

## OUTPUT FEEDBACK POLE ASSIGNMENT FOR TRANSFER FUNCTIONS WITH SYMMETRIES\*

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**Abstract.** This paper studies the problem of pole assignment for symmetric and Hamiltonian transfer functions. A necessary and sufficient condition for pole assignment by complex symmetric output feedback transformations is given. Moreover, in the case where the McMillan degree coincides with the number of parameters appearing in the symmetric feedback transformations, we derive an explicit combinatorial formula for the number of pole assigning symmetric feedback gains. The proof uses intersection theory in projective space as well as a formula for the degree of the complex Lagrangian Grassmann manifold.

**Key words.** output feedback, pole placement, inverse eigenvalue problems, Lagrangian Grassmannian, symmetric or Hamiltonian realizations, degree of a projective variety

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**1. Introduction.** One of the best known inverse eigenvalue problems from linear system theory is that of pole assignment, i.e., finding a static output feedback gain for a given linear system such that the closed loop poles of the system coincide with a specified subset of the complex plane. Moreover, in the case of finitely many solutions, a formula for the number of pole assigning feedback transformations is desirable. Early contributions on the subject were obtained by, e.g., Davison and Wang [7] and Kimura [20], who derived sufficient conditions for the solvability. However, these conditions were far from being necessary as well. In a series of pioneering papers [16, 24, 25], Hermann and Martin applied tools from algebraic geometry to obtain necessary and sufficient conditions, valid for a generic class of systems and for *complex* feedback transformations. Their approach was based on the dominant morphism theorem [2, Chapter AG, section 17, Theorem 17.3] from complex algebraic geometry. A second breakthrough was subsequently made by Brockett and Byrnes [3], who used intersection theoretic arguments and the Schubert calculus on Grassmann manifolds to count the number of pole assigning complex feedback transformations. By refining these algebraic-geometric approaches of Hermann and Martin and Brockett and Byrnes, a number of fundamental contributions on the subject were made that finally led to a solution of the problem in the *real* case, with important contributions due to [8, 21, 29, 37, 38]. For an excellent survey paper on this subject, written from a control-theoretic point of view, see, e.g., [4]. More recently, various intersection theoretic tasks related to the Schubert calculus have been studied in the algebraic geometry literature; see, e.g., [12, 18, 34]. The focus of most of the investigations so far has been on the unstructured case, where no underlying symmetries

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for the involved transfer function or for the associated feedback transformations are imposed. However, transfer functions with symmetries occur naturally in various application areas, such as in network theory or mechanics. For example, the transfer functions  $G(s)$  of linear *RLC*-circuits, consisting solely of resistors, capacitors, and inductive elements, are symmetric; i.e., they satisfy  $G(s)^\top = G(s)$ . In mechanics, the transfer functions of linear Hamiltonian systems are characterized by the symmetry relation  $G(-s)^\top = G(s)$ , while second order mechanical systems of the form

$$M\ddot{x} = Nx + Bu, \quad y = B^\top x$$

yield symmetric Hamiltonian transfer functions, satisfying

$$G(s) = H(s^2), \quad H(s) = H(s)^\top;$$

see, e.g., [1, 5, 6, 9]. For such structured systems it is reasonable to restrict the class of admissible feedback transformations to those that preserve the symmetry properties of the transfer functions. Therefore the known results on pole placement on unstructured systems do not apply in these cases and instead require a new approach.

In this paper we start an investigation of the pole placement problem for  $n \times n$  symmetric transfer functions  $G(s) = G(s)^\top$ , arising in electrical network theory, and Hamiltonian transfer functions. For both types of systems the natural class of admissible output feedback transformations are the symmetric ones  $F = F^\top$ , yielding a symmetric closed loop transfer function

$$G_F(s) := (I_n - G(s)F)^{-1}G(s).$$

As the number of free parameters occurring in the symmetric feedback matrices  $F$  is  $n(n + 1)/2$ , a necessary condition for generic solvability of this output feedback problem is that the McMillan degree  $\delta$  of the transfer function  $G$  satisfies  $\delta \geq \binom{n+1}{2}$  in the symmetric case, and  $\delta \geq n(n+1)$  in the Hamiltonian case. In fact, we show that generically for *complex* symmetric output feedback transformations this condition is also sufficient. Moreover, for the limit case  $\delta = \binom{n+1}{2}$  (or  $\delta = n(n + 1)$ ), we derive an explicit combinatorial formula for the number of complex symmetric output feedback gains that place the poles at given points. Our formula coincides with that of the degree for the complex Lagrangian manifold, given in [36].

In the real case such complete results cannot be expected. In fact, the symmetry of the transfer functions then imposes a priori limitations for the possible pole locations of such systems. This has been observed in [23], where it is shown for symmetric transfer functions that—in the special case that the Cauchy index of  $G$  coincides with the McMillan degree—generically real symmetric output feedback pole assignability holds if and only if  $n \geq \delta$ . Of course, in most applications we have  $n \leq \delta$ , and therefore the description of the set of poles that can be achieved by real symmetric output feedback becomes a complicated and nontrivial task.

**2. Complex symmetric and Hamiltonian realizations.** In this section we recall some basic facts concerning complex symmetric and Hamiltonian transfer functions and associated signature symmetric and Hamiltonian realizations. Let  $\mathbb{C}$  denote the field of complex numbers. A complex rational transfer function  $G(s) \in \mathbb{C}(s)^{n \times n}$  of McMillan degree  $\delta$  is called *symmetric or Hamiltonian, respectively*, if

$$G(s) = G(s)^\top \quad \text{or} \quad G(s) = G(-s)^\top, \quad \text{respectively,}$$

holds for all  $s \in \mathbb{C}$ . A *complex symmetric realization* is a linear system of the form

$$\dot{x} = Ax + Bu, \quad y = B^\top x,$$

where  $A \in \mathbb{C}^{\delta \times \delta}$  is symmetric, i.e.,  $A^\top = A$ , and  $B \in \mathbb{C}^{\delta \times n}$ . Similarly, a *Hamiltonian realization* is a linear system

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where  $A \in \mathbb{C}^{\delta \times \delta}$ ,  $B \in \mathbb{C}^{\delta \times n}$ ,  $C \in \mathbb{C}^{n \times \delta}$  satisfies

$$AJ = (AJ)^\top, \quad C^\top = JB$$

and

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

denotes the standard symplectic form on  $\mathbb{C}^{\delta \times \delta}$ . In particular, Hamiltonian systems always have even McMillan degree  $\delta$ .

Complex symmetric realizations are the natural class of realizations for complex symmetric transfer functions. In fact, they are the proper analogue of signature symmetric realizations of real rational transfer functions, appearing in network theory. Here a signature symmetric realization  $(A, B, C)$  is one that is symmetric relative to the bilinear form  $I_{pq} := \text{diag}(I_p, -I_q)$ ; i.e.,  $(A, B, C)$  satisfies  $(AI_{pq})^\top = AI_{pq}$ ,  $C^\top = I_{pq}B$ . Such realizations always exist for real symmetric transfer functions  $G$ , and the integer  $p - q$  coincides with the so-called Cauchy–Maslov index of  $G$ . If  $q = 0$ , then this definition coincides with the one above for symmetric realizations. In particular, over  $\mathbb{R}$ , real symmetric realizations correspond to linear models of *RC*-networks, constructed entirely using capacitors and resistors. The real symmetric transfer functions defined by them are characterized by the property that the Cauchy–Maslov index coincides with the McMillan degree [1, 9, 14].

