Computing a Minimal Input-State-Output Representation of Convolutional Codes

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Outline

Overview

Motivation

- Obtain an algorithm which provide a minimal ISO representation of a k/n-rate convolutional code.
- Apply the previous algorithm to provide a minimal ISO representation of a convolutional product code.


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Linear Block codes

Let $\mathbb{F}$ be a finite field and $n \in \mathbb{N}$. A linear block code of length n over $\mathbb{F}$ is a linear subspace $\mathcal{B}$ of $\mathbb{F}^n$. If $\dim \mathcal{B} = k$, then $\mathcal{B}$ is called an $[n, k]$-code and

$$\mathcal{B} = \left\{ \boldsymbol{v} \in \mathbb{F}^n \mid \boldsymbol{v} = \boldsymbol{u}G \text{ with } \boldsymbol{u} \in \mathbb{F}^k \right\}$$

where

- $G \in \mathbb{F}^{k \times n}$ with $\operatorname{rank}(G) = k$ is the **generator matrix** of $\mathcal{B}$,
- $\boldsymbol{u} \in \mathbb{F}^k$ is called **information word**,
- $\boldsymbol{v} \in \mathcal{B}$ is called **codeword**.

Equivalently, an $[n, k]$-code can be described as

$$\mathcal{B} = \left\{ \boldsymbol{v} \in \mathbb{F}^n \mid H\boldsymbol{v}^T = 0 \right\}$$

where $H \in \mathbb{F}^{(n-k) \times n}$ with $\operatorname{rank}(H) = n - k$ is called a **parity-check matrix** of $\mathcal{B}$.

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Linear Block Codes

Theorem

Let $G \in \mathbb{F}^{k\times n}$ and $H \in \mathbb{F}^{(n-k)\times n}$ such that $rank(G) = k$ and $rank(H) = n - k$.

1. *Assume that G is a generator matrix of an $[n, k]$-code $\mathcal{B}$. Then, H is a parity-check matrix of $\mathcal{B}$ if and only if $HG^T = 0$.*
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Linear Block Codes

We say that $G \in \mathbb{F}^{k \times n}$ is a **systematic generator matrix** of $\mathcal{B}$ if $G = [I_k \ A]$ for some $A \in \mathbb{F}^{k \times (n-k)}$.

Theorem

Let $\mathcal{B}$ be an $[n, k]$-code and consider $A \in \mathbb{F}^{k \times (n-k)}$. Then $G = [I_k \ A]$ is a systematic generator matrix of $\mathcal{B}$ if and only if $H = [-A^T \ I_{n-k}]$ is a systematic parity-check matrix of $\mathcal{B}$.

For any non-systematic generator matrix $G \in \mathbb{F}^{k \times n}$, there exists a permutation matrix $P \in \mathbb{F}^{n \times n}$ such that $GP = [I_k \ A]$, for some $A \in \mathbb{F}^{k \times (n-k)}$. Therefore, $H = [-A^T \ I_{n-k}]P^T$ is a non-systematic parity-check matrix.

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Overview

Convolutional Codes

Definition

A convolutional code $\mathcal{C}$ is a $\mathbb{F}[z]$-module of $\mathbb{F}[z]^n$.

A matrix $G(z) \in \mathbb{F}[z]^{n \times k}$ whose rows form a basis for $\mathcal{C}$ is called an encoder or generator matrix. If $\mathcal{C}$ has rank k then we say that $\mathcal{C}$ has rate k/n.

$$\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z]} G(z) = \left\{ \boldsymbol{v}(z) = \boldsymbol{u}(z)G(z) \mid \boldsymbol{u}(z) \in \mathbb{F}[z]^k \right\}$$

where $\boldsymbol{u}(z)$ is the **information vector**, and $\boldsymbol{v}(z)$ is the corresponding **codeword**.

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Convolutional Codes

Assume that

$$G(z) = \begin{bmatrix} \boldsymbol{g}_1(z) & \boldsymbol{g}_2(z) & \cdots & \boldsymbol{g}_k(z) \end{bmatrix} \in \mathbb{F}[z]^{n\times k}$$

$$\boldsymbol{g}_j(z) = \boldsymbol{g}_j^{(0)} + \boldsymbol{g}_j^{(1)} z + \cdots + \boldsymbol{g}_j^{(\nu_j-1)} z^{\nu_j-1} + \boldsymbol{g}_j^{(\nu_j)} z^{\nu_j},$$

$$\boldsymbol{g}_j^{(\ell)} \in \mathbb{F}^n \quad j = 1, 2, \ldots, k, \ \ell = 0, 1, \ldots, \nu_j,$$

$$\boldsymbol{g}_j^{(\nu_j)} \neq 0, \ j = 1, 2, \ldots, k,$$

where ν_j is the j-th column degree of $G(z)$, for $j = 1, 2, \ldots, k$.

