1. Let $A = A_j$ be an apex of Ω and set $\mu := \mu_j$. Let $\mathcal{T} = \mathcal{T}(h,\sigma,L)$ be a geometric γ-shape regular triangulation of Ω and denote by $\mathcal{S}$ the elements $K \in \mathcal{T}$ such that $\operatorname{dist}(A,K) \leq h$ and $A \notin \overline{K}$. Then there exist constants $b,c,C > 0$, depending on μ,ν, γ, and c_{geo} (cf. (1.1.6)) but independent of k,h, and p, such that there exists a function $v \in S^p(\mathcal{S})$ with*

$$\left(\sum_{K\in\mathcal{S}} k^2\|u-v\|^2_{L^2(K)} + |u-v|^2_{H^1(K)} + (k+h_K^{-1})\|u-v\|^2_{L^2(\partial K)} + h_K\|\nabla((u-v)|_K)\|^2_{L^2(\partial K)}\right)^{1/2}$$
$$\leq CC_u\Big((k+1)\left(h+h^{1-\mu}\right) + (k+1)^2\left(h^2+h^{2-\mu}\right)\Big)e^{-bp}e^{ch(k+1)}.$$

Proof. Let $K \in \mathcal{S}$. Let $\Pi^\infty_{p,\mathcal{T}} : H^1(\Omega) \to S^p(\mathcal{T})$ be the operator defined in [23, (3.3.3)]. By choosing 'ε' $= 1/(k+1)$ in [23, Lemma 3.4.7], we get as in the proof of [23, Lemma 3.4.7] that for some constants $b,c > 0$ there holds

$$\|u|_K - \Pi^\infty_{p,\mathcal{T}}u|_K\|_{L^\infty(K)} + h_K\|\nabla(u|_K - \Pi^\infty_{p,\mathcal{T}}u|_K)\|_{L^\infty(K)}$$
$$\leq CC_u(k+1)\left(h_K(k+1) + (h_K(k+1))^{1-\mu}\right)e^{-bp}e^{ch_K(k+1)}.$$

Let $e_K := u|_K - \Pi^{\infty}_{p,\mathcal{T}} u|_K$. Then, with [23, Lemma 3.4.6] we get

$$\begin{aligned}\sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} h_K\|\nabla(e_K)\|^2_{L^2(\partial K)} &\lesssim \sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} h_K^2\|\nabla(e_K)\|^2_{L^\infty(K)}\\ &\lesssim \sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} C^2C_u^2(k+1)^4\left(h_K^2+h_K^{2-2\mu}\right)e^{-2bp}e^{2ch_K(k+1)}\\ &\lesssim C_u^2(k+1)^4\left(h^2+h^{2-2\mu}\right)e^{-2bp}e^{2ch(k+1)}.\end{aligned}$$

Similar computations lead to the estimates

$$\begin{aligned}\sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} k^2\|e_K\|^2_{L^2(K)} &\le CC_u^2(k+1)^4\left(h^4+h^{4-2\mu}\right)e^{-2bp}e^{2ch(k+1)},\\ \sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} |e_K|^2_{H^1(K)} &\le CC_u^2(k+1)^2\left(h^2+h^{2-2\mu}\right)e^{-2bp}e^{2ch(k+1)},\\ \sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} k\|e_K\|^2_{L^2(\partial K)} &\le CC_u^2(k+1)^3\left(h^3+h^{3-2\mu}\right)e^{-2bp}e^{2ch(k+1)},\\ \sum_{\{K\in\mathcal{T}:A\notin\overline{K}\}} h_K^{-1}\|e_K\|^2_{L^2(\partial K)} &\le CC_u^2(k+1)^2(h^2+h^{2-2\mu})e^{-2bp}e^{2ch(k+1)}.\end{aligned}$$

Summing over all elements concludes the proof. □

Next, we approximate the function on elements in the uniform part of the mesh.

Lemma 2.4.6. *Let $\mathcal{T}=\mathcal{T}(h,\sigma,L)$ be a geometric γ-shape regular triangulation of Ω, let $p\in\mathbb{N}$, and assume that (2.4.1) is satisfied. Let $u\in\mathcal{B}_{\boldsymbol{\mu},k}(C_u,\nu)$ and set $\mu:=\min_j\mu_j$. Then there exist constants $c,C,b,\lambda>0$ independent of k,h and p, but depending on μ,γ,ν, and Ω, and a function $v\in S^p(\mathcal{T}_{unif})$ such that*

$$\begin{aligned}&\left(\sum_{K\in\mathcal{T}_{unif}} k^2\|u-v\|^2_{L^2(K)}+|u-v|^2_{H^1(K)}+(k+h_K^{-1})\|u-v\|^2_{L^2(\partial K)}+h_K\|\nabla(u-v)\|^2_{L^2(\partial K)}\right)^{1/2}\\ &\quad\le CC_uk\left((hk)^{1-\mu}e^{-bp}+\left(\frac{kh}{\lambda p}\right)^p\right)\end{aligned}$$

Proof. We proceed along the lines of the proof of [27, Proposition 5.6]. Fix $K\in\mathcal{T}_{unif}$. Then $d:=\operatorname{dist}(K,A_j)\ge ch$ for all vertices A_j, and some fixed $c>0$. We define

$$C_K^2:=\sum_{n\ge0}\left(\frac{1}{2\nu\max\{k,n\}}\right)^{2(n+2)}\|\Phi_{n,\boldsymbol{\mu},k+1}\nabla^{n+2}u\|^2_{L^2(K)},$$

and observe that due to (1.2.6)

$$\sum_{K\in\mathcal{T}} C_K^2 \le 2C_u^2. \tag{2.4.6}$$

Similar as in the proof of [27, Proposition 5.6], by distinguishing several cases for the values of n and k, and by using the Definition of $\Phi_{n,\boldsymbol{\mu},k+1}$, it can be shown that there exists $C_1(\Omega) > 0$ such that for all $n \in \mathbb{N}_0$

$$\|\nabla^{n+2}u\|_{L^2(K)} \lesssim C_K \min\{1, kd\}^{2-\mu}(C_1(\Omega)\nu)^{n+2} \max\left\{k, \frac{n}{d}\right\}^{n+2},$$

and therefore

$$\|\nabla^{n+2}(u\circ F_K)\|_{L^2(\hat{K})} \lesssim \frac{C_K}{h} \min\{1, kd\}^{2-\mu}(C_1(\Omega)\nu h)^{n+2} \max\left\{k, \frac{n}{d}\right\}^{n+2}.$$

With this, Lemma C.2 in [26] gives a polynomial $v \in \mathcal{P}_p(K)$, and constants $C, \lambda > 0$, depending on ν, such that for $m \in \{0, 1, 2\}$, due to $\|u - v\|_{W^{m,2}(K)} \le Ch\|u - v\|_{W^{m,\infty}(K)}$, it holds that

$$h^m|u - v|_{H^m(K)} \le CC_K \min\{1, kd\}^{2-\mu}\left(\left(\frac{h/d}{\lambda + h/d}\right)^{p+1} + \left(\frac{kh}{\lambda p}\right)^{p+1}\right). \tag{2.4.7}$$

Distinguishing between the cases $d \ge 1/k$ and $d < 1/k$, one can check that

$$(k + h^{-1}) \min\{1, kd\}^{2-\mu}\left(\frac{h/d}{\lambda + h/d}\right)^{p+1} \lesssim k \min\{1, kh\}^{1-\mu}\left(\frac{1}{c\lambda + 1}\right)^p$$

(for details see again the proof of [27, Proposition 5.6]). Hence we obtain with (2.4.7) and (2.4.1)

$$\begin{aligned}(k + h^{-1})\|u - v\|_{L^2(K)} &+ |u - v|_{H^1(K)} + h|u - v|_{H^2(K)} \\ &\lesssim C_K k\left(\min\{1, kh\}^{1-\mu}\left(\frac{1}{c\lambda + 1}\right)^p + \left(\frac{kh}{\lambda p}\right)^p\right).\end{aligned} \tag{2.4.8}$$

Together with the trace inequality

$$\|w\|^2_{L^2(\partial K)} \le C\left(\|w\|_{L^2(K)}|w|_{H^1(K)} + h_K^{-1}\|w\|^2_{L^2(K)}\right) \tag{2.4.9}$$

(see, e.g., Lemma 3.1.4), we get

$$\sqrt{h}\|\nabla((u - v)|_K)\|_{L^2(\partial K)} \le CC_K k\left(\min\{1, kh\}^{1-\mu}\left(\frac{1}{c\lambda + 1}\right)^p + \left(\frac{kh}{\lambda p}\right)^p\right),$$

and similary, using either the estimate for $k\|u-v\|_{L^2(K)}$ or $h^{-1}\|u-v\|_{L^2(K)}$ from (2.4.8), we obtain

$$\sqrt{k}\|u - v\|_{L^2(\partial K)} \le CC_K k\left(\min\{1, kh\}^{1-\mu}\left(\frac{1}{c\lambda + 1}\right)^p + \left(\frac{kh}{\lambda p}\right)^p\right),$$

$$h^{-1/2}\|u - v\|_{L^2(\partial K)} \le CC_K k\left(\min\{1, kh\}^{1-\mu}\left(\frac{1}{c\lambda + 1}\right)^p + \left(\frac{kh}{\lambda p}\right)^p\right).$$

Summing over all elements and using (2.4.6) concludes the proof. □

We are now in a position to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. With Theorem 2.2.1 we decompose $u = u_{\mathcal{A}} + u_{H^2}$ and approximate both parts separately.
1st Step: We start with the approximation of u_{H^2}. A more general proof with the use of an element by element construction can be found in [25, Theorem 4.11] (the assumption that Ω has analytic boundary is not necessary for this part of the proof). However, since we construct another suitable interpolation operator later on (with stricter assumptions on the domain than in [25]), we show the estimate anyway. Theorem 3.1.10 gives an operator $I_2^{hp,0} : H^2(\Omega) \to S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ with

$$\|v - I_2^{hp,0} v\|_{H^m(K)} \leq C \left(\frac{h_K}{p_K}\right)^{2-m} |v|_{H^2(\omega_K)} \qquad \forall m \in \{0,1,2\},$$

$$\|v - I_2^{hp,0} v\|_{L^2(e)} + \frac{h_e}{p_e} \|\nabla((v - I_2^{hp,0} v)|_K)\|_{L^2(e)} \leq C \left(\frac{h_e}{p_e}\right)^{3/2} |v|_{H^2(\omega_e)}$$

for every $v \in H^2(\Omega)$, $K \in \mathcal{T}$, and $e \in \mathcal{E}(K)$. With $(k\alpha)^{-1}|_e \sim h_e/p^2$, $\beta/k|_e \sim h_e/p$, $\delta/k|_e \sim h_e/p$, and $h_e \lesssim h$ we get

$$\begin{aligned}
\|\nabla(u_{H^2} - I_2^{hp,0} u_{H^2})\|_{L^2(K)}^2 &\leq C \frac{h^2}{p^2} |u_{H^2}|_{H^2(\omega_K)}^2, \\
k^2 \|u_{H^2} - I_2^{hp,0} u_{H^2}\|_{L^2(K)}^2 &\leq C \frac{k^2 h^2}{p^2} \frac{h^2}{p^2} |u_{H^2}|_{H^2(\omega_K)}^2, \\
\|(\delta/k)^{1/2} \partial_{\boldsymbol{n}} (u_{H^2} - I_2^{hp,0} u_{H^2})\|_{L^2(e)}^2 &\leq C \frac{h^2}{p^2} |u|_{H^2(\omega_e)}^2, \\
k \|u_{H^2} - I_2^{hp,0} u_{H^2}\|_{L^2(e)}^2 &\leq C \frac{kh}{p} \frac{h^2}{p^2} |u_{H^2}|_{H^2(\omega_e)}^2, \\
\|(\beta/k)^{1/2} [\![\nabla_{\mathcal{T}} (u_{H^2} - I_2^{hp,0} u_{H^2})]\!]_N\|_{L^2(e)} &\leq C \frac{h^2}{p^2} |u_{H^2}|_{H^2(\omega_e)}^2, \\
\|(k\alpha)^{-1/2} \{\nabla_{\mathcal{T}} (u_{H^2} - I_2^{hp,0} u_{H^2})\}\|_{L^2(e)}^2 &\leq C \frac{h^2}{p^3} |u_{H^2}|_{H^2(\omega_e)}^2,
\end{aligned}$$

where, depending on the term, e is either an edge on the boundary or an interior edge. The error of the jump of $u_{H^2} - I_2^{hp,0} u_{H^2}$ across an edge is zero because $I_2^{hp,0} u_{H^2} \in C^0(\Omega)$. Hence all terms in (2.3.4) are accounted for, and by summing over all elements and edges and with $kh/p \leq C_1$ and (2.2.4c) we obtain

$$\|u_{H^2} - I_2^{hp,0} u_{H^2}\|_{DG^+} \leq C \left(\frac{h}{p} + \left(\frac{h}{p}\right)^2\right) \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right).$$

2nd Step: We approximate $u_{\mathcal{A}}$. First, let us introduce the following norm:

$$\|w\|_{\tilde{DG}(K)}^2 := k^2 \|w\|_{L^2(K)}^2 + |w|_{H^1(K)}^2 + \left(\frac{1}{h_K} + k\right) \|w\|_{L^2(\partial K)}^2 + \frac{h_K}{p} \|\nabla w\|_{L^2(\partial K)}^2.$$

Then for $v_1, v_2 \in S^p(\mathcal{T})$ we have

$$\|v_1 - v_2\|^2_{DG^+} \lesssim p^2 \sum_{K\in\mathcal{T}} \|v_1 - v_2\|^2_{\tilde{DG}(K)},$$

where p^2 comes from the weight α, and we generously multiply it with all terms. Let us start with the elements touching an apex. With (2.2.4c) and (2.2.1a) we get

$$\begin{aligned}|u_{\mathcal{A}}|_{H^2(\Omega)} &= |u - u_{H^2}|_{H^2(\Omega)} \leq |u|_{H^2(\Omega)} + |u_{H^2}|_{H^2(\Omega)} \\ &\leq C\Bigg((1+k)\underbrace{\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right) + \|g\|_{H^{1/2}(\partial\Omega)}}_{=:C_{f,g}}\Bigg).\end{aligned}$$

Now let $K \in \mathcal{T}$ and $A_j \in \overline{K}$. With Theorem 3.1.10 there exists a polynomial $v_K \in \mathcal{P}_3(K)$ such that

$$\begin{aligned}&\|u_{\mathcal{A}} - v_K\|_{L^2(K)} \lesssim C_{f,g}(k+1)h_K^2, \qquad |u_{\mathcal{A}} - v_K|_{H^1(K)} \lesssim C_{f,g}(k+1)h_K, \\ &\|u_{\mathcal{A}} - v_K\|_{L^2(\partial K)} \lesssim C_{f,g}(k+1)h_K^{3/2}, \qquad \|\nabla(u_{\mathcal{A}} - v_K)\|_{L^2(\partial K)} \leq C_{f,g}(k+1)h_K^{1/2},\end{aligned}$$

and thus with $h_K \sim h\sigma^L$ in this case

$$\|u_{\mathcal{A}} - v_K\|_{\tilde{DG}(K)} \leq Cpk\left(1 + \sqrt{kh\sigma^L} + kh\sigma^L\right) h\sigma^L \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right), \tag{2.4.10}$$

where we have used $k \sim k+1$ because of $k > 1$.

Next, we consider approximation on $\mathcal{T}_{geo}$. According to Theorem 2.2.1 and Remark 2.2.2 we have $u_{\mathcal{A}} \in \mathcal{B}_{\boldsymbol{\mu},\nu}(C_{u_{\mathcal{A}}}, \nu)$, where $\mu_j := \mu$ for all $j \in \{1, \ldots, J\}$, μ is arbitrary in $(0,1)$, and

$$C_{u_{\mathcal{A}}} = \left(\frac{\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}}{k}\right).$$

Denote the set of elements touching an apex by $\mathcal{T}_A$. Then, with Lemma 2.4.5 we get a function $v_{geo} \in S^p(\mathcal{T}_{geo}\backslash\mathcal{T}_A)$ such that

$$p\left(\sum_{K\in\mathcal{T}_{geo}\backslash\mathcal{T}_A} \|u_{\mathcal{A}} - v_{geo}\|^2_{\tilde{DG}(K)}\right)^{1/2} \leq Cpk(h^{1-\mu} + h^2)e^{chk-bp}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right). \tag{2.4.11}$$

Finally, we consider approximation on $\mathcal{T}_{unif}$. With Lemma 2.4.6 we obtain $v_{unif} \in S^p(\mathcal{T}_{unif})$ such that

$$p\left(\sum_{K\in\mathcal{T}_{unif}} \|u_{\mathcal{A}} - v_{unif}\|^2_{\tilde{DG}(K)}\right)^{1/2} \leq Cp\left((hk)^{1-\mu}e^{-bp} + \left(\frac{kh}{\lambda p}\right)^p\right)\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right). \tag{2.4.12}$$

Combining the estimates from Step 1 with (2.4.10), (2.4.11), and (2.4.12) concludes the proof. □

Chapter 3

Approximation Theorems

This chapter is devoted to two types of approximation results. The first type of approximation we consider, is Clément type interpolation. That is, we approximate Sobolev functions in $W^{n,q}(\Omega)$ with piecewise polynomials. The approximant is constructed such that it simultaneously approximates higher derivatives up to order $n-1$ and is H^2-conforming. We have already used the first property in the proof of Theorem 2.4.1. For the purposes of this thesis, the advantage of the second property lies in the fact that the normal component of the gradient does not jump.

The second result is concerned with the approximation of discontinuous piecewise polynomials by globally continuous piecewise polynomials. Typically in the a posteriori analysis of DG methods, the error is estimated by inserting a continuous function and using the triangle inequality. Therefore such a result needs to be established.

3.1 C^1 Clément Type Interpolant

In order to obtain our interpolant, we will follow the proof for the *hp*-Clément interpolant of $W^{1,q}$ functions given in [24] and adapt it where necessary. To begin with, we introduce some notation and construct element mappings as well as reference elements that we use in the following lemmata. They differ from the usual setting, in that we will have a set of reference elements and an infinite set of reference patches rather than a finite one. In this section we adhere to the convention $1/\infty := 0$. Moreover, in what follows, we shall assume that $\mathcal{T}$ is an admissible γ-shape regular triangulation of Ω.

3.1.1 Notation

Let $K \in \mathcal{T}$ and $e \in \mathcal{E}(K)$. Then there exists a unique map $\tilde{F}_{K,e} : \mathbb{R}^2 \to \mathbb{R}^2$ and a unique similar triangle $\tilde{K}$ with the properties that $\tilde{K} \subseteq \big\{(x,y) \in \mathbb{R}^2 : y \geq 0\big\}$, $\tilde{F}_{K,e}$ is a composition of a dilation, a rotation, and a translation such that $\tilde{F}_{K,e}([-1,1] \times 0) = e$ and $\tilde{F}_{K,e}(\tilde{K}) = K$. Clearly $\tilde{K}$ and $\tilde{F}_{K,e}$ are then well-defined. In case e is not of interest to us, we pick an arbitrary but fixed $e \in \mathcal{E}(K)$ and use the notation $\tilde{F}_K := \tilde{F}_{K,e}$. The triangle $\tilde{K}$ has the three vertices $\tilde{V}_1(\tilde{K})$, $\tilde{V}_2 := (-1,0)$ and $\tilde{V}_3 := (1,0)$. Since $\mathcal{T}$ is γ-shape regular, there is a lower bound for the interior angles of $\tilde{K}$.

Therefore there exists $R(\gamma) > 0$ and $0 < \Theta(\gamma) < \pi/2$ such that $V_1 \in \tilde{a}(\gamma)$ with

$$\tilde{a}(\gamma) := \left\{ r(\sin(\theta), \cos(\theta)) \in \mathbb{R}^2 : \frac{1}{R(\gamma)} \leq r \leq R(\gamma) \text{ and } \theta \in \left[\frac{\pi}{2} - \Theta(\gamma), \frac{\pi}{2} + \Theta(\gamma)\right] \right\}.$$

Of course $\tilde{a}(\gamma)$ is a compact subset of $\big\{(x, y) \in \mathbb{R}^2 : y > 0\big\}$. Now let ω_V be the patch belonging to $V \in \mathcal{N}(\mathcal{T})$. Then we pick an edge $e \in \mathcal{E}(V)$ and denote by $\tilde{F}_V : \mathbb{R}^2 \to \mathbb{R}^2$ the unique function that is a composition of a dilation, a rotation, and a translation such that $\tilde{F}_V(0) = V$ and $\tilde{F}_V([0, 1] \times \{0\}) = e$. The reference patch belonging to V is then defined as $\tilde{\omega}_V := \tilde{F}_V^{-1}(\omega_V)$. The patch $\tilde{\omega}_V$ naturally inherits a triangulation $\mathcal{T}_{\tilde{\omega}_V}$ from the triangulation $\mathcal{T}|_{\omega_V}$ of ω_V. The situation and the difference between the two patch maps F_V and $\tilde{F}_V$ are depicted in Figure 3.1.

Definition 3.1.1 (Reference patches of the second type). *Let the set of* **reference triangles**, *denoted by $\tilde{A}(\gamma)$, be given as the set of triangles $\tilde{K}$ with the three vertices $\tilde{V}_2 = (-1, 0)$, $\tilde{V}_3 = (1, 0)$, and $\tilde{V}_1 \in \tilde{a}(\gamma)$ arbitrary. Moreover, we introduce the infinite set $\tilde{B}(\gamma)$ of* **reference patches** *of the second type: Let Υ be the set of tupels $(\Omega, \mathcal{T})$, such that $\Omega \subseteq \mathbb{R}^2$ is a polygonal domain with* (1.0.1), *and $\mathcal{T}$ is an admissible γ-shape regular triangulation of Ω. Then with the above notation*

$$\tilde{B}(\gamma) := \bigcup_{(\Omega,\mathcal{T})\in\Upsilon} \bigcup_{V\in\mathcal{N}(\mathcal{T})} \{(\tilde{\omega}_V, \mathcal{T}_{\tilde{\omega}_V})\}.$$

If we do not care about the (not necessarily unique) triangulation of a patch we write $\tilde{\omega} \in \tilde{B}(\gamma)$, and mean that $(\tilde{\omega}, \mathcal{T}_{\tilde{\omega}}) = (\tilde{\omega}_V, \mathcal{T}_{\tilde{\omega}_V})$ for some $(\tilde{\omega}_V, \mathcal{T}_{\tilde{\omega}_V}) \in \tilde{B}(\gamma)$.

Per definition, we have for every γ-shape regular mesh $\mathcal{T}$, $K \in \mathcal{T}$, and $V \in \mathcal{N}(\mathcal{T})$, that $\tilde{K} \in \tilde{A}(\gamma)$ and $(\tilde{\omega}_V, \mathcal{T}_{\tilde{\omega}_V}) \in \tilde{B}(\gamma)$. The next lemma gathers some elementary facts.

Lemma 3.1.2. *We employ the above notation.*

(i) There exists a reference square $\hat{S}(\gamma) \subseteq \mathbb{R}^2$ with center $\mathbf{0} \in \mathbb{R}^2$ such that $\tilde{\omega} \in \hat{S}(\gamma)$ for every $\tilde{\omega} \in \tilde{B}(\gamma)$.

(ii) The maps $\tilde{F}_V$ and $\tilde{F}_{K,e}$ are conformal, they are in $C^\infty(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$, and they have the properties

$$h_V^{-1} \|\tilde{F}_V'\|_{L^\infty(\mathbb{R}^2)} + h_V \|(\tilde{F}_V')^{-1}\|_{L^\infty(\mathbb{R}^2)} \sim 1, \tag{3.1.1}$$

$$h_V^{-2} |\det \tilde{F}_V'| + h_V^2 |\det(\tilde{F}_V')^{-1}| \sim 1, \tag{3.1.2}$$

where the constants hidden in the $\sim$-notation only depend on γ and q.

(iii) Let $v \in \mathcal{P}_p(\mathbb{R}^2)$. Then $v \circ \tilde{F}_V$ and $v \circ \tilde{F}_V^{-1}$ are in $\mathcal{P}_p(\mathbb{R}^2)$.

(iv) Let $n \in \mathbb{N}$, $q \in [1, \infty]$ and $v \in W^{n,q}(\omega_V)$ for $V \in \mathcal{N}(\mathcal{T})$. Then $\tilde{v} := v \circ \tilde{F}_V \in W^{n,q}(\tilde{\omega}_V)$ and for every $m \in \{0, \ldots, n\}$

$$|\tilde{v}|_{W^{m,q}(\tilde{\omega}_V)} \sim h_V^{m-2/q} |v|_{W^{m,q}(\omega_V)}, \tag{3.1.3}$$

where the constant hidden in the $\sim$-notation only depends on n, γ, and q.

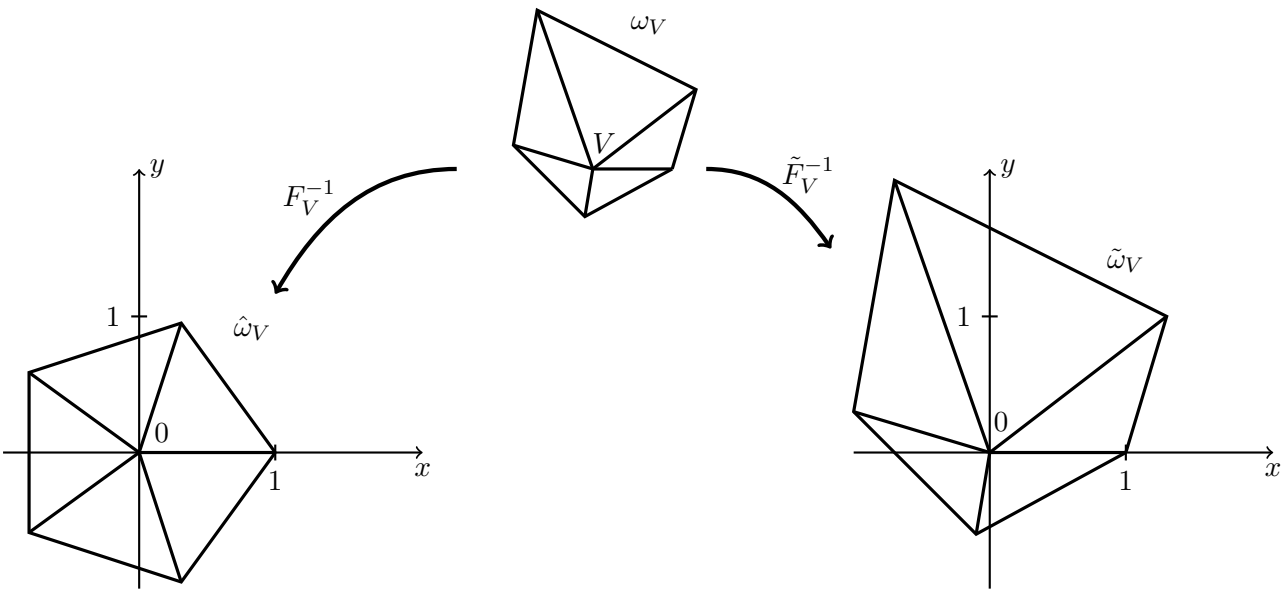


Figure 3.1: Behaviour of the two patch maps $F_V : \hat{\omega}_V \to \omega_V$ and $\tilde{F}_V : \tilde{\omega}_V \to \omega_V$.

Proof.

(i) The property of a triangulation $\mathcal{T}$ being γ-shape regular implies that the ratio of the diameter of neighbouring elements is bounded from below and from above by constants depending on γ only. Also, the lengths of different edges of the same triangle are comparable in size. The patch $\tilde{\omega}$ is a scaled version of some patch ω_V with these properties, and thus they also hold for $\tilde{\omega}$. Since one (possibly interior) edge of the patch $(\tilde{\omega}, \mathcal{T}_{\tilde{\omega}})$ has length one, $\hat{S}(\gamma)$ exists.

(ii) Per definition the maps $\tilde{F}_V$ and $\tilde{F}_{K,e}$ consist of a dilation, a rotation, and a translation, such that some edge e with length h_e is mapped to $[0,1] \times \{0\}$. Therefore the dilation must be of size $h_e \sim h_V$, and (3.1.1) as well as (3.1.2) hold. It is also clear that these maps are conformal (as a composition of conformal maps) and that they are in $C^\infty(\mathbb{R}^2)$.

(iii) The composition of a dilation, a rotation, or a translation with a polynomial of degree p is a polynomial of degree p.

(iv) For the last item let $v \in W^{n,q}(\omega_V)$. First assume that $\tilde{F}_V$ is the composition of a rotation and a translation. Then for $m \in \{0, \dots, n\}$ we obviously have

$$|\tilde{v}|_{W^{m,q}(\tilde{\omega})} = |v|_{W^{m,q}(\omega_V)}.$$

Thus it is sufficient to consider $\tilde{F}_V(\boldsymbol{x}) := h\boldsymbol{x}$ with $h \sim h_V$. Let $\alpha \in \mathbb{N}_0^2$ be a multiindex with $|\alpha| = m$, and denote by $D^\alpha v$ the respective (weak) partial derivative of v. Then we get with

$D^\alpha \tilde{v} = h^{|\alpha|}(D^\alpha v) \circ \tilde{F}_V$ and for $q < \infty$

$$\begin{aligned}\int_{\tilde{\omega}_V} (D^\alpha \tilde{v})^q \,\mathrm{d}\boldsymbol{x} &= \int_{\tilde{\omega}_V} ((D^\alpha v) \circ \tilde{F}_V)^q h^{|\alpha| q} \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\omega_V} (D^\alpha v)^q h^{mq} |\det(\tilde{F}_V^{-1})'| \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\omega_V} (D^\alpha v)^q h^{mq-2} \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

This gives the assertion for $q \in [1, \infty)$. The case $q = \infty$ follows immediately from $D^\alpha \tilde{v} = h^{|\alpha|}(D^\alpha v) \circ \tilde{F}_V$.

□

3.1.2 Estimates Independent of $\tilde{\omega}$

For the proof of our C^1 interpolant we have to apply standard extension and trace results as well as the Deny-Lions Lemma on the reference patches $\tilde{\omega} \in \tilde{B}(\gamma)$. The constants appearing in these estimates should then be independent of these (infinitely many) patches. We now give proofs, that constants depending solely on γ and the smallest outer angle of the domain can be chosen. Let us motivate with an intuitive argument why this should work: The polygon $\tilde{\omega}$ is determined by its finitely many vertices. The set of possible vertices is compact in a suitable space. If the constants now depend continuously on these vertices, the desired upper bounds must exist.

First, we introduce an extension operator for Sobolev spaces.

Lemma 3.1.3 (Extension). *Let $\tilde{\omega} \in \tilde{B}(\gamma)$ with $\theta > 0$ being the smallest outer angle of this polygon. Let $v \in W^{n,q}(\tilde{\omega})$ for $q \in [1, \infty]$ and let $n \in \mathbb{N}$. Then there exists an extension $Ev \in W^{n,q}(\mathbb{R}^2)$ of v and a constant $C > 0$ depending only on n, q, γ, and θ such that*

$$\|Ev\|_{W^{n,q}(\mathbb{R}^2)} \leq C \|v\|_{W^{n,q}(\tilde{\omega})}.$$

Moreover,

$$E_{\tilde{\omega}} := \begin{cases} W^{n,q}(\tilde{\omega}) \to W^{n,q}(\mathbb{R}^2) \\ v \mapsto Ev \end{cases}$$

is a linear operator.

Proof. An inspection of the derivation of the extension operator in Section VI.3 in [32] shows that this is true (we now switch to their notation): There it is proved, that for a bounded Lipschitz domain $D \subseteq \mathbb{R}^n$, $n \geq 2$, there exists a bounded linear extension operator from $W^{k,p}(D)$ to $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and non-negative integral k . In a first step this is shown for so called special Lipschitz domains D, which are of the type $D = \{(\boldsymbol{x}, y) \in \mathbb{R}^{n+1} : y \geq \varphi(\boldsymbol{x})\}$ for some Lipschitz continuous function φ (see Theorem 5' on page 181 in [32]). It is also remarked that the region D enters the

continuity constant only via its Lipschitz bound. In a second step a covering argument is used to generalize this. With the same notation as in [32] let U_i, $i \in \mathbb{N}$ be such a covering of the boundary of D. Assume further, that D_i with $D_i \cap U_i = D \cap U_i$ are the associated local special Lipschitz domains with Lipschitz bounds M_i for $i \in \mathbb{N}$. The bounds appearing in this second part of the proof then depend on the constants ε, N, and M, which fulfill the following properties: The constant $\varepsilon > 0$ must be small enough such that $B(\boldsymbol{x}, \varepsilon) \subseteq U_i$ for some $i \in \mathbb{N}$ for every $\boldsymbol{x} \in \partial D$. The integer $N \in \mathbb{N}$ must be large enough such that no point of $\mathbb{R}^n$ lies in more than N members U_i of the covering. Finally, M must be larger than any M_i. The assertion follows, if we prove that ε, N, and M can be chosen, depending only on γ and θ such that these constants are relevant for all possible domains $\tilde{\omega}$. We observe that $\tilde{\omega}$ has the following properties:
1. It is a polygon, where the number of outer edges is bounded by some $C_1(\gamma) < \infty$.
2. The length of an edge is bounded from above by some constant $C_2(\gamma) > 0$ and from below by $1/C_2(\gamma)$.
3. The inner angles of $\tilde{\omega}$ and the outer angles of $\tilde{\omega}$ are bounded from below by some constant $C_3(\gamma, \theta) > 0$.
4. The distance of two vertices $\tilde{V}' \neq \tilde{V}''$ of $\tilde{\omega}$ is bounded from below by a constant $C_4(\gamma, \theta) > 0$.
5. The distance of a vertex $\tilde{V}' \neq \mathbf{0}$ and an outer edge e of $\tilde{\omega}$ not sharing the vertex $\tilde{V}'$, is bounded from below by a constant $C_5(\gamma, \theta) > 0$.
The first two points follow from the γ-shape regularity of $\mathcal{T}$. For the third point we remark that interior angles of elements of shape regular triangulations are bounded from both directions according to Lemma 1.1.3, which entails the same for $\tilde{\omega}$. The fourth point is a consequence of the second and the third point: Suppose this is not true. It holds that $\tilde{V}'$ and $\tilde{V}''$ are either connected via an edge e or they are not, in which case $\mathbf{0}$, $\tilde{V}'$, and $\tilde{V}''$ form a triangle (note that the vertex $\mathbf{0}$ is connected with all others by an edge). Since the length of the edges connecting $\mathbf{0}$ and $\tilde{V}'$ respectively $\mathbf{0}$ and $\tilde{V}''$ are bounded from below, the angle $\sphericalangle \tilde{V}' \mathbf{0} \tilde{V}''$, and thus some inner or outer angle (if it already is the outer angle), would have to become arbitrarily small, which contradicts point three. Now let $\delta_1 := C_5/3$, and for every vertex on the boundary we choose a ball of radius δ_1, with center the respective vertex. It is easy to see that there exists a constant $0 < \delta_2 < \delta_1/2, 1/(2C_2)$ depending on C_1 and δ_1 such that a ball of radius δ_2 with a center that lies on an edge $e \subseteq \partial\tilde{\omega}$ and is more than $\delta_1/2$ from each vertex away, does not intersect or touch any other boundary edge. We cover the part of the edges with distance from each vertex more than $\delta_1/2$ with balls of radius δ_2 such that their centers are no farther than $\delta_2/2$ and no less than $\delta_2/3$ along the boundary apart. The set of all these balls is denoted by $(U_i)_{i \in I}$, and they cover the whole boundary per construction. Now we can choose $\varepsilon > 0$ depending only on δ_2 such that $B(\boldsymbol{x}, \varepsilon) \subseteq U_i$ for every $\boldsymbol{x} \in \partial\tilde{\omega}_V$ and some $i \in I$. Also, the first two points show that the length of the boundary of $\tilde{\omega}_V$ is bounded from above. Together with the condition that the center of two balls covering the boundary are no closer than $\delta_2/3$, we then observe that there exists $N \in \mathbb{N}$ only depending on γ and θ such that $|I| < N$. It remains to check the condition on M. Let $D_i := U_i \cap D$ for every $i \in I$. Note that we chose δ_1, δ_2 small enough such that the sets D_i are connected. Upon rotating and extending D_i, we may assume without loss of generality that D_i is a special Lipschitz domain, with Lipschitz constant M_i either zero (if U_i does not contain a vertex) or bounded from above by a constant depending on the smallest possible outer angle. Hence any M_i is bounded by a global constant M depending on γ and θ only. The fact that $E_{\tilde{\omega}}$ is a linear operator is part of

the formulation of Theorem 5 in [32, p. 181]. This concludes the proof. □

Lemma 3.1.4 (Trace). *Let $\tilde{\omega} \in \tilde{B}(\gamma)$ and let $q \in [1, \infty]$. Then there exists a constant $C > 0$ depending only on γ and q (but not on $\tilde{\omega}$) such that for all $v \in W^{1,q}(\tilde{\omega})$*

$$\|v\|_{L^q(\partial\tilde{\omega})} \leq C\Big(\|v\|_{L^q(\tilde{\omega})}^{1-1/q}|v|_{W^{1,q}(\tilde{\omega})}^{1/q} + \|v\|_{L^q(\tilde{\omega})}\Big). \tag{3.1.4}$$

Proof. It is well-known that (3.1.4) holds for polygonal star shaped domains, with a constant depending on the respective domain. For a proof in the case $q = 2$, see for example [14, Lemma 3.1], the proof for $q \neq \infty$ allows the same lines. For $q = \infty$, we point out that $W^{1,\infty}$ functions are Lipschitz continuous [2, Lemma 4.28], and hence the estimate is trivial with the constant not depending on the domain since it is one. We are left with the case $q < \infty$ and want to show, that a uniform constant valid for all domains $\tilde{\omega} \in \tilde{B}(\gamma)$ exists.

