NOTES ON THE SELF-LINKING NUMBER

ALBERTO S. CATTANEO

The aim of this note is to review the results of [4, 8, 9] in view of integrals over configuration spaces and also to discuss a generalization to other 3-manifolds.

Recall that the linking number of two non-intersecting curves in $\mathbb{R}^3$ can be expressed as the integral (Gauss formula) of a 2-form on the Cartesian product of the curves. It also makes sense to integrate over the product of an embedded curve with itself and define its self-linking number this way. This is not an invariant as it changes continuously with deformations of the embedding; in addition, it has a jump whenever a crossing is switched.

There exists another function, the integrated torsion, which has opposite variation under continuous deformations but is defined on the whole space of immersions (and consequently has no discontinuities at a crossing change), but depends on a choice of framing.

The sum of the self-linking number and the integrated torsion is therefore an invariant of embedded curves with framings and turns out simply to be the linking number between the given embedding and its small displacement by the framing. This also allows one to compute the jump at a crossing change.

The main application is that the self-linking number or the integrated torsion may be added to the integral expressions of [3] to make them into knot invariants.

We also discuss on the geometrical meaning of the integrated torsion.

1. The trivial case

Let us start recalling the case of embeddings and immersions in $\mathbb{R}^3$. This will help us fixing the notations and preparing for the more complicated case of a rational homology sphere. As in [3], we denote by $C_2(\mathbb{R}^3) \doteq \{(x, y) \in \mathbb{R}^3 : x \neq y\}$ the configuration space[1] of two points

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[1] In [3], this and the similar notations appearing in this Section are actually used for the compactifications of the corresponding spaces, but in this Section it makes no difference using a compactification or the space itself.

in $\mathbb{R}^3$ and by $\Phi\colon\ \mathrm{C}_2(\mathbb{R}^3) \to S^2$ the map

$$\Phi\colon (x,y) \mapsto \frac{y-x}{||y-x||} \tag{1.1}$$

that associates to each pair of distinct points the unit vector (in the Euclidean norm) joining them. If ω is the SO(3)-invariant volume form

$$\omega = x\,\mathrm{d}y\,\mathrm{d}z + \text{cyclic permutations}$$

on S^2, then one defines $\vartheta = \Phi^*\omega/4\pi$. The Gauss formula then states that the linking number can be written as

$$\mathrm{lk}(\gamma_1,\gamma_2) = \int_{S^1\times S^1} (\gamma_1\times\gamma_2)^*\vartheta, \tag{1.2}$$

where γ_1 and γ_2 are any two mutually non-intersecting closed curves. One defines the self-linking number

$$\mathrm{slk}(K) \doteq \int_{C^K_{2,0}} \vartheta = \int_{C_2(S^1)} K^*\vartheta, \tag{1.3}$$

where $K\colon S^1 \to \mathbb{R}^3$ is an embedding, $C_2(S^1) \doteq \{(x,y)\in S^1 : x \neq y\}$ is the configuration space of two points on S^1 and $C^K_{2,0} \doteq K(C_2(S^1))$, using the notations of [3]. It is an easy exercise to verify that the last integral converges (the high-brow way would be to consider compactification and to show that the form extends there).

A framing $\mathbf{w}$ of an immersion $K\colon S^1 \to \mathbb{R}^3$ is a nowhere vanishing vector field normal to K. For $\epsilon > 0$ small enough, $K+\epsilon\mathbf{w}$ (i.e., the map $x \mapsto K(x)+\epsilon\mathbf{w}(x)$) will be another closed curve that does not intersect K and that can be deformed to K without meeting any intersection point. This way, the linking number between K and $K+\epsilon\mathbf{w}$ is well defined and independent of ϵ.

The choice of a framing $\mathbf{w}$ is also equivalent to the choice of a section $\sigma_{\mathbf{w}}$ of the normal bundle $\mathrm{N}K$. More precisely, denoting by $\mathbf{t}(x)$ the unit tangent vector at $x \in K$, the choice of the vector field $\mathbf{w}$ determines the oriented frame $(\mathbf{t}(x), \mathbf{w}(x)\times\mathbf{t}(x), \mathbf{w}(x))$ at x, and consequently the section

$$\sigma_{\mathbf{w}}\colon x \mapsto (x, \mathbf{w}(x)\times\mathbf{t}(x), \mathbf{w}(x)) \tag{1.4}$$

of $\mathrm{N}K$.