We can express $G(z) = G_{LD}X(z) + G_{\infty}Y(z)$, where:

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We define the **less-degree coefficients matrix**, as follows

$$G_{LD} = \left[\boldsymbol{g}_1^{(0)} \cdots \boldsymbol{g}_1^{(\nu_1 - 1)} \mid \boldsymbol{g}_2^{(0)} \cdots \boldsymbol{g}_2^{(\nu_2 - 1)} \mid \cdots \mid \boldsymbol{g}_k^{(0)} \cdots \boldsymbol{g}_k^{(\nu_k - 1)} \right] \in \mathbb{F}^{n \times \delta}.$$

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$$X(z) = \begin{bmatrix} X_1(z) & & & \\ & X_2(z) & & \\ & & \ddots & \\ & & & X_k(z) \end{bmatrix}, \; X_j(z) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\nu_j - 1} \end{bmatrix}, \text{ with } j = 1, 2, \ldots, k$$

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$$Y(z) = \begin{bmatrix} z^{\nu_1} & & & \\ & z^{\nu_2} & & \\ & & \ddots & \\ & & & z^{\nu_k} \end{bmatrix}.$$

Convolutional Code

Theorem

If $\mathcal{C}$ is a rate k/n convolutional code, then $\mathcal{C}$ admits a generator matrix $G(z)$ which is:

- *right-prime*
- *column reduced*

Therefore, $\delta = \sum_{j=1}^{k} \nu_j$ is the **degree** or **complexity** of $\mathcal{C}$.

Overview

Convolutional code as a linear system

A rate k/n convolutional code can be described by a time invariant linear system:

$$\left.\begin{aligned} \boldsymbol{x}_{t+1} &= A\boldsymbol{x}_t + B\boldsymbol{u}_t \\ \boldsymbol{y}_t &= C\boldsymbol{x}_t + D\boldsymbol{u}_t \end{aligned}\right\}, \quad \boldsymbol{v}_t = \begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{u}_t \end{bmatrix}, \quad t \geq 0, \quad \boldsymbol{x}_0 = \mathbf{0},$$

where $A \in \mathbb{F}^{m\times m}$, $B \in \mathbb{F}^{m\times k}$, $C \in \mathbb{F}^{(n-k)\times m}$ and $D \in \mathbb{F}^{(n-k)\times k}$.

- $\boldsymbol{x}_t \in \mathbb{F}^m$, **state vector**,
- $\boldsymbol{u}_t \in \mathbb{F}^k$, **input vector**,
- $\boldsymbol{y}_t \in \mathbb{F}^{n-k}$, **output vector (redundancy)**,
- (A, B, C, D) is a **realization** of $G(x)$ of dimension m, known as the **Input-state-output (ISO) representation**.

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ISO representation

- A realization (A, B, C, D) of $G(z)$ is **minimal** if $m = \delta$.
- (A, B, C, D) is minimal $\leftrightarrow$ (A, B) is reachable and (A, C) is observable.

(A, B) is called **reachable** if

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{\delta-1}B \end{bmatrix} = \delta.$$

(A, C) is **observable** $\leftrightarrow$ (A^T, C^T) is reachable.

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First order representation

A minimal ISO representation (A, B, C, D) can be obtained from a first order representation (K, L, M).

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First order representation

Theorem (Theorem 3.1 [1])

Assume $\mathcal{C} \subseteq \mathbb{F}[z]^n$ is a rate k/n convolutional code of complexity δ. Then, there exist $(\delta + n - k) \times \delta$ matrices K, L and a $(\delta + n - k) \times n$ matrix M (all defined over $\mathbb{F}$) such that the code $\mathcal{C}$ is described by

$$\mathcal{C} = \left\{ v(z) \in \mathbb{F}[z]^n \mid \exists x(z) \in \mathbb{F}[z]^\delta : zKx(z) + Lx(z) + Mv(z) = 0 \right\}$$

(K, L, M) is known as a **first order representation** *of $\mathcal{C}$. Moreover, it is minimal, if it satisfies:*

