Maximal regularity and quasilinear evolution equations
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1. Abstract theory

Let E_0 and E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$, set $J := J_{T_0} := [0, T_0)$ for some fixed positive T_0, and suppose that $1 < p < \infty$. Put

$$\mathbb{W}_p^1(J) := \mathbb{W}_p^1\big(J, (E_1, E_0)\big) := L_p(J, E_1) \cap W_p^1(\mathring{J}, E_0).$$

Then

$$B \in L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$$

possesses the property of **maximal** L_p **regularity** on J with respect to (E_1, E_0) if the map

$$\mathbb{W}_p^1(J) \to L_p(J, E_0) \times E, \quad u \mapsto \big(\dot{u} + Bu, u(0)\big)$$

is a bounded isomorphism, where E is the real interpolation space $(E_0, E_1)_{1/p',p}$ and the overdot denotes the distributional derivative on $\mathring{J}$. Since (e.g., [1, Theorem III.4.10.2])

$$\mathbb{W}_p^1(J) \hookrightarrow C(\overline{J}, E), \tag{1}$$

$u(0)$ is well defined. The set of all such maps B is denoted by

$$\mathcal{MR}_p(J) := \mathcal{MR}_p\big(J, (E_1, E_0)\big).$$

We also write $\mathcal{MR} := \mathcal{MR}(E_1, E_0)$ for the set of all $C \in \mathcal{L}(E_1, E_0)$ such that the constant map $t \mapsto C$ belongs to $\mathcal{MR}_p(J)$. Since the latter property is independent of p and the (bounded) interval (e.g., [3]), this notation is justified.

We are interested in quasilinear evolution equations of the form

$$\dot{u} + A(u)u = f(u) \text{ on } \mathring{J}, \qquad u(0) = u^0. \tag{2}$$

By a solution on J_T, where $0 < T \leq T_0$, we mean a $u \in \mathbb{W}_{p,\mathrm{loc}}^1(J_T)$ satisfying (2) in the sense of distributions on $\mathring{J}_T$ or, equivalently, a.e. on J_T.

Henceforth, we write C^{1-} for spaces of locally Lipschitz continuous maps, and $\mathcal{C}^{1-}$ if the Lipschitz continuity is uniform on bounded subsets of the domain (which is always the case if the latter is finite dimensional).

Due to (1) it is natural to assume that

$$(A, f) \in \mathcal{C}^{1-}\big(E, \mathcal{L}(E_1, E_0) \times E\big). \tag{3}$$

Indeed, this type of assumption has been used in practically all investigations of (2). In particular, Clément and Li [11] were the first to study (2) — in a concrete setting — by imposing the maximal regularity hypothesis that $A(e) \in \mathcal{MR}$ for each $e \in E$. Recently, Prüss [13] has extended this method to a nonautonomous abstract setting.

An assumption like (3) uses only part of the information contained in the statement: $u \in \mathbb{W}_p^1(J)$. Consequently, it imposes stronger restrictions on (A, f) than the hypothesis that this map be defined on $\mathbb{W}_p^1(J)$, which, after all, is the space in which solutions live.

Considering a map

$$(A,f):\ \mathbb{W}_p^1(J)\to L_\infty\bigl(J,\mathcal{L}(E_1,E_0)\bigr)\times L_p(J,E_0)$$

we say that it possesses the Volterra property if, given $u\in\mathbb{W}_p^1(J)$ and $0<T<T_0$, the restriction of $(A,f)(u)$ to J_T depends on $u\,|\,J_T$ only. Now we can formulate our main result, whose proof is found in [2].

Theorem *Suppose that*

- $A\in\mathcal{C}^{1-}\bigl(\mathbb{W}_p^1(J),\mathcal{MR}_p(J)\bigr)$;
- $f-f(0)\in\mathcal{C}^{1-}\bigl(\mathbb{W}_p^1(J),L_r(J,E_0)\bigr)$ *for some* $r\in(p,\infty]$, *and* $f(0)\in L_p(J,E_0)$;
- (A,f) *possesses the Volterra property;*
- $u^0\in E$.

Then:

- *there exist a maximal* $T^*\in(0,T_0]$ *and a unique solution* u *of* (2) *on* $J^*:=J_{T^*}$;
- *the map* $(A,f,u^0)\mapsto u$ *is locally Lipschitz continuous with respect to the natural Fréchet topologies of the spaces occurring above;*
- *if* $u\in\mathbb{W}_p^1(J^*)$, *then* $J^*=J$, *that is,* u *is global.*

The following proposition gives two important sufficient conditions for maximal regularity in the nonautonomous case.

Proposition (i) *If* $B\in C(\overline{J},\mathcal{MR})$, *then* $B\in\mathcal{MR}_p(J)$.

(ii) *Let* $V\overset{d}{\hookrightarrow}H\overset{d}{\hookrightarrow}V'$ *be real Hilbert spaces and let* $B\in L_\infty\bigl(J,\mathcal{L}(V,V')\bigr)$ *be such that there exist constants* $\alpha>0$ *and* $\beta\geq 0$ *with*

$$\bigl\langle v,B(t)v\bigr\rangle+\beta\,\|v\|_H^2\geq\alpha\,\|v\|_V^2,\qquad a.a.\ t\in J,\quad v\in V,$$

where $\langle\cdot,\cdot\rangle:\ V\times V'\to\mathbb{R}$ *is the duality pairing. Then* $B\in\mathcal{MR}_2\bigl(J,(V,V')\bigr)$.

Proof (i) has been shown in [14] by constructing an evolution family. A simple direct proof is given in [3].

