Coding Theory and Cryptography:
A Conference in Honor of
Joachim Rosenthal's 60th Birthday

Universtität Zürich, July 11–15, 2022

Quantum Convolutional Codes

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11 July 2022

European Funds Smart Growth
Republic of Poland
FNP Foundation for Polish Science
European Union European Regional Development Fund

Overview

- qubits and qudits
- quantum codes
- operational view on quantum convolutional codes
- stabilizer formalism
- basic operations
- encoding circuit
- open problems

Quantum Information

Quantum-bit (qubit)

basis states:

$$\text{“0”} \mathrel{\hat{=}} |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2, \quad \text{“1”} \mathrel{\hat{=}} |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

general state:

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \text{where } \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1$$

measurement (read-out):

result “0” with probability $|\alpha|^2$

result “1” with probability $|\beta|^2$

Quantum Information

Quantum register

basis states:

$$|b_1\rangle \otimes \ldots \otimes |b_n\rangle =: |b_1 \ldots b_n\rangle = |\boldsymbol{b}\rangle \qquad \text{where } b_i \in \{0,1\}$$

general state:

$$|\psi\rangle = \sum_{\boldsymbol{x} \in \{0,1\}^n} c_{\boldsymbol{x}} |\boldsymbol{x}\rangle \qquad \text{where } \textstyle\sum_{\boldsymbol{x} \in \{0,1\}^n} |c_{\boldsymbol{x}}|^2 = 1$$

$\longrightarrow$ normalized vector in $(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$

Qudits

generalization to $(\mathbb{C}^q)^{\otimes n}$: basis states $|\boldsymbol{b}\rangle$ labelled by vectors $\boldsymbol{b} \in \mathbb{F}_q^n$

Quantum Error-Correcting Block Codes

- **subspace** $\mathcal{C}$ of a complex vector space $\mathcal{H} \cong \mathbb{C}^N$
  usually: $\mathcal{H} \cong \mathbb{C}^q \otimes \mathbb{C}^q \otimes \ldots \otimes \mathbb{C}^q =: (\mathbb{C}^q)^{\otimes n}$ "n qudits"

- **errors:** described by linear transformations acting on
  - some of the subsystems (local errors)
  - many subsystems in the same way (correlated errors)

- **notation:** $\mathcal{C} = [\![n, k, d]\!]_q$
  q^k-dimensional subspace $\mathcal{C}$ of $(\mathbb{C}^q)^{\otimes n}$

- **minimum distance** d:
  - detection of errors acting on $d-1$ subsystems
  - correction of errors acting on $\lfloor (d-1)/2 \rfloor$ subsystems
  - correction of erasures acting on $d-1$ known subsystems

Quantum Error-Correcting Codes

quantum error-correction is "linear"

If the errors A and B can be corrected,
then all errors $\lambda A + \mu B$ $(\lambda, \mu \in \mathbb{C})$ can be corrected.

$\Longrightarrow$ consider only a vector space basis of the errors

Error Basis for Qudits

[A. Ashikhmin & E. Knill, Nonbinary quantum stabilizer codes, IEEE-IT **47**, pp. 3065–3072 (2001)]