Using this section, we can pull back to K any connection on this SO(2)-bundle. In particular, we will denote by $\tau_{\mathbf{w}}$ the pullback of the connection on $\mathrm{N}K$ induced by the immersion K from the Levi-Civita connection on $\mathbb{R}^3$ with Euclidean metric. Then we have the following

Theorem 1.1. *With the above notations,*

$$\operatorname{lk}(K, K+\epsilon\mathbf{w}) = \operatorname{slk}(K) + T_{\mathbf{w}}(K), \tag{1.5}$$

where

$$T_{\mathbf{w}}(K) = \frac{1}{2\pi}\int_K \tau_{\mathbf{w}}. \tag{1.6}$$

Remark 1.2. Since the linking number is an invariant, we have

$$\mathrm{d}\operatorname{slk} = -\mathrm{d}T_{\mathbf{w}}, \tag{1.7}$$

where d is the exterior derivative on the space of embeddings.

In [3], the exterior derivative of the self-linking number is computed as

$$\mathrm{d}\operatorname{slk}(K) = -\frac{1}{2\pi}\int_K \Psi^*\omega,$$

where Ψ is the map $K \to S^2$ that associates to each point its unit tangent vector $\mathbf{t}$, and the integration is assumed to act from the left. Thus, the previous result yields

$$\mathrm{d}\tau_{\mathbf{w}} = -\Psi^*\omega. \tag{1.8}$$

In view of our geometric interpretation, it should be clear why the right-hand side does not depend on the framing $\mathbf{w}$: it is just the well-known fact that the curvature of an abelian connection is basic.

Remark 1.3. $T_{\mathbf{w}}$ is defined on immersions and not only on embeddings, and in particular it behaves smoothly under crossing changes; the price to pay with respect to the self-linking number is the introduction of a framing.

Remark 1.4. By our definition of $\tau_{\mathbf{w}}$ it is clear that

$$\exp[2\pi\mathrm{i}\, T_{\mathbf{w}}(K)] = \operatorname{Hol}(K),$$

where Hol denotes the holonomy in $\mathrm{N}K$, viewed as a $U(1)$-bundle. This immediately shows that $T_{\mathbf{w}}$ must jump by an integer under a change of framing.

1.1. **Proof of Theorem 1.1.**

We split the proof into two Lemmata. The first is

Lemma 1.5 (Călugareanu, Pohl)**.** *Equations* (1.5) *and* (1.6) *hold with*

$$\tau_{\mathbf{w}} = \mathbf{t}\times\mathbf{w}\cdot\mathrm{d}\mathbf{w}, \tag{1.9}$$

where $\mathbf{t}$ *is the unit vector field tangent to* K*.*

The original result of Călugareanu [4] covered the case when the curvature of K never vanishes so that there is a well-defined Frenet frame. In this case, denoting the binormal vector field by $\mathbf{b}$, one gets

$$\tau_{\mathbf{b}} = \tau \, \mathrm{d}s,$$

where s is the arc length parameter and τ is the torsion (this also explains the name integrated torsion for T). Pohl [8] then found the generalization for any framing.

The second Lemma of our proof contains the geometric interpretation of $\tau_{\mathbf{w}}$:

Lemma 1.6. *The one-form $\tau_{\mathbf{w}}$ defined in* (1.9) *is the pullback via $\sigma_{\mathbf{w}}$, see* (1.4)*, of the connection on* $\mathrm{N}K$ *induced by the immersion K from the Levi-Civita connection on $\mathbb{R}^3$ with Euclidean metric.*

Proof of Lemma 1.5. We closely follow the proof given by White in [9]. We repeat it to fix the notations and to prepare for the proof of the analogous Lemma in the case when M is a rational homology sphere.

Let $C_{1,1}(S^1, S^1 \times [0,\epsilon])$ be the compactification of the configuration space $C^0_{1,1}(S^1, S^1 \times [0,\epsilon])$ of two points on $(S^1 \times [0,\epsilon]) \times (S^1 \times [0,\epsilon])$ with the first constrained to lie on $S^1 \times \{0\}$ and with orientation induced from that of $S^1 \times (S^1 \times [0,\epsilon])$. This is a manifold with corners defined by blowing up the diagonal in the spirit of [6, 1, 3]; explicitly, we can construct it as follows.

