NONLOCAL QUASILINEAR PARABOLIC EQUATIONS

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Abstract. We give a survey of the most common approaches to quasilinear parabolic evolution equations, discuss their advantages and drawbacks, and present an entirely new approach based on maximal L_p regularity. Our general results apply, above all, to parabolic initial boundary value problems being nonlocal in time. This is illustrated by indicating their relevance for quasilinear parabolic equations with memory and, in particular, for time regularized versions of the Perona-Malik equation of image processing.

Dedicated to S.M. Nikols'kii on the occasion of his 100th birthday

Introduction

In this paper we discuss a new approach to the abstract quasilinear parabolic equation

$$\dot{u} + A(u)u = F(u) \quad \text{on } (0,T), \qquad u(0) = u^0, \tag{0.1}$$

where T is a fixed positive number. Formulation (0.1) encompasses a great variety of concrete problems, most prominently parabolic initial boundary value problems of the form

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \big(\boldsymbol{a}(u)\nabla u\big) &= \boldsymbol{f}(u) && \text{on } \Omega \times (0,T), \\ \chi \vec{\nu} \cdot \boldsymbol{a}(u)\nabla u + (1-\chi)u &= \chi \boldsymbol{g}(u) && \text{on } \Gamma \times (0,T), \\ u(\cdot,0) &= u^0 && \text{on } \Omega, \end{aligned}\right\} \tag{0.2}$$

where $\chi \in \{0,1\}$, Ω is a bounded smooth domain with boundary Γ, and $\vec{\nu}$ is the exterior normal on Γ. In contrast to standard classical settings we are particularly interested in situations where $\boldsymbol{a}$, $\boldsymbol{f}$, and $\boldsymbol{g}$ are nonlocal functions of u. Of course, the matrix $\boldsymbol{a}(u)$ has to be uniformly positive definite on $\overline{\Omega}$ for every admissible choice of u. It should also be mentioned that (0.2) encompasses systems as well if obvious interpretations are employed.

In the following section we review the standard known approaches to (0.1) and (0.2). Then, in the next section, we briefly discuss concepts of maximal regularity. In Section 3 we present our main new abstract theorem, and in the last section we indicate some of its applications illustrating the strength and novelty of our approach.

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1. Usual approaches

The following, somewhat vague remarks refer predominantly to (0.2) in the simple scalar case where $\boldsymbol{a}$, $\boldsymbol{f}$, and $\boldsymbol{g}$ are local functions.

The perhaps best known approach to (linear) parabolic problems is the

Galerkin approximation method

To the best of our knowledge, it has first been used in the context of nonlinear evolution equations by E. Hopf [24], who employed it to prove the existence of weak solutions to the Navier-Stokes equations. It has then been widely popularized by M.I. Višik and O.A. Ladyzhenskaya [40], O.A. Ladyzhenskaya [26], and others, in particular by J.-L. Lions [30]. In the context of linear problems the latter author developed it to a powerful abstract theory. Extensions to nonlinear problems have since been carried out by many writers (e.g., [13], [42], and numerous research papers), where in most cases only semilinear equations are studied. (Here and in the following, we restrict ourselves to giving easily accessible references, mostly books. This does not mean that we provide a correct historical account of the development. We leave it to the reader to trace the literature for the origins of a specific theory.)

The Galerkin method has the important advantage that

- it is relatively simple.

However, it has a number of serious shortcomings. Namely:

- It is restricted to a Hilbert space setting, which amounts to the weak H^1 setting in case of problem (0.2).
- It can be applied to coercive problems only, thus general systems cannot be handled.
- Due to the Hilbert space setting, strong growth restrictions for the nonlinearities f and g have to be imposed.

Another well known abstract method is based on

Monotone operators and accretive semigroups

In the context of evolution equations this has been developed, in particular, by Ph. Bénilan [10], H. Brezis [11], and M. Crandall [16], and has been refined and extended in numerous papers. Although it gives rather good results, mainly for degenerate problems, its range is severely restricted by the fact that it is crucially based on monotonicity properties and maximum principles. Thus

- it does apply neither to general systems nor to higher order problems,

unless very particular structural conditions are satisfied.

