The Density of Extremal Codes With Sublinearity

Nadja Willenborg

University of St. Gallen, Switzerland

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Joachim Rosenthal's 60th birthday

Joint work with Anina Gruica, Anna-Lena Horlemann and Alberto Ravagnani

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- $$\delta_q^D(\mathbb{F}_{q^m}^n, S, 0, d) := \frac{|\{\mathscr{C} \subseteq \mathbb{F}_{q^m}^n \colon |\mathscr{C}| = S,\ D(\mathscr{C}) \geq d\}|}{|\{\mathscr{C} \subseteq \mathbb{F}_{q^m}^n \colon |\mathscr{C}| = S\}|}$$

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  the density of $\mathbb{F}_{q^\ell}$-linear codes with $D(\mathscr{C}) \geq d$.

Theorem

Let $(S_q)_{q\in Q}$ be a sequence of integers with $S_q \geq 4$ for all $q \in Q$ and for which $\lim_{q\to+\infty} S_q$ exists. We fix $1 \leq d \leq \max\{D(x,y)\colon x,y \in \mathbb{F}_q^n\}$ and let $(\mathbf{v}_q^D(\mathbb{F}_q^n, d-1))_{q\in Q}$ be a sequence where each $\mathbf{v}_q^D(\mathbb{F}_q^n, d-1)$ describes the size of the ball of radius $d-1$ in $\mathbb{F}_q^n$. We have

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(i)

$$\max\left\{\liminf_{q\to+\infty}\left(1-\frac{\mathbf{v}_q^D(\mathbb{F}_q^n, d-1)S_q^2}{2q^n}\right), 0\right\} \leq \liminf_{q\to+\infty} \delta_q^D(\mathbb{F}_q^n, S_q, 0, d).$$

The Asymptotic Density of Non-Linear Codes

Theorem

Let $(S_q)_{q\in Q}$ be a sequence of integers with $S_q \geq 4$ for all $q \in Q$ and for which $\lim_{q\to+\infty} S_q$ exists. We fix $1 \leq d \leq \max\{D(x,y)\colon x,y \in \mathbb{F}_q^n\}$ and let $(\mathbf{v}_q^D(\mathbb{F}_q^n, d-1))_{q\in Q}$ be a sequence where each $\mathbf{v}_q^D(\mathbb{F}_q^n, d-1)$ describes the size of the ball of radius $d-1$ in $\mathbb{F}_q^n$. We have

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In particular, if $S_q \in o\left(\sqrt{\frac{q^n}{\mathbf{v}_q^D(\mathbb{F}_q^n,d-1)}}\right)$ as $q\to+\infty$, then

$$\lim_{q\to+\infty}\delta_q^D(\mathbb{F}_q^n,S_q,0,d)=1.$$

i.e. the class of non-linear codes with $D(\mathscr{C}) \geq d$ is **dense**.

Theorem (cont.)

(ii) If $S_q \in \Omega\left(\sqrt{\frac{q^n}{\mathbf{v}_q^D(\mathbb{F}_q^n, d-1)}}\right)$ as $q \to +\infty$, then

$$\limsup_{q \to +\infty} \delta_q^D(\mathbb{F}_q^n, S_q, 0, d) \leq \limsup_{q \to +\infty} \left(1 - \frac{\mathbf{v}_q^D(\mathbb{F}_q^n, d-1) S_q^2}{2q^n + \mathbf{v}_q^D(\mathbb{F}_q^n, d-1) S_q^2}\right).$$

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In particular, if $S_q \in \omega\left(\sqrt{\frac{q^n}{\mathbf{v}_q^D(\mathbb{F}_q^n, d-1)}}\right)$ as $q \to +\infty$, then

$$\lim_{q \to +\infty} \delta_q^D(\mathbb{F}_q^n, S_q, 0, d) = 0,$$

i.e. the class of non-linear codes with $D(\mathscr{C}) \geq d$ is **sparse**.

Theorem

We fix $1 \leq d \leq \max\{D(x,y) \colon x, y \in \mathbb{F}_{q^{\ell s}}^n\}$ and let $(\mathbf{v}_q^D(\mathbb{F}_{q^{\ell s}}^n, d-1))_{q \in Q}$ be a sequence where each $\mathbf{v}_q^D(\mathbb{F}_{q^{\ell s}}^n, d-1)$ describes the size of the ball of radius $d-1$ in $\mathbb{F}_{q^{\ell s}}^n$. We have

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$$\max\left\{\liminf_{q\to+\infty}\left(1-\mathbf{v}_q^D(\mathbb{F}_{q^{\ell s}}^n, d-1)q^{\ell(k-ns-1)}\right), 0\right\} \leq \liminf_{q\to+\infty} \delta_q^D(\mathbb{F}_{q^{\ell s}}^n, q^{\ell k}, \ell, d).$$

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In particular, if $q^{\ell k} \in o\left(\frac{q^{\ell(ns+1)}}{\mathbf{v}^D_q(\mathbb{F}^n_{q^{\ell s}}, d-1)}\right)$ as $q \to +\infty$, then

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$$\lim_{q\to+\infty} \delta_q^D(\mathbb{F}_{q^{\ell s}}^n, q^{\ell k}, \ell, d) = 0.$$

Asymptotic Densities of Extremal Codes

Table: The parameters we consider are q (the field size), ℓ (the index of sublinearity) and s (the $\mathbb{F}_{q^\ell}$-dimension of $\mathbb{F}_{q^m}$).