Let q denote the projection from $C^0_{1,1}(S^1, S^1 \times [0,\epsilon]) = S^1 \times (S^1 \times [0,\epsilon]) \setminus \{(x,x,0),\ x \in S^1\}$ to the first factor S^1. For every $x \in S^1$, we regard the preimage $q^{-1}(x)$ as the rectangle $R \doteq [-\pi,\pi] \times [0,\epsilon] \setminus \{(0,0)\}$ with identified vertical sides (the removed point $(x,0)$ corresponds in this description to $(0,0)$). In other words, for a point $(x,y,t) \in C^0_{1,1}(S^1, S^1 \times [0,\epsilon])$ we use the parametrization $y = x + z$ with $(z,t) \in R$. We can use polar coordinates on R; i.e., we write $(z,t) \in R$ as $z = -r\cos\phi$, $t = r\sin\phi$ with

$$\begin{aligned} 0 < r &\leq \frac{\epsilon}{\sin\phi} \quad \text{for } |\tan\phi| \geq \frac{\epsilon}{\pi}, \\ 0 < r &\leq \frac{\pi}{\cos\phi} \quad \text{for } |\tan\phi| < \frac{\epsilon}{\pi}, \end{aligned}$$

and $\phi \in [0,\pi]$. The compactification $C_{1,1}(S^1, S^1 \times [0,\epsilon])$ of $C^0_{1,1}(S^1, S^1 \times [0,\epsilon])$ is defined by allowing $r = 0$ in each fiber.

Let $K_{\mathbf{w}}$ be the map $C^0_{1,1}(S^1, S^1 \times [0,\epsilon]) \to C_2(\mathbb{R}^3)$ defined by

$$K_{\mathbf{w}}(x,y,t) \doteq (K(x), K(y) + t\mathbf{w}(y)).$$

Since ϑ is closed, we have

$$0 = \int_{C_{1,1}(S^1, S^1\times[0,\epsilon])} \mathrm{d}K_{\mathbf{w}}^*\vartheta = \int_{\partial C_{1,1}(S^1,S^1\times[0,\epsilon])} K_{\mathbf{w}}^*\vartheta,$$

where $\partial C_{1,1}(S^1, S^1\times[0,\epsilon])$ denotes the codimension-one boundary stratum of $C_{1,1}(S^1, S^1 \times [0,\epsilon])$. It is not too difficult to see that $K_{\mathbf{w}}^*\vartheta$ smoothly extends to this boundary, as we will see in a moment.

Observe that $\partial C_{1,1}(S^1, S^1\times[0,\epsilon])$ actually has three disjoint components: The first corresponds to the second point approaching the upper boundary and is given by $S^1 \times (S^1 \times \{\epsilon\})$. The second, to be taken with reversed orientation, corresponds to the second point approaching the lower boundary away form the first point and is given by $C_2(S^1)$. The third, also to be taken with reversed orientation, corresponds to the blow up of the diagonal; we will denote this face by Bl. Therefore, we get

$$0 = \mathrm{lk}(K, K + \epsilon\mathbf{w}) - \mathrm{slk}(K) - T_{\mathbf{w}}(K),$$

with

$$T_{\mathbf{w}}(K) = \int_{\mathrm{Bl}} K_{\mathbf{w}}^*\vartheta.$$

To finish the proof, we thus only have to simplify the expression for $T_{\mathbf{w}}$.

Remark that in a neighborhood of Bl we can choose coordinates $x \in S^1$ and $(r\xi, r\eta) \in R$ with $\xi^2+\eta^2 = 1$, $\eta \geq 0$ and r small. The boundary component Bl is a bundle over S^1 with fiber the upper half circle S^1_+ obtained as $r \to 0^+$. In this neighborhood we have

$$K_{\mathbf{w}}(x, x + r\xi, r\eta) = (K(x), K(x) + r(\xi\mathbf{t}(x) + \eta\mathbf{w}(x))) + O(r^2),$$

where, to simplify the computation, we have assumed that x is the arc length parameter. It follows that

$$\Phi \circ K_{\mathbf{w}}(x, x + r\xi, r\eta) = \xi\mathbf{t}(x) + \eta\mathbf{w}(x) + O(r).$$

