Coding Theory and Cryptography:
A Conference in Honor of Joachim Rosenthal's 60th Birthday

The Marginal Distribution of the Lee Channel and its Applications

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joint work with Hannes Bartz and Gianluigi Liva
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Outline

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1. Preliminaries and Motivation
2. The Lee Channel and its Properties
3. Information Set Decoding
4. Information Set Decoding using Restricted Spheres
   - Bounded Minimum Distance Decoding
   - Decoding Beyond the Minimum Distance

DLR

Syndrome Decoding Problem

Assume we send a codeword $x \in C$ and receive a vector $y = x + e \in (\mathbb{Z}/p^s\mathbb{Z})^n$.

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Given an $(n-k) \times n$ parity-check matrix H of C and a syndrome $s = yH^\top$, find the length-n vector e such that

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- The security of the McEliece cryptosystem relies on the hardness of the syndrome decoding problem
  - Is an NP-hard problem (in the Hamming metric, Lee metric, . . .)
  - generic decoding has a large cost in the Lee metric

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  - Is an NP-hard problem (in the Hamming metric, Lee metric, . . .)
  - generic decoding has a large cost in the Lee metric
- Has a unique solution for a relatively small weight (w.r.t. the GV bound)

Ring-Linear Codes

Let p a prime number and s and n two positive integers.

Definition

A *linear code* C $(Z=p^s Z)^n$ is a $Z=p^s Z$-submodule of $(Z=p^s Z)^n$.

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Parameters:

- n is called the *length* of C
- $k := \log_{p^s} j\, C\, j$ is the $Z{=}p^s Z$-*dimension* of C
- $R := k{=}n$ denotes the *rate* of C.

The Lee Metric

Definition

For $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $e = (e_1, \dots, e_n) \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we define their *Lee weight*, respectively, by

$$\mathrm{wt_L}(a) := \min(a, p^s - a),$$

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Example over $\mathbb{Z}/5\mathbb{Z}$

$0: \quad \mathrm{wt}_\mathrm{L}(0) = 0$

$1: \quad \mathrm{wt}_\mathrm{L}(1) = 1$

$2: \quad \mathrm{wt}_\mathrm{L}(2) = 2$

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Properties:

For every $a\ 2\ \mathbb{Z}{=}p^s\mathbb{Z}$ and $e\ 2\ (\mathbb{Z}{=}p^s\mathbb{Z})^n$

$$\mathrm{wt_L}(a) = \mathrm{wt_L}(j\,p^s \quad a\,j)$$

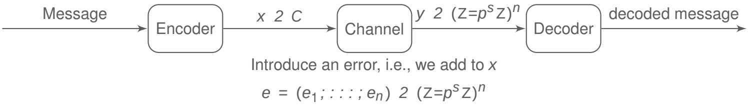
$$\mathrm{wt_H}(a) \quad \mathrm{wt_L}(a) \quad bp^s{=}2c =: M$$

$$\mathrm{wt_H}(e) \quad \mathrm{wt_L}(e) \quad nM$$

Outline

The Constant-Weight Lee Channel

Take a linear code $C \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$.



Here: Take e uniformly at random from $e \in S_{t,p^s}^{(n)} := \{ z \in (\mathbb{Z}/p^s\mathbb{Z})^n \mid \mathrm{wt}_L(z) = t \}$.

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Question: What can we say about the entries of the error term?

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Lemma

Let $a \in \mathbb{Z}/p^s\mathbb{Z}$ be chosen uniformly at random. Then

$$\mu_{p^s} := \mathbb{E}(\mathrm{wt}_L(a)) = \begin{cases} \frac{(p^s)^2 - 1}{4p^s} & \text{if } p^s \text{ is odd}, \\ \frac{p^s}{4} & \text{if } p^s \text{ is even}. \end{cases}$$

The Marginal Distribution

Let E be the random variable corresponding to the realization of a random entry of e.

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Theorem [1]

Assume that the asymptotic relative Lee weight is $T := \lim_{n \to \infty} \frac{t(n)}{n}$. For every $i \in \mathbb{Z}/p^s\mathbb{Z}$ the marginal distribution of E is given by

$$p_i := \mathbb{P}(E = i) = \frac{1}{\sum_{j=0}^{p^s - 1} \exp(\quad \mathrm{wt}_{\mathrm{L}}(j))} \exp(\quad i)$$

where is the solution to $T = \sum_{i=0}^{M} \mathrm{wt}_{\mathrm{L}}(i) p_i$.