$$\mathcal{E} = \{X^\alpha Z^\beta : \alpha, \beta \in \mathbb{F}_q\},$$

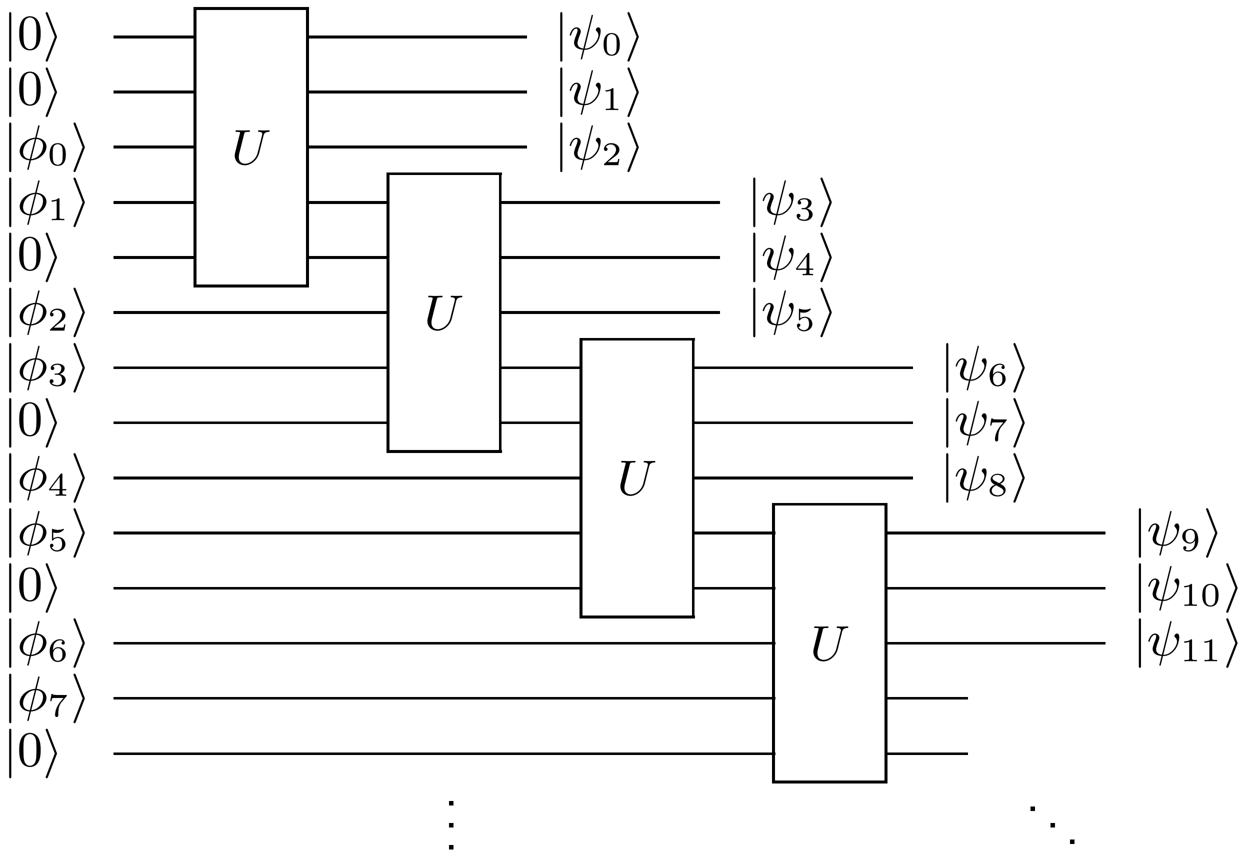
where (you may think of $\mathbb{C}^q \cong \mathbb{C}[\mathbb{F}_q]$)

$$X^\alpha := \sum_{x \in \mathbb{F}_q} |x + \alpha\rangle\langle x| \qquad \text{for } \alpha \in \mathbb{F}_q$$

$$\text{and} \quad Z^\beta := \sum_{z \in \mathbb{F}_q} \omega^{\mathrm{Tr}(\beta z)} |z\rangle\langle z| \quad \text{for } \beta \in \mathbb{F}_q \ (\omega := \omega_p = \exp(2\pi i/p))$$