1. *K has full column rank,*
2. *$[K \ M]$ has full row rank,*
3. *$[zK + L \mid M]$ is left-prime.*

From First order representation to ISO representation

If $G(z)$ is a generator matrix of $\mathcal{C}$, then (K, L, M) satisfies that

$$[K\ L\ M] \begin{bmatrix} zX(z) \\ X(z) \\ G(z) \end{bmatrix} = 0.$$

Rosenthal's Algorithm

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1. Let $G(z)$ be a generator matrix of a k/n-rate convolutional code.
2. Given $[zX(z)^T, X(z)^T, G(z)^T] \in \mathbb{F}^{(\delta+k)\times(2\delta+n)}$, compute its scalar matrix which describes a linear map $\Phi : \mathbb{F}^{2\delta+k} \rightarrow \mathbb{F}^{\delta+k}$. The vectors of the basis of the Kernel of Φ define a matrix from which they can obtain a minimal first-order representation $(K,\ L,\ M)$.
3. *After a suitable similarity transformation and a permutation (if needed)*, they can obtain

$$[K \mid L \mid M] \sim \left[\begin{array}{c|c|cc} -I & A & 0 & B \\ 0 & C & -I & D \end{array}\right]$$

4. (A, B, C, D) is a minimal ISO representation of $\mathcal{C}$.

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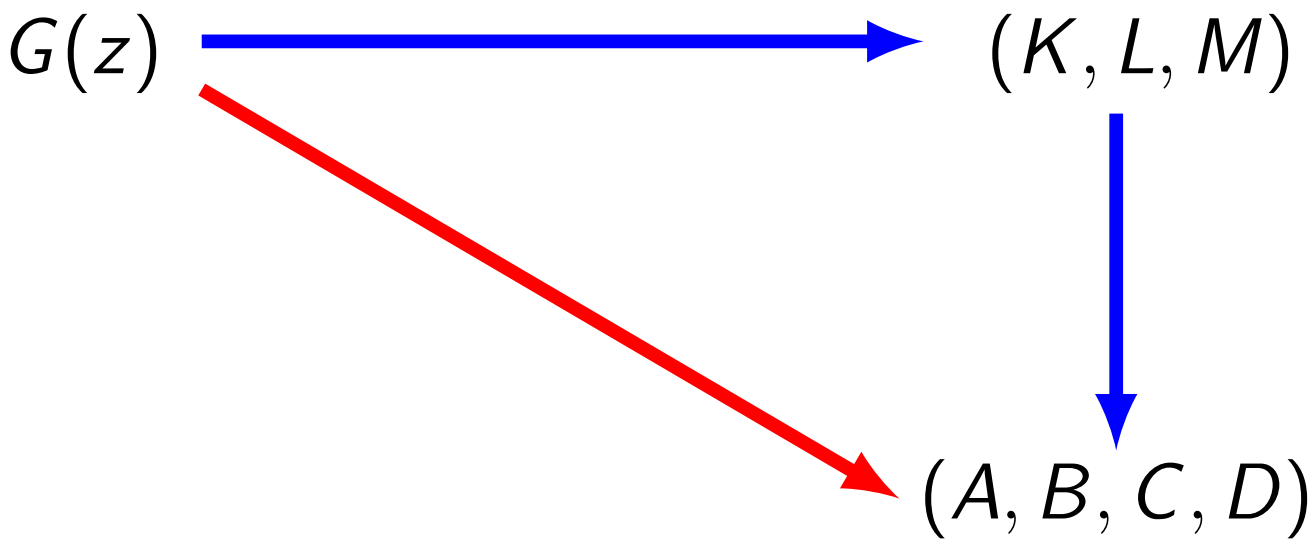
From First order representation to ISO representation



Our minimal ISO representation

Aim

We provide an algorithm to obtain a minimal ISO representation (A, B, C, D), using its generator matrix (not (K, L, M)).

Conditions for $G(z)$, generator matrix of a (n, k) convolutional code $\mathcal{C}$:

- $G(z)$ is right-prime.
- $G(z)$ is column reduced.
- Its complexity $\delta = \sum_{i=1}^{k} \nu_i$.
- Assume that the rows which determine the column degrees are in the last k rows of $G(z)$.

