Coproducts of q-Matroids: The Quest for a Direct Sum

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Outline

Matroids

$\mathcal{L} = 2^{[n]}$ = subset lattice of $[n] := \{1, \ldots, n\}$.

Matroid

A **matroid** on $[n]$ is a pair $M = ([n], r)$, where

$$r : \mathcal{L} \longrightarrow \mathbb{N}_0,$$

satisfies

(R1) $0 \leq r(A) \leq |A|$ for all $A \in \mathcal{L}$;

(R2) $A \subseteq B \Longrightarrow r(A) \leq r(B)$ for all $A, B \in \mathcal{L}$;

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \in \mathcal{L}$.

The map r is called a **rank function**.

q-Matroids

$\mathcal{L} := \mathcal{L}(\mathbb{F}^n) =$ subspace lattice of $\mathbb{F}^n$ $\quad (\mathbb{F} = \mathbb{F}_q)$

q-Matroid (Crapo '63, Jurrius/Pellikaan '18)

A ***q*-matroid** on $\mathbb{F}^n$ is a pair $\mathcal{M} = (\mathbb{F}^n, \rho)$, where

$$\rho : \mathcal{L} \longrightarrow \mathbb{N}_0,$$

satisfies

(R1) $0 \leq \rho(V) \leq \dim V$ for all $V \in \mathcal{L}$;

(R2) $V \leq W \Longrightarrow \rho(V) \leq \rho(W)$ for all $V, W \in \mathcal{L}$;

(R3) $\rho(V + W) + \rho(V \cap W) \leq \rho(V) + \rho(W)$ for all $V, W \in \mathcal{L}$.

The map ρ is called a **rank function**.

Outline

Codes give rise to Matroids/q-Matroids

- *Matroid Theory:* Block codes induce matroids.
- *Jurrius/Pellikaan '18:* $\mathbb{F}_{q^m}$-linear rank-metric codes induce q-matroids.

$G \in \mathbb{F}_q^{k\times n}$	$r: \ 2^{[n]} \longrightarrow \mathbb{N}_0, \quad r(\{i_1,\dots,i_s\}) = \operatorname{rk}\left(G_{i_1},\dots,G_{i_s}\right)$	M_G
$G \in \mathbb{F}_{q^m}^{k\times n}$	$\rho: \mathcal{L}(\mathbb{F}_q^n) \longrightarrow \mathbb{N}_0, \quad \rho(\operatorname{im} Y) = \operatorname{rk} GY^{\mathsf{T}}$	$\mathcal{M}_G$

M_G and $\mathcal{M}_G$ only depend on the code generated by G.

Many code invariants can be recovered from the (q-)matroid.

Gorla/Jurrius/López/Ravagnani '20: $\mathbb{F}_q$-linear rank-metric codes induce q-polymatroids.

Codes give rise to Matroids/q-Matroids

Uniform Matroids/q-Matroids

- If $\mathcal{C}$ is an $[n,k]_q$-MDS code, then $M_{\mathcal{C}} = ([n], r)$ is the **uniform matroid** of rank k, i.e.,

$$r(A) = \min\{|A|, k\} \quad \text{for all} \quad A \subseteq [n].$$

- If $\mathcal{C}$ is an $[n,k]_{q^m}$-MRD code, then $\mathcal{M}_{\mathcal{C}} = (\mathbb{F}_q^n, \rho)$ is the **uniform** q**-matroid** of rank k, i.e.,

$$\rho(V) = \min\{\dim V, k\} \quad \text{for all} \quad V \leq \mathbb{F}_q^n.$$

Outline

Direct Sum of Matroids

Direct Sum of Matroids

Let $M_i = (S_i, r_i)$ be matroids where S_1, S_2 are finite sets and $S_1 \cap S_2 = \emptyset$. Define

$$r : 2^{S_1 \cup S_2} \longrightarrow \mathbb{N}_0, \quad r(A) = r_1(A \cap S_1) + r_2(A \cap S_2).$$

Then $M = (S_1 \cup S_2, r)$ is a matroid, denoted as $M = M_1 \oplus M_2$.

- $M_1 \oplus M_2$ can also be defined via independent sets, circuits, flats, ...
- The direct sum behaves well with restriction and contraction.
- Matroids are the direct sums of their *connected components*.