As far as the standard (local) quasilinear parabolic problem (0.2) is concerned, there is the fundamental and deep work based on

A priori estimates and Leray-Schauder continuation techniques

developed at the beginning of the second half of the last century especially by O.A. Ladyzhenskaya, N.N. Ural'ceva, and V.A. Solonnikov, and exposed in their monumental work [27]. More recent extensions and refinements, concerning certain degenerate problems as well, are given in G. Lieberman's book [29]. The results obtained by this method are optimal. However, their shortcomings are that

- this technique is essentially restricted to the scalar case and cannot be applied to general systems,
- it is difficult to treat problems where blow-up occurs since global existence is essentially built in the proofs.

The most general and flexible approach to the abstract parabolic evolution equation (0.1), applying, in particular, to its concrete realization (0.2), is based on

Analytic semigroup techniques

This development has been initiated independently by T. Kato [25] and P.E. Sobolevskii [39]. These authors established mainly the linear theory, being the basis for treating quasilinear problems, and derived preliminary results for quasilinear problems. Analytic semigroups play a decisive role in D. Henry's geometric theory of semilinear parabolic problems [22]. A satisfactory general abstract theory for quasilinear parabolic evolution equations has been established only relatively recently by the author [2], [3], and by A. Lunardi [31], where the latter restricts herself to a Hölder space theory.

The advantages of the approach by analytic semigroup techniques are manifold. In particular:

- It applies to weak as well as classical settings of quasilinear parabolic problems [2], [8].
- It applies to general (Petrowski parabolic) systems [2], [28].
- It allows for a geometric theory of quasilinear parabolic evolution equations in the spirit of the geometric theory of ordinary differential equations [1], [2], [37], [38].
- It is of great flexibility applying to a wide variety of nonstandard models like free and moving boundary value problems [19], equations with dynamic boundary conditions [18], singular Cauchy problems [20], equations with infinitely — even uncountably — many equations [4], etc., problems which are all out of reach of the other methods.

In principle, the approach by analytic semigroup techniques to (0.1) is quite simple and straightforward. It comprises two basic steps, namely:

- First, one acquires a good knowledge of existence, uniqueness, and continuity properties for the nonautonomous linear problem
$$\dot{u} + A(t)u = f(t) \quad \text{on } (0,T), \qquad u(0) = u^0. \tag{1.1}$$
- Second, on the basis of the first step one specifies appropriate classes of functions v to which a prospective solution of (0.1) is likely to belong. Then, denoting by $u(v)$ the unique solution of
$$\dot{u} + A\big(v(t)\big)u = F\big(v(t)\big) \quad \text{on } (0,T), \qquad u(0) = u^0,$$
in this class, it remains to establish a fixed point of the map $v \mapsto u(v)$.

Of course, the preceding outline is drastically oversimplified and there are many difficulties in the realization of this scheme some of which we discuss now.

2. Maximal regularity

The semigroup approach to problem (0.1), outlined above, employs the following basic hypotheses:

$$\left.\begin{array}{l}\bullet\ E,\ E_0, \text{ and } E_1 \text{ are Banach spaces such that } E_1 \hookrightarrow E \hookrightarrow E_0;\\ \bullet\ \big(x \mapsto A(x)\big) \text{ is a locally Lipschitz continuous map}\\ \quad \text{from } E \text{ into } \mathcal{L}(E_1,E_0);\\ \bullet\ -A(x) \text{ generates for each } x\in E \text{ a strongly continuous}\\ \quad \text{analytic semigroup on } E_0;\\ \bullet\ F \text{ is a locally Lipschitz continuous map from } E \text{ into } E_0.\end{array}\right\} \qquad (2.1)$$

As usual, $\hookrightarrow$ denotes continuous injection and $\mathcal{L}(E_1, E_0)$ is the Banach space of all bounded linear operators from E_1 into E_0.

Under these hypotheses, and assuming that E is an interpolation space between E_0 and E_1 (adding some slight technical refinements), it can be shown that (0.1) is well posed and generates a local semiflow on E (cf. [2]). Although this result has numerous deep applications, indicated above, the fact that E has to be an intermediate space between E_0 and E_1 is a minor but, in some cases, unwanted restriction. It is intuitively clear that the optimal setting occurs if $E = E_1$.