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[1] "On the Properties of Error Patterns in the Constant Lee Weight Channel". In: *International Zurich Seminar on Information and Communication (IZS)*. 2022, pp. 44–48.

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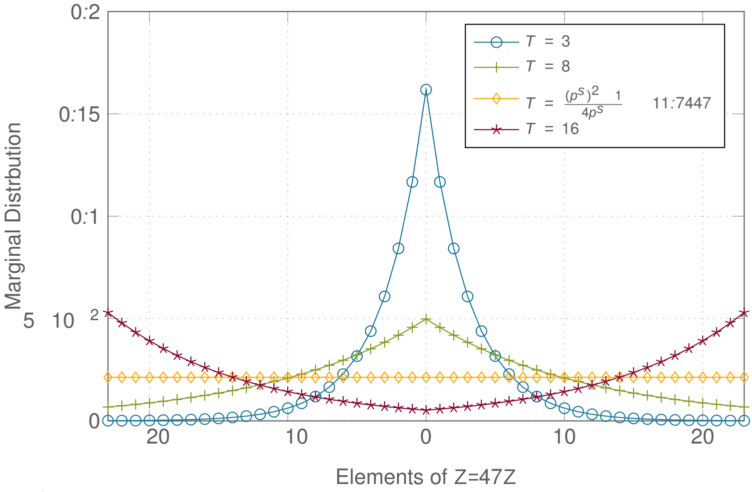
Note $T <$ $_{p^s}$ ( ) > 0

The Marginal Distribution - Example over Z=47Z



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Information Set Decoding in the Lee Metric

Consider an instance of the Lee Syndrome Decoding Problem (LSDP):

$$\text{Given } H \in (\mathbb{Z}/p^s\mathbb{Z})^{(n-k)\times n},\ s \in (\mathbb{Z}/p^s\mathbb{Z})^{n-k} \text{ and } t \in \mathbb{N};$$
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→ Recent improvements: using partial Gaussian elimination[1]

[1] Matthieu Finiasz and Nicolas Sendrier. “Security bounds for the design of code-based cryptosystems”. In: *International Conference on the Theory and Application of Cryptology and Information Security*. Springer. 2009, pp. 88–105.

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[1] Anja Becker et al. "Decoding random binary linear codes in $2^{n/20}$: How $1+1=0$ improves information set decoding". In: *Annual international conference on the theory and applications of cryptographic techniques.* Springer. 2012, pp. 520–536.

[2] Alexander May, Alexander Meurer, and Enrico Thomae. "Decoding Random Linear Codes in $\tilde{O}(2^{0.054n})$". In: *International Conference on the Theory and Application of Cryptology and Information Security.* Springer. 2011

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… BJMM on 2 Levels is fastest in the Lee metric (non-amortized)[1]

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[1] Violetta Weger et al. "On the hardness of the Lee syndrome decoding problem". In: *Advances in Mathematics of Communications* (2019). DOI: 10.3934/amc.2022029.

[2] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

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The cost of an ISD algorithm is given by

$$\underbrace{\text{nr. of iterations}}_{\frac{1}{\text{success probability per iter.}}} \cdot \text{cost per iteration}$$

General Framework

We use the idea of partial Gaussian elimination to solve the problem:

1. Find $U \in \mathrm{GL}_{n-k}(\mathbb{Z}/p^s\mathbb{Z})$ such that

$$UH^\top = \begin{pmatrix} \mathrm{I}_{n-k-\ell} & 0 \\ A^\top & B^\top \end{pmatrix}$$

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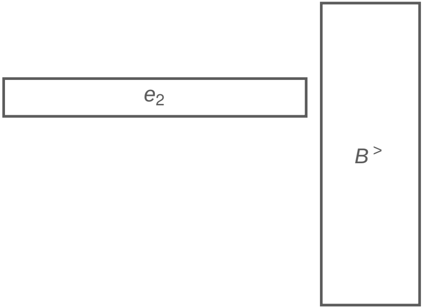
4. Solve the smaller instance of the LSDP. Immediately check whether $e_1 = s_1 - e_2A^\top$ has Lee weight $t - v$.

