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Fully Discrete Version of the AL Basis for Elliptic Problems with General L^{∞}-Coefficient

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Introduction

In this report we consider elliptic problems in heterogeneous media, whose accurate numerical simulation is of fundamental importance in applications such as diffusion in composite materials or porous media and turbulent transport. The abstract mathematical formulation is as follows: Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d \in \{1,2,3\})$, a uniformly elliptic diffusion matrix $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$, and a function $f \in L^2(\Omega)$, we are seeking $u \in H^1_0(\Omega)$ such that

$$a(u,v) := \int_\Omega \langle A\nabla u, \nabla v\rangle dx = \int_\Omega fv dx =: F(v) \qquad \forall v \in H^1_0(\Omega).$$

If the coefficient A is highly oscilllatory on microscopic scales or even non-smooth, the classical polynomial based finite element methods (FEM) become prohibitively expensive, since the usual piecewise polynomial spaces cannot resolve the essential features of the solution unless the mesh size h is chosen small enough (i.e. smaller than the smallest scale in the coefficient). However, the computational work involved becomes too costly – especially for three-dimensional problems.
To overcome this difficulty many types of generalized finite element methods (GFEM) have been developed in the recent years. They are based on the partition of unity method (PUM). This method is explained in detail e.g. in [6]. The PUM constructs a global conforming finite element space using a set of local approximation spaces. Therefore the key ingredient of a PUM is to find good local approximation spaces for a given problem.
GFEM is introduced in [4] as a method for the numerical solution of elliptic PDEs with rough or highly oscillating coefficients. The method is further elaborated and extended to other applications e.g. in [3], [5], [28], and [29]. In the GFEM approach the computational domain Ω is partitioned into a collection of subsets ω_i, $i = 1, \dots, n$. Employing these subsets, local approximation spaces Ψ_i are constructed over each subset ω_i using local information, e.g. in [5] these local approximation spaces are constructed via the solution of certain eigenvalue problems. Then a finite dimensional subspace S of the solution space has to be constructed employing these local approximation spaces. This finite dimensional space can then be employed in a finite element method, e.g. in the Galerkin method.
The GFEM approach allows to significantly reduce the computational work involved in the numerical modeling of large heterogeneous problems, since it is based on general (non-polynomial) ansatz functions whose shape contains some information about the characteristic physical behaviour of the solution. Therefore the

scales of the coefficient A may not be resolved by the finite element mesh and thus the full global solution can be obtained by solving a macro system which is an order of magnitude smaller than the system corresponding to a direct application of the finite element method to the full structure. Moreover, the basis functions to construct the spaces Ψ_i can be computed independently and therefore their computation allows parallelization.
If the heterogeneity in the material is distributed periodically over the domain, the very efficient method presented in [24], [25], [22] can be applied. A further approach is the heterogenous multiscale method (HMM), which is a framework for linking models at different scales (see e.g. [1], [16]). An application of HMM are multiscale finite element methods (cf. [2], [17]). These methods use a fine mesh for computing locally and independently a finite element basis and a coarse mesh for computing globally and at low cost the solution. The computation of the basis functions can be done in parallel.
In [18] a semidiscrete method for elliptic problems with general L^∞-coefficient is presented. It is shown that for this problem class there exists a local finite element basis (AL basis) consisting of $O\Big(\big(\log\frac{1}{h}\big)^{d+1}\Big)$ basis functions per nodal point such that the convergence rates of the classical FEM for Poisson-type problems are preserved. The method is based on PUM and is closely related to GFEM.
The goal of this report is to develop a fully discrete version of the AL basis such that the linear convergence rate is preserved. The report is divided into six parts. In Chapter 1 some elementary FEM-notations and basic definitions are introduced. Afterwards the precise abstract mathematical formulation of the model problem considered in this report is stated. Finally, the Galerkin discretization for this problem is formulated and it is illustrated why for the problem class under consideration linear finite elements are not appropriate.
Chapter 2 is devoted to the description of the semidiscrete method which has been proposed in [18]. Before we explain the construction of the generalized basis functions in detail, the main idea of it is pointed out. Since locally L-harmonic functions are a key tool of this method, Subsection 2.2.2 presents some important properties of this function class. At the end of Chapter 2 the definition of the AL basis is given and the linear convergence property of it is stated.
A fully discrete algorithm for the construction of the AL basis is presented in Chapter 3, whereas Chapter 4 is devoted to the error analysis of this fully discrete method under some appropriate assumption, which will be further investigated in Chapter 5. It is shown that the method converges linearly with respect to the H^1-norm.
In comparison to the result presented in Chapter 4, Chapter 5 presents the results of two papers which also examine a non-periodic setting. Whereas in [27] the coefficient is assumed to be smooth but highly varying, in [21] the case of a general L^∞-coefficient without any smoothness assumptions is investigated.
The last chapter of this report consists of the Appendix. It contains some error estimates which are used in the error analysis described in Chapter 4.

Chapter 1

Setting

1.1 FEM-Notations and Basic Definitions

In this section we introduce some basic FEM-notations, which will be often used in this report.

Definition 1.1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain. We define the Hilbert space $L^2(\Omega)$ over the field of real numbers by*

$$L^2(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \,\textit{Lebesgue measurable and} \int\limits_\Omega |u|^2 dx < \infty\}.$$

The scalar product on $L^2(\Omega)$ is given by

$$\langle u, v \rangle_{L^2(\Omega)} := \int\limits_\Omega u(x)v(x)dx$$

and the associated norm is denoted by $\|\cdot\|_{L^2(\Omega)}$. Furthermore, we introduce the spaces of test functions as well as the usual Sobolev spaces:

Definition 1.1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain. Then the spaces of test functions are given by*

$$\begin{aligned} C^\infty(\Omega) &:= \{u : \Omega \to \mathbb{R} \mid u^{(k)} \,\textit{exists and is continuous for all}\, k \in \mathbb{N}_0\} \\ C_0^\infty(\Omega) &:= \{u \in C^\infty(\Omega) \mid \operatorname{supp} u \subset\subset \Omega\}, \end{aligned}$$

where $\operatorname{supp} u := \overline{\{x \in \Omega \,|\, u(x) \neq 0\}}$ *is the* ***support*** *of u and $K \subset\subset \Omega :\Longleftrightarrow K$ is a compact subset of Ω.*

Definition 1.1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain. For $m \in \mathbb{N}_0$ the Sobolev spaces are given by*

$$H^m(\Omega) := \{u \in L^2(\Omega) \mid \forall\, |\alpha| \leq m : D^\alpha u \in L^2(\Omega)\},$$

where $D^\alpha u \in L^2(\Omega)$ *satisfies*

$$\int\limits_\Omega u D^\alpha v = (-1)^{|\alpha|} \int\limits_\Omega D^\alpha u v \quad \forall v \in C_0^\infty(\Omega).$$

Equipped with the norm

$$\|u\|_{H^m(\Omega)} := \sqrt{\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)}} \tag{1.1}$$

the space $H^m(\Omega)$ is a Hilbert space. The seminorm on $H^m(\Omega)$ is given by

$$|u|_{H^m(\Omega)} := \sqrt{\sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(\Omega)}}.$$

Definition 1.1.4. *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *be a domain. For* $m \in \mathbb{N}_0$ *the space of functions denoted by* $H_0^m(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *with respect to the norm* $\|\cdot\|_{H^m(\Omega)}$.

The dual space of $H_0^1(\Omega)$ is denoted by $H^{-m}(\Omega)$ and its norm is defined by

$$\|u\|_{H^{-m}(\Omega)} := \sup_{0\neq v\in H_0^m(\Omega)} \frac{(u,v)}{\|v\|_{H^m(\Omega)}},$$

where the functional

$$\begin{aligned}(u,\cdot) : H_0^m(\Omega) &\to \mathbb{R}\\ v &\mapsto (u,v)\end{aligned}$$

is the duality mapping, i.e. u defines a linear functional on $H_0^m(\Omega)$.

Definition 1.1.5. *A domain* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *is bounded, if it is contained in a ball of finite radius, i.e.* $\exists\, x \in \mathbb{R}^n$ *and* $r > 0$ *(*$r \in \mathbb{R}$*) such that* $\forall\, \omega \in \Omega$ *we have* $\|x - \omega\| < r$, *where* $\|\cdot\|$ *denotes the Euclidean norm on* $\mathbb{R}^n$.

Remark 1.1.6. *If the domain* Ω *is bounded, there exists a constant* C_Ω *such that*

$$\|u\|_{L^2(\Omega)} \le C_\Omega |u|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega). \tag{1.2}$$

The inequality (1.2) *is known as the Poincaré-Friedrichs' inequality and it states that, when the domain* Ω *is bounded, the seminorm* $|\cdot|_{H^1(\Omega)}$ *is a norm over the space* $H_0^1(\Omega)$ *which is equivalent to the norm* $\|\cdot\|_{H^1(\Omega)}$ *(cf. [13, p. 12]). Thus there exist positive constants* $\underline{C}_{\Omega,1}$, $\bar{C}_{\Omega,1} < \infty$ *such that*

$$\underline{C}_{\Omega,1}\|u\|_{H^1(\Omega)} \le |u|_{H^1(\Omega)} \le \bar{C}_{\Omega,1}\|u\|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega). \tag{1.3}$$

For the formulation of our problem, which is given in the subsequent chapter, the following definition plays an important role:

Definition 1.1.7. *We say a domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, has a Lipschitz boundary or Ω is a Lipschitz domain if there exist $M \in \mathbb{N}$ and a collection of open sets $O_1, \dots, O_M \subset \mathbb{R}^n$ with the following two properties:*

(i) $\partial\Omega \subset \bigcup_{i=1}^M O_i$

(ii) $\partial\Omega \cap O_i$ *can be represented as graph of a Lipschitz continuous function for all* $1 \leq i \leq M$.

1.2 Problem Formulation

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded Lipschitz domain and let the diffusion matrix $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$ be uniformly elliptic, i.e.

$$\begin{aligned} 0 < \alpha(A, \Omega) &:= \operatorname*{ess\,inf}_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle A(x)v, v\rangle}{\langle v, v\rangle} \\ \infty > \beta(A, \Omega) &:= \operatorname*{ess\,sup}_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle A(x)v, v\rangle}{\langle v, v\rangle}. \end{aligned} \tag{1.4}$$

Here $\langle\cdot, \cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. Its associated norm will be denoted by $\|\cdot\|$.

We consider the following problem: For a given function $f \in L^2(\Omega)$, we are seeking $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega \langle A\nabla u, \nabla v\rangle dx = \int_\Omega f v dx =: F(v) \qquad \forall v \in H_0^1(\Omega). \tag{1.5}$$

Before we can state the Lax-Milgram-Theorem, which ensures that the solution of (1.5) exists and is unique (cf. Remark 1.2.3), we need to characterize some properties of bilinear forms.

Definition 1.2.1. *Let V be a normed linear space and $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ a bilinear form.*

(i) a is bounded (or continuous) if there exists $C < \infty$ such that

$$|a(v, w)| \leq C\|v\|_V \|w\|_V \quad \forall v, w \in V. \tag{1.6}$$

(ii) a is coercive on V if there exists $\gamma > 0$ such that

$$a(v, v) \geq \gamma\|v\|_V^2 \quad \forall v \in V. \tag{1.7}$$

Theorem 1.2.2 (Lax-Milgram). *Let V be a Hilbert space, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ a continuous, coercive bilinear form, and let $f : V \to \mathbb{R}$ be a continuous linear functional. Then the abstract variational problem: Find an element $u \in V$ such that*

$$a(u,v) = f(v) \quad \forall\, v \in V \tag{1.8}$$

has a unique solution.

The proof can be found e.g. in [11, Theorem 2.7.7], [13, Theorem 1.1.3.].

Remark 1.2.3. *Assumption (1.4) on the diffusion matrix A implies*

$$\alpha\|w\|^2 \le \langle Aw, w\rangle = \|A^{1/2}w\|^2 \le \beta\|w\|^2 \quad \forall w \in \mathbb{R}^d. \tag{1.9}$$

Due to the Cauchy-Schwarz inequality and (1.9) we get the following upper bound for all u, $v \in H^1(\Omega)$:

$$\begin{aligned}\langle A\nabla u, \nabla v\rangle &= \langle A^{1/2}\nabla u, A^{1/2}\nabla v\rangle \\ &\le \|A^{1/2}\nabla u\|\|A^{1/2}\nabla v\| \\ &\le \beta\|\nabla u\|\|\nabla v\|. \end{aligned}\tag{1.10}$$

By (1.10) and the definition of the Sobolev space $H^1(\Omega)$ equipped with the norm $\|\cdot\|_{H^1(\Omega)}$ (cf. Definition 1.1.3 and (1.1)) we obtain

$$|a(u,v)| \le \beta\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall u, v \in H^1(\Omega),$$

which means that the bilinear form a is bounded, i.e. continuous on $H^1_0(\Omega)$. Moreover, due to (1.4) we also have

$$\langle A\nabla u, \nabla u\rangle \ge \alpha\|\nabla u\|^2.$$

Integrating the last inequality over Ω and using Poincaré-Friedrichs' inequality (cf. (1.3)) we get

$$a(u,u) \ge \alpha\|\nabla u\|^2_{L^2(\Omega)} = \alpha|u|^2_{H^1(\Omega)} \ge \alpha\, \underline{C}^2_{\Omega,1}\|u\|^2_{H^1(\Omega)} \quad \forall u \in H^1_0(\Omega), \tag{1.11}$$

where $\underline{C}_{\Omega,1}$ is the constant from (1.3). Thus, besides being continuous, the bilinear form a is also coercive and therefore the differential equation (1.5) has a unique solution by Theorem 1.2.2.

Remark 1.2.4. *Let $L : H^1_0(\Omega) \to H^{-1}(\Omega)$ be the operator associated to the bilinear form $a(\cdot,\cdot)$ defined in (1.5). Since L is uniformly elliptic, $L^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$ exists and the assumptions on A imply*

$$\|L^{-1}\|_{H^1_0(\Omega)\leftarrow H^{-1}(\Omega)} \le \frac{1}{\alpha\, \underline{C}^2_{\Omega,1}},$$

with $\underline{C}_{\Omega,1}$ as in (1.3) and α as in (1.4).

1.3 Galerkin Discretization

We want to solve problem (1.5) numerically using a Galerkin finite element method. For simplicity we assume that the domain Ω is either a one-dimensional interval, or a two-dimensional polygonal domain, or a three-dimensional polyhedron. In order to discretize Ω we need some definitions concerning finite element meshes.

Definition 1.3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded domain. A simplicial finite element mesh $\mathcal{T}$ on Ω is a partition of Ω into d-dimensional disjoint open simplices satisfying the following properties:*

(i) For any two elements $\tau_1, \tau_2 \in \mathcal{T}$ with $\tau_1 \neq \tau_2$ the intersection $\bar{\tau}_1 \cap \bar{\tau}_2$ is either empty, a common vertex (an interval endpoint if $d = 1$), a common edge (for $d \geq 2$), or a common face (for $d = 3$).

(ii) $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}} \bar{\tau}$.