Let $\tilde{K}$ be an element of the triangulation $\mathcal{T}_{\tilde{\omega}}$ of $\tilde{\omega}$, and let $F_{\tilde{K}} : \tilde{K} \to \hat{K}$ be an affine function with $F_{\tilde{K}}(\hat{K}) = \tilde{K}$. The crucial point is, that there exist constants $C_1, C_2 > 0$ depending only on γ with

$$C_1 \leq \operatorname{diam}(\tilde{K}') \leq C_2,$$

for any element $\tilde{K}'$ of $\tilde{\omega}'$ for an arbitrary $\tilde{\omega}' \in \tilde{B}(\gamma)$. This holds due to Lemma 1.1.3 and because one (possibly interior) edge of $(\tilde{\omega}, \mathcal{T}_{\tilde{\omega}})$ is $[0, 1] \times \{0\}$ and therefore of size one. Since interior angles of these triangles are bounded from below as well, we have constants $C_3, C_4 > 0$ depending only on γ with

$$C_3 \leq \inf_{\hat{K}} |\det F'_{\tilde{K}}| \leq \sup_{\hat{K}} |\det F'_{\tilde{K}}| \leq C_4,$$

and

$$C_3 \leq \|F'_{\tilde{K}}\|_{L^\infty(\hat{K})} + \|{F'_{\tilde{K}}}^{-1}\|_{L^\infty(\hat{K})} \leq C_4.$$

Using this, we observe for an edge $\tilde{e} = F_{\tilde{K}}(\hat{e})$ of $\tilde{K}$ by transforming the integrals

$$\begin{aligned}
\|v\|_{L^q(\tilde{e})} &\leq C\|v \circ F_{\tilde{K}}|_{\hat{e}}\|_{L^q(\hat{e})} \leq C\|v \circ F_{\tilde{K}}|_{\partial\hat{K}}\|_{L^q(\partial\hat{K})} \\
&\leq C\left(\|v \circ F_{\tilde{K}}\|_{L^q(\hat{K})}^{1-1/q}\|\nabla(v \circ F_{\tilde{K}})\|_{L^q(\hat{K})}^{1/q} + \|v \circ F_{\tilde{K}}\|_{L^q(\hat{K})}\right) \\
&\leq C\left(\|v\|_{L^q(\tilde{K})}^{1-1/q}\|\nabla v\|_{L^q(\tilde{K})}^{1/q} + \|v\|_{L^q(\tilde{K})}\right).
\end{aligned}$$

We have also used that $v \circ F_{\tilde{K}} \in W^{1,q}(\hat{K})$ if $v \in W^{1,q}(\tilde{K})$, which can be checked directly. We conclude the argument by summing over all edges $\tilde{e} \in \mathcal{E}(\tilde{\omega})$ respectively all elements $\tilde{K} \in \mathcal{T}_{\tilde{\omega}}$ and applying Cauchy-Schwarz for sums. □

Next, we consider the Poincaré inequality. Let $\tilde{\omega} \in \tilde{B}(\gamma)$ and $\hat{\omega} \in \hat{B}(\gamma)$ such that $F_{\tilde{\omega}}(\hat{\omega}) = \tilde{\omega}$, where $F_{\tilde{\omega}}$ is a continuous piecewise (restricted to an element) affine function that fulfills this property. This means $F_{\tilde{\omega}}^{-1}$ restricted to an element $\tilde{K} \in \mathcal{T}_{\tilde{\omega}}$ can be seen as $F_{\tilde{K}}^{-1}$ from the above proof. Then let $v \in W^{1,q}(\tilde{\omega})$ and

$$\overline{v \circ F_{\tilde{\omega}}} := \frac{1}{|\hat{\omega}|} \int_{\hat{\omega}} v \circ F_{\tilde{\omega}} \, \mathrm{d}\boldsymbol{x},$$

which is well-defined, because $L^q(\hat{\omega})$ is embedded in $L^1(\hat{\omega})$. With the Poincaré inequality there then exists a constant $C_P(\hat{\omega}, q)$ such that for $q \in [1, \infty]$

$$\|v \circ F_{\tilde{\omega}} - \overline{v \circ F_{\tilde{\omega}}}\|_{L^q(\hat{\omega})} \leq C_P(\hat{\omega}, q) |v \circ F_{\tilde{\omega}}|_{W^{1,q}(\hat{\omega})}.$$

Lemma 3.1.5 (Poincaré inequality). *Let $\tilde{\omega} \in \tilde{B}(\gamma)$ and let $q \in [1, \infty]$. Then there exists a constant $C_P > 0$ depending only on γ and q (but not on $\tilde{\omega}$) such that we have for all $v \in W^{1,q}(\tilde{\omega})$*

$$\|v - \overline{v \circ F_{\tilde{\omega}}}\|_{L^q(\tilde{\omega})} \leq C_P \|\nabla v\|_{L^q(\tilde{\omega})}. \tag{3.1.5}$$

Proof. Let $q < \infty$. We follow the arguments of [34, Proposition 3.11]. There holds

$$\begin{aligned}
\int_{\tilde{\omega}} |v - \overline{v \circ F_{\tilde{\omega}}}|^q \, \mathrm{d}\boldsymbol{x} &= \int_{\hat{\omega}} |v \circ F_{\tilde{\omega}} - \overline{v \circ F_{\tilde{\omega}}}|^q |\det F'_{\tilde{\omega}}| \, \mathrm{d}\boldsymbol{x} \\
&\leq C_P(\hat{\omega}, q)^q \|\det F'_{\tilde{\omega}}\|_{L^\infty(\hat{\omega})} \int_{\hat{\omega}} |\nabla(v \circ F_{\tilde{\omega}})|^q \, \mathrm{d}\boldsymbol{x} \\
&\leq C_P(\hat{\omega}, q)^q \|\det F'_{\tilde{\omega}}\|_{L^\infty(\hat{\omega})} \|\det(F_{\tilde{\omega}}^{-1})'\|_{L^\infty(\tilde{\omega})} \|F'_{\tilde{\omega}}\|^q_{L^\infty(\hat{\omega})} |v|^q_{W^{1,q}(\tilde{\omega})},
\end{aligned}$$

where we transformed the integral back in the last step. As in the above proof it is clear that the quantity

$$\|\det F'_{\tilde{\omega}}\|_{L^\infty(\hat{\omega})} \|\det(F_{\tilde{\omega}}^{-1})'\|_{L^\infty(\tilde{\omega})} \|F'_{\tilde{\omega}}\|^q_{L^\infty(\hat{\omega})}$$

is bounded from above and below by a constant depending on γ, because $F_{\tilde{\omega}}$ restricted to an element behaves like $F_{\tilde{K}}^{-1}$ from above. Since there is only a finite number of reference patches in $\hat{B}(\gamma)$, the desired constant, depending only on γ and q, must exist.

Now let $q = \infty$. Then with $\overline{v \circ F_{\tilde{\omega}}}$ being the integral mean of this function on $\hat{\omega}$, and $\nabla(v \circ F_{\tilde{\omega}}) = (\nabla v \circ F_{\tilde{\omega}}) F'_{\tilde{\omega}}$ in a weak sense, we obtain for a constant $C(\hat{\omega}) > 0$

$$\begin{aligned}
\|v - \overline{v \circ F_{\tilde{\omega}}}\|_{L^\infty(\tilde{\omega})} &= \|v \circ F_{\tilde{\omega}} - \overline{v \circ F_{\tilde{\omega}}}\|_{L^\infty(\hat{\omega})} \\
&\leq C(\hat{\omega}, \infty) |v \circ F_{\tilde{\omega}}|_{W^{1,\infty}(\hat{\omega},\infty)} \leq C(\hat{\omega}) |v|_{W^{1,\infty}(\tilde{\omega})} |F_{\tilde{\omega}}|_{W^{1,\infty}(\hat{\omega})}.
\end{aligned}$$

With similar arguments as in the case $q < \infty$ we get the assertion for $q = \infty$. □

Since we want to obtain higher order approximation results, we will need a similar statement as the above one for the Deny-Lions Lemma. The arguments we just used cannot work in this case. The reason is that the map $F_{\tilde{\omega}}$ in the above proof has the properties that $v \circ F_{\tilde{\omega}}$ is in $W^{1,q}$ if $v \in W^{1,q}$ and $\overline{v} \circ F_{\tilde{\omega}}$ is still a polynomial of degree zero, namely a constant. With the use of $F_{\tilde{\omega}}$ we can therefore simply map functions on $\tilde{\omega}$ to functions on $\hat{\omega}$, approximate them there, and pull them back. The distortion of the approximation error is then described by the derivative of $F_{\tilde{\omega}}$ and its inverse, respectively their determinants, all of which can be controlled. Since $F_{\tilde{\omega}}$ is merely Lipschitz, an equivalent statement does not hold if $v \in W^{n,q}$ for some $n > 1$ and the approximating polynomial is not a constant but of degree larger than zero. We will circumvent this by relating the constant from the Deny-Lions Lemma, in the following referred to as Deny-Lions constant, to the Poincaré constant. Since we already have bounded the Poincaré constant, this will give us an upper bound for the other one. The next lemma achieves this. Together with the Poincaré inequality, it also gives an alternative proof of the Deny-Lions Lemma.

Lemma 3.1.6 (Bound of the Deny-Lions constant). *Let $q \in [1, \infty]$. Let $n, d \in \mathbb{N}$ and let $\omega \subseteq \mathbb{R}^d$. Assume that there exists a map $v \mapsto \overline{v} \in \mathbb{R}$ such that for every $v \in W^{1,q}(\omega)$ there holds*

$$\|v - \overline{v}\|_{L^q(\omega)} \leq C_P(\omega, q)\|\nabla v\|_{L^q(\omega)}.$$

Then there exists a map $v \mapsto r_v \in \mathcal{P}_{n-1}(\mathbb{R}^d)$ and a constant $C_{DL}(\omega, n, q) > 0$ such that for every $v \in W^{n,q}(\omega)$

$$\|v - r_v\|_{W^{n,q}(\omega)} \leq C_{DL}(\omega, n, q)|v|_{W^{n,q}(\omega)}$$

and

$$C_{DL}(\omega, n, q) \leq \begin{cases} d^{n/q}(n+1)^{(d+1)/q}C_P(\omega, q)^n & \text{if } q < \infty \\ d^n(n+1)^{d+1}C_P(\omega, q)^n & \text{if } q = \infty. \end{cases}$$

Moreover, if $v \mapsto \overline{v}$ is linear, then the map $v \mapsto r_v$ is linear.

Proof. If $n = 1$, then $\overline{v}$ is the desired approximant. The idea is to construct a polynomial of higher degree in an analogue way.
1st Step: Let $v \in W^{n,q}(\omega)$ for some $n \in \mathbb{N}$. Then we define

$$v_{n,n} := v$$

and for $l \in \{0, \dots, n-1\}$

$$v_{l,n} := v_{l+1,n} - \sum_{\substack{j_1,\dots,j_d \in \mathbb{N}_0 \\ \sum_m j_m \leq l}} \frac{\left(\overline{D_{x_1}^{j_1} \dots D_{x_d}^{j_d} v_{l+1,n}}\right) x_1^{j_1} \dots x_d^{j_d}}{j_1! \dots j_d!}. \tag{3.1.6}$$

With $v_n := v_{0,n}$ we obviously have $v_n \in \left\{v + r : r \in \mathcal{P}_{n-1}(\mathbb{R}^d)\right\}$. Hence the task is now to prove

$$\|v_n\|_{W^{n,q}(\omega)} \leq \begin{cases} d^{n/q}(n+1)^{(d+1)/q}C_P(\omega, q)^n|v|_{W^{n,q}(\omega)} & \text{if } q < \infty \\ d^n(n+1)^{d+1}C_P(\omega, q)^n|v|_{W^{n,q}(\omega)} & \text{if } q = \infty. \end{cases}$$

2nd Step: Let $\alpha \in \mathbb{N}_0^d$ be a multiindex with $0 \le |\alpha| \le n$. We want to show that $D^\alpha v_n = (D^\alpha v)_{n-|\alpha|}$. This is trivial if $|\alpha| = 0$. For the remaining cases it is sufficient to prove $D_{x_m}(v_n) = (D_{x_m} v)_{n-1}$ for every $m \in \{1, \dots, d\}$. Without loss of generality we consider the case $m = 1$. To this end let us compute

$$\begin{aligned}
D_{x_1} v_{n-1,n} &= D_{x_1} v - \sum_{\substack{j_1,\dots,j_d \in \mathbb{N}_0 \\ \sum_m j_m \le n-1}} D_{x_1} \left(\frac{\left(\overline{D_{x_1}^{j_1} \dots D_{x_d}^{j_d} v}\right) x_1^{j_1} \dots x_d^{j_d}}{j_1! \dots j_d!} \right) \\
&= D_{x_1} v - \sum_{\substack{j_1 \in \mathbb{N}, j_2,\dots,j_d \in \mathbb{N}_0 \\ \sum_m j_m \le n-1}} \frac{\left(\overline{D_{x_1}^{j_1-1} \dots D_{x_d}^{j_d} (D_{x_1} v)}\right) x_1^{j_1-1} \dots x_d^{j_d}}{(j_1-1)! \dots j_d!} \\
&= D_{x_1} v - \sum_{\substack{j_1,\dots,j_d \in \mathbb{N}_0 \\ \sum_m j_m \le n-2}} \frac{\left(\overline{D_{x_1}^{j_1} \dots D_{x_d}^{j_d} (D_{x_1} v)}\right) x_1^{j_1} \dots x_d^{j_d}}{j_1! \dots j_d!} \\
&= (D_{x_1} v)_{n-2,n-1}.
\end{aligned}$$

We now proceed by induction on l. With (3.1.6) and the induction hypothesis $D_{x_1} v_{l+1,n} = (D_{x_1} v)_{l,n-1}$ we get

$$\begin{aligned}
D_{x_1} v_{l,n} &= D_{x_1} v_{l+1,n} - \sum_{\substack{j_1 \in \mathbb{N}, j_2,\dots,j_d \in \mathbb{N}_0 \\ \sum_m j_m \le l}} \frac{\left(\overline{D_{x_1}^{j_1-1} \dots D_{x_d}^{j_d} (D_{x_1} v_{l+1,n})}\right) x_1^{j_1-1} \dots x_d^{j_d}}{(j_1-1)! \dots j_d!} \\
&= (D_{x_1} v)_{l,n-1} - \sum_{\substack{j_1,\dots,j_d \in \mathbb{N}_0 \\ \sum_m j_m \le l-1}} \frac{\left(\overline{D_{x_1}^{j_1} \dots D_{x_d}^{j_d} (D_{x_1} v)_{l,n-1}}\right) x_1^{j_1} \dots x_d^{j_d}}{j_1! \dots j_d!} \\
&= (D_{x_1} v)_{l-1,n-1}.
\end{aligned}$$

Thus for $l = 1$

$$D_{x_1} v_n = D_{x_1} v_{1,n} = (D_{x_1} v)_{0,n-1} = (D_{x_1} v)_{n-1}.$$

3rd Step: We claim that for $l \in \mathbb{N}$ and $q < \infty$ there holds $\|v_l\|_{L^q(\omega)} \le C_P(\omega, q)^l d^{l/q} |v|_{W^{l,q}(\omega)}$. Our assumptions and the second step imply

$$\begin{aligned}
\|v_l\|_{L^q(\omega)}^q &= \|v_{1,l} - \overline{v_{1,l}}\|_{L^q(\omega)}^q \le C_P(\omega, q)^q |v_{1,l}|_{W^{1,q}(\omega)}^q = C_P(\omega, q)^q \sum_{m \in \{1,\dots,d\}} \|D_{x_m} v_{1,l}\|_{L^q(\omega)}^q \\
&= C_P(\omega, q)^q \sum_{m \in \{1,\dots,d\}} \|D_{x_m} v_l\|_{L^q(\omega)}^q = C_P(\omega, q)^q \sum_{m \in \{1,\dots,d\}} \|(D_{x_m} v)_{l-1}\|_{L^q(\omega)}^q,
\end{aligned}$$

where we have also used that $v_l = v_{0,l} = v_{1,l} - \overline{v_{1,l}}$, which yields $D_{x_m} v_{1,l} = D_{x_m} v_l$. By repeating the same argument we get all derivatives of order l of v (some of them several times) on the right-hand

side and hence with a crude estimate the claim.
4th Step: It remains to prove the assertion. Let $q < \infty$. Using what we already know we get

$$\begin{aligned}
\|v_n\|^q_{W^{n,q}(\omega)} &= \sum_{l=0}^{n} |v_n|^q_{W^{l,q}(\omega)} = \sum_{l=0}^{n} \sum_{\{\alpha\in\mathbb{N}_0^d:|\alpha|=l\}} \|D^\alpha v_n\|^q_{L^q(\omega)} \\
&= \sum_{l=0}^{n} \sum_{\{\alpha\in\mathbb{N}_0^d:|\alpha|=l\}} \|(D^\alpha v)_{n-|\alpha|}\|^q_{L^q(\omega)} \\
&\leq \sum_{l=0}^{n} C_P(\omega,q)^{q(n-l)} d^{n-l} \sum_{\{\alpha\in\mathbb{N}_0^d:|\alpha|=l\}} |D^\alpha v|^q_{W^{n-|\alpha|,q}(\omega)} \\
&\leq \sum_{l=0}^{n} C_P(\omega,q)^{q(n-l)} d^{n-l} \sum_{\{\alpha\in\mathbb{N}_0^d:|\alpha|=l\}} |v|^q_{W^{n,q}(\omega)} \\
&\leq \sum_{l=0}^{n} C_P(\omega,q)^{q(n-l)} d^{n-l} (l+1)^d |v|^q_{W^{n,q}(\omega)} \leq C_P(\omega,q)^{qn} d^n (n+1)^{d+1} |v|^q_{W^{n,q}(\omega)}.
\end{aligned}$$

Taking the q-th root concludes the proof for $q < \infty$. In the case $q = \infty$ steps 3 and 4 need to be repeated without taking the q-th power. □

The following corollary states what we actually aspire. It is a direct consequence of the previous two lemmata.

Corollary 3.1.7. *Let $\tilde{\omega} \in \tilde{B}(\gamma)$, $q \in [1,\infty]$, and $n \in \mathbb{N}$. Then there exists a linear map $v \mapsto r_v \in \mathcal{P}_{n-1}(\mathbb{R}^2)$ and a constant $C_{DL} > 0$ depending only on γ, n, and q such that for all $v \in W^{n,q}(\tilde{\omega})$*

$$\|v - r_v\|_{W^{n,q}(\tilde{\omega})} \leq C_{DL} |v|_{W^{n,q}(\tilde{\omega})}.$$

3.1.3 C^1-hp-Interpolant

We now construct our interpolant. As always we work with a γ-shape regular admissible triangulation $\mathcal{T}$. In particular, we do not have hanging nodes. This has the important consequence that it is easy to find a C^l partition of one for $l \in \{0,1\}$ and functions with local support. In the case $l = 0$, the hat functions provide this partition of one. If $l = 1$ the Argyris element ensures its existence. The next lemma will discuss this in more detail. With this at hand, we can then reduce the problem of finding an interpolant with the desired approximation properties on Ω, to a local problem.

Lemma 3.1.8 (Partition of unity). *Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of Ω. Then there exist piecewise polynomials $\varphi_V \in S^5(\mathcal{T}) \cap C^1(\Omega)$, $V \in \mathcal{N}(\mathcal{T})$ such that* $\operatorname{supp}(\varphi_V) \subseteq \omega_V$

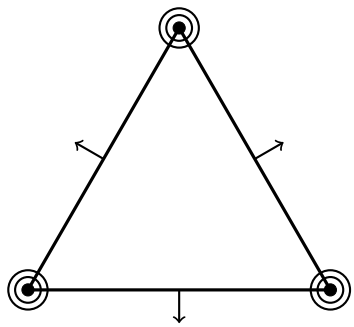


Figure 3.2: The Argyris element and its 21 degrees of freedom as depicted in [10, Fig. 3.9]: The point denotes an evaluation of the function, the inner circle stands for the evaluation of the two first derivatives, the outer circle stands for the evaluation of the three second derivatives and the arrows represent evaluations of the normal derivative at the center of each edge.

for every $V \in \mathcal{T}$ *and*

$$\sum_{V \in \mathcal{N}(\mathcal{T})} \varphi_V \equiv 1 \text{ in } \Omega. \tag{3.1.7}$$

Moreover, there exists a constant $C = C(\gamma) > 0$ *such that for every* $V \in \mathcal{N}(\mathcal{T})$

$$\|\varphi_V\|_{L^\infty(\Omega)} + h_V |\varphi_V|_{W^{1,\infty}(\Omega)} + h_V^2 |\varphi_V|_{W^{2,\infty}(\Omega)} \leq C, \tag{3.1.8}$$

and for every $K \in \mathcal{T}$, $n \in \mathbb{N}$

$$h_V^n \|\varphi_V|_K\|_{W^{n,\infty}(K)} \leq C. \tag{3.1.9}$$

Proof. *1st Step:* In order to construct the functions φ_V we employ the Argyris element as described in [10, Example 3.2.10]. The Argyris element is of type C1-P5, i.e. polynomials of total degree five on a triangle are considered, such that matching values of the shared degrees of freedom of adjacent elements result in continuously differentiable functions. Its 21 degrees of freedom consist of the six derivatives up to order two in each vertex, which makes eighteen, and the three normal derivatives at the midpoint of each edge, which makes 21 in total (see Figure 3.2). These values determine a polynomial of degree five and its gradient on the boundary of the triangle: The restriction of the polynomial on an edge is again a polynomial of degree five. Since the function value, as well as the first and second derivative in the endpoint of this edge are known, so is the polynomial (this is easily checked). Similarly, the function values of the normal derivative in the endpoints and the midpoint of the edge, together with the first derivative of the normal derivative in the endpoints are sufficient to determine the normal derivative, which is a polynomial of degree four, along this edge. Now we denote by $N_i^K : \mathcal{P}_5(K) \to \mathbb{R}$, $i \in \{1, \ldots, 21\}$ the functionals that evaluate one of those 21 degrees of freedom for a polynomial of degree five on an arbitrary

triangle K. We assume that N_1^K, N_2^K, N_3^K are the point evaluations at the three vertices V_1, V_2, V_3 of K. Additionally, let

$$N^K := \begin{cases} \mathcal{P}_5(K) \to \mathbb{R}^{21} \\ r \mapsto (N_i^K(r))_{i=1}^{21}. \end{cases}$$

We claim that N^K is injective. Suppose that $N^K(r) = 0$ and $r \neq 0$. Then, on account of the above deliberations, there holds $r \equiv 0 \equiv \nabla r$ on ∂K. Thus $r \in \mathcal{P}^5(K)$ is of the type $r(\boldsymbol{x}) = L_1^2 L_2^2 L_3^2 \tilde{r}(\boldsymbol{x})$, where L_i is a linear function that is zero along the edge e_i and constant along parallels of e_i, $i \in \{1,2,3\}$, and $\tilde{r}$ is another polynomial. This is a contradiction since r is only of degree five. The dimension of $\mathcal{P}_5(K)$ is 21 as well and therefore N^K is clearly bijective. This implies that polynomials $r_i^K \in \mathcal{P}^5(K)$ with $N_j^K(r_i^K) = \delta_{i,j}$ exist. In particular

$$r_1^K + r_2^K + r_3^K \equiv 1 \text{ in } K, \tag{3.1.10}$$

because $r_1^K + r_2^K + r_3^K$ is the unique polynomial $r \in \mathcal{P}_5$ with $N_j^K(r) = \delta_{1,j} + \delta_{2,j} + \delta_{3,j}$ for all $j \in \{1, \dots, 21\}$.

2nd Step: We now construct φ_V. Let $V \in \mathcal{N}(\mathcal{T})$ and $K \in \mathcal{K}(V)$. With the notation from the beginning of this section recall that $\tilde{F}_K(\tilde{K}) = K$. The triangle $\tilde{K} \in \tilde{A}(\gamma)$ has the three vertices $\tilde{V}_1, \tilde{V}_2$, and $\tilde{V}_3$ where, without loss of generality, $\tilde{V}_1 \in \tilde{a}(\gamma)$, $\tilde{V}_2 = (-1,0)$, $\tilde{V}_3 = (0,1)$, and $\tilde{F}_K(\tilde{V}_1) = V$. It is easily seen that

$$N_j^K(r_i^{\tilde{K}} \circ \tilde{F}_K^{-1}) = \delta_{i,j} \qquad \forall i \in \{1,2,3\}\ , j \in \{1, \dots, 21\},$$

and therefore $r_i^{\tilde{K}} \circ \tilde{F}_K^{-1} = r_i^K$ for every $i \in \{1,2,3\}$. Hence, due to (3.1.10),

$$\varphi_V := \sum_{K \in \mathcal{K}(V)} r_1^{\tilde{K}} \circ \tilde{F}_K^{-1} \tag{3.1.11}$$

results in a partition of unity if we repeat this process for every $V \in \mathcal{N}(\mathcal{T})$.

3rd Step: We show (3.1.8) and (3.1.9). Let $\tilde{K} \in \tilde{A}(\gamma)$ and let again $\tilde{V}_1 \in \tilde{a}(\gamma)$, $\tilde{V}_2 = (-1,0)$, and $\tilde{V}_3 = (0,1)$ be its vertices. To indicate that $\tilde{K} \in \tilde{A}(\gamma)$ is determined by the vertex $\tilde{V}_1 \in \tilde{a}(\gamma)$, we write $\tilde{K}(\tilde{V}_1)$, and to simplify some notation we define

$$r_i(\tilde{V}_1) := r_i^{\tilde{K}(\tilde{V}_1)},$$
$$N_i(\tilde{V}_1) := N_i^{\tilde{K}(\tilde{V}_1)}.$$

We claim that $r_i(\tilde{V}_1)$ depends continuously on $\tilde{V}_1 \in \tilde{a}(\gamma)$ as a function of the space $C^1(B_R(\mathbf{0}))$, where $R > 0$ is large enough such that the closed ball $B_R(\mathbf{0}) \subseteq \mathbb{R}^2$ contains $\tilde{a}(\gamma)$. To this end let $\boldsymbol{c}(r) \in \mathbb{R}^{21}$ be the vector containing the coefficients of $r \in \mathcal{P}_5$ w.r.t. the basis $1, x, y, \dots, x^5, y^5$ in some fixed order. We also write $r = r(\boldsymbol{c})$. Let $M(\tilde{V}_1) \in \mathbb{R}^{21 \times 21}$ be the matrix, such that the i-th row represents the linear functional

$$N_i(\tilde{V}_1) := \begin{cases} \mathbb{R}^{21} \to \mathbb{R} \\ \boldsymbol{c} \mapsto N_i(\tilde{V}_1)(r(\boldsymbol{c})). \end{cases}$$

Then $M(\tilde{V}_1)$ obviously depends continuously on $\tilde{V}_1 \in \tilde{a}(\gamma)$ in $C^0(\mathbb{R}^{21\times 21})$. With $\boldsymbol{c}_i(\tilde{V}_1) := \boldsymbol{c}(r_i(\tilde{V}_1))$ we get

$$M(\tilde{V}_1) \cdot \boldsymbol{c}_i(\tilde{V}_1) = (\delta_{i,j})_{j=1}^{21}, \ \forall i \in \{1, \dots, 21\}.$$

Therefore $M(\tilde{V}_1)$ is regular and $\boldsymbol{c}_i(\tilde{V}_1) = M(\tilde{V}_1)^{-1}(\delta_{i,j})_{j=1}^{21}$. All occuring maps are continuous and therefore $\boldsymbol{c}_i(\tilde{V}_1)$ depends continuously on $\tilde{V}_1$. Thus the map

$$\begin{cases} \tilde{a}(\gamma) \to \mathbb{R} \\ \tilde{V}_1 \mapsto \|r_i(\tilde{V}_1)\|_{W^{5,\infty}(B_R(0))} \end{cases}$$

is well-defined, continuous, and it has a maximum C in the compact set $\tilde{a}(\gamma)$. The fact that $\tilde{F}_V$ is linear (which means that derivatives of order higher than one are zero), together with (3.1.1) and (3.1.11), proves (3.1.8) and (3.1.9), because $|\varphi_V|_{W^{6,\infty}(K)} = 0$ since $\varphi_V|_K \in \mathcal{P}_5(K)$. □

The next theorem is a bit more involved. It states the needed approximation property of $W^{n,q}$ functions by polynomials [24, Theorem A.3]:

Theorem 3.1.9. *Let $d \in \mathbb{N}$. Set $I := I_1 \times \cdots \times I_d$, with I_i being a bounded interval for every $i \in \{1, \dots, d\}$. Let $R \in \mathbb{N}$. Then for each $N \in \mathbb{N}_0$ there exists a bounded linear operator $J_{R,N} : L^1(I) \to \mathcal{Q}_N(I)$ with the following properties: For each $q \in [1, \infty]$ there exists a constant $C > 0$, which depends only on R, q and I, such that for all $N \geq R - 1$ and all $0 \leq r \leq R$*

$$J_{R,N} u = u \qquad \forall u \in \mathcal{Q}_{R-1}, \tag{3.1.12}$$

$$\|u - J_{R,N} u\|_{W^{l,q}(I)} \leq C(N+1)^{-(r-l)} |u|_{W^{r,q}(I)}, \qquad l = 0, \dots, r. \tag{3.1.13}$$

These preliminaries constitute sufficient preparation such that we are now able to prove the main result of this section. Let us first highlight the differences to Theorem 2.1 in [24] on which our construction is based: The interpolant in [24] approximates $W^{1,q}$ functions with continuous piecewise polynomials. The interpolant in the following theorem locally inherits the approximation properties of $J_{R,N}$ from Theorem 3.1.9. This means, we observe better approximation rates for smoother functions, for example $(h/p)^n$ instead of h/p in the L^q-norm for $W^{n,q}$ functions. Moreover, we get approximation of higher derivatives. Additionally, the interpolant in item (i) maps to C^1-functions rather than C^0-functions. Therefore it is H^2-conforming. At the same time we emphasize that slit domains are excluded, and the occuring constant behaves worse than in [24] (cf. Remark 3.1.14). The proof follows the one in [24, Theorem 2.1] and the changes are mostly owed to the fact that we work with higher derivatives and different reference patches. Furthermore, we point out that the following operators I_n^{hp} and $I_n^{hp,0}$ are no projectors.

Theorem 3.1.10 (Clément type quasi-interpolation). *Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of the polygonal domain $\Omega \subseteq \mathbb{R}^2$. Denote the smallest outer angle of Ω by θ and assume that $\theta > 0$. Let $\boldsymbol{p}$ be a polynomial degree distribution on $\mathcal{T}$ satisfying* (1.2.1)*. Assume that $q \in [1, \infty]$ and let $n \in \mathbb{N}$.*

(i) Assume that $\lfloor (p_{\mathcal{T}}-5)/2 \rfloor \geq n-1$. Then there exists a bounded linear operator $I_n^{hp} : W^{n,q}(\Omega) \to S^{\boldsymbol{p}}(\mathcal{T}) \cap C^1(\Omega)$ such that for every $K \in \mathcal{T}$

$$|u - I_n^{hp} u|_{W^{m,q}(K)} \leq C \left(\frac{h_K}{p_K}\right)^{n-m} |u|_{W^{n,q}(\omega_K)} \qquad \forall m \in \{0, \dots, n\}, \tag{3.1.14}$$

and for every $e \in \mathcal{E}(K)$

$$\|D^m((u - I_n^{hp} u)|_K)\|_{L^q(e)} \leq C \left(\frac{h_e}{p_e}\right)^{n-m-1/q} |u|_{W^{n,q}(\omega_e)} \qquad \forall m \in \{0, \dots, n-1\}, \tag{3.1.15}$$

where $D^m((u - I_n^{hp} u)|_K)$ stands for all partial derivatives of order m of $u - I_n^{hp} u$ restricted to K, and $C > 0$ only depends on n, q, γ, and θ.

(ii) Assume that $\lfloor (p_{\mathcal{T}} - 1)/2 \rfloor \geq n - 1$. Then there exists a bounded linear operator $I_n^{hp,0} : W^{n,q}(\Omega) \to S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ such that (3.1.14) and (3.1.15) hold with $I_n^{hp,0} u$ instead of $I_n^{hp} u$ for a constant $C > 0$ solely depending on n, q, γ, and θ.