	$\lim_{q\to+\infty} \delta_q^D(\mathbb{F}_{q^m}^n, S, \ell, d)$	$\lim_{\ell\to+\infty} \delta_q^D(\mathbb{F}_{q^m}^n, S, \ell, d)$	$\lim_{s\to+\infty} \delta_q^D(\mathbb{F}_{q^m}^n, S, \ell, d)$
sublinear MDS	1	1	$\leq \left(1 + q^{-\ell}\binom{n}{d-1}\right)^{-1}$
non-linear MDS	0[1]	-	-

[1] A. Gruica, A. Ravagnani; "The Typical Non-Linear Code over Large Alphabets"; 2021 IEEE Information Theory Workshop (ITW)t

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sublinear MRD	0[2]	1[3]	$\leq \left(1 + q^{-\ell}\begin{bmatrix} n \\ d-1 \end{bmatrix}_q\right)^{-1}$
non-linear MRD	0[4]	-	-

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[2] A. Gruica, A. Ravagnani; "Common complements of linear subspaces and the sparseness of MRD codes"; SIAM Journal on Applied Algebra and Geometry; 2022

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sublinear MSRD	various	1[5]	$\leq \left(1 + q^{-\ell} \sum_{\substack{(j_1,\dots,j_t)\in\mathbb{N}_0^t, \\ j_1+\dots+j_t=d-1}} \prod_{i=1}^{t} \begin{bmatrix} n_i \\ j_i \end{bmatrix}_q\right)^{-1}$
non-linear MSRD	0	-	-

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[4] A. Gruica, A. Ravagnani; "The Proportion of (Non-) Linear MRD Codes".

[5] C.Ott, S. Puchinger, M. Bossert; "Bounds and genericity of sum-rank-metric codes"; 2021 XVII International Symposium" Problems of Redundancy in Information and Control Systems"

Codes in the Sum-Rank Metric[1]

- Let $x \in \mathbb{F}_{q^m}^n$ consisting of t blocks $x^{(i)} \in \mathbb{F}_{q^m}^{n_i}$ for $i \in [t]$. The **sum-rank weight** of x is defined as $\omega^{sr,t}(x) := \sum_{i=1}^{t} \omega^{rk}(x^{(i)})$, where $\omega^{\mathrm{rk}}(x^{(i)}) := \dim_{\mathbb{F}_q} \langle x_1^{(i)}, \ldots, x_{n_i}^{(i)} \rangle$. For $x, y \in \mathbb{F}_{q^m}^n$ the **sum-rank distance** is defined via $D^{\mathrm{sr,t}}(x, y) = \omega^{sr,t}(x - y)$.

[1] E. Byrne, H. Gluesing-Luerssen **and** A. Ravagnani, "Fundamental properties of sum-rank-metric codes," *IEEE Transactions on Information Theory*, **jourvol** 67, **number** 10, **pages** 6456–6475, 2021.

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- $\mathbf{v}_q^{\mathrm{sr,t}}(\mathbb{F}_{q^m}^n, r) := \sum_{j=0}^{r} \sum_{\substack{(j_1,\ldots,j_t) \in \mathbb{N}_0^t, \\ j_1+\cdots+j_t=j}} \prod_{i=1}^{t} \begin{bmatrix} n_i \\ j_i \end{bmatrix}_q \prod_{k=0}^{j_i-1} (q^m - q^k)$ denotes the **volume of the sum-rank metric ball** of radius $0 \le r \le n$ in $\mathbb{F}_{q^m}^n$.

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- **Singleton-type bound** : $|\mathscr{C}| \le q^{m(n-D(\mathscr{C})+1)}$

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- **Singleton-type bound** : $|\mathscr{C}| \le q^{m(n-D(\mathscr{C})+1)}$
- If $\mathscr{C} \subseteq \mathbb{F}_{q^m}^n$ attains the Singleton-type bound or if $|\mathscr{C}| = 1$, then $\mathscr{C}$ is called **MSRD**- (maximum sum-rank distance) **code**.

[1] E. Byrne, H. Gluesing-Luerssen **and** A. Ravagnani, "Fundamental properties of sum-rank-metric codes," *IEEE Transactions on Information Theory*, **jourvol** 67, **number** 10, **pages** 6456–6475, 2021.

