

Algebraic Coding Theory e-Summer School - ACT21  
June 9, 2021

# Analysis of Low-Density Parity-Check Codes over Finite Integer Rings for the Lee Channel

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joint work with Hannes Bartz, Gianluigi Liva  
and Joachim Rosenthal



Knowledge for Tomorrow

# Outline

- 1 Introduction
- 2 The Lee Channel
- 3 LDPC Codes: Performance in the Lee Channel



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# Linear Block Codes

Let  $F_q$  be a finite field of order  $q$  and let  $n$  be a positive integer. We will denote by  $Z_q$  the ring of integers modulo  $q$ .

## Definition [Linear Code]

An  $[n, k]_q$ -linear code  $C \subseteq F_q^n$  is a  $k$ -dimensional subspace of  $F_q^n$ . The elements of  $C$  are called codewords.



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### Definition [Hamming Weight/Distance]

For any two codewords  $x, y \in C$  we define

the *Hamming weight* of  $x$ ,  $\text{wt}_H(x) = |\{i \in \{1, \dots, n\} \mid x_i \neq 0\}|$

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An  $[n, k]_q$ -linear code  $C$  can be represented by an  $(n - k) \times n$  matrix  $H$  satisfying

$$C = \ker(H).$$

We call  $H$  a *parity-check matrix* of  $C$ .



# LDPC Codes over Finite Integer Rings

According to Sridhara and Fuja

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An  $[n, k]_q$  LDPC code over  $Z_q$  is defined by a sparse parity-check matrix  $H$ , whose nonzero entries lie in the set of units  $Z_q^\times$ .



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VN  $v_j$  is connected to CN  $c_i$  if and only if  $h_{ij} \neq 0$ .





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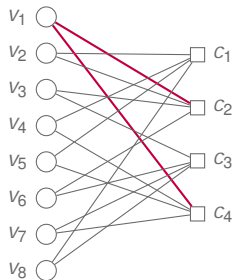
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$$H = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 & 0 & 0 & 1 \\ \mathbf{1} & 2 & 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 3 & 1 \\ \mathbf{1} & 0 & 0 & 1 & 4 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{Z}_5^{4 \times 8}$$



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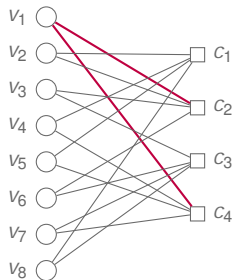
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An LDPC code is  $(k, \ell)$ -regular, if every VN connects to  $k$  CNs and every CN connects to  $\ell$  VNs, for some fixed positive integer  $k$  and  $\ell$ .



## The Lee Metric

### Definition [Lee weight]

For any integer  $a \in \mathbb{Z}_q$  we define its *Lee weight* as

$$\text{wt}_L(a) := \min(a, q - a). \quad (1)$$

The Lee weight of a vector  $x \in \mathbb{Z}_q^n$  is the sum of the Lee weights of its entries, i.e.,

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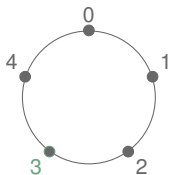
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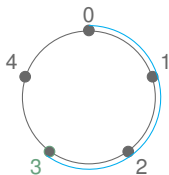
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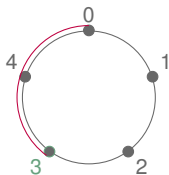
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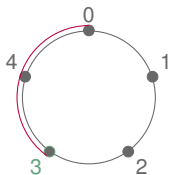
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$$\Rightarrow \text{wt}_L(3) = 2$$





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For every  $a \in \mathbb{Z}_q$  it holds:

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If  $q \geq 2, 3q$ , the Lee weight is equivalent to the Hamming weight.