Definition 1.3.2. *Let $\mathcal{T} := \{\tau_i : 1 \leq i \leq n\}$ be a finite element mesh on the domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. The mesh width is given by*

$$H := \max_{1 \leq i \leq n} \{\operatorname{diam}(\tau_i) : \tau_i \in \mathcal{T}\},$$

where $\operatorname{diam}(\tau_i)$ *denotes the diameter of an element $\tau_i \in \mathcal{T}$.*

Definition 1.3.3. *Let $\mathcal{T} := \{\tau_i : 1 \leq i \leq n\}$ be a finite element mesh on $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$.*

(i) $\mathcal{T}$ is said to be regular (or non-degenerate) if there exists a constant $C > 0$ such that

$$\frac{\operatorname{diam}(\tau)}{\rho_\tau} \leq C \quad \forall \tau \in \mathcal{T},$$

where ρ_τ is the diameter of the maximal inscribed ball in τ.

(ii) $\mathcal{T}$ is called quasi-uniform (or shape regular) if it is regular and there exists a constant $C > 0$ such that

$$\max_{1 \leq i \leq n} \{\operatorname{diam}(\tau_i)\} \leq C \operatorname{diam}(\tau) \quad \forall \tau \in \mathcal{T}.$$

Remark 1.3.4. *The shape-regularity of a finite element mesh $\mathcal{T}$ is described by the constant*

$$\kappa := \max\left\{\frac{\operatorname{diam}(\tau)}{\rho_\tau} : \tau \in \mathcal{T}\right\}. \tag{1.12}$$

The constant κ is always bounded, since $\mathcal{T}$ consists of finitely many elements. Note that κ becomes large, if the simplices are degenerated (e.g. if they are flat or needle-shaped).

Let $\mathcal{T} := \{\tau_i : 1 \leq i \leq N\}$ be a regular finite element mesh (cf. Definition 1.3.3) with mesh width H. The space of continuous, piecewise linear finite elements is given by

$$S := \{u \in H_0^1(\Omega) \mid \forall \tau \in \mathcal{T} : u_{|\tau} \in \mathbb{P}_1\}, \tag{1.13}$$

where $\mathbb{P}_1$ is the space of polynomials with degree ≤ 1. Furthermore, let $(b_i)_{i=1}^n$ denote the usual local nodal basis of S and denote their support by

$$\omega_i := \operatorname{supp} b_i. \tag{1.14}$$

Then the abstract conforming Galerkin method to problem (1.5) can be formulated as: Find $u_S \in S$ such that

$$a(u_S, v) = F(v) \quad \forall v \in S \tag{1.15}$$

with $a(\cdot,\cdot)$ and $F(\cdot)$ as in (1.5).

Remark 1.3.5. *S is a closed subspace of $H_0^1(\Omega)$ and therefore S is a Hilbert space. Moreover, the bilinear form $a(\cdot,\cdot)$ defined in* (1.5) *is continuous and coercive on S. Hence, the Lax-Milgram theorem (cf. Theorem 1.2.2) guarantees the existence and uniqueness of the solution u_S of problem* (1.15)*.*

The following theorem, called Céa's lemma, shows that the error $\|u - u_S\|_{H^1(\Omega)}$ is proportional to the best approximation of u in the space S.

Theorem 1.3.6 (Céa's lemma). *Let V be a Hilbert space and $W \subset V$ be a finite dimensional subspace. Furthermore, let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous, coercive bilinear form and $f : V \to \mathbb{R}$ a continuous linear functional. Assume u solves the variational problem* (1.8)*. Then the solution u_W of the problem: Find $u_W \in W$ such that*

$$a(u_W, v) = f(v) \quad \forall v \in W$$

satisfies the following error estimate

$$\|u - u_W\|_V \leq \frac{C}{\gamma} \inf_{v \in W} \|u - v\|_V,$$

where C is the continuity constant (cf. (1.6)*) and γ is the coercivity constant (cf.* (1.7)*) of $a(\cdot,\cdot)$ on V.*

The proof can be found e.g. in [10, 4.2], [11, Theorem 2.8.1], [13, Theorem 2.4.1.].

By Céa's lemma we know that for the solution u_S of problem (1.15) the estimate

$$\|u - u_S\|_{H^1(\Omega)} \leq \frac{\beta}{\underline{C}_{\Omega,1}^2 \alpha} \inf_{v \in S} \|u - v\|_{H^1(\Omega)}$$

holds with α, β as in (1.4) and $\underline{C}_{\Omega,1}$ as in (1.3). If the diffusion coefficient A, the right-hand side f as well as the domain Ω in problem (1.5) are sufficiently smooth such that the solution u is in $H^2(\Omega) \cap H_0^1(\Omega)$, then there exists an interpolant $\Pi : H^2(\Omega) \to S$ such that

$$\inf_{v \in S} \|u - v\|_{H^1(\Omega)} \leq \|u - \Pi v\|_{H^1(\Omega)} \leq \frac{C_{\Omega,1,\kappa}}{\underline{C}_{\Omega,1}} H |u|_{H^2(\Omega)}, \tag{1.16}$$

where $\underline{C}_{\Omega,1}$ is the constant from (1.3) and the constant $C_{\Omega,1,\kappa}$ depends only on Ω and the shape-regularity constant κ (cf. (1.12)). Furthermore, the regularity estimate

$$|u|_{H^2(\Omega)} \leq C_{reg} \|f\|_{L^2(\Omega)}$$

holds. Therefore the unique Galerkin approximation u_S satisfies the error estimate

$$\|u - u_S\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)},$$

where $C := \frac{\beta}{\alpha} \frac{C_{\Omega,1,\kappa}}{\underline{C}^3_{\Omega,1}} C_{reg}$. This estimate states linear convergence of the classical finite element method as H tends to zero. However, the regularity assumption is not realistic for the problem class under consideration. It is well known that as long as the mesh $\mathcal{T}$ does not resolve the discontinuities and oscillations of A the convergence rates of linear finite elements are substantially reduced.

Even if the coefficient is smooth, it might oscillate at a frequency ϵ^{-1}, for some small parameter ϵ. The smooth, periodic case was investigated in [22] and for a hp-finite element discretization in the finite element space of piecewise polynomials of degree $p \geq 1$ on a quasiuniform mesh of mesh width H the error estimate

$$\|u - u_H\|_{H^1(\Omega)} \leq C \min \left\{ 1, \left(\frac{H}{\epsilon} \right)^p \right\}$$

was presented, where C is a constant independent of ϵ and H but dependent of p, Ω, f, and A. Also the non-smooth, periodic case has been analyzed in the literature (see e.g. [15]).

Note that we did not impose any periodicity assumptions on the coefficient A. We consider the more general case $A \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{sym})$. In [18] is proven that for the problem class under consideration there exists a local generalized finite element basis with the following property: For any shape regular finite element mesh of step size H there exist $O\big(\log(\frac{1}{H})^{d+1}\big)$ local basis functions per nodal point such that the corresponding Galerkin solution u_{AL} satisfies

$$\|u - u_{AL}\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)}.$$

The construction of these basis functions is described in Chapter 2.

Also in [27] and [21] a non-periodic setting has been considered. In [27] a regularity theory for smooth but highly varying diffusion matrices has been developed, whereas in [21] a construction of local generalized basis functions without any smoothness assumptions on the coefficient was presented. The main results of these two papers are summarized in Chapter 5.

Chapter 2

Semidiscrete Method

As we have already pointed out in Section 1.3, linear finite elements are not appropriate for the problem class under consideration. Therefore many types of GFEM have been developed in the recent years. In [18] a semidiscrete method for the construction of generalized local basis functions has been proposed. The approach presented there is based on PUM and is closely related to GFEM. In this chapter I will describe the semidiscrete method of [18] in order to present a fully discrete version of it in Chapter 3.
The goal is to construct a finite dimensional subspace $V_{AL} \subset H_0^1(\Omega)$ such that the error estimate $\|u - u_{AL}\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)}$ holds, where u denotes the solution of (1.5), u_{AL} is the solution of the Galerkin discretization (1.15) using V_{AL} instead of S, H is the mesh width (cf. Definition 1.3.2), $f \in L^2(\Omega)$, and the constant C depends only on α, β (cf. 1.4).
In [18] a set of basis functions $b_{i,j} \in H_0^1(\Omega)$, $1 \leq j \leq p$, $1 \leq i \leq n := \dim(S)$ with S as in (1.13), called AL basis, is constructed such that

$$\operatorname{supp} b_{i,j} \subset \omega_i,$$

where ω_i is defined in (1.14).
Moreover, it is shown that by choosing the number p in (2.1) proportionally to $O(\log^{d+1} \frac{1}{H})$ the linear convergence property (cf. Definition 2.0.1) holds.

Definition 2.0.1 (Linear convergence property). *Let* $a(\cdot,\cdot)$ *be as in* (1.5) *and* S *be as in* (1.13) *with supports* ω_i *of basis functions as in* (1.14). *Let* $\tilde{S} \subset H_0^1(\Omega)$ *be the finite dimensional subspace given by the span of some linearly independent functions* $b_{i,j} \in H_0^1(\Omega)$

$$\tilde{S} = \operatorname{span}\{b_{i,j} \,|\, 1 \leq j \leq p,\ 1 \leq i \leq n \text{ and } \operatorname{supp} b_{i,j} \subset \omega_i\}. \tag{2.1}$$

$\tilde{S}$ *has the* ***linear convergence property*** *if, for any* $f \in L^2(\Omega)$, *the solution to the problem of finding* $u_{\tilde{S}} \in \tilde{S}$ *such that*

$$a(u_{\tilde{S}}, v) = \int_\Omega fv \quad \forall v \in \tilde{S}$$

satisfies the error estimate

$$\|u - u_{\tilde{S}}\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)},$$

where C *only depends on* α *and* β *(cf.* (1.4)*).*

Before the construction of the AL basis will be explained in detail, the main idea of it will be briefly summarized.

2.1 Idea

The definition of the AL basis is patchwise. For a nodal patch ω_i the set of indices $\mathcal{I} := \{1, 2, \dots, n\}$ is split into a nearfield and a farfield, denoted by $\mathcal{I}_i^{near}$ respectively $\mathcal{I}_i^{far}$. The nearfield $\mathcal{I}_i^{near}$ contains those indices j which correspond to basis functions with support close to ω_i, whereas the farfield $\mathcal{I}_i^{far}$ consists of the remaining indices. The construction of the AL basis is divided into two steps.

step 1 For an index $i \in \mathcal{I}$, one part of the AL basis is given by $b_i L^{-1} b_j,\ j \in \mathcal{I}_i^{near}$, where $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ denotes the operator associated to the continuous bilinear form $a(\cdot,\cdot)$ in (1.5) and b_i is the usual nodal basis. We set $V_i^{near} := \text{span}\{b_i L^{-1} b_j \ : \ j \in \mathcal{I}_i^{near}\}$.
For the other part of the AL basis one needs to set up an auxiliary space $X_i^{far} := \text{span}\{L^{-1} b_j|_{\omega_i^*} : j \in \mathcal{I}_i^{far}\}$ in a certain neighbourhood $\omega_i^* \supset \omega_i$. It turns out that the space X_i^{far} can be approximated by a low dimensional space, which is done in the second step.

step 2 Introduce intermediate neighbourhoods $\omega_i = D_{i,\ell} \subset D_{i,\ell-1} \subset \cdots \subset D_{i,0} \subset \omega_i^*$, where $\ell = O(\log \frac{1}{H})$. For any $D_{i,j}$, a mesh $\mathcal{G}_{ij}$ is constructed by intersecting $D_{i,j}$ with a regular Cartesian mesh of width $O(H/\log \frac{1}{H})$. Then the farfield part of the AL basis for the patch ω_i is given by $b_i P \chi_\tau,\ \tau \in \mathcal{G}_{i,j},\ 0 \leq j \leq \ell$, where P is the L^2-orthogonal projection onto X_i^{far} and χ_τ denotes the characteristic function for $\tau \in \mathcal{G}_{i,j}$.

2.2 Construction in Detail

As mentioned in the previous Section 2.1, the construction of the AL basis consists of two steps. In the first step one needs to set up the spaces V_i^{near} and X_i^{far}. The second step is dedicated to the approximation of X_i^{far} by a low dimensional space.

2.2.1 Construction of V_i^{near} and X_i^{far}

Let $\mathcal{G} := \{\tau_i \ : \ 1 \leq i \leq N\}$ be a regular finite element mesh (cf. Definition 1.3.3) with mesh width H (cf. Definition 1.3.2) and set

$$B_i := L^{-1} b_i, \qquad i \in \mathcal{I} := \{1, 2, \dots, n\}, \tag{2.2}$$

where $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ denotes the operator associated to the continuous bilinear form $a(\cdot,\cdot)$ in (1.5), i.e. it holds $a(u,v) = \langle Lu, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$, and b_i is the standard finite element basis of S (cf. (1.13)).
We define recursively simplex layers around ω_i by

$$\begin{aligned} \omega_{i,0} &:= \omega_i \\ \omega_{i,j+1} &:= \bigcup\{\bar\tau \mid \tau \in \mathcal{G} \text{ and } \omega_{i,j} \cap \bar\tau \neq \emptyset\}, \quad j = 0,1,2,\dots \end{aligned} \tag{2.3}$$

Depending on the parameter $\eta \in (0,1)$ we define $\omega_i^* := \omega_{i,m}$, where m is chosen such that the condition

$$0 < \eta \operatorname{diam}(\omega_i) \leq \operatorname{dist}(\omega_i, \partial\omega_i^*) \tag{2.4}$$

is satisfied.
Since the mesh is locally quasi-uniform (cf. Definition 1.3.3 and Remark 2.2.1), we can choose $0 < \eta$ sufficiently small but independent of H such that $m = m(\eta) = O(1)$.

We split the set of indices $\mathcal{I} := \{1, 2, \dots, n\}$ into a nearfield and a farfield by setting

$$\mathcal{I}_i^{near} := \{j \in \mathcal{I} : 0 < |\omega_i^* \cap \operatorname{supp} b_j|\} \qquad \mathcal{I}_i^{far} := \mathcal{I}\backslash\mathcal{I}_i^{near}, \tag{2.5}$$

where for a measurable subset $M \subset \mathbb{R}^d$, we set $|M| := \int_M 1$.
Finally, we define

$$X_i^{far} := \operatorname{span}\left\{B_j|_{\omega_i^*} \ : \ j \in \mathcal{I}_i^{far}\right\} \tag{2.6}$$

and

$$V_i^{near} := \operatorname{span}\left\{b_i B_j \ : \ j \in \mathcal{I}_i^{near}\right\}. \tag{2.7}$$

Remark 2.2.1. *For all $1 \leq i \leq n$ and $m \in \mathbb{N}_0$, there exists a constant $c_{m,\kappa}$ depending only on m and κ (cf. (1.12)) such that*

$$\rho_\tau \geq c_{m,\kappa} \operatorname{diam}(t) \qquad \forall \tau,\, t \in \omega_{i,m},$$

where ρ_τ denotes the diameter of the maximal inscribed ball in τ.

2.2.2 Locally L-Harmonic Functions

The key element for the approximation of X_i^{far} is the space of locally L-harmonic functions. Therefore its definition as well as some important facts about locally L-harmonic functions are introduced in this subsection, before the construction of X_i^{far} can be given in the next subsection.

Definition 2.2.2 (Locally L-harmonic functions). *Let $D \subseteq \mathbb{R}^d$, $d \in \{1,2,3\}$, be a domain (that may be unrelated to Ω) and $a(\cdot,\cdot)$ as in (1.5). A function $u \in$*

$L^2(D)$ is called locally L-harmonic on D if for all $K \subseteq D$ with $\operatorname{dist}(K, \partial D) > 0$ *the following conditions hold:*

$$u|_K \in H^1(K), \tag{2.8a}$$
$$a(v, u|_\Omega) = 0 \quad \forall v \in H_0^1(\Omega) \text{ with } \operatorname{supp} v \subseteq K, \tag{2.8b}$$
$$u|_{D\setminus\Omega} = 0. \tag{2.8c}$$

The space of all locally L-harmonic functions on D is denoted by $X(D)$.