The following variant of the Kalman realization theorem is well known; see, e.g., [9, 10]. Recall that the complex orthogonal group  $O(\delta, \mathbb{C})$  is the matrix group consisting of all complex  $\delta \times \delta$  matrices  $S$ , satisfying  $SS^\top = I_\delta$ . Given any complex realization  $(A, B, C)$  of a symmetric transfer function  $G(s) = C(sI - A)^{-1}B$ , note that  $(A^\top, C^\top, B^\top)$  is also a realization.

**PROPOSITION 2.1.** *Let  $G(s) = G(s)^\top$  be an  $n \times n$  strictly proper, complex rational transfer function of McMillan degree  $\delta$ . Then the following hold:*

- (1)  $G(s)$  has a controllable and observable complex symmetric realization

$$(A, B, C) = (A^\top, C^\top, B^\top).$$

- (2) If  $(A_i, B_i, C_i)$ ,  $i=1, 2$ , are two controllable and observable complex symmetric realizations of  $G(s)$ , then there exists a unique complex orthogonal transformation  $S \in O(\delta, \mathbb{C})$  such that  $(A_2, B_2, C_2) = (SA_1S^{-1}, SB_1, C_1S^{-1})$ .

In the literature usually only the real case of the above result is proved, where the statement is actually slightly different due to the presence of signature symmetric realizations. In the complex case the result simplifies to the one given here. For the sake of completeness we include the proof; see also [10].

*Proof.* If  $(A, B, C)$  is a minimal realization of  $G(s)$ , then, by symmetry of  $G$ ,  $(A^\top, C^\top, B^\top)$  also is a minimal realization. Applying Kalman’s realization theorem implies the existence of a unique invertible complex  $\delta \times \delta$  matrix  $S$  with

$$(A^\top, C^\top, B^\top) = (SAS^{-1}, SB, CS^{-1}).$$

By transposing this equation and using the uniqueness of  $S$  we conclude that  $S = S^\top$ . It is a well-known fact from linear algebra that every complex symmetric invertible matrix has a representation  $S = XX^\top$  by a complex invertible matrix  $X$ . Moreover,  $X$  is uniquely determined up to right factors  $XT$ , where  $T \in O(\delta, \mathbb{C})$ . Then  $(XAX^{-1}, XB, CX^{-1})$  is a complex symmetric realization, which completes the proof.  $\square$

There is a similar realization theorem for Hamiltonian systems, for which we refer to the literature; see, e.g., [6, 9]. Static linear output feedback can be meaningfully defined for such systems only through symmetric gain matrices. Note that

$$\text{adj}(sI - A) = s^{n-1}I + s^{n-2}(A - \text{trace}(A)I) + \text{lower order terms.}$$

Thus, for symmetric  $A$ , the rational matrix

$$B^\top (sI - A - BFB^\top)^{-1}B = B^\top \frac{1}{\det(sI - A - BFB^\top)} \text{adj}(sI - A - BFB^\top)B$$

is symmetric only if  $B^\top BFB^\top B$  is symmetric. Therefore, if  $B$  has full column rank, an output feedback transformation

$$u = Fy + v$$

with the closed system

$$\dot{x} = (A + BFB^\top)x + Bu, \quad y = B^\top x$$

preserves the complex symmetry of the realizations if and only if  $F = F^\top$ . Thus we define two complex symmetric realizations  $(A_i, B_i, C_i)$ ,  $i = 1, 2$ , to be *symmetric output feedback equivalent* if and only if there exist  $S \in O(\delta, \mathbb{C})$ ,  $F = F^\top \in \mathbb{C}^{n \times n}$  with

$$(A_2, B_2, C_2) = (S(A_1 + B_1FB_1^\top)S^{-1}, SB_1, C_1S^{-1}).$$

Equivalently, they are symmetric output feedback equivalent if and only if the associated transfer functions satisfy

$$G_2(s) = (I_n - G_1(s)F)^{-1}G_1(s).$$

Similarly, output feedback for Hamiltonian systems

$$\dot{x} = (A + BFC)x + Bu, \quad y = Cx$$

preserves the Hamiltonian properties of the realization if and only if  $F = F^\top$ . Thus in both cases we have to focus on symmetric output feedback.

We note some elementary geometric properties of the set of complex symmetric transfer functions that will be important in the subsequent development; see, e.g., [6] for some of the details for the proof of the subsequent theorem. We omit a full proof as it would take us too far from the subject.

**PROPOSITION 2.2.** *Let  $SRat_{\delta,n}(\mathbb{C})$  and  $Ham_{\delta,n}(\mathbb{C})$ , respectively, denote the sets of strictly proper, complex symmetric and Hamiltonian  $n \times n$  transfer functions of McMillan degree  $\delta$ . Then  $SRat_{\delta,n}(\mathbb{C})$  and  $Ham_{\delta,n}(\mathbb{C})$  are, respectively, a smooth complex manifold of complex dimension  $\delta(n + 1)$  and dimension  $\delta n$ . Moreover, they are nonsingular irreducible quasi-affine varieties.*

In particular, there is a canonical notion of “genericity” for complex symmetric or Hamiltonian transfer functions. Explicitly, a property  $E$  of complex symmetric transfer functions is called generic if the set defined by  $E$ ,

$$\{G \in SRat_{\delta,n}(\mathbb{C}) \mid G \text{ has property } E\},$$

is a Zariski open subset of  $SRat_{\delta,m}(\mathbb{C})$ . Equivalently, this also can be expressed in terms of complex symmetric realizations.

**3. Main result.** After these preliminaries we can now formulate and prove the main technical results of this paper. Let  $G(s)$  be an  $n \times n$  complex symmetric or Hamiltonian transfer function, i.e.,  $G(s)^\top = G(s)$  or  $G(-s)^\top = G(s)$ , respectively. Assume that  $G(s)$  is strictly proper and has McMillan degree  $\delta$ . The complex symmetric eigenvalue assignment problem then asks the following question.

**PROBLEM 3.1.** *For a given arbitrary monic polynomial  $\varphi(s) \in \mathbb{C}[s]$  of degree  $\delta$  ( $\varphi(s) = \varphi(-s)$  is assumed to be even in the Hamiltonian case), is there an  $n \times n$  complex symmetric matrix  $F$  such that the closed loop transfer function*

$$G_F(s) := (I_n - G(s)F)^{-1}G(s)$$

*has the characteristic polynomial  $\varphi(s)$ , i.e., such that the poles of  $G_F(s)$  are the zeros of  $\varphi(s)$ ?*

If for a particular symmetric (Hamiltonian) transfer function  $G(s)$  Problem 3.1 has an affirmative answer, we will say that  $G(s)$  is *pole assignable in the class of complex symmetric (Hamiltonian) feedback compensators*. We say that  $G(s)$  is *generically pole assignable* if the problem is solvable for a generic choice of admissible polynomials  $\varphi(s)$ .