We let $\tilde{\Phi}_{\mathbf{w}}$ denote the restriction of $\Phi\circ K_{\mathbf{w}}$ to Bl: i.e., $\tilde{\Phi}_{\mathbf{w}}(x,\xi,\eta) = \lim_{r\to 0^+}\Phi\circ K_{\mathbf{w}}(x, x+r\xi, r\eta)$. Hence we get the map

$$\begin{aligned} \tilde{\Phi}_{\mathbf{w}}\colon \quad & \mathrm{Bl} \to S^2 \\ & (x,\xi,\eta) \mapsto \mathbf{x} := (\xi\mathbf{t}(x) + \eta\mathbf{w}(x)) \end{aligned} \tag{1.10}$$

The restriction of $K_{\mathbf{w}}^*\vartheta$ to Bl then reads

$$\begin{aligned} 4\pi\, K_{\mathbf{w}}^*\vartheta = \tilde{\Phi}_{\mathbf{w}}^*\omega = \frac{1}{2}(\mathbf{x}\times\mathrm{d}\mathbf{x})\cdot\mathrm{d}\mathbf{x} = \\ = -(\mathbf{t}\times\mathbf{w})\cdot(\xi\mathbf{t}' + \eta\mathbf{w}')\,\mathrm{d}x\,(\xi\mathrm{d}\eta - \eta\mathrm{d}\xi), \end{aligned}$$

where the prime denotes derivation with respect to x. If we write

$$\xi = -\cos\phi, \quad \eta = \sin\phi, \qquad \phi \in [0,\pi],$$

then the orientation of Bl is given by the top form $\mathrm{d}x\,\mathrm{d}\phi$. Moreover, we can simplify ϑ to

$$4\pi\,\vartheta = \mathrm{d}x\,(\mathbf{t}\times\mathbf{w})\cdot(\mathbf{t}'\cos\phi + \mathbf{w}'\sin\phi)\,\mathrm{d}\phi.$$

Integrating over ϕ finally yields $T_{\mathbf{w}}$ as in (1.6) with $\tau_{\mathbf{w}}$ as in (1.9). □

Remark 1.7. If we had used the compactification $C_2(\mathbb{R}^3)$ of [3], then we could have defined $\mathrm{C}(K,\Sigma_{\mathbf{w}})$ as the image of $C_{1,1}(S^1, S^1\times[0,\epsilon])$ under $K_{\mathbf{w}}$ (here $\Sigma_{\mathbf{w}}$ denotes the strip $K+t\mathbf{w}$, $t\in[0,\epsilon]$, i.e., the image of $S^1\times[0,\epsilon]$ under $K_{\mathbf{w}}$). Then we could have directly used the formula

$$0 = \int_{\mathrm{C}(K,\Sigma_{\mathbf{w}})} \mathrm{d}\vartheta = \int_{\partial\,\mathrm{C}(K,\Sigma_{\mathbf{w}})} \vartheta.$$

The study of the boundary components of $\mathrm{C}(K,\Sigma_{\mathbf{w}})$ is however the same as before, even though we could understand the blow-up boundary more geometrically as the points of the form $(v, v + r(\xi\mathbf{t}+\eta\mathbf{w}))$, with v in the image of K.

Proof of Lemma 1.6. If K is an immersion (and not necessarily an embedding), its unit tangent vector $\mathbf{t}$ is defined everywhere. By $\mathrm{O}\mathbb{R}^3|_K$ we denote the pullback of the frame bundle $\mathrm{O}\mathbb{R}^3$ of $\mathbb{R}^3$. An oriented frame $(\mathbf{b}_1,\mathbf{b}_2)$ in NK_x consists of unit vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ orthogonal to $\mathbf{t}(x)$ with $\mathbf{b}_2=\mathbf{t}\times\mathbf{b}_1$. It yields the adapted frame $(x,\mathbf{t}(x),\mathbf{b}_1,\mathbf{b}_2)\in\mathrm{O}\mathbb{R}^3|_K$.

