SPHERE PACKING LOWER BOUNDS: NEW DEVELOPMENTS

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INTRODUCTION: SPHERE PACKING

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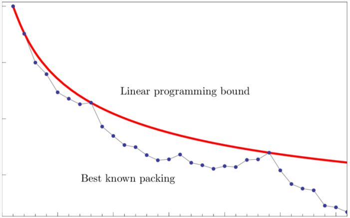
Figure: SP bounds by dimension (Hartman et al., log-scale)



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High-dimensional sphere packings are a fascinating subject (error-correcting codes, mystery,...) but the best known packings achieve exponentially less than upper bounds such as

$$\Delta_n \leq \frac{2^{n \cdot 0.401\ldots}}{2^n}$$

due to Kabatiansky-Levenshtein. We embark on a quest for **structure** (or lack thereof..) !

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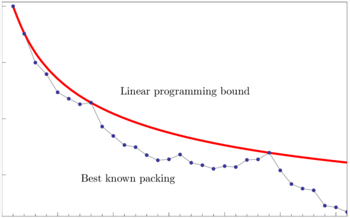

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N.B.: All of these results are **existential/nonconstructive** and most relate to some Siegel Mean Value Theorem: for random lattices in $\mathbb{R}^n$ (fix covolume one) the average lattice sum for a nice function f satisfies

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Q: What is optimal Δ_n? Explicit constructions?

Effective lower bounds

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Construction A-style: consider sets of pre-images of codes via reduction maps like:

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One can then often show that the existential lower bounds on $\mathbf{\Delta}_n$ are approached (up to arbitrary precision) by a lattice in a finite set (alas exp. size in n) of pre-images $\phi_p^{-1}(C)$ of codes C.

DIVISION RINGS AND EFFECTIVE RESULTS

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Or consider instead of $\mathbb{Q}$ the (non-commutative) quaternion algebra

$$\left(\frac{-1,-1}{\mathbb{Q}}\right) = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Q}, i^2 = j^2 = -1, ij = -ji = k\}$$

and the subring of Hurwitz integers $\mathcal{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}\}$.

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above primes p split in A (here we take $t \geq 2$ copies).

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We consider subsets of the sets of codes:

$$\mathcal{C}_{k,p} = \{C \subset M_n(\mathbb{F}_q)^t : C \cong \mathbb{F}_q^{nk} \text{ as } \mathbb{F}_q - \text{modules}\}.$$

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The preimages via ϕ_p of $C \in \mathcal{C}_{k,p}$ are sublattices of $\mathcal{O}^t$. We consider the rescaled set:

$$\mathbb{L}_p = \{\beta_p \phi_p^{-1}(C) \mid C \in \mathcal{C}_{k,p}\} \text{ for } \beta_p = q^{\frac{nk-n^2t}{n^2mt}}.$$

Main technical result:

Theorem (Gargava, S.)
Let $f : \mathbb{R}^{n^2 mt} \to \mathbb{R}$ *be a nice (integrable, rapid decay) function. With the notations as above, provided* $(n-1)t < k < nt$*, we have that*

$$\lim_{p\to\infty} \mathbb{E}_{\mathbb{L}_{k,p}} \left(\sum_{x \in (\beta_p \phi_p^{-1}(C))'} f(i(x)) \right) \leq (\zeta(n^2 mt) \cdot \mathrm{Vol}(\mathcal{O}^t))^{-1} \cdot \int_{\mathbb{R}^{n^2 mt}} f(x) dx.$$

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We take as codes k copies of the simple $M_n(\mathbb{F}_q)$-modules $\mathbb{F}_q^n$ and as bad set $B_p \subset M_n(\mathbb{F}_q)^t$ the points with at least one coordinate non-invertible in $M_n(\mathbb{F}_q)$.

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There exists $\forall \varepsilon > 0$ a n^2mt-dimensional sub-lattice $\Lambda_\varepsilon \subset \mathcal{O}^t$ with packing density

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Proposition (Gargava, S.)

Let $m_k = \prod_{\substack{p \leq k \text{ prime} \\ 2 \nmid \mathrm{ord}_2 p}} p$ and set $n_k := 8\varphi(m_k)$. Then $\forall \varepsilon > 0$, $\exists$ effective c_ε such that for $k > c_\varepsilon$ a lattice Λ in dimension n_k with density $\Delta(\Lambda) \geq (1-\varepsilon)\frac{24 \cdot m_k}{2^{n_k}}$ can be constructed in $e^{4.5 \cdot n_k \log(n_k)(1+o(1))}$ binary operations. This construction leads to

$$\Delta(\Lambda) \geq (1 - e^{-n_k}) \frac{3 \cdot n_k (\log\log n_k)^{7/24}}{2^{n_k}}$$

asymptotically in dimension n_k.

Conclusion

The improved constant means we get the best effective bounds up to dimension $\approx 1.98 \cdot 10^{46}$ roughly. Can also recover a full $n \log \log n$ improvement with non-commutative symmetry.

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Thank you!