Similar to the situation of the static pole placement problem [3, 37] and the dynamic pole placement problem [28], Problem 3.1 turns out to be highly nonlinear, and techniques from algebraic geometry will be required to study the problem. The first main result is in the spirit of Hermann and Martin [16], by deriving a generic necessary and sufficient condition via the dominant morphism theorem.

We prove some lemmas first. Let  $\pi(A) = (a_{11}, \dots, a_{\delta\delta})$  be the projection onto the diagonal entries of an  $\delta \times \delta$  matrix  $A$ . In what follows we will identify  $\mathbb{C}^\delta$  with the complex vector space of row vectors. For any symmetric matrix  $L$ , define  $\theta^L : O(\delta, \mathbb{C}) \rightarrow \mathbb{C}^\delta$  through

$$\theta^L(S) = \pi(SLS^{-1}).$$

As  $O(\delta, \mathbb{C})$  is a Lie group, its tangent space at the identity matrix  $I$  is given by the Lie algebra of complex skew-symmetric matrices

$$so(\delta, \mathbb{C}) = \{X \in \mathbb{C}^{\delta \times \delta} \mid X + X^\top = 0\}.$$

Moreover, the Jacobian  $d\theta_I^L$  of  $\theta^L$  at  $I$  is given by

$$d\theta_I^L : so(\delta, \mathbb{C}) \rightarrow V, \quad d\theta_I^L(X) = \pi(XL - LX),$$

where

$$V = \left\{ (x_1, \dots, x_\delta) \in \mathbb{C}^\delta \mid \sum_1^\delta x_i = 0 \right\}.$$

For any  $\delta \times \delta$  matrix  $L$ , the graph  $\mathcal{G}(L)$  of  $L$  is defined as a graph with  $\delta$  vertices such that there is a path from vertex  $i$  to vertex  $j$  if and only if the  $ij$ th entry of  $L$  is nonzero. It is a well-known fact from linear algebra that the graph  $\mathcal{G}(L)$  is connected if and only if  $L$  is irreducible, i.e., if and only if there exists no permutation matrix  $P$  such that  $PLP^{-1}$  is block diagonal. We use this fact together with an idea developed in [15, Lemma 2.5] to prove the following equivalent characterization.

LEMMA 3.2. *The Jacobian  $d\theta_I^L$  is surjective if and only if the associated graph  $\mathcal{G}(L)$  is connected.*

*Proof.* By inspection, the derivative  $d\theta_I^L$  is not surjective if and only if there exists a nonzero diagonal matrix  $Z$  of trace zero such that for all  $X \in so(\delta, \mathbb{C})$

$$\text{trace}(Z(XL - LX)) = \text{trace}((LZ - ZL)X) = 0.$$

By symmetry of  $L, Z$  we have  $LZ - ZL \in so(\delta, \mathbb{C})$ . Since the trace function defines a nondegenerate bilinear form on  $so(\delta, \mathbb{C})$ , the condition  $\text{trace}((LZ - ZL)X) = 0$  is equivalent to  $LZ = ZL$ . Since  $Z$  is a nonzero diagonal matrix of trace zero, there is a permutation matrix  $P$  such that  $\hat{Z} := PZP^{-1} = \text{block diag}(a_1 I_1, \dots, a_k I_k)$  with  $k \geq 2$  and  $a_i$ 's distinct. Let  $\hat{L} = PLP^{-1}$ . Then  $LZ = ZL$  is equivalent to  $\hat{L}\hat{Z} = \hat{Z}\hat{L}$ , which is equivalent to  $\hat{L}$  being block diagonal. But from the above remark this is equivalent to the graph  $\mathcal{G}(L)$  being disconnected. The result follows.  $\square$

LEMMA 3.3. *Let  $L$  be a nonzero complex symmetric matrix such that  $\pi(L) = 0$ . Then there is a family of orthogonal matrices  $S(\epsilon) \in O(\delta, \mathbb{C})$ ,  $\epsilon \geq 0$ , with  $S(0) = I$  such that the matrix  $\hat{L}(\epsilon) := S(\epsilon)LS(\epsilon)^{-1}$  has the properties that  $\pi(\hat{L}(\epsilon)) = 0$  and  $d\varphi_I^{\hat{L}(\epsilon)}$  is surjective for all  $\epsilon \in (0, \pi/2)$ .*

*Proof.* If  $\mathcal{G}(L)$  is connected, then by the previous lemma the choice  $S(\epsilon) := I$  does the job. Thus it suffices to prove that  $\mathcal{G}(L)$  not connected implies that then we can find a family of transformations  $S(\epsilon)$  such that  $\pi(S(\epsilon)LS(\epsilon)^{-1}) = 0$  and the largest connected subgraph of  $\mathcal{G}(S(\epsilon)LS(\epsilon)^{-1})$  contains more vertices than that of  $\mathcal{G}(L)$  for all  $0 < \epsilon < \pi/2$ .

Note that  $\pi(L) = 0$  and  $L \neq 0$  imply that the largest connected subgraph of  $\mathcal{G}(L)$  must contain at least 2 vertices. Assume that the largest connected subgraph of  $\mathcal{G}(L)$  contains  $k$  vertices,  $2 \leq k < \delta$ . Without loss of generality, assume

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix},$$

where the graph of the  $k \times k$  submatrix  $L_1$  is connected. Write

$$L_1 = \begin{bmatrix} L_{11} & \alpha \\ \alpha^\top & 0 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & \beta^\top \\ \beta & L_{22} \end{bmatrix},$$

where  $L_{11}$  and  $L_{22}$  are sizes  $(k-1) \times (k-1)$  and  $(\delta-k-1) \times (\delta-k-1)$ , respectively. By irreducibility of  $L_1$  we have  $\alpha \neq 0$ . Thus  $L$  has the form

$$\begin{bmatrix} L_{11} & \alpha & 0 & 0 \\ \alpha^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^\top \\ 0 & 0 & \beta & L_{22} \end{bmatrix}.$$

Let

$$S(\epsilon) = \begin{bmatrix} I_{(k-1) \times (k-1)} & 0 & 0 & 0 \\ 0 & \cos \epsilon & -\sin \epsilon & 0 \\ 0 & \sin \epsilon & \cos \epsilon & 0 \\ 0 & 0 & 0 & I_{(\delta-k-1) \times (\delta-k-1)} \end{bmatrix}.$$

Then

$$S(\epsilon)LS^{-1}(\epsilon) = \begin{bmatrix} L_{11} & (\cos \epsilon)\alpha & (\sin \epsilon)\alpha & 0 \\ (\cos \epsilon)\alpha^\top & 0 & 0 & (-\sin \epsilon)\beta^\top \\ (\sin \epsilon)\alpha^\top & 0 & 0 & (\cos \epsilon)\beta^\top \\ 0 & (-\sin \epsilon)\beta & (\cos \epsilon)\beta & L_{22} \end{bmatrix}.$$