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ISO representation

$$[K\ L\ M]\begin{bmatrix} zX(z) \\ X(z) \\ G(z) \end{bmatrix} = 0 \longleftrightarrow [K\ L\ M]\,\tilde{G}^T = 0$$

$$\tilde{G} = \left[\begin{array}{cccccccc}
\mathbf{0}^T\ 0 & & & 1\ \mathbf{0}^T & & & & \boldsymbol{g}_1^{(0)T} \\
I_{\nu_1-1}\mathbf{0} & & & \mathbf{0}I_{\nu_1-1} & & & & \bar{\boldsymbol{g}}_1^T \\
\mathbf{0}^T\ 1 & & & 0\ \mathbf{0}^T & & & & \boldsymbol{g}_1^{(\nu_1)T} \\
\hline
 & \mathbf{0}^T\ 0 & & & 1\ \mathbf{0}^T & & & \boldsymbol{g}_2^{(0)T} \\
 & I_{\nu_2-1}\mathbf{0} & & & \mathbf{0}I_{\nu_2-1} & & & \bar{\boldsymbol{g}}_2^T \\
 & \mathbf{0}^T\ 1 & & & 0\ \mathbf{0}^T & & & \boldsymbol{g}_2^{(\nu_2)T} \\
\hline
 & & \ddots & & & \ddots & & \vdots \\
\hline
 & & & \mathbf{0}^T\ 0 & & & 1\ \mathbf{0}^T & \boldsymbol{g}_k^{(0)T} \\
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 & & & \mathbf{0}^T\ 1 & & & 0\ \mathbf{0}^T & \boldsymbol{g}_k^{(\nu_k)T}
\end{array}\right]$$

with $\bar{\boldsymbol{g}}_j = [\boldsymbol{g}_j^{(1)}\ \boldsymbol{g}_j^{(2)}\ \cdots\ \boldsymbol{g}_j^{(\nu_j-1)}]$, for $j = 1, 2, \ldots, k$

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$$\tilde{G} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \begin{matrix}\mathbf{0}^T & 0\\ I_{\nu_1-1}\mathbf{0} & \\ \mathbf{0}^T & 1\end{matrix} & & & & \begin{matrix}1 & \mathbf{0}^T\\ \mathbf{0} & I_{\nu_1-1}\\ 0 & \mathbf{0}^T\end{matrix} & & & & \begin{matrix}\boldsymbol{g}_1^{(0)T}\\ \bar{\boldsymbol{g}}_1^T\\ \boldsymbol{g}_1^{(\nu_1)T}\end{matrix} \\ \hline & \begin{matrix}\mathbf{0}^T & 0\\ I_{\nu_2-1}\mathbf{0} & \\ \mathbf{0}^T & 1\end{matrix} & & & & \begin{matrix}1 & \mathbf{0}^T\\ \mathbf{0} & I_{\nu_2-1}\\ 0 & \mathbf{0}^T\end{matrix} & & & \begin{matrix}\boldsymbol{g}_2^{(0)T}\\ \bar{\boldsymbol{g}}_2^T\\ \boldsymbol{g}_2^{(\nu_2)T}\end{matrix} \\ \hline & & \ddots & & & & \ddots & & \vdots \\ \hline & & & \begin{matrix}\mathbf{0}^T & 0\\ I_{\nu_k-1}\mathbf{0} & \\ \mathbf{0}^T & 1\end{matrix} & & & & \begin{matrix}1 & \mathbf{0}^T\\ \mathbf{0} & I_{\nu_k-1}\\ 0 & \mathbf{0}^T\end{matrix} & \begin{matrix}\boldsymbol{g}_k^{(0)T}\\ \bar{\boldsymbol{g}}_k^T\\ \boldsymbol{g}_k^{(\nu_k)T}\end{matrix} \end{array}\right]$$

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$$[K\ L\ M]\begin{bmatrix} zX(z) \\ X(z) \\ G(z) \end{bmatrix} = 0 \longleftrightarrow [K\ L\ M]\,\tilde{G}^T = 0.$$

$\widetilde{G}$ is a generator matrix of a $[2\delta + n, \delta + k]$ block code $\mathcal{B}$ where $[K\ L\ M]$ is a parity-check of $\mathcal{B}$.