The following lemma shows that for any function $u \in X(D)$ and for any measurable subset $K \subset D$ with $\operatorname{dist}(K, \partial D) > 0$ we can bound $\|\nabla u\|_{L^2(K)}$ in terms of the L^2-norm over D (cf. [8, Lemma 2.4] and [9, Lemma 1]).

Lemma 2.2.3 (Caccioppoli inequality). *Let $u \in X(D)$ and let $K \subseteq D$ be a domain with* $\operatorname{dist}(K, \partial D) > 0$. *Then we have $u|_K \in H^1(K)$ and*

$$\|\nabla u\|_{L^2(K)} \leq \sqrt{\frac{\beta}{\alpha}} \frac{4}{\operatorname{dist}(K, \partial D)} \|u\|_{L^2(D)}$$

with α, β as in (1.4).

The proof can be found in [8, Lemma 2.4].

Using Lemma 2.2.3 the following important property of $X(D)$ can be shown:

Lemma 2.2.4. *The space $X(D)$ is closed in $L^2(D)$.*

For a proof see [8, Lemma 2.2].

Lemma 2.2.5 (Finite-dimensional approximation I). *Let $D \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a convex domain and X a closed subspace of $L^2(D)$. Then for any $k \in \mathbb{N}$ there is a subspace $V_k \subset X$ satisfying* $\dim V_k \leq k$ *such that*

$$\inf_{v \in V_k} \|u - v\|_{L^2(D)} \leq c_{appr} \frac{\operatorname{diam}(D)}{\sqrt[d]{k}} \|\nabla u\|_{L^2(D)} \quad \forall u \in X \cap H^1(D),$$

where the constant c_{appr} only depends on the spatial dimension d.

The proof can be found in [8, Lemma 2.1].

Remark 2.2.6. *The closedness of X is important, since the proof of Lemma 2.2.5 uses orthogonal projections to map functions from $L^2(D)$ onto X. This construction is only straightforward if X is closed in $L^2(D)$. By Lemma 2.2.4 $X(D)$ is closed in $L^2(D)$ and thus we can use Lemma 2.2.5 with $X(D)$ instead of X.*

The following lemma tells us a sufficient condition for the dimension of a finite dimensional subspace to approximate a function from $X(D)$ in a subdomain D_2 of D up to a certain error.

Lemma 2.2.7 (Finite-dimensional approximation II). *Let $D \subset \Omega$ and $X(D)$ the space of locally L-harmonic functions on D. Furthermore, let $D_2 \subset D$ be a convex domain such that*

$$\operatorname{dist}(D_2, \partial D) \geq \eta \operatorname{diam}(D_2) > 0$$

for some constant η (cf. Remark 2.2.8). Then for any $M > 1$ there is a subspace $W \subset X(D_2)$ so that

$$\inf_{w \in W} \|u - w\|_{L^2(D_2)} \leq \frac{1}{M} \|u\|_{L^2(D)} \quad \forall u \in X(D)$$

and

$$\dim W \leq c_\eta^d \lceil \log M \rceil^{d+1} + \lceil \log M \rceil, \quad c_\eta = 4ec_{appr} \sqrt{\frac{\beta}{\alpha}} \frac{1+2\eta}{\eta}, \tag{2.9}$$

where c_{appr} only depends on the spatial dimension d and α, β as in (1.4).

The proof can be found in [8, Lemma 2.6].

Remark 2.2.8. *The factor $\frac{1+2\eta}{\eta}$ in (2.9) shows that η should be of order $O(1)$.*

2.2.3 Approximation of X_i^{far}

In the following we always assume that ω_i and ω_i^* are convex sets for all $1 \leq i \leq n$.

Remark 2.2.9. *Any function $v \in X_i^{far}$ satisfies*

$$\int_{\omega_i^*} \langle A \nabla v, \nabla w \rangle = 0 \qquad \forall w \in H_0^1(\omega_i^*),$$

i.e. the functions in X_i^{far} are locally L-harmonic on ω_i^* (cf. Definition 2.2.2). The space X_i^{far} can be approximated by a low dimensional space. More precisely, applying Lemma 2.2.7 with $X(D) \leftarrow X_i^{far}$, $D \leftarrow \omega_i^*$ and $D_2 \leftarrow \omega_i$ we see that for any $M > 1$ there is a subspace $W \subset X_i^{far}|_{\omega_i}$ such that the estimate

$$\inf_{w \in W} \|u - w\|_{L^2(\omega_i)} \leq \frac{1}{M} \|u\|_{L^2(\omega_i^*)} \quad \forall u \in X_i^{far}$$

holds and for the dimension of W we have

$$\dim W \leq C_\eta^d \lceil \log M \rceil^{d+1} + \lceil \log M \rceil, \quad C_\eta = 4eC_d \sqrt{\frac{\beta}{\alpha}} \frac{1+2\eta}{\eta}.$$

The constant C_d only depends on the spatial dimension d and α, β are defined in (1.4).

Our goal is to approximate the space X_i^{far} by a low dimensional space. For this purpose we use the construction which has been suggested in [8, proofs of Lemma 2.1 and Lemma 2.6].
First, we introduce intermediate layers between ω_i and ω_i^*. Therefore we set $r_{i,1} := \operatorname{dist}(\omega_i, \partial\omega_i^*)$ and

$$r_{i,j} := \Big(1 - \frac{j-1}{\ell-1}\Big) r_{i,1}, \qquad 2 \le j \le \ell, \tag{2.10}$$

where ℓ will be fixed later. It holds $r_{i,1} > r_{i,2} > \cdots > r_{i,\ell} = 0$. The intermediate layers are given by

$$\begin{aligned} D_{i,0} &:= \omega_i^* \\ D_{i,j} &:= \{x \in \omega_i^* \mid \operatorname{dist}(x, \omega_i) \le r_{i,j}\}, \qquad 1 \le j \le \ell, \end{aligned}$$

and satisfy $\omega_i = D_{i,\ell} \subset D_{i,\ell-1} \subset \cdots \subset D_{i,1} \subset D_{i,0} = \omega_i^*$. The domains $D_{i,j}$ are convex for all j and by Lemma 2.2.4 $X(D_{i,j})$ is closed in $L^2(D_{i,j})$. Therefore we know by Lemma 2.2.5 that for any $\kappa_j \in \mathbb{N}$ there exists a subspace $V_{\kappa_j} \subset X(D_{i,j})$ such that $\dim V_{\kappa_j} \le \kappa_j$. In order to construct these subspaces $V_{\kappa_j} =: \tilde{V}_{i,j}^{far}$ for $0 \le j \le \ell - 1$ we use L^2-orthogonal projections onto X_i^{far} (cf. [8, proof of Lemma 2.1]).
We set $\kappa_j =: k^d$, where $k \in \mathbb{N}$ will be fixed later. For $\rho > 0$ let $\mathcal{G}_\rho$ denote a Cartesian tensor mesh on $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, which consists of d-dimensional elements with side length ρ. Then define

$$\mathcal{G}_{i,j} := \Big\{ D_{i,j} \cap \tau \mid \tau \in \mathcal{G}_\rho \text{ with } \rho := \frac{\operatorname{diam}(D_{i,j})}{k} \Big\}, \quad 0 \le j \le \ell - 1. \tag{2.11}$$

For $t \in \mathcal{G}_{i,j}$, we denote the characteristic function for t by $\chi_t : \Omega \to \mathbb{R}$. We define

$$\tilde{V}_{i,j}^{far} := \operatorname{span}\{(P\chi_t)|_{\omega_i} \ : \ t \in \mathcal{G}_{i,j}\},$$

where $P : L^2(\Omega) \to X_i^{far}$ is the L^2-orthogonal projection. We set

$$\tilde{V}_i^{far} := \tilde{V}_{i,0}^{far} + \tilde{V}_{i,1}^{far} + \cdots + \tilde{V}_{i,\ell-1}^{far} \tag{2.12}$$

and finally,

$$V_i^{far} := \{b_i v \ : \ v \in \tilde{V}_i^{far}\}. \tag{2.13}$$

Remark 2.2.10. *We have $b_i v \in H_0^1(\omega_i)$ for all $v \in \tilde{V}_i^{far}$, since $b_i \in W_0^{1,\infty}(\omega_i)$ and $X_i^{far} \subset H^1(\omega_i^*)$. Thus, we can identify $b_i v$ by its extension by zero to a function (again denoted by $b_i v$) in $H_0^1(\Omega)$. In this sense we have*

$$V_i^{far} \subset H_0^1(\Omega), \qquad \dim V_i^{far} \le \sum_{j=0}^{\ell-1} \#\mathcal{G}_{i,j} \le \sum_{j=0}^{\ell-1} k^d = \ell k^d.$$

2.3 Definition of the AL Basis and some Properties

Definition 2.3.1 (AL basis). *For any support* ω_i *(cf.* (1.14)*) the set of AL basis functions consists of the functions* $b_i B_j$, $j \in \mathcal{I}_i^{near}$, *and of the functions*

$$b_i P \chi_t \quad \forall t \in \mathcal{G}_{i,q} \quad 0 \le q \le \ell - 1.$$

The general notation is $b_{i,j}$, $1 \le j \le p$, $1 \le i \le n$, *where* $p := \dim(V_i^{far} + V_i^{near})$. *The corresponding generalized finite element space is given by*

$$V_{AL} := (V_1^{near} + V_1^{far}) + (V_2^{near} + V_2^{far}) + \cdots + (V_n^{near} + V_n^{far}). \tag{2.14}$$

The Galerkin method for the generalized finite element space V_{AL} reads: Find $u_{AL} \in V_{AL}$ such that

$$a(u_{AL}, v) = F(v) \qquad \forall v \in V_{AL}, \tag{2.15}$$

where $a(\cdot,\cdot)$ and $F(\cdot)$ are defined in (1.5).

Under the following three assumptions the Galerkin method (2.15) converges linearly as Theorem 2.3.2 shows.

(i) The domains ω_i and ω_i^* (cf. (1.14), (2.3) and (2.4)) are convex and satisfy (2.4) for some $\eta \gtrsim 1$.

(ii) The constant

$$C_\# := \max_{i \in \mathcal{I}} \# \mathcal{I}_i^{near}$$

depends only on the shape-regularity of the finite element mesh $\mathcal{G}$ and the number $m = O(1)$ in the definition of ω_i^* (cf. (2.4)).

(iii) There exists a constant C_q such that

$$\# \mathcal{I} \le C_q H^{-d}.$$

Theorem 2.3.2. *Let* u *denote the solution of* (1.5). *Let the parameters* ℓ *and* k *in the definition of the farfield part of* V_{AL} *be chosen according to*

$$\ell := \max\left\{2, \left\lceil \frac{2+d}{2\log 2} \log \frac{1}{H} \right\rceil \right\} \quad \text{and} \quad k := \left\lceil \frac{2c_0 \ell^2}{(\ell - 1)} \right\rceil \tag{2.16}$$

for some $c_0 = O(1)$. *Let* u_{AL} *be the solution of* (2.15). *Then the error estimate*

$$\|u - u_{AL}\|_{H^1(\Omega)} \le CH \|f\|_{L^2(\Omega)}$$

holds and

$$\dim V_{AL} \le C_d n \ell^{d+1} \le \tilde{C}_d H^{-d} \log^{d+1} \frac{1}{H}. \tag{2.17}$$

The proof can be found in [18, Theorem 7].

Remark 2.3.3

(i) Estimate (2.17) *for the dimension of* V_{AL} *given in Theorem 2.3.2 can be seen as follows: From the definition of* V_{AL} *(cf.* (2.14)*) it is clear that*

$$\dim V_{AL} \leq n(\dim V_i^{near} + \dim V_i^{far})$$

holds. The choice $k := \left\lceil \frac{2c_0\ell^2}{(\ell-1)} \right\rceil$ *yields*

$$\begin{aligned} k &\leq \frac{2c_0\ell^2}{\ell-1} + 1 \\ &= \frac{2c_0\ell(\ell-1)}{\ell-1} + \frac{2c_0(\ell-1)}{\ell-1} + \frac{2c_0}{\ell-1} + 1 \\ &= 2c_0\ell + 2c_0 + \frac{2c_0}{\ell-1} + 1 \\ &\leq 2c_0\ell + 4c_0 + 1 \\ &\leq \ell\Big(4c_0 + \frac{1}{2}\Big). \end{aligned}$$

In the last inequality we used that $\ell \geq 2$. *Remark 2.2.10 and the above computation show that*

$$\dim V_i^{far} \leq \ell k^d \leq \Big(4c_0 + \frac{1}{2}\Big)\ell^{d+1}.$$

Moreover, since the index m *in the definition of* ω_i^* *is independent of* H *(cf.* (2.3) *and* (2.4)*), we have* $\dim V_i^{near} = O(1)$. *Hence,*

$$\dim V_{AL} \leq C_d n\ell^{d+1} \leq \tilde{C}_d H^{-d} \log^{d+1}\frac{1}{H}.$$

The last inequality follows by assumption (iii) and the choice of ℓ.

(ii) Inequality (2.17) *shows that the number* p *in the definition of the AL basis has to be chosen proportionally to* $O(\log^{d+1}\frac{1}{H})$.

Chapter 3

Fully Discrete Method

The aim of this chapter is to develop a fully discrete version of the method presented in Chapter 2.
Let $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator associated to the bilinear form $a(\cdot,\cdot)$ defined in (1.5) (cf. also Remark 1.2.4). In the definition of the AL basis the inverse of the continuous differential operator L is involved and thus the construction of the basis functions $b_{i,j}$ (cf. Definition 2.3.1) which has been proposed in [18] is only semidiscrete.
To get a fully discrete method one has to approximate the action of L^{-1} to the nodal basis $(b_i)_{i=1}^n$ in an appropriate way. One possible approach to obtain such an approximation is to impose some scale assumptions on the diffusion matrix A and to employ a Galerkin discretization with a conforming finite dimensional subspace $V \subset H_0^1(\Omega)$ on a sufficiently fine mesh $\mathcal{T}_h$. In the following we assume that the approximation of L^{-1} is computed by this method. We denote it by $\tilde{L}^{-1}$.
By replacing L^{-1} by $\tilde{L}^{-1}$ in the definition of the spaces V_i^{near} (cf. (2.7)) and X_i^{far} (cf. (2.6)) we obtain the spaces

$$\tilde{X}_i^{far} := \operatorname{span}\left\{ \tilde{L}^{-1} b_j|_{\omega_i^*} \ : \ j \in \mathcal{I}_i^{far} \right\} \tag{3.1}$$

and

$$\tilde{V}_i^{near} := \operatorname{span}\left\{ b_i \tilde{L}^{-1} b_j \ : \ j \in \mathcal{I}_i^{near} \right\}, \tag{3.2}$$

where the nearfield and the farfield $\mathcal{I}_i^{near}$ resp. $\mathcal{I}_i^{far}$ are defined in (2.5).