For the graph  $\mathcal{G}(S(\epsilon)LS^{-1}(\epsilon))$  with  $0 < \epsilon < \pi/2$ , the vertices  $\{1, \dots, k\}$  are still connected and the vertex  $k + 1$  is symmetrically connected to at least one of the first  $k$  vertices. Thus the vertices  $\{1, \dots, k + 1\}$  are connected.  $\square$

LEMMA 3.4. *Let  $\mathcal{L}$  be a linear subspace of complex symmetric matrices of dimension  $\delta$ , and  $\mathcal{L} \not\subset sl(\delta, \mathbb{C})$ . Then there exists an orthogonal matrix  $S \in O(\delta, \mathbb{C})$  such that  $\pi|_{S\mathcal{L}S^{-1}}$  is one to one, and onto.*

*Proof.* The proof goes by recursively constructing a basis  $\{L_1, \dots, L_\delta\}$  of  $\mathcal{L}$  such that  $\{\pi(SL_1S^{-1}), \dots, \pi(SL_\delta S^{-1})\}$  are linearly independent for a suitable complex orthogonal matrix  $S \in O(\delta, \mathbb{C})$ . First note that we can modify any basis of  $\mathcal{L}$  into a basis  $\mathcal{L}^{(1)} := \{L_1, \dots, L_\delta\}$  of  $\mathcal{L}$  such that  $L_1 \notin sl(\delta, \mathbb{C})$  and  $L_i \in sl(\delta, \mathbb{C})$  for  $i = 2, \dots, \delta$ . In fact, if  $\{K_1, \dots, K_\delta\}$  denotes any basis of  $\mathcal{L}$  with  $\text{trace}(K_1) \neq 0$ , then  $\{L_1 := K_1, L_2 := K_2 - c_2K_1, \dots, L_\delta := K_\delta - c_\delta K_1\}$ ,  $c_i := \text{trace}(K_i)/\text{trace}(K_1)$ , is as desired. By construction of  $L_1$ , we have  $\pi(L_1) \neq 0$ .

Let  $\{L_1, \dots, L_\delta\}$  be a basis of  $\mathcal{L}$  such that  $L_1 \notin sl(\delta, \mathbb{C})$  and  $L_i \in sl(\delta, \mathbb{C})$  for  $i = 2, \dots, \delta$ . Then  $\dim \text{span}\{\pi(L_1), \dots, \pi(L_\delta)\} := k \geq 1$ . If  $k < \delta$ , then by re-ordering the indices we can assume that  $\{\pi(L_1), \dots, \pi(L_k)\}$  are linearly independent and

$$\pi(L_j) = \sum_{i=1}^k c_{ij}\pi(L_i) \text{ for } j = k + 1, \dots, \delta.$$

By replacing  $L_j$  with  $L_j - \sum_{i=1}^k c_{ij}L_i$ , we can further assume that  $\pi(L_j) = 0$  for  $j = k + 1, \dots, \delta$ . It is thus sufficient to show that if there is an orthogonal matrix  $\hat{S}$  such that the matrices  $\{M_j := \hat{S}L_j\hat{S}^{-1}, j = 1, \dots, \delta\}$  have the property that  $\{\pi(M_1), \dots, \pi(M_k)\}$  are linearly independent and  $\pi(M_j) = 0, j = k + 1, \dots, \delta$ , for some  $k < n$ , then we can find an orthogonal  $S$  such that

$$\{\pi(SM_1S^{-1}), \dots, \pi(SM_kS^{-1}), \pi(SM_{k+1}S^{-1})\}$$

are linearly independent.

By Lemma 3.3, there exists  $S_\epsilon \in O(\delta, \mathbb{C})$  arbitrarily close to the identity matrix such that  $\pi(S_\epsilon M_1 S_\epsilon^{-1}), \dots, \pi(S_\epsilon M_k S_\epsilon^{-1})$  are linearly independent and the graph  $\mathcal{G}(S_\epsilon M_{k+1} S_\epsilon^{-1})$  is connected. By replacing  $M_i$  with  $S_\epsilon M_i S_\epsilon^{-1}$ , we can assume further that  $d\theta_I^{M_{k+1}}$  is onto  $V$ . Then there exists a skew-symmetric matrix  $X$  such that

$$\pi(XM_{k+1} - M_{k+1}X) \notin \text{span}\{\pi(M_1), \dots, \pi(M_k)\}.$$

Let

$$S(\epsilon) = \exp(\epsilon X).$$

Then  $S(\epsilon)$  is orthogonal for all  $\epsilon$ , and

$$S(\epsilon) = I + \epsilon X + \text{higher order terms.}$$

The Taylor series expansions of  $\{\pi(S(\epsilon)M_iS(\epsilon)^{-1})\}$  have the forms

$$\pi(S(\epsilon)M_iS(\epsilon)^{-1}) = \pi(M_i) + \beta_i(\epsilon), \quad i = 1, \dots, k,$$

and

$$\pi(S(\epsilon)M_{k+1}S(\epsilon)^{-1}) = \epsilon(\pi(XM_{k+1} - M_{k+1}X) + \beta_{k+1}(\epsilon)),$$

where  $\beta_i(\epsilon)$  are continuous with respect to  $\epsilon$  and  $\beta_i(\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  for  $i = 1, \dots, k+1$ . Since  $\{\pi(M_1), \dots, \pi(M_k), \pi(XM_{k+1} - M_{k+1}X)\}$  are linearly independent, for sufficient small  $\epsilon > 0$ ,  $\{\pi(M_1) + \beta_1(\epsilon), \dots, \pi(M_k) + \beta_k(\epsilon), \pi(XM_{k+1} - M_{k+1}X) + \beta_{k+1}(\epsilon)\}$  are also linearly independent; i.e.,

$$\{\pi(S(\epsilon)M_1S(\epsilon)^{-1}, \dots, \pi(S(\epsilon)M_kS(\epsilon)^{-1}, \pi(S(\epsilon)M_{k+1}S(\epsilon)^{-1})\}$$

are linearly independent.  $\square$

**THEOREM 3.5.** *If  $G(s)$  is a symmetric (or Hamiltonian) transfer function of McMillan degree  $\delta > \binom{n+1}{2}$  (or  $\delta > n(n+1)$ ), then  $G(s)$  is not pole assignable in the class of (real or) complex symmetric feedback compensators.*

*When  $\delta \leq \binom{n+1}{2}$  (or  $\delta \leq n(n+1)$ ), then there is a generic set of  $n \times n$  symmetric (or Hamiltonian) transfer functions of degree  $\delta$  which are generically pole assignable via complex symmetric feedback compensators.*

*Proof.* We give only a sketch of the proof, as the arguments based on the dominant morphism theorem are well known from [16, 24]. Note, however, that there is a serious gap in the proof of [24] for the pole placement result on Hamiltonian systems because it is not proved that the set of generically pole assignable Hamiltonian systems is nonempty. In fact, a construction of such an example is not completely trivial and depends on our previous lemmas.