Proof. *1st Step:* We start with the proof of (i) and fix $n \in \mathbb{N}$ and $q \in [1, \infty]$. Let $v \in W^{n,q}(\tilde{\omega}_V)$ for a fixed vertex V and recall that $p_V = \min_{K \in \omega_V} p_K$. Furthermore we choose $N_V := \lfloor (p_V - 5)/2 \rfloor$, and denote by $Ev \in W^{n,q}(\hat{S})$ the extension from Lemma 3.1.3 with $\hat{S}$ being the reference square from Lemma 3.1.2. Corollary 3.1.7 gives a polynomial $r_v \in \mathcal{P}_n(\mathbb{R}^2)$ such that

$$\|v - r_v\|_{W^{n,q}(\tilde{\omega}_V)} \leq C_{DL} |v|_{W^{n,q}(\tilde{\omega}_V)}. \tag{3.1.16}$$

We now define for every $V \in \mathcal{N}(\mathcal{T})$ the operator

$$J_V := \begin{cases} W^{n,q}(\tilde{\omega}_V) \to \mathcal{P}_{p_V - 5}(\hat{S}) \\ v \mapsto r_v + J_{n,N_V}(E(v - r_v)). \end{cases} \tag{3.1.17}$$

This operator is linear according to Lemma 3.1.3 and Corollary 3.1.7. Moreover, it maps to $\mathcal{P}_{p_V-5}(\hat{S})$ because $J_{n,N_V} : W^{n,q}(\hat{S}) \to \mathcal{Q}_{N_V} \subseteq \mathcal{P}_{2N_V}$, $r_v \in \mathcal{P}_n$ and by assumption

$$N_V = \lfloor (p_V - 5)/2 \rfloor \geq \lfloor (p_{\mathcal{T}} - 5)/2 \rfloor \geq n - 1, \tag{3.1.18}$$

which gives $n \leq 2n \leq p_V - 5$ and obviously $2N_V \leq p_V - 5$. Equation (3.1.18) also implies that we can apply the approximation property (3.1.13) of the operator J_{n,N_V} from Theorem 3.1.9 and obtain for every $m \in \mathbb{N}$ with $0 \leq m \leq n$ ($r, R := n$ in Theorem 3.1.9)

$$\begin{aligned} \|v - J_V(v)\|_{W^{m,q}(\tilde{\omega}_V)} &= \|v - r_v - J_{n,N_V}(E(v - r_v))\|_{W^{m,q}(\tilde{\omega}_V)} \\ &\leq C \|E(v - r_v) - J_{n,N_V}(E(v - r_v))\|_{W^{m,q}(\hat{S})} \\ &\leq C (N_V + 1)^{-(n-m)} \|E(v - r_v)\|_{W^{n,q}(\hat{S})} \\ &\leq C (N_V + 1)^{-(n-m)} \|v - r_v\|_{W^{n,q}(\tilde{\omega}_V)} \\ &\leq C p_V^{-(n-m)} |v|_{W^{n,q}(\tilde{\omega}_V)}, \end{aligned} \tag{3.1.19}$$

where we have used $N_V + 1 \sim p_V$, Lemma 3.1.3 and (3.1.16). Note that the occuring constant only depends on n, q, γ, and θ due to Corollary 3.1.7 and Lemma 3.1.3: Any outer angle ϑ of $\tilde{\omega}_V$ is either an outer angle of Ω or bounded from below by an interior angle of an element in $\mathcal{T}$, and hence ϑ is bounded from below by a constant depending on γ and θ.
2nd Step: Let us now consider the function $u \in W^{n,q}(\Omega)$, and we define $u_V := u|_{\omega_V}$, $\tilde{u}_V := u_V \circ \tilde{F}_K$, and $u_{V,p_V} := J_V(\tilde{u}_V) \circ \tilde{F}_V^{-1} \in \mathcal{P}_{p_V-5}(\omega_V)$ for $V \in \mathcal{N}(\mathcal{T})$. This is well-defined, because according to Lemma 3.1.2 it holds that $\tilde{u}_V \in W^{n,q}(\tilde{\omega}_V)$. Let $m \in \mathbb{N}$ and $0 \le m \le n$. With $v := \tilde{u}_V$ in (3.1.19) we obtain using item (iv) of Lemma 3.1.2

$$\begin{aligned}
|u - u_{V,p_V}|_{W^{m,q}(\omega_V)} &= |u - J_V(\tilde{u}_V) \circ \tilde{F}_V^{-1}|_{W^{m,q}(\omega_V)} \le C|\tilde{u}_V - J_V(\tilde{u}_V)|_{W^{m,q}(\tilde{\omega}_V)} h_V^{2/q-m} \\
&\le C p_V^{-(n-m)} h_V^{2/q-m} |\tilde{u}_V|_{W^{n,q}(\tilde{\omega}_V)} \le C \left(\frac{h_V}{p_V}\right)^{n-m} |u_V|_{W^{n,q}(\omega_V)} \\
&= \left(\frac{h_V}{p_V}\right)^{n-m} |u|_{W^{n,q}(\omega_V)}. \qquad (3.1.20)
\end{aligned}$$

For the edge estimate, let m be an integer with $0 \le m \le n-1$ and let $e \in \mathcal{E}(K)$ be some edge of the element K that belongs to the patch ω_V. Moreover, let $\tilde{e} := \tilde{F}_V^{-1}(e)$, then $|\det((\tilde{F}_V|_{\tilde{e}})')|^{1/q} \sim h_V^{1/q}$. Similar as in Lemma 3.1.2, the fact that $\tilde{F}_V$ is a composition of a scaling of size h_V, a rotation, and a translation, shows that for an integer l and $v \in W^{l,q}(\omega_V)$

$$\begin{aligned}
\|D^l(v \circ \tilde{F}_V)\|_{L^q(\tilde{\omega}_V)} &\sim h_V^l \|(D^l v) \circ \tilde{F}_V\|_{L^q(\tilde{\omega}_V)}, \\
\|D^{l+1}(v \circ \tilde{F}_V)\|_{L^q(\tilde{\omega}_V)} &\sim h_V^l |(D^l v) \circ \tilde{F}_V|_{W^{1,q}(\tilde{\omega}_V)}.
\end{aligned}$$

We now make use of this observation, Lemma 3.1.4, (3.1.19), and again item (iv) of Lemma 3.1.2. Then we obtain

$$\begin{aligned}
\|D^m(u - u_{V,p_V})\|_{L^q(e)} &= \|D^m(u - u_{V,p_V}) \circ \tilde{F}_V\|_{L^q(\tilde{e})} |\det((\tilde{F}_V|_{\tilde{e}})')|^{1/q} \\
&\le C h_V^{1/q} \Big(\|D^m(u - u_{V,p_V}) \circ \tilde{F}_V\|_{L^q(\tilde{\omega}_V)}^{1-1/q} |D^{m+1}(u - u_{V,p_V}) \circ \tilde{F}_V|_{W^{1,q}(\tilde{\omega}_V)}^{1/q} \\
&\quad + \|D^m(u - u_{V,p_V}) \circ \tilde{F}_V\|_{L^q(\tilde{\omega}_V)} \Big) \\
&\le C h_V^{1/q} \Big(h_V^{-m} \|D^m(\tilde{u}_V - J_V(\tilde{u}_V))\|_{L^q(\tilde{\omega}_V)}^{1-1/q} \|D^{m+1}(\tilde{u}_V - J_V(\tilde{u}_V))\|_{L^q(\tilde{\omega}_V)}^{1/q} \\
&\quad + h_V^{-m} \|D^m(\tilde{u}_V - J_V(\tilde{u}_V))\|_{L^q(\tilde{\omega}_V)} \Big) \\
&\le C |\tilde{u}_V|_{W^{n,q}(\tilde{\omega}_V)} h_V^{-m+1/q} \left(p_V^{-(n-m)+1/q} + p_V^{-(n-m)} \right) \\
&\le C \left(\frac{h_V}{p_V}\right)^{n-m-1/q} |u_V|_{W^{n,q}(\omega_V)} = C \left(\frac{h_V}{p_V}\right)^{n-m-1/q} |u|_{W^{n,q}(\omega_V)}. \qquad (3.1.21)
\end{aligned}$$

3rd Step: With the partition of unity $(\varphi_V)_{V \in \mathcal{N}(\mathcal{T})}$ from Lemma 3.1.8, we put together the interpolant as follows:

$$I_n^{hp}(u) := \sum_{V \in \mathcal{N}(\mathcal{T})} \varphi_V u_{V,p_V}.$$

The first observation is that $I_n^{hp}u \in S^{\boldsymbol{p}}(\mathcal{T})$: Per construction (see (3.1.17)), we have $u_{V,p_V} \in \mathcal{P}_{p_V-5}$. This together with $\varphi_V \in S^5(\mathcal{T})$ readily gives $u \in S^{\boldsymbol{p}}(\mathcal{T})$. Moreover, $I_n^{hp}u \in C^1(\Omega)$ because u_{V,p_V} is a polynomial, $\varphi_V \in C^1(\Omega)$, $\operatorname{supp}(\varphi_V) \subseteq \omega_V$, and $\nabla\varphi_V|_{\partial\omega_V} = \varphi_V|_{\partial\omega_V} \equiv 0$. The approximation properties are now easily obtained. We keep in mind that

$$\left(\sum_{V\in\mathcal{N}(K)} \varphi_V\right)\Bigg|_K \equiv 1.$$

Therefore, for $m \in \mathbb{N}$, $0 \le m \le n$ and with (3.1.9)

$$\begin{aligned}
|u - I_n^{hp}u|_{W^{m,p}(K)} &\le C \sum_{V\in\mathcal{N}(K)} |\varphi_V(u - I_n^{hp}u)|_{W^{m,p}(K)} \\
&\le C \sum_{V\in\mathcal{N}(K)} \sum_{\substack{m_1,m_2\in\mathbb{N}_0\\ m_1+m_2=m}} |\varphi_V|_{W^{m_1,\infty}(K)} |u - I_n^{hp}u|_{W^{m_2,q}(K)} \\
&\le C \sum_{V\in\mathcal{N}(K)} \sum_{\substack{m_1,m_2\in\mathbb{N}_0\\ m_1+m_2=m}} h_V^{-m_1} \left(\frac{h_V}{p_V}\right)^{n-m_2} |u|_{W^{n,q}(\omega_K)} \\
&\le C \left(\frac{h_V}{p_V}\right)^{n-m} |u|_{W^{n,q}(\omega_K)},
\end{aligned}$$

where we have used (3.1.20). This proves (3.1.14). For the edge estimate let $e \in \mathcal{E}(K)$. We use (3.1.21) and once more (3.1.9). Then for every $m \in \{0, \dots, n-1\}$

$$\begin{aligned}
\|D^m((u - I^{hp}u)|_K)\|_{L^q(e)} &= \left\| \sum_{V\in\mathcal{N}(e)} D^m((u_V - u_{V,p_V})\varphi_V|_K) \right\|_{L^q(e)} \\
&\le C \sum_{V\in\mathcal{N}(e)} \sum_{\substack{m_1,m_2\in\mathbb{N}_0\\ m_1+m_2=m}} \|D^{m_1}\varphi_V|_K\|_{L^\infty(e)} \|D^{m_2}(u_V - u_{V,p_V})\|_{L^q(e)} \\
&\le C \sum_{V\in\mathcal{N}(e)} \sum_{\substack{m_1,m_2\in\mathbb{N}_0\\ m_1+m_2=m}} h_e^{-m_1} \left(\frac{h_e}{p_e}\right)^{n-m_2-1/q} |u|_{W^{n,q}(\omega_V)} \\
&\le C \left(\frac{h_e}{p_e}\right)^{n-m-1/q} |u|_{W^{n,q}(\omega-e)}.
\end{aligned}$$

We also used that $\varphi_V|_K$ is a polynomial, which yields

$$\|D^{m_1}\varphi_V|_K\|_{L^\infty(e)} \le \|D^{m_1}\varphi_V|_K\|_{L^\infty(K)} \lesssim h_K^{-m_1} \lesssim h_e^{-m_1}.$$

Retracing the steps of the construction, we see that I_n^{hp} is in fact linear. We mention once more that all occuring constants depend solely on n, q, γ, and θ.

4th step: Finally, we consider (ii). This result is obtained by using hat functions in steps 1-3

instead of the functions φ_V. This will only affect the smoothness of $I_n^{hp,0}u$ but it does not change the approximation properties on the elements or edges. $\square$

We conclude this section with a few remarks on the above proof.

Remark 3.1.11. The core of the proof of Theorem 3.1.10 is Theorem 3.1.9. In one dimension this theorem is a consequence of trigonometric approximation results (for details see Chapter 7 of [12]). The d-dimensional interpolation operator is then the d-fold tensor product of the one dimensional interpolation operator.

Remark 3.1.12. Locally the operator I_n^{hp} has the same properties as the operator in Theorem 3.1.9. Therefore, we may have stated Theorem 3.1.9 more generally by relaxing the condition $\lfloor (p_{\mathcal{T}}-5)/2 \rfloor \geq n-1$ to a local one (with a local smoothness parameter n_V, and the local polynomial degree p_V). This would result in approximation properties that are linked with the local smoothness of the function.

Remark 3.1.13. The same line of reasoning as above leads to an interpolant which lies in $C^l(\Omega)$ for some $l \in \mathbb{N}$, if a partition of unity with functions from $C^l(\Omega) \cap S^p(\mathcal{T})$ is available for a $p \in \mathbb{N}$. In particular, the use of a Cl-Pp element, which contains all polynomials of degree p for some $p \in \mathbb{N}$, rather than the Argyris element, would not change steps two and three in the proof of Lemma 3.1.8. Moreover, arguments similar to the ones we have used should work to generalize the theorem to spaces with higher dimension $d > 2$. The condition on the polynomial degree then reads $\lfloor (p_{\mathcal{T}} - p)/d \rfloor \geq n - 1$.

Remark 3.1.14. The largest outer angle θ only played a role in the proof of Lemma 3.1.3. Since we assumed $\theta > 0$, unlike the operator derived in [24], our interpolation operator does not allow slit domains. Throughout this thesis, constants depending on Ω will only depend on the smallest outer angle of Ω. In particular if Ω is convex, then the constant can be chosen independent of the domain.

3.2 Conforming Error

In this section we construct a conforming approximant. That means, we approximate a discontinuous piecewise polynomial $v \in S^{\boldsymbol{p}}(\mathcal{T})$ with a continuous piecewise polynomial $v^* \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$. We then wish to bound the error by a constant multiplied with the L^2-norm of the jump of v across the edges. The question is then, how this constant depends on the local meshwidth and polynomial degree. Theorem 3.2.7 will give an answer to this, and it turns out that this weight is the same one that is used for the jump terms in the DG-norm (see Corollary 3.2.9). Results of this type have been established in various papers and for several different situations (hanging nodes, quadrilaterals, 3D), see for example [35, Section 4.3], [36, Remark 4.5] and the references therein. The method of proof is very straightforward: Lifting operators are used to correct the

discontinuities along the edges and at the vertices. Nonetheless the proof will be lengthy and also a bit technical. Here we follow the ideas of [19, Section 5]. The difference is, that whereas in [19] the error in the H^1-seminorm was of interest, we desire simultaneous approximation in the L^2-norm and in the L^2-norm on the skeleton. To achieve this, an optimal vertex lifting will be constructed in Lemma 3.2.5. Such liftings and some elementary inverse inequalities are the purpose of the next subsection. After that we have enough at hand to prove Theorem 3.2.7.

3.2.1 Inverse Inequalities and Liftings

Inverse inequalities

We quote two basic theorems distinguishing between one- and two-dimensional results. They will prove useful throughout the rest of this thesis.

Theorem 3.2.1 (1D Inverse inequalities). *Let $I = (a, b)$ be a bounded interval and let $h = b - a$. Then for every polynomial $v \in \mathcal{P}_p(I)$ we have*

$$\|v'\|_{L^2(I)} \leq 2\sqrt{3}\frac{p^2}{h}\|v\|_{L^2(I)}, \tag{3.2.1a}$$

$$\|v'\|_{L^\infty(I)} \leq \frac{2p^2}{h}\|v\|_{L^\infty(I)}, \tag{3.2.1b}$$

$$\|v\|_{L^\infty(I)} \leq 4\sqrt{3}\frac{p}{\sqrt{h}}\|v\|_{L^2(I)}. \tag{3.2.1c}$$

Proof. A proof is given in Theorem 3.91 and Theorem 3.92 in [30]. □

Theorem 3.2.2 (2D Inverse inequalities). *Let $\hat{K}$ be the reference triangle and let $\hat{e} \in \mathcal{E}(\hat{K})$ be an edge of $\hat{K}$. Then there exists a constant $C > 0$ not depending on p such that for every polynomial $v \in \mathcal{P}_p(\hat{K})$ it holds that*

$$\|v\|_{L^\infty(\hat{K})} \leq Cp^2\|v\|_{L^2(\hat{K})}, \tag{3.2.2a}$$

$$\|v\|_{L^2(\hat{e})} \leq Cp\|v\|_{L^2(\hat{K})}, \tag{3.2.2b}$$

$$\|\nabla v\|_{L^2(\hat{K})} \leq Cp^2\|v\|_{L^2(\hat{K})}. \tag{3.2.2c}$$

Proof. A proof can be found in [30, Theorem 4.76]. □

Trace Liftings

Lifting operators allow us to find polynomials with prescribed values on either the boundary or at a vertex of an element K. The crucial point is, that the L^2- and H^1-norm of these polynomials are

small. For some of the original work on these polynomial liftings we refer to [5, 6]. The following theorem provides an H^1-stable lifting from the boundary.

Theorem 3.2.3 (Edge lifting). *Let $\hat{K}$ be the reference triangle and let Γ be the closed union of the edges of $\hat{K}$. Then there exists a linear extension operator $L : C(\Gamma) \to H^1(\hat{K})$ and a constant $C > 0$ with the property that if $v \in C(\Gamma)$ is a polynomial of degree p on every edge $e \in \mathcal{E}(\hat{K})$, then $L(v) \in \mathcal{P}_p(\hat{K})$ and for every $v \in C(\Gamma)$*

$$\|L(v)\|_{L^2(\hat{K})} \leq C\|v\|_{L^2(\Gamma)}, \tag{3.2.3a}$$

$$\|L(v)\|_{H^1(\hat{K})} \leq C\|v\|_{H^{1/2}(\Gamma)}. \tag{3.2.3b}$$

Proof. A proof is given in Theorem B.4 in [24]. □

To control the $H^{1/2}$-norm in (3.2.3b), we will use the following well-known Lemma.

Lemma 3.2.4. *Let $v \in \mathcal{P}_p([0,1])$ and let $p \in \mathbb{N}$. Then there exists a constant $C > 0$ independent of p such that*

$$\|v\|_{H^{1/2}(0,1)} \leq Cp\|v\|_{L^2(0,1)}. \tag{3.2.4}$$

Proof. This is an interpolation result. With the notation of [12, Chapter 6] it holds (on $\mathbb{R}$ or on closed intervals) that $(L^2, W^{1,2})_{1/2,2} = H^{1/2}$, where the space on the left-hand side is an interpolation space (see [12, p. 196] and use the well-known fact that $B^2_{1/2,2} = H^{1/2}$). According to [33, Section 1.3.3], for an interpolation couple (X, Y) (see [33, Section 1.2.1]) it holds that

$$\|v\|_{(X,Y)_{\theta,q}} \leq C_{\theta,q}\|v\|_X^{1-\theta}\|f\|_Y^{\theta} \quad \forall v \in (X,Y)_{\theta,q}.$$

Thus with the inverse estimate (3.2.1a), $X := L^2$, $Y := W^{1,2}$, $\theta := 1/2$, and $q := 1$ there exists a constant $C > 0$ such that we can bound

$$\|v\|_{H^{1/2}(0,1)} \leq C\|v\|^{1/2}_{L^2(0,1)}\|v\|^{1/2}_{H^1(0,1)} \leq Cp\|v\|_{L^2(0,1)},$$

and hence (3.2.4) holds. □

Item (iii) in the next lemma gives us a vertex lifting. Equation (3.2.2a) shows that it is optimal in the L^2-norm. Conversely, item (i) basically states that (3.2.1c) is sharp (this was also shown in [31, Lemma 9.1]).

Lemma 3.2.5 (Vertex lifting). *Let $n, p \in \mathbb{N}$ and $p > n$. Let*

$$\hat{K} = \left\{(x,y) \in \mathbb{R}^2 : x, y > 0 \ \wedge \ x + y < 1\right\}$$

and let $\hat{e} = [0,1]$.

(i) There exists a constant $C > 0$ independent of p and a polynomial $\pi_p \in \mathcal{P}_p(\hat{e})$ with the properties

$$\pi_p(0) = 1, \qquad \pi_p(1) = 0, \qquad \|\pi_p\|_{L^2(\hat{e})} \leq \frac{C}{p}. \tag{3.2.5}$$

(ii) There exists a constant $C(n) > 0$ independent of p such that if $q \in \mathcal{P}_p(\hat{e})$ with $q(0) = 1$ then

$$\|\pi_{\lceil p/n \rceil}\|_{L^2(\hat{e})} + \|\pi_{p-n}\|_{L^2(\hat{e})} \leq C(n)\|q\|_{L^2(\hat{e})}. \tag{3.2.6}$$

(iii) There exist constants $C_1, C_2 > 0$ independent of p and a polynomial $\Pi_p \in \mathcal{P}_p(\hat{K})$ with the properties

$$\begin{aligned} &\Pi_p(0,0) = 1, \qquad \Pi_p|_{\{x+y=1\}} \equiv 0, \qquad \|\Pi_p\|_{L^2(\partial\hat{K})} \leq C_1\|\pi_p\|_{L^2(\hat{e})} \leq C_2\frac{1}{p} \\ &\|\Pi_p\|_{L^2(\hat{K})} \leq \frac{C_1}{p^2}, \qquad \|\nabla\Pi_p\|_{L^2(\hat{K})} \leq C_1 p\|\pi_p\|_{L^2(\hat{e})} \leq C_2. \end{aligned} \tag{3.2.7}$$

Proof.

(i) We work with the Legendre polynomials $(P_l)_{l\in\mathbb{N}_0}$. According to [1, 22.2.10] they form a system of L^2-orthogonal polynomials on $[-1,1]$ with

$$\|P_l\|^2_{L^2(-1,1)} = \frac{2}{2l+1}, \qquad P_l(1) = 1 \ \forall l \in \mathbb{N}_0.$$

Therefore, if $v = \sum_{l=1}^p a_l P_l$ and $\sum_{l=1}^p a_l = 1$, we also have $v(1) = 1$. We define $\pi_1(x) := (1-x) \in \mathcal{P}_1(\hat{e})$ and now consider the case $p > 1$. Since

$$\sum_{l=1}^{p-1}\left(\frac{l}{p^3}\right)^{1/2} \sim \int_1^{p-1} \frac{x^{1/2}}{p^{3/2}}\,\mathrm{d}x \sim 1$$

and

$$\Bigg\|\underbrace{\sum_{l=1}^{p-1}\left(\frac{l}{p^3}\right)^{1/2} P_l}_{=:v_p}\Bigg\|^2_{L^2(-1,1)} = \sum_{l=1}^{p-1}\frac{l}{p^3}\frac{2}{2l+1} \sim \frac{1}{p^2},$$

the assertion follows, if we multiply the polynomial v_p with $c_p(x+1)$ for a suitable constant c_p (which is bounded independent of p from both directions) and transform it from $[-1,1]$ to the intervall $[0,1]$.

(ii) With $q(0) = 1$ and the inverse estimate (3.2.1c), we have $1/p \leq C\|q\|_{L^2(\hat{e})}$. Hence

$$\|\pi_{p-n}\|_{L^2(\hat{e})} \leq \frac{C}{p-n} \leq C\frac{p}{p-n}\frac{1}{p} \leq \underbrace{Cn}_{=:C(n)} \|q\|_{L^2(\hat{e})}$$

and upon adjusting $C(n)$

$$\|\pi_{\lceil p/n \rceil}\|_{L^2(\hat{e})} \leq \frac{Cn}{p} \leq C(n)\|q\|_{L^2(\hat{e})}.$$

(iii) First, we define $\Pi_p(x, y) := 1 - (x + y)$ for all $p \in \{1, 2, 3\}$. If $p > 3$ let

$$\Pi_p(x, y) := (1 - x - y)\pi_{\lfloor p/2-1 \rfloor}(x)\pi_{\lfloor p/2-1 \rfloor}(y).$$

We have

$$\|\Pi_p\|_{L^2(\hat{K})} \leq \|\pi_{\lfloor p/2-1 \rfloor}\|^2_{L^2(\hat{e})} \leq \frac{C}{p^2}. \tag{3.2.8}$$

To obtain the H^1-bound we combine (3.2.8) with (3.2.2c). The other properties can be checked easily as well.

□

3.2.2 Conforming Approximation

To state the local approximation properties of the conforming approximant, we first introduce a local set of edges:

Definition 3.2.6. *Let $\mathcal{T}$ be an admissible triangulation of $\Omega \subseteq \mathbb{R}^2$. Let $e \in \mathcal{E}(\mathcal{T})$ be an edge of $\mathcal{T}$ and let $K \in \mathcal{T}$ be an element of $\mathcal{T}$. Then we denote the union of interior edges e' belonging to the respective patch by ρ_K and ρ_e:*

$$\rho_K := \bigcup_{e' \in \mathcal{E}^I(\omega_K)} e', \qquad \rho_e := \bigcup_{e' \in \mathcal{E}^I(\omega_e)} e'. \tag{3.2.9}$$

The construction of the conforming approximant is an important step to prove reliability of a DGFEM a posteriori estimator. The approximant will be constructed as a function in $S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$. The fact that the function is in $S^{\boldsymbol{p}}(\mathcal{T})$ is not important in order to prove reliability of the error estimator. However, it will allow us to bound the jump terms by other error terms later on (see Lemma 4.1.13).

Theorem 3.2.7 (Conforming approximant). *Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of Ω. Let $v \in S^{\boldsymbol{p}}(\mathcal{T})$ and let $\boldsymbol{p}$ be a polynomial degree distribution satisfying* (1.2.1) *and $p_{\mathcal{T}} \geq 1$.*

Then there exists a constant $C > 0$ solely depending on γ, and a function $v^* \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ such that for every edge $e \in \mathcal{E}(\mathcal{T})$ and for every element $K \in \mathcal{T}$ there holds

$$\|v - v^*\|^2_{L^2(e)} \leq C\|[\![v]\!]\|^2_{L^2(\rho_e)}, \tag{3.2.10a}$$

$$\|v - v^*\|^2_{L^2(K)} \leq Ch_K\|[\![v]\!]\|^2_{L^2(\rho_K)}, \tag{3.2.10b}$$

$$\|\nabla(v - v^*)\|^2_{L^2(K)} \leq C\frac{p_K^2}{h_K}\|[\![v]\!]\|^2_{L^2(\rho_K)}. \tag{3.2.10c}$$

Proof. Let us first outline the main steps to be taken. In order to prove this theorem we will split the function v into a nodal part, an edge part, and an interior part:

$$v = v^N + v^E + v^I.$$

The nodal part v^N, which is generally not continuous, is chosen such that its limit coincides with the one of v in each vertex on each element. The edge part v^E will coincide with $v - v^N$ on each element on each edge (thus it is also generally not continuous). The interior part is then given by $v^I = v - v^N - v^E$, and it is continuous. After this splitting is defined, we will approximate each part separately with continuous functions v_*^N, v_*^E, and v_*^I and put everything back together in the end. The general idea is to construct the edge and nodal part in such a way, that they can be approximated with a continuous function and the respective error is bounded by appropriate jump terms only. This is done via the lifting operators from above. The interior part does not need to be approximated since it is already continuous. Throughout this proof we will use the fact that the diameter h_K and the polynomial degree p_K of the element K are comparable with the ones of its neighbours, without mentioning this at every instance. Moreover, we recall that

$$p_V = \min_{K \in \omega_V} p_K.$$

Working with the polynomial degree p_V on the patch ω_V will guarantee that we stay in $S^{\boldsymbol{p}}(\mathcal{T})$. The proof proceeds in six steps.

1st Step: We establish some notation used throughout this proof. To begin with, we define for every $V \in \mathcal{N}(\mathcal{T})$

$$m_V := \max_{\{K,K' \in \mathcal{K}(V): \exists e \in \mathcal{E}(K) \cap \mathcal{E}(K')\}} \big|v|_K(V) - v|_{K'}(V)\big|, \tag{3.2.11}$$

$$M_V := \max_{\{K,K' \in \mathcal{K}(V)\}} \big|v|_K(V) - v|_{K'}(V)\big|. \tag{3.2.12}$$

The number of elements sharing the vertex V is bounded by a constant C depending on γ and therefore

$$m_V \leq M_V \leq Cm_V. \tag{3.2.13}$$

We denote by K_V and K'_V the two elements for which the maximum m_V is reached in (3.2.11), and write e_V for the edge shared by those two elements. We remark that $e_V \in \mathcal{E}^I$ is always an inner edge, regardless of whether V is on the boundary or not. Then

$$q_V := [\![v]\!]|_{e_V}$$

is a polynomial of degree $\max(p_{K_V}, p_{K'_V})$ on e_V, and we choose its sign such that $q_V(V) \geq 0$. Moreover, we define for every $V \in \mathcal{N}(\mathcal{T})$ the values

$$a_V := \frac{1}{|\mathcal{K}(V)|} \sum_{K \in \mathcal{K}(V)} v|_K(V) \tag{3.2.14}$$

and for every $K \in \mathcal{K}(V)$

$$j_V(K) := v|_K(V) - a_V. \tag{3.2.15}$$

Then a_V is the arithmetic mean of the different limits of v at the vertex V, and $j_V(K)$ is its difference with the limit on the element K. It is easy to see that

$$|j_V(K)| \leq M_V \leq C m_V, \tag{3.2.16}$$

for all $V \in \mathcal{N}(\mathcal{T})$ and $K \in \mathcal{K}(V)$. Furthermore, we set with $\Pi_{p_V} \in \mathcal{P}_{p_V}(\hat{K})$ from Lemma 3.2.5

$$r_V^K := \Pi_{p_V} \circ F_{K,V}^{-1} \in \mathcal{P}_{p_V}(K), \tag{3.2.17}$$

for every $K \in \mathcal{T}$ and $V \in \mathcal{N}(K)$. Recall that $F_{K,V} : \hat{K} \to K$ is the affine map with $F_{K,V}(\mathbf{0}) = V$. Therefore the function r_V^K fulfills $r_V^K(V) = 1$ and $r_V^K(V') = 0$ for all $V' \neq V$.
2nd Step: Having disposed of this preliminary step, we turn our attention to the construction of the nodal part v^N. We start by defining local node functions $v_V^N \in S^{\boldsymbol{p}}(\mathcal{T})$ such that $\operatorname{supp}(v_V^N) \subseteq \omega_V$, $v_V^N|_K(V') = 0$ for every $K \in \mathcal{T}$, $V' \neq V$, and the limits of v_V^N coincide with the ones of v on each element $K \in \mathcal{K}(V)$ at the vertex V: Let

$$v_V^N|_K := \begin{cases} v|_K(V) r_V^K & \text{if } K \in \omega_V \\ 0 & \text{else,} \end{cases} \tag{3.2.18}$$

for every $K \in \mathcal{T}$. It is easy to see that v_V^N fulfills the above properties. Now we define the nodal part $v^N \in S^{\boldsymbol{p}}(\mathcal{T})$ as

$$v^N := \sum_{V \in \mathcal{N}(\mathcal{T})} v_V^N. \tag{3.2.19}$$

A few observations are in order. First of all, it holds indeed that

$$(v - v^N)|_K(V) = v|_K(V) - v_V^N|_K(V) = v|_K(V) - v|_K(V) \underbrace{r_V^K(V)}_{=1} = 0,$$

for every $K \in \mathcal{T}$ and every $V \in \mathcal{N}(K)$. Moreover, for $K \in \mathcal{K}(V)$ we want to estimate some norms of these functions on K, which will be important later.

- Using (3.2.6), (3.2.7), and (3.2.16), we observe

$$\begin{aligned}\|j_V(K)r_V^K\|_{L^2(K)} &\le Ch_V|j_V(K)|\|r_V^K\circ F_{K,V}\|_{L^2(\hat{K})} \le Ch_V m_V\|\Pi_{p_V}\|_{L^2(\hat{K})}\\ &\le Cm_V\frac{h_V}{p_V}\|\pi_{p_V}\|_{L^2(\hat{e})} \le C\frac{h_V^{1/2}}{p_V}\|q_V\|_{L^2(e_V)}\\ &= C\frac{h_V^{1/2}}{p_V}\|[\![v]\!]_N\|_{L^2(e_V)} \le C\frac{h_K^{1/2}}{p_K}\|[\![v]\!]\|_{L^2(\rho_K)}. \end{aligned} \tag{3.2.20}$$

For the inequality in the second line we transfered the integral to the edge e_V and used $q_V(V) = |[\![v]\!]|_{e_V}(V)| = m_V$, as well as (3.2.6) with $q = q_V \circ F_{e_V}$, where $F_{e_V} : \hat{e} \to e_V$. This gives

$$m_V\|\pi_{p_V}\circ F_{e_V}^{-1}\|_{L^2(e_V)} \le C\|q_V\|_{L^2(e_V)} = C\|[\![v]\!]\|_{L^2(e_V)}. \tag{3.2.21}$$

- Per construction we have that for some edge e, $r_V^K|_e$ is either zero or fulfills $\|r_V^K|_e\|_{L^2(e)} \le \|\pi_p\circ F_{e_V}\|_{L^2(e_V)}$ (cf. (3.2.7) and (3.2.17)). With (3.2.6), and because $|[\![v]\!]|_e(V)| \le m_V = |[\![v]\!]|_{e_V}(V)|$ for every edge e with one endpoint being the vertex V, we thus get for every $e \in \mathcal{E}$

$$\|j_V(K)r_V^K\|_{L^2(e)} \le C\|[\![v]\!]_N\|_{L^2(e_V)} \le C\|[\![v]\!]\|_{L^2(\rho_e)}. \tag{3.2.22}$$

Note that if $e \notin \mathcal{E}(V)$, then (3.2.22) is trivial since $r_V^K|_e \equiv 0$.