Given a connection ψ on $\mathrm{O}\mathbb{R}^3$, which we view as an $\mathfrak{so}(3)$-valued one-form on $\mathrm{O}\mathbb{R}^3$, the induced connection on $\mathrm{N}K$ is obtained (see, e.g., [7]) by first restricting ψ to the adapted frame and then selecting the $\mathfrak{so}(2)$-component corresponding to normal vectors. In other words, let j denote the map $\mathrm{O}\mathbb{R}^3|_K\to\mathrm{O}\mathbb{R}^3$ determined by our immersion and let i be the injection $\mathrm{N}K\to\mathrm{O}\mathbb{R}^3|_K$ defined by

$$i\colon (x,\mathbf{b}_1,\mathbf{b}_2)\mapsto(x,\mathbf{t}(x),\mathbf{b}_1,\mathbf{b}_2).$$

Then the induced connection is $\varpi(i^*j^*\psi)$, where ϖ is the projection to the lower right 2×2 block.

The Levi-Civita connection on $\mathbb{R}^3$ with Euclidean metric is trivial; i.e., $\psi(x,g)=g^{-1}\mathrm{d}g$, $x\in\mathbb{R}^3, g\in\mathrm{SO}(3)$. By writing $g=(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)$, where $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)$ is an orthonormal oriented frame, we get

$$\psi = \begin{pmatrix} 0 & \mathbf{a}_1\cdot\mathrm{d}\mathbf{a}_2 & \mathbf{a}_1\cdot\mathrm{d}\mathbf{a}_3 \\ -\mathbf{a}_1\cdot\mathrm{d}\mathbf{a}_2 & 0 & \mathbf{a}_2\cdot\mathrm{d}\mathbf{a}_3 \\ -\mathbf{a}_1\cdot\mathrm{d}\mathbf{a}_3 & -\mathbf{a}_2\cdot\mathrm{d}\mathbf{a}_3 & 0 \end{pmatrix}. \tag{1.11}$$

Thus, the induced connection on NK is

$$i^*j^*\psi = -(\mathbf{b}_1 \cdot \mathrm{d}\mathbf{b}_2)\, R = \hat\tau\, R$$

where $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the generator of the Lie algebra $\mathfrak{so}(2) \simeq \mathbb{R}$, and $\hat\tau = -(\mathbf{b}_1 \cdot \mathrm{d}\mathbf{b}_2)$ is the corresponding real-valued connection. Therefore,

$$\sigma_{\mathbf{w}}^*\hat\tau(x) = -\mathbf{w}(x) \times \mathbf{t}(x) \cdot \mathrm{d}\mathbf{w}(x) = \tau_{\mathbf{w}}(x),$$

where $\sigma_{\mathbf{w}}$ is the section defined in (1.4). □

Theorem 1.1 is finally an immediate consequence of the two Lemmata.

2. The case of rational homology spheres

In [2], a generalization of the two-form ϑ described in the previous Section was considered. From now on we assume familiarity with [3]. In particular, $C_2(M)$ will denote the compactified configuration space of two points in a compact manifold M.

Let M be a rational homology sphere in this Section. Then it is possible to construct a form $\hat\eta \in \Omega^2(\mathrm{C}_2(M))$ with the following properties:[2]

$$\mathrm{d}\hat\eta = v_1 - v_2, \tag{2.1a}$$

$$\iota_\partial^*\hat\eta = \eta, \tag{2.1b}$$

$$T^*\hat\eta = -\hat\eta, \tag{2.1c}$$

where v_1 and v_2 are the pullbacks to $\mathrm{C}_2(M)$ of an arbitrary unit generator of $\mathrm{H}^3(M)$; T is the involution that exchanges the factors in $\mathrm{C}_2(M)$; ι_∂ is the inclusion map of the boundary of $\mathrm{C}_2(M)$, and η is an odd global angular form for the sphere bundle $\partial\,\mathrm{C}_2(M) \to M$ (odd with respect to the antipodal map).

Next an odd global angular form $\eta[\theta]$ is constructed for a given choice of a Riemannian metric and of a metric connection θ. More precisely, a Riemannian metric allows us to write $\partial C_2(M) \simeq \mathrm{O}M \times_{\mathrm{SO}(3)} S^2$. We will let θ^i denote the components of θ in the basis $\{\xi_i, i = 1, 2, 3\}$ of $\mathfrak{so}(3)$ given by $(\xi_i)_{jk} = \epsilon_{ijk}$. To simplify the notation, we will also let $\boldsymbol{\theta}$ denote the vector whose components are $(\theta^1, \theta^2, \theta^3)$. Then (see [2] for the details) the two-form[3]

$$\bar\eta[\theta] = \frac{\omega + \mathrm{d}(\boldsymbol{\theta} \cdot \mathbf{x})}{4\pi}$$

[2]We choose here the opposite sign convention of [2].