The outline in the beginning of the preceding section makes it clear that a deep understanding of the linear problem (1.1) is decisive for the whole approach. Indeed, in order to carry out step 2, that is, to guarantee that the fixed point map $v \mapsto u(v)$ is well defined, it turns out that one has to know that

$$\dot{u} + A(t)u = f(t) \quad \text{on } (0,T), \qquad u(0) = 0, \qquad (2.2)$$

has for each $f : (0,T) \to E$ a unique solution, provided f has appropriate time regularity and the solution is in the same regularity class. More precisely, put $J := [0,T)$ and suppose that

$$\left.\begin{array}{l}\bullet\ X_j(J) \text{ are Banach spaces such that } X_j(J) \hookrightarrow L_1(J,E_j), \quad j=0,1;\\ \bullet\ A \in L_\infty\big(J,\mathcal{L}(E_1,E_0)\big) \text{ and } (u\mapsto Au) \in \mathcal{L}\big(X_1(J),X_0(J)\big),\end{array}\right\} \qquad (2.3)$$

where, of course, $(Au)(t) := A(t)u(t)$ for a.a. $t \in J$. Then A is said to possess **maximal regularity with respect to** $\big(X_1(J), X_0(J)\big)$ if (2.2) has for each $f \in X_0(J)$ a unique (distributional) solution $u \in X_1(J)$ such that $\dot{u} \in X_0(J)$. Note that, due to the second part of (2.3), this means that all three terms in (2.2) possess the same regularity, that is, $\dot{u}, Au, f \in X_0(J)$.

In general, it is unfortunately not true that maximal regularity holds, even if A and $X_j(J)$ are 'rather nice'. However, there are some important cases where it is satisfied. They are described in the following.

Maximal continuous regularity

The most natural choice is, of course,

$$X_j(J) := C(J, E_j), \qquad j = 0, 1.$$

In this case we obtain classical solutions

$$u \in C(J, E_1) \cap C^1(J, E_0)$$

whenever

$$f \in C(J, E_0).$$

Clearly, we have to assume that $t \mapsto A(t)$ is also continuous and not merely bounded and measurable. Sadly, in this simple and natural setting maximal regularity is only true under severe restrictions on the Banach spaces E_1 and E_0 which make concrete applications rather difficult. In particular, it does never hold if $E_1 \neq E_0$ and E_0 is reflexive. Thus it rules out so important choices in the theory of partial differential equations as $L_q(\Omega)$ or $H_q^{-1}(\Omega)$ for $1 < q < \infty$. On the other hand, (little) Nikols'kii or Hölder spaces are admissible. We refer to [3, Section III.3] for details, and to [15] for the corresponding nonlinear theory pertaining to (0.1), where references to concrete applications can be found as well.

Maximal Hölder regularity

Restrictions on the basic Banach spaces E_0 and E_1 can be completely dropped if we consider Hölder continuous solutions, that is, if we set

$$X_j(J) := C^{\alpha}(J, E_j), \qquad j = 0, 1,$$

for some $\alpha \in (0,1)$. (In fact, more complicated subspaces of $C^{\alpha}(J, E_j)$ have to be chosen, but we refrain from giving precise details.) In this case it is possible to develop a satisfactory theory even for fully nonlinear equations $\dot{u} = \Phi(u)$, provided it is assumed that $\Phi \in C^1(E_1, E_0)$ and $\Phi'(e)$ generates for each $e \in E_1$ a strongly continuous analytic semigroup on E_0. This approach has been carried through by G. Da Prato and A. Lunardi, in particular, and is exposed in [31]. However, it turns out that it is not too well adapted to the quasilinear case where $\Phi(u) = -A(u)u + F(u)$. Furthermore, due to the high regularity involved it is not easy to apply it to concrete equations or to get global existence results since this setting requires a priori bounds in rather strong norms.

Maximal L_p regularity

From the modern viewpoint of partial differential equations an L_p theory is most desirable. More precisely, suppose that $1 < p < \infty$ and choose

$$X_j(J) := L_p(J, E_j), \qquad j = 0, 1.$$

Thus of interest are now $\mathcal{H}_p^1$ **solutions**, where

$$\mathcal{H}_p^1 := \mathcal{H}_p^1\big(J, (E_1, E_0)\big) := L_p(J, E_1) \cap H_p^1\big(\mathring{J}, E_0\big)$$

and $H_p^1\big(\mathring{J}, E_0\big)$ is the Sobolev space of E_0 valued distributions u on $\mathring{J}$ such that u and $\dot{u}$ belong to $X_0(J) = L_p(J, E_0)$. In this case maximal regularity with respect to $\big(X_1(J), X_0(J)\big)$ is called **maximal L_p regularity on J with respect to** (E_1, E_0). In order to guarantee that maximal L_p regularity holds one has to put assumptions on A as well as on the underlying Banach spaces E_1 and E_0. In the autonomous case a complete characterization, due to L. Weis [41], is based on the concept of R-boundedness, a new Mikhlin type Fourier multiplier theorem for operator valued symbols, and on results from the theory of Banach spaces. We refrain from giving details here and refer instead to the comprehensive presentation [17]. May it suffice to say somewhat informally that