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### Definition [Lee Distance]

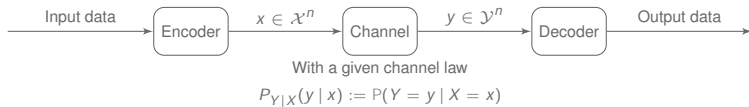
The Lee distance of two scalars  $a, b \in \mathbb{Z}_q$  is  $d_L(a, b) := \text{wt}_L(a - b)$ . The Lee distance between two vectors  $x, y \in \mathbb{Z}_q^n$  is

$$d_L(x, y) = \sum_{i=1}^n d_L(x_i, y_i).$$



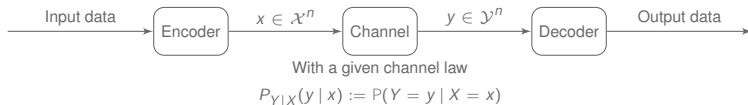
## Channel Coding

Let  $X$  and  $Y$  the input and output alphabet of the channel, respectively.



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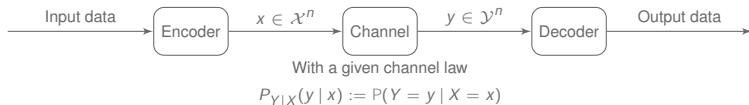
A channel is called *discrete memoryless*, if the input and output alphabets are discrete, finite sets and the output  $Y = y$  at time  $t$  only depends on the input  $X = x$  at that time  $t$ , i.e.,

$$P(Y_1 = y_1, \dots, Y_n = y_n | X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(Y_i = y_i | X_i = x_i)$$



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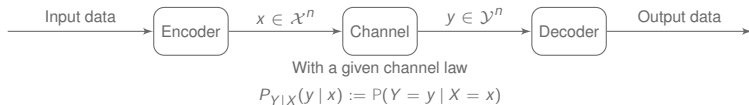
**Example:**  $q$ -ary Symmetric Channel

Let  $x \in \mathbb{Z}_q^n$  sent and  $y \in \mathbb{Z}_q^n$  received.



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### Example: $q$ -ary Symmetric Channel

Let  $x \in \mathcal{Z}_q^n$  sent and  $y \in \mathcal{Z}_q^n$  received. Then  $P_{Y_i|X_i}(y_i | x_i) := \begin{cases} 1 - \epsilon & \text{if } y_i = x_i, \\ \frac{\epsilon}{q-1} & \text{else.} \end{cases} \quad \forall i.$





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## The Lee Channel

Define the “Lee Channel” over  $Z_q$  as proposed by Chiang and Wolf:

$$p_i := P(ij0) = P(ij0), \text{ for } i = 0, \dots, bq/2c. \quad (3)$$

Due to symmetry:  $P(ijj) = P(i - j \bmod qj0)$



## The Lee Channel

Define the “Lee Channel” over  $Z_q$  as proposed by Chiang and Wolf:

$$p_i := P(ij0) = P(-ij0), \text{ for } i = 0, \dots, bq/2c. \quad (3)$$

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### Theorem (Chiang and Wolf)

*The channel described in (3) is strictly matched to the Lee metric for maximum likelihood decoding if and only if the following two properties hold.*

$$p_0 > p_1 \quad \text{and} \quad p_i = \frac{p_1^i}{p_0^{i-1}} \quad \text{for all } i = 2, \dots, bq/2c.$$



# The Lee Channel

For  $y, x, e \in \mathbb{Z}_q$ , consider a discrete memoryless channel (DMC)

$$\underset{\text{channel output}}{y} = \underset{\text{channel input}}{x} + \underset{\text{additive error term}}{e} \quad (4)$$



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The channel law is given by

$$P(Y = y | X = x) =: P_{Y|X}(y|x) = \frac{1}{Z} \exp(-\lambda d_L(x, y)), \quad (5)$$

where  $Z := \sum_{e=0}^{q-1} \exp(-\lambda \text{wt}_L(e))$  and  $\lambda > 0$ .

### Note:

The channel defined in (5) is the DMC matched to the Lee metric.

The conditional distribution (5) arises (in the limit of large  $n$ ) as the marginal distribution of a channel.



## The Constant-Weight Lee Channel

Let  $y, x, e \in \mathbb{Z}_q^n$ , where  $\text{wt}_L(e) = t$  for some fixed positive integer  $t$ . Consider again

$$y = x + e.$$



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**Note:** The error vector  $e$  is chosen uniformly at random from the set of all length- $n$  vectors of Lee weight  $t$ :

$$S_t^n := \{x \mid x \in \mathbb{Z}_q^n, \text{wt}_L(x) = t\}.$$



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**Question:** What would  $P_{Y|X}(y|x)$  look like?