[3]By abuse of notation, we will write θ and ω also for their pullbacks to $OM \times S^2$.

is an odd global angular form for the trivial bundle $\mathrm{O}M \times S^2 \to \mathrm{O}M$ and is basic on the principal bundle

$$p\colon \mathrm{O}M \times S^2 \to \mathrm{O}M \times_{\mathrm{SO}(3)} S^2.$$

Finally, the odd global angular form $\eta[\theta]$ is implicitly defined by the equation

$$\bar{\eta}[\theta] = p^*\eta[\theta].$$

In the rest of the Section, we will let $\hat{\eta}[\theta]$ denote the two-form $\hat{\eta}$ defined in (2.1) for the given odd global angular form $\eta[\theta]$.

Using this two-form, it is possible to write down a formula for the linking number of two curves in a rational homology sphere as

$$\mathrm{lk}(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} (\gamma_1 \times \gamma_2)^* \hat{\eta}, \tag{2.2}$$

where we drop the argument θ since the left-hand side turns out to be independent of it (as well as of all the other choices involved in the construction of $\hat{\eta}$). Moreover, lk defines a link invariant which, in general, is not integer valued, as in $\mathbb{R}^3$, but rational valued (one still has an integer-valued linking number on integral homology spheres).

To show this, first observe that switching any crossing between γ_1 and γ_2 changes lk by one. In the case when γ_1 is homologically trivial, it is possible, after switching a certain number of crossings, to reduce γ_1 to a circle that can be contracted without intersecting γ_2. So lk turns out to be an integer. In the general case, one has just to observe that lk is additive on γ_1 and γ_2, so $\mathrm{lk}(\gamma_1, \gamma_2) = (1/n_1)\mathrm{lk}(n_1\, \gamma_1, \gamma_2)$. Choosing n_1 so that $n_1[\gamma_1]$ is homologically trivial yields the previous case.[4]

The self-linking number, which is not an invariant and depends on θ, is then defined by

$$\mathrm{slk}(K)[\theta] \doteq \int_{C_2(S^1)} K^* \hat{\eta}[\theta], \tag{2.3}$$

where $K\colon S^1 \to M$ is an embedding.

By choosing a framing $\mathbf{w}$ for K, we can generalize Theorem 1.1 as follows:

[4]With a little more effort, one can also show that lk coincides with the geometrical linking number

$$\mathrm{lk}_{\mathrm{geom}}(\gamma_1, \gamma_2) = \frac{1}{n_1}\, \#(C_1, \gamma_2) = \frac{1}{n_2}\, \#(\gamma_1, C_2),$$

where C_i is any surface that intersects γ_j $(i \neq j)$ transversally and such that $\partial[C_i] = n_i[\gamma_i]$ for a suitable integer n_i.

Theorem 2.1. *With the above notations (and recalling the considerations preceding Theorem 1.1), we have*

$$\operatorname{lk}(K, K+\epsilon\mathbf{w}) = \operatorname{slk}(K)[\theta] + T_{\mathbf{w}}(K)[\theta], \tag{2.4}$$

with

$$T_{\mathbf{w}}(K)[\theta] = \frac{1}{2\pi}\int_K \tau_{\mathbf{w}}[\theta], \tag{2.5}$$

where $\tau_{\mathbf{w}}[\theta]$ is the pullback to K via the framing $\mathbf{w}$ of the connection induced from θ by the immersion K.

Remark 2.2. All the remarks following Theorem 1.1 obviously generalize to this case. In particular, (1.7) still holds and (1.8) is replaced by

$$\mathrm{d}\tau_{\mathbf{w}}[\theta] = -4\pi\,\Psi^*\eta[\theta], \tag{2.6}$$

where Ψ is now the map $K \to \partial\,\mathrm{C}_2(M)$ that sends a point $x \in K$ to $(x, x + r\mathbf{t}(x))$, $r \to 0^+$.

Notice that (2.6) is in complete agreement with the interpretation given in [2] of η as half the Euler class of $\mathrm{T}_{S^2}\,\partial\,\mathrm{C}_2(M)$, where T_{S^2} denotes the tangent bundle along the fiber S^2.