- For the H^1-seminorm on the element K we compute similarly and again with (3.2.7)

$$\begin{aligned}\|\nabla(j_V(K)r_V^K)\|_{L^2(K)} &\le Cm_V\|\nabla\Pi_{p_V}\|_{L^2(\hat{K})} \le Cm_V p_V\|\pi_{p_V}\|_{L^2(\hat{e})}\\ &\le C\frac{p_V}{h_V^{1/2}}\|[\![v]\!]_N\|_{L^2(e_V)} \le C\frac{p_K}{h_K^{1/2}}\|[\![v]\!]\|_{L^2(\rho_K)}. \end{aligned} \tag{3.2.23}$$

- Finally we will bound the jump of v^N across an interior edge $e \in \mathcal{E}^I$. Let e be shared by the two elements K, K' and let V, V' be its two endpoints. Since $\mathrm{supp}(v_{V''}^N) \subseteq \omega_{V''}$, $\forall V'' \in \mathcal{N}(\mathcal{T})$, only the functions v_V^N and $v_{V'}^N$ can be nonzero along e. Hence, the combination of the definitions (3.2.18), (3.2.17), (3.2.11) of v^N, r_V^K, and m_V, together with (3.2.21), imply

$$\begin{aligned}\|[\![v^N]\!]\|_{L^2(e)} &\le \|[\![v_V^N]\!]\|_{L^2(e)} + \|[\![v_{V'}^N]\!]\|_{L^2(e)}\\ &= \|(v|_K(V) - v|_{K'}(V))\pi_{p_V}\circ(F_{K,e}|_{\hat{e}})^{-1}\|_{L^2(e)}\\ &\quad + \|(v|_K(V') - v|_{K'}(V'))\pi_{p_{V'}}\circ(F_{K,e}|_{\hat{e}})^{-1}\|_{L^2(e)}\\ &\le m_V\|\pi_{p_V}\circ(F_{K,e}|_{\hat{e}})^{-1}\|_{L^2(e)} + m_{V'}\|\pi_{p_{V'}}\circ(F_{K,e}|_{\hat{e}})^{-1}\|_{L^2(e)}\\ &\le C\left(\|[\![v]\!]\|_{L^2(e_V)} + \|[\![v]\!]\|_{L^2(e_{V'})}\right)\\ &\le C\|[\![v]\!]\|_{L^2(\rho_e)}. \end{aligned} \tag{3.2.24}$$

Inspection of the steps shows that all appearing constants only depend on γ.
3rd Step: We want to define v^E and v^I. To begin with, we construct local lifting operators associated with an edge and an element. Let $e \in \mathcal{E}(\mathcal{T})$ and let $q \in \mathcal{P}_p(e)$ with $q(V) = q(V') = 0$ for both endpoints V, V' of e. We can extend q continuously by zero on the whole boundary ∂K of an element $K \in \mathcal{K}(e)$ and denote this function with $q^{\partial K}$. With this, the element map $F_K : \hat{K} \to K$, and L from Theorem 3.2.3 we define for every $e \in \mathcal{E}(\mathcal{T})$ and $K \in \mathcal{K}(e)$ the lifting operator

$$L_{K,e} := \begin{cases} \{q \in \mathcal{P}_p(e) : q(V) = 0 \ \forall V \in \mathcal{N}(e)\} \to H^1(K) \\ q \mapsto L(q^{\partial K} \circ F_K|_{\partial \hat{K}}) \circ F_K^{-1}. \end{cases}$$

Now let $w := v - v^N$, then $w|_K(V) = 0$ for all $K \in \mathcal{T},\ V \in \mathcal{N}(\mathcal{T})$. With the notation $w_K := w|_K$ and $w_{K,e} := w_K|_e$, the function $L_{K,e}(w_{K,e}) \in \mathcal{P}_p(K)$ is well-defined. We call v^E with

$$v^E|_K := \sum_{e \in \mathcal{E}(K)} L_{K,e}(w_{K,e}) \tag{3.2.25}$$

for every $K \in \mathcal{T}$, the edge part of v. Finally, we define the interior part v^I of v by

$$v^I := v - v^N - v^E.$$

Note that $v^I|_e \equiv 0$ for every $e \in \mathcal{E}(\mathcal{T})$ and therefore $v^I \in C^0(\Omega)$. We also point out that v^N, v^E, and v^I are all in $S^{\boldsymbol{p}}(\mathcal{T})$.
4th Step: We construct the approximant $v_*^N \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ of v^N. With (3.2.14) and (3.2.17) let

$$v_{*,V}^N|_K := \begin{cases} a_V r_V^K & \text{if } K \in \mathcal{K}(V) \\ 0 & \text{else,} \end{cases}$$

for every $K \in \mathcal{T}$. It is clear that $v_{*,V}^N$ is a continuous function on Ω with

$$\operatorname{supp}(v_{*,V}^N) \subseteq \omega_V, \qquad v_{*,V}^N(V') = \delta_{V,V'} \cdot a_V.$$

The approximant

$$v_*^N := \sum_{V \in \mathcal{N}(\mathcal{T})} v_{*,V}^N \tag{3.2.26}$$

is then in $S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ and has the property $v_*^N(V) = a_V$. Now let us bound the L^2-error of our approximation. With definition (3.2.19) of v^N and (3.2.26) of v_*^N we get

$$\begin{aligned} \|v^N - v_*^N\|_{L^2(K)} &\leq \sum_{V \in \mathcal{N}(\mathcal{T})} \|v_V^N - v_{*,V}^N\|_{L^2(K)} = \sum_{V \in \mathcal{N}(K)} \|v_V^N - v_{*,V}^N\|_{L^2(K)} \\ &= \sum_{V \in \mathcal{N}(K)} \|v|_K(V) r_V^K - a_V r_V^K\|_{L^2(K)}. \end{aligned}$$

Further with $j_V(K) = v|_K(V) - a_V$ and (3.2.20)

$$\begin{aligned}\|v^N - v_*^N\|_{L^2(K)} &\leq \sum_{V\in\mathcal{N}(K)} \|j_V(K) r_V^K\|_{L^2(K)} \leq C \sum_{V\in\mathcal{N}(K)} \frac{{h_K}^{1/2}}{p_K} \|[\![v]\!]\|_{L^2(\rho_K)} \\ &\leq C \frac{h_K^{1/2}}{p_K} \|[\![v]\!]\|_{L^2(\rho_K)}, \end{aligned} \tag{3.2.27}$$

where ρ_K is as in Definition 3.2.6. In a similar fashion and using (3.2.23), we compute

$$\begin{aligned}\|\nabla(v^N - v_*^N)\|_{L^2(K)} &\leq \sum_{V\in\mathcal{N}(\mathcal{T})} \|\nabla(v_V^N - v_{*,V}^N)\|_{L^2(K)} = \sum_{V\in\mathcal{N}(K)} \|\nabla(v_V^N - v_{*,V}^N)\|_{L^2(K)} \\ &= \sum_{V\in\mathcal{N}(K)} \|\nabla(v|_K(V) r_V^K - a_V r_V^K)\|_{L^2(K)} = \sum_{V\in\mathcal{N}(K)} \|j_V(K)\nabla(r_V^K)\|_{L^2(K)} \\ &\leq C \sum_{V\in\mathcal{N}(K)} \frac{p_K}{\sqrt{h_K}} \|[\![v]\!]\|_{L^2(\rho_K)} \leq C \frac{p_K}{\sqrt{h_K}} \|[\![v]\!]\|_{L^2(\rho_K)}. \end{aligned} \tag{3.2.28}$$

5th Step: In this step we construct the approximant $v_*^E \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ of v^E. Recall that $w_{K,e} = ((v - v^N)|_K)|_e$. For every edge $e \in \mathcal{T}$ we then define the polynomial $q_e \in \mathcal{P}_p(e)$ as

$$q_e := \begin{cases} w_{K,e} & \text{if } e = \mathcal{E}(K) \cap \mathcal{E}(K') \text{ and } p_K \leq p_{K'}, \\ w_{K',e} & \text{if } e = \mathcal{E}(K) \cap \mathcal{E}(K') \text{ and } p_K > p_{K'}, \\ w_{K,e} & \text{if } e \in \mathcal{E}^B(K). \end{cases} \tag{3.2.29}$$

We have $q_e(V) = 0$ for $V \in \mathcal{N}(e)$, since $w_{K,e}$ respectively $w_{K',e}$ fulfill this. Thus we are able to extend q_e continuously with zero on the other edges of an element $K \in \mathcal{K}(e)$, and $L_{K,e}(q_e) \in \mathcal{P}_p(K)$ is well-defined and vanishes on $\partial K \backslash e$. The approximant defined as

$$v_*^E|_K := \sum_{e\in\mathcal{E}(K)} L_{K,e}(q_e), \tag{3.2.30}$$

for every $K \in \mathcal{T}$, is then continuous and in $S^{\boldsymbol{p}}(\mathcal{T})$ because, for interior edges, q_e is defined as the polynomial with the smaller polynomial degree of the two adjacent elements. We begin with the estimate in the L^2-norm. Recalling that functions with superscript ∂K denote extensions by zero on ∂K, we obtain with the linearity of the operator $L_{K,e}$

$$\begin{aligned}\|v^E - v_*^E\|_{L^2(K)} &= \sum_{e\in\mathcal{E}(K)} \|L_{K,e}(w_{K,e}) - L_{K,e}(q_e)\|_{L^2(K)} \\ &= \sum_{e\in\mathcal{E}(K)} \|L_{K,e}(w_{K,e} - q_e)\|_{L^2(K)} \\ &= \sum_{e\in\mathcal{E}(K)\backslash\mathcal{E}^B} \|L((w_{K,e}^{\partial K} - q_e^{\partial K}|_{\partial\hat{K}}) \circ F_K|_{\partial\hat{K}}) \circ F_K^{-1}\|_{L^2(K)}, \end{aligned}$$

since per definition (3.2.29) it holds that $q_e = w_{K,e}$ if $e \in \mathcal{E}^B$. Now we use a scaling argument, (3.2.3a), and $|w_{K,e}^{\partial K}|_e - q_e^{\partial K}|_e| \leq |w_{K,e} - w_{K',e}|$, where K' is the element sharing the common interior edge e with K. This allows to bound the L^2-error with

$$\begin{aligned}
\|v^E - v_*^E\|_{L^2(K)}^2 &= \sum_{e\in\mathcal{E}^I(K)} \int_K \left(L((w_{K,e}^{\partial K} - q_e^{\partial K}) \circ F_K|_{\partial\hat{K}}) \circ F_K^{-1}\right)^2 \,\mathrm{d}\boldsymbol{x} \\
&= \sum_{e\in\mathcal{E}^I(K)} \int_{\hat{K}} \left(L((w_{K,e}^{\partial K} - q_e^{\partial K}) \circ F_K|_{\partial\hat{K}})\right)^2 |\det F_K'| \,\mathrm{d}\boldsymbol{x} \\
&\leq C \sum_{e\in\mathcal{E}^I(K)} h_K^2 \int_{\hat{K}} \left(L((w_{K,e}^{\partial K} - q_e^{\partial K}) \circ F_K|_{\partial\hat{K}})\right)^2 \,\mathrm{d}\boldsymbol{x} \\
&\leq C \sum_{e\in\mathcal{E}^I(K)} h_K^2 \int_{F_K^{-1}(e)} \left((w_{K,e}^{\partial K} - q_e^{\partial K}) \circ F_K\right)^2 \,\mathrm{d}S \\
&\leq C \sum_{e\in\mathcal{E}^I(K)} h_K^2 \int_{F_K^{-1}(e)} \left(|w_{K,e} - w_{K',e}| \circ F_K\right)^2 \,\mathrm{d}S \\
&\leq C \sum_{e\in\mathcal{E}^I(K)} h_K \left\|w_{K,e} - w_{K',e}\right\|_{L^2(e)}^2 .
\end{aligned} \tag{3.2.31}$$

In the last step the one-dimensional integral was transfered back to the edge e. With $(v - v^N)_{K,e} := ((v - v^N)|_K)|_e = w_{K,e}$ and similar definitions for $v_{K,e}$ and $v_{K,e}^N$, we get

$$\begin{aligned}
\|w_{K,e} - w_{K',e}\|_{L^2(e)} &= \|(v - v^N)_{K,e} - (v - v^N)_{K',e}\|_{L^2(e)} \\
&\leq \|v_{K,e} - v_{K',e}\|_{L^2(e)} + \|v_{K,e}^N - v_{K',e}^N\|_{L^2(e)} \\
&= \|[\![v]\!]\|_{L^2(e)} + \|[\![v^N]\!]\|_{L^2(e)}.
\end{aligned} \tag{3.2.32}$$

The jump of v^N was bounded in (3.2.24) and hence

$$\|v^E - v_*^E\|_{L^2(K)}^2 \leq C \sum_{e\in\mathcal{E}^I(K)} h_e \|[\![v]\!]\|_{L^2(e)}^2 \leq C h_K \|[\![v]\!]\|_{L^2(\rho_K)}^2. \tag{3.2.33}$$

It remains to estimate the error in the H^1-seminorm. Similar calculations as above, a scaling

argument, and (3.2.3b) lead to

$$\begin{aligned}
\|\nabla(v^E - v_*^E)\|_{L^2(K)}^2 &\leq \sum_{e\in\mathcal{E}^I(K)} \|\nabla(L_{K,e}(w_{K,e} - q_e))\|_{L^2(K)}^2 \\
&= \sum_{e\in\mathcal{E}^I(K)} \|\nabla(L((w_{K,e}^{\partial K} - q_e^{\partial K}) \circ F_K|_{\partial\hat{K}}) \circ F_K^{-1})\|_{L^2(K)}^2 \\
&\leq C \sum_{e\in\mathcal{E}^I(K)} \|\nabla(L((w_{K,e}^{\partial K} - q_e^{\partial K}) \circ F_K|_{\partial\hat{K}}))\|_{L^2(\hat{K})}^2 \\
&\leq C \sum_{e\in\mathcal{E}^I(K)} \|(w_{K,e} - q_e) \circ F_K\|_{H^{1/2}(F_K^{-1}(e))}^2 \\
&= C \sum_{e\in\mathcal{E}^I(K)} \left\|(w_{K,e} - w_{K',e}) \circ F_K\right\|_{H^{1/2}(F_K^{-1}(e))}^2 \\
&= C \sum_{e\in\mathcal{E}^I(K)} \left\|((v - v^N)_{K,e} - (v - v^N)_{K',e}) \circ F_K\right\|_{H^{1/2}(F_K^{-1}(e))}^2, \qquad (3.2.34)
\end{aligned}$$

where K' is again the element with $e = \mathcal{E}(K) \cap \mathcal{E}(K')$. With Lemma 3.2.4, we obtain

$$\begin{aligned}
&\|((v - v^N)_{K,e} - (v - v^N)_{K',e}) \circ F_K\|_{H^{1/2}(F_K^{-1}(e))} \\
&\qquad \leq \|[\![v]\!] \circ F_K\|_{H^{1/2}(F_K^{-1}(e))} + \|[\![v^N]\!] \circ F_K\|_{H^{1/2}(F_K^{-1}(e))} \\
&\qquad \leq C p_K \left(\|[\![v]\!] \circ F_K\|_{L^2(F_K^{-1}(e))} + \|[\![v^N]\!] \circ F_K\|_{L^2(F_K^{-1}(e))}\right),
\end{aligned}$$

and conclude with (3.2.24)

$$\left\|\nabla(v^E - v_*^E)\right\|_{L^2(K)}^2 \leq C\frac{p_K^2}{h_K}\|[\![v]\!]\|_{L^2(\rho_K)}^2. \qquad (3.2.35)$$

6th Step: Finally, we prove the assertion. Let $v_*^I := v^I \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ and consider the approximant

$$v^* := v_*^N + v_*^E + v_*^I \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega).$$

The desired estimates (3.2.10b) and (3.2.10c) follow with (3.2.27) and (3.2.33):

$$\begin{aligned}
\|v - v^*\|_{L^2(K)}^2 &\leq \left(\|v^N - v_*^N\|_{L^2(K)}^2 + \|v^E - v_*^E\|_{L^2(K)}^2 + \|v^I - v_*^I\|_{L^2(K)}^2\right) \\
&\leq C h_K \|[\![v]\!]\|_{L^2(\rho_K)}^2,
\end{aligned}$$

respectively with (3.2.28) and (3.2.35):

$$\begin{aligned}
\|\nabla(v - v^*)\|_{L^2(K)}^2 &\leq C\left(\|\nabla(v^N - v_*^N)\|_{L^2(K)}^2 + \|\nabla(v^E - v_*^E)\|_{L^2(K)}^2 + \|\nabla(v^I - v_*^I)\|_{L^2(K)}^2\right) \\
&\leq C\frac{p_K^2}{h_K}\|[\![v]\!]\|_{L^2(\rho_K)}^2.
\end{aligned}$$

It remains to prove (3.2.10a). Let $e \in \mathcal{E}^B$ be a boundary edge with endpoints V, V', and belonging to the element K. Then with the notation from the first step and (3.2.22)

$$\begin{aligned}\|v^N - v_*^N\|_{L^2(e)} &= \|(v|_K(V)r_V^K - a_V r_V^K) + (v_K(V')r_{V'}^K - a_{V'}r_{V'}^K)\|_{L^2(e)} \\ &\le |j_V(K)|\|r_V^K\|_{L^2(e)} + |j_{V'}(K)|\|r_{V'}^K\|_{L^2(e)} \\ &\le C\left(\|[\![v]\!]\|_{L^2(e_V)} + \|[\![v]\!]\|_{L^2(e_{V'})}\right).\end{aligned}$$

Recall that $v^I|_e = v_*^I|_e \equiv 0$ and $v^E|_e = v_*^E|_e$ for boundary edges (cf. (3.2.29)). Therefore

$$\begin{aligned}\|v - v^*\|_{L^2(e)}^2 &\le C\left(\|v^N - v_*^N\|_{L^2(e)}^2 + \|v^E - v_*^E\|_{L^2(e)}^2 + \|v^I - v_*^I\|_{L^2(e)}^2\right) \\ &\le C\|[\![v]\!]\|_{L^2(\rho_e)}^2,\end{aligned}$$

which gives (3.2.10a) for boundary edges. To prove the estimate for interior edges, the contribution of the edge part additionally needs to be taken into account. Assume that $e \in \mathcal{E}^I(K) \cap \mathcal{E}^I(K')$ and without loss of generality let $v^E|_K|_e \neq v_*^E|_K|_e$. Then (cf. (3.2.29) and (3.2.25))

$$\begin{aligned}\|(v^E - v_*^E)|_K\|_{L^2(e)} &= \|v^E|_K - v^E|_{K'}\|_{L^2(e)} = \|(v - v^N)|_K - (v - v^N)|_{K'}\|_{L^2(e)} \\ &\le \|[\![v]\!]\|_{L^2(e)} + \|[\![v^N]\!]\|_{L^2(e)}. \qquad (3.2.36)\end{aligned}$$

The first term after the last equality sign is the one we want to get, and the second one was bounded in (3.2.24). This concludes the proof. □

Remark 3.2.8. In light of (3.2.2b), the weight h_K instead of h_K/p_K^2 in (3.2.10b) seems to be suboptimal. In fact, whereas the error of the nodal part would give h_K/p_K^2 (cf. (3.2.27)), the problem is that the edge lifting (3.2.3a) is only stable (but not more) in L^2. However, as the next corollary shows, the weights are good enough for us. For tensor product spaces (on quadrilaterals) an optimal edge lifting can be constructed (for example with the use of π_p from (3.2.5)). In this case the optimal weight is obtained (see [36], where Lagrange polynomials associated with Gauss-Lobatto nodes are used).

We now state the above theorem in the way it will be used it in the next chapter.

Corollary 3.2.9 (Conforming error). *With the same assumptions as in the previous theorem, there exists a constant $C > 0$ solely depending on γ such that for every $v \in S^{\boldsymbol{p}}(\mathcal{T})$ there is a function $v^* \in S^{\boldsymbol{p}}(\mathcal{T}) \cap C^0(\Omega)$ with*

$$\begin{aligned}k^2\|v - v^*\|_{L^2(\Omega)}^2 + \|\nabla(v - v^*)\|_{L^2(\Omega)}^2 + k\|v - v^*\|_{L^2(\partial\Omega)}^2 \\ \le \frac{C}{\mathfrak{a}} \sum_{e \in \mathcal{E}^I} \left(1 + \frac{k^2h_e^2 + kh_e}{p_e^2}\right) \|(k\alpha)^{1/2}[\![v]\!]\|_{L^2(e)}^2, \qquad (3.2.37)\end{aligned}$$

where α is as in (2.3.5).

Proof. Recall that $k\alpha|_e = \mathfrak{a}p_e^2/h_e$. Summing over all elements and with the use of Theorem 3.2.7 we readily obtain (3.2.37). □

Chapter 4

A Posteriori Error Estimation

The goal of this chapter is to find an a posteriori error estimate for the solution of (2.3.1). We make use of some general techniques as described in [3, 28, 34] and apply ideas from [13] and [15]. To start with, we will define non-weighted error indicators and prove that they can be used to estimate the error from above. This means that the estimator is **reliable**. Afterwards we consider weighted error indicators, which then allows to prove efficiency in the succeeding section.

4.1 Reliability

Two problems will have to be dealt with to obtain a reliability estimate: The first one is that the sesquilinear form $a_{\mathcal{T}}$ is not coercive on $H^{3/2}(\Omega)$ (and does not even fulfill a Gårding inequality), which is a general concern of DG methods. The second one is the lower order term $-k^2(\cdot,\cdot)_{L^2(\Omega)}$ contained in $a_{\mathcal{T}}$, and this is characteristic for the Helmholtz equation. The first problem will be overcome by introducing a new sesquilinear form, the second one can be resolved with an Aubin-Nitsche-type argument. Before we start, as a preparatory result, we compute an alternative representation of the term $a_{\mathcal{T}}(u-u_{\mathcal{T}},v)$, which will be needed frequently in the following.

Lemma 4.1.1. *Let $u \in H_{\mathcal{T}}^{3/2+\varepsilon}(\Omega)$ be the solution of* (2.1.2) *for some $\varepsilon > 0$ and let $u_{\mathcal{T}}$ be the solution of* (2.3.1)*. Then we have for every $v \in H_{\mathcal{T}}^{3/2+\varepsilon}(\Omega)$*

$$\begin{aligned} a_{\mathcal{T}}(u-u_{\mathcal{T}},v) = &\sum_{K\in\mathcal{T}} \int_K (f+\Delta_{\mathcal{T}}u_{\mathcal{T}}+k^2u_{\mathcal{T}})\overline{v}\ \mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}^I} [\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\{\overline{v}\}\ \mathrm{d}S + \int_{\mathcal{E}^I} [\![u_{\mathcal{T}}]\!]_N\{\overline{\nabla_{\mathcal{T}}v}\}\ \mathrm{d}S \\ &+ \int_{\mathcal{E}^B} (1-\delta)(g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}})\overline{v}\ \mathrm{d}S - \frac{1}{\mathrm{i}k}\int_{\mathcal{E}^B} \delta(g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}}v}\ \mathrm{d}S \\ &- \mathrm{i}k\int_{\mathcal{E}^I} \alpha[\![u_{\mathcal{T}}]\!]_N[\![\overline{v}]\!]_N\ \mathrm{d}S + \frac{1}{\mathrm{i}k}\int_{\mathcal{E}^I} \beta[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N[\![\overline{\nabla_{\mathcal{T}}v}]\!]_N\ \mathrm{d}S. \end{aligned} \tag{4.1.1}$$

Proof. With definition (2.3.2) of $a_{\mathcal{T}}$ we get

$$\begin{aligned}
a_{\mathcal{T}}(u-u_{\mathcal{T}},v) =& (\nabla_{\mathcal{T}}(u-u_{\mathcal{T}}),\nabla_{\mathcal{T}}v)_{L^2(\Omega)} - k^2(u-u_{\mathcal{T}},v)_{L^2(\Omega)} \\
&- \int_{\mathcal{E}^I} [\![u-u_{\mathcal{T}}]\!]_N \cdot \{\overline{\nabla_{\mathcal{T}}v}\} \,\mathrm{d}S - \int_{\mathcal{E}^I} \{\nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\} \cdot [\![\overline{v}]\!]_N \,\mathrm{d}S \\
&- \int_{\mathcal{E}^B} \delta(u-u_{\mathcal{T}})\overline{\nabla_{\mathcal{T}}v\cdot\boldsymbol{n}} \,\mathrm{d}S - \int_{\mathcal{E}^B} \delta\nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\cdot\boldsymbol{n}\overline{v} \,\mathrm{d}S \\
&- \frac{1}{\mathrm{i}k}\int_{\mathcal{E}^I} \beta[\![\nabla_{\mathcal{T}}(u-u_{\mathcal{T}})]\!]_N[\![\overline{\nabla_{\mathcal{T}}v}]\!]_N \,\mathrm{d}S - \frac{1}{\mathrm{i}k}\int_{\mathcal{E}^B} \delta\nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\cdot\boldsymbol{n}\overline{\nabla_{\mathcal{T}}v\cdot\boldsymbol{n}} \,\mathrm{d}S \\
&+ \mathrm{i}k\int_{\mathcal{E}^I} \alpha[\![u-u_{\mathcal{T}}]\!]_N[\![\overline{v}]\!]_N \,\mathrm{d}S + \mathrm{i}k\int_{\mathcal{E}^B} (1-\delta)(u-u_{\mathcal{T}})\overline{v} \,\mathrm{d}S. \qquad (4.1.2)
\end{aligned}$$

Moreover, $-\Delta u - k^2 u = f$ in Ω. Integrating by parts, we then obtain for the term $(\nabla_{\mathcal{T}}(u-u_{\mathcal{T}}),v) - k^2(u-u_{\mathcal{T}},v)$ in (4.1.2) with the so called DG magic formula

$$\begin{aligned}
(\nabla_{\mathcal{T}}(u-u_{\mathcal{T}}),\nabla_{\mathcal{T}}v) - k^2(u-u_{\mathcal{T}},v)_{L^2(\Omega)} =& \\
&\sum_{K\in\mathcal{T}}\left(\int_K -\Delta_{\mathcal{T}}(u-u_{\mathcal{T}})\overline{v}\,\mathrm{d}\boldsymbol{x} + \sum_{e\in\partial K}\int_e \nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\cdot\boldsymbol{n}_K\overline{v}\,\mathrm{d}S\right) \\
&- k^2(u-u_{\mathcal{T}},v)_{L^2(\Omega)} \\
=& \sum_{K\in\mathcal{T}}\int_K (f+\Delta_{\mathcal{T}}u_{\mathcal{T}}+k^2u_{\mathcal{T}})\overline{v}\,\mathrm{d}\boldsymbol{x} + \sum_{e\in\mathcal{E}^B}\int_e \nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\cdot\boldsymbol{n}\overline{v}\,\mathrm{d}S \\
&+ \sum_{e\in\mathcal{E}^I}\int_e \Big([\![\nabla_{\mathcal{T}}(u-u_{\mathcal{T}})]\!]_N\{\overline{v}\} + \{\nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\}[\![\overline{v}]\!]_N\Big)\,\mathrm{d}S, \qquad (4.1.3)
\end{aligned}$$

where $\boldsymbol{n}_K$ is the outer normal of the domain K, and $\boldsymbol{n}$ is the outer normal of Ω. Now we insert (4.1.3) into (4.1.2) and use $\partial_{\boldsymbol{n}}(u-u_{\mathcal{T}}) + \mathrm{i}k(u-u_{\mathcal{T}}) = g - \partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}}$ on $\partial\Omega$. This yields for the integrals on the boundary

$$\begin{aligned}
&- \int_{\mathcal{E}^B} \delta(u-u_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}}v}\,\mathrm{d}S - \frac{1}{\mathrm{i}k}\int_{\mathcal{E}^B} \delta\partial_{\boldsymbol{n}}(u-u_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}}v}\,\mathrm{d}S - \int_{\mathcal{E}^B} \delta\partial_{\boldsymbol{n}}(u-u_{\mathcal{T}})\overline{v}\,\mathrm{d}S \\
&+ \mathrm{i}k\int_{\mathcal{E}^B} (1-\delta)(u-u_{\mathcal{T}})\overline{v}\,\mathrm{d}S + \int_{\mathcal{E}^B} \partial_{\boldsymbol{n}}(u-u_{\mathcal{T}})\overline{v}\,\mathrm{d}S = \\
&\int_{\mathcal{E}^B} (1-\delta)(g-\partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\overline{v}\,\mathrm{d}S - \frac{1}{\mathrm{i}k}\int_{\mathcal{E}^B} \delta(g-\partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}}v}\,\mathrm{d}S.
\end{aligned}$$

The regularity of the solution $u \in H^{3/2+\varepsilon}$, $\varepsilon > 0$ implies $[\![u]\!]_N = [\![\nabla_{\mathcal{T}}u]\!]_N = 0$ for interior edges, which is why these terms vanish. Finally, we put everything together and aggregate the right terms which gives (4.1.1). □

4.1.1 Non-weighted Error Indicator

Definition 4.1.2 (Error indicators). *Let $u \in H^{3/2+\varepsilon}(\Omega)$ be the solution of* (2.1.2) *and let $u_{\mathcal{T}} \in S^{\boldsymbol{p}}(\mathcal{T})$ be the solution of* (2.3.1)*. Then we introduce for every $K \in \mathcal{T}$ the* **local error indicators** *of $u_{\mathcal{T}}$:*

$$\eta_{R_K}(u_{\mathcal{T}})^2 := \left(\frac{h_K}{p_K}\right)^2 \|\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f\|^2_{L^2(K)}, \tag{4.1.4a}$$

$$\eta_{E_K}(u_{\mathcal{T}})^2 := \sum_{e\in\mathcal{E}^I(K)} \frac{1}{2}\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}^B(K)} h_e\|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|^2_{L^2(e)}, \tag{4.1.4b}$$

$$\eta_{J_K}(u_{\mathcal{T}})^2 := \sum_{e\in\mathcal{E}^I(K)} \frac{1}{2}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(e)}, \tag{4.1.4c}$$

$$\eta_K(u_{\mathcal{T}})^2 := \eta_{R_K}(u_{\mathcal{T}})^2 + \eta_{E_K}(u_{\mathcal{T}})^2 + \eta_{J_K}(u_{\mathcal{T}})^2. \tag{4.1.4d}$$

The first one is also called the **internal residual** *and the second term in* (4.1.4b) *is the* **edge residual**. *The* **global error indicator** *of $u_{\mathcal{T}}$ is given by*

$$\eta(u_{\mathcal{T}})^2 := \sum_{K\in\mathcal{T}} \eta_K(u_{\mathcal{T}})^2.$$

When no confusion can arise we omit the argument $u_{\mathcal{T}}$ and write $\eta_K := \eta_K(u_{\mathcal{T}})$, and similar for the other quantities.

We start our analysis of these error indicators with the following lemma that bounds the error, with respect to a norm that is linked to the DG-norm, by parts of the estimator plus the k-weighted L^2-norm of the actual error. As we have already mentioned, neither $a_{\mathcal{T}}$ nor $a_{\mathcal{T}} + k^2(\cdot,\cdot)_{L^2(\Omega)}$ is coercive on $H^{3/2}(\Omega)$. To avoid this problem, one has to define a new sesquilinear form. Oftentimes this is done via lifting operators. We choose a different path, and simply drop those integrals in $a_{\mathcal{T}}$, which destroy coercivity. This idea has been used similarly for an interior penalty h-DGFEM in [13, Section 5.6.2.2]. At this point, the C^1 interpolant from Chapter 3 will come into play and allow us to avoid terms containing jumps of the gradient.

Lemma 4.1.3. *Let $u \in H^{3/2+\varepsilon}(\Omega)$ be the solution of* (2.1.2) *for some $\varepsilon > 0$. Let $u_{\mathcal{T}} \in S^{\boldsymbol{p}}(\mathcal{T})$ be the solution of* (2.3.1) *with $S = S^{\boldsymbol{p}}(\mathcal{T})$, where $\mathcal{T}$ is an admissible γ-shape regular triangulation of Ω and $p_{\mathcal{T}}$ fulfills $p_{\mathcal{T}} \geq 5$. Further assume $\mathfrak{a} \geq 1$. Then there exists a constant $C > 0$ solely depending on $\gamma, \mathfrak{b}, \mathfrak{d}$, and Ω such that*

$$\begin{aligned}\|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} &+ k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)} + \sqrt{k}\|u - u_{\mathcal{T}}\|_{L^2(\partial\Omega)} \\ &\leq CC_{conf}\left(\sum_{K\in\mathcal{T}}\left(\eta^2_{R_K} + \eta^2_{J_K}\right) + \sum_{e\in\mathcal{E}^B} h_e\|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|^2_{L^2(e)}\right)^{1/2} + 2k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)},\end{aligned} \tag{4.1.5}$$

where, with $h_{\mathcal{T}}$, $p_{\mathcal{T}}$ as in (1.1.3), (1.2.2),

$$C_{conf} := 1 + \frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}. \tag{4.1.6}$$

Proof. We introduce

$$\tilde{a}_{\mathcal{T}}(v_1, v_2) := (\nabla_{\mathcal{T}} v_1, \nabla_{\mathcal{T}} v_2)_{L^2(\Omega)} + k^2 (v_1, v_2)_{L^2(\Omega)} + \mathrm{i}k \int_{\partial\Omega} v_1 \overline{v_2} \, \mathrm{d}S,$$

and the associated norm

$$\|v\|_{\tilde{a}}^2 := |\tilde{a}_{\mathcal{T}}(v, v)|, \tag{4.1.7}$$

where $v_1, v_2, v \in H^1_{\mathcal{T}}(\Omega)$. This norm is equivalent to the left-hand side of (4.1.5). Integrating by parts we observe for sufficiently smooth functions v_1, v_2

$$\begin{aligned}
\tilde{a}_{\mathcal{T}}(v_1, v_2) &= \sum_{K \in \mathcal{T}} \int_K \nabla_{\mathcal{T}} v_1 \overline{\nabla_{\mathcal{T}} v_2} + k^2 v_1 \overline{v_2} \, \mathrm{d}\boldsymbol{x} + \mathrm{i}k \int_{\partial\Omega} v_1 \overline{v_2} \, \mathrm{d}S \\
&= \sum_{K \in \mathcal{T}} \left(\int_K -\Delta_{\mathcal{T}} v_1 \overline{v_2} + k^2 v_1 \overline{v_2} \, \mathrm{d}\boldsymbol{x} + \sum_{e \in \mathcal{E}(K)} \int_e \nabla_{\mathcal{T}} v_1 \cdot \boldsymbol{n} \overline{v_2} \, \mathrm{d}S \right) + \sum_{e \in \mathcal{E}^B} \mathrm{i}k \int_e v_1 \overline{v_2} \, \mathrm{d}S \\
&= \sum_{K \in \mathcal{T}} \left(\int_K (-\Delta_{\mathcal{T}} v_1 - k^2 v_1) \overline{v_2} \, \mathrm{d}\boldsymbol{x} + 2k^2 \int_K v_1 \overline{v_2} \, \mathrm{d}\boldsymbol{x} \right) + \sum_{e \in \mathcal{E}^B} \int_e (\mathrm{i}k v_1 + \partial_{\boldsymbol{n}} v_1) \overline{v_2} \, \mathrm{d}S \\
&\quad + \sum_{e \in \mathcal{E}^I} \int_e [\![\nabla_{\mathcal{T}} v_1]\!]_N \{\overline{v_2}\} + \{\nabla_{\mathcal{T}} v_1\} [\![\overline{v_2}]\!]_N \, \mathrm{d}S.
\end{aligned}$$

Since u is a solution of (2.1.2), it holds that $-\Delta_{\mathcal{T}}(u - u_{\mathcal{T}}) - k^2(u - u_{\mathcal{T}}) = \Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f$ in Ω and $\partial_{\boldsymbol{n}}(u - u_{\mathcal{T}}) + \mathrm{i}k(u - u_{\mathcal{T}}) = g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}$ on $\partial\Omega$. Let $\varphi \in H^1(\Omega)$. Then we have $[\![\varphi]\!] = 0$ and, because of $u \in H^{3/2+\varepsilon}(\Omega)$, we also have $[\![u]\!] = [\![\nabla_{\mathcal{T}} u]\!] = 0$ on interior edges. Therefore,

$$\begin{aligned}
\tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi) = &\int_\Omega (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f) \overline{\varphi} \, \mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}^I} [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N \overline{\varphi} \, \mathrm{d}S + \int_{\mathcal{E}^B} (g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}) \overline{\varphi} \, \mathrm{d}S \\
&+ 2k^2 (u - u_{\mathcal{T}}, \varphi)_{L^2(\Omega)}.
\end{aligned} \tag{4.1.8}$$

With $u^*_{\mathcal{T}}$ denoting the conforming approximant of $u_{\mathcal{T}}$ constructed in Theorem 3.2.7, Corollary 3.2.9 implies due to $\mathfrak{a} \geq 1$

$$\begin{aligned}
\|u - u_{\mathcal{T}}\|_{\tilde{a}} &\leq \|u - u^*_{\mathcal{T}}\|_{\tilde{a}} + \|u^*_{\mathcal{T}} - u_{\mathcal{T}}\|_{\tilde{a}} \\
&\leq \|u - u^*_{\mathcal{T}}\|_{\tilde{a}} + C C_{conf} \left(\sum_{e \in \mathcal{E}^I} \|(k\alpha)^{1/2} [\![u_{\mathcal{T}}]\!]\|^2_{L^2(e)} \right)^{1/2}.
\end{aligned} \tag{4.1.9}$$

To estimate the first term in (4.1.9) we define the set

$$\Phi := \left\{ \varphi \in H^1(\Omega) \cap H_{\mathcal{T}}^{3/2+\varepsilon}(\Omega) : \|\varphi\|_{\tilde{a}} \leq 1 \right\}.$$