Solving the Smaller Instance - Finding e_2

Focus on $e_2 B^\top = s_2$, with $\mathrm{wt}_L(e_2) = v$



Solving the Smaller Instance - Finding e_2

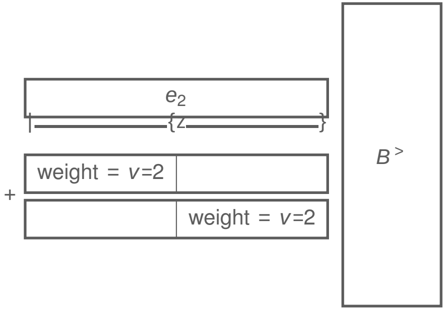
Focus on $e_2 B^> = s_2$, with $\mathrm{wt}_L(e_2) = v$

Stern/Dumer

Represent e_2 as

$$e_2 = y_1 + y_2;$$

where
$\mathrm{wt}_L(y_1) = \mathrm{wt}_L(y_2) = v$=2.



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Enumerate the following sets

$$L_1 := \left\{ y_1 B_1^{\top} \mid \mathrm{wt}(y_1) = v{=}2 \right\}$$
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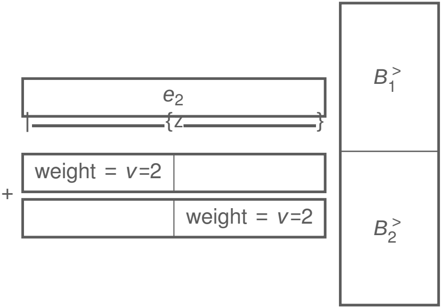

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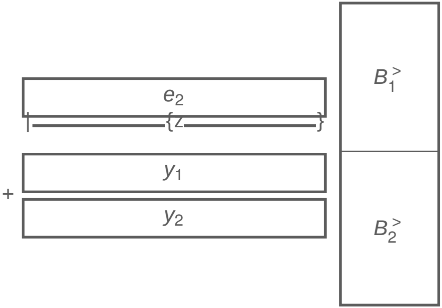

Note: The two vectors $y_1 \in L_1$ and $y_2 \in L_2$ share ε nonzero positions. The expected weight of $y_1 + y_2$ is still v.

Outline

New Idea: Using Restricted Spheres

Focus on the small instance of the Lee syndrome decoding problem.

$$\text{Given } B \in (\mathbb{Z}/p^s\mathbb{Z})^{\ell \times (k+\ell)},\ s_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{\ell} \text{ and } v, t \in \mathbb{N}$$
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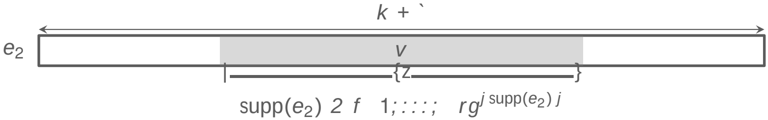
- With high probability the least probable entries of e lie **outside** the information set, hence are not in e_2.
- We will restrict e_2 to live either in $\{0, \pm 1, \ldots, \pm r\}^{k+\ell}$ or in $\{\pm r, \ldots, \pm M\}^{k+\ell}$, respectively.

Bounded Minimum Distance Decoding - Representation of e_2



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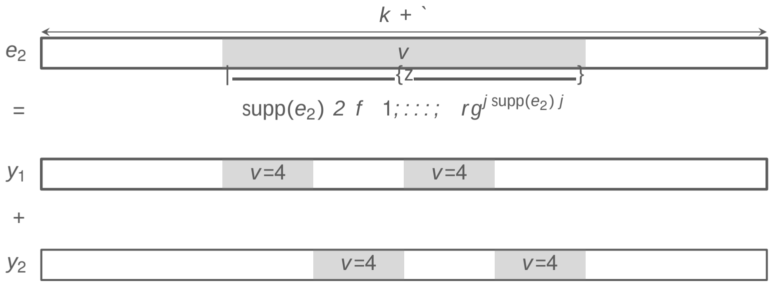