Again the proof of the Theorem is an immediate consequence of two Lemmata that generalize 1.5 and 1.6.

Lemma 2.3. *Equations* (2.4) *and* (2.5) *hold with*

$$\tau_{\mathbf{w}}[\theta] = \mathbf{t}\times\mathbf{w}\cdot\mathrm{d}\mathbf{w} - \hat{\Phi}^*_{\mathbf{w}}\boldsymbol{\theta}\cdot\mathbf{t}, \tag{2.7}$$

where $\mathbf{t}$ is the unit vector field tangent to K, and

$$\begin{aligned}\hat{\Phi}_{\mathbf{w}}\colon\ & K \to \mathrm{O}M\\ & x \mapsto (x, \mathbf{t}(x), \mathbf{w}(x)\times\mathbf{t}(x), \mathbf{w}(x))\end{aligned}$$

Proof. The first part of the proof is exactly as in the proof of Lemma 1.6 with ϑ replaced by $\hat{\eta}[\theta]$. We only have to remark that, though $\hat{\eta}$ is not closed, by (2.1a) the integral of the pullback of its differential over $C_{1,1}(S^1, S^1\times[0,\epsilon])$ still vanishes by dimensional reasons. In this way we prove (2.4) with

$$T_{\mathbf{w}}(K)[\theta] = \int_{\mathrm{Bl}} \Psi^*_{\mathbf{w}}\eta[\theta],$$

where $\Psi_{\mathbf{w}}\colon \mathrm{Bl} \to \partial\,\mathrm{C}_2(M) \simeq \mathrm{O}M \times_{\mathrm{SO}(3)} S^2$ is the restriction to the indicated boundary of the embedding $C_{1,1}(S^1, S^1\times[0,\epsilon]) \hookrightarrow \mathrm{C}_2(M)$ induced from $K \hookrightarrow M$. The map

$$\hat{\Psi}_{\mathbf{w}} = (\hat{\Phi}_{\mathbf{w}}, \tilde{\Phi}_{\mathbf{w}})\colon \mathrm{Bl} \to \mathrm{O}M \times S^2,$$

with $\tilde{\Phi}_{\mathbf{w}}$ defined in (1.10), is a lift of $\Psi_{\mathbf{w}}$, as can be easily checked. Therefore,

$$T_{\mathbf{w}}(K)[\theta] = \int_{\mathrm{Bl}} \hat{\Psi}^*_{\mathbf{w}} \bar{\eta}[\theta],$$

and

$$4\pi\, \hat{\Psi}^*_{\mathbf{w}}\bar{\eta}[\theta] = \tilde{\Phi}^*_{\mathbf{w}}\omega + \mathrm{d}(\hat{\Phi}^*_{\mathbf{w}}\boldsymbol{\theta}\cdot(\xi\mathbf{t}+\eta\mathbf{w})) = \\ = \tilde{\Phi}^*_{\mathbf{w}}\omega - \hat{\Phi}^*_{\mathbf{w}}\boldsymbol{\theta}\cdot(\mathbf{t}\sin\phi + \mathbf{w}\cos\phi)\mathrm{d}\phi + \cdots,$$

where the dots denote forms that integrate to zero along the fiber of Bl. Finally, (2.5) is obtained by performing this integration. □

Lemma 2.4. *The one-form $\tau_{\mathbf{w}}[\theta]$ defined in* (2.7) *is the pullback via $\sigma_{\mathbf{w}}$, see* (1.4)*, of the connection on* $\mathrm{N}K$ *induced from θ by the immersion K.*

Proof. We proceed as in the proof of Lemma 1.6. First of all we have to write the connection θ in the adapted frame $(x, \mathbf{t}(x), \mathbf{b}_1, \mathbf{b}_2) \in \mathrm{O}M|_K$. Denoting by $g \in \mathrm{SO}(3)$ the matrix whose columns are the vectors $\mathbf{a}_1 = \mathbf{t}, \mathbf{a}_2 = \mathbf{b}_1, \mathbf{a}_3 = \mathbf{b}_2$, we get