Remark 3.0.1

(i) The ellipticity of L, the assumptions (1.4) *on the coefficient A, and the conformity of the finite element space V imply that the approximation $\tilde{L}^{-1}$ is elliptic and*

$$\|\tilde{L}^{-1}\|_{H_0^1(\Omega)\leftarrow H^{-1}(\Omega)} \le \frac{1}{\alpha C_{\Omega,1}^2}, \tag{3.3}$$

where α is defined in (1.4) *and $C_{\Omega,1}$ is the Friedrichs' constant (cf.* (1.3)*).*

(ii) Since $\forall\, \tilde{v} \in \tilde{X}_i^{far}$ *we have*

$$\int_{\omega_i^*} \langle A\nabla\tilde{v}, \nabla w\rangle = 0 \quad \forall w \in V(\omega_i^*) := \{w \in V \ :\ \operatorname{supp} w \subset \omega_i^*\},$$

the functions in $\tilde{X}_i^{far}$ *are locally L-harmonic on* ω_i^* *(cf. Definition 2.2.2) and thus* $\tilde{X}_i^{far}$ *can be approximated by a low dimensional space* $\tilde{V}_i^{far}$ *using the same construction as for the approximation of* X_i^{far}*.*

In the next step we want to approximate the space $\tilde{X}_i^{far}$ by a low dimensional space. For $t \in \mathcal{G}_{i,j}$ (cf. (2.11)), we denote the characteristic function for t by $\chi_t : \Omega \to \mathbb{R}$. We define

$$\hat{V}_{i,j}^{far} := \operatorname{span}\{(\tilde{P}\chi_t)|_{\omega_i} \ :\ t \in \mathcal{G}_{i,j}\},$$

where $\tilde{P} : L^2(\Omega) \to \tilde{X}_i^{far}$ is the L^2-orthogonal projection. We set

$$\hat{V}_i^{far} := \hat{V}_{i,0}^{far} + \hat{V}_{i,1}^{far} + \cdots + \hat{V}_{i,\ell-1}^{far} \tag{3.4}$$

and get the approximation

$$\hat{V}_i^{far} := \{b_i v \ :\ v \in \hat{V}_i^{far}\}. \tag{3.5}$$

Finally, we end up with a (computable) approximate AL basis and corresponding generalized finite element space

$$\tilde{V}_{AL} := (\tilde{V}_1^{near} + \hat{V}_1^{far}) + (\tilde{V}_2^{near} + \hat{V}_2^{far}) + \cdots + (\tilde{V}_n^{near} + \hat{V}_n^{far}). \tag{3.6}$$

The Galerkin discretization for the generalized finite element space $\tilde{V}_{AL}$ is given by seeking $\tilde{u}_{AL} \in \tilde{V}_{AL}$ such that

$$a(\tilde{u}_{AL}, v) = F(v) \qquad \forall v \in \tilde{V}_{AL}, \tag{3.7}$$

where $a(\cdot,\cdot)$ and $F(\cdot)$ are defined in (1.5).

The above algorithm is fully discrete, since the inverse of the differential operator L^{-1} is replaced by some finite-dimensional approximation. However, at the current stage of methodological development, the focus lies on the definition of a fully discrete method. The efficient algorithmic realization will be the topic of our future research. To get a fast algorithm one needs efficient methods for computing the approximation $\tilde{L}^{-1}$ as well as for the evaluation of the L^2-projections $\tilde{P}$ onto the spaces $\tilde{X}_i^{far}$.

Chapter 4

Error Analysis

This chapter is devoted to the error analysis of the method which has been described in Chapter 3. It is shown that the accuracy of the arising Galerkin finite element method with respect to the H^1-norm is of order $O(H)$ under the Assumption 4.0.1.

Let $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the differential operator associated to the bilinear form $a(\cdot,\cdot)$ defined in (1.5). As in Chapter 3 we assume that the approximation of L^{-1}, denoted by $\tilde{L}^{-1}$, is computed by a Galerkin discretization with a conforming finite-dimensional subspace $V \subset H_0^1(\Omega)$ on a sufficiently fine mesh $\mathcal{T}_h$. For the error analysis L is supposed to satisfy the following condition, whose validity will be further investigated in Chapter 5.

Assumption 4.0.1.

$$\sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v \in V} \frac{\|L^{-1}f - v\|_{H^1(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq C_{apx} H, \tag{4.1}$$

where H is the mesh width (cf. Definition 1.3.2) and the constant C_{apx} is independent of H and f.

Corollary 4.0.2. *Céa's lemma (cf. Theorem 1.3.6) implies*

$$\|L^{-1}f - \tilde{L}^{-1}f\|_{H^1(\Omega)} \leq \frac{\beta}{\alpha \underset{\sim}{C}_{\Omega,1}^2} C_{apx} H \|f\|_{L^2(\Omega)}, \tag{4.2}$$

where α, β are the constants from (1.4)*, C_{apx} is defined in* (4.1)*, and $\underset{\sim}{C}_{\Omega,1}$ is the Friedrichs' constant (cf.* (1.3)*).*

Proof. Applying Céa's lemma (cf. Theorem 1.3.6) and due to Assumption 4.0.1 we get:

$$\begin{aligned}\|L^{-1}f - \tilde{L}^{-1}f\|_{H^1(\Omega)} &\leq \frac{\beta}{\alpha \underline{C}_{\Omega,1}^2} \inf_{v\in V} \|L^{-1}f - v\|_{H^1(\Omega)} \\ &\leq \frac{\beta}{\alpha \underline{C}_{\Omega,1}^2} \sup_{f\in L^2(\Omega)\setminus\{0\}} \inf_{v\in V} \|L^{-1}f - v\|_{H^1(\Omega)} \\ &\leq \frac{\beta}{\alpha \underline{C}_{\Omega,1}^2} C_{apx} H \|f\|_{L^2(\Omega)}.\end{aligned}$$

□

Before the main result can be stated, we need to introduce some notations and lemmas, which will be used in the proof of Theorem 4.0.8.

Let $P_S : L^2(\Omega) \to S$ denote the L^2-orthogonal projection onto S, where S is defined in (1.13). For $f \in L^2(\Omega)$ and an index $i \in \mathcal{I} := \{1, 2, \ldots, n\}$ we define the nearfield and the farfield parts of f in the following way:

$$f_i^{near} := \sum_{j\in\mathcal{I}_i^{near}} (P_S f)_j b_j \quad \text{and} \quad f_i^{far} := \sum_{j\in\mathcal{I}_i^{far}} (P_S f)_j b_j, \tag{4.3}$$

where the index sets $\mathcal{I}_i^{near}$ resp. $\mathcal{I}_i^{far}$ are defined in (2.5), $(P_S f)_j := (P_S f)(x_j)$, and x_j is the nodal point corresponding to b_j. Then,

$$\tilde{L}^{-1} P_S f = \sum_{i=1}^{n} \underbrace{b_i \tilde{L}^{-1} f_i^{near}}_{v_i^{near}} + \sum_{i=1}^{n} b_i \underbrace{\tilde{L}^{-1} f_i^{far}}_{v_i^{far}}. \tag{4.4}$$

Remark 4.0.3. *Let ω_i^* be defined as in* (2.3)*,* (2.4)*. Since it holds $f_i^{near}|_{\omega_i^*} = P_S f|_{\omega_i^*}$, we have*

$$\|f_i^{near}\|_{L^2(\omega_i^*)} = \|P_S f\|_{L^2(\omega_i^*)} \quad \forall f \in L^2(\Omega).$$

Remark 4.0.4. *By construction we have $v_i^{near} \in \tilde{V}_i^{near}$ (cf.* (3.2)*) and $v_i^{far}|_{\omega_i^*} \in \tilde{X}_i^{far}$ (cf.* (3.1)*). Therefore, to get an approximation of $v := \tilde{L}^{-1} P_S f$ in the space $\tilde{V}_{AL}$ (cf. 3.6), we need to approximate v_i^{far} by a function $\tilde{v}_i^{far} \in \hat{V}_i^{far}$ (cf.* (3.5)*).*

The following lemma gives us an error bound for the approximation of v_i^{far}.

Lemma 4.0.5. *Let ω_i as in* (1.14)*, ω_i^* as in* (2.3)*,* (2.4) *and define the local mesh width by $H_i := \max\{\operatorname{diam}(\tau) \mid \tau \in \omega_i^*\}$. Furthermore, let $v_i^{far} := \tilde{L}^{-1} f_i^{far}$ and $\hat{V}_i^{far}$ as defined in* (3.5)*. There exists $\tilde{v}_i^{far} \in \hat{V}_i^{far}$ such that the approximation estimate*

$$\|v_i^{far} - \tilde{v}_i^{far}\|_{H^m(\omega_i)} \leq C_1 H_i^{2+\frac{d}{2}-m} \|\nabla(\tilde{L}^{-1} f_i^{far})\|_{L^2(\omega_i^*)} \quad m = 0, 1 \tag{4.5}$$

holds.

Proof. Set

$$\ell := \max\left\{2, \left\lceil \frac{2+d}{2\log 2} \log \frac{1}{H_i} \right\rceil\right\} \quad \text{and} \quad k := \left\lceil \frac{2c_0\ell^2}{(\ell-1)} \right\rceil \tag{4.6}$$

for some $c_0 = O(1)$. Choosing $p \leftarrow \ell$, $\ell \leftarrow k$, $i \leftarrow \ell$, $c \leftarrow c_0$, and $\delta \leftarrow O(H_i)$ in the second estimate of [9, p. 172] yields

$$\|v_i^{far} - \tilde{v}_i^{far}\|_{L^2(\omega_i)} \leq C_2 H_i \Big(c_0 \frac{\ell}{k}\Big)^\ell \|\nabla(\tilde{L}^{-1} f_i^{far})\|_{L^2(\omega_i^*)}. \tag{4.7}$$

Similarly, choosing $p \leftarrow \ell$, $\ell \leftarrow k$, and $c \leftarrow c_0$ in the second last estimate of [9, p. 172] we get

$$\|\nabla(v_i^{far} - \tilde{v}_i^{far})\|_{L^2(\omega_i)} \leq \Big(c_0 \frac{\ell}{k}\Big)^\ell \|\nabla(\tilde{L}^{-1} f_i^{far})\|_{L^2(\omega_i^*)}. \tag{4.8}$$

According to the definition of ℓ we have to distinguish the following two cases:

- Case 1: $\left\lceil \frac{2+d}{2\log 2} \log \frac{1}{H_i} \right\rceil \leq 2$
  By definition of ℓ we know that $\ell = 2$ and after some simple calculations we see that $H_i \geq \left(\frac{1}{4}\right)^{\frac{2}{2+d}}$. Therefore we obtain

$$\Big(c_0 \frac{\ell}{k}\Big)^\ell = \Big(\frac{\ell-1}{2\ell}\Big)^\ell = \frac{1}{16} < \frac{1}{4} \leq H_i^{1+\frac{d}{2}}. \tag{4.9}$$

- Case 2: $\left\lceil \frac{2+d}{2\log 2} \log \frac{1}{H_i} \right\rceil > 2$
  Set $\alpha := \frac{2+d}{2\log 2}$. Then $\ell = \lceil -\alpha \log H_i \rceil \geq -\alpha \log H_i$ and furthermore we have

$$\Big(c_0 \frac{\ell}{k}\Big)^\ell = \Big(\frac{\ell-1}{2\ell}\Big)^\ell \leq 2^{-\ell} = e^{-\ell \log 2} = e^{-\lceil -\alpha \log H_i \rceil \log 2} \leq H_i^{\alpha \log 2} = H_i^{1+\frac{d}{2}}. \tag{4.10}$$

The assertion follows by combining (4.7), (4.8), (4.9), and (4.10). $\square$

Lemma 4.0.6. *Let b_i denote the usual nodal basis functions with support ω_i (cf.* (1.14)*) and v_i^{far} as well as $\tilde{v}_i^{far}$ as in Lemma 4.0.5. Then the estimate*

$$\Big\| \sum_{i=1}^n b_i(v_i^{far} - \tilde{v}_i^{far}) \Big\|^2_{H^1(\Omega)} \leq C_3^2 \sum_{i=1}^n \big\| b_i(v_i^{far} - \tilde{v}_i^{far}) \big\|^2_{H^1(\omega_i)} \tag{4.11}$$

holds.

Proof. Using the definition and the bilinearity of the scalar product and by Cauchy-Schwarz inequality we get:

$$\begin{aligned}
\Big\| \sum_{i=1}^{n} b_i(v_i^{far} - \tilde{v}_i^{far}) \Big\|_{H^1(\Omega)}^2 &= \Big\langle \sum_{i=1}^{n} b_i(v_i^{far} - \tilde{v}_i^{far}), \sum_{i=1}^{n} b_i(v_i^{far} - \tilde{v}_i^{far}) \Big\rangle_{H^1(\Omega)} \\
&= \sum_{i=1}^{n} \big\langle b_i(v_i^{far} - \tilde{v}_i^{far}), \sum_{i=1}^{n} b_i(v_i^{far} - \tilde{v}_i^{far}) \big\rangle_{H^1(\omega_i)} \\
&\leq \sum_{i=1}^{n} \Big(\|b_i(v_i^{far} - \tilde{v}_i^{far})\|_{H^1(\omega_i)} \Big\| \sum_{j=1}^{n} b_j(v_j^{far} - \tilde{v}_j^{far}) \Big\|_{H^1(\omega_i)} \Big) \\
&\leq \sqrt{\sum_{i=1}^{n} \|b_i(v_i^{far} - \tilde{v}_i^{far})\|_{H^1(\omega_i)}^2} \sqrt{\sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} b_j(v_j^{far} - \tilde{v}_j^{far}) \Big\|_{H^1(\omega_i)}^2}.
\end{aligned}$$

Set

$$g := \sum_{j=1}^{n} b_j(v_j^{far} - \tilde{v}_j^{far})$$

and for $\tau \in \mathcal{G}$

$$V(\tau) := \{i \in \{1, \ldots, n\} \mid x_i \text{ vertex of } \tau\}.$$

Then we have:

$$\sqrt{\sum_{i=1}^{n} \|g\|_{H^1(\omega_i)}^2} = \sqrt{\sum_{\tau \in \mathcal{G}} \sum_{i \in V(\tau)} \|g\|_{H^1(\tau)}^2} \leq C_3 \sqrt{\sum_{\tau \in \mathcal{G}} \|g\|_{H^1(\tau)}^2} = C_3 \|g\|_{H^1(\Omega)},$$

where $C_3 := \max_{\tau \in \mathcal{G}} \#V(\tau)$ (this constant exists since $\mathcal{G}$ is assumed to be regular). Hence,

$$\Big\| \sum_{i=1}^{n} b_i(v_i^{far} - \tilde{v}_i^{far}) \Big\|_{H^1(\Omega)}^2 \leq C_3 \sqrt{\sum_{i=1}^{n} \|b_i(v_i^{far} - \tilde{v}_i^{far})\|_{H^1(\omega_i)}^2} \Big\| \sum_{i=1}^{n} b_i(v_i^{far} - \tilde{v}_i^{far}) \Big\|_{H^1(\Omega)}$$

and the claim follows from the last inequality. □

Lemma 4.0.7. *Let $\tilde{L}^{-1}$ be an approximation of L^{-1} computed by a Galerkin discretization with a conforming finite-dimensional subspace $V \subset H_0^1(\Omega)$, f_i^{near} as in (4.3), $P_S f$ is the L^2-orthogonal projection of f onto S, and ω_i^* as in (2.3), (2.4). Then it holds that*

$$\|\nabla(\tilde{L}^{-1} f_i^{near})\|_{L^2(\omega_i^*)}^2 \leq \frac{K \bar{C}_{\Omega,1}^2}{\alpha^2 \underline{C}_{\Omega,1}^4} \|P_S f\|_{L^2(\Omega)}^2 \tag{4.12}$$

and

$$\|\nabla(\tilde{L}^{-1} P_S f)\|_{L^2(\omega_i^*)}^2 \leq \frac{\bar{C}_{\Omega,1}^2}{\alpha^2 \underline{C}_{\Omega,1}^4} \|P_S f\|_{L^2(\Omega)}^2 \tag{4.13}$$

with α as in (1.4), $\bar{C}_{\Omega,1}$ and $\underline{C}_{\Omega,1}$ as in (1.3), and K depending on the spatial dimension (cf. Lemma 6.1.1, Lemma 6.1.2, and Lemma 6.1.3).

