Universal Decoding of Interleaved Linearized Reed–Solomon Codes in the Sum-Rank Metric

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Outline

1. Introduction
2. Interleaved Linearized Reed–Solomon Codes
3. Universal Decoding of ILRS Codes
4. Summary & Outlook
Introduction

Motivation

- Interleaving of codes allows for decoding errors beyond the unique decoding radius
- Usually interleaving of codewords of the same code (or a subcode thereof) is considered
- Interpolation-based decoding of vertically interleaved linearized Reed–Solomon (ILRS) codes is considered in [BP22]

Our contribution:

- We consider interleaving of different linearized Reed–Solomon codes by generalizing the universal decoding concept for Gabidulin codes from [SLK20]
- We propose an efficient universal decoding framework for ILRS codes
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Some Definitions

- Finite field $\mathbb{F}_q$ of order $q$ (prime power) and extension field $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ of degree $m$
- $\theta : \mathbb{F}_{q^m} \mapsto \mathbb{F}_{q^m}$: field automorphism (e.g. Frobenius automorphism: $\sigma(\cdot) = \cdot^q$)
- The $i$-th generalized norm of $a \in \mathbb{F}_{q^m}$ is
  \[ N_i^\theta(a) = \theta^{i-1}(a)\theta^{i-2} \cdots \theta(a)a \]
- Operator $D_a(b) := \theta(b)a$ for $a, b \in \mathbb{F}_{q^m}$. The $i$-fold application of $D$ to $b$ w.r.t. $a$ is given by
  \[ D_a^i(b) = D_a(D_a^{i-1}(b)) = \theta^i(b)N_i^\theta(a) \]
- Two elements $a, b \in \mathbb{F}_{q^m}$ are called conjugates if
  \[ \exists c \in \mathbb{F}_{q^m}^* : a^c := \theta(c)ac^{-1} = b \]
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- Let $n \in (\mathbb{N}^*)^\ell$ be a **length partition** such that $\sum_{i=1}^\ell n_i = n$ and let $\xi = (\xi_1, \ldots, \xi_\ell) \in \mathbb{F}_q^n$.

- The $d \times n$ **generalized Moore matrix** of $x = (x^{(1)} | \cdots | x^{(\ell)}) \in \mathbb{F}_q^n$ with $x^{(i)} \in \mathbb{F}_{q^{n_i}}$ for all $i = 1, \ldots, \ell$ is defined as

  $$\lambda_d(x)\xi := (V_d(x^{(1)})\xi_1 | V_d(x^{(2)})\xi_2 | \cdots | V_d(x^{(\ell)})\xi_\ell) \in \mathbb{F}_{q^m}^{d \times n}$$

  where

  $$V_d(x^{(i)})\xi_i = \begin{pmatrix} x_1^{(i)} & x_2^{(i)} & \cdots & x_{n_i}^{(i)} \\ D_{\xi_i}(x_1^{(i)}) & D_{\xi_i}(x_2^{(i)}) & \cdots & D_{\xi_i}(x_{n_i}^{(i)}) \\ \vdots & \vdots & \ddots & \vdots \\ D_{\xi_i}^{d-1}(x_1^{(i)}) & D_{\xi_i}^{d-1}(x_2^{(i)}) & \cdots & D_{\xi_i}^{d-1}(x_{n_i}^{(i)}) \end{pmatrix} \in \mathbb{F}_{q^m}^{d \times n_i}, \ \forall i = 1, \ldots, \ell.$$  

- We have $\text{rk}_{q^m}(\lambda_d(x)\xi) = \min\{n, d\}$ iff. $x_1^{(i)}, \ldots, x_{n_i}^{(i)}$ are $\mathbb{F}_q$-linearly independent for all $i = 1, \ldots, \ell$ and $\xi_1, \ldots, \xi_\ell$ belong to different conjugacy classes of $\mathbb{F}_{q^m}$. 
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"DLR"
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- The **sum-rank weight** of a vector $x = (x^{(1)} | x^{(2)} | \cdots | x^{(\ell)}) \in \mathbb{F}_{q^m}^n$ with $x^{(i)} \in \mathbb{F}_{q^m}$ for all $i = 1, \ldots, \ell$ is defined as [NUF10]

  $$\text{wt}_{\Sigma R}(x) := \sum_{i=1}^{\ell} \text{rk}_q(x^{(i)})$$

  where $\text{rk}_q(x^{(i)})$ denotes the rank of the matrix obtained by expanding $x^{(i)}$ over $\mathbb{F}_q$.

- The **sum-rank distance** between two vectors $x = (x^{(1)} | x^{(2)} | \cdots | x^{(\ell)}) \in \mathbb{F}_{q^m}^n$ and $y = (y^{(1)} | y^{(2)} | \cdots | y^{(\ell)}) \in \mathbb{F}_{q^m}^n$ is then defined as

  $$d_{\Sigma R}(x, y) := \text{wt}_{\Sigma R}(x - y) = \sum_{i=1}^{\ell} \text{rk}_q(x^{(i)} - y^{(i)})$$

- The sum-rank metric is a hybrid between the Hamming metric and the rank metric:
  - $\ell = n$: Hamming metric
  - $\ell = 1$: rank metric

- There is an **isometry** between the sum-rank metric and the skew metric [MP18]
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Skew Polynomials
Definition

- A *skew polynomial* over $\mathbb{F}_{q^m}$ is a polynomial of the form [Ore33]

$$f(x) = \sum_i f_i x^i, \quad f_i \in \mathbb{F}_{q^m} \quad (1)$$

- **Addition**: ordinary monomial-wise polynomial addition
- **Multiplication**: 
  $$xa = \theta(a)x$$

- The set of skew polynomials $\mathbb{F}_{q^m}[x; \theta]$ over $\mathbb{F}_{q^m}$ forms a *non-commutative* ring under addition “+” and multiplication “·.”
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Example:

\[ x^2 \cdot (ax