Moreover, let $I^{hp} := I_1^{hp}$ be the interpolation operator from Theorem 3.1.10. Then $(u - u_{\mathcal{T}}^*)/\|u - u_{\mathcal{T}}^*\|_{\tilde{a}} \in \Phi$ and we obtain again with Corollary 3.2.9 and Cauchy-Schwarz

$$\begin{aligned}
\|u - u_{\mathcal{T}}^*\|_{\tilde{a}} &\leq \sup_{\varphi \in \Phi} |\tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}^*, \varphi)| \\
&\leq \sup_{\varphi \in \Phi} |\tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi)| + \sup_{\varphi \in \Phi} |\tilde{a}_{\mathcal{T}}(u_{\mathcal{T}} - u_{\mathcal{T}}^*, \varphi)| \\
&\leq \sup_{\varphi \in \Phi} |\tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi)| + \sup_{\varphi \in \Phi} \|u_{\mathcal{T}} - u_{\mathcal{T}}^*\|_{\tilde{a}} \|\varphi\|_{\tilde{a}} \\
&\leq \sup_{\varphi \in \Phi} \Big| \tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi) - \underbrace{a_{\mathcal{T}}(u - u_{\mathcal{T}}, I^{hp}\varphi)}_{=0} \Big| + C C_{conf} \left(\sum_{e \in \mathcal{E}^I} \|(k\alpha)^{1/2} [\![u_{\mathcal{T}}]\!]\|_{L^2(e)}^2 \right)^{1/2}.
\end{aligned} \tag{4.1.10}$$

Next, we use representation (4.1.8) of $\tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi)$ and representation (4.1.1) of $a_{\mathcal{T}}(u - u_{\mathcal{T}}, I^{hp}\varphi)$ to evaluate the term in the remaining supremum

$$\begin{aligned}
&\tilde{a}_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi) - a_{\mathcal{T}}(u - u_{\mathcal{T}}, I^{hp}\varphi) = \\
&\quad \int_\Omega (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f)\overline{\varphi} \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}^I} [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N \overline{\varphi} \,\mathrm{d}S + \int_{\mathcal{E}^B} (g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})\overline{\varphi} \,\mathrm{d}S \\
&\quad + 2k^2 (u - u_{\mathcal{T}}, \varphi)_{L^2(\Omega)} \\
&\quad - \Bigg(\int_\Omega (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f)\overline{I^{hp}\varphi} \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}^I} [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N \{\overline{I^{hp}\varphi}\} \,\mathrm{d}S + \int_{\mathcal{E}^I} [\![u_{\mathcal{T}}]\!]_N \{\overline{\nabla_{\mathcal{T}} I^{hp}\varphi}\} \,\mathrm{d}S \\
&\quad + \int_{\mathcal{E}^B} (1 - \delta)(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})\overline{I^{hp}\varphi} \,\mathrm{d}S - \int_{\mathcal{E}^B} \frac{\delta}{\mathrm{i}k}(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}} I^{hp}\varphi} \,\mathrm{d}S \\
&\quad - \mathrm{i}k \int_{\mathcal{E}^I} \alpha [\![u_{\mathcal{T}}]\!]_N \underbrace{[\![\overline{I^{hp}\varphi}]\!]_N}_{=0} \,\mathrm{d}S + \frac{1}{\mathrm{i}k} \int_{\mathcal{E}^I} \beta [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N \underbrace{[\![\overline{\nabla_{\mathcal{T}} I^{hp}\varphi}]\!]_N}_{=0} \,\mathrm{d}S \Bigg) \\
&= \int_\Omega (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f)(\overline{\varphi - I^{hp}\varphi}) \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}^I} [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N (\overline{\varphi - I^{hp}\varphi}) \,\mathrm{d}S \\
&\quad + \int_{\mathcal{E}^B} (g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})(\overline{\varphi - I^{hp}\varphi}) \,\mathrm{d}S + \int_{\mathcal{E}^B} \delta(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})(\overline{I^{hp}\varphi}) \,\mathrm{d}S \\
&\quad - \int_{\mathcal{E}^I} [\![u_{\mathcal{T}}]\!]_N \{\overline{\nabla_{\mathcal{T}} I^{hp}\varphi}\} \,\mathrm{d}S + \int_{\mathcal{E}^B} \frac{\delta}{\mathrm{i}k}(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}} I^{hp}\varphi} \,\mathrm{d}S + 2k^2 (u - u_{\mathcal{T}}, \varphi)_{L^2(\Omega)}.
\end{aligned} \tag{4.1.11}$$

We denote the terms after the last equality sign in (4.1.11) from left to right with $T_1, \ldots, T_7$ and seperately investigate the integrals $T_1, \ldots, T_6$ in the sequel. For T_1, we use (3.1.14) and obtain

$$\left| \int_K (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f)(\varphi - I^{hp}\varphi) \,\mathrm{d}\boldsymbol{x} \right| \leq C \left\| \Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f \right\|_{L^2(K)} \|\nabla \varphi\|_{L^2(\omega_K)} \frac{h_K}{p_K}.$$

With the inverse estimate (3.2.2b) and the stability of I^{hp} in $H^1(K)$ it holds for $K \in \mathcal{K}(e)$, due to a scaling argument, that

$$\|\nabla_{\mathcal{T}} I^{hp}\varphi|_K\|_{L^2(e)} \leq C\frac{p_e}{h_e^{1/2}}\|\nabla I^{hp}\varphi\|_{L^2(K)} \leq C\frac{p_e}{h_e^{1/2}}\|\nabla\varphi\|_{L^2(\omega_K)}\,. \tag{4.1.12}$$

Let now $e \in \mathcal{E}^I$. Recall that $\beta/k|_e \sim h_e/p_e$. Thus we get for T_2 with (3.1.15)

$$\begin{aligned}\left|\int_e [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N(\overline{\varphi - I^{hp}\varphi})\,\mathrm{d}S\right| &= \left|\int_e \left((\beta/k)^{1/2}[\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N\right)\left((\beta/k)^{-1/2}(\overline{\varphi - I^{hp}\varphi})\right)\,\mathrm{d}S\right| \\ &\leq C\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e)}.\end{aligned}$$

With $\alpha k|_e \sim p_e^2/h_e$ and (4.1.12) we observe for T_5

$$\begin{aligned}\left|\int_e [\![u_{\mathcal{T}}]\!]_N\{\overline{\nabla_{\mathcal{T}} I^{hp}\varphi}\}\,\mathrm{d}S\right| &= \left|\int_e \left((\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\right)\left((\alpha k)^{-1/2}\{\overline{\nabla_{\mathcal{T}} I^{hp}\varphi}\}\right)\,\mathrm{d}S\right| \\ &\leq C\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|\{\overline{\nabla_{\mathcal{T}} I^{hp}\varphi}\}\|_{L^2(e)}\frac{h_e^{1/2}}{\mathfrak{a}^{1/2}p_e} \\ &\leq C\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e^2)}.\end{aligned}$$

Next we consider boundary edges. Let $e \in \mathcal{E}^B$. Then we obtain for T_3 with the approximation property (3.1.15)

$$\left|\int_e (g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})(\overline{\varphi - I^{hp}\varphi})\,\mathrm{d}S\right| \leq C\|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e)}\left(\frac{h_e}{p_e}\right)^{1/2}.$$

Recall that $\delta|_e \sim kh_e/p_e$. In particular, $\delta \leq Ckh_{\mathcal{T}}/p_{\mathcal{T}}$ and moreover

$$\frac{\delta}{k^{1/2}} \sim \frac{k^{1/2}h_e}{p_e} \leq C\left(\frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}\right)^{1/2}\left(\frac{h_e}{p_e}\right)^{1/2}.$$

Again we apply (3.1.15) and therefore get for T_4

$$\begin{aligned}\left|\int_e \delta(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})(\overline{I^{hp}\varphi})\,\mathrm{d}S\right| &= \left|\int_e \delta(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})(\overline{I^{hp}\varphi - \varphi + \varphi})\,\mathrm{d}S\right| \\ &\lesssim \|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\|\delta(I^{hp}\varphi - \varphi)\|_{L^2(e)} \\ &\quad + \|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\|\delta\varphi\|_{L^2(e)} \\ &\lesssim \|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\left(1 + \frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}\right) \\ &\quad \times \left(\|\nabla\varphi\|_{L^2(\omega_e)} + \|\sqrt{k}\varphi\|_{L^2(e)}\right)\left(\frac{h_e}{p_e}\right)^{1/2}.\end{aligned}$$

With (4.1.12) and $\delta/k|_e \sim h_e/p_e$, we bound T_6

$$\left|\int_e \frac{\delta}{\mathrm{i}k}(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})\overline{\partial_{\boldsymbol{n}} I^{hp}\varphi}\,\mathrm{d}S\right| \leq C\|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e^2)}h_e^{1/2}. \tag{4.1.13}$$

These estimates now imply for the supremum in (4.1.10)

$$\begin{aligned}
&\sup_{\varphi\in\Phi}\left|\tilde{a}_{\mathcal{T}}(u-u_{\mathcal{T}},\varphi)-a_{\mathcal{T}}(u-u_{\mathcal{T}},I^{hp}\varphi)\right|\lesssim\\
&\quad\sum_{K\in\mathcal{T}}\left\|\Delta_{\mathcal{T}}u_{\mathcal{T}}+k^2u_{\mathcal{T}}+f\right\|_{L^2(\omega_K)}\|\nabla\varphi\|_{L^2(\omega_K)}\frac{h_K}{p_K}+\sum_{e\in\mathcal{E}^I}\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e)}\\
&\quad+\sum_{e\in\mathcal{E}^B}\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}}\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e)}\left(\frac{h_e}{p_e}\right)^{1/2}+\sum_{e\in\mathcal{E}^B}\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}}\|_{L^2(e)}\\
&\quad\times\left(\frac{h_e}{p_e}\right)^{1/2}\left(1+\frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}\right)\left(\|\sqrt{k}\varphi\|_{L^2(e)}+\|\nabla\varphi\|_{L^2(\omega_e)}\right)+\sum_{e\in\mathcal{E}^I}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e^2)}\\
&\quad+\sum_{e\in\mathcal{E}^B}\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}}\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e^2)}h_e^{1/2}+2k^2\|u-u_{\mathcal{T}}\|_{L^2(K)}\|\varphi\|_{L^2(K)}\,. \qquad (4.1.14)
\end{aligned}$$

Putting together (4.1.9), (4.1.10), and (4.1.14) and using Cauchy-Schwarz for sums, we have

$$\begin{aligned}
\|u-u_{\mathcal{T}}\|_{\tilde{a}}\lesssim\Bigg(&\sum_{K\in\mathcal{T}}\left\|\Delta_{\mathcal{T}}u_{\mathcal{T}}+k^2u_{\mathcal{T}}+f\right\|^2_{L^2(K)}\left(\frac{h_K}{p_K}\right)^2\\
&+\sum_{e\in\mathcal{E}^B}\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}}\|^2_{L^2(e)}\left(\frac{h_e}{p_e}+h_e\right)\left(1+\frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}\right)^2\\
&+\sum_{e\in\mathcal{E}^I}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(e)}\Bigg)^{1/2}+2k\|u-u_{\mathcal{T}}\|_{L^2(\Omega)}\\
&+CC_{conf}\left(\sum_{e\in\mathcal{E}^I}\|(k\alpha)^{1/2}[\![v]\!]\|^2_{L^2(e)}\right)^{1/2},
\end{aligned}$$

and therefore

$$\|u-u_{\mathcal{T}}\|_{\tilde{a}}\lesssim CC_{conf}\left(\sum_{K\in\mathcal{T}}\left(\eta^2_{R_K}+\eta^2_{J_K}\right)+\sum_{e\in\mathcal{E}^B}h_e\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-\mathrm{i}ku_{\mathcal{T}}\|^2_{L^2(e)}\right)^{1/2}+2k\|u-u_{\mathcal{T}}\|_{L^2(\Omega)},$$

which gives (4.1.5). □

Remark 4.1.4. Lemma 4.1.3 also holds for $p_{\mathcal{T}}\geq 1$ if we add the sum over the terms $\eta^2_{E_K}$ on the right-hand side of (4.1.5). This follows, if we work with $I_1^{hp,0}$ instead of I_1^{hp} from Theorem 3.1.10 for the values $1\leq p_{\mathcal{T}}<5$. In this case we bound for $e\in\mathcal{E}^I(K)$ with (4.1.12)

$$\frac{1}{\mathrm{i}k}\int_e\beta[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N[\![\overline{\nabla_{\mathcal{T}}I^{hp}\varphi}]\!]_N\,\mathrm{d}S\leq C\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\|_{L^2(e)}p_e^{1/2}\|\nabla\varphi\|_{L^2(\omega_K)}$$

since this term does not vanish in (4.1.11). Whereas this results in slightly worse constants (namely by the factor $\sqrt{p_e}$) for $p_{\mathcal{T}} < 5$, it will merely alter the overall constant in Lemma 4.1.3 (after adding the terms $\eta_{E_K}^2$).

The next corollary shows that the remaining terms in the DG-norm are, up to some factor, bounded by $\eta(u_{\mathcal{T}}) + k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)}$.

Corollary 4.1.5. *With the same assumptions as in the previous lemma, it holds that*

$$\|u - u_{\mathcal{T}}\|_{DG} \leq C \left(C_{conf}^{3/2} \eta(u_{\mathcal{T}}) + C_{conf}^{1/2} k \, \|u - u_{\mathcal{T}}\|_{L^2(\Omega)} \right),$$

for some $C = C(\gamma, \mathfrak{b}, \mathfrak{d}, \Omega) > 0$.

Proof. If we compare the DG-norm (2.3.3) with the left-hand side of (4.1.5), we see that the only terms missing are the ones containing the jumps of the function and the gradient on interior edges, and the one containing the normal derivative of $u_{\mathcal{T}}$ on the boundary. To estimate the latter, we remark that $\partial_{\boldsymbol{n}} u = g - \mathrm{i}ku$ on $\partial\Omega$ and thus it holds on $e \in \mathcal{E}^B$ that

$$\left(\frac{h_e}{p_e}\right)^{1/2} \partial_{\boldsymbol{n}}(u - u_{\mathcal{T}}) = \left(\frac{h_e}{p_e}\right)^{1/2} (g - \mathrm{i}ku_{\mathcal{T}} - u_{\mathcal{T}}) + \mathrm{i}k^{1/2} \left(\frac{kh_e}{p_e}\right)^{1/2} (u - u_{\mathcal{T}})$$

which in turn yields

$$\begin{aligned} \|(\delta/k)^{1/2} \partial_{\boldsymbol{n}}(u - u_{\mathcal{T}})\|_{L^2(e)} &\lesssim \|(h_e/p_e)^{1/2} \partial_{\boldsymbol{n}}(u - u_{\mathcal{T}})\|_{L^2(e)} \\ &\lesssim \|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}}\|_{L^2(e)} \left(\frac{h_e}{p_e}\right)^{1/2} + \left(\frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}}\right)^{1/2} \|k^{1/2}(u - u_{\mathcal{T}})\|_{L^2(e)} \end{aligned}$$

and

$$\|(\delta/k)^{1/2} \partial_{\boldsymbol{n}}(u - u_{\mathcal{T}})\|_{L^2(\partial\Omega)}^2 \lesssim \sum_{e \in \mathcal{E}^B} \|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}}\|_{L^2(e)}^2 \frac{h_e}{p_e} + \frac{kh_{\mathcal{T}}}{p_{\mathcal{T}}} \sum_{e \in \mathcal{E}^B} \|k^{1/2}(u - u_{\mathcal{T}})\|_{L^2(e)}^2. \tag{4.1.15}$$

Since $C_{conf}^{1/2}$ times the left-hand side of (4.1.5) contains the square root of the second term on the right-hand side of (4.1.15), and $\eta(u_{\mathcal{T}})^2$ contains (an upper bound for) the first term on the right-hand side of (4.1.15), the square root of both terms is bounded by

$$C \left(C_{conf}^{3/2} \eta(u_{\mathcal{T}}) + C_{conf}^{1/2} k \|u - u_{\mathcal{T}}\|_{L^2(\Omega)} \right),$$

due to, and with the notation of, the previous lemma. For the jump terms let $e \in \mathcal{E}^I$. Then

$$\begin{aligned} \|(\alpha k)^{1/2} [\![u - u_{\mathcal{T}}]\!]\|_{L^2(e)} &= \|(\alpha k)^{1/2} [\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}, \\ \|(\beta/k)^{1/2} [\![u - u_{\mathcal{T}}]\!]_N\|_{L^2(e)} &= \|(\beta/k)^{1/2} [\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}, \end{aligned}$$

and they are already contained in $\eta(u_{\mathcal{T}})$ as well. By summing over all interior edges, we conclude

$$\|u - u_{\mathcal{T}}\|_{DG} \leq C \left(C_{conf}^{3/2} \eta(u_{\mathcal{T}}) + C_{conf}^{1/2} k \, \|u - u_{\mathcal{T}}\|_{L^2(\Omega)} \right)$$

for some appropriate $C > 0$. □

In order to bound the error with our estimator, it remains to treat the term $k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)}$. We will show that $k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)}$ is, up to some constant, bounded by $\eta(u_{\mathcal{T}})$ multiplied with the adoint approximability constant $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))$ from (2.3.8). If the space $S^{\boldsymbol{p}}(\mathcal{T})$ is rich enough (cf. Theorem 2.4.2), C_{conf} and σ_k^* will be bounded, which implicates that η estimates the error from above. The key idea of this proof comes from [15, Lemma 4.7].

Lemma 4.1.6. *Let the assumptions of Lemma 4.1.3 be fulfilled. Then there exists a constant $C > 0$ solely depending on $\gamma, \mathfrak{b}, \mathfrak{d}$, and Ω such that with $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))$ from* (2.3.8) *we have*

$$k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)} \leq C\eta(u_{\mathcal{T}})\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T})). \tag{4.1.16}$$

Proof. *1st Step:* The first step is to show that for every $\varphi \in H^1(\Omega) \cap H_{\mathcal{T}}^{3/2+\varepsilon}(\Omega)$ there holds

$$|a_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi)| \leq C\eta(u_{\mathcal{T}})\|\varphi\|_{DG^+}. \tag{4.1.17}$$

To this end we employ (4.1.1) to evaluate $a_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi) = a_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi - I^{hp}\varphi)$. Then

$$\begin{aligned}
&|a_{\mathcal{T}}(u - u_{\mathcal{T}}, \varphi - I^{hp}\varphi)| = \\
&\quad \Bigg| \int_\Omega (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f)(\overline{\varphi - I^{hp}\varphi}) \, \mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}^I} [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N \{\overline{\varphi - I^{hp}\varphi}\} \, \mathrm{d}S \\
&\quad + \int_{\mathcal{E}^I} [\![u_{\mathcal{T}}]\!]_N \{\overline{\nabla_{\mathcal{T}}(\varphi - I^{hp}\varphi)}\} \, \mathrm{d}S - \mathrm{i}k \int_{\mathcal{E}^I} \alpha [\![u_{\mathcal{T}}]\!]_N [\![\overline{\varphi - I^{hp}\varphi}]\!]_N \, \mathrm{d}S \\
&\quad + \frac{1}{\mathrm{i}k} \int_{\mathcal{E}^I} \beta [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N [\![\overline{\nabla_{\mathcal{T}}(\varphi - I^{hp}\varphi)}]\!]_N \, \mathrm{d}S + \int_{\mathcal{E}^B} (1 - \delta)(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})(\overline{\varphi - I^{hp}\varphi}) \, \mathrm{d}S \\
&\quad - \frac{1}{\mathrm{i}k} \int_{\mathcal{E}^B} \delta(g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}) \overline{\partial_{\boldsymbol{n}}(\varphi - I^{hp}\varphi)} \, \mathrm{d}S \Bigg|.
\end{aligned}$$

Taking into account $[\![I^{hp}\varphi]\!]_N = [\![\nabla I^{hp}\varphi]\!]_N \equiv 0$ on interior edges, we compute similarly as in the

proof of Lemma 4.1.3

$$
\begin{aligned}
&|a_{\mathcal{T}}(u-u_{\mathcal{T}},\varphi-I^{hp}\varphi)| \leq \\
&\quad C\sum_{K\in\mathcal{T}}\|\Delta_{\mathcal{T}}u_{\mathcal{T}}+k^2u_{\mathcal{T}}+f\|_{L^2(K)}\|\nabla\varphi\|_{L^2(\omega_K)}\frac{h_K}{p_K} \\
&\quad + C\sum_{e\in\mathcal{E}^I}\Bigg(\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e)}\|(\beta/k)^{-1/2}\|_{L^\infty(e)}\left(\frac{h_e}{p_e}\right)^{1/2} \\
&\quad + \|(k\alpha)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\Big(\|(k\alpha)^{-1/2}\{\nabla\varphi\}\|_{L^2(e)}+\|(k\alpha)^{-1/2}\{\nabla I^{hp}\varphi\}\|_{L^2(e)}\Big) \\
&\quad + \|(k\alpha)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|(k\alpha)^{1/2}[\![\varphi]\!]\|_{L^2(e)}+\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\|(\beta/k)^{1/2}[\![\nabla\varphi]\!]_N\|_{L^2(e)}\Bigg) \\
&\quad + C\sum_{e\in\mathcal{E}^B}\Bigg(\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-u_{\mathcal{T}}\|_{L^2(e)}\|\nabla\varphi\|_{L^2(\omega_e)}\left(\frac{h_e}{p_e}\right)+\|(\delta/k)^{1/2}(g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-u_{\mathcal{T}})\|_{L^2(e)} \\
&\quad \times\Big(\|(\delta/k)^{1/2}\nabla\varphi\cdot\boldsymbol{n}\|_{L^2(e)}+\|(\delta/k)^{1/2}\nabla I^{hp}\varphi\cdot\boldsymbol{n}\|_{L^2(e)}\Big)\Bigg).
\end{aligned}
$$

As in (4.1.12) we bound $\|\nabla I^{hp}\varphi\|_{L^2(e)} \leq Cp_e/h_e^{1/2}\,\|\nabla\varphi\|_{L^2(\omega_e^2)}$ for every $e\in\mathcal{E}$. With

$$
(\alpha k)^{1/2}|_e \sim p_e/h_e^{1/2}, \qquad (\beta/k)^{1/2}|_e \sim (h_e/p_e)^{1/2}, \qquad (\delta/k)^{1/2}|_e \sim (h_e/p_e)^{1/2},
$$

definition (2.3.4) of the DG^+-norm, and Cauchy-Schwarz for sums, we then deduce (4.1.17):

$$
\begin{aligned}
|a_{\mathcal{T}}(u-u_{\mathcal{T}},\varphi)| &\leq C\Bigg(\sum_{K\in\mathcal{T}}\|\Delta_{\mathcal{T}}u_{\mathcal{T}}+k^2u_{\mathcal{T}}+f\|^2_{L^2(K)}\left(\frac{h_K}{p_K}\right)^2 \\
&\qquad + \sum_{e\in\mathcal{E}^I}\Big(\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(e)}+\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\|^2_{L^2(e)}\Big) \\
&\qquad + \sum_{e\in\mathcal{E}^B}h_e\|g-\partial_{\boldsymbol{n}}u_{\mathcal{T}}-u_{\mathcal{T}}\|^2_{L^2(e)}\Bigg)^{1/2}\|\varphi\|_{DG^+} \\
&\leq C\eta(u_{\mathcal{T}})\|\varphi\|_{DG^+}.
\end{aligned}
$$

2nd Step: We prove the assertion. The technique used in the following to bound the L^2-norm is an Aubin-Nitsche-type argument. Consider the solution z of the adjoint problem (2.3.7) with right-hand side $k^2(u-u_{\mathcal{T}})$, i.e.

$$
a(v,z) = (v,k^2(u-u_{\mathcal{T}}))_{L^2(\Omega)} \quad \forall v\in H^1(\Omega).
$$

Moreover, let $z_S^{\varepsilon'}\in S^{\boldsymbol{p}}(\mathcal{T})$ be such that

$$
\|z-z_S^{\varepsilon'}\|_{DG^+} \leq \inf_{\Psi_S\in S^{\boldsymbol{p}}(\mathcal{T})}\|z-\Psi_S\|_{DG^+}+\varepsilon',
$$

for a given $\varepsilon' > 0$. Then with Lemma 2.3.4

$$k^2\|u - u_{\mathcal{T}}\|^2_{L^2(\Omega)} = (u - u_{\mathcal{T}}, k^2(u - u_{\mathcal{T}}))_{L^2(\Omega)} = a_{\mathcal{T}}(u - u_{\mathcal{T}}, z) = a_{\mathcal{T}}(u - u_{\mathcal{T}}, z - z_S^{\varepsilon'}).$$

Due to the adjoint approximation property (2.3.8) we have

$$\|z - z_S^{\varepsilon'}\|_{DG^+} \leq \sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))\frac{\|k^2(u - u_{\mathcal{T}})\|_{L^2(\Omega)}}{k} + \varepsilon'.$$

By using (4.1.17), we get

$$\begin{aligned} k^2\|u - u_{\mathcal{T}}\|^2_{L^2(\Omega)} &= a_{\mathcal{T}}(u - u_{\mathcal{T}}, z - z_S^{\varepsilon'}) \leq C\eta(u_{\mathcal{T}})\|z - z_S^{\varepsilon'}\|_{DG^+} \\ &\leq C\eta(u_{\mathcal{T}})\left(\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))\|k(u - u_{\mathcal{T}})\|_{L^2(\Omega)} + \varepsilon'\right), \end{aligned}$$

and whence (4.1.16) for $\varepsilon' \to 0$. □

4.1.2 Weighted Error Indicator

Before we gather the previous results in one theorem, we generalize the error indicators by multiplying the residuals as well as the gradient jump with weight functions, and add data oscillation terms as it is done in [28]. Let $\hat{K}$ be the reference element. Then we define the weight functions

$$\begin{aligned} \Phi_{\hat{e}}(x) &:= x(1 - x), && x \in \hat{e} = [0, 1], \\ \Phi_{\hat{K}}(\boldsymbol{x}) &:= \operatorname{dist}(\boldsymbol{x}, \partial\hat{K}), && \boldsymbol{x} \in \hat{K}. \end{aligned}$$

We get weight functions Φ_K and Φ_e associated with elements and edges, by scaling $\Phi_{\hat{K}}$ and $\Phi_{\hat{e}}$:

$$\Phi_K = c_K\Phi_{\hat{K}} \circ F_K^{-1}, \qquad \Phi_e = c_e\Phi_{\hat{e}} \circ F_e^{-1}, \tag{4.1.18}$$

where the scaling factors c_K and c_e are such that

$$\int_K \Phi_K \,\mathrm{d}\boldsymbol{x} = \int_K 1 \,\mathrm{d}\boldsymbol{x}, \qquad \int_e \Phi_e \,\mathrm{d}S = \int_e 1 \,\mathrm{d}S.$$

The following inverse estimates then hold.

Theorem 4.1.7. *Let $-1 < \zeta < \xi$ and $\nu \in [0, 1]$. Then there exist $C_1(\zeta, \xi), C_2(\nu) > 0$ such that for all $p \in \mathbb{N}$, $q \in \mathcal{P}_p(\hat{e})$, and $r \in \mathcal{P}_p(\hat{K})$*

$$\int_{\hat{e}} \Phi_{\hat{e}}^{\zeta}(x)q^2 \,\mathrm{d}x \leq C_1 p^{2(\xi-\zeta)} \int_{\hat{e}} \Phi_{\hat{e}}^{\xi} q^2 \,\mathrm{d}x, \tag{4.1.19a}$$

$$\int_{\hat{K}} \Phi_{\hat{K}}^{\zeta} r^2 \,\mathrm{d}\boldsymbol{x} \leq C_1 p^{2(\xi-\zeta)} \int_{\hat{K}} \Phi_{\hat{K}}^{\xi} r^2 \,\mathrm{d}\boldsymbol{x}, \tag{4.1.19b}$$

$$\int_{\hat{K}} \Phi_{\hat{K}}^{2\nu} |\nabla r|^2 \,\mathrm{d}\boldsymbol{x} \leq C_2 p^{2(2-\nu)} \int_{\hat{K}} \Phi_{\hat{K}}^{\nu} r^2 \,\mathrm{d}\boldsymbol{x}. \tag{4.1.19c}$$

PROOF. A proof is given in Lemma 2.4 and Theorem 2.5 in [28]. □

Definition 4.1.8 (Weighted error indicators). *Denote by f_{p_K} the L^2-orthogonal projection of $f|_K \in L^2(K)$ onto $\mathcal{P}_{p_K}(K)$, and let g_{p_K} be the L^2-orthogonal projection of $g|_e \in L^2(e)$ onto $\mathcal{P}_{p_K}(e)$, where $e \in \mathcal{E}^B(K)$. With the same terminology as in Definition 4.1.2 and for $\zeta \in [0,1]$ the* **weighted error indicators** *are given by*

$$\eta_{\zeta;R_K}(u_\mathcal{T})^2 := \left(\frac{h_K}{p_K}\right)^2 \|(\Delta_\mathcal{T} u_\mathcal{T} + k^2 u_\mathcal{T} + f_{p_K})\Phi_K^{\zeta/2}\|^2_{L^2(K)}, \tag{4.1.20a}$$

$$\begin{aligned}\eta_{\zeta;E_K}(u_\mathcal{T})^2 := &\sum_{e\in\mathcal{E}^I(K)} \frac{1}{2}\|(\beta/k)^{1/2}[\![\nabla_\mathcal{T} u_\mathcal{T}]\!]_N \Phi_e^{\zeta/2}\|^2_{L^2(e)} \\ &+ \sum_{e\in\mathcal{E}^B(K)} h_e\|(g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T})\Phi_e^{\zeta/2}\|^2_{L^2(e)},\end{aligned} \tag{4.1.20b}$$

$$\eta_{\zeta;K}(u_\mathcal{T})^2 := \eta_{\zeta;R_K}(u_\mathcal{T})^2 + \eta_{\zeta;E_K}(u_\mathcal{T})^2 + \eta_{J_K}(u_\mathcal{T})^2. \tag{4.1.20c}$$

When no confusion can arise we omit the argument $u_\mathcal{T}$ and write $\eta_{\zeta;K} := \eta_{\zeta;K}(u_\mathcal{T})$, and similar for the other quantities. Furthermore

$$\mathrm{osc}_K^2 := \frac{h_K^2}{p_K^2}\|f - f_{p_K}\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}^B(K)} h_e\|g - g_{p_K}\|^2_{L^2(e)},$$

represents the oscillation of the data.

Remark 4.1.9. In Defintion 4.1.8, f and g is locally projected on polynomials of degree p_K. Another polynomial degree may be chosen as long as it is of size $\mathcal{O}(p_K)$. Such a choice would not influence subsequent results.

Now that we have our weighted error indictators, we can state the next theorem, which is the main result of this section on reliability. It summarizes the above estimates.

Theorem 4.1.10 (Reliability estimate). *Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of Ω and let $\boldsymbol{p}$ be a polynomial degree distribution on $\mathcal{T}$ satisfying* (1.2.1)*. Moreover, let $u \in H^{3/2+\varepsilon}(\Omega)$ be the solution of* (2.1.2) *for some $\varepsilon > 0$, and let $u_\mathcal{T} \in S^{\boldsymbol{p}}(\mathcal{T})$ be the solution of* (2.3.1) *with $S = S^{\boldsymbol{p}}(\mathcal{T})$ and $p_\mathcal{T} \geq 1$. Let $\zeta \in [0,1]$ and assume that $\mathfrak{a} \geq 1$. Then there exists a constant $C > 0$ solely depending on $\gamma, \mathfrak{b}, \mathfrak{d}$, and Ω (cf. Remark 3.1.14), such that with $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))$ from* (2.3.8) *and C_{conf} as in* (4.1.6)

$$\|u - u_\mathcal{T}\|_{DG} \leq C C_{conf}^{3/2}\Big(1 + \sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))\Big)\eta(u_\mathcal{T}), \tag{4.1.21}$$

and

$$\|u - u_{\mathcal{T}}\|_{DG} \leq C C_{conf}^{3/2}\Big(1 + \sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))\Big) \times \left(\sum_{K\in\mathcal{T}} p_K^{2\zeta}\Big(\eta_{\zeta;R_K}(u_{\mathcal{T}})^2 + \eta_{\zeta;E_K}(u_{\mathcal{T}})^2\Big) + \eta_{J_K}(u_{\mathcal{T}})^2 + \operatorname{osc}_K^2\right)^{1/2}. \tag{4.1.22}$$

Proof. Let at first $p_{\mathcal{T}} \geq 5$. Estimate (4.1.21) follows directly from Corollary 4.1.5 and Lemma 4.1.6. For (4.1.22), we set 'ζ' $= 0$ and 'ξ' $= \zeta$ in the inverse estimate (4.1.19b) and obtain

$$\begin{aligned}
\eta_{R_K}^2 &\leq \frac{h_K^2}{p_K^2}\|f - f_{p_K}\|_{L^2(K)}^2 + \frac{h_K^2}{p_K^2}\|\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K}\|_{L^2(K)}^2 \\
&\leq \frac{h_K^2}{p_K^2}\|f - f_{p_K}\|_{L^2(K)}^2 + \frac{h_K^2}{p_K^2} p_K^{2\zeta}\|(\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})\Phi_K^{\zeta/2}\|_{L^2(K)}^2 \\
&= \frac{h_K^2}{p_K^2}\|f - f_{p_K}\|_{L^2(K)}^2 + p_K^{2\zeta}\eta_{\zeta;R_K}^2.
\end{aligned}$$

Similarly with (4.1.19a)

$$\eta_{E_K}^2 \leq p^{2\zeta}\eta_{\zeta;E_K}^2 + \sum_{e\in\mathcal{E}^B(K)} h_e\|g - g_{p_K}\|_{L^2(e)}^2.$$

Therefore

$$\eta^2 = \sum_{K\in\mathcal{T}} \eta_{R_K}^2 + \eta_{E_K}^2 + \eta_{J_K}^2 \leq \sum_{K\in\mathcal{T}} p_K^{2\zeta}\eta_{\zeta;R_K}^2 + p_K^{2\zeta}\eta_{\zeta;E_K}^2 + \eta_{J_K}^2 + \operatorname{osc}_K^2 .$$

Together with the first estimate, this proves (4.1.22) for $p_{\mathcal{T}} \geq 5$.

The case $1 \leq p_{\mathcal{T}} < 5$ is shown similarly and with the use of Remark 4.1.4. □

Remark 4.1.11. The squared edge residual in (4.1.20b) has the weight h_e instead of h_e/p_e, which is what could be expected. The problem occurs in (4.1.13), where we used an inverse inequality to bound the normal derivative of $I^{hp}\varphi$ on the edge $e \in \mathcal{E}^B(K)$ by

$$\|\partial_{\boldsymbol{n}} I^{hp}\varphi\|_{L^2(e)} \leq C\frac{p_e}{\sqrt{h_e}}\|\nabla_{\mathcal{T}} I^{hp}\varphi\|_{L^2(K)} \leq C\frac{p_e}{\sqrt{h_e}}\|\nabla\varphi\|_{L^2(\omega_K)},$$

which is off by $\sqrt{p_e}$. Whereas a similar issue could be resolved on interior edges by using a C^1 interpolant, we would, for example, have to use an interpolant with a stable normal derivative to get rid of this suboptimality. With a general interpolation operator this can only be achieved if at least the $H^{3/2}$-norm, instead of the H^1-norm, of φ can be controlled. All subsequent results are affected by this, and overcoming this problem would improve most constants in the theorems of this and the following section on efficiency by the factor $\sqrt{p_e}$. However, at least in practice, where $kh_K/p_K \leq C$ and $p_K \sim \log(k)$, this should only be of minor importance since p_K is then almost a logarithmic term of h_K.