## The Constant-weight Lee Channel

Let  $\mathbf{p} = (p_0, \dots, p_{q-1})$ , with  $p_i := P(i|j_0)$  for all  $i \in \mathbb{Z}_q$ .

### Lemma

The constant-weight Lee channel over  $\mathbb{Z}_q$  has channel distribution

$$p_i^* = \kappa \exp(-\lambda \text{wt}_L(i)), \quad \kappa := \frac{1}{\sum_{j=0}^{q-1} \exp(-\lambda \text{wt}_L(j))},$$

such that it matches under maximum likelihood decoding.



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### Sketch of proof

We want that  $\mathbf{p} = (p_0, \dots, p_{q-1})$  maximizes the entropy function

$$H_e(\mathbf{p}) := \sum_{i=0, p_i \neq 0}^{q-1} p_i \log p_i$$

under the constraint that  $\sum_{i=0}^{q-1} \text{wt}_L(i) p_i = t/n =: \delta$ .



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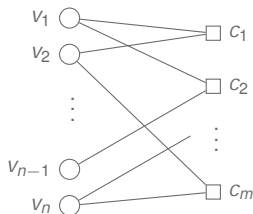
## Symbol Message Passing

Consider a nonbinary LDPC code  $C$  with VNs  $v_1, \dots, v_n$  and CNs  $c_1, \dots, c_m$  and parity-check matrix  $H$ . Denote by  $N(v_j)$  and  $N(c_i)$  the set of all connecting elements to VN  $v_j$  and CN  $c_i$ , respectively.



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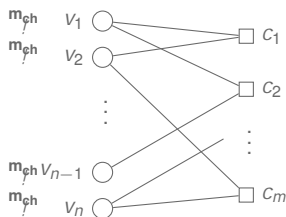
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Every VN  $v$  receives the channel observation  $\mathbf{m}_{\text{ch}} := (P_{Y|X}(y_j=0), \dots, P_{Y|X}(y_j=q-1))$

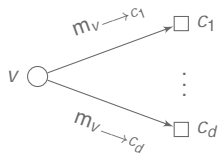


# Symbol Message Passing

## Initialization.

Each VN  $v$  sends channel observation to the neighboring CNs  $c \in N(v)$

$$m_{v \rightarrow c} = \mathbf{m}_{\text{ch}}.$$



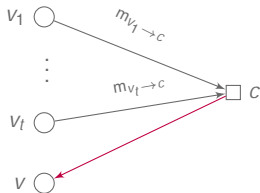
## Symbol Message Passing

### CN-to-VN step.

Each CN computes for every  $v \in N(c)$

$$m_{c \rightarrow v} = h_{c,v}^{-1} \sum_{v' \in N(c) \setminus \{v\}} h_{c,v'} m_{v' \rightarrow c}$$

**Note:**  $h_{c,v}^{-1}$  exists, since we said the nonzero entries of  $H$  are units.





## Symbol Message Passing

### VN-to-CN step.

Define the aggregated extrinsic  $L$ -vector

$$E = L(y) + \sum_{c' \in \mathcal{N}(v) \setminus \{c\}} L(m_{c' \rightarrow v}),$$

where  $y$  is the channel output and

$L(y) = (L_0(y), \dots, L_{q-1}(y))$  with

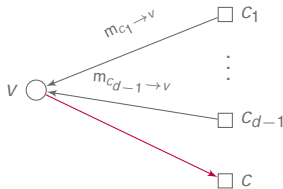
$L_x(y) = \log(P_{Y|X}(y|x))$ .