Proof. Using Friedrichs' inequality and the property (3.3) and because of the fact that there exists a constant K such that $\|f_i^{near}\|_{L^2(\Omega)} \le K\|P_S f\|_{L^2(\Omega)}$ (cf. Lemma 6.1.1, Lemma 6.1.2, and Lemma 6.1.3) we get

$$\begin{aligned}\|\nabla(\tilde{L}^{-1} f_i^{near})\|^2_{L^2(\omega_i^*)} &\le \|\nabla(\tilde{L}^{-1} f_i^{near})\|^2_{L^2(\Omega)} \\ &\le \bar{C}^2_{\Omega,1}\|\tilde{L}^{-1} f_i^{near}\|^2_{H^1(\Omega)} \\ &\le \frac{\bar{C}^2_{\Omega,1}}{\alpha^2 \underline{C}^4_{\Omega,1}}\|f_i^{near}\|^2_{H^{-1}(\Omega)} \\ &\le \frac{\bar{C}^2_{\Omega,1}}{\alpha^2 \underline{C}^4_{\Omega,1}}\|f_i^{near}\|^2_{L^2(\Omega)} \\ &\le \frac{K\bar{C}^2_{\Omega,1}}{\alpha^2 \underline{C}^4_{\Omega,1}}\|P_S f\|^2_{L^2(\Omega)}.\end{aligned}$$

The inequality (4.13) can be proven analogously. □

Theorem 4.0.8. *Let u denote the solution of* (1.5)*. Let the parameters ℓ and k in the definition of the farfield part of $\tilde{V}_{AL}$ (cf.* (3.6)*) be chosen according to*

$$\ell := \max\left\{2, \left\lceil \frac{2+d}{2\log 2}\log\frac{1}{H_i}\right\rceil\right\} \quad \text{and} \quad k := \left\lceil \frac{2c_0\ell^2}{(\ell-1)}\right\rceil \tag{4.14}$$

for some $c_0 = O(1)$, where H_i is as in Lemma 4.0.5. Let $\tilde{u}_{AL}$ be the solution of (3.7) *and assume that the condition*

$$H^d n \lesssim 1 \tag{4.15}$$

is satisfied, where d is the spatial dimension and $n = \#\mathcal{I}$. Then the error estimate

$$\|u - \tilde{u}_{AL}\|_{H^1(\Omega)} \le CH\|f\|_{L^2(\Omega)} \tag{4.16}$$

holds. Moreover,

$$\dim \tilde{V}_{AL} \le C_d n \ell^{d+1} \le \tilde{C}_d H^{-d}\log^{d+1}\frac{1}{H}. \tag{4.17}$$

Proof. For $f \in L^2(\Omega)$ let $u := L^{-1}f$. Then, using a triangle inequality, Corollary 4.0.2, and (3.3) we have

$$\begin{aligned}\|u - \tilde{L}^{-1}P_S f\|_{H^1(\Omega)} &\le \|u - \tilde{L}^{-1}f\|_{H^1(\Omega)} + \|\tilde{L}^{-1}(f - P_S f)\|_{H^1(\Omega)} \\ &\le \frac{\beta}{\alpha \underline{C}^2_{\Omega,1}} C_{apx} H\|f\|_{L^2(\Omega)} + \frac{1}{\alpha \underline{C}^2_{\Omega,1}}\|f - P_S f\|_{H^{-1}(\Omega)}.\end{aligned} \tag{4.18}$$

According to [18, inequality 18] there exists a constant C_4 such that

$$\|f - P_S f\|_{H^{-1}(\Omega)} \le C_4 H\|f\|_{L^2(\Omega)}. \tag{4.19}$$

Setting $C_5 := \frac{2}{\alpha \underline{C}_{\Omega,1}^2} \max\{\beta C_{apx}, C_4\}$ and using (4.18) and (4.19) we get

$$\|u - \tilde{L}^{-1} P_S f\|_{H^1(\Omega)} \leq C_5 H \|f\|_{L^2(\Omega)}. \tag{4.20}$$

Define v_i^{far} and v_i^{near} as in (4.4). By Lemma 4.0.5 we know that there exists $\tilde{v}_i^{far} \in \hat{V}_i^{far}$ such that

$$\|v_i^{far} - \tilde{v}_i^{far}\|_{H^m(\omega_i)} \leq C_1 H_i^{2+\frac{d}{2}-m} \|\nabla(\tilde{L}^{-1} f_i^{far})\|_{L^2(\omega_i^*)} \quad m = 0, 1, \tag{4.21}$$

where $\hat{V}_i^{far}$ is the space defined in (3.5). Finally, the approximation of u is given by

$$\tilde{v} := \sum_{i=1}^n v_i^{near} + \sum_{i=1}^n b_i \tilde{v}_i^{far} \in \tilde{V}_{AL}.$$

Applying a triangle inequality, (4.20), and due to (4.4) we get

$$\begin{aligned} \|u - \tilde{v}\|_{H^1(\Omega)} &\leq \|u - \tilde{L}^{-1} P_S f\|_{H^1(\Omega)} + \|\tilde{L}^{-1} P_S f - \tilde{v}\|_{H^1(\Omega)} \\ &\leq C_5 H \|f\|_{L^2(\Omega)} + \Big\| \sum_{i=1}^n b_i (v_i^{far} - \tilde{v}_i^{far}) \Big\|_{H^1(\Omega)}. \end{aligned} \tag{4.22}$$

Due to Lemma 4.0.6 the square of the second term can be estimated as following:

$$\Big\| \sum_{i=1}^n b_i (v_i^{far} - \tilde{v}_i^{far}) \Big\|_{H^1(\Omega)}^2 \leq C_3^2 \sum_{i=1}^n \|b_i (v_i^{far} - \tilde{v}_i^{far})\|_{H^1(\omega_i)}^2. \tag{4.23}$$

Applying Friedrichs' inequality, the Leibniz rule for products, a triangle inequality, a Hölder's inequality, the fact that $\|b_i\|_{L^\infty(\omega_i)} = 1$, $\|\nabla b_i\|_{L^\infty(\omega_i)} \leq \frac{C_6}{H_i}$, and (4.21) yields

$$\begin{aligned} \|b_i (v_i^{far} - \tilde{v}_i^{far})\|_{H^1(\omega_i)}^2 &\leq \frac{1}{\underline{C}_{\Omega,1}^2} \|\nabla(b_i (v_i^{far} - \tilde{v}_i^{far}))\|_{L^2(\omega_i)}^2 \\ &\leq \frac{2}{\underline{C}_{\Omega,1}^2} \Big(\|(\nabla b_i)(v_i^{far} - \tilde{v}_i^{far})\|_{L^2(\omega_i)}^2 + \|b_i \nabla (v_i^{far} - \tilde{v}_i^{far})\|_{L^2(\omega_i)}^2 \Big) \\ &\leq \frac{2}{\underline{C}_{\Omega,1}^2} \Big(\|\nabla b_i\|_{L^\infty(\omega_i)}^2 \|v_i^{far} - \tilde{v}_i^{far}\|_{L^2(\omega_i)}^2 \\ &+ \|b_i\|_{L^\infty(\omega_i)}^2 \|\nabla (v_i^{far} - \tilde{v}_i^{far})\|_{L^2(\omega_i)}^2 \Big) \\ &\leq \frac{2}{\underline{C}_{\Omega,1}^2} \Big(\frac{C_6^2}{H_i^2} \|v_i^{far} - \tilde{v}_i^{far}\|_{L^2(\omega_i)}^2 + \|\nabla (v_i^{far} - \tilde{v}_i^{far})\|_{L^2(\omega_i)}^2 \Big) \\ &\leq C_7 H_i^{2+d} \|\nabla(\tilde{L}^{-1} f_i^{far})\|_{L^2(\omega_i^*)}^2 \\ &\leq C_7 H^{2+d} \|\nabla(\tilde{L}^{-1} f_i^{far})\|_{L^2(\omega_i^*)}^2. \end{aligned} \tag{4.24}$$

Due to the splitting $P_S f = f_i^{far} + f_i^{near}$ and a triangle inequality we have

$$\begin{aligned}\|\nabla(\tilde{L}^{-1} f_i^{far})\|^2_{L^2(\omega_i^*)} &= \|\nabla(\tilde{L}^{-1}(P_S f - f_i^{near}))\|^2_{L^2(\omega_i^*)} \\ &\leq 2\big(\|\nabla(\tilde{L}^{-1} P_S f)\|^2_{L^2(\omega_i^*)} + \|\nabla(\tilde{L}^{-1} f_i^{near})\|^2_{L^2(\omega_i^*)}\big). \qquad (4.25)\end{aligned}$$

Because of (4.24), (4.25), and Lemma 4.0.7 we have:

$$\begin{aligned}\sum_{i=1}^{n} \|b_i(v_i^{far} - \tilde{v}_i^{far})\|^2_{H^1(\omega_i)} &\leq C_7 H^{2+d} \sum_{i=1}^{n} \|\nabla(\tilde{L}^{-1} f_i^{far})\|^2_{L^2(\omega_i^*)} \\ &\leq 2C_7 H^{2+d} \sum_{i=1}^{n} \big(\|\nabla(\tilde{L}^{-1} P_S f)\|^2_{L^2(\omega_i^*)} + \|\nabla(\tilde{L}^{-1} f_i^{near})\|^2_{L^2(\omega_i^*)}\big) \\ &\leq 2C_7 H^{2+d} \sum_{i=1}^{n} \Big(\frac{\bar{C}^2_{\Omega,1}}{\alpha^2 \underline{C}^4_{\Omega,1}} \|P_S f\|^2_{L^2(\Omega)} + \frac{K\bar{C}^2_{\Omega,1}}{\alpha^2 \underline{C}^4_{\Omega,1}} \|P_S f\|^2_{L^2(\Omega)}\Big) \\ &\leq \frac{4n}{\alpha^2} C_8 H^{2+d} \|f\|^2_{L^2(\Omega)}.\end{aligned}$$

By (4.15) we get

$$\sum_{i=1}^{n} \|b_i(v_i^{far} - \tilde{v}_i^{far})\|^2_{H^1(\omega_i)} \leq \frac{4}{\alpha^2} C_9 H^2 \|f\|^2_{L^2(\Omega)} \qquad (4.26)$$

and the assertion (4.16) follows combining (4.22), (4.23), and (4.26).
For a proof of (4.17) see Remark 2.3.3(i). □

Chapter 5

On Assumption 4.0.1

To prove the linear approximation property of the Galerkin approximation in the space $\tilde{V}_{AL}$ we had to assume that the differential operator L satisfies the condition

$$\sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v \in V} \frac{\|L^{-1}f - v\|_{H^1(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq C_{apx} H$$

(cf. Assumption 4.0.1). We recall that no periodicity assumption was imposed on the diffusion matrix A. Also in [27] and [21] a non-periodic setting is considered. Whereas in [27] the coefficient is assumed to be smooth but highly varying, in [21] the case of a general L^∞-coefficient without any smoothness assumptions is investigated.
In this chapter I will summarize the theory and results presented in [27] and [21] in order to give a comparison to the result stated in Theorem 4.0.8. In both papers problem (1.5) is considered and the diffusion matrix A is assumed to be uniformly elliptic, i.e. A satisfies (1.4).

5.1 Theory and Results of [27]

In [27] a regularity theory for elliptic problems has been developed for smooth but highly varying diffusion matrices. The coefficient is allowed to oscillate on very different scales and the distribution of these oscillations is not assumed to be periodic. Using weighted Sobolev norms (cf. Definition 5.1.13) all the constants in those regularity estimates are independent of derivatives of A, i.e. the constants are independent of the scales of the oscillations. They only depend on global lower and upper bounds of the diffusion matrix, more precisely on the constants α, β defined in (1.4).
In this section I will only point out the main results presented in [27].

5.1.1 Oscillation Adapted Sobolev Norms

In order to give the main regularity result, we need to define some properly weighted Sobolev norms. For this purpose an oscillation adapted partition of Ω has to

be constructed. Besides fulfilling condition (1.4), A is assumed to satisfy $A \in C^p(\bar{\Omega}, \mathbb{R}^{d\times d}_{sym})$ for some smoothness parameter $p \in \mathbb{N}$.
In the following definition the smoothness of the coefficient is quantified relative to subdomains of Ω.

Definition 5.1.1 (Oscillation condition). *Let $A \in C^p(\bar{\Omega}, \mathbb{R}^{d\times d}_{sym})$ for some $p \in \mathbb{N}$. A subset $\omega \subset \Omega$ fulfills the oscillation condition of order p if*

$$\operatorname{osc}(A, \omega, p) := \max_{1\leq q\leq p} \left\{ \frac{1}{q!} (\operatorname{diam} \omega)^q \|\nabla^q A\|_{L^\infty(\omega)} \right\} \leq 1. \tag{5.1}$$

The above definition can be extended to the case $p = \infty$ by replacing $\max\limits_{1\leq q\leq p}$ by $\sup\limits_{q\in\mathbb{N}}$.

Remark 5.1.2. *The oscillation condition is fulfilled if and only if*

$$\operatorname{diam} \omega \max_{1\leq q\leq p} \left\{ \left(\frac{1}{q!} \|\nabla^q A\|_{L^\infty(\omega)} \right)^{1/q} \right\} \leq 1. \tag{5.2}$$

Moreover, (5.2) *implies that ω resolves the scales of ∇A in the sense that* $\operatorname{diam} \omega \leq \|\nabla A\|^{-1}_{L^\infty(\omega)}$ *holds.*

In the next step a density function $H_{p,A} : \Omega \to (0, \infty)$ is constructed. This function measures the "variation" of the regularity for problem (1.5) from a standard Poisson problem and will depend on the smoothness parameter p. The constuction is as follows. We subdivide some bounding box $Q_0 \supset \bar{\Omega}$ into hypercubes such that the oscillation condition is satisfied for every such cube. A cube $Q := \{x \in \mathbb{R}^d : \|x - c_Q\|_\infty \leq R_Q\}$ is represented by its center c_Q and its radius R_Q (its halved width). For any parameter $\rho > 0$,

$$B_\rho(Q) := \{x \in \mathbb{R}^d : \|x - c_Q\|_\infty \leq \rho R_Q\}$$

defines a ρ-scaled version of the cube Q. Clearly, $B_1(Q) = Q$.