Convolutional Quantum Encoder

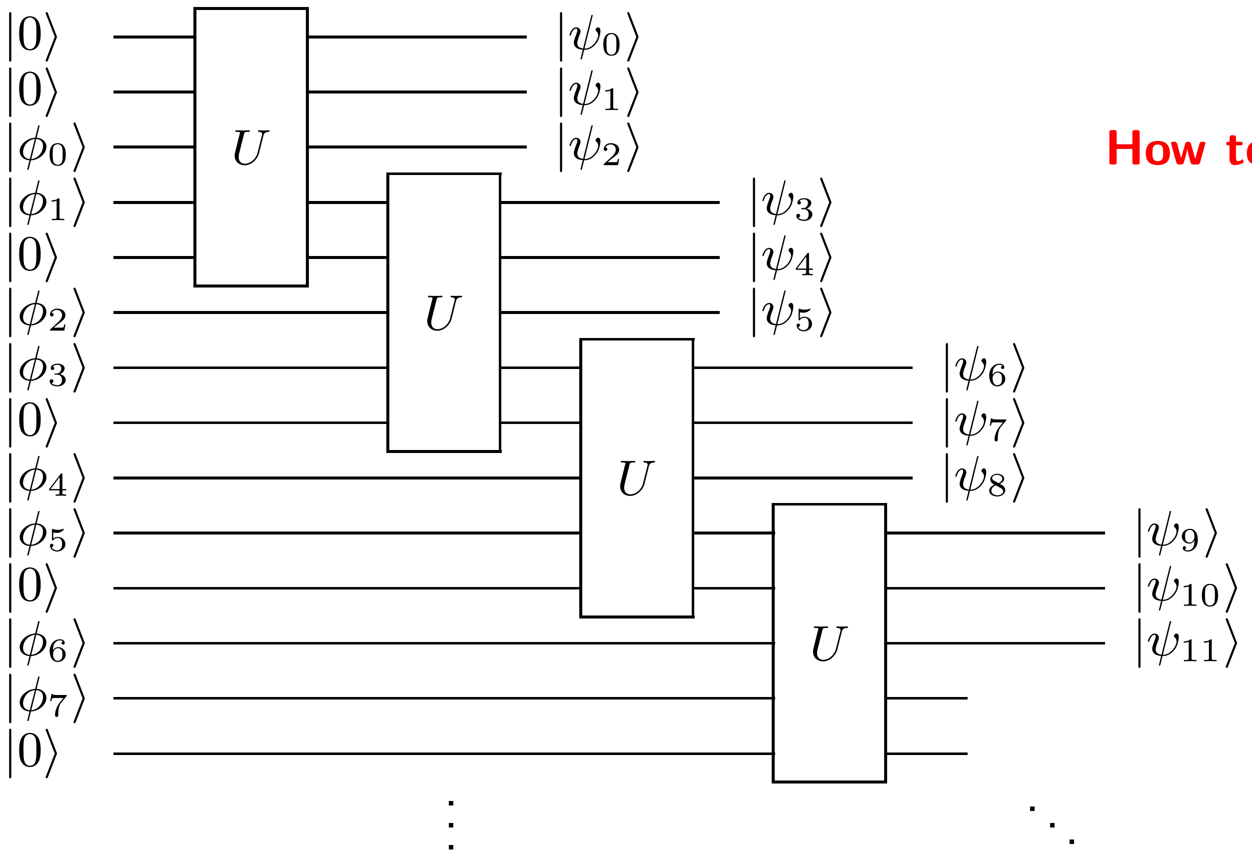
encoding a stream of qudits



unitary transformation U

Convolutional Quantum Encoder

encoding a stream of qudits



How to invert this circuit?

unitary transformation U

Quantum Convolutional Stabilizer Codes

Quantum Block (Stabilizer) Codes

The code is the common eigenspace of the stabilizers.

Quantum Convolutional Codes

Idea: impose local constraints by stabilizers

Example:

$$
\begin{aligned}
s_1 &= \ldots\ III\ XXX\ XZY\ III\ III\ \ldots \\
s_2 &= \ldots\ III\ ZZZ\ ZYX\ III\ III\ \ldots
\end{aligned}
$$

shift the stabilizers by three qubits:

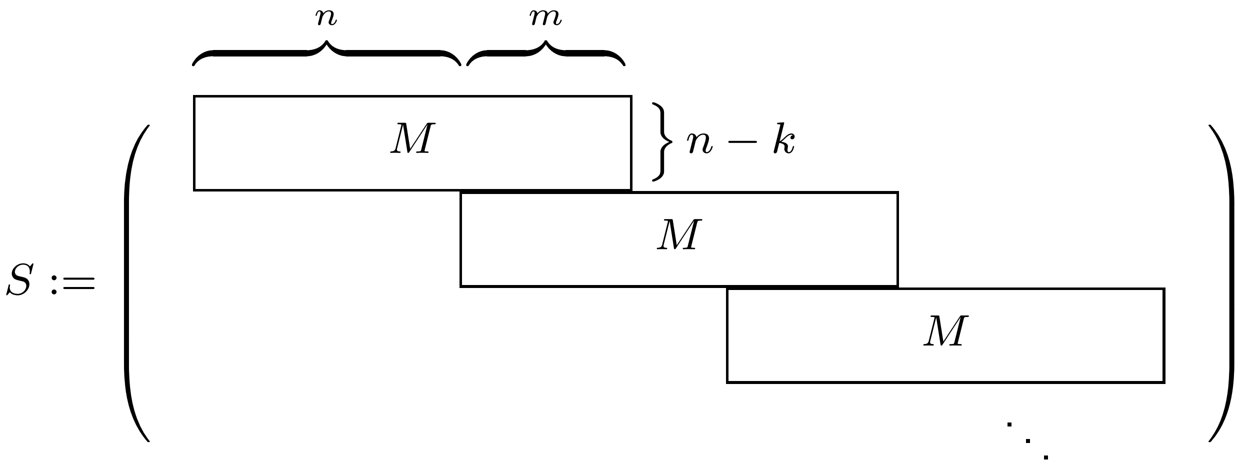
$$
\begin{aligned}
s_1' &= \ldots\ III\ III\ XXX\ XZY\ III\ \ldots \\
s_2' &= \ldots\ III\ III\ ZZZ\ ZYX\ III\ \ldots
\end{aligned}
$$