> maximal L_p regularity holds for 'nice' spaces like $L_q(\Omega)$, $H_q^{-1}(\Omega)$, and for 'good' operators, like those induced in these spaces by elliptic differential operators.

Finally, it should be mentioned that in all cases of maximal regularity discussed above, it is necessary that $-A(t)$ generates for each $t \in J$ an analytic semigroup on E_0.

3. Quasilinear parabolic problems

Now we return to problem (0.1) and discuss its well posedness in the framework of maximal L_p regularity. Thus we suppose that

- E_0 and E_1 are Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$,

where $\overset{d}{\hookrightarrow}$ means: dense injection. Having fixed (E_1, E_0), we simply write

$$\mathcal{H}_p^1(J) := \mathcal{H}_p^1\big(J, (E_1, E_0)\big).$$

We also set

$$E := (E_0, E_1)_{1/p',p},$$

where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor of exponent $\theta \in (0,1)$ and parameter p. Thus E is the trace space of $\mathcal{H}_p^1(J)$, and

$$E_1 \overset{d}{\hookrightarrow} E \overset{d}{\hookrightarrow} E_0. \tag{3.1}$$

It is known that

$$\mathcal{H}_p^1(J) \hookrightarrow C\big(\overline{J}, E\big)$$

(e.g., [3, Theorem III.4.10.2]).

Due to (3.1) it is possible and not too difficult to carry out the semigroup approach based on assumption (2.1) under the additional hypothesis

- $A(x)$ possesses for each $x \in E$ maximal L_p regularity.

Indeed, this has been done by Ph. Clément and Sh. Li [14] in a concrete setting, and by J. Prüss [36] in an abstract framework. The proofs in this restricted setting are much simpler than the ones in the general case [2] not imposing maximal regularity hypotheses. However, the improvement for applications to concrete quasilinear parabolic problems is marginal.

Thus there arises the question:

what is the optimal setting for a maximal L_p regularity theory for problem (0.1)*?*

Clearly, seeking solutions in $\mathcal{H}_p^1(J)$, a minimal requirement is that

- A and F are defined on $\mathcal{H}_p^1(J)$ and u^0 belongs to E. (3.2)

This assumption is fundamentally different from the standard hypotheses (2.1) where A and F are defined on the Banach space E and, given a function $u : J \to E$, the map

$$\big(A(u), F(u)\big) : J \to \mathcal{L}(E_1, E_0) \times E_0$$

is defined via

$$\big(A(u), F(u)\big)(t) := \big(A(u(t)), F(u(t))\big), \qquad t \in J.$$

In other words, A and F are *local* operators (with respect to t) in the standard situation, whereas A and F can be *nonlocal* maps in case (3.2) holds.

Since we are concerned with evolution equations, in addition to (3.2) we need a further assumption guaranteeing that any time no information from the future is used. Thus we require the nonlocal maps to possess the **Volterra property**:

- for every $S \in (0,T)$ and $u \in \mathcal{H}_p^1(J)$,
  $\big(A(u), F(u)\big) \,\big|\, J_S = \big(A(u \,|\, J_S), F(u \,|\, J_S)\big)$.

In other words: The restriction of $\big(A(u), F(u)\big)$ to any subinterval J_S depends on the restriction of u to the same interval only.

After these preparations we can formulate the following existence, uniqueness, and continuity result.