Example

Let $G_i \in \mathbb{F}_q^{k_i \times n_i}$. Then $M_{G_1} \oplus M_{G_2} = M_G$, where

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}.$$

Outline

Toward a Direct Sum of q-Matroids

None of the above definitions leads to a well-defined q-analogue.

Example:

Let $G_1 = \begin{pmatrix} 1 & a \end{pmatrix}$, $G_2 = \begin{pmatrix} 1 & b \end{pmatrix} \in \mathbb{F}_{q^m}^{1\times 2}$ and set

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}.$$

Then

- $\mathcal{M}_{G_i}$ do not depend on $a, b \in \mathbb{F}_{q^m} \setminus \mathbb{F}_q$.
- $\mathcal{M}_G$ depends on the choice of a, $b \in \mathbb{F}_{q^m} \setminus \mathbb{F}_q$.

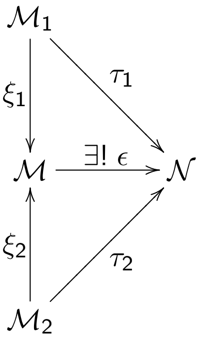
Definition of direct sum by Ceria/Jurrius: a few slides later!

A Categorical Approach

Theorem (Crapo/Rota '70)

The direct sum of matroids is a coproduct in the category of matroids with strong maps as morphisms (also true for weak maps).

Coproduct of $\mathcal{M}_1$, $\mathcal{M}_2$ is a triple $(\mathcal{M}, \xi_1, \xi_2)$...



Examples:

- direct sum of vector spaces,
- direct sum of abelian groups,

...

A Categorical Approach

Morphisms of q-Matroids

Let $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i)$ be q-matroids and let $\phi : \mathbb{F}^{n_1} \longrightarrow \mathbb{F}^{n_2}$ be a linear map.

- ϕ is a **weak map** from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $\rho_2(\phi(V)) \leq \rho_1(V)$ for all $V \in \mathcal{L}(\mathbb{F}^{n_1})$.
- ϕ is a **strong map** from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $\phi^{-1}(V)$ is a flat in $\mathcal{M}_1$ for all flats V in $\mathcal{M}_2$.
  (A **flat** of a q-matroid is an inclusion-maximal space for its rank.).

Theorem (Jany '22)

Strong maps are weak map.

Theorem (GL/Jany '21)

There is no coproduct in the category of q-matroids with strong maps.

Direct Sum of q-Matroids

What about the category with weak maps?

Direct Sum of q-Matroids

What about the category with weak maps?

While starring at this, Ceria/Jurrius came to our rescue.

Direct Sum of q-Matroids

Definition (Ceria/Jurrius '21)

Let $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i)$ for $i = 1, 2$. Define the projections $\pi_i : \mathbb{F}^{n_1+n_2} \longrightarrow \mathbb{F}^{n_i}$ and

$$\begin{aligned} \rho : \mathcal{L}(\mathbb{F}^{n_1+n_2}) &\longrightarrow \mathbb{N}_0 \\ V \quad &\longmapsto \min_{X \leq V} (\rho_1(\pi_1(X)) + \rho_2(\pi_2(X)) + \dim V - \dim X). \end{aligned}$$

Then $\mathcal{M} = (\mathbb{F}^{n_1+n_2}, \rho)$ is a q-matroid, denoted as $\mathcal{M}_1 \oplus \mathcal{M}_2$.

Compatibility with Restriction and Contraction (Ceria/Jurrius '21)

Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Then

$$\mathcal{M}|_{\mathbb{F}^{n_1}} \cong \mathcal{M}_1 \cong \mathcal{M}/\mathbb{F}^{n_2} \quad \text{and} \quad \mathcal{M}|_{\mathbb{F}^{n_2}} \cong \mathcal{M}_2 \cong \mathcal{M}/\mathbb{F}^{n_1}.$$

Direct Sum of q-Matroids

Theorem (GL/Jany '21)

$\mathcal{M}_1 \oplus \mathcal{M}_2$ is a coproduct in the category of q-matroids with weak maps.

Corollary

Let $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i)$. Consider the collection

$$\mathcal{S} = \{\mathcal{N} \mid \mathcal{N} \text{ } q\text{-matroid on } \mathbb{F}^{n_1+n_2},\ \mathcal{N}|_{\mathbb{F}^{n_i}} \cong \mathcal{M}_i\}.$$

Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the unique q-matroid in $\mathcal{S}$ with the most independent spaces.

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Thank You!