ISO representation

With some row operations in $\widetilde{G}$

$$\widetilde{G} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \mathbf{0}^T\ 0 & & & & 1\ \mathbf{0}^T & & & & \boldsymbol{g}_1^{(0)^T} \\ I_{\nu_1-1}\mathbf{0} & & & & \mathbf{0}I_{\nu_1-1} & & & & \bar{\boldsymbol{g}}_1^T \\ \mathbf{0}^T\ 1 & & & & 0\ \mathbf{0}^T & & & & \boldsymbol{g}_1^{(\nu_1)^T} \\ \hline & \mathbf{0}^T\ 0 & & & & 1\ \mathbf{0}^T & & & \boldsymbol{g}_2^{(0)^T} \\ & I_{\nu_2-1}\mathbf{0} & & & & \mathbf{0}I_{\nu_2-1} & & & \bar{\boldsymbol{g}}_2^T \\ & \mathbf{0}^T\ 1 & & & & 0\ \mathbf{0}^T & & & \boldsymbol{g}_2^{(\nu_2)^T} \\ \hline & & \ddots & & & & \ddots & & \vdots \\ \hline & & & \mathbf{0}^T\ 0 & & & & 1\ \mathbf{0}^T & \boldsymbol{g}_k^{(0)^T} \\ & & & I_{\nu_k-1}\mathbf{0} & & & & \mathbf{0}I_{\nu_k-1} & \bar{\boldsymbol{g}}_k^T \\ & & & \mathbf{0}^T\ 1 & & & & 0\ \mathbf{0}^T & \boldsymbol{g}_k^{(\nu_k)^T} \end{array}\right]$$

ISO representation

$$\widetilde{\widetilde{G}} = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
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ISO representation

We have that $[g_1^{(\nu_1)} \; g_2^{(\nu_2)} \cdots g_k^{(\nu_k)}]$ is full rank, then using row operations in $\widetilde{\widetilde{G}}$, we can obtain an equivalent generator matrix

$$G = \left[\begin{array}{cccc} A^T & I_\delta & C^T & 0_{\delta \times k} \\ B^T & 0_{k \times \delta} & D^T & I_k \end{array} \right].$$

For the matrix

$$H = \left[\begin{array}{cccc} I & -A & 0 & -B \\ 0 & -C & I & -D \end{array} \right] \in \mathbb{F}^{(\delta+n-k) \times (2\delta+n)},$$

it follows that $HG^T = 0$. Therefore, as G has full rank, we have that H is the parity check matrix of $\mathcal{B}$. Furthermore,

$$H = [K \mid L \mid M]$$

then, we have that (A, B, C, D) is a minimal ISO representation of a convolutional code $\mathcal{C}$.

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Algorithm to obtain an ISO representation

Algorithm:

- **Input**: $G(z) \in \mathbb{F}[z]^{n\times k}$. Consider

$$G_{LD} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad \text{and} \quad G_{\infty} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where $U_1 \in \mathbb{F}^{(n-k)\times\delta}$, $U_2 \in \mathbb{F}^{k\times\delta}$, $V_1 \in \mathbb{F}^{(n-k)\times k}$, with $V_2 \in \mathbb{F}^{k\times k}$ an invertible matrix.

- **Step 1**: Compute $V_2^{-1} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_k \end{bmatrix}$, where $\boldsymbol{v}_j \in \mathbb{F}^{1\times k}$, for $j = 1, 2, \ldots, k$.

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Algorithm to obtain an ISO representation

Algorithm:

- **Step 2**: Compute $D = V_1 V_2^{-1}$.
- **Step 3**: Compute $C = U_1 - DU_2$.
- **Step 4**: Consider the matrix

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix} \quad \text{with } B_j = \begin{bmatrix} 0_{(\nu_j - 1) \times k} \\ \boldsymbol{v}_j \end{bmatrix}$$

  for $j = 1, 2, \ldots, k$.

- **Step 5**: Compute

$$\boldsymbol{w}_j = \boldsymbol{v}_j U_2 = \begin{bmatrix} \boldsymbol{v}_j U_2^{(\nu_1)} & \boldsymbol{v}_j U_2^{(\nu_2)} & \ldots & \boldsymbol{v}_j U_2^{(\nu_k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_j^{(\nu_1)} & \boldsymbol{w}_j^{(\nu_2)} & \ldots & \boldsymbol{w}_j^{(\nu_k)} \end{bmatrix}$$

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Algorithm to obtain an ISO representation

Algorithm:

- **Step 6**: Consider the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}$$

with

$$A_{jj} = \begin{bmatrix} \mathbf{0} & I_{\nu_j - 1} \\ \boldsymbol{w}_j^{(\nu_j)} & \end{bmatrix}, \quad \text{for } j = 1, 2, \ldots, k$$

$$A_{ij} = \begin{bmatrix} 0_{(\nu_i - 1) \times \nu_j} \\ \boldsymbol{w}_i^{(\nu_j)} \end{bmatrix}, \quad \text{for } j = 1, 2, \ldots, k \quad \text{with } i \neq j.$$

- **Output**: (A, B, C, D) minimal.