3.1 Theorem *Suppose that* $\mathsf{J} := J_{\mathsf{T}} := [0,\mathsf{T})$ *for some* $\mathsf{T} \in (0,\infty)$*, that* $p \in (1,\infty)$*, and that* E_0 *and* E_1 *are Banach spaces such that* $E_1 \overset{d}{\hookrightarrow} E_0$*. Also suppose that*

(i) *A is a Volterra map from $\mathcal{H}_p^1(\mathsf{J})$ into $L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$ being uniformly Lipschitz continuous on bounded sets.*

(ii) *For each $u \in \mathcal{H}_p^1(\mathsf{J})$ and each $T \in \mathsf{J}$ the linear map $A(u) \,|\, J_T$ possesses maximal L_p regularity on J_T with respect to (E_1, E_0).*

(iii) *F is a Volterra map from $\mathcal{H}_p^1(\mathsf{J})$ into $L_p(\mathsf{J}, E_0)$ and there exists $r \in (p,\infty]$ such that $F - F(0)$ is uniformly Lipschitz continuous on bounded subsets of $\mathcal{H}_p^1(\mathsf{J})$ with values in $L_r(\mathsf{J}, E_0)$.*

(iv) $u^0 \in (E_0, E_1)_{1/p',p}$.

Then:

(1) *There exist a maximal $T^* \in (0,\mathsf{T}]$ and a unique solution u^* on J_{T^*} of* (0.1) *such that $u^* \in \mathcal{H}_p^1(J_T)$ for $0 < T < T^*$, the maximal $\mathcal{H}_p^1$ solution.*

(2) *If $T^* < \mathsf{T}$, then $u^* \notin \mathcal{H}_p^1(J_{T^*})$.*

(3) *Suppose that $\big((A_j, F_j, u_j^0)\big)$ is a sequence satisfying hypotheses* (i)–(iv) *and converging towards (A, F, u^0). Let u_j^* on $(0, T_j^*)$ be the maximal $\mathcal{H}_p^1$ solution of*

$$\dot{u} + A_j(u)u = F_j(u) \quad \text{on } (0,\mathsf{T}), \qquad u(0) = u_j^0.$$

If $u^* \in \mathcal{H}_p^1(\mathsf{J})$, then set $S := \mathsf{T}$. Otherwise, fix any $S \in (0, T^*)$. Then there exists $N \in \mathbb{N}$ such that $T_j^* > S$ for $j > N$ and the sequence (u_j^*) converges in $\mathcal{H}_p^1(J_S)$ towards u^*.

3.2 Remarks **(a)** In order to make the sense of convergence of the sequence data $\big((A_j, F_j, u_j^0)\big)$ precise we put

$$[g]_{X,\mathcal{B}} := \sup_{\substack{v,w \in \mathcal{B} \\ v \neq w}} \frac{\|g(v) - g(w)\|_X}{\|v - w\|_{\mathcal{H}_p^1(\mathsf{J})}},$$

where $\mathcal{B}$ is a bounded subset of $\mathcal{H}_p^1(\mathsf{J})$ and $X := L_\infty\big(\mathsf{J}, (E_1, E_0)\big)$ or $X := L_r(\mathsf{J}, E_0)$. Then $\big((A_j, F_j, u_j^0)\big) \to (A, F, u^0)$ means that $A_j(0) \to A(0)$ in $L_\infty\big(\mathsf{J}, \mathcal{L}(E_1, E_0)\big)$ and $[A_j - A]_{L_\infty(\mathsf{J}, \mathcal{L}(E_1,E_0)),\mathcal{B}} \to 0$, that $F_j(0) \to F(0)$ in $L_p(\mathsf{J}, E_0)$ and

$$[F_j - F]_{L_r(\mathsf{J}, E_0),\mathcal{B}} \to 0$$

for every bounded subset $\mathcal{B}$ of $\mathcal{H}_p^1(\mathsf{J})$, as well as $u_j^0 \to u^0$ in $(E_0, E_1)_{1/p',p}$.

(b) The proof of Theorem 3.1 is given in [6] (see Theorems 2.1 an 3.1 therein). In that paper it is only assumed that $A(u)$ possesses for every $u \in \mathcal{H}_p^1(\mathsf{J})$ the property of maximal L_p regularity on J with respect to (E_1, E_0). The assumption that this is true for every subinterval J_T of J is missing. However, the latter is needed since the proofs in [6] use [5, Lemma 4.1], which is not complete. In fact, given any $B \in L_\infty\big(\mathsf{J}, \mathcal{L}(E_1, E_0)\big)$ possessing maximal L_p regularity on J, the proof of that lemma shows that the linear problem

$$\dot u + Bu = f \quad \text{on } (0,T), \qquad u(0) = 0$$

has for each $T \in (0, \mathsf{T})$ and each $f \in L_p(J_T, E_0)$ a solution $u \in \mathcal{H}_p^1(J_T)$, but uniqueness remains open. (The importance of [5, Lemma 4.1] lies in the uniform estimates given there, however.)