Algorithm 5.1.3 (Oscillation adapted covering). *Let $Q_0 \supset \bar{\Omega}$ be some closed bounding box of Ω. For $p \in \mathbb{N}$, a subdivision $\mathcal{Q} = \mathcal{Q}_p(A)$ of Q_0 into closed cubes is defined by:*

$\mathcal{Q} = \{Q_0\}$, $\mathcal{Q}^* := \emptyset$
while $\mathcal{Q}^* \neq \mathcal{Q}$ ***do***
  $\mathcal{Q}^* := \mathcal{Q}$
  for $Q \in \mathcal{Q}^*$ ***do***
    if $\operatorname{osc}(A, B_2(Q) \cap \Omega, p) > 1$ ***then***
      Q is subdivided into 2^d disjoint, congruent cubes $q_1, \ldots, q_{2^d}$ and
      $\mathcal{Q} = (\mathcal{Q}\backslash Q) \cup \{q_1, \ldots, q_{2^d}\}$
    end if
  end for
end while

Concerning the local quasi-uniformity of the subdivisions $\mathcal{Q}_p(A)$ the following observation is important.

Proposition 5.1.4. *There exists $C_{ol} \in \mathbb{N}$ depending only on d such that for all $Q \in \mathcal{Q}_p(A)$ and for all $\eta \in [0, 1[$ it holds*

$$\#\{P \in \mathcal{Q}_p(A) \,:\, |P \cap B_{1+\eta}(Q)| > 0\} \leq C_{ol} M_d(\eta),$$

where $M_1(\eta) = \log(1 - \eta)$ and $M_d(\eta) = (1 - \eta)^{1-d}$ if $d \geq 2$.

The proof can be found in [27, Proposition 3.6.].
A density function can now be defined by the local element size in $\mathcal{Q}_p(A)$.

Definition 5.1.5 (Oscillation adapted density). *Let $\mathcal{Q}_p(A)$, $p \in \mathbb{N}$, be some covering of Ω generated by Algorithm 5.1.3. Then $\mathcal{Q}_p(A)$-piecewise constant functions $H_{p,A} : \cup\mathcal{Q}_p(A) \to (0, \infty)$ are defined by*

$$H_{p,A}(x) := \min\{\mathrm{diam}(Q) : Q \in \mathcal{Q}_p(A) \text{ with } x \in Q\} \quad \text{for } x \in \cup\mathcal{Q}_p(A).$$

The function $H_{p,A}$ contains important information of the diffusion matrix A for higher order regularity estimates. Since the construction of $H_{p,A}$ via subdivisions into (overlapping) cubes is not appropriate for the representation of the geometry of Ω, a regular finite element mesh is constructed. The distribution of the simplices in this mesh is controlled by the oscillation adapted density $H_{p,A}$.
First an initial coarse mesh that resolves the geometry is introduced. This mesh is then refined according to the oscillations of the coefficient.

Definition 5.1.6 (Macro triangulation, refinement, parametrization)

a) We assume that there exists a polyhedral (polygonal in two dimensions) domain $\tilde{\Omega}$ along with a bi-Lipschitz mapping $\chi : \tilde{\Omega} \to \Omega$. Let $\tilde{\mathcal{T}}^{macro} = \{\tilde{K}_i^{macro} : 1 \leq i \leq q\}$ denote some conforming finite element mesh for $\tilde{\Omega}$ consisting of simplices which are regular in the sense of [13]. $\tilde{\mathcal{T}}^{macro}$ is considered as a coarse partition of $\tilde{\Omega}$, i.e., the diameters of the elements in $\tilde{\mathcal{T}}^{macro}$ are of order 1. We assume that the restrictions $\chi_i := \chi|_{\tilde{K}_i^{macro}}$ are analytic for all $1 \leq i \leq q$. The macromesh for Ω is then given by

$$\mathcal{T}^{macro} := \left\{K = \chi(\tilde{K}^{macro}) : \tilde{K}^{macro} \in \tilde{\mathcal{T}}^{macro}\right\}.$$

b) Using the macromesh as the initial mesh we introduce a recursive refinement procedure REFINE. The input of REFINE is a finite element mesh $\mathcal{T}$, where some elements are marked for refinement, and the output is a new conforming finite element mesh $\mathcal{T}^{refine}$ in the sense of [13]. The output is derived by refining the corresponding simplicial mesh $\tilde{\mathcal{T}}$ in a standard way (e.g., in two dimensions, by first connecting the midpoints of the marked triangle edges and then eliminating hanging nodes by some suitable closure algorithm). The resulting mesh is denoted by $\tilde{\mathcal{T}}^{refine} = \{\tilde{K}_i : 1 \leq i \leq N\}$. The corresponding *finite element mesh for* Ω *is denoted by* $\mathcal{T}^{refine} = \{K = \chi(\tilde{K}) : \tilde{K} \in \tilde{\mathcal{T}}^{refine}\}$*. As a simplifying assumption on the refinement strategy we assume that the elimination of hanging nodes causes refinement of nonmarked triangles only in the first layer around marked triangles. In certain cases this strategy generates meshes with some "flat" triangles, i.e., the constant measuring the shape-regularity of the mesh increases.*

c) There exists an affine bijection $J_K : \hat{K} \to \tilde{K}$ *which maps the reference element* $\hat{K} := \{x \in ([0,\infty))^d : \sum_{i=1}^d x_i \le 1\}$ *to the simplex* $\tilde{K}$ *for any* $K = \chi(\tilde{K}) \in \mathcal{T}$*, where* $\mathcal{T}$ *is derived from* $\mathcal{T}^{macro}$ *by repeated application of REFINE. A parametrization* $F_K : \hat{K} \to K$ *can be written as* $F_K = R_K \circ J_K$*, where* J_K *is an affine map and the maps* R_K *and* J_K *satisfy for constants* C_{affine}*,* C_{metric}*,* $\gamma > 0$*:*

$$\begin{aligned} \|J_K'\|_{L^\infty(\hat{K})} &\le C_{affine}\operatorname{diam}(K), \\ \|(J_K')^{-1}\|_{L^\infty(\hat{K})} &\le C_{affine}\operatorname{diam}(K)^{-1}, \\ \|(R_K')^{-1}\|_{L^\infty(\tilde{K})} &\le C_{metric}, \\ \|\nabla^n R_K\|_{L^\infty(\tilde{K})} &\le C_{metric}\gamma^n n! \quad \text{for } n \in \mathbb{N}_0. \end{aligned} \tag{5.3}$$

Using the density function $H_{p,A}$, the oscillation adapted finite element meshes can be constructed by successively refining the macromesh as follows.

Algorithm 5.1.7 (Oscillation adapted finite element mesh). *Let* $\mathcal{T}^{macro}$ *be a subdivision of* $\bar{\Omega}$ *in the sense of Definition 5.1.6 and let* $p \in \mathbb{N}$*. A subdivision* $\mathcal{T}_p(A)$ *of* Ω *that reflects the regularity of the coefficient is defined by the following:*

$\mathcal{T} := \mathcal{T}^{macro}$
for $q = 1, 2, \ldots, p$ ***do***
  $\mathcal{M} := \mathcal{T}$
  while $\mathcal{M} \ne \emptyset$ ***do***
    $\mathcal{M} := \{K \in \mathcal{T} : \operatorname{diam}(K) > \min_{x \in K} H_{q,A}(x)\}$
    $\mathcal{T} = REFINE(\mathcal{T}, \mathcal{M})$
  end while
end for

Notation 5.1.8. *If an element* $K \in \mathcal{T}$ *is refined during Algorithm 5.1.7, the resulting new elements* $K_1', \ldots, K_m'$ *(*$m \in \mathbb{N}$*) are called sons of* K*, denoted by* $\operatorname{sons}(K)$*. Correspondingly,* K *is said to be the father of* $K_1', \ldots, K_m'$*.*

Remark 5.1.9. $\mathcal{T}_p(A)$ *has analog properties as the mesh* $\mathcal{Q}_p(A)$ *- it also satisfies Proposition 5.1.4. Moreover, it is a simplicial finite element mesh (cf. Definition 1.3.1).*

Definition 5.1.10. *For* $K \in \mathcal{T}$ *and* $\rho \ge 1$*, some scaled neighbourhood of* K *is defined by*

$$K_\rho := \{x \in \mathbb{R}^d : \exists y \in K : \|y - x\| \le \frac{\rho}{2}\operatorname{diam}(K)\}.$$

Remark 5.1.11. *Let $K \in \mathcal{T}_p(A)$ and $Q \in \mathcal{Q}_p(A)$, for some $p \in \mathbb{N}$, be given such that $K \cap Q \neq \emptyset$. Then, depending on the actual realization of the procedure REFINE, there exists a constant $\theta > 0$ such that*

$$\theta\, \mathrm{diam}(Q) \leq \mathrm{diam}(K) \leq \mathrm{diam}(Q). \tag{5.4}$$

Lemma 5.1.12. *For all $K \in \mathcal{T}_p(A)$ with diameter h_K the lower bound*

$$h_K \geq c \min\Big\{\tau, \Big(\max_{1\leq q\leq p}\Big\{\Big(\frac{\|\nabla^q A\|_{L^\infty(K^*)}}{q!}\Big)^{1/q}\Big\}\Big)^{-1}\Big\}$$

holds with a constant τ that represents the minimal mesh size in the initial macromesh $\mathcal{T}^{macro}$ (cf. Definition 5.1.6) and with a constant $c > 0$ that depends only on the shape parameters in $\mathcal{T}^{macro}$ and, through (5.4)*, the procedure REFINE; $K^* := K_C$ denotes the C-scaled version of K (cf. Definition 5.1.10), where the constant C depends only on the shape parameters in $\mathcal{T}^{macro}$ and the procedure REFINE.*

The proof can be found in [27, Lemma 3.12].

Finally, we introduce some properly weighted (mesh-dependent) Sobolev norms.

Definition 5.1.13 (Oscillation adapted Sobolev norms). *Let $\mathcal{T}_p(A)$, $p \in \mathbb{N}$, be the subdivision of Ω generated by Algorithm 5.1.7. A weighted seminorm $|\cdot|_{p+1,A}$ in $H^{p+1}(\Omega)$ is defined by*

$$|u|_{p+1,A} := \frac{1}{p!}\sqrt{\sum_{K\in\mathcal{T}_p(A)} \mathrm{diam}(K)^{2p}\|\nabla^{p+1}u\|^2_{L^2(K)}},$$

while corresponding full norms are given by

$$\|u\|_{p+1,A} := \sqrt{\|u\|^2_{H^1(\Omega)} + \sum_{\ell=2}^{p+1}|u|^2_{\ell,A}}.$$

5.1.2 Main Regularity Result

The following theorem states the main result concerning the regularity estimates in weighted Sobolev norms:

Theorem 5.1.14. *Let $A \in C^p(\Omega, \mathbb{R}^{d\times d}_{sym})$ satisfy* (1.4) *for some $p \in \mathbb{N}$, and assume $f \in H^{p-1}(\Omega)$. The corresponding solution of* (1.5) *is denoted by u. Further assume that the mesh $\mathcal{T}_p(A)$ is generated by Algorithm 5.1.7. Let the boundary $\partial\Omega$ be of class C^p. Then the solution satisfies $u \in H^{p+1}(\Omega)$ and the estimate*

$$\|u\|_{p+1,A} \leq C_{10}C_{11}^p\|f\|_{H^{p-1}(\Omega)}, \tag{5.5}$$

where $\|\cdot\|_{p+1,A}$ is the oscillation adapted Sobolev norm as in Definition 5.1.13. The constants C_{10} and C_{11} are independent of p and the variation of A but depend on α, β as in (1.4)*, on C_{ol} (cf. Proposition 5.1.4), on the constants in Definition 5.1.6 (c), on the spatial dimension d, and on the geometry of the domain Ω through its diameter and the constants describing the regularity of the boundary $\partial\Omega$.*

The proof is based on local interior regularity estimates and can be found in [27, Theorem 4.1].

5.1.3 Oscillation Adapted Finite Elements

As an application of the above regularity estimates, problem-adapted *hp*-finite elements have been developed and error estimates for Galerkin *hp*-finite element discretizations of (1.5) have been derived in [27].
Let $\mathcal{T}_p(A)$ be generated by Algorithm 5.1.7 and assume that the mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_p(A)$ according to Definition 5.1.6 and satisfies (5.3) with moderate constants.
The *hp*-finite element space for the mesh $\mathcal{T}_h$ with polynomial degree p is given by

$$S_h^p := \{u \in H_0^1(\Omega) \mid \forall \tau \in \mathcal{T}_h \,:\, u_{|\tau} \circ F_\tau \in \mathbb{P}_p\},$$

where F_τ is as in Definition 5.1.6 (c). Furthermore, the Galerkin discretization of (1.5) reads: Find $u_h^p \in S_h^p$ such that

$$a(u_h^p, v) = F(v) \quad \forall v \in S_h^p. \tag{5.6}$$

According to the Lax-Milgram Theorem (cf. Theorem 1.2.2) the Galerkin solution exists and is unique. Moreover, Céa's lemma (cf. Lemma 1.3.6) implies the quasi-optimal error estimate

$$\|u - u_h^p\|_{H^1(\Omega)} \leq \frac{1}{\alpha} \inf_{v \in S_h^p} \|u - v\|_{H^1(\Omega)}.$$

In order to get explicit convergence estimates in terms of h and p the construction of an *hp*-interpolation operator plays an essential role.

Theorem 5.1.15. *There exists an interpolation operator* $\Pi_{h,p} : H^{p+1}(\Omega) \to S_h^p$ *such that*

$$\|u - \Pi_{h,p}\|_{H^1(K)} \leq C_{apx}\left(\frac{h_K}{p}\right)^p \|u\|_{H^{p+1}(K)}$$

holds for all $K \in \mathcal{T}_h$. *The constant* C_{apx} *depends only on the constants in* (5.3) *and is independent of* p, u, K, *and the diameter* $h_K := \operatorname{diam}(K)$.

A construction for the interpolation operator $\Pi_{h,p}$ and the proof of the theorem can be found e.g. in [7, Lemma 4.5], [26, Lemma 17].

The error estimate for the Galerkin solution can be obtained by the combination of Theorem 5.1.15, Theorem 5.1.14, and Céa's Lemma (cf. Theorem 1.3.6).

Theorem 5.1.16. *Let the assumption of Theorem 5.1.14 be satisfied. Let the hp-finite element discretization be as in* (5.6). *Then the Galerkin solution* u_h *exists, is unique, and satisfies the error estimate*

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C_{10} C_{apx}}{c\alpha} (C_{12} h_{eff})^p \|f\|_{H^{p-1}(\Omega)}, \tag{5.7a}$$

where

$$h_{eff} := \max_{K \in \mathcal{T}_p(A)} \left\{ \left(1 + \max_{1 \le q \le p} \left(\frac{\|\nabla^q A\|_{L^\infty(K^*)}}{q!} \right)^{1/q} \right) \max_{K' \in \mathrm{sons}(K)} h_{K'} \right\} \tag{5.7b}$$

with K^* and c as in Lemma 5.1.12.

The proof of the theorem can be found in [27, Theorem 5.2].

Corollary 5.1.17. *Let the assumption of Theorem 5.1.16 be satisfied. Assume that the coefficient A satisfies*

$$\frac{1}{q!}\|\nabla^q A\|_{L^\infty(\Omega)} \le \frac{C}{\epsilon^q} \tag{5.8}$$

for some (small) $\epsilon > 0$ and for all $1 \le q \le p$. Let p and h be chosen such that

$$p = \left\lceil \frac{\log h}{\log(C_{13}h/\epsilon)} \right\rceil \quad \textit{and} \quad C_{13}h < \epsilon$$

holds. Then the Galerkin discretization with the corresponding hp-finite element space S_h^p has a unique solution u_h which converges linearly:

$$\|u - u_h\|_{H^1(\Omega)} \le Ch\|f\|_{H^{p-1}(\Omega)}, \tag{5.9}$$

where C is independent of ϵ, h, and f.