4.1.3 Error Indicator Without Jump Term

We shall now consider yet another version of the error estimator. The next lemma will show, that the jump term in the error estimator can be omitted. The price we pay is twofold: a) As a requirement of the lemma, the constant $\mathfrak{a}$ needs to be large enough and we do not give a precise lower bound for this condition. b) The weight belonging to the jump of the gradient in $\eta_{0;E_K}$ will increase by $\sqrt{p_e}$, and this is likely to be suboptimal.

Such a result is important if one wants to prove convergence of an adaptive algorithm, which we will not do here however. The reason is, that the error estimator η ideally should monotonically decrease when refining the triangulation $\mathcal{T}$. The increasing factor $\alpha k = \mathfrak{a} p_e^2/h_e$ of the jump term η_{J_K} in the error estimator from Definition 4.1.2 can then be an obstacle. On the other hand, the observation that the jump term is not a necessary contribution to the error indicator, at least if we weight the jump strong enough, is of course interesting in itself.

Definition 4.1.12 (Error indicators without jump term). *With $\eta_{0;R_K}$ from Definition 4.1.8 we introduce the local error indicators*

$$\begin{aligned}\tilde{\eta}_{E_K}(u_{\mathcal{T}})^2 :=& \sum_{e\in\mathcal{E}^I(K)} p_e \frac{1}{2}\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N\|^2_{L^2(e)} \\ &+ \sum_{e\in\mathcal{E}^B(K)} h_e\|(g_{p_K} - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}})\|^2_{L^2(e)}, \end{aligned} \tag{4.1.23a}$$

$$\tilde{\eta}_K(u_{\mathcal{T}})^2 := \eta_{0;R_K}(u_{\mathcal{T}})^2 + \tilde{\eta}_{E_K}(u_{\mathcal{T}})^2, \tag{4.1.23b}$$

and the global error indicator

$$\tilde{\eta}(u_{\mathcal{T}})^2 := \sum_{K\in\mathcal{T}} \tilde{\eta}_K(u_{\mathcal{T}})^2.$$

When no confusion can arise we omit the argument $u_{\mathcal{T}}$ and write $\tilde{\eta}_K := \tilde{\eta}_K(u_{\mathcal{T}})$, and similar for the other quantities.

Lemma 4.1.13. *With η_{J_K} from Definition 4.1.2 there exist constants $C = C(\gamma, \mathfrak{b}, \mathfrak{d}) > 0$ and $C_{\mathfrak{a}} = C_{\mathfrak{a}}(\gamma) > 0$ such that if $\mathfrak{a} > C_{\mathfrak{a}}$, then*

$$\sum_{K\in\mathcal{T}} \eta^2_{J_K} \le C \sum_{K\in\mathcal{T}} \left(\tilde{\eta}^2_K + \mathrm{osc}^2_K\right). \tag{4.1.24}$$

Proof. Recall that

$$\eta^2_{J_K} = \eta_{J_K}(u_{\mathcal{T}})^2 = \sum_{e\in\mathcal{E}^I(K)} \frac{1}{2}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(e)}.$$

The idea is to test our variational formulation (2.3.1) with a properly selected test function, as it was similarly done in [13]. Due to Galerkin orthogonality, it holds with the conforming approximant

$u^*_{\mathcal{T}} \in S^{\boldsymbol{p}}(\mathcal{T})$ from Theorem 3.2.7 that

$$a_{\mathcal{T}}(u - u_{\mathcal{T}}, u_{\mathcal{T}} - u^*_{\mathcal{T}}) = 0.$$

Hence with

$$k\alpha(\boldsymbol{x})|_e \equiv \mathfrak{a}\frac{p_e^2}{h_e}, \qquad (\beta/k)(\boldsymbol{x})|_e \equiv \mathfrak{b}\frac{h_e}{p_e}, \qquad (\delta/k)(\boldsymbol{x})|_e \equiv \mathfrak{d}\frac{h_e}{p_e},$$

$[\![u_{\mathcal{T}} - u^*_{\mathcal{T}}]\!]_N = [\![u_{\mathcal{T}}]\!]_N$, and (4.1.1) we obtain

$$\begin{aligned}
\sum_{e\in\mathcal{E}^I} \|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(e)} \le &\sum_{K\in\mathcal{T}} \frac{h_K}{p_K}\|\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f\|_{L^2(K)}\frac{p_K}{h_K}\|u_{\mathcal{T}} - u^*_{\mathcal{T}}\|_{L^2(K)} \\
&+ \sum_{e\in\mathcal{E}^I} p_e^{1/2}\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\left(\frac{1}{\mathfrak{b}h_e}\right)^{1/2}\|\{u_{\mathcal{T}} - u^*_{\mathcal{T}}\}\|_{L^2(e)} \\
&+ \sum_{e\in\mathcal{E}^B} h_e^{1/2}\|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\frac{1}{h_e^{1/2}}\|u_{\mathcal{T}} - u^*_{\mathcal{T}}\|_{L^2(e)} \\
&+ \sum_{e\in\mathcal{E}^B} h_e^{1/2}\|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)}\frac{\mathfrak{d}h_e^{1/2}}{p_e}\|\partial_{\boldsymbol{n}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})\|_{L^2(e)} \\
&+ \sum_{e\in\mathcal{E}^I} p_e^{1/2}\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\left(\frac{\mathfrak{b}h_e}{p_e^2}\right)^{1/2}\|[\![\nabla_{\mathcal{T}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})]\!]_N\|_{L^2(e)} \\
&+ \sum_{e\in\mathcal{E}^I} \|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(e)}\left(\frac{h_e}{\mathfrak{a}p_e^2}\right)^{1/2}\|\{\nabla_{\mathcal{T}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})\}\|_{L^2(e)}. \qquad (4.1.25)
\end{aligned}$$

Let $e \in \mathcal{E}(\mathcal{T})$. Then either e is shared by some elements K and K', or e is a boundary edge belonging to the element K. In the first case we define $K_e := \overline{K} \cup \overline{K'}$, and recall that $\rho_K = \mathcal{E}^I(\omega_K)$, $\rho_e = \mathcal{E}^I(\omega_e)$. The bounds in Theorem 3.2.7 together with the inverse inequality (3.2.2b) yield

$$\begin{aligned}
\frac{p_K^2}{h_K^2}\|u_{\mathcal{T}} - u^*_{\mathcal{T}}\|^2_{L^2(K)} \le &\frac{C}{\mathfrak{a}}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(\rho_K)}, \\
\frac{1}{h_e}\|\{u_{\mathcal{T}} - u^*_{\mathcal{T}}\}\|^2_{L^2(e)} \le &C\frac{p_e^2}{h_e^2}\|u_{\mathcal{T}} - u^*_{\mathcal{T}}\|^2_{L^2(K_e)} \le \frac{C}{\mathfrak{a}}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(\rho_e)}, \\
\frac{1}{h_e}\|u_{\mathcal{T}} - u^*_{\mathcal{T}}\|^2_{L^2(e)} \le &C\frac{p_e^2}{h_e^2}\|u_{\mathcal{T}} - u^*_{\mathcal{T}}\|^2_{L^2(K)} \le \frac{C}{\mathfrak{a}}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(\rho_e)}, \\
\frac{h_e}{p_e^2}\|\partial_{\boldsymbol{n}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})\|^2_{L^2(e)} \le &C\|\nabla_{\mathcal{T}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})\|^2_{L^2(K)} \le \frac{C}{\mathfrak{a}}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(\rho_e)}, \\
\frac{h_e}{p_e^2}\|[\![\nabla_{\mathcal{T}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})]\!]_N\|^2_{L^2(e)} \le &C\|\nabla_{\mathcal{T}}(u_{\mathcal{T}} - u^*_{\mathcal{T}})\|^2_{L^2(K_e)} \le \frac{C}{\mathfrak{a}}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|^2_{L^2(\rho_e)}.
\end{aligned}$$

By using these estimates, each term except the last one in (4.1.25) can be bounded by either $\tilde{\eta}_{E_K} + \mathrm{osc}_K$ or $\eta_{R_K} + \mathrm{osc}_K$ multiplied with $\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]_N\|_{L^2(\rho_K)}$. For the remaining term, we use

again the inverse inequality (3.2.2b) as well as (3.2.10c) which gives

$$\sum_{e\in\mathcal{E}^I} \|(\alpha k)^{1/2}[\![u_\mathcal{T}]\!]_N\|_{L^2(e)} \left(\frac{h_e}{\mathfrak{a}p_e^2}\right) \|\{\nabla_\mathcal{T}(u_\mathcal{T}-u_\mathcal{T}^*)\}\|_{L^2(e)} \leq \frac{C}{\mathfrak{a}} \sum_{e\in\mathcal{E}^I} \|(\alpha k)^{1/2}[\![u_\mathcal{T}]\!]_N\|^2_{L^2(\rho_e)}.$$

If $\mathfrak{a} > C_\mathfrak{a}$ for some constant $C_\mathfrak{a} > 0$, which only depends on γ, we can absorb this term in the left-hand side of (4.1.24). Applying Cauchy-Schwarz for sums and dividing both sides by $\|(\alpha k)^{1/2}[\![u_\mathcal{T}]\!]_N\|_{L^2(\mathcal{E}^I)}$ finishes the proof. □

Let us state the reliability estimate in this case:

Theorem 4.1.14 (Reliability estimate). *Let $\mathfrak{a} > C_\mathfrak{a}$ as in Lemma 4.1.13. With the assumptions from Theorem 4.1.10 and C_{conf} as in* (4.1.6)*, we have*

$$\|u-u_\mathcal{T}\|_{DG} \leq C C_{conf}^{3/2}\Big(1+\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))\Big)\left(\sum_{K\in\mathcal{T}}\big(\tilde{\eta}_K(u_\mathcal{T})^2+\mathrm{osc}_K^2\big)\right)^{1/2}, \tag{4.1.26}$$

for some $C = C(\gamma,\mathfrak{b},\mathfrak{d})$.

Proof. Theorem 4.1.10, Lemma 4.1.13, and a comparison of the error indicators $\tilde{\eta}_{E_K}$ and $\eta_{0;E_K}$ readily give (4.1.26). The reason why we didn't include dependence on Ω in the constant, is that using the continuous interpolation operator from [24] (where the constant does not depend on Ω) instead of I_1^{hp} from Theorem 3.1.10, would alter the constants in the proofs leading up to Theorem 4.1.10 according to the situation of Definition 4.1.12. That is, we would obtain Theorem 4.1.10 in the case $\zeta = 0$ with $\tilde{\eta}_{E_K}$ instead of $\eta_{0;E_K}$. □

4.2 Efficiency

Besides being reliable, it is also important that the error indicators are bounded by the actual error on the element. This (local) property is called **efficiency**, and it means that the error indicators do not overestimate the error. Typically in *hp*-FEM, the local error indicators can be shown to be efficient only up to a constant depending on the polynomial degree p_K (at least with the proofs that are available up to date). With the use of the weighted residuals it is possible to compromise between efficiency and reliability. Enhancing the parameter ζ from above causes the reliability constant to grow and the efficiency constant to drop, and vice versa. The proof for the efficiency of the error estimator is similar as in the conforming FEM, for which it is given in [28]. Therefore, in order to bound the residuals, we keep close to this paper and also to [19, Theorem 3.2]. Let us start with the internal residual.

Lemma 4.2.1 (Efficiency of the internal residual). *Let $\zeta \in [0,1]$ and $\varepsilon > 0$. Let $\eta_{\zeta;R_K}$ be as*

in Definition 4.1.8. Then there exists $C_\varepsilon = C_\varepsilon(\varepsilon,\gamma) > 0$ independent of k, h_K, p_K such that

$$\begin{aligned}\eta_{\zeta;R_K}^2 \leq & C_\varepsilon \Bigg(p_K^{2(1-\zeta)} |u - u_{\mathcal{T}}|_{H^1(K)}^2 \\ & + p_K^{\max\{1+2\varepsilon-2\zeta,0\}} \left(\left(\frac{kh_K}{p_K} \right)^2 \|k(u-u_{\mathcal{T}})\|_{L^2(K)}^2 + \frac{h_K^2}{p_K^2} \|f - f_{p_K}\|_{L^2(K)}^2 \right) \Bigg). \end{aligned} \tag{4.2.1}$$

Proof. First let $1/2 < \zeta \leq 1$. We define $v_K := (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})\Phi_K^\zeta$ and point out that $\|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)} = (p_K/h_K)\eta_{\zeta;R_K}$. With $-\Delta u - k^2 u = f$ in K we get

$$\begin{aligned}\|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)}^2 &= \int_K (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K}) v_K \,\mathrm{d}\boldsymbol{x} \\ &= \int_K (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f) v_K + (f_{p_K} - f) v_K \,\mathrm{d}\boldsymbol{x} \\ &= \int_K \Delta_{\mathcal{T}}(u_{\mathcal{T}} - u) v_K + k^2 (u_{\mathcal{T}} - u) v_K + (f_{p_K} - f) v_K \,\mathrm{d}\boldsymbol{x} \\ &= \int_K \nabla_{\mathcal{T}}(u - u_{\mathcal{T}}) \nabla_{\mathcal{T}} v_K + k^2 (u_{\mathcal{T}} - u) v_K + (f_{p_K} - f) v_K \,\mathrm{d}\boldsymbol{x} \\ &\leq |u_{\mathcal{T}} - u|_{H^1(K)} |v_K|_{H^1(K)} + \|k^2 (u_{\mathcal{T}} - u)\Phi_K^{\zeta/2}\|_{L^2(K)} \|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)} \\ &\quad + \|(f - f_{p_K})\Phi_K^{\zeta/2}\|_{L^2(K)} \|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)}, \end{aligned} \tag{4.2.2}$$

since Φ_K is zero on the boundary of K. Now we take a look at the H^1-seminorm of v_K. With the inverse estimates (4.1.19c), (4.1.19b) (with 'ζ' $= 2(\zeta-1)$ and 'ν' = 'ξ' $= \zeta$), and $|\nabla \Phi_K^\zeta| \lesssim |\Phi_K^{\zeta-1}|/h_K$ we find

$$\begin{aligned}|v_K|_{H^1(K)}^2 &\leq 2 \int_K \Phi_K^{2\zeta} |\nabla_{\mathcal{T}}(\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})|^2 \,\mathrm{d}\boldsymbol{x} + 2 \int_K (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})^2 |\nabla \Phi_K^\zeta|^2 \,\mathrm{d}\boldsymbol{x} \\ &\leq C \frac{p_K^{2(2-\zeta)}}{h_K^2} \int_K \Phi_K^\zeta (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})^2 \,\mathrm{d}\boldsymbol{x} + \frac{C}{h_K^2} \int_K \Phi_K^{2(\zeta-1)} (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})^2 \,\mathrm{d}\boldsymbol{x} \\ &\leq C \frac{p_K^{2(2-\zeta)}}{h_K^2} \int_K \Phi_K^\zeta (\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K})^2 \,\mathrm{d}\boldsymbol{x} \\ &\leq C p_K^{2(1-\zeta)} \frac{p_K^2}{h_K^2} \|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)}^2. \end{aligned}$$

In order to apply the inverse estimate, we needed $\zeta > 1/2$ since only then the requirement $2\zeta - 2 > -1$ is satisfied. Now we use $\|\Phi_K\|_{L^\infty(K)} \leq C$ and $\|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)} = (p_K/h_K)\eta_{\zeta;R_K}$. Canceling the term $\|v_K \Phi_K^{-\zeta/2}\|_{L^2(K)}$ in (4.2.2) then leads to the assertion in the first case:

$$\eta_{\zeta;R_K} \leq C \left(p_K^{1-\zeta} |u - u_{\mathcal{T}}|_{H^1(K)} + \frac{kh_K}{p_K} \|k(u - u_{\mathcal{T}})\|_{L^2(K)} + \frac{h_K}{p_K} \|f - f_{p_K}\|_{L^2(K)} \right). \tag{4.2.3}$$

For the second case let $0 \le \zeta \le 1/2$. With (4.1.19b) and then (4.2.3) we find for $\xi := 1/2 + \varepsilon$

$$\begin{aligned}
\eta_{\zeta;R_K} &\le C p_K^{\xi-\zeta} \eta_{\xi,R_K} \\
&\le C p_K^{\xi-\zeta} \left(p_K^{1-\xi} |u - u_{\mathcal{T}}|_{H^1(K)} + \frac{k h_K}{p_K} \|k(u - u_{\mathcal{T}})\|_{L^2(K)} + \frac{h_K}{p_K} \|f - f_{p_K}\|_{L^2(K)} \right) \\
&= C \left(p_K^{1-\zeta} |u - u_{\mathcal{T}}|_{H^1(K)} + p_K^{1/2+\varepsilon-\zeta} \left(\frac{k h_K}{p_K} \|k(u - u_{\mathcal{T}})\|_{L^2(K)} + \frac{h_K}{p_K} \|f - f_{p_K}\|_{L^2(K)} \right) \right),
\end{aligned}$$

where C depends on ε only. □

The above lemma shows that the weighted internal residual on each element is, up to data oscillations and constants depending on p_K, bounded by the actual error on this element in a suitable norm. Since we consider the error in the DG-norm, the remaining terms in the local error estimator $\eta_{\zeta;K}$ essentially already are error terms associated with this element (in order to see that, the weighted edge residual needs to be slightly modified, as we will do below). Nonetheless, we continue with our analyis and proceed with the edge part $\eta_{\zeta;E_K}$. First, an extension result is needed to treat the edge error indicator [28, Lemma 2.6]:

Lemma 4.2.2. *Let $\hat{K}$ be the reference element, and let $\hat{e} = [0,1] \times \{0\}$. Let $\zeta \in (1/2, 1]$. Then there exists a constant $C = C(\zeta) > 0$, such that for every $\epsilon \in (0,1]$, $p \in \mathbb{N}$, and $q \in \mathcal{P}_p(\hat{e})$ there exists an extension $v_{\hat{e}} \in H^1(\hat{K})$ of $q\Phi_{\hat{e}}^{\zeta}$ with*

$$v_{\hat{e}}|_{\hat{e}} = q\Phi_{\hat{e}}^{\zeta} \text{ and } v_{\hat{e}}|_{\partial\hat{K}\setminus\hat{e}} \equiv 0, \tag{4.2.4a}$$

$$\|v_{\hat{e}}\|_{L^2(\hat{K})}^2 \le C\epsilon \|q\Phi_{\hat{e}}^{\zeta/2}\|_{L^2(\hat{e})}^2, \tag{4.2.4b}$$

$$\|\nabla v_{\hat{e}}\|_{L^2(\hat{K})}^2 \le C(\epsilon p^{2(2-\zeta)} + \epsilon^{-1}) \|q\Phi_{\hat{e}}^{\zeta/2}\|_{L^2(\hat{e})}^2. \tag{4.2.4c}$$

Lemma 4.2.3 (Efficiency of the edge residual and gradient jumps). *Let $\zeta \in [0,1]$ and $\varepsilon > 0$. Let $\eta_{\zeta;E_K}$ be as in Definition 4.1.8. Then there exists $C_\varepsilon = C_\varepsilon(\varepsilon, \gamma) > 0$ independent of k, h_K, p_K such that for every $e \in \mathcal{E}^I$*

$$\begin{aligned}
\|(\beta/k)^{1/2} [\![\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!]_N \Phi_e^{\zeta/2}\|_{L^2(e)}^2 \le & C_\varepsilon p_e^{\max\{1-2\zeta+2\varepsilon, 0\}} \Bigg(p_e |u - u_{\mathcal{T}}|_{H^1(\omega_e)}^2 \\
& + p_e^{2\varepsilon} \left(\left(\frac{k h_e}{p_e} \right)^2 \|k(u - u_{\mathcal{T}})\|_{L^2(\omega_e)}^2 + \frac{h_e^2}{p_e^2} \|f - f_{p_e}\|_{L^2(\omega_e)}^2 \right) \Bigg)
\end{aligned} \tag{4.2.5}$$

and

$$\eta_{\zeta;E_K}^2 \leq C_\varepsilon p_K^{\max\{1-2\zeta+2\varepsilon,0\}} \Bigg(\sum_{e\in\mathcal{E}^B(K)} \Big(h_e k \|\sqrt{k}(u-u_\mathcal{T})\|_{L^2(e)}^2 + h_e \|g-g_{p_K}\|_{L^2(e)}^2 \Big) + p_K^2 |u-u_\mathcal{T}|_{H^1(\omega_K)}^2 + p_K^{1+2\varepsilon} \left(\left(\frac{kh_K}{p_K}\right)^2 \|k(u-u_\mathcal{T})\|_{L^2(\omega_K)}^2 + \frac{h_K^2}{p_K^2} \|f-f_{p_K}\|_{L^2(\omega_K)}^2 \right) \Bigg). \tag{4.2.6}$$

Proof. Recall that

$$\eta_{\zeta;E_K}(u_\mathcal{T})^2 = \sum_{e\in\mathcal{E}^I(K)} \frac{1}{2} \|(\beta/k)^{1/2} [\![\nabla_\mathcal{T} u_\mathcal{T}]\!]_N \Phi_e^{\zeta/2}\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}^B(K)} h_e \|(g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T}) \Phi_e^{\zeta/2}\|_{L^2(e)}^2. \tag{4.2.7}$$

We merely prove the second estimate, i.e. we accept (4.2.5) and only consider the edge residual $\|(g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T}) \Phi_e^{\zeta/2}\|_{L^2(e)}$. The proof of (4.2.5) is very similar, and both estimates follow by the arguments from the proof given in [28, Lemma 3.5]. Let at first $\zeta \in (1/2, 1]$ and let $e \in \mathcal{E}^B(K)$. Furthermore, let $w_e \in H^1(K)$ be the pullback of $v_{\hat{e}}$ with $q := (g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T}) \circ F_e$ in Lemma 4.2.2 (where $F_e : \hat{e} \to e$). Because of (4.2.4a), we have $w_e|_{\partial K \setminus e} \equiv 0$. Together with $\partial_{\boldsymbol{n}} u + \mathrm{i}ku = g$ on $\partial\Omega$ we obtain therefore

$$\|(g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T}) \Phi_e^{\zeta/2}\|_{L^2(e)}^2 = \int_{\partial K} \partial_{\boldsymbol{n}}(u-u_\mathcal{T}) w_e \,\mathrm{d}S + \int_e (\mathrm{i}k(u-u_\mathcal{T}) + (g_{p_K} - g))\, w_e \,\mathrm{d}S. \tag{4.2.8}$$

For the second integral we get

$$\begin{aligned} \int_e (\mathrm{i}k(u-u_\mathcal{T}) + (g_{p_K} - g))\, w_e \,\mathrm{d}S &\leq \big(\|k(u-u_\mathcal{T})\|_{L^2(e)} + \|g_{p_K} - g\|_{L^2(e)} \big) \|w_e\|_{L^2(e)} \\ &\leq \big(\|k(u-u_\mathcal{T})\|_{L^2(e)} + \|g_{p_K} - g\|_{L^2(e)} \big) \\ &\quad \times \|(g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T}) \Phi_e^{\zeta/2}\|_{L^2(e)}, \end{aligned} \tag{4.2.9}$$

where we have used the definition of w_e and the fact that Φ_e is pointwise bounded by some constant (independent of h_e, cf. (4.1.18)), which implies $\Phi_e^\zeta \lesssim \Phi_e^{\zeta/2}$. For the first integral on the right-hand

side of (4.2.8) we compute

$$\begin{aligned}\int_e \partial_{\boldsymbol{n}}(u-u_{\mathcal{T}})w_e \,\mathrm{d}S &= \int_{\partial K} \nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\cdot \boldsymbol{n} w_e \,\mathrm{d}S \\ &= \int_K \nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\nabla_{\mathcal{T}} w_e \,\mathrm{d}\boldsymbol{x} + \int_K \Delta_{\mathcal{T}}(u-u_{\mathcal{T}})w_e \,\mathrm{d}\boldsymbol{x} \\ &= \int_K \nabla_{\mathcal{T}}(u-u_{\mathcal{T}})\nabla_{\mathcal{T}} w_e \,\mathrm{d}\boldsymbol{x} + \int_K k^2(u_{\mathcal{T}}-u)w_e \,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_K (\Delta_{\mathcal{T}}u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f)w_e \,\mathrm{d}\boldsymbol{x} \\ &\le |u-u_{\mathcal{T}}|_{H^1(K)}|w_e|_{H^1(K)} + \|k^2(u_{\mathcal{T}}-u)\|_{L^2(K)}\|w_e\|_{L^2(K)} \\ &\quad + \|\Delta_{\mathcal{T}}u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f_{p_K}\|_{L^2(K)}\|w_e\|_{L^2(e)} + \|f_{p_K} - f\|_{L^2(K)}\|w_e\|_{L^2(K)}.\end{aligned} \tag{4.2.10}$$

Lemma 4.2.2 gives upper bounds for $|w_e|_{H^1(K)}$ and $\|w_e\|_{L^2(K)}$:

$$\begin{aligned}|w_e|^2_{H^1(K)} &\le C\frac{1}{h_K}(\epsilon p_K^{2(2-\zeta)} + \epsilon^{-1})\|(g_{p_K} - \partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\Phi_e^{\zeta/2}\|^2_{L^2(e)}, \\ \|w_e\|^2_{L^2(K)} &\le Ch_K\epsilon\|(g_{p_K} - \partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\Phi_e^{\zeta/2}\|^2_{L^2(e)},\end{aligned}$$

where ϵ will be chosen below. Note that the above constant does not depend on ϵ. Combining this with (4.2.10), (4.2.9), and (4.2.8) and canceling the term $\|(g_{p_K} - \partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\Phi_e^{\zeta/2}\|_{L^2(e)}$ yields

$$\begin{aligned}\|(g_{p_K} - \partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\Phi_e^{\zeta/2}\|_{L^2(e)} \le & C\Bigg(\left(\frac{1}{h_K}(\epsilon p_K^{2(2-\zeta)} + \epsilon^{-1})\right)^{1/2}|u-u_{\mathcal{T}}|_{H^1(K)} + (h_K\epsilon)^{1/2} \\ & \times\Bigg(\|\Delta_{\mathcal{T}}u_{\mathcal{T}} + k^2u_{\mathcal{T}} + f_{p_K}\|_{L^2(K)} + \|f_{p_K} - f\|_{L^2(K)} \\ & + \|k^2(u-u_{\mathcal{T}})\|_{L^2(K)}\Bigg) + \|k(u-u_{\mathcal{T}})\|_{L^2(e)} + \|g_{p_K} - g\|_{L^2(e)}\Bigg).\end{aligned}$$

Now we use $\eta_{0;R_K}$ in (4.2.1) to bound $\|\Delta u_{\mathcal{T}} + k^2u_{\mathcal{T}} + f_{p_K}\|_{L^2(K)}$. This gives

$$\begin{aligned}h_e\|(g - \partial_{\boldsymbol{n}}u_{\mathcal{T}} - \mathrm{i}ku_{\mathcal{T}})\Phi_e^{\zeta/2}\|^2_{L^2(e)} \le & h_e\|g - g_{p_K}\|^2_{L^2(e)} + h_ek\|\sqrt{k}(u-u_{\mathcal{T}})\|^2_{L^2(e)} \\ & + \left(\epsilon p_K^{2(2-\zeta)} + \epsilon^{-1} + \epsilon p_K^4\right)|u-u_{\mathcal{T}}|^2_{H^1(K)} \\ & + \epsilon p_K^{3+2\varepsilon}\left(\frac{h_K^2}{p_K^2}\|f - f_{p_K}\|^2_{L^2(K)} + \left(\frac{kh_K}{p_K}\right)^2\|k(u-u_{\mathcal{T}})\|^2_{L^2(K)}\right) \\ & + \epsilon h_K^2k^2\|k(u-u_{\mathcal{T}})\|^2_{L^2(K)}.\end{aligned}$$

The assertion for $\zeta \in (1/2, 1]$ then follows by summing over all edges, using (4.2.5), and choosing $\epsilon = 1/p_K^2$. Now let $\zeta \in [0, 1/2]$. Then, with $\xi = 1/2 + \varepsilon$ in (4.1.19b), we obtain $\eta_{\zeta;E_K} \le C(\varepsilon)p_K^{1/2+\varepsilon-\zeta}\eta_{\xi;E_K}$ and furthermore (4.2.6) in this case. □

So far we have seen that $\eta_{\zeta;R_K} + \eta_{\zeta;E_K}$ is bounded by data oscillations, the k-weighted L^2-error, the error in the H^1-seminorm, and the $k^{1/2}$-weighted L^2-error on the boundary. Now we take a closer look at the jump term η_{J_K} from Definition 4.1.2. We already absorbed the jump terms in the rest of the estimator in Section 4.1.3, thus the jumps are easily bounded with the above error terms. However, the statement is not a local one.

Lemma 4.2.4 (Efficiency of the jumps). *Let $\varepsilon > 0$, and let $\mathfrak{a} > C_{\mathfrak{a}}$ as in Lemma 4.1.13. Then there exists a constant $C_\varepsilon = C_\varepsilon(\varepsilon, \gamma) > 0$, such that*

$$\begin{aligned}\sum_{K\in\mathcal{T}} \eta_{J_K}^2 &= \sum_{e\in\mathcal{E}^I} \|(\alpha k)^{1/2}[\![u_\mathcal{T}]\!]_N\|_{L^2(e)}^2 \\ &\le C_\varepsilon \sum_{e\in\mathcal{E}^B(K)} p_K^{1+2\varepsilon}\Big(h_e k\|\sqrt{k}(u-u_\mathcal{T})\|_{L^2(e)}^2 + h_e\|g-g_{p_K}\|_{L^2(e)}^2\Big) \\ &\quad + C_\varepsilon \sum_{K\in\mathcal{T}} \left(p_K^{2+4\varepsilon}\left(\left(\frac{kh_K}{p_K}\right)^2 \|k(u-u_\mathcal{T})\|_{L^2(K)}^2 + \frac{h_K^2}{p_K^2}\|f-f_{p_K}\|_{L^2(K)}^2\right)\right. \\ &\quad \left. + p_K^{3+2\varepsilon}|u-u_\mathcal{T}|_{H^1(K)}^2\right). \end{aligned} \tag{4.2.11}$$

Proof. Lemmata 4.2.1 and 4.2.3 imply the estimates

$$\begin{aligned}\frac{h_K^2}{p_K^2}\|\Delta_\mathcal{T} u_\mathcal{T} + k^2 u_\mathcal{T} + f_{p_K}\|_{L^2(K)}^2 \le C_\varepsilon\Bigg(&p_K^2|u-u_\mathcal{T}|_{H^1(K)}^2 + p_K^{1+2\varepsilon} \\ &\times\left(\left(\frac{kh_K}{p_K}\right)^2\|k(u-u_\mathcal{T})\|_{L^2(K)}^2 + \frac{h_K^2}{p_K^2}\|f-f_{p_K}\|_{L^2(K)}^2\right)\Bigg),\end{aligned}$$

and

$$\begin{aligned}&\sum_{e\in\mathcal{E}^B(K)} h_e\|g_{p_K} - \partial_{\boldsymbol{n}} u_\mathcal{T} - \mathrm{i}k u_\mathcal{T}\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}^I(K)} p_e\|(\beta/k)^{1/2}[\![\nabla_\mathcal{T} u_\mathcal{T}]\!]_N\|_{L^2(e)}^2 \\ &\le C_\varepsilon p_K^{1+2\varepsilon}\Bigg(\sum_{e\in\mathcal{E}^B(K)}\Big(kh_K\|\sqrt{k}(u-u_\mathcal{T})\|_{L^2(e)}^2 + h_e\|g-g_{p_K}\|_{L^2(e)}^2\Big) \\ &\quad + p_K^2|u-u_\mathcal{T}|_{H^1(\omega_K)}^2 + p_K^{1+2\varepsilon}\left(\left(\frac{kh_K}{p_K}\right)^2\|k(u-u_\mathcal{T})\|_{L^2(\omega_K)}^2 + \frac{h_K^2}{p_K^2}\|f-f_{p_K}\|_{L^2(\omega_K)}^2\right)\Bigg).\end{aligned}$$

Equation (4.2.11) is therefore an immediate consequence of Lemma 4.1.13. □

Remark 4.2.5. In contrast to the residuals, there is not really a point in considering a weighted version of the jump. This is because η_{J_K} is both: a part of the error in the DG-norm, and an error

indicator. Of course, the same can be said about the jump of the gradient since it is also contained in the DG-norm. However, since we constructed the conforming approximant as a function of $C^0(\Omega)$ and not $C^1(\Omega)$, the arguments in Section 4.1.3 do not work for the jump of the gradient, although they possibly might be extended to this case with an appropriate C^1 approximant at hand. Therefore we used a weighted version to prove efficiency for this term.

At the end of this section we summarize our results in one theorem.