**Note:** We assume the CN-to-VN messages are modelled as a qSC,

$$P_{M|X}(m|x) = \begin{cases} 1 & \xi & \text{if } m = x \\ \xi/(q-1) & 1 & \text{otherwise} \end{cases}$$

Then the VN-to-CN messages are

$$m_{v \rightarrow c} = \arg \max_{x \in \mathcal{q}} E_x.$$



## Symbol Message Passing

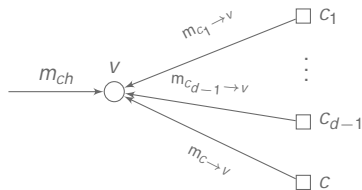
### Final decision.

The final decision at each VN  $v$  is

$$\hat{x} = \arg \max_{x \in \mathbb{Z}_q} L_x^{\text{FIN}}$$

where

$$L^{\text{FIN}} = L(m_{\text{ch}}) + \sum_{c \in \mathcal{N}(v)} L(m_{c \rightarrow v}).$$



## The $q$ SC-Assumption for SMP

Following Lechner, Pedersen, and Kramer, the CN to VN messages are modelled as observations from a  $q$ -ary symmetric channel.



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- The use of the  $q$ SC approximation is important from a practical viewpoint, i.e., decoding becomes particularly simple.



# Simulations

Decoding performance for both BP and SMP over both the Lee channel and the constant-weight Lee channel using

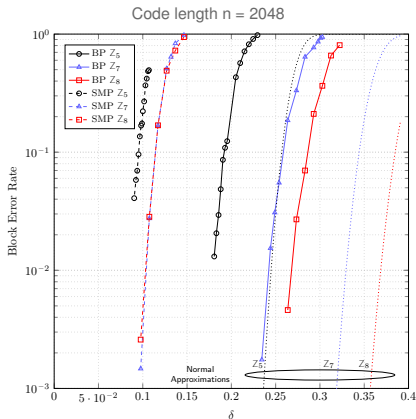
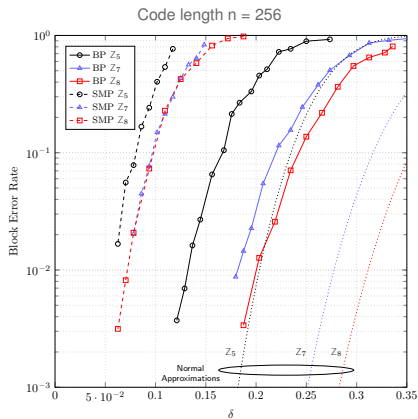
(3, 6) regular nonbinary LDPC codes of length 256 and 2048,

For the constant-weight Lee channel, the error vectors are drawn uniformly at random from the set of vectors with a given weight.



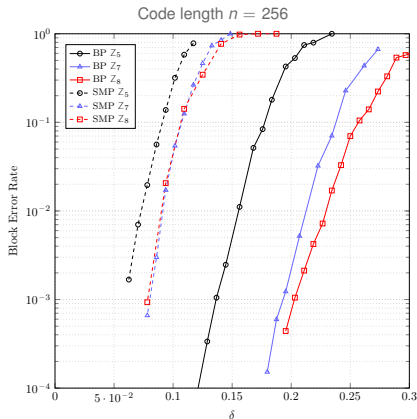
# Simulations

Block error rate vs. average Lee weight  $\delta$  for regular (3, 6) nonbinary LDPC codes in the Lee channel for BP and SMP decoding.



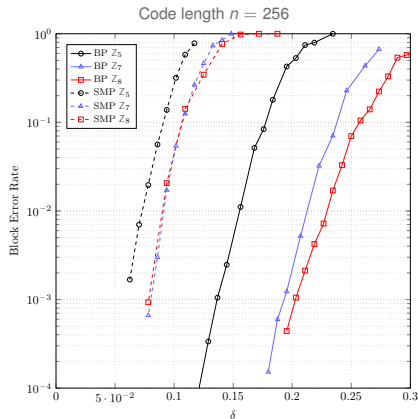
# Simulations

Block error rate vs. average Lee weight  $\delta$  for regular (3, 6) nonbinary LDPC codes in the **constant-weight Lee channel** for BP and SMP decoding.



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Block error rate vs. average Lee weight  $\delta$  for regular (3, 6) nonbinary LDPC codes in the **constant-weight Lee channel** for BP and SMP decoding.



Thank you very much for your attention!