The first claim follows immediately from a standard dimension argument, as the vector space  $\text{Sym}(n)$  of complex  $n \times n$  symmetric matrices has dimension  $\binom{n+1}{2}$ . For the second claim we note that the set of generically pole assignable systems is a Zariski open subset of the nonsingular, irreducible quasi-affine variety of symmetric or Hamiltonian transfer functions, respectively. Therefore we need only show that this Zariski open subset is nonempty. By the dominant morphism theorem, it suffices to find one system whose Jacobian of the pole placement map at one point is onto.

Note, by the Newton formula, that the coefficients of the characteristic polynomial  $\det(sI - A) = s^\delta + \alpha_{\delta-1}s^{\delta-1} + \dots + \alpha_1s + \alpha_0$  are related to the traces of powers of  $A$  as follows:

$$\begin{aligned} \alpha_{\delta-1} &= -\text{trace}(A), \\ \alpha_{\delta-2} &= -\frac{1}{2}(\text{trace}(A^2) + \alpha_{\delta-1}\text{trace}(A)), \\ &\vdots \\ \alpha_0 &= -\frac{1}{\delta}(\text{trace}(A^\delta) + \alpha_{\delta-1}\text{trace}(A^{\delta-1}) + \dots + \alpha_1\text{trace}(A)). \end{aligned}$$

Therefore for the case of symmetric transfer functions, the pole placement map is equivalent to the map

$$(3.1) \quad \begin{array}{ccc} \phi: \text{Sym}(n) & \longrightarrow & \mathbb{C}^\delta \\ F & \longmapsto & (\text{trace}(A + BFB^\top), \dots, \text{trace}(A + BFB^\top)^\delta), \end{array}$$

and its Jacobian at 0 is given by

$$d\phi_0(F) = (\text{trace}(BFB^\top), 2 \text{trace}(ABFB^\top), \dots, \delta \text{trace}(A^{\delta-1}BFB^\top)).$$

For the case of Hamiltonian transfer functions, since  $JAJ = A^\top$  and  $J^2 = -I$ , we have  $(-1)^{k-1}JA^kJ = (A^k)^\top$  for  $k = 1, 2, \dots$ , which implies that the characteristic polynomial of  $A$  is even and

$$\text{trace}(A^k) = 0 \text{ holds for all odd } k\text{'s.}$$

Therefore the pole placement map is equivalent to the map

$$(3.2) \quad \begin{aligned} \psi : \text{Sym}(n) &\longrightarrow \mathbb{C}^{\delta/2} \\ F &\longmapsto (\text{trace}(A + BFC)^2, \text{trace}(A + BFC)^4, \dots, \text{trace}(A + BFC)^\delta), \end{aligned}$$

and its Jacobian at 0 is given by

$$d\psi_0(F) = (2 \text{trace}(ABFC), 4 \text{trace}(A^3BFC) \dots, \delta \text{trace}(A^{\delta-1}BFC)).$$

We first consider the case of symmetric transfer functions. Let  $B$  be any real nonzero matrix and  $\mathcal{L} = \{BFB^\top \mid F \in \text{Sym}(n)\}$ . Then  $\mathcal{L} \not\subset sl(\delta, \mathbb{C})$  and  $\dim \mathcal{L} \geq \delta$ . By Lemma 3.4 there exists an orthogonal matrix  $S \in O(\delta, \mathbb{C})$  such that  $\pi|_{S\mathcal{L}S^{-1}}$  is surjective. Let  $D = \text{diag}(1, 2, \dots, \delta)$  and  $A = S^{-1}DS$ . Then

$$\begin{aligned} d\phi_0(F) &= (\text{trace}(BFB^\top), 2 \text{trace}(ABFB^\top), \dots, \delta \text{trace}(A^{\delta-1}BFB^\top)) \\ &= (\text{trace}(SBFB^\top S^{-1}), 2 \text{trace}(DSBFB^\top S^{-1}), \dots, \delta \text{trace}(D^{\delta-1}SBFB^\top S^{-1})) \\ &= \pi(SBFB^\top S^{-1})V, \end{aligned}$$

where

$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 2^{\delta-1} \\ \vdots & \vdots & & \vdots \\ 1 & \delta & \dots & \delta^{\delta-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \delta \end{bmatrix}.$$

Since  $\pi|_{S\mathcal{L}S^{-1}}$  is surjective and  $V$  is nonsingular,  $d\phi_0$  is onto.

For the case of Hamiltonian transfer functions, let

$$B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \text{ and } C = [ B_1^\top \quad 0 ],$$

where  $B_1$  is any real nonzero  $\frac{\delta}{2} \times n$  matrix and  $\mathcal{L} = \{B_1FB_1^\top \mid F \in \text{Sym}(n)\}$ . Then  $\mathcal{L} \not\subset sl(\delta/2, \mathbb{C})$ . By Lemma 3.4 there exists an orthogonal matrix  $S_1 \in O(\delta/2, \mathbb{C})$  such that  $\pi : S_1\mathcal{L}S_1^{-1} \mapsto \mathbb{C}^{\frac{\delta}{2}}$  is surjective. Let  $D_1 = \text{diag}(1, 2, \dots, \delta/2)$ ,

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & D_1 \\ D_1 & 0 \end{bmatrix},$$

and  $A = S^{-1}DS$ . Note that  $D$  and  $S$  are Hamiltonian and symplectic matrices, respectively. In particular,  $A$  is Hamiltonian. Then

$$\begin{aligned} d\psi_0(F) &= (2 \text{trace}(ABFC), 4 \text{trace}(A^3BFC), \dots, \delta \text{trace}(A^{\delta-1}BFC)) \\ &= (2 \text{trace}(DSBFCS^{-1}), 4 \text{trace}(D^3SBFCS^{-1}), \dots, \delta \text{trace}(D^{\delta-1}SBFCS^{-1})) \\ &= (2 \text{trace}(D_1S_1B_1FB_1^\top S_1^{-1}), \dots, \delta \text{trace}(D_1^{\delta-1}S_1B_1FB_1^\top S_1^{-1})) \\ &= \pi(S_1B_1FB_1^\top S_1^{-1})U, \end{aligned}$$

where

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^3 & \cdots & 2^{\delta-1} \\ \vdots & \vdots & & \vdots \\ \frac{\delta}{2} & (\frac{\delta}{2})^3 & \cdots & (\frac{\delta}{2})^{\delta-1} \end{bmatrix} \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 4 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \delta \end{bmatrix}.$$

Since  $\pi|_{S_1 B_1 F B_1^\top S_1^{-1}}$  is surjective and  $U$  is nonsingular,  $d\psi_0$  is onto.  $\square$

The second main theorem in this paper deals with the limit case  $\delta = \binom{n+1}{2}$ , where we can prove a more precise statement.

**THEOREM 3.6.** *Let  $\delta = \binom{n+1}{2}$  in the symmetric case, and  $\delta = n(n+1)$  for Hamiltonian systems. Then for a generic set of  $n \times n$  symmetric (or Hamiltonian) transfer functions of degree  $\delta$ , the number of pole assigning complex symmetric feedback compensators is finite, and when counted with multiplicities there are exactly*

$$(3.3) \quad d(n) := 2^{\binom{n}{2}} \frac{\binom{n+1}{2}! 1! 2! \cdots (n-1)!}{1! 3! \cdots (2n-1)!} = \frac{\binom{n+1}{2}!}{\prod_{i=0}^{n-1} (2i+1)^{n-i}}$$

many symmetric compensators as solution.