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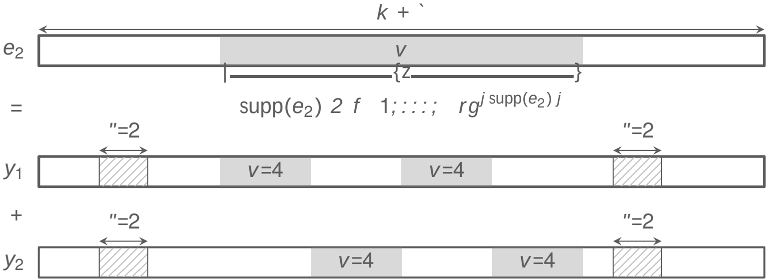
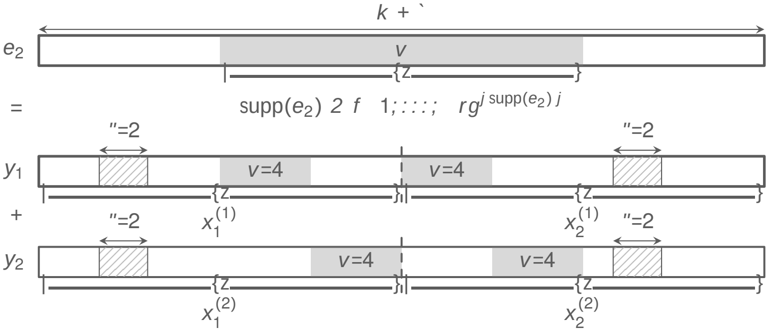
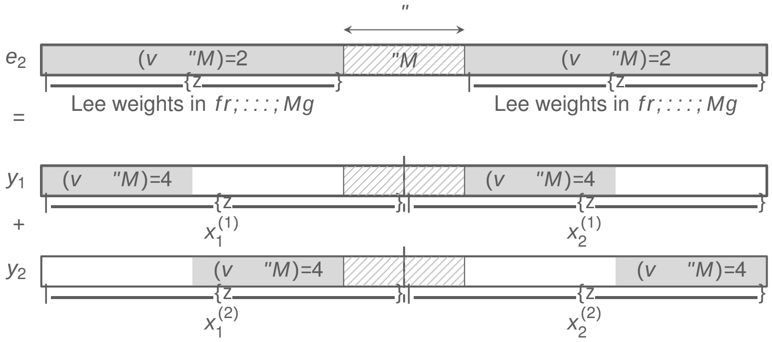


Bounded Minimum Distance Decoding - Representation of e_2



$$B_i = \{(x) \mid x_{E_i^c} \in \{0, \dots, r\}^{(k+\ell-\varepsilon)/2}; \mathrm{wt}_L(x_{E_i^c}) = v/4; x_{E_i} \in (\mathbb{Z}/p^s\mathbb{Z})^{\varepsilon/2}\} \in S_{(k+\ell)/2}$$

Decoding Beyond the Minimum Distance



Bounded Minimum Distance Decoding - BJMM Approach

Recall, $s_2 = e_2 B^{>}$, where $e_2 = y_1 + y_2 = (x_1^{(1)}; x_2^{(1)}) + (x_1^{(2)}; x_2^{(2)})$.

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1. Splitting $B = (B_1 \; B_2)$, for $i = 1; 2$ concatenate all $x_1^{(i)}; x_2^{(i)} \; 2 \; B_i$ satisfying

$$x_1^{(1)} B_1^{>} =_u \quad x_2^{(1)} B_2^{>};$$
$$x_1^{(2)} B_1^{>} =_u s_2 \quad x_2^{(2)} B_2^{>}:$$

They imply the syndrome equations for y_1 and y_2, respectively.

$$y_1 B^{>} = 0 \;\text{ and }\; y_2 B^{>} = s_2$$

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   b) the original LSDP is fulfilled as well
   $$\mathrm{wt_L}(s_1 \quad (y_1 + y_2) A^{>}) = t \quad v$$

DLR

Comparison - Bounded Minimum Distance Decoding in Z=47Z

[1]

Algorithm	$e(R\ ;p^S)$	R
Lee-BJMM	0.1618	0.451
Restricted Lee-BJMM for $r = 5$	0.1539	0.408
Amortized Lee-BJMM	0.1205	0.396
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Amortized Lee-Wagner	0.1441	0.445
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[1] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

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Thank you for your attention!

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