Quantum Convolutional Codes (QCCs)

[H. Ollivier and J.-P. Tillich, “Quantum convolutional codes: fundamentals,” 2004, preprint quant-ph/0401134]

quantum convolutional code with parameters (n, k, m):

- semi-infinite stabilizer with block band structure



- S generates a self-orthogonal classical convolutional code
- M generates a self-orthogonal classical block code

Semi-infinite Stabilizer

Compact representation of the semi-infinite stabilizer matrix

$$
\begin{pmatrix}
XXX & XZY & & \\
ZZZ & ZYX & & \\
 & XXX & XZY & \\
 & ZZZ & ZYX & \\
 & & & \ddots
\end{pmatrix}
$$

$$
\hat{=} \left(\begin{array}{ccc|ccc}
111 & 101 & & 000 & 011 & \\
000 & 011 & & 111 & 110 & \\
 & 111 & 101 & & 000 & 011 \\
 & 000 & 011 & & 111 & 110 \\
 & & \ddots & & & \ddots
\end{array}\right)
$$

$$
\hat{=} \left(\begin{array}{ccc|ccc}
1+D & 1 & 1+D & 0 & D & D \\
0 & D & D & 1+D & 1+D & 1
\end{array}\right) = \mathbf{S}(D)
$$

Quantum Convolutional Codes

Quantum Block Codes

The stabilizer $\mathcal{S}$ corresponds to a self-orthogonal additive code over $\mathbb{F}_2 \times \mathbb{F}_2$ generated by the stabilizer matrix $(\mathbf{X}|\mathbf{Z})$.

Quantum Convolutional Codes

The semi-infinite stabilizer corresponds to an additive self-orthogonal convolutional code generated by $\big(\mathbf{X}(D) \bigm| \mathbf{Z}(D)\big)$ with

$$\mathbf{X}(D)\mathbf{Z}(1/D)^t - \mathbf{Z}(D)\mathbf{X}(1/D)^t = \mathbf{0}$$

Example:

$$\mathbf{S}(D) = \left(\begin{array}{ccc|ccc} 1+D & 1 & 1+D & 0 & D & D \\ 0 & D & D & 1+D & 1+D & 1 \end{array}\right)$$

Catastrophic (Quantum) Convolutional Codes

Bad example:

$$\begin{pmatrix} Z & Z & & & \\ & Z & Z & & \\ & & Z & Z & \\ & & & & \ddots \end{pmatrix} \hat{=} \big(\, 0 \, | 1 + D \big) = \mathbf{S}(D)$$

Quantum code with basis states $|\underline{0}\rangle = |000\ldots\rangle$ and $|\underline{1}\rangle = |111\ldots\rangle$, contains in particular "infinite cat state"

$\Longrightarrow$ local errors spread unboundedly

$\Longrightarrow$ further constraints on $\mathbf{S}(D)$

Elementary Operations on $\mathrm{S}(D) = (\mathbf{X}(D)|\mathbf{Z}(D))$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \in \mathbb{C}^{2\times 2}$	$\overline{H} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \in \mathbb{F}_2^{2\times 2}$
$P = \begin{pmatrix}1 & 0\\ 0 & \exp(i\pi/2)\end{pmatrix} \in \mathbb{C}^{2\times 2}$	$\overline{P} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \in \mathbb{F}_2^{2\times 2}$
$\mathrm{CNOT}^{(i,j+\ell n)}, i \not\equiv j \pmod n$	$\overline{\mathrm{CNOT}} = \left(\begin{array}{cc\|cc} 1 & D^{\ell} & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & D^{-\ell} & 1\end{array}\right)$
$P_\ell := \mathrm{CSIGN}^{(i,i+\ell n)}, \ell \neq 0$	$\overline{P_\ell} = \begin{pmatrix}1 & D^{-\ell} + D^{\ell}\\ 0 & 1\end{pmatrix}$

Computing an Inverse Encoding Circuit

[M. Grassl and M. Rötteler, "On encoders for quantum convolutional codes", ITW 2010]