(c) It should be noted that hypothesis (ii) can be weakened to:

- $A(u)$ possesses for each $u \in \mathcal{H}_p^1(\mathsf{J})$ maximal L_p regularity on J with respect to (E_1, E_0), and zero is for each $T \in (0, \mathsf{T})$ the only solution in $\mathcal{H}_p^1(J_T)$ of
$$\dot v + A(u)v = 0 \quad \text{on } (0,T), \qquad v(0) = 0. \tag{3.3}$$

Proof. Clearly, assumption (ii) implies (3.3). Conversely, it follows from (3.3) and (the proof of) [5, Lemma 4.1] that, given any $T \in (0, \mathsf{T})$ and any $f \in L_p(J_T, E_0)$, the linear problem

$$\dot v + A(u)v = f \quad \text{on } (0,T), \qquad v(0) = 0$$

possesses a solution $v \in \mathcal{H}_p^1(J_T)$. □

4. Model applications

Theorem 3.1 has numerous applications to a wide variety of problems. For simplicity, we restrict ourselves here to indicate briefly some of them pertaining to parabolic problems being nonlocal in time.

Problems with memory

First we consider reaction-diffusion type equations of the form

$$\left.\begin{aligned} \partial_t\big(e(u)\big) + \operatorname{div} \vec{\jmath}(u) &= f(u) && \text{on } \Omega \times (0,\infty), \\ \chi \vec{\nu} \cdot \vec{\jmath}(u) + (1-\chi)u &= \chi g(u) && \text{on } \Gamma \times (0,\infty), \\ u(\cdot, 0) &= u^0 && \text{on } \Omega, \\ u &= \overline{u} && \text{on } \Omega \times (-\infty, 0). \end{aligned}\right\} \tag{4.1}$$

We assume that

$$\alpha, \beta \in L_{s,\mathrm{loc}}(\mathbb{R}^+) \qquad \text{for some } s > 1$$

and

$$a \in C^{2\text{-}}\big(\mathbb{R}, (0,\infty)\big).$$

As usual, $C^{(k+1)\text{-}}$ is the space of all C^k functions whose k^{th} derivatives are locally Lipschitz continuous. We also suppose that

$$e(u)(\cdot, t) := u + \int_{-\infty}^t \alpha(t-\tau)u(\tau)\,d\tau =: u + \alpha *_t u$$

and that either

$$\vec{\jmath}(u) := -a(\beta *_t u)\nabla u$$

or

$$\vec{\jmath}(u) := -a(u)\nabla u - \beta *_t \big(b(u)\nabla u\big) \tag{4.2}$$

or

$$\vec{\jmath}(u) := -a(u)\nabla u - b\big(u(t-\tau)\big)\nabla u(t-\tau),$$

where $\tau > 0$ is a fixed number and $b \in C^{2-}(\mathbb{R})$. Then (4.1) is a quasilinear parabolic delay initial boundary value problem. Systems of this type occur in various applications, for example, in the theory of heat conducting rigid bodies with memory. In that case e is the initial energy, $\vec{\jmath}$ the heat flux vector, and u the temperature (see e.g., [21], [32], [34]). Other instances of problem (4.1) come up in the mathematical theory of climate models (cf. [23]), in mathematical biology, and in control theory, for example. It will be shown in another publication how Theorem 3.1 can be applied to (4.1) resulting in considerable improvements and generalizations of known results as well as opening up ways to investigate problems having been so far out of reach.