$$j^*\theta = g^{-1}\mathrm{d}g + g^{-1}\,\theta\, g = j^*\psi + \begin{pmatrix} 0 & \boldsymbol{\theta}\cdot\mathbf{a}_3 & -\boldsymbol{\theta}\cdot\mathbf{a}_2 \\ -\boldsymbol{\theta}\cdot\mathbf{a}_3 & 0 & \boldsymbol{\theta}\cdot\mathbf{a}_1 \\ \boldsymbol{\theta}\cdot\mathbf{a}_2 & -\boldsymbol{\theta}\cdot\mathbf{a}_1 & 0 \end{pmatrix},$$

with ψ defined in (1.11). Restricting then to the SO(2)-subbundle $\mathrm{N}K$ yields the connection

$$\varpi(i^*j^*\theta) = \varpi(i^*j^*\psi) - i^*\boldsymbol{\theta}\cdot\mathbf{t}\,R = \hat{\tau}[\theta]\,R,$$

with

$$\hat{\tau}[\theta] = -\mathbf{b}_1\cdot\mathrm{d}\mathbf{b}_2 - i^*\boldsymbol{\theta}\cdot\mathbf{t}.$$

Finally, observe that $\hat{\Phi}_{\mathbf{w}} = i\circ\sigma_{\mathbf{w}}$; so

$$\sigma^*_{\mathbf{w}}\hat{\tau}[\theta] = \tau_{\mathbf{w}}[\theta].$$

□

3. The general case

In this Section, we consider the generalizations of Theorems 1.1 and 2.1 to the case of a compact, oriented manifold M.

First of all observe that the construction of [2] leading to (2.1) can be easily generalized to any connected, compact, closed, oriented 3-manifold (for details, see [5]) by replacing (2.1a) by

$$\mathrm{d}\hat{\eta} = -\pi^*\chi_\Delta. \tag{2.1a'}$$

Here χ_Δ is an odd (with respect to the involution T) representative of the Poincaré dual of the diagonal Δ in $M \times M$, and π is the projection $\mathrm{C}_2(M) \to M \times M$.

If $\{[\omega^i]\}$ is a basis of $\mathrm{H}^1(M)$ and $\{[\tau^i]\}$ its dual basis (that is, the τ^is are closed two forms on M, and $\int_M \omega^i \, \tau^j = \delta^{ij}$), then an odd representative of the Poincaré dual of Δ is

$$\chi_\Delta = v_2 - \sum\nolimits_i \omega_1^i \, \tau_2^i + \sum\nolimits_i \tau_1^i \, \omega_2^i - v_1.$$

Here the indices 1 and 2 denote the pullbacks of the forms to $M \times M$ using the left and the right projection respectively.

We can then define an analogue of the linking number by using formula (2.2). This, however, is in general not an invariant. Let us then introduce

$$P(\gamma, \Sigma) \doteq \sum\nolimits_i \int_K \omega^i \int_\Sigma \tau^i,$$

where γ is a curve and Σ is a surface. It is not difficult to show that, if γ_1 and γ_2 are homologous, then

$$\widetilde{\mathrm{lk}}(\gamma_1, \gamma_2) \doteq \mathrm{lk}(\gamma_1, \gamma_2) - P(\gamma_1, \Sigma)$$

is a link invariant for any surface Σ such that $\partial[\Sigma] = [\gamma_1] - [\gamma_2]$.

Next we consider an embedded loop K and define its self-linking number as in (2.3). We finally have the following

Theorem 3.1. *Theorem 2.1 holds for any connected, compact, closed, oriented* 3*-manifold if* lk *is replaced by* $\widetilde{\mathrm{lk}}$.

Proof. In the proof of (the analogue of) Lemma 2.3 one only has to notice that now

$$\int_{C_{1,1}(S^1, S^1 \times [0,\epsilon])} K_{\mathbf{w}}^* \mathrm{d}\hat{\eta}[\theta] = \int_{K \times \Sigma_{\mathbf{w}}} \left(\sum\nolimits_i \omega_1^i \, \tau_2^i - \sum\nolimits_i \tau_1^i \, \omega_2^i \right) = P(K, \Sigma_{\mathbf{w}}),$$

where $\Sigma_{\mathbf{w}}$ denotes the strip $K + t\mathbf{w}$, $t \in [0, \epsilon]$, i.e., the image of $S^1 \times [0, \epsilon]$ under $K_{\mathbf{w}}$). The rest of this proof as well as the proof of Lemma 2.4 are unchanged. □

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Institut für Mathematik, Universität Zürich–Irchel, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
E-mail address: alberto.cattaneo@math.uzh.ch