Using the previous elementary operations on columns and (free) row operations $A(D)$, we can compute the Smith normal form of $\mathbf{S}(D) = (\mathbf{X}(D)|\mathbf{Z}(D))$:

$$A(D)(\mathbf{X}(D)|\mathbf{Z}(D))T(D) = (0|I0)$$

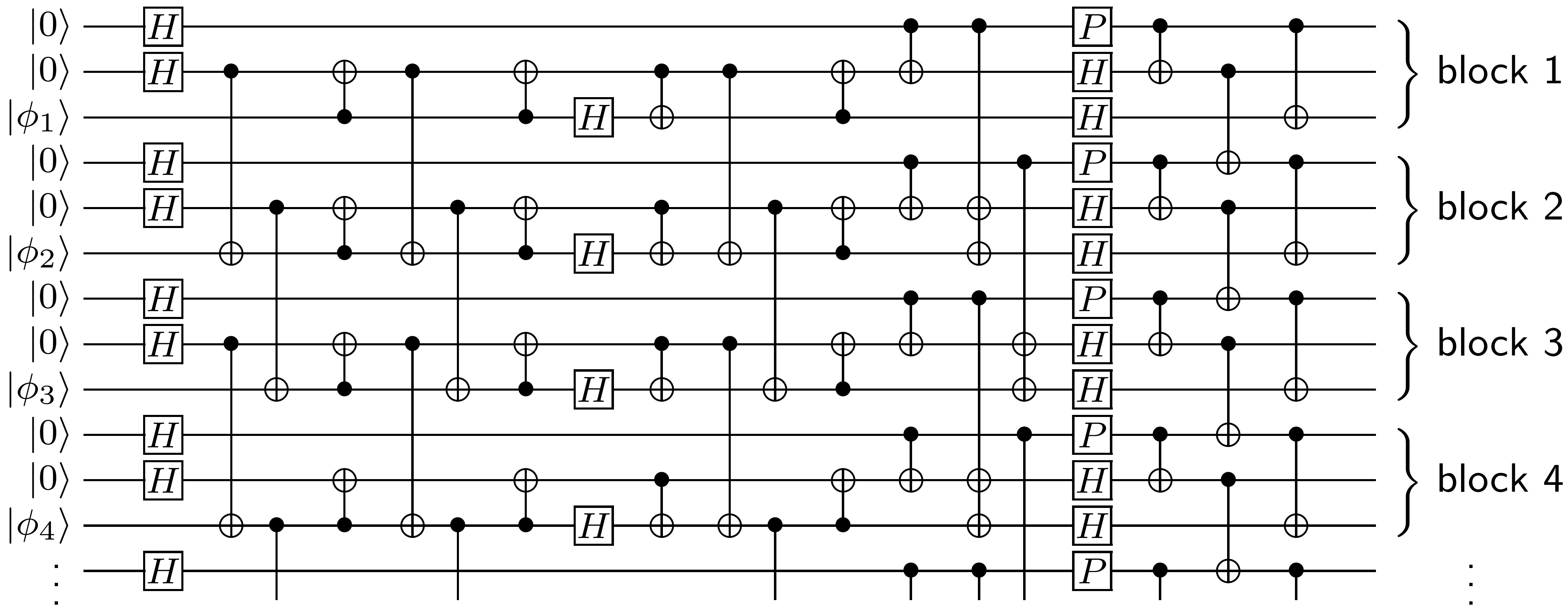
- if $\mathbf{S}(D)$ has non-trivial elementary divisors, replace them by 1
- the stabilizer matrix $(0|I0)$ corresponds to a trivial code with no encoding
- the factorisation of $T(D)$ into elementary operations yields an inverse encoding circuit

open problem:

How many elementary operations are needed to implement $T(D)$?

So far, only exponential upper bound known (but see [Kannan 1985]).

Example: Rate $1/3$ Quantum Convolutional Code



Every gate has to be repeatedly applied shifted by one block.

Outlook

- "pearl-necklace" encoder with finite depth for quantum convolutional codes
- How much memory is required?
  (see work by [Houshmand, Hosseini-Khayat & Wilde])
- Is there a pearl-necklace encoder with polynomial depth?
- When does a convolutional encoder with matrix U have an inverse with a similar convolutional structure?
- find optimal encoders
- develop bounds on the parameters
- develop "practical" decoding algorithms with good performance

Thank you!

Danke! Merci!

Dziekuje!

Acknowledgment

The 'International Centre for Theory of Quantum Technologies' project (contract no. 2018/MAB/5) is carried out within the International Research Agendas Programme of the Foundation for Polish Science co-financed by the European Union from the funds of the Smart Growth Operational Programme, axis IV: Increasing the research potential (Measure 4.3).

European Funds Smart Growth

Republic of Poland

FNP Foundation for Polish Science

European Union European Regional Development Fund

Uniwersytet Gdański

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