Time regularization of ill posed problems

Suppose that

$$a \in C^{2-}\big(\mathbb{R}^+, (0,\infty)\big), \quad f \in C^{1-}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \tag{4.3}$$

and consider the model problem

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \big(a(|\nabla u|^2)\nabla u\big) &= f(x,u,\nabla u) && \text{on } \Omega \times (0,\infty), \\ \chi\partial_{\vec{\nu}} u + (1-\chi)u &= 0 && \text{on } \Gamma \times (0,\infty), \\ u(\cdot,0) &= u^0 && \text{on } \Omega. \end{aligned}\right\} \tag{4.4}$$

Assume that $S := S(t)$ is a regular level set of an appropriately smooth solution u at some time instant $t \geq 0$. Then, near S,

$$\begin{aligned} \nabla \cdot \big(a(|\nabla u|^2)\nabla u\big) &= a(|\nabla u|^2)\Delta u + 2a'(|\nabla u|^2)D^2 u \nabla u \cdot \nabla u \\ &= a(|\nabla u|^2)\Delta_S u + b(|\nabla u|^2)\partial_\xi^2 u, \end{aligned} \tag{4.5}$$

where $D^2 u$ is the Hessian of u, Δ_S is the Laplace-Beltrami operator of S, the vector field $\xi := -\nabla u/|\nabla u|$ gives the direction of steepest descent of u, and

$$b(s) := a(s) + 2sa'(s), \qquad s \geq 0.$$

Thus it may happen that $b(s) < 0$ for s belonging to some interval I of $(0,\infty)$. Consequently, if $|\nabla u(x)| \in I$ then (4.4) is a backward parabolic problem perpendicular to S, hence ill posed. This is the case, in particular, for

$$a(s) := 1/(1+s), \qquad s \geq 0,$$

where $b(s) < 0$ for $s > 1$. With this choice of a, with $f = 0$, and with the Neumann boundary condition, (4.4) is the Perona-Malik equation [35], well known in image processing and giving surprisingly good numerical results. However, there is no sound mathematical theory for this equation providing a theoretical justification of those results. For this reason various modifications of it have been proposed, most notably the space regularization method of F. Catté, P.-L. Lions, J.-M. Morel, and T. Coll [12]. These authors — and afterwards practically everybody considering modifications of the original Perona-Malik model — replace $|\nabla u|^2$ in the argument of a by $|\nabla u_\sigma|^2$, where u_σ denotes convolution in the space variable with a Gaussian of variance $\sigma > 0$. In this case it is not difficult to see that (4.4) is replaced by a well posed problem which is close to the equation of P. Perona and J. Malik. Unfortunately, space realization results in smoothing of sharp edges and thus produces an unwanted blurring of pictures (see [7] for numerical illustrations and a more detailed discussion).

In [7] we have proposed a time regularization method for the Perona-Malik and related ill posed problems, justifying, in particular, a model introduced by M. Nitzberg and T. Shiota [33] (see [9] for a Hölder space theory of the latter system). Our model does not smear sharp edges, is rather flexible, and gives extremely good numerical results. Its mathematical justification is a consequence of the following theorem pertaining to the time delayed version of (4.4):

$$\left.\begin{aligned} \partial_t u - \nabla\cdot\big(a(\theta *_t |\nabla u|^2)\nabla u\big) &= f(x,u,\nabla u) && \text{on } \Omega\times(0,\infty),\\ \chi\partial_{\vec{\nu}} u + (1-\chi)u &= 0 && \text{on } \Gamma\times(0,\infty),\\ u &= u^0 && \text{on } \Omega\times(-S,0], \end{aligned}\right\} \tag{4.6}$$

where

$$\theta\in L_s\big((0,S)\big) \text{ for some } s\in(1,\infty] \text{ and } S\in(0,\infty). \tag{4.7}$$

4.1 Theorem *Suppose that $p,q\in(1,\infty)$ satisfy $2/p+n/q<1$ and that (4.3) and (4.7) hold. Then, given $u^0\in H_q^2$ satisfying the boundary conditions, there exists a maximal $T^*\in(0,\infty]$ such that (4.6) possesses a unique solution u^* belonging to*

$$\mathcal{H}^{2,1}_{q,p}\big(\Omega\times(0,T)\big) := L_p\big((0,T),H_q^2\big)\cap H_p^1\big((0,T),L_q\big)$$

for every $T\in(0,T^*)$. If $T^*<\infty$, then $u^*\notin\mathcal{H}^{2,1}_{q,p}\big(\Omega\times(0,T^*)\big)$. If f is linearly bounded in u and ∇u and $\operatorname{supp}(\theta)\subset(\sigma,S)$ for some $\sigma\in(0,S)$, then u^* exists globally, that is, $T^*=\infty$.

Proof. The first two assertions follow from the more general Theorem 4.1 of [7], whose proof is based on Theorem 3.1. For the last claim we refer to Remark 5.1(b) of [7]. □

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