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Overview

Minimal ISO representation of Convolutional Product Codes

Main goal:

Let (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) be minimal ISO representations of a $\mathcal{C}_h$ horizontal and a $\mathcal{C}_v$ vertical codes. Our aim is to obtain a minimal ISO representations of the convolutional product code from the minimal ISO representations (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) without using the generator matrix of the product.


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Climent, J.-J., Herranz, M., and Perea, C. (2015).
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ISO representations of Convolutional Product Codes

Theorem (Theorem 4 in Climent2015)

Let (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) be minimal ISO representations of a $\mathcal{C}_h$ horizontal and a $\mathcal{C}_v$ vertical codes. Then,

$$A = \begin{bmatrix} I_{n_h-k_h} \otimes A_v & C_h \otimes B_v & 0 \\ 0 & A_h \otimes I_{k_v} & 0 \\ 0 & 0 & I_{k_h} \otimes A_v \end{bmatrix} B = \begin{bmatrix} D_h \otimes B_v \\ B_h \otimes I_{k_v} \\ I_{k_h} \otimes B_v \end{bmatrix}$$

$$C = \begin{bmatrix} I_{n_h-k_h} \otimes C_v & C_h \otimes D_v & 0 \\ 0 & C_h \otimes I_{k_v} & 0 \\ 0 & 0 & I_{k_h} \otimes C_v \end{bmatrix} D = \begin{bmatrix} D_h \otimes D_v \\ D_h \otimes I_{k_v} \\ I_{k_h} \otimes D_v \end{bmatrix}$$

is an ISO representation of the convolutional product code $\mathcal{C} = \mathcal{C}_h \otimes \mathcal{C}_v$ of rate $k_h k_v / n_h n_v$.

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Let (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) be minimal ISO representations of a $\mathcal{C}_h$ horizontal and a $\mathcal{C}_v$ vertical codes. Then,

$$A = \begin{bmatrix} A_h \otimes I_{n_v - k_v} & B_h \otimes C_v & 0 \\ 0 & I_{k_h} \otimes A_v & 0 \\ 0 & 0 & A_h \otimes I_{k_v} \end{bmatrix} B = \begin{bmatrix} B_h \otimes D_v \\ I_{k_h} \otimes B_v \\ B_h \otimes I_{k_v} \end{bmatrix}$$

$$C = \begin{bmatrix} C_h \otimes I_{n_v - k_v} & D_h \otimes C_v & 0 \\ 0 & I_{k_h} \otimes C_v & 0 \\ 0 & 0 & C_h \otimes I_{k_v} \end{bmatrix} D = \begin{bmatrix} D_h \otimes D_v \\ I_{k_h} \otimes D_v \\ D_h \otimes I_{k_v} \end{bmatrix}$$

is an ISO representation of the convolutional product code $\mathcal{C} = \mathcal{C}_h \otimes \mathcal{C}_v$ of rate $k_h k_v / n_h n_v$.

A minimal ISO representation of a Convolutional Product Code

Let (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) be minimal ISO representations of a $\mathcal{C}_h$ horizontal and a $\mathcal{C}_v$ vertical convolutional codes. Assume that $G_h = G_{LD}^{(h)} X_h(z) + G_{\infty}^{(h)} Y_h(z)$ and $G_v = G_{LD}^{(v)} X_v(z) + G_{\infty}^{(v)} Y_v(z)$ are their generator matrices.

$$
\begin{aligned}
G(z) =& G_h(z) \otimes G_v(z) \\
=& \left[G_{LD}^{(h)} \otimes G_{LD}^{(v)} G_{LD}^{(h)} \otimes G_{\infty}^{(v)} G_{\infty}^{(h)} \otimes G_{LD}^{(h)} \right] \begin{bmatrix} X_h(z) \otimes X_v(z) \\ X_h(z) \otimes Y_v(z) \\ Y_h(z) \otimes X_v(z) \end{bmatrix} \\
&+ G_{\infty}^{(h)} \otimes G_{\infty}^{(v)} \left(Y_h(z) \otimes Y_v(z) \right) \\
=& \widetilde{G}_{LD} \widetilde{X}(z) + G_{\infty} Y(z)
\end{aligned}
$$

$$
\widetilde{G}_{LD} \widetilde{X}(z) >> G_{LD} X(z)
$$

$$
\widetilde{G}_{LD} P^{-1} = \begin{bmatrix} G_{LD} & \bar{G}_{LD} \end{bmatrix} \quad P \widetilde{X}(z) = \begin{bmatrix} X(z) \\ 0 \end{bmatrix}
$$