The Asymptotic Density of Non-Linear MSRD Codes

Theorem

For $2 \leq d \leq n$ we have

$$\lim_{q \to +\infty} \delta_q^{\mathrm{sr,t}}(\mathbb{F}_q^n, q^{n-d+1}, 0, d) = 0.$$

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Proof.

Using the estimate

$$\mathbf{v}_q^{\mathrm{sr,t}}(\mathbb{F}_q^n, d-1) \sim q^{(d-1)} \sum_{\substack{(j_1,\dots,j_t) \in \mathbb{N}_0^t, \\ j_1+\dots+j_t = d-1}} q^{(j_1(n_1-j_1)+\dots+j_t(n_t-j_t))}$$

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Therefore $q^{n-d+1} \in \omega\left(\sqrt{\frac{q^n}{\mathbf{v}_q^{\mathrm{sr,t}}(\mathbb{F}_q^n, d-1)}}\right)$ as $q \to +\infty$. □

Theorem

Let $m, d, n_1, \dots, n_t \geq 1$ be integers such that $n = \sum_{i=1}^{t} n_i$ and $1 \leq d \leq n$.

(i) If $d = 1$ or $t = n$ and $n_i = 1$ for all $i \in [n]$ we have

$$\lim_{q \to +\infty} \delta_q^{sr,t}(\mathbb{F}_{q^m}^n, q^{m(n-d+1)}, 1, d) = 1.$$

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$$\limsup_{q \to +\infty} \delta_q^{sr,t}(\mathbb{F}_{q^m}^n, q^{m(n-d+1)}, 1, d) \leq \frac{1}{1 + \binom{t-1}{d-2}}.$$

(iii) If $d = 2, n_{i_1} = \cdots = n_{i_r} = 2$ and $n_{i_{r+1}} = \cdots = n_{i_t} = 1$ for $r \leq t$ we have

$$\limsup_{q \to +\infty} \delta_q^{sr,t}(\mathbb{F}_{q^m}^n, q^{m(n-d+1)}, 1, d) \leq \frac{1}{1 + r}.$$

Theorem (cont.)

(iv) In all other cases, that is, if $d \geq 3$ and $n \geq t+2$ or $d = 2$ and $\max\{n_1, \ldots, n_t\} \geq 3$, we have

$$\lim_{q \to +\infty} \delta_q^{sr,t}(\mathbb{F}_{q^m}^n, q^{m(n-d+1)}, 1, d) = 0.$$

Theorem (cont.)

(iv) In all other cases, that is, if $d \geq 3$ and $n \geq t+2$ or $d = 2$ and $\max\{n_1, \dots, n_t\} \geq 3$, we have

$$\lim_{q \to +\infty} \delta_q^{sr,t}(\mathbb{F}_{q^m}^n, q^{m(n-d+1)}, 1, d) = 0.$$

Proof.

Using the estimate

$$\mathbf{v}_q^{\mathrm{sr,t}}(\mathbb{F}_{q^m}^n, d-1) \sim q^{m(d-1)} \sum_{\substack{(j_1,\dots,j_t) \in \mathbb{N}_0^t, \\ j_1+\cdots+j_t = d-1}} q^{(j_1(n_1-j_1)+\cdots+j_t(n_t-j_t))}$$

gives us

Theorem (cont.)

(iv) In all other cases, that is, if $d \geq 3$ and $n \geq t+2$ or $d = 2$ and $\max\{n_1, \dots, n_t\} \geq 3$, we have

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gives us

$$\lim_{q\to\infty} \frac{q^{m(n-d+1)} \cdot \mathbf{v}_q^{\mathrm{sr,t}}(\mathbb{F}_{q^m}^n, d-1)}{q^{(nm+1)}} = \lim_{q\to\infty} \frac{1}{q} \left(\sum_{\substack{(j_1,\dots,j_t)\in\mathbb{N}_0^t,\\ j_1+\dots+j_t=d-1}} q^{(j_1(n_1-j_1)+\dots+j_t(n_t-j_t))} \right).$$

Theorem (cont.)

(iv) In all other cases, that is, if $d \geq 3$ and $n \geq t+2$ or $d = 2$ and $\max\{n_1, \dots, n_t\} \geq 3$, we have

$$\lim_{q \to +\infty} \delta_q^{sr,t}(\mathbb{F}_{q^m}^n, q^{m(n-d+1)}, 1, d) = 0.$$

Proof.

Using the estimate

$$\mathbf{v}_q^{\mathrm{sr,t}}(\mathbb{F}_{q^m}^n, d-1) \sim q^{m(d-1)} \sum_{\substack{(j_1,\dots,j_t)\in\mathbb{N}_0^t,\\ j_1+\dots+j_t=d-1}} q^{(j_1(n_1-j_1)+\dots+j_t(n_t-j_t))}$$

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Now analyze the four cases separately by using combinatorial arguments.

The End

Thank you for your attention!