One immediately computes  $d(1) = 1$ ,  $d(2) = 2$ ,  $d(3) = 2^4$ ,  $d(4) = 3 \cdot 2^8$ ,  $d(5) = 11 \cdot 13 \cdot 2^{11}$ , and  $d(6) = 13 \cdot 17 \cdot 19 \cdot 2^{18}$ . The integer sequence  $d(n)$  is sequence A005118 in Sloane’s data bank of integer sequences [33]. The sequence has several combinatorial and geometric interpretations. For the context of this paper it will be important that  $d(n)$  is equal to the degree of the Lagrangian Grassmannian, the projective variety of all maximal isotropic subspaces in a complex vector space of dimension  $2n$ .

As for the Grassmann variety, classical Schubert calculus [30] (see [11, Chapter 14] for a modern treatment of Schubert calculus) provides the tools to compute the degree of the Lagrangian Grassmannian. An explicit formula for the integers  $d(n)$  was probably first given by Hiller [17], who computed

$$(3.4) \quad d(n) = \begin{cases} 2^{\binom{n}{2}} \frac{\binom{n+1}{2}! 2! 4! \cdots (n-2)!}{(n+1)1! (n+3)! \cdots (2n-1)!} & \text{if } n \text{ is even,} \\ 2^{\binom{n}{2}} \frac{\binom{n+1}{2}! 2! 4! \cdots (n-2)!}{(n+1)1! (n+3)! \cdots (2n-1)!} & \text{if } n \text{ is odd.} \end{cases}$$

The hard combinatorial work to derive formula (3.4) is actually due to Schur [31]. It has been pointed out by Totaro [36] that  $d(n)$  is equal to the Kostka number  $K_{\lambda, (1^N)}$ , where  $\lambda$  is the partition  $\lambda = (n, n-1, \dots, 1)$  and  $N = \binom{n+1}{2}$ . Totaro derived an explicit formula for more general Kostka numbers, and Totaro’s formula specializes to formula (3.3). Readers interested in combinatorial aspects of Kostka numbers should consult the book by Macdonald [22].

It can be seen from the formulas that  $d(n)$  is always even, except for  $n = 1$ . This is related to the fact that the symmetric output feedback pole placement problem is *not* generically solvable over the reals. Actually, more can be said about  $d(n)$ . The sequence

$$\tilde{d}(n) := d(n)2^{-\binom{n}{2}}$$

is the degree of the spinor variety, the complex projective variety  $SO(2n+1)/U(n)$  [17]; in particular,  $\tilde{d}(n)$  represents an integer sequence again. The sequence  $\tilde{d}(n)$  appears under the number A003121 in Sloane’s data bank [33].

The proof of Theorem 3.6 will occupy the rest of this section. The proof will necessitate a geometric reformulation and several technical lemmas.

First we will describe the closed loop characteristic equation in a slightly more convenient way. Consider a left coprime factorization  $D^{-1}(s)N(s) = G(s)$  of the symmetric or Hamiltonian transfer function  $G(s)$ . Let  $F \in \text{Sym}(n)$  be an  $n \times n$  complex symmetric matrix. When the feedback law  $y = -Fu + v$  is applied, then up to a constant factor the characteristic polynomial  $\varphi(s)$  is also equal to

$$(3.5) \quad \det \begin{bmatrix} D(s) & N(s) \\ F & I_n \end{bmatrix}.$$

The vector space  $\text{Sym}(n)$  describing the set of  $n \times n$  complex symmetric matrices is not very well suited to invoking strong theorems from algebraic geometry and intersection theory [11], as these usually require compactness assumptions on the underlying spaces. A similar difficulty exists for the static output pole placement problem. Brockett and Byrnes showed in [3] how to translate the static pole placement problem into a geometric problem. This then resulted in an intersection problem on a compact Grassmann variety, and methods from classical Schubert calculus [30, 35] could be invoked.

We will follow this compactification strategy for Problem 3.1 as well. This will lead us to an intersection problem on some projective variety. In order to do so we therefore need a good compactification of  $\text{Sym}(n)$ . For this identify the row span  $\text{rowsp}[F \ I_n]$  of any symmetric matrix  $F$  with an element of the Grassmann variety  $\text{Grass}(n, \mathbb{C}^{2n})$ . Using the Plücker embedding

$$\text{Grass}(n, \mathbb{C}^{2n}) \longrightarrow \mathbb{P}(\wedge^n \mathbb{C}^{2n}) = \mathbb{P}^N, \quad N = \binom{2n}{n} - 1,$$

we can then identify  $\text{Sym}(n)$  with a quasi-projective subset of the complex projective variety  $\mathbb{P}^N$ .

DEFINITION 3.7. The algebraic closure of the set

$$\{\text{rowsp}[F \ I_n] \mid F \in \text{Sym}(n)\}$$

is called the *complex Lagrangian Grassmann manifold*. It will be denoted by  $\mathbb{L}G(n)$ .

It is well known that  $\mathbb{L}G(n)$  is a smooth projective variety of dimension  $\binom{n+1}{2}$ , the dimension of  $\text{Sym}(n)$ . Note that every element in  $\mathbb{L}G(n)$  can be simply represented by a subspace of the form  $\text{rowsp}[F_1 \ F_2]$ , where  $F_1(F_2)^\top$  is a symmetric matrix, i.e.,  $F_1(F_2)^\top = F_2(F_1)^\top$ . The elements of  $\mathbb{L}G(n)$  are thus exactly the Lagrangian subspaces of  $\mathbb{C}^{2n}$ . The subspace  $\text{rowsp}[F_1 \ F_2]$  coincides with the subspace  $\text{rowsp}[S \ I_n]$  associated with an element  $S$  of  $\text{Sym}(n)$  if and only if  $F_2$  is invertible. Moreover, then  $S = (F_2)^{-1}F_1$ . When  $F_2$  is singular one can still define a characteristic polynomial through

$$(3.6) \quad \varphi(s) := \det \begin{bmatrix} D(s) & N(s) \\ F_1 & F_2 \end{bmatrix}.$$

Note that in the Hamiltonian case,  $\varphi(s)$  is necessarily even; i.e.,

$$\varphi(s) = \varphi(-s).$$

Let  $f_i, i = 0, \dots, N$ , be the Plücker coordinates of  $\text{rowsp}[F_1 \ F_2]$ . In terms of the Plücker coordinates the characteristic equation can then be written as

$$(3.7) \quad \det \begin{bmatrix} D(s) & N(s) \\ F_1 & F_2 \end{bmatrix} = \sum_{i=0}^N p_i(s) f_i,$$

where  $p_i(s)$  is the cofactor of  $f_i$  in the determinant (3.7).