Theorem 4.2.6 (Efficiency estimate). *Let $\mathcal{T}$ be an admissible γ-shape regular triangulation of Ω with a polynomial degree distribution $\boldsymbol{p}$ satisfying* (1.2.1)*. Let $u \in H^{3/2+\varepsilon}(\Omega)$ be the solution of* (2.1.2) *for some $\varepsilon > 0$, and let $u_{\mathcal{T}} \in S^{\boldsymbol{p}}(\mathcal{T})$ be the solution of* (2.3.1) *with $S = S^{\boldsymbol{p}}(\mathcal{T})$. Additionally, let the error indicators $\eta_{\zeta;K}$ be as in Definition 4.1.8 for some $\zeta \in [0,1]$. Then, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon = C_\varepsilon(\varepsilon,\gamma) > 0$ such that*

$$\begin{aligned}\eta_{\zeta;K}^2 \leq & C_\varepsilon p_K^{\max\{1-2\zeta+2\varepsilon,0\}}\Bigg(\sum_{e\in\mathcal{E}^B(K)}\Big(h_e k\|\sqrt{k}(u-u_{\mathcal{T}})\|_{L^2(e)}^2 + h_e\|g-g_{p_K}\|_{L^2(e)}^2\Big)\\ &+ p_K^2|u-u_{\mathcal{T}}|_{H^1(\omega_K)}^2 + p_K^{1+2\varepsilon}\left(\left(\frac{kh_K}{p_K}\right)^2\|k(u-u_{\mathcal{T}})\|_{L^2(\omega_K)}^2 + \frac{h_K^2}{p_K^2}\|f-f_{p_K}\|_{L^2(\omega_K)}^2\right)\Bigg)\\ &+ \frac{1}{2}\sum_{e\in\mathcal{E}^I(K)}\|(\alpha k)^{1/2}[\![u_{\mathcal{T}}]\!]\|_{L^2(e)}^2. \end{aligned} \tag{4.2.12}$$

Moreover, if $\mathfrak{a} > C_{\mathfrak{a}}$ for $C_{\mathfrak{a}}$ as in Lemma 4.2.4, then with $\tilde{\eta}_K$ from Definition 4.1.12

$$\begin{aligned}\tilde{\eta}_K^2 \leq & C_\varepsilon p_K^{1+2\varepsilon}\Bigg(\sum_{e\in\mathcal{E}^B(K)}\Big(h_e k\|\sqrt{k}(u-u_{\mathcal{T}})\|_{L^2(e)}^2 + h_e\|g-g_{p_K}\|_{L^2(e)}^2\Big)\\ &+ p_K^2|u-u_{\mathcal{T}}|_{H^1(\omega_K)}^2 + p_K^{1+2\varepsilon}\left(\left(\frac{kh_K}{p_K}\right)^2\|k(u-u_{\mathcal{T}})\|_{L^2(\omega_K)}^2 + \frac{h_K^2}{p_K^2}\|f-f_{p_K}\|_{L^2(\omega_K)}^2\right)\Bigg), \end{aligned} \tag{4.2.13}$$

and

$$\begin{aligned}\sum_{K\in\mathcal{T}}\eta_{0;K}^2 \leq & C_\varepsilon\sum_{e\in\mathcal{E}^B} p_e^{1+2\varepsilon}\Big(h_e k\|\sqrt{k}(u-u_{\mathcal{T}})\|_{L^2(e)}^2 + h_e\|g-g_{p_K}\|_{L^2(e)}^2\Big)\\ &+ C_\varepsilon\sum_{K\in\mathcal{T}}\Bigg(p_K^{2+4\varepsilon}\left(\left(\frac{kh_K}{p_K}\right)^2\|k(u-u_{\mathcal{T}})\|_{L^2(K)}^2 + \frac{h_K^2}{p_K^2}\|f-f_{p_K}\|_{L^2(K)}^2\right)\\ &+ p_K^{3+2\varepsilon}|u-u_{\mathcal{T}}|_{H^1(K)}^2\Bigg). \end{aligned} \tag{4.2.14}$$

Proof. Estimates (4.2.12) and (4.2.14) are a direct consequence of the above discussion on efficiency. For (4.2.13) we remark that according to (4.2.5) it holds that $p_e\|(\beta/k)^{1/2}[\![\nabla_{\mathcal{T}}u_{\mathcal{T}}]\!]_N\|_{L^2(e)}^2$ is

bounded by the right-hand side of (4.2.6) for $\zeta = 0$. Since this term is the only difference between $\tilde{\eta}^2_{E_K}$ and $\eta^2_{0;E_K}$, Lemmata 4.2.1 and 4.2.3 conclude the proof. $\square$

Remark 4.2.7. The constant C_ε in Theorem 4.2.6 does not depend on the wavenumber k. If Ω is a convex polygonal domain, and the finite element space fulfills the resolution condition from Theorem 2.4.2, we have $h_e k \lesssim p$, where p is considered to be the global polynomial degree as in Theorem 2.4.2. Therefore the error estimators $\tilde{\eta}_K$ and $\eta_{\zeta;K}$ are bounded by oscillation terms and the actual error on the element multiplied with a power of p. Let us now assume $p \lesssim \log(k)$, which Theorem 2.4.2 suggests to be reasonable. In this case we get an efficiency estimate with a constant depending linearly on said power of $\log(k)$, and, moreover, reliability holds with a constant independent of k due to the Theorems 4.1.10 and 4.1.14.

Chapter 5

Numerical Experiments

In this chapter we perform numerical experiments to test parts of the theory from the previous chapters. Several issues are of interest. We will test the reliability of the error estimator and compare uniform vs. adaptive refinement for a nonsmooth problem. Moreover, we will observe how the algorithm behaves in settings that are not covered by our theory, namely for non-constant k and non-convex domains. All of the following experiments were conducted in MATLAB. The program used for testing is based on the finite element toolbox LehrFEM, documentation of which can be found online [1]. Before we begin, let us very briefly explain the adaptive algorithm and some aspects of the implementation.

5.1 Adaptive Algorithm

We shortly describe the typical adaptive scheme (see, e.g., [29] for more details). The standard adaptive algorithm consists of the repeated realization of the four modules

$$\boxed{\text{SOLVE}} \longrightarrow \boxed{\text{ESTIMATE}} \longrightarrow \boxed{\text{MARK}} \longrightarrow \boxed{\text{REFINE}}$$

Solve

The module Solve finds the solution $u_{\mathcal{T}}$ of (2.3.1) for a given mesh $\mathcal{T}$, data f, g, a wavenumber k, and a polynomial degree p. In practice, all integrals in (2.3.1) are computed by quadrature on the edges and elements. This means that integrals including the functions f and g are generally not evaluated exactly, in contrast to the other ones (up to rounding errors).

Estimate

The ability to estimate the error on each element is the crucial prerequisite for any adaptive algorithm. As an error estimator, we use an adapted version of $\tilde{\eta}_K$ from Definition 4.1.12: For simplicity the oscillation terms are omitted, and we work with the functions f, g instead of f_{p_K}, g_{p_K}.

[1] `http://www.sam.math.ethz.ch/~hiptmair/tmp/LehrFEMManual.pdf`

Nonetheless, this simplified version will also be referred to as $\tilde{\eta}_K$ in the following. Again, integrals are computed by quadrature. We have seen that the estimator is reliable and efficient if the resolution condition from Theorem 2.4.2 is fulfilled.

Mark

After having computed the local estimators $\tilde{\eta}_K$, one needs to decide which elements to refine. Here, this is done via Dörfler's marking strategy: Fix the triangulation $\mathcal{T}$ and let $u_{\mathcal{T}}$ be the respective discrete solution. Denote by $\mathcal{S}$ some subset of $\mathcal{T}$. We write

$$\tilde{\eta}(u_{\mathcal{T}}, \mathcal{S})^2 := \sum_{K \in \mathcal{S}} \tilde{\eta}_K^2. \tag{5.1.1}$$

Now let $\theta \in (0, 1]$. Then the set of marked elements $\mathcal{M} \subseteq \mathcal{T}$ is defined to be such that

$$|\mathcal{M}| = \min_{\{\mathcal{S} \subseteq \mathcal{T} : \theta \tilde{\eta}(u_{\mathcal{T}}, \mathcal{S}) \geq \tilde{\eta}(u_{\mathcal{T}}, \mathcal{T})\}} |\mathcal{S}|, \tag{5.1.2}$$

where $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. In general, $\mathcal{M}$ can also be taken as any set fulfilling $\theta \tilde{\eta}(u_{\mathcal{T}}, \mathcal{M}) \geq \tilde{\eta}(u_{\mathcal{T}}, \mathcal{T})$. However, sorting the entries of $(\tilde{\eta}_K)_{K \in \mathcal{T}}$ does not really add to the complexity of the algorithm, so one can make use of this optimization without relinquishing too much computational time.

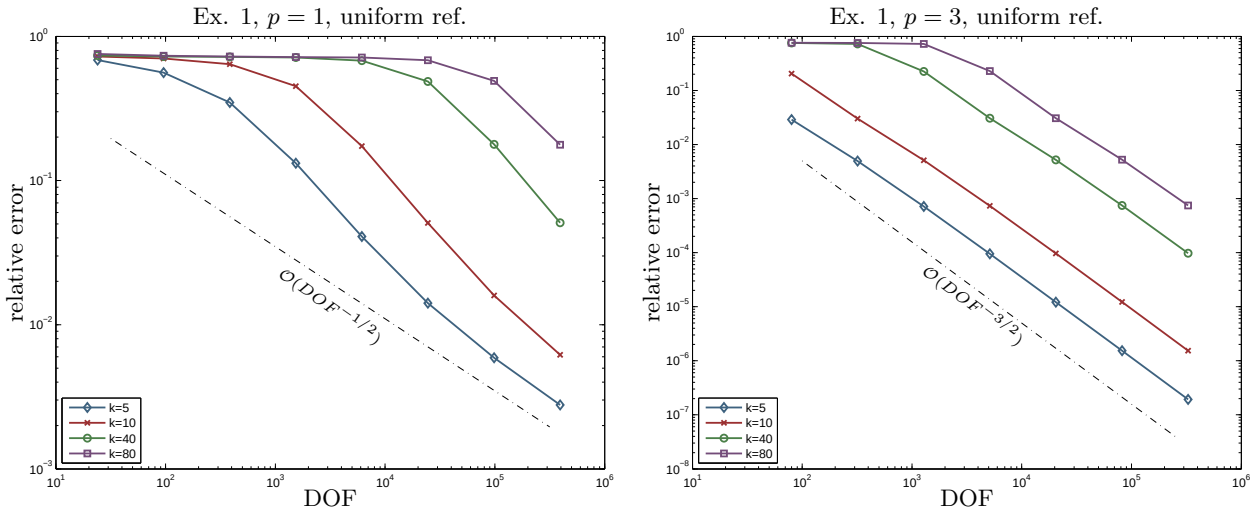
Refine

In this step at least all marked elements are refined. For the purpose of eliminating hanging nodes, possibly other elements need to be refined as well. The program achieves this with largest edge bisection and a recursive refinement procedure.

We also use our estimator as a stopping criterion for the algorithm: If, after the estimation step, the estimated error falls below some given tolerance, the loop consisting of the four modules is aborted. Of course, particularly in a practical situation, it needs to be taken into account that reliability merely holds up to an unknown constant.

Remark 5.1.1. Here, we only consider h-refinement for fixed polynomial degree p on all elements. With our a priori knowledge, p can be chosen accordingly, i.e. $p \sim \log(k)$, and the adaptive algorithm takes care of the mesh refinement. Naturally, it would be desirable that the algorithm also recognizes on which elements the polynomial degree should be increased. Such strategies aim for larger polynomial degrees on elements where the function is assumed to be smooth (see, e.g., [20]).

Remark 5.1.2. For a large class of problems, convergence and even quasi-optimality of the adaptive FEM can be proved. In the case of an interior penalty discontinuous Galerkin (IPDG) FEM, quasi-optimality was shown for an elliptic problem in [8]. Moreover, convergence of an adaptive IPDG method applied to the Helmholtz equation was established in [18].



(a) $p = 1$ (b) $p = 3$

Figure 5.1: Comparison of the relative error in the norm $k\|\cdot\|_{L^2} + |\cdot|_{H^1}$, for the polynomial degrees $p = 1$ and $p = 3$ for different values of k in Example 1.

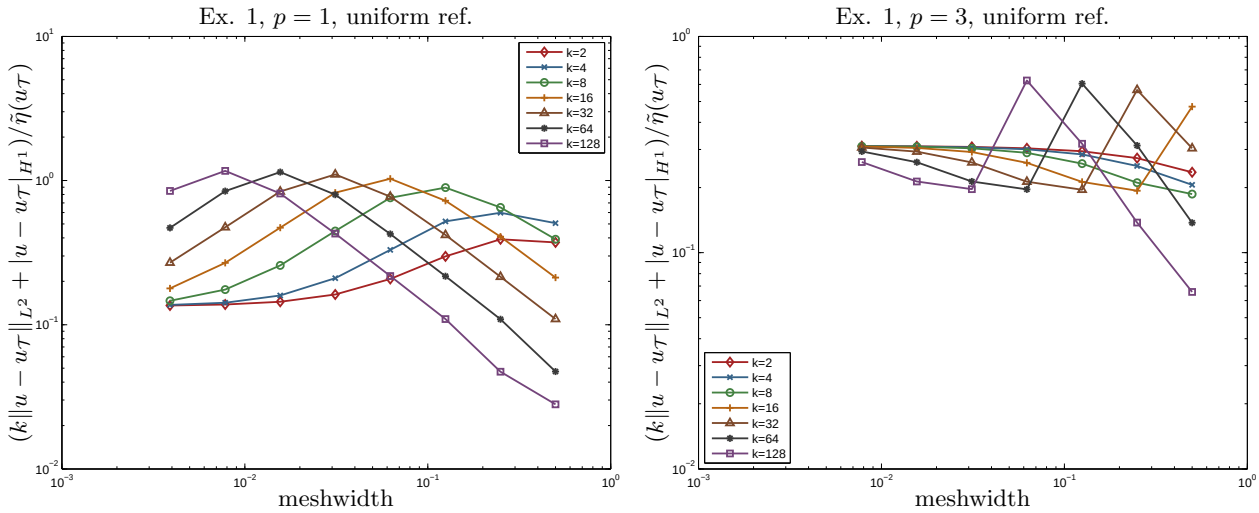
5.2 Plane Wave

The parameters $\mathfrak{a} = 30$, $\mathfrak{b} = 1$, and $\mathfrak{d} = 1/4$ from (2.3.5) are fixed for all experiments in this chapter. The error will be measured in the norm $k\|\cdot\|_{L^2(\Omega)} + |\cdot|_{H^1(\Omega)}$. If we apply adaptive mesh refinement to a problem in the following, the cardinality of the initial mesh is always $\mathcal{O}(1)$ independent of k and p. We start our experiments with two problems, where the analytic solution is a plane wave in both cases.

5.2.1 Example 1

Consider (2.1.1) on the domain $\Omega := (0,1)^2$. The data f, g is chosen such that $u(x,y) = \exp(\mathrm{i}k(x+y))$ is the exact solution. As u does not have any singularities, it is reasonable to refine the mesh uniformly. In Figure 5.1, we compare the relative error for different wavenumbers and $p = 1, 3$. As expected, we observe pollution for polynomial degree 1, which causes the rounded bump in the convergence plot. With the moderate choice $p = 3$, this can be avoided ($k = 5, 10$) respectively strongly reduced ($k = 40, 80$) for the plotted wavenumbers, as the optimal decay rate of the error is obtained (almost) immediately after convergence starts. Moreover, we notice that for large k convergence does not start until a certain meshwidth is reached. This critical meshwidth is larger in the case $p = 1$ since the j-th point in the plots corresponds to the j-th uniformly refined mesh.

Next we test the reliability of the error estimator. Figure 5.2 shows the ratio $(k\|u - u_{\mathcal{T}}\|_{L^2} + |u - u_{\mathcal{T}}|_{H^1})/\tilde{\eta}(u_{\mathcal{T}})$ of the actual error and the estimated error for different polynomial degrees and wavenumbers. Recall that in the reliability estimate (4.1.26), the factor $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))$ and an unknown constant occured. We only bounded $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))$ under certain requirements on the space



(a) $p = 1$ (b) $p = 3$

Figure 5.2: Ratio of the exact error $k\|u - u_{\mathcal{T}}\|_{L^2} + |u - u_{\mathcal{T}}|_{H^1}$ and the estimated error $\tilde{\eta}(u_{\mathcal{T}})$ for different values of k in Example 1.

(see Theorem 2.4.2) and this factor may be large for large element sizes, which is why the estimator differs considerably from the actual error in this case. Moreover, in the range of large meshwidths, σ_k^* also grows with the wavenumber k, which is hinted in the plots. We observe that for larger polynomial degree, the peak of the ratio occurs earlier, is smaller, and the curves drop faster afterwards. It can be seen that at some point the curves settle on a value between 1/10 and 1. For higher polynomial degrees, we noticed that the corrider in which this ratio has its values, further tightens around the presumed limit of the curves.

5.2.2 Example 2

Now we consider (2.1.1) on the domain $\Omega := (0, 2\pi)^2$ with the analytic solution $u(x, y) = \exp(\mathrm{i}kx)$. The appropriate data is in this case

$$f \equiv 0, \qquad g(x,y) = \begin{cases} 0 & \text{if } x = 0 \\ 2\mathrm{i}k & \text{if } x = 2\pi \\ \mathrm{i}ke^{\mathrm{i}kx} & \text{otherwise} \end{cases} \qquad \forall x, y \in \partial\Omega. \tag{5.2.1}$$

Notice that the data near the edge contained in $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ corresponds to the case $u \equiv 0$. The imaginary part of the solution is $\sin(kx)$. In Figure 5.4(b), the imaginary part of the DGFEM solution is plotted for a uniformly refined mesh, polynomial degree $p = 1$, and wavenumber $k = 5$. The discrete solution is close to zero on an area close to the left boundary and plausible only at the far right of the domain. The meshwidth is $h_{\mathcal{T}} = 0.098$ as compared to $k^2 = 25$ and would principally suffice to resolve the solution but is still within the range of the pollution effect.

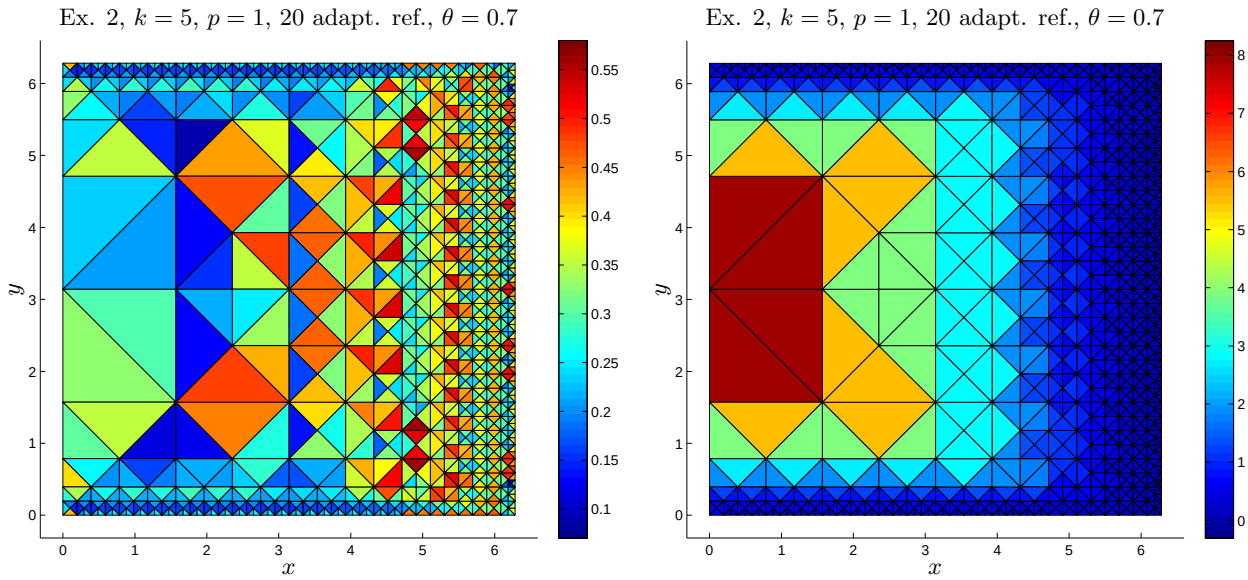
Let us take a look how the adaptive algorithm refines the mesh. In Figure 5.3, the error estimator and the exact error are plotted on each element after 20 adaptive refinements. We observe, that the estimator fails to recognize the error on the large elements in the left part of the domain. The reason is this: As we have seen in Figure 5.4(b), the solution is almost zero in the left part of the domain if $h_{\mathcal{T}}$ is too large. As $f \equiv 0$ and $g \equiv 0$ on the edge where $x = 0$, the terms

$$\|\Delta_{\mathcal{T}} u_{\mathcal{T}} + k^2 u_{\mathcal{T}} + f\|_{L^2(K)} \quad \text{and} \quad \|g - \partial_{\boldsymbol{n}} u_{\mathcal{T}} - \mathrm{i}k u_{\mathcal{T}}\|_{L^2(e)},$$

are almost zero for an element K and a boundary edge e in this part of the domain. In addition to that, the jump of the gradient and the function are small as well, and hence $\tilde{\eta}_K$ is small (cf. (4.1.23)). As a result, the adaptive algorithm at first refines the mesh mainly in the right part of the domain, as can be seen in Figure 5.4(a). In Figure 5.5(b), we observe that at some point this difference in element size vanishes and the mesh is practically uniform. Uniform refinement is what one would expect, knowing the smoothness of the solution. Moreover, we see in Figure 5.5(a) that the element size at which the mesh becomes uniform is close to the critical meshwidth at which significant decrease of the error begins. It therefore seems, that elements in the right part of the domain are refined until the error estimator starts do drop, at which point refinement of the mesh shifts towards the left part of the domain. It is not too surprising that refinement in the right part stops at a reasonable point, since we expect the error estimator to decrease, as soon as the discrete solution starts to resemble the actual solution, or in other words, as soon as the critical meshwidth is reached. Two things should be noted:

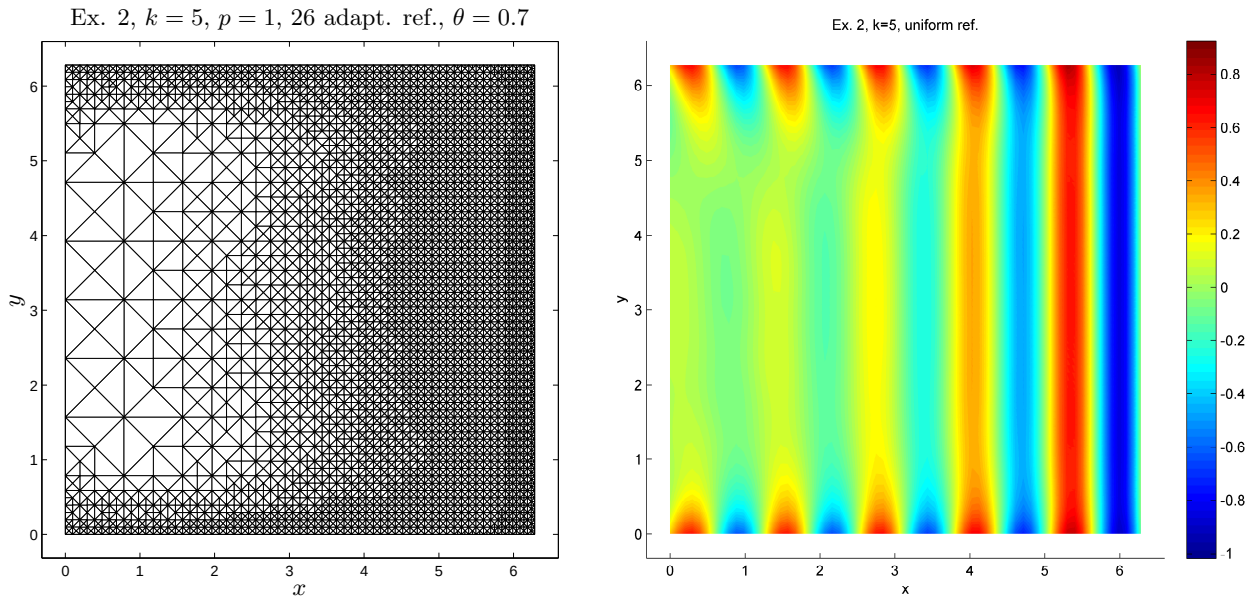
- It is clear that reliability is not a local property, and here we have an example where $\tilde{\eta}_K$ differs significantly from the actual error on some elements in the left part of the domain. Of course, one should keep in mind that these elements are relatively large, which means that σ_k^* could be large (cf. Theorems 2.4.2 and 4.1.14). Therefore, the gap between the estimator and the error is not just a consequence of the fact that reliability does not hold locally, but in accordance with our theory, which suggests that this gap may also occur for the global estimator $\tilde{\eta}$ if the FEM space is too small. Figure 5.5(c) shows that this is indeed the case for large meshwidths.

- At least in this example, the adaptive algorithm apparently perceives what the critical meshwidth is: Even though the exact error is underestimated at first, and elements with large differences in size are generated, further decrease in element size slows down at the meshwidth where convergence is observed for uniform refinement, and the adaptively refined mesh becomes almost uniform as well. Before this meshwidth is reached, no significant convergence can be observed for uniform refinement, and hence for such element sizes it does not really matter whether the mesh is uniform or not. In Figure 5.5(c) we observe that, even though the refinement seems to be suboptimal at first, there is practically no difference in the convergence of the error for uniform or adaptive refinement. Of course, with the knowledge of the pollution effect and the requirements in the previous chapters, one could simply start the adaptive algorithm with an appropriately refined mesh (e.g., as in Theorem 2.4.2). But this and the following examples indicate that the adaptive algorithm is capable to take care of this initial refinement by itself.



(a) $\tilde{\eta}_K$ (b) Exact error on each element

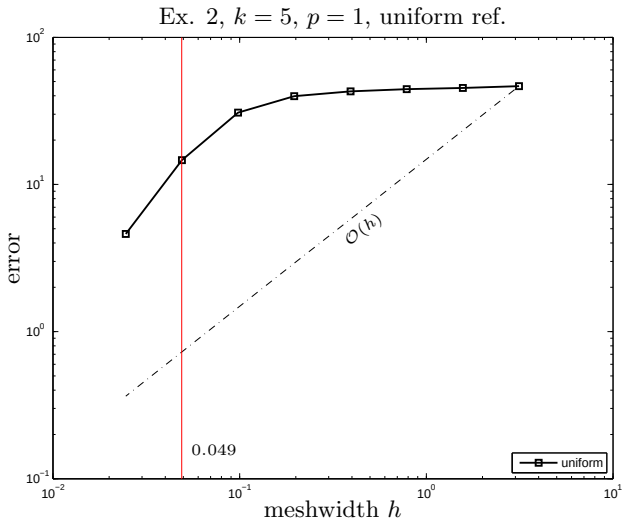
Figure 5.3: Comparison of the local estimator $\tilde{\eta}_K$ and the exact error in the norm $k\|\cdot\|_{L^2} + |\cdot|_{H^1}$ on each element for Example 2 and 1248 elements after 20 adaptive refinements. The error in the left part of the domain is not yet properly recognized by the estimator.



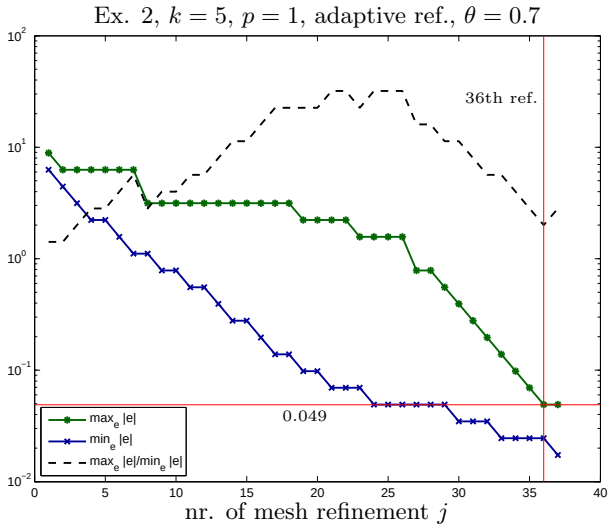
(a) Mesh after 26 refinements, 9044 elements, $\min_K h_K = 0.049$

(b) $\Im(u_{\mathcal{T}})$ for a uniformly refined mesh with 8192 elements and $h_{\mathcal{T}} = 0.098$.

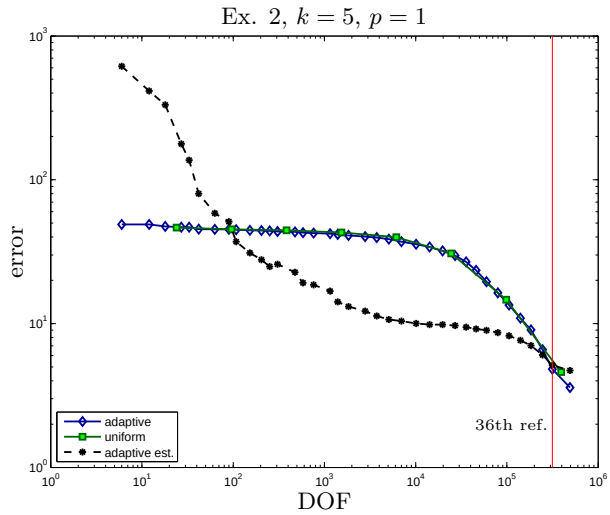
Figure 5.4: Adaptive mesh and imaginary part of the DGFEM solution for a uniform mesh with large meshwidth, $k = 5$, and $p = 1$ in Example 2. The exact solution is $u(x, y) = \exp(\mathrm{i}kx)$, and therefore $\Im(u(x, y)) = \sin(kx)$.



(a) Error in the norm $k\|\cdot\|_{L^2} + |\cdot|_{H^1}$ for uniform refinement



(b) In this plot $|e|$ denotes the length of the edge e. The plot shows the maximum length of an edge, the minimum length of an edge, and the ratio for the j-th adaptively refined mesh.



(c) Error $k\|u - u_{\mathcal{T}}\|_{L^2} + |u - u_{\mathcal{T}}|_{H^1}$ for uniform and adaptive refinement (with $\theta = 0.7$) and the estimated error $\tilde{\eta}$ for adaptive refinement

Figure 5.5: In Figure (b) it can be seen that the adaptive algorithm, applied to Example 2 with $k = 5$ and $p = 1$, at first generates a mesh with very diverse element sizes, which then turns into an almost uniform mesh at about the 36th refinement. This refinement corresponds to a maximum edge length of 0.049. In all three plots, the red line marks this meshwidth, respectively the point at which this adaptive refinement takes place. We observe that convergence for uniform refinement starts shortly before this mesh size is reached. Moreover, at this refinement, the error estimator surpasses the actual error in this example.

5.3 L-shaped Domain

5.3.1 Example 3

For the third example, we consider the L-shaped domain $\Omega := (-1,1)\times(0,1)\cup(-1,0)\times(-1,1)$. The DG method is applied to (2.1.1) with f, g such that the exact solution is given by $u(x,y) = J_{1/2}(kr)$, where $J_{1/2}$ denotes the respective Bessel function of the first kind, and $r = \sqrt{x^2+y^2}$ is the distance to $\mathbf{0}$. The Bessel function and u are plotted in Figure 5.6. The problem is chosen such that the solution has a singularity at the reentrant corner situated at $\mathbf{0}$, which will serve to illustrate the advantages of adaptive over uniform refinement. This example has also been used in [18].

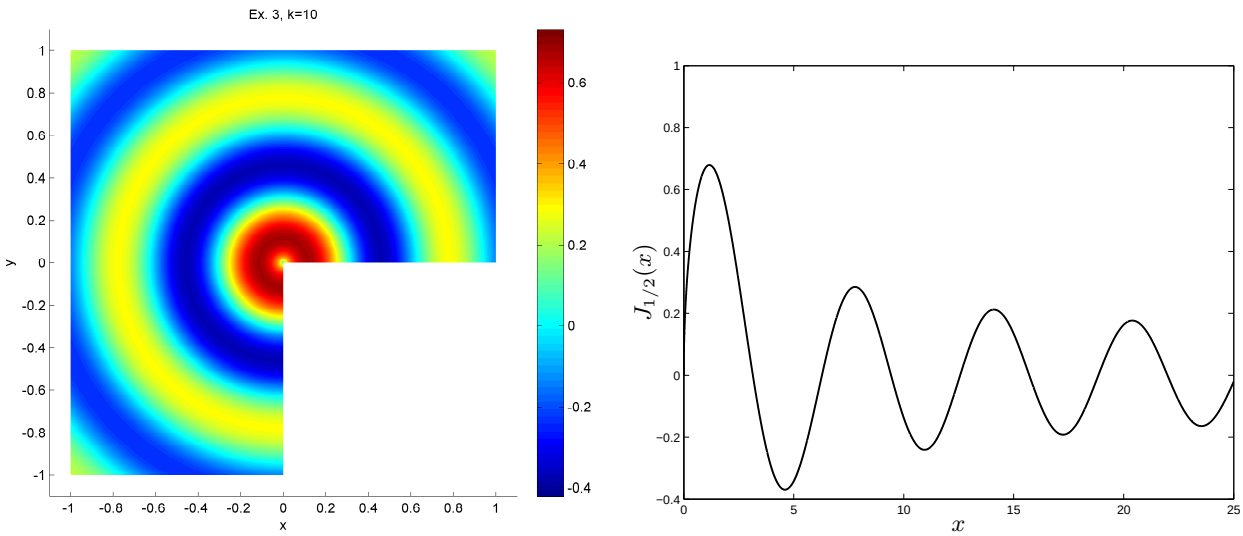
Figure 5.7 shows two adaptive meshes generated by the algorithm for polynomial degree $p = 1$ and wavenumber $k = 10$. We observe that the mesh is refined towards the singularity at $\mathbf{0}$. Moreover, the oscillating nature of the solution u is reflected in the structure of the mesh. It is coarser in areas where u is close to linear (cf. Figure 5.6) and can thus be approximated well by the linear basis functions.

In Figure 5.8, we compare uniform with adaptive refinement for different values of k and p. The singularity causes suboptimal convergence rates for uniform mesh refinement, and the superiority of the adaptive method is evident. We observe optimal convergence rates for the adaptive algorithm applied with the polynomial degrees $p = 2, 4$. Again, it takes some initial refinements until the asymptotic regime is reached. In addition to that, the plots confirm once more that larger wavenumbers require more refinements in order for the error to catch up with the estimator. The pollution effect seems to be perceived a bit weaker by the estimator. Recall, that the space $S^{\boldsymbol{p}}(\mathcal{T})$ suggested in Theorem 2.4.2 avoids pollution and allows to bound the factor $\sigma_k^*(S^{\boldsymbol{p}}(\mathcal{T}))$. In this case, the estimated error should describe the behaviour of the actual error up to some constant. In practice, the occuring constants in Theorem 2.4.2 are unknown, and it is not clear at which point the FEM space is rich enough. However, it can be seen that this seems to be the case for $k = 5$ and $p = 2$ in Figure 5.8 as well as for $k = 10$ and $p = 4$, since no delay is observed for either curve (estimator or error).

In Figure 5.9, the adaptivity parameter θ is varied and compared to the case of uniform refinement. We see that smaller θ amounts to better approximations, but there is almost no difference for the values $\theta = 0.3, 0.5, 0.7$, all of which yield optimal convergence rates. The only case standing out is $\theta = 0.99$, and even then the optimal rate can be preserved.

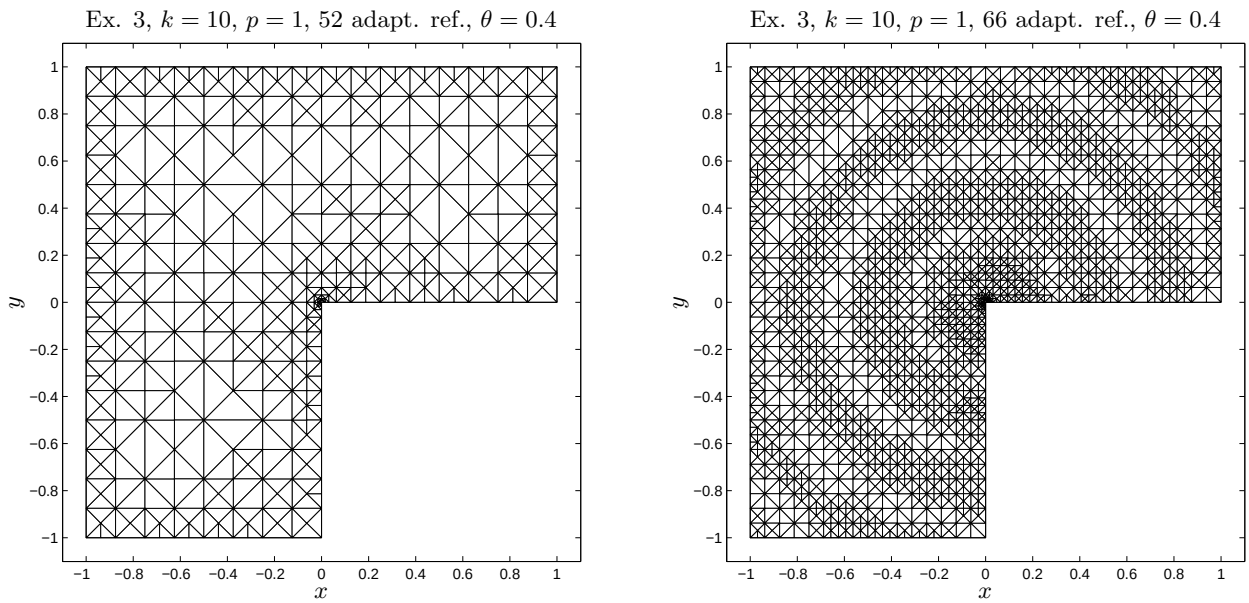
5.4 Non-constant Wavenumber

The examples in this section deal with non-constant k in (2.1.1), i.e. $k = k(x,y)$. All discrete solutions that are plotted in this section, seem to accurately describe the essence of the exact solution, in so far, as no substantial change of appearance is observed for further mesh refinement or increase of the polynomial degree.