The assertion follows using (5.7) and (5.8) (cf. [27, Corollary 5.3]).

Remark 5.1.18. *If the assumptions of Corollary 5.1.17 are satisfied and we set $V := S_h^p$, then Assumption 4.0.1 is satisfied.*

5.2 Method and Results Presented in [21]

Another approach which is quite similar to the method described in Chapter 3 has been presented in [21]. As in the present report, this article considers elliptic problems with a general L^∞ diffusion matrix without any periodicity assumptions. For this problem class, local generalized basis functions have been constructed via solutions of local problems on vertex patches.
In the following I will briefly summarize the method which has been developed in [21] and sum up the main error estimates presented there.

5.2.1 Construction of the Local Basis Functions

The goal is to construct a set of local basis functions for the multiscale problem (1.5). The construction is based on a regular finite element mesh $\mathcal{T}$ of Ω into closed triangles ($d = 2$) or tetrahedra ($d = 3$). After the presentation of some properties of quasi interpolation, a modified (coefficient dependent) nodal basis is introduced which is then localized.

Let $\mathcal{T} = \{\tau_i : 1 \leq i \leq N\}$ be a regular finite element mesh (cf. Definition 1.3.3) and the space S as defined in (1.13). Moreover let $\mathcal{N}$ be the index set of interior vertices of $\mathcal{T}$. For every index $i \in \mathcal{N}$ let $b_i \in S$ denote the usual nodal basis function with support ω_i (cf. (1.14)). Further let $H : \bar{\Omega} \to \mathbb{R}_{>0}$ denote the $\mathcal{T}$-piecewise constant mesh size function with $H|_\tau = \operatorname{diam}(\tau) := H_\tau$ for all $\tau \in \mathcal{T}$.

The key tool in the construction of the local basis functions is some bounded linear (quasi-)interpolation operator $\mathfrak{I}_H : H_0^1(\Omega) \to S$. Its choice is not unique and a different choice leads to a different multiscale method. We will use the following modification of Clément's interpolation [14] which is presented and analyzed in [12, Section 6]. Given $v \in H_0^1(\Omega)$, we define a modified Clément operator $H_0^1(\Omega) \to S$ by

$$\mathfrak{I}_H v := \sum_{\substack{x_i \\ i \in \mathcal{N}}} (\mathfrak{I}_H v)(x_i) b_i, \tag{5.10}$$

where the nodal values are given by

$$(\mathfrak{I}_H v)(x_i) := \Big(\int_\Omega v b_i dx \Big) \Big/ \Big(\int_\Omega b_i dx \Big) \quad \text{for } i \in \mathcal{N}.$$

Note that the summation is taken only with respect to the interior vertices. Therefore this operator matches homogeneous Dirichlet boundary conditions.
The following lemma characterizes the local approximation and stability properties of the interpolation operator $\mathfrak{I}_H$.

Lemma 5.2.1. *There exists a constant $C_{\mathfrak{I}_H}$ such that for all $v \in H_0^1(\Omega)$ and for all $\tau \in \mathcal{T}$ it holds*

$$H_\tau^{-1} \|v - \mathfrak{I}_H v\|_{L^2(\tau)} + \|\nabla(v - \mathfrak{I}_H v)\|_{L^2(\tau)} \leq C_{\mathfrak{I}_H} \|\nabla v\|_{L^2(\omega_\tau)} \tag{5.11}$$

with $\omega_\tau := \cup\{T \in \mathcal{T} \mid \tau \cap T \neq \emptyset\}$. The constant $C_{\mathfrak{I}_H}$ depends only on κ (cf. (1.12)), but not on $\operatorname{diam}(\tau)$.

The proof can be found in [12, Lemma 6.2]. In [21, Lemma 1] the following lemma is shown:

Lemma 5.2.2. *There exists a generic constant $C'_{\mathfrak{I}_H}$ which only depends on κ (cf. (1.12)) but not on the local mesh size H, such that for all $u \in S$ there exists $v \in H_0^1(\Omega)$ such that*

$$\mathfrak{I}_H(v) = u \quad \textit{and} \quad \|A^{1/2}\nabla v\|_{L^2(\Omega)} \leq C'_{\mathfrak{I}_H} \|A^{1/2}\nabla u\|_{L^2(\Omega)} \quad \textit{and} \quad \operatorname{supp} v \subset \operatorname{supp} u. \tag{5.12}$$

Remark 5.2.3. *In what follows the interpolation operator* (5.10) *can be replaced by any linear bounded surjective operator that satisfies* (5.11) *and* (5.12)*.*

Let $\mathfrak{I}_H : H_0^1(\Omega) \to S$ be as in (5.10) and define the space

$$S^f := \{v \in H_0^1(\Omega) \mid \mathfrak{I}_H v = 0\}.$$

The space S^f represents the microscopic features of $H_0^1(\Omega)$ that are not captured by S. Furthermore, for $v \in S$ we define a fine-scale projection operator $\mathfrak{F} : S \to S^f$ by

$$a(\mathfrak{F}v, w) = a(v, w) \quad \forall w \in S^f, \tag{5.13}$$

where $a(\cdot, \cdot)$ is as in (1.5).
We set $S^{ms} := (S - \mathfrak{F}S)$. The operator $\mathfrak{F}$ leads to an orthogonal splitting with respect to the scalar product a

$$H_0^1(\Omega) = S^{ms} \oplus S^f,$$

i.e. every function $u \in H_0^1(\Omega)$ can be written as $u = u^{ms} + u^f$, where $u^{ms} \in S^{ms}$, $u^f \in S^f$, and $a(u^{ms}, u^f) = 0$. The space S^{ms} can be regarded as a modified coarse space, since $\dim S^{ms} = \dim S$. The corresponding Galerkin method for problem (1.5) reads: Find $u^{ms} \in S^{ms}$ such that

$$a(u^{ms}, v) = F(v) \quad \forall v \in S^{ms}. \tag{5.14}$$

A basis of S^{ms} is given by the set of modified nodal basis functions

$$\{b_i - \phi_i \mid i \in \mathcal{N}\}, \tag{5.15}$$

where $\phi_i := \mathfrak{F}b_i \in S^f$, i.e. by (5.13) ϕ_i satisfies

$$a(\phi_i, w) = a(b_i, w) \quad \forall w \in S^f. \tag{5.16}$$

The functions ϕ_i usually have global support and therefore the basis functions defined in (5.15) also have global support and thus are of limited use in practice. In the next step, these basis functions will be localized. This can be done by a simple truncation, since ϕ_i decays exponentially away from the vertex x_i (cf. Lemma 5.2.7).

Remark 5.2.4. *The construction of basis functions as in* (5.15) *and* (5.16) *is first introduced in [19] and [20] in a variational multiscale framework.*

Let $k \in \mathbb{N}$ and define patches of k-th order around ω_i by

$$\tilde{\omega}_{i,k} := \omega_{i,k-1}$$

with $\omega_{i,k}$ as in (2.3). For an index $i \in \mathcal{N}$ we define the localized fine-scale spaces

$$S^f(\tilde{\omega}_{i,k}) := \{v \in S^f \ : \ v|_{\Omega \setminus \tilde{\omega}_{i,k}} = 0\}$$

by intersecting S^f with those functions that vanish outside $\tilde{\omega}_{i,k}$. The solutions $\phi_{i,k} \in S^f(\tilde{\omega}_{i,k})$ of

$$a(\phi_{i,k}, w) = a(b_i, w) \quad \forall w \in S^f(\tilde{\omega}_{i,k})$$

are approximations of ϕ_i from (5.16) with local support. The localized multiscale finite element spaces are defined by

$$S_k^{ms} := \operatorname{span}\{b_i - \phi_{i,k} \mid i \in \mathcal{N}\} \subset H_0^1(\Omega).$$

The corresponding multiscale approximation of (1.5) is given by seeking $u_k^{ms} \in S_k^{ms}$ such that

$$a(u_k^{ms}, v) = F(v) \quad \forall v \in S_k^{ms}. \tag{5.17}$$

Remark 5.2.5. *Note that for the dimension it holds* $\dim S_k^{ms} = |\mathcal{N}| = \dim S$. *Hence, the number of degrees of freedom of the method* (5.17) *is the same as for the classical method* (1.15). *The basis functions spanning the space* S_k^{ms} *have local support. Their overlap is proportional to the parameter* k. *The error estimates of Subsection 5.2.2 show that* k *should be chosen proportionally to* $O(\log \frac{1}{H})$.

5.2.2 Error Estimates

This subsection is devoted to the error analysis of the multiscale method which has been described in the previous subsection. In a first step an error bound for the idealized method (5.14) is presented. Then the error of truncation to local patches is analyzed and finally, the main result, i.e. an error bound for (5.17), is stated. The error analysis below depends as usual on the shape-regularity constant κ (cf. (1.12)).

The following lemma shows that the idealized method (5.14) converges linearly.

Lemma 5.2.6. *Let* $u \in H_0^1(\Omega)$ *solve* (1.5) *and* $u^{ms} \in S^{ms}$ *solve* (5.14). *Then it holds*

$$\|A^{1/2}\nabla(u - u^{ms})\|_{L^2(\Omega)} \leq C_{ol}^{1/2} \frac{C_{\mathfrak{I}_H}}{\alpha} \|Hf\|_{L^2(\Omega)} \tag{5.18}$$

with constants C_{ol} *and* $C_{\mathfrak{I}_H}$ *that only depend on* κ.

For a proof see [21, Lemma 3].
In order to bound the error of the localized multiscale FEM we need the following lemma which measures the error between ϕ_i and its approximation $\phi_{i,\ell k} \in S^f(\tilde{\omega}_{i,\ell k})$ in the energy norm. It illustrates that the functions ϕ_i decay exponentially away from the vertex x_i and therefore the approximation obtained by a simple truncation makes sense.

Lemma 5.2.7. *For all* $i \in \mathcal{N}$, k, $\ell \geq 2 \in \mathbb{N}$ *the estimate*

$$\|A^{1/2}\nabla(\phi_i - \phi_{i,\ell k})\|_{L^2(\Omega)} \leq C_{15}\Big(\frac{C_{14}}{\ell}\Big)^{\frac{k-2}{2}} \|A^{1/2}\nabla\phi_i\|_{L^2(\tilde{\omega}_{i,\ell})}$$

holds with constants C_{14}, C_{15} *that only depend on* κ *but not on* i, k, ℓ, *or* H.

The proof can be found in [21, Lemma 6].
Furthermore, the following lemma plays an essential role for the error bound of the localized multiscale FEM.

Lemma 5.2.8. *There is a constant C_{16} that depends only on κ and β/α, but not on $|\mathcal{N}|$, k, or ℓ such that*

$$\Big\|A^{1/2}\nabla\Big(\sum_{\substack{x_i\\ i\in\mathcal{N}}} v(x_i)(\phi_i-\phi_{i,\ell k})\Big)\Big\|^2_{L^2(\Omega)} \leq C_{16}(\ell k)^d \sum_{\substack{x_i\\ i\in\mathcal{N}}} v^2(x_i)\|A^{1/2}\nabla(\phi_i-\phi_{i,\ell k})\|^2_{L^2(\Omega)}.$$

For a proof see [21, Lemma 7].
The following theorem gives an error estimate for the localized multiscale FEM.

Theorem 5.2.9. *Let $u\in H^1_0(\Omega)$ solve* (1.5) *and, given ℓ, $k\geq 2\in\mathbb{N}$, let $u^{ms}_{\ell k}\in S^{ms}_{\ell k}$ solve* (5.17). *Then*

$$\|A^{1/2}\nabla(u-u^{ms}_{\ell k})\|_{L^2(\Omega)} \leq \frac{C_{17}}{\alpha}\|H_\tau^{-1}\|_{L^\infty(\Omega)}(\ell k)^{d/2}\Big(\frac{C_{14}}{\ell}\Big)^{\frac{k-2}{2}}\|f\|_{L^2(\Omega)}+\frac{C_{\mathfrak{I}_H}}{\alpha}\|Hf\|_{L^2(\Omega)}$$

holds with C_{14} from Lemma 5.2.7 and a constant C_{17} that depends on β/α, and κ but not on H, k, ℓ, f, or u.

The proof can be found in [21, Theorem 8].

5.2.3 Computation of the Localized Basis Functions

In this subsection it is shown how the numerical approximations of the local basis functions $b_i-\phi_{i,\ell k}$ and thereby also of the multiscale solution $u^{ms}_{\ell k}$ can be computed. In order to approximate these local basis functions, the error analysis of the previous subsection needs to be extended to a fully discrete setting. The computation of the approximations is assumed to be done using subgrids of a fine-scale reference mesh which is a (possibly space adaptive) refinement of the coarse grid $\mathcal{T}$.

Let $\mathcal{T}_h$ be the result of one uniform refinement and several conforming but possibly non-uniform refinements of the coarse mesh $\mathcal{T}$. Let $h:\bar{\Omega}\to\mathbb{R}_{>0}$ be the $\mathcal{T}_h$-piecewise constant mesh size function with $h_\tau := h|_\tau = \operatorname{diam}(\tau)$ for all $\tau\in\mathcal{T}_h$.
Define the finite element space

$$S_h := \{u\in H^1_0(\Omega)\mid \forall\tau\in\mathcal{T}_h,\ u|_\tau\in\mathbb{P}_1\}$$

and let $u_h\in S_h$ solve

$$a(u_h,v)=F(v)\quad \forall v\in S_h \tag{5.19}$$

with $a(\cdot,\cdot)$ and $F(\cdot)$ as in (1.5). Locally on each patch we let

$$S^f_h(\tilde{\omega}_{i,k}) := \{v\in S_h \mid \mathfrak{I}_H v = 0 \text{ and } v|_{\Omega\setminus\tilde{\omega}_{i,k}}=0\}.$$

Then the numerical approximation $\phi_{i,k}^h \in S_h^f(\tilde{\omega}_{i,k})$ of the corrector ϕ_i^h is defined by

$$a(\phi_{i,k}^h, w) = a(b_i, w) \quad \forall w \in S_h^f(\tilde{\omega}_{i,k}).$$

The discrete multiscale finite element space is given by

$$S_k^{ms,h} := \operatorname{span}\{b_i - \phi_{i,k}^h \mid i \in \mathcal{N}\}.$$

The corresponding discrete multiscale approximation $u_k^{ms,h} \in S_k^{ms,h}$ is determined by

$$a(u_k^{ms,h}, v) = F(v) \quad \forall v \in S_k^{ms,h}. \tag{5.20}$$

The error between the exact solution and the discrete multiscale approximation can be bounded as follows:

Theorem 5.2.10. *Let $u \in H_0^1(\Omega)$ solve* (1.5) *and let $u_{\ell k}^{ms,h} \in S_k^{ms,h}$ solve* (5.20)*. Then*

$$\begin{aligned}\|A^{1/2}\nabla(u - u_{\ell k}^{ms,h})\|_{L^2(\Omega)} \leq & \tilde{C}_{17}\|H^{-1}\|_{L^\infty(\Omega)}(\ell k)^{d/2}\Big(\frac{\tilde{C}_{14}}{\ell}\Big)^{\frac{k-2}{2}}\|f\|_{L^2(\Omega)} \\ & + \frac{C_{\mathfrak{I}_H}}{\alpha}\|Hf\|_{L^2(\Omega)} + \|A^{1/2}\nabla(u - u_h)\|_{L^2(\Omega)}.\end{aligned}$$

Proof. By the triangle inequality it holds that

$$\begin{aligned}\|A^{1/2}\nabla(u - u_{\ell k}^{ms,h})\|_{L^2(\Omega)} \leq & \|A^{1/2}\nabla(u - u_h)\|_{L^2(\Omega)} + \|A^{1/2}\nabla(u_h - u_k^{ms,h})\|_{L^2(\Omega)} \\ & + \|A^{1/2}\nabla(u_k^{ms,h} - u_{\ell k}^{ms,h})\|_{L^2(\Omega)}.\end{aligned}$$

The same arguments as in the proof of Theorem 5.2.9 can be applied replacing $H_0^1(\Omega)$ by S_h. In order to bound the last two terms one can use Lemmas 5.2.11, 5.2.12, and 5.2.13 below (cf. [21, Theorem 10]). □

Lemma 5.2.11 (Discrete version of Lemma 5.2.6). *Let $u_h \in S_h$ solve* (5.19) *and $u_k^{ms,h} \in S_k^{ms,h}$ solve* (5.20) *with k large enough so that $\tilde{\omega}_{i,k} = \Omega$ for all $i \in \mathcal{N}$. Then*

$$\|A^{1/2}\nabla(u_h - u_k^{ms,h})\|_{L^2(\Omega)} \leq C_{ol}^{1/2}\frac{C_{\mathfrak{I}_H}}{\alpha}\|Hf\|_{L^2(\Omega)}$$

holds with constants C_{ol} and $C_{\mathfrak{I}_H}$ that only depend on κ.