Let  $\mathcal{Z} \subset \mathbb{P}^N$  be the linear subspace defined by

$$(3.8) \quad \mathcal{Z} = \left\{ z \in \mathbb{P}^N \mid \sum_{i=0}^N p_i(s) z_i = 0 \right\}.$$

Following [19, 27, 28, 37] we identify a closed loop characteristic polynomial  $\varphi(s)$  with a point in  $\mathbb{P}^\delta$ . In analogy to the situation of the static pole placement problem considered in [3, 37] (compare also with [28, section 5]), one has a well-defined characteristic map

$$(3.9) \quad \begin{aligned} \chi : \quad \mathbb{L}G(n) - \mathcal{Z} &\longrightarrow \mathbb{P}^\delta \\ \text{rowsp}[F_1 \ F_2] &\longmapsto \sum_{i=0}^N f_i p_i(s) \end{aligned}$$

in the complex symmetric case and

$$(3.10) \quad \begin{aligned} \chi' : \quad \mathbb{L}G(n) - \mathcal{Z} &\longrightarrow \mathbb{P}^{\delta/2} \\ \text{rowsp}[F_1 \ F_2] &\longmapsto \text{even part of } \sum_{i=0}^N f_i p_i(s) \end{aligned}$$

in the Hamiltonian case. In the latter case the reduction in dimension of the projective space arises due to the evenness of the closed loop characteristic polynomial, so that in the second map only the coefficients of the even terms of  $\sum_{i=0}^N f_i p_i(s)$  appear.

Recall the notion of degree of a variety [13, Chapter I, section 7] and the notion of a central projection (see [32, Chapter I, section 4]). The geometric properties of the map  $\chi$  are as follows.

**THEOREM 3.8.** *The maps  $\chi, \chi'$  define central projections. In particular, if  $\mathcal{Z} \cap \mathbb{L}G(n) = \emptyset$  and  $\dim \mathbb{L}G(n) = \binom{n+1}{2} = \delta$ , then  $\chi$  is surjective, and there are  $\deg \mathbb{L}G(n)$  many preimages (counted with multiplicity) for each point in  $\mathbb{P}^\delta$ , where  $\deg \mathbb{L}G(n)$  is the degree of the Lagrangian manifold  $\mathbb{L}G(n)$  in  $\mathbb{P}^N$ . Similarly, if  $\dim \mathbb{L}G(n) = \binom{n+1}{2} = \delta/2$ , then  $\chi'$  is surjective with exactly  $\deg \mathbb{L}G(n)$  many preimage points in each fiber.*

*Proof.* By definition (see, e.g., [26, 32]),  $\chi$  represents a central projection of  $\mathbb{L}G(n)$  from the center  $\mathcal{Z}$  to  $\mathbb{P}^\delta$ . When  $\mathcal{Z} \cap \mathbb{L}G(n) = \emptyset$  and  $\dim \mathbb{L}G(n) = \binom{n+1}{2} = \delta$ , then  $\chi$  is a finite morphism [32, Chapter I, section 5, Theorem 7] and onto of degree  $\deg \mathbb{L}G(n)$  [26, Corollary 5.6]. Similar argument can be applied to  $\chi'$ .  $\square$

The set  $\mathcal{Z} \cap \mathbb{L}G(n)$  is sometimes referred to as the *base locus*. The interesting part of the theorem occurs when the base locus  $\mathcal{Z} \cap \mathbb{L}G(n) = \emptyset$  since in this situation very specific information on the number of solutions is provided. If  $\mathcal{Z} \cap \mathbb{L}G(n) = \emptyset$  and  $\binom{n+1}{2} = \delta$  (or  $n(n+1) = \delta$ ), then one says that  $\chi$  (or  $\chi'$ ) describes a *finite morphism* from the projective variety  $\mathbb{L}G(n)$  onto the projective space  $\mathbb{P}^\delta$  (or  $\mathbb{P}^{\delta/2}$ ).

This last situation is most desirable, and this motivates the following definition.

**DEFINITION 3.9.** A particular symmetric transfer function  $G(s)$  is called *nondegenerate* if  $\mathcal{Z} \cap \mathbb{L}G(n) = \emptyset$ . A system which is not nondegenerate will be called degenerate.

In terms of matrices a symmetric transfer function  $G(s) = D(s)^{-1}N(s)$  is degenerate as soon as there is a Lagrangian subspace  $\text{rowsp}[F_1 \ F_2] \in \mathbb{L}G(n)$ , such that

$$\det \begin{bmatrix} D(s) & N(s) \\ F_1 & F_2 \end{bmatrix} = 0.$$

In more geometric language this means that the Hermann–Martin curve [25] defined by  $\text{rowsp}[D(s) \ N(s)]$  is fully contained in a Lagrangian hyperplane defined by  $\text{rowsp}[F_1 \ F_2]$ . In the study of the static pole placement problem [3] and the dynamic pole placement problem [28], definitions analogous to Definition 3.9 played an important role.

The next lemmas give specific information as to under which conditions  $\mathcal{Z} \cap \mathbb{L}G(n) = \emptyset$ , i.e., under which conditions a symmetric transfer function is nondegenerate. Similar results were crucial in proving the pole placement results in [3, 19, 28].

LEMMA 3.10. *If  $\delta < \binom{n+1}{2} = \dim \mathbb{L}G(n)$ , then every  $n \times n$  symmetric transfer function of McMillan degree  $\delta$  is degenerate. Similarly, any  $n \times n$  Hamiltonian transfer function of McMillan degree  $\delta$  is degenerate if  $\delta < n(n+1) = 2 \dim \mathbb{L}G(n)$ .*

*Proof.*  $\dim \mathcal{Z} \geq N - \delta - 1$  as  $\mathcal{Z}$  is defined by  $\delta + 1$  linear equations ( $\delta/2 + 1$  in the Hamiltonian case). If  $\dim \mathbb{L}G(n) > \delta$  (or  $\dim \mathbb{L}G(n) > \delta/2$  in the Hamiltonian case), then  $\mathcal{Z} \cap \mathbb{L}G(n)$  is nonempty by the (projective) dimension theorem (see, e.g., [13, Chapter I, Theorem 7.2]).  $\square$

LEMMA 3.11. *If  $\delta = \binom{n+1}{2} = \dim \mathbb{L}G(n)$  (or  $\delta = n(n+1)$ ), then a generic set of  $n \times n$  symmetric (or Hamiltonian) transfer functions of McMillan degree  $\delta$  is nondegenerate.*

*Proof.* Let  $\mathcal{Q}$  be the set of all  $n \times n$  symmetric transfer functions of McMillan degree  $n$ .  $\mathcal{Q}$  can be given the structure of a quasi-projective variety. For this, recall the definition of the projective variety  $K_{n,n}^\delta$  that was introduced in [28] and which compactifies the set of all  $n \times n$  transfer functions of McMillan degree  $\delta$ . An element (Hermann–Martin curve)  $\text{rowsp}[D(s) \ N(s)] \in K_{n,n}^\delta$  describes an element of  $\mathcal{Q}$  as soon as  $\deg \det D(s) = \delta$  and  $D(s)N(s)^\top = N(s)D(s)^\top$ . The last condition translates into some linear conditions to be satisfied among the Plücker coordinates of  $K_{n,n}^\delta$ . The resulting subvariety of  $K_{n,n}^\delta$  constitutes a natural compactification of  $\mathcal{Q}$ , and  $\mathcal{Q}$  itself is a quasi-projective subset.