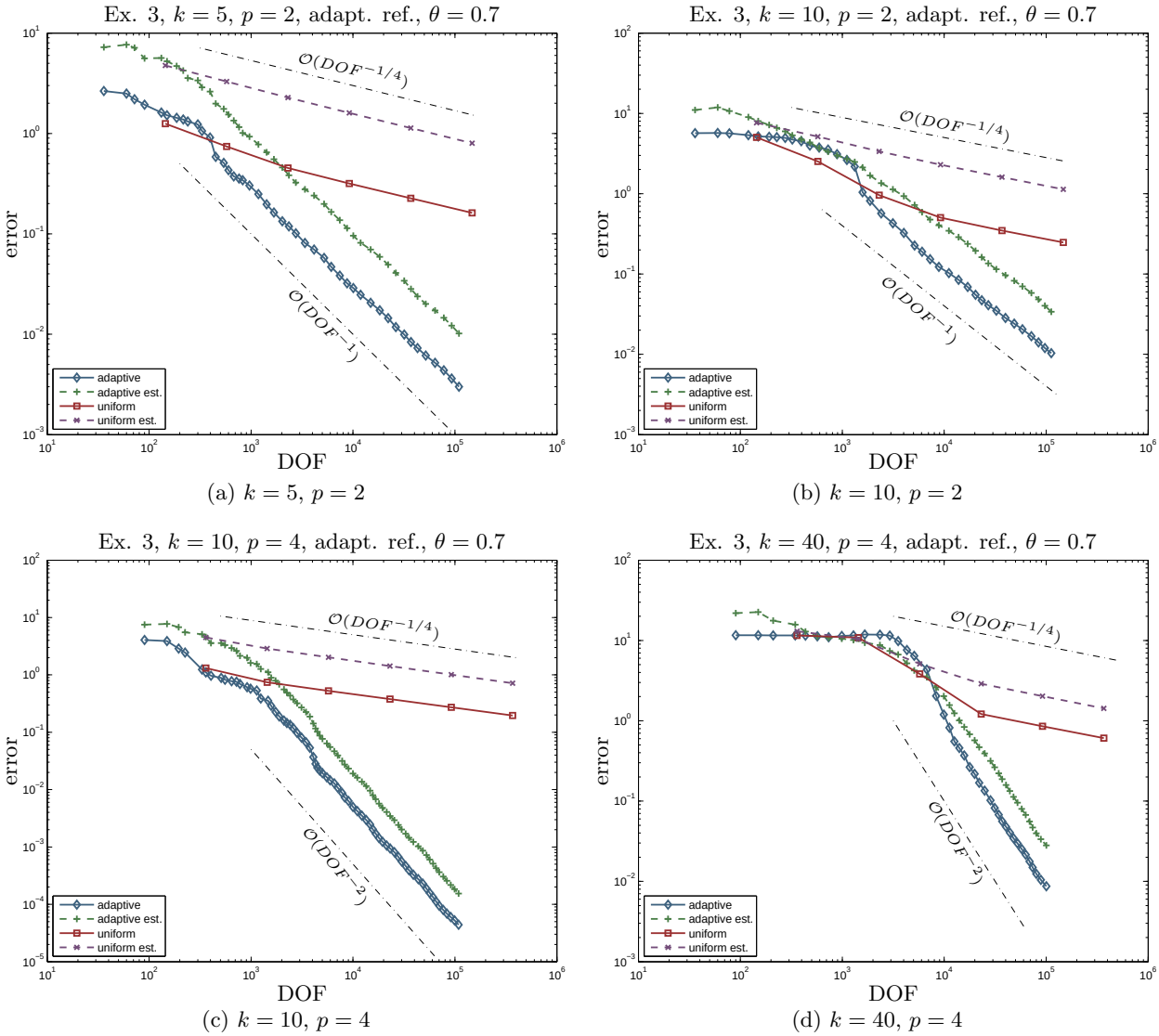
(a) $J_{1/2}(10\|\boldsymbol{x}\|)$ (b) Besselfunction $J_{1/2}$ of the first kind

Figure 5.6: The solution $u = J_{1/2}(kr)$ in Example 3 for $k = 10$, and the Besselfunction $J_{1/2}(x)$, whose derivative goes to infinity for $x \to 0$.



(a) Mesh after 52 refinements, 677 elements, $\min_K h_K = 0.00024$ (b) Mesh after 66 refinements, 3677 elements, $\min_K h_K = 0.000022$

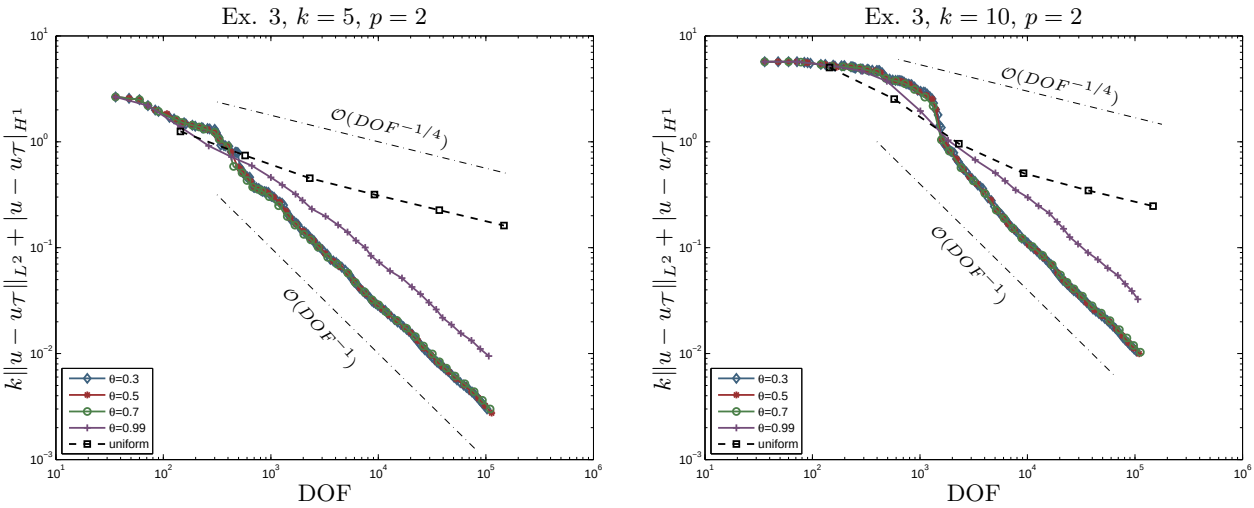
Figure 5.7: Meshes obtained by the adaptive algorithm for Example 3.



(a) $k = 5$, $p = 2$ (b) $k = 10$, $p = 2$

(c) $k = 10$, $p = 4$ (d) $k = 40$, $p = 4$

Figure 5.8: Comparison of the actual error $k\|u - u_{\mathcal{T}}\|_{L^2(\Omega)} + |u - u_{\mathcal{T}}|_{H^1(\Omega)}$ and the estimated error $\tilde{\eta}(u_{\mathcal{T}})$, using uniform and adaptive refinement with $\theta = 0.7$ in Example 3 for different values of k and p.



(a) $k = 5$, $p = 2$ (b) $k = 10$, $p = 2$

Figure 5.9: Comparison of the error for adaptive refinement and different values of θ in Example 3.

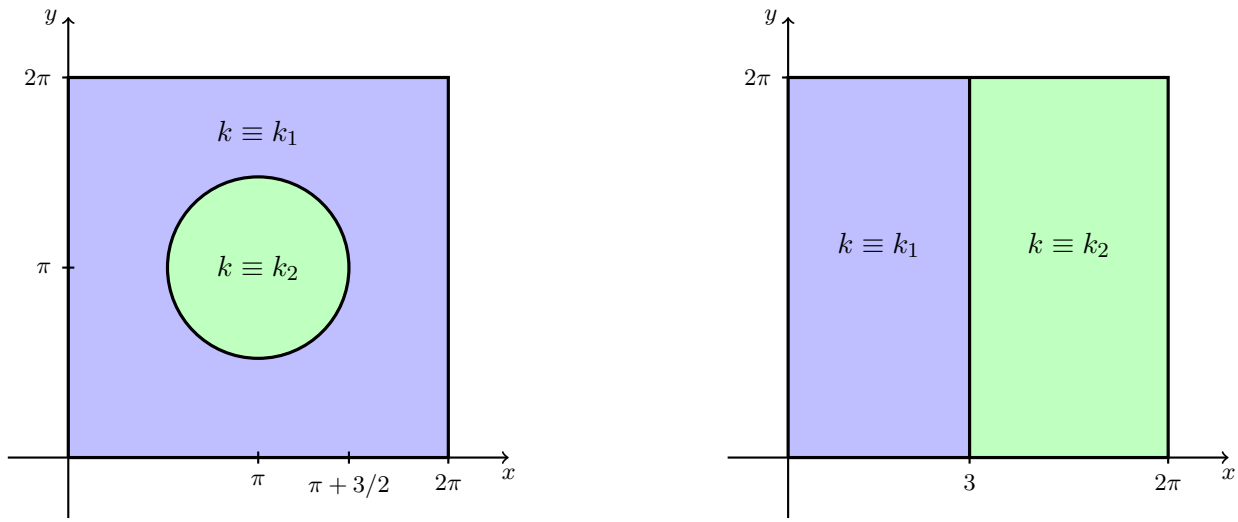
5.4.1 Example 4

Consider the domain $\Omega := (0, 2\pi)^2$. We partition Ω into the ball $\Omega_2 := B_{3/2}(\pi, \pi)$ and its complement $\Omega_1 := \Omega \backslash \Omega_2$. Let $k_1, k_2 > 0$. The function k is now defined to be piecewise constant $k(x, y) := k_1 \mathbb{1}_{\Omega_1} + k_2 \mathbb{1}_{\Omega_2}$ (see Figure 5.10). As the right-hand side in (2.1.1) we take $f \equiv 0$. Moreover let g_2 be as in (5.2.1) with wavenumber k_1, and let

$$g_1(x, y) := \begin{cases} -1 & \text{if } x = 0 \\ \mathrm{i} & \text{if } x = 2\pi \\ 0 & \text{otherwise} \end{cases} \qquad \forall x, y \in \partial\Omega. \tag{5.4.1}$$

In Figure 5.11, the adaptively refined mesh and the real part of the DGFEM solution are plotted for $k_1 = 1$, $k_2 = 10$, and the boundary data g_1. Pronounced refinement can be seen in the vicinity of the circle across which the wavenumber jumps. Moreover, the meshwidth is much smaller inside the circle where the wavenumber is high. The plot suggests that this refinement is in accordance with the smoothness properties of the solution. Figure 5.12 implies, that strong refinement close to the jump of the wavenumber is not necessarily obtained. The plot shows the mesh and the real part of the discrete solution for $k_1 = 10$, $k_2 = 1$, and the boundary data g_2. In this case, the solution appears to be smooth respectively almost zero near the left part of the inner circle on which we have $k \equiv k_2$. This is recognized by the algorithm and results in a coarse mesh in the corresponding area.

Figure 5.15 reflects the convergence of the error estimator in these examples for several values of θ and uniform refinement. Once more, adaptive refinement with $\theta \neq 0.99$ appears to deliver asymptotically optimal behaviour, and hardly any difference can be seen for these values of θ. The curve for $\theta = 0.99$ has a slightly worse convergence rate, but is evidently still superior to the one



(a) Example 4

(b) Example 5

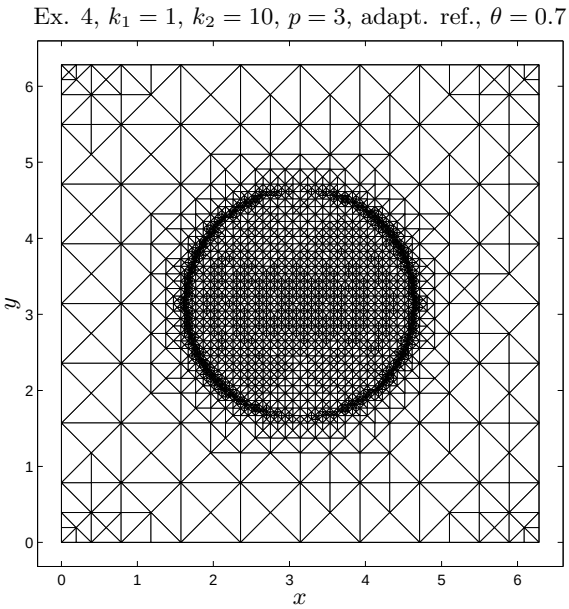
Figure 5.10: Domain and wavenumber in Examples 4 and 5.

belonging to uniform mesh refinement.

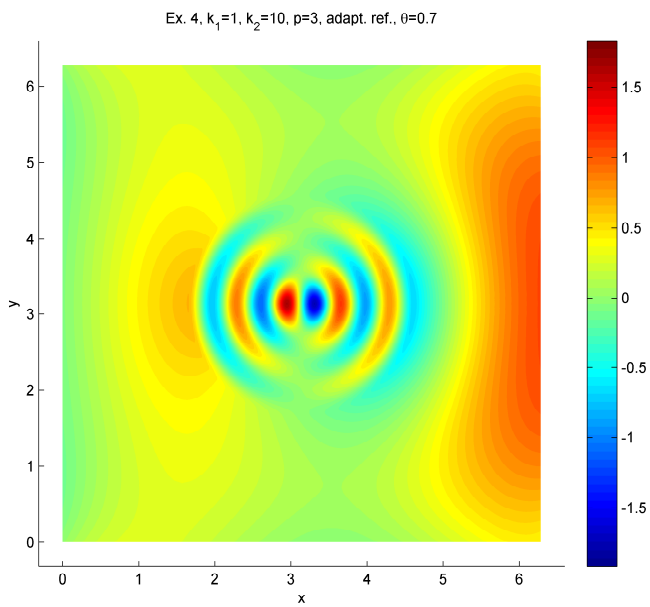
5.4.2 Example 5

For the last example, we consider the same domain as above, together with the data $f \equiv 0$ and g as in (5.4.1). Let again $k_1, k_2 > 0$ and let $k(x, y) := k_1 \mathbb{1}_{[0,3)}(x) + k_2 \mathbb{1}_{[3,2\pi]}(x)$ (see Figure 5.10). The initial mesh on the domain consists of two elements. The position of the jump is chosen in such a way that edges of elements will never lie on this line. Figure 5.13 shows the mesh and the real part of the discrete solution for this setting. The adaptive algorithm proceeds as expected, and refines the mesh according to the local wavenumber and close to where k jumps. We also observe enhanced refinement in the corners and some characteristics of the solution are insinuated by the mesh. Finally, in Figure 5.14, the mesh and the real part of the discrete solution with the continuous wavenumber $k(x, y) = 1/2 + 4x$ are plotted, and Figure 5.16 displays the convergence of the estimator for Example 5. In the first case, where k is discontinuous, we expect the adaptive algorithm to perform better. However, within the plotted range, we only notice a slight advantage of the adaptive method, which again yields an optimal convergence rate. In the second case, where $k(x, y) = 1/2 + 4x$, the rate appears to be optimal for both refinement methods, which is not surprising since k is smooth. However, adaptive refinement achieves a better distribution of the elements and their sizes, which is why the adaptive curve is superior up to a constant.

These examples indicate that the adaptive algorithm, applied with the error estimator $\tilde{\eta}_K$, properly accomplishes the task of refining the mesh according to the properties of the solution. Singularities and wave characteristics are recognized by the estimator, and we observed optimal convergence rates.

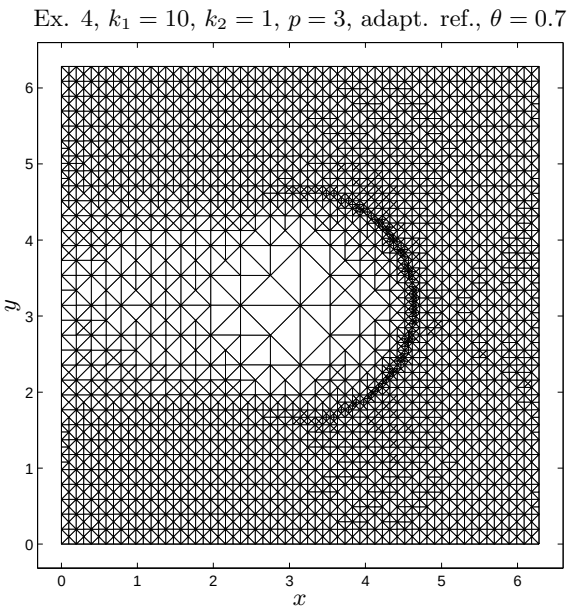


(a) Mesh after 28 refinements, 6076 elements, $\min_K h_K = 0.012$

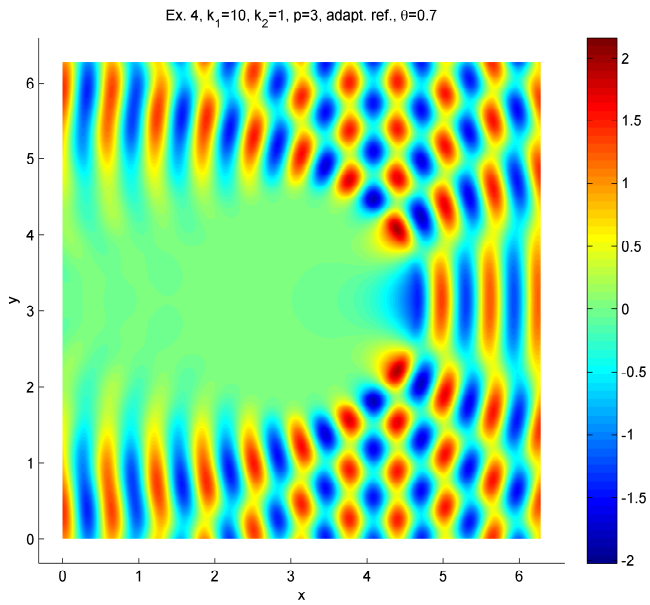


(b) $\Re(u_{\mathcal{T}})$

Figure 5.11: Adaptively refined mesh with $\theta = 0.7$ and real part of the DGFEM solution on this mesh for Example 4 with $k_1 = 1$, $k_2 = 10$, and the boundary data g_1.

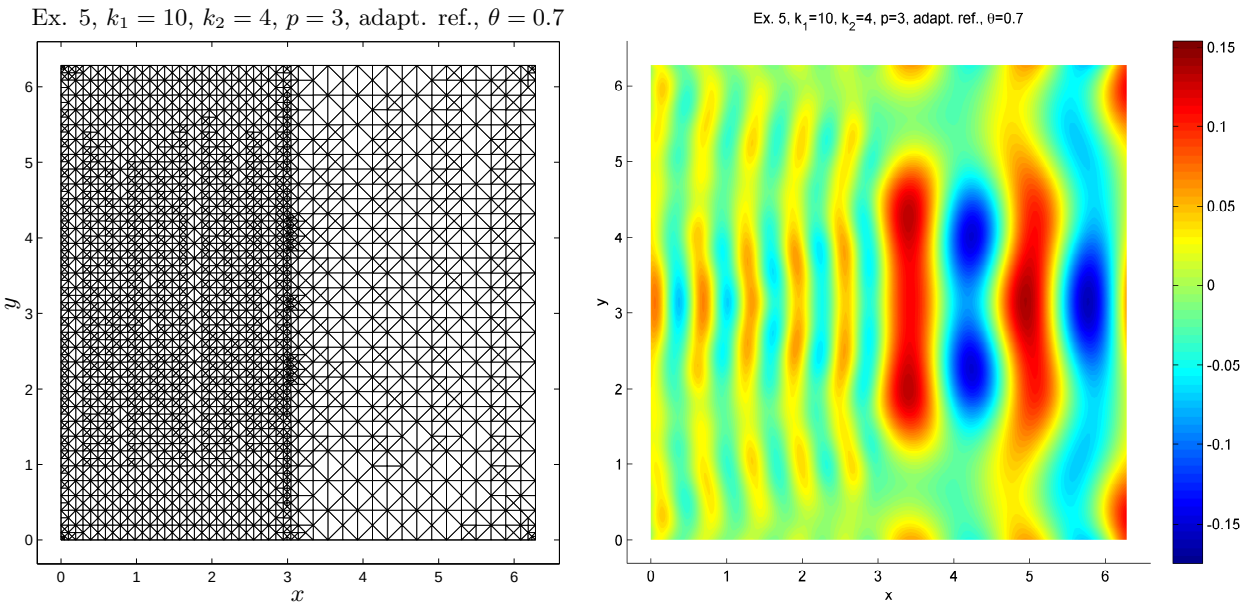


(a) Mesh after 29 refinements, 5892 elements, $\min_K h_K = 0.034$



(b) $\Re(u_{\mathcal{T}})$

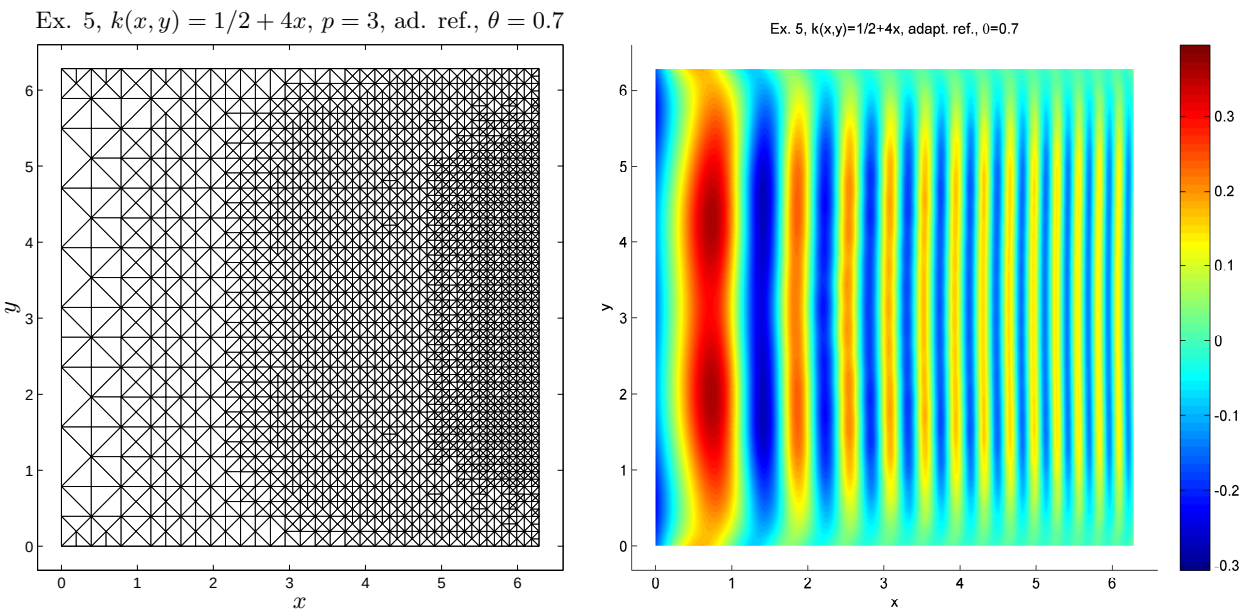
Figure 5.12: Adaptively refined mesh with $\theta = 0.7$ and real part of the DGFEM solution on this mesh for Example 4 with $k_1 = 10$, $k_2 = 1$, and the boundary data g_2.



(a) Mesh after 27 refinements, 5758 elements, $\min_K h_K = 0.034$

(b) $\Re(u_{\mathcal{T}})$

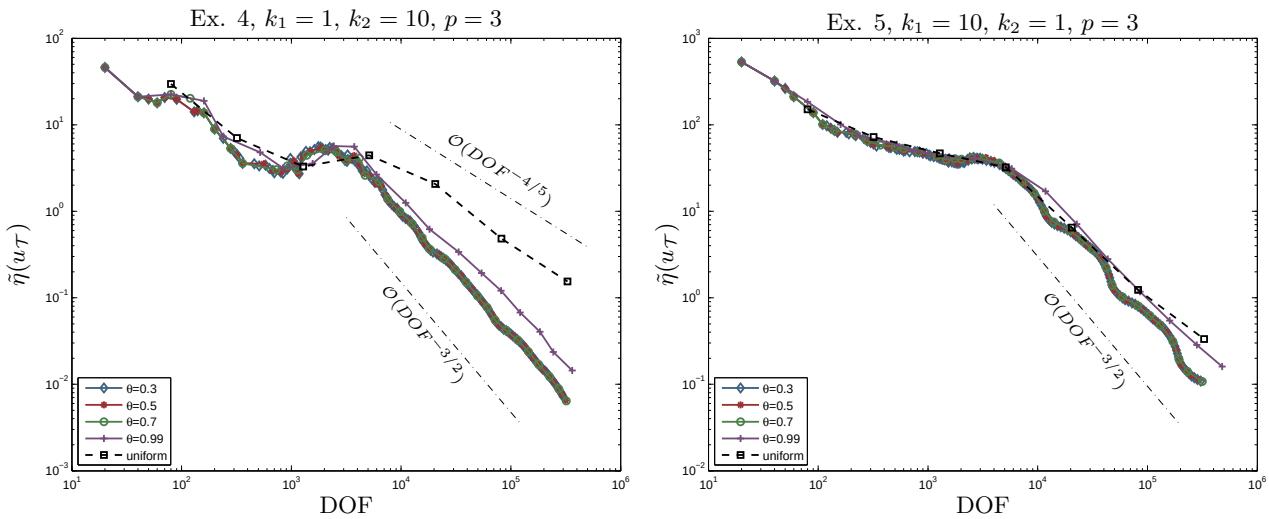
Figure 5.13: Adaptively refined mesh with $\theta = 0.7$ and real part of the DGFEM solution on this mesh for Example 5 with $k_1 = 10$ and $k_2 = 4$.



(a) Mesh after 30 refinements, 5357 elements, $\min_K h_K = 0.069$

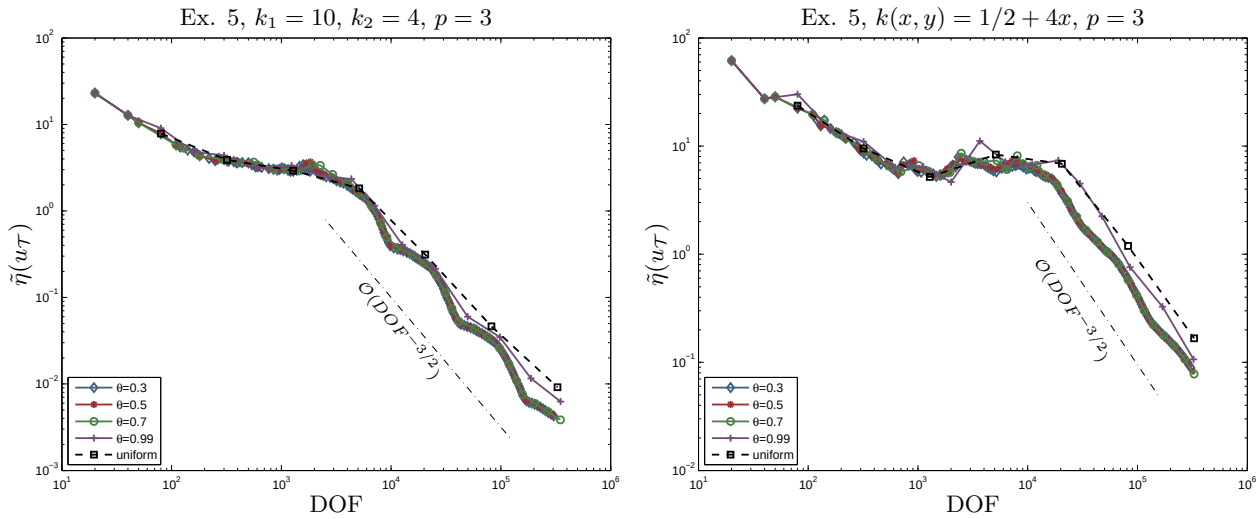
(b) $\Re(u_{\mathcal{T}})$

Figure 5.14: Adaptively refined mesh with $\theta = 0.7$ and real part of the DGFEM solution on this mesh for Example 5 with $k(x, y) = 1/2 + 4x$.



(a) Boundary data g_1, $k_1 = 1$, $k_2 = 10$ (b) Boundary data g_2, $k_1 = 10$, $k_2 = 1$

Figure 5.15: The convergence of the error estimator $\tilde{\eta}(u_{\mathcal{T}})$ in Example 4 for two non-constant functions $k(x, y)$ and the boundary data g_1, g_2, respectively.



(a) $k_1 = 10$, $k_2 = 4$ (b) $k(x, y) = 1/2 + 4x$

Figure 5.16: The convergence of the error estimator $\tilde{\eta}(u_{\mathcal{T}})$ in Example 5 for two non-constant functions $k(x, y)$.

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[2] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[3] M. Ainsworth and J. T. Oden. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000.

[4] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2001/02.

[5] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficient preconditioning for the p-version finite element method in two dimensions. *SIAM J. Numer. Anal.*, 28(3):624–661, 1991.

[6] I. Babuška and M. Suri. The h-p version of the finite element method with quasi-uniform meshes. *RAIRO Modél. Math. Anal. Numér.*, 21(2):199–238, 1987.

[7] I. M. Babuška and S. A. Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? *SIAM Rev.*, 42(3):451–484 (electronic), 2000. Reprint of SIAM J. Numer. Anal. **3**4 (1997), no. 6, 2392–2423 [MR1480387 (99b:65135)].

[8] A. Bonito and R. H. Nochetto. Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method. *SIAM J. Numer. Anal.*, 48(2):734–771, 2010.

[9] D. Braess. *Finite elements*. Cambridge University Press, Cambridge, third edition, 2007.

[10] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[11] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38(5):1676–1706 (electronic), 2000.

[12] R. A. DeVore and G. G. Lorentz. *Constructive approximation*, volume 303 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993.

[13] D. Di Pietro and A. Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*. Mathematiques & Applications. Springer, 2011.

[14] V. Dolejši, M. Feistauer, and C. Schwab. A finite volume discontinuous Galerkin scheme for nonlinear convection-diffusion problems. *Calcolo*, 39(1):1–40, 2002.

[15] W. Dörfler and S. Sauter. A Posteriori Error Estimation for Highly Indefinite Helmholtz Problems. *Comput. Methods Appl. Math.*, 13(3):333–347, 2013.

[16] C. J. Gittelson, R. Hiptmair, and I. Perugia. Plane wave discontinuous Galerkin methods: analysis of the h-version. *M2AN Math. Model. Numer. Anal.*, 43(2):297–331, 2009.

[17] R. Hiptmair, A. Moiola, and I. Perugia. Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version. *SIAM J. Numer. Anal.*, 49(1):264–284, 2011.

[18] R. H. W. Hoppe and N. Sharma. Convergence analysis of an adaptive interior penalty discontinuous Galerkin method for the Helmholtz equation. *IMA J. Numer. Anal.*, 33(3):898–921, 2013.

[19] P. Houston, D. Schötzau, and T. P. Wihler. Energy norm a posteriori error estimation of hp-adaptive discontinuous Galerkin methods for elliptic problems. *Math. Models Methods Appl. Sci.*, 17(1):33–62, 2007.

[20] P. Houston and E. Süli. A note on the design of hp-adaptive finite element methods for elliptic partial differential equations. *Comput. Methods Appl. Mech. Engrg.*, 194(2-5):229–243, 2005.

[21] F. Ihlenburg. *Finite element analysis of acoustic scattering*, volume 132 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[22] J. M. Melenk. *On generalized finite-element methods*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–University of Maryland, College Park.

[23] J. M. Melenk. *hp-finite element methods for singular perturbations*, volume 1796 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[24] J. M. Melenk. hp-interpolation of nonsmooth functions and an application to hp-a posteriori error estimation. *SIAM J. Numer. Anal.*, 43(1):127–155, 2005.

[25] J. M. Melenk, A. Parsania, and S. Sauter. General DG-Methods for Highly Indefinite Helmholtz Problems. *J. Sci. Comput.*, 57(3):536–581, 2013.

[26] J. M. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Math. Comp.*, 79(272):1871–1914, 2010.

[27] J. M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49(3):1210–1243, 2011.

[28] J. M. Melenk and B. I. Wohlmuth. On residual-based a posteriori error estimation in *hp*-FEM. *Adv. Comput. Math.*, 15(1-4):311–331 (2002), 2001. A posteriori error estimation and adaptive computational methods.

[29] R. H. Nochetto, K. G. Siebert, and A. Veeser. Theory of adaptive finite element methods: an introduction. In *Multiscale, nonlinear and adaptive approximation*, pages 409–542. Springer, Berlin, 2009.

[30] C. Schwab. *p- and hp-finite element methods.* Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.

[31] C. Schwab and M. Suri. The optimal *p*-version approximation of singularities on polyhedra in the boundary element method. *SIAM J. Numer. Anal.*, 33(2):729–759, 1996.

[32] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[33] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1978.

[34] R. Verfürth. *A posteriori error estimation techniques for finite element methods.* Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2013.

[35] L. Zhu, S. Giani, P. Houston, and D. Schötzau. Energy norm a posteriori error estimation for *hp*-adaptive discontinuous Galerkin methods for elliptic problems in three dimensions. *Math. Models Methods Appl. Sci.*, 21(2):267–306, 2011.

[36] L. Zhu and D. Schötzau. A robust *a posteriori* error estimate for *hp*-adaptive DG methods for convection-diffusion equations. *IMA J. Numer. Anal.*, 31(3):971–1005, 2011.

ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Eigenständigkeitserklärung

Die unterzeichnete Eigenständigkeitserklärung ist Bestandteil jeder während des Studiums verfassten Semester-, Bachelor- und Master-Arbeit oder anderen Abschlussarbeit (auch der jeweils elektronischen Version).

Die Dozentinnen und Dozenten können auch für andere bei ihnen verfasste schriftliche Arbeiten eine Eigenständigkeitserklärung verlangen.

Ich bestätige, die vorliegende Arbeit selbständig und in eigenen Worten verfasst zu haben. Davon ausgenommen sind sprachliche und inhaltliche Korrekturvorschläge durch die Betreuer und Betreuerinnen der Arbeit.

Titel der Arbeit (in Druckschrift):

A Posteriori Error Estimation of hp-DG Finite Element Methods for Highly Indefinite Helmholtz Problems

Verfasst von (in Druckschrift):
Bei Gruppenarbeiten sind die Namen aller Verfasserinnen und Verfasser erforderlich.

Name(n):	**Vorname(n):**
Zech	Jakob

Ich bestätige mit meiner Unterschrift:
- Ich habe keine im Merkblatt „Zitier-Knigge" beschriebene Form des Plagiats begangen.
- Ich habe alle Methoden, Daten und Arbeitsabläufe wahrheitsgetreu dokumentiert.
- Ich habe keine Daten manipuliert.
- Ich habe alle Personen erwähnt, welche die Arbeit wesentlich unterstützt haben.

Ich nehme zur Kenntnis, dass die Arbeit mit elektronischen Hilfsmitteln auf Plagiate überprüft werden kann.

Ort, Datum

Zürich, 28.3.2014

Unterschrift(en)

Bei Gruppenarbeiten sind die Namen aller Verfasserinnen und Verfasser erforderlich. Durch die Unterschriften bürgen sie gemeinsam für den gesamten Inhalt dieser schriftlichen Arbeit.