The proof can be found in [21, Lemma 15].

Lemma 5.2.12 (Discrete version of Lemma 5.2.7). *For all $i \in \mathcal{N}$, $k, \ell \geq 2 \in \mathbb{N}$ the estimate*

$$\|A^{1/2}\nabla(\phi_i^h - \phi_{i,\ell k}^h)\|_{L^2(\Omega)} \leq \tilde{C}_{15}\Big(\frac{\tilde{C}_{14}}{\ell}\Big)^{\frac{k-2}{2}}\|A^{1/2}\nabla\phi_i^h\|_{L^2(\tilde{\omega}_{i,\ell})}$$

holds with constants $\tilde{C}_{13}$, $\tilde{C}_{14}$ that only depend on κ but not on i, k, ℓ, h, or H.

The proof can be found in [21, Lemma 17].

Lemma 5.2.13 (Discrete version of Lemma 5.2.8). *There is a constant $\tilde{C}_{16}$ depending only on κ and β/α, but not on $|\mathcal{N}|$, k, or ℓ such that*

$$\Big\| A^{1/2}\nabla\Big(\sum_{\substack{x_i\\ i\in\mathcal{N}}} v(x_i)(\phi_i^h - \phi_{i,\ell k}^h)\Big)\Big\|_{L^2(\Omega)}^2 \leq \tilde{C}_{16}(\ell k)^d \sum_{\substack{x_i\\ i\in\mathcal{N}}} v^2(x_i)\|A^{1/2}\nabla(\phi_i^h - \phi_{i,\ell k}^h)\|_{L^2(\Omega)}^2.$$

Chapter 6

Appendix

6.1 Estimates of $\|f_i^{near}\|_{L^2(\Omega)}$ in Terms of $\|P_S f\|_{L^2(\Omega)}$

Let $\mathcal{G} := \{\tau_i : 1 \leq i \leq N\}$ be a regular finite element mesh (cf. Definition 1.3.3) and $S := \{u \in H_0^1(\Omega) : \forall \tau \in \mathcal{G} : u|_\tau \in \mathbb{P}_1\}$ be the space of continuous, piecewise linear finite elements. Moreover, let $P_S f : L^2(\Omega) \to S$ denote the L^2-orthogonal projection onto S. We define

$$f_i^{near} := \sum_{j \in \mathcal{I}_i^{near}} (P_S f)_j b_j, \tag{6.1}$$

where $(b_i)_{i=1}^n$ denotes the usual local nodal basis of S, $(P_S f)_j := (P_S f)(x_j)$ with nodal point x_j corresponding to b_j, and $\mathcal{I}_i^{near}$ is the nearfield defined in (2.5).

Lemma 6.1.1 (1d). *Let $\Omega \subset \mathbb{R}$ be a bounded Lipschitz domain, $f \in L^2(\Omega)$, and f_i^{near} be defined as in* (6.1)*. Then the estimate*

$$\|f_i^{near}\|_{L^2(\Omega)} \leq \sqrt{3}\|P_S f\|_{L^2(\Omega)}$$

holds.

Proof. Let $\mathcal{G} := \{\tau_i : 1 \leq i \leq N\}$ be a regular finite element mesh. Since $\mathcal{G}$ is a partition of Ω, i.e. $\Omega = \bigcup_{j=1}^{N} \tau_j$, we have[1]

$$\|f_i^{near}\|_{L^2(\Omega)} \leq \sum_{j=1}^{N} \|f_i^{near}\|_{L^2(\tau_j)}. \tag{6.2}$$

Let $\hat{\tau} := [0, 1]$ be the unit interval and $\tau := [a, b] \in \mathcal{G}$ such that $\operatorname{diam}(\tau) = \max_{1 \leq j \leq N} \{\operatorname{diam}(\tau_j) : \tau_j \in \mathcal{G}\}$. Define a linear map $F_\tau : \hat{\tau} \to \tau$ by

$$F_\tau(x) := (b - a)x + a.$$

[1] Note that only $O(1)$ summands on the right-hand side of (6.2) are non-zero.

F_τ transforms $\hat{\tau}$ into τ and its inverse $F_\tau^{-1} : \tau \to \hat{\tau}$ is given by

$$F_\tau^{-1}(y) := \frac{y-a}{b-a}.$$

Let b_0, b_1 denote the usual nodal basis functions on $\hat{\tau}$. Then the hat functions on τ are given by

$$\begin{aligned}\phi_0(y) &= b_0(F_\tau^{-1}(y)) = \frac{b-y}{b-a}\\ \phi_1(y) &= b_1(F_\tau^{-1}(y)) = \frac{y-a}{b-a}.\end{aligned}$$

By definition of $P_S f$ we have

$$P_S f|_\tau = K_0\phi_0(y) + K_1\phi_1(y) \quad \text{with } K_0 = P_S f(a) \text{ and } K_1 = P_S f(b).$$

Let M_τ be the matrix such that $(m_\tau)_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2(\tau)}$. If we denote the length of τ by ℓ, then M_τ is given by

$$M_\tau = \frac{\ell}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and the eigenvalues of M_τ are $\lambda_{max} = \frac{\ell}{2}$ and $\lambda_{min} = \frac{\ell}{6}$.
Furthermore, it holds that

$$\|P_S f\|^2_{L^2(\tau)} = \langle K, M_\tau K\rangle_{L^2(\tau)}, \quad \text{where } K = (K_0, K_1)^T$$

and

$$\|f_i^{near}\|^2_{L^2(\tau)} = \langle \tilde{K}, M_\tau \tilde{K}\rangle_{L^2(\tau)}, \quad \text{where } \tilde{K} = (\tilde{K}_0, \tilde{K}_1)^T \text{ and } \tilde{K}_m \in \{0, K_m\} \text{ for } m = 0, 1.$$

Hence,

$$\begin{aligned}\|f_i^{near}\|^2_{L^2(\tau)} &= \langle \tilde{K}, \tilde{K} M_\tau\rangle_{L^2(\tau)} \le \lambda_{max}\|\tilde{K}\|^2_{L^2(\tau)} \le \lambda_{max}\|K\|^2_{L^2(\tau)}\\ &\le \frac{\lambda_{max}}{\lambda_{min}}\langle K, M_\tau K\rangle_{L^2(\tau)} = 3\|P_S f\|^2_{L^2(\tau)}.\end{aligned}$$

Thus from the last inequality we obtain

$$\|f_i^{near}\|_{L^2(\tau)} \le \sqrt{3}\|P_S f\|_{L^2(\tau)}. \tag{6.3}$$

Combining (6.2) and (6.3) yields the assertion. □

Lemma 6.1.2 (2d). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $f \in L^2(\Omega)$, and f_i^{near} be defined as in* (6.1)*. Then the estimate*

$$\|f_i^{near}\|_{L^2(\Omega)} \le 2\|P_S f\|_{L^2(\Omega)}.$$

holds.

Proof. Let $\mathcal{G} := \{\tau_i : 1 \leq i \leq N\}$ be a regular finite element mesh. Furthermore, let $\hat{\tau} := \{(x,y) \in [0,1]^2 \,|\, x+y \leq 1\}$ be the unit triangle and $\tau \in \mathcal{G}$ a triangle with vertices z_1, z_2, $z_3 \in \mathbb{R}^2$ such that $\operatorname{diam}(\tau) = \max_{1\leq j\leq N}\{\operatorname{diam}(\tau_j) : \tau_j \in \mathcal{G}\}$. We define an affine map $F_\tau : \hat{\tau} \to \tau$ by

$$F_\tau(x,y) := z_1 + (z_2 - z_1)x + (z_3 - z_1)y = z_1 + J_\tau(x,y)^T,$$

where $J_\tau := (z_2 - z_1 | z_3 - z_1) \in \mathbb{R}^{2\times 2}$. We set $(\xi,\eta)^T = F_\tau(x,y)$. The usual nodal basis functions on τ are given by

$$\begin{aligned}
\phi_1(\xi,\eta) &= b_1(F_\tau^{-1}(\xi,\eta)) \\
\phi_2(\xi,\eta) &= b_2(F_\tau^{-1}(\xi,\eta)) \\
\phi_3(\xi,\eta) &= b_3(F_\tau^{-1}(\xi,\eta)),
\end{aligned}$$

where b_1, b_2, b_3 denote the hat functions on $\hat{\tau}$, i.e. $b_1(x,y) = 1 - x - y$, $b_2(x,y) = x$, $b_3(x,y) = y$.
Assume that $P_S f(z_1) = K_1$, $P_S f(z_2) = K_2$ and $P_S f(z_3) = K_3$. Then by definition of $P_S f$ it holds that

$$\begin{aligned}
P_S f|_\tau &= P_S f(z_1)\phi_1(\xi,\eta) + P_S f(z_2)\phi_2(\xi,\eta) + P_S f(z_3)\phi_3(\xi,\eta) \\
&= K_1 b_1(F_\tau^{-1}(\xi,\eta)) + K_2 b_2(F_\tau^{-1}(\xi,\eta)) + K_3 b_3(F_\tau^{-1}(\xi,\eta)).
\end{aligned}$$

Let M_τ denote the Gram matrix of the scalar product $\langle\cdot,\cdot\rangle_{L^2(\tau)}$, i.e. $(m_\tau)_{ij} = \int_\tau \langle\phi_i(\xi,\eta), \phi_j(\xi,\eta)\rangle_{L^2(\tau)} d\xi d\eta$. Due to the formula for the change of variables we obtain

$$\begin{aligned}
\int_\tau \langle\phi_i(\xi,\eta), \phi_j(\xi,\eta)\rangle_{L^2(\tau)} d\xi d\eta &= \int_{\hat{\tau}} \langle\phi_i(F_\tau(x,y)), \phi_j(F_\tau(x,y))\rangle_{L^2(\hat{\tau})} det J_\tau dxdy \\
&= \int_{\hat{\tau}} \langle b_i(x,y), b_j(x,y)\rangle_{L^2(\hat{\tau})} det J_\tau dxdy \\
&= det J_\tau \int_{\hat{\tau}} \langle b_i(x,y), b_j(x,y)\rangle_{L^2(\hat{\tau})} dxdy \\
&= det J_\tau (\hat{m}_{\hat{\tau}})_{i,j}.
\end{aligned}$$

Here $\hat{M}_{\hat{\tau}}$ is the mass matrix for the unit triangle. We denote the area of τ by $|\tau|$. Since $|\tau| = \frac{1}{2} det J_\tau$, M_τ is given by

$$M_\tau = \frac{|\tau|}{12}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of M_τ are $\lambda_1 = \lambda_2 = \frac{|\tau|}{12} =: \lambda_{min}$ and $\lambda_3 = \frac{|\tau|}{3} =: \lambda_{max}$.

Moreover,

$$\|P_S f\|^2_{L^2(\tau)} = \langle K, M_\tau K\rangle_{L^2(\tau)}, \quad \text{where } K = (K_1, K_2, K_3)^T$$

and

$$\|f_i^{near}\|^2_{L^2(\tau)} = \langle \tilde{K}, M_\tau \tilde{K}\rangle, \quad \text{where } \tilde{K} = (\tilde{K}_1, \tilde{K}_2, \tilde{K}_3)^T \text{ and } \tilde{K}_i \in \{0, K_i\},\ i = 1, 2, 3.$$

Thus we have

$$\begin{aligned} \|f_i^{near}\|^2_{L^2(\tau)} &= \langle \tilde{K}, M_\tau \tilde{K}\rangle \le \lambda_{max}\|\tilde{K}\|^2 \le \lambda_{max}\|K\|^2 \\ &\le \frac{\lambda_{max}}{\lambda_{min}}\langle K, M_\tau K\rangle = 4\|P_S f\|^2_{L^2(\tau)}. \end{aligned}$$

And therefore

$$\|f_i^{near}\|_{L^2(\tau)} \le 2\|P_S f\|_{L^2(\tau)}. \tag{6.4}$$

Since the estimate

$$\|f_i^{near}\|_{L^2(\Omega)} \le \sum_{j=1}^{N} \|f_i^{near}\|_{L^2(\tau_j)}$$

holds (see also footnote 1 on p. 43) the claim follows from (6.4). □

Lemma 6.1.3 (3d). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $f \in L^2(\Omega)$, and f_i^{near} be defined as in* (6.1)*. Then there exists a constant Θ such that the estimate*

$$\|f_i^{near}\|_{L^2(\Omega)} \le \Theta\|P_S f\|_{L^2(\Omega)}$$

holds.

Proof. Let $\hat{\tau}$ be the unit tetrahedron, i.e. the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. For a tetrahedron $\tau \in \mathcal{G}$ with vertices z_1, z_2, z_3, $z_4 \in \mathbb{R}^3$ the affine map $F_\tau : \hat{\tau} \to \tau$ defined by

$$F_\tau(x, y, z) := z_1 + (z_2 - z_1)x + (z_3 - z_1)y + (z_4 - z_1)z = z_1 + J_\tau(x, y, z)^T,$$

where $J_\tau := (z_2 - z_1 | z_3 - z_1 | z_4 - z_1) \in \mathbb{R}^{3\times 3}$, is a transformation from the unit tetrahedron into an arbitrary tetrahedron $\tau \in \mathcal{G}$. Using the fact that the nodal basis functions on $\hat{\tau}$ are given by $b_1(x,y,z) = 1 - x - y - z$, $b_2(x,y,z) = x$, $b_3(x,y,z) = y$, $b_4(x,y,z) = z$ and applying the same arguments as in the proof of Lemma 6.1.2 the assertion follows. □

Remark 6.1.4. *The condition number of the local mass matrix M_τ for general simplicial finite elements of degree p depends on the choice of basis functions as well as of the choice of the nodal points. For further details we refer the reader to [23].*

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