Consider now the coincidence set

$$\mathcal{S} := \left\{ (D(s)^{-1}N(s); F_1, F_2) \in \mathcal{Q} \times \mathbb{L}G(n) \mid \det \begin{bmatrix} D(s) & N(s) \\ F_1 & F_2 \end{bmatrix} = 0 \right\}.$$

Since  $\mathbb{L}G(n)$  is projective, the projection onto  $\mathcal{Q}$  is an algebraic set by the main theorem of elimination theory (see, e.g., [26]). The set of nondegenerate systems therefore forms a Zariski open subset of  $\mathcal{Q}$ . We have shown the result if we can exhibit one  $n \times n$  transfer function of McMillan degree  $\binom{n+1}{2}$  which is nondegenerate. The next lemma gives such an example, and the claim therefore follows. Note that the previous arguments run in a completely similar manner for the Hamiltonian case, and it therefore remains to construct one example as well. However, the symmetric Hamiltonian transfer function  $G(s^2)$  is exactly such an example.  $\square$

LEMMA 3.12. *The symmetric transfer function*

$$G(s) := \begin{bmatrix} \frac{1}{s} & & & & \\ & \frac{1}{s^2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{1}{s^n} \end{bmatrix}$$

is nondegenerate.

*Proof.* First it is clear that  $G(s)$  has McMillan degree  $\delta = \binom{n+1}{2}$  and that

$$[D(s) \ N(s)] = \left[ \begin{array}{cccc|cccc} s & & & & 1 & & & \\ & s^2 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & s^n & & & & 1 \end{array} \right]$$

forms a left coprime factorization of  $G(s)$ . Let

$$R := \left[ \begin{array}{cccc} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{array} \right]$$

and assume by contradiction that  $G(s)$  is degenerate. Then there exists rowsp  $[F_1 \ F_2] \in \mathbb{L}G(n)$ , such that

$$(3.11) \quad 0 = \det \begin{bmatrix} D(s) & N(s) \\ F_1 & F_2 \end{bmatrix} = \det \begin{bmatrix} D(s) & N(s)R \\ F_1 & F_2R \end{bmatrix}.$$

Let  $S \in Gl_n$  be the matrix which transforms the  $n \times 2n$  matrix  $[F_1 \ F_2R]$  into row reduced echelon form, i.e.,

$$(3.12) \quad [(SF_1) \ (SF_2R)] = \left[ \begin{array}{cccccccccccccccc} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots & * & \cdots & * & 1 & 0 & \cdots & 0 \end{array} \right] =: [\tilde{F}_1 \ \tilde{F}_2].$$

Let

$$i_1 < \cdots i_k \leq n < i_{k+1} < \cdots i_n \leq 2n$$

be the pivot indices. We claim that the first  $k$  pivot indices determine the last  $n - k$  pivot indices uniquely. For this let  $\hat{i}_1 < \cdots < \hat{i}_{n-k}$  be the complementary indices of the indices  $\{i_1, \dots, i_k\}$  inside the set  $\{1, \dots, n\}$ . Then we claim that

$$\begin{aligned} i_{k+1} &= 2n - \hat{i}_{n-k} + 1 \\ &\vdots \\ i_n &= 2n - \hat{i}_1 + 1. \end{aligned}$$

Indeed, if this is not the case, then it follows that  $\tilde{F}_1R(\tilde{F}_2)^\top$  cannot be symmetric for any choice of values in the row reduced echelon form (3.12). On the other hand, the matrix  $\tilde{F}_1R(\tilde{F}_2)^\top$  has to be symmetric since by assumption  $F_1(F_2)^\top$  is symmetric.

The indices  $i_1, \dots, i_n$  describe the maximal Plücker coordinate (with regard to the Bruhat order) of rowsp  $[F_1 \ F_2R]$  which is nonzero, and the corresponding cofactor of  $[D(s) \ N(s)R]$  is computed as  $\pm s^\alpha$ , where  $\alpha = \sum_{\ell=1}^{n-k} \hat{i}_\ell$ . In general there are other full-size minors (Plücker coordinates) of  $[D(s) \ N(s)R]$  which have the form  $\pm s^\alpha$ . All

other Plücker coordinates with this value, however, are not comparable with regard to the Bruhat order, and since  $i_1, \dots, i_n$  was the maximal nonzero Plücker coordinate of  $\text{rowsp} [F_1 \ F_2 R]$ , it follows that the determinant expansion in (3.11) cannot be zero. This is a contradiction, and it follows that  $G(s)$  is nondegenerate.  $\square$

*Remark 3.13.* For the static pole placement problem Brockett and Byrnes [3] showed that the osculating normal curve

$$\text{rowsp} \begin{bmatrix} 1 & s & s^2 & \dots & \dots & s^{m+p-1} \\ 0 & 1 & 2s & \dots & \dots & \binom{m+p-1}{1} s^{m+p-2} \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 0 & 1 & \dots & \binom{m+p-1}{m-1} s^p \end{bmatrix} \in \text{Grass}(m, \mathbb{K}^{m+p})$$

is nondegenerate. Also in this situation the Plücker coordinates have the simple form  $\pm s^\beta$ , where  $\beta = \sum_{\ell=1}^m i_\ell - \ell$  and there are no two Plücker coordinates which are comparable in the Bruhat order and give rise to the same monomial  $s^\beta$ .

We have now all pieces together in order to prove the main result.

*Proof of Theorem 3.6.* Without loss of generality we focus on the case of symmetric transfer functions. The arguments for the Hamiltonian case run in a completely similar manner. Note, however, that the closed loop characteristic polynomial of a Hamiltonian system is always an *even* polynomial. Therefore our definition of generic pole assignability for Hamiltonian systems restricts to the space of even polynomials. Since the dimension of the space of even monic polynomials of degree  $\delta$  is  $\delta/2$ , the appropriate condition for Hamiltonian systems is  $\delta/2 \leq \binom{n+1}{2}$ . With these comments in mind, we return to the proof for symmetric transfer functions.

When  $\delta > \binom{n+1}{2}$ , then a simple dimension argument shows that the image of the characteristic map  $\chi$  described in (3.9) has dimension at most  $\binom{n+1}{2}$  and therefore there is a Zariski open set in  $\mathbb{P}^\delta$  not in the image of  $\chi$ .

When  $\delta = \binom{n+1}{2}$ , then Lemmas 3.11 and 3.12 show that there is a generic set of  $n \times n$  symmetric transfer functions of McMillan degree  $\delta$  which are nondegenerate. The characteristic map (3.9) therefore has no base locus, and every point in the image of  $\chi$  has  $\deg \mathbb{L}G(n)$  preimage points when counted with multiplicities. The degree of the variety  $\mathbb{L}G(n)$  was recently computed by Totaro [36] and resulted in the number (3.3).

A priori the geometric formulation only predicts  $\deg \mathbb{L}G(n)$  many solutions inside  $\mathbb{L}G(n)$ , and it is not clear if all these solutions correspond to regular feedback laws of the form  $u = -Fy + v$ . If  $G(s)$  is a strictly proper symmetric transfer function, then this is indeed the case and the same argument applies as in [3].  $\square$

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