A minimal ISO representation of a Convolutional Product Code

Let (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) be minimal ISO representations of a $\mathcal{C}_h$ horizontal and a $\mathcal{C}_v$ vertical convolutional codes. Assume that $G_h = G_{LD}^{(h)} X_h(z) + G_{\infty}^{(h)} Y_h(z)$ and $G_v = G_{LD}^{(v)} X_v(z) + G_{\infty}^{(v)} Y_v(z)$ are their generator matrices.

$$
\begin{aligned}
G(z) =& G_h(z) \otimes G_v(z) \\
=& \left[G_{LD}^{(h)} \otimes G_{LD}^{(v)} G_{LD}^{(h)} \otimes G_{\infty}^{(v)} G_{\infty}^{(h)} \otimes G_{LD}^{(h)} \right] \begin{bmatrix} X_h(z) \otimes X_v(z) \\ X_h(z) \otimes Y_v(z) \\ Y_h(z) \otimes X_v(z) \end{bmatrix} \\
&+ G_{\infty}^{(h)} \otimes G_{\infty}^{(v)} \left(Y_h(z) \otimes Y_v(z) \right) \\
=& \widetilde{G}_{LD} \widetilde{X}(z) + G_{\infty} Y(z)
\end{aligned}
$$

$$\widetilde{G}_{LD} \widetilde{X}(z) >> G_{LD} X(z)$$

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A minimal ISO representation of a Convolutional Product Code

Let (A_h, B_h, C_h, D_h) and (A_v, B_v, C_v, D_v) be minimal ISO representations of a $\mathcal{C}_h$ horizontal and a $\mathcal{C}_v$ vertical convolutional codes. Assume that $G_h = G_{LD}^{(h)} X_h(z) + G_{\infty}^{(h)} Y_h(z)$ and $G_v = G_{LD}^{(v)} X_v(z) + G_{\infty}^{(v)} Y_v(z)$ are their generator matrices.

$$
\begin{aligned}
G(z) =& G_h(z) \otimes G_v(z) \\
=& \left[G_{LD}^{(h)} \otimes G_{LD}^{(v)} G_{LD}^{(h)} \otimes G_{\infty}^{(v)} G_{\infty}^{(h)} \otimes G_{LD}^{(h)} \right] \begin{bmatrix} X_h(z) \otimes X_v(z) \\ X_h(z) \otimes Y_v(z) \\ Y_h(z) \otimes X_v(z) \end{bmatrix} \\
& + G_{\infty}^{(h)} \otimes G_{\infty}^{(v)} \left(Y_h(z) \otimes Y_v(z) \right) \\
=& \widetilde{G}_{LD} \widetilde{X}(z) + G_{\infty} Y(z)
\end{aligned}
$$

$$
\widetilde{G}_{LD} \widetilde{X}(z) >> G_{LD} X(z)
$$

$$
\widetilde{G}_{LD} P^{-1} = \begin{bmatrix} G_{LD} & \bar{G}_{LD} \end{bmatrix} \quad P \widetilde{X}(z) = \begin{bmatrix} X(z) \\ 0 \end{bmatrix}
$$

Summary of the basic ideas

- We provide a method to obtain a minimal ISO representation (A, B, C, D), using the generator matrix of a convolutional code $\mathcal{C}$, without the need to compute the first-order representation (K, L, M).
- We will use this algorithm to apply in convolutional product podes to obtain a minimal ISO representation from minimal ISO representations of a vertical and a horizontal convolutional code.

Summary of the basic ideas

- We provide a method to obtain a minimal ISO representation (A, B, C, D), using the generator matrix of a convolutional code $\mathcal{C}$, without the need to compute the first-order representation (K, L, M).
- We will use this algorithm to apply in convolutional product podes to obtain a minimal ISO representation from minimal ISO representations of a vertical and a horizontal convolutional code.


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Thank you