(ii) is a consequence of the well known Galerkin approach to evolution equations in a variational setting, essentially due to J.-L. Lions (see [2] for details). ■

2. Applications

To give an idea of the scope of the Theorem we consider two model problems. For this we suppose that

- Ω is a bounded Lipschitz domain;
- $a\in C^{1-}(\mathbb{R},\mathbb{R})$ and $a(\xi)\geq\alpha>0$ for $\xi\in\mathbb{R}$.

We also set $Q:=\Omega\times J$ and $\Sigma:=\partial\Omega\times J$.

Example 1 (nonlocal problems) Let $a_0, m \in L_\infty(\Omega)$ and $1 \le \lambda < 1 + 4/n$. Denote by Ω' a measurable subset of Ω. Then the nonlocal parabolic problem

$$\begin{aligned} \partial_t u - \nabla \cdot \bigl(a\bigl(m \textstyle\int_{\Omega'} u(x,\cdot)\,dx\bigr)\nabla u\bigr) &= a_0\,|u|^{\lambda-1}\,u + f_0 && \text{on } Q,\\ u &= 0 && \text{on } \Sigma,\\ u(\cdot,0) &= u^0 && \text{on } \Omega, \end{aligned}$$

has for each $f_0 \in L_2(Q)$ and $u^0 \in L_2(\Omega)$ a unique maximal weak solution u. If f_0 and u^0 are positive, then so is u. It is global if a is bounded and $a_0 \le 0$. ∎

By a maximal weak solution we mean, of course, a

$$u \in \mathbb{W}^1_{2,\mathrm{loc}}\bigl(J^*, \bigl(\mathring{H}^1(\Omega), H^{-1}(\Omega)\bigr)\bigr)$$

satisfying $u(0) = u^0$ and

$$\langle v, \dot u\rangle + \bigl\langle \nabla v, a\bigl(m \textstyle\int_{\Omega'} u(x,\cdot)\,dx\bigr)\nabla u\bigr\rangle = \bigl\langle v, a_0\,|u|^{\lambda-1}\,u + f_0\bigr\rangle$$

a.e. on J^* and for every $v \in \mathcal{D}(\Omega)$.

We mention that an application of the results in [13], based on hypothesis (3), would require $\lambda = 1$.

Problems of this type have been intensively studied by M. Chipot and coworkers (cf. [4]–[10] and [15]–[18]). More precisely, in those papers the differential equations are either of the form

$$\partial_t u - a\bigl(\langle v, u\rangle\bigr)\Delta u = f_0,$$

where $v \in L_2(\Omega)$, or they are semilinear with nonlocal lower order terms. (The Laplace operator can be replaced by a general second order elliptic operator.) It is crucial that $a\bigl(\langle v, u(\cdot,t)\rangle\bigr)$ is a pure function of t, that is, independent of $x \in \Omega$. The proofs, except the ones in [18], rely on Schauder's fixed point theorem and are completely different from our approach.

Example 2 (equations with memory) Assume that Ω has a C^2 boundary, that $b, f \in C^{1-}(\mathbb{R},\mathbb{R})$, that $k \in L_\rho(\mathbb{R}^+,\mathbb{R})$ for some $\rho > 1$, and μ is a bounded Radon measure on $[0,\infty)$ with $\operatorname{supp}\mu \subset [0,S)$ for some $0 < S \le \infty$. Also suppose that $2/p + n/q < 1$. Then

$$\begin{aligned} \partial_t u - \nabla \cdot \bigl(a(\mu * u)\nabla u\bigr) + k * \bigl(\nabla \cdot (b(u)\nabla u)\bigr) &= f(u) + f_0 && \text{on } Q,\\ u &= 0 && \text{on } \Sigma,\\ u &= \overline{u} && \text{on } \Omega \times (-S, 0], \end{aligned}$$

has for each $f_0 \in L_p\bigl(J, H_q^{-1}(\Omega)\bigr)$ and each

$$\overline{u} \in \mathbb{W}^1_p\bigl((-S,0), \bigl(\mathring{H}^1(\Omega), H^{-1}(\Omega)\bigr)\bigr)$$

a unique maximal weak solution

$$u \in \mathbb{W}^1_{p,\mathrm{loc}}\bigl((-S,T^*), \bigl(\mathring{H}^1_q(\Omega), H^{-1}_q(\Omega)\bigr)\bigr). \tag{4}$$

If there exists $r > 0$ such that $\operatorname{supp}\mu \subset [r,S)$, then the unique maximal weak solution in (4) of

$$\partial_t u - \nabla \cdot \bigl(a(\mu * u)\nabla u\bigr) = f_0 \quad \text{on } Q, \qquad u = 0 \quad \text{on } \Sigma$$

with $u\,|\,(-S,0] = \overline{u}$ is global. ■

Choosing for μ the Dirac measure supported on $\{r\}$ for some $r > 0$, it follows that the retarded quasilinear parabolic problem

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a\big(u(t-r)\big)\nabla u\big) &= f_0 \quad \text{on } Q, \\ u &= 0 \quad \text{on } \Sigma, \end{aligned}$$

has for each $f_0 \in L_p\big(J, H_q^{-1}(\Omega)\big)$ and each $\overline{u} \in \mathbb{W}_p^1\big((-S,0], \big(\mathring{H}_q^1(\Omega), H_q^{-1}(\Omega)\big)\big)$, where $S > r$, a unique global weak solution in (4) (with $T^* = T$).

It should be remarked that problems like the one of Example 2 cannot be treated at all by theorems invoking hypotheses of type (3).

There is a large literature on parabolic equations involving delays and memory terms. However, most of it concerns semilinear equations. Very little seems to be known about an L_p theory for quasilinear equations with memory terms in the top order part (see [12] and the references therein, and [19]). In fact, we do not know of any result for quasilinear equations in which (nondistributed) delay terms occur within the diffusion matrix.

For proofs of the above facts and many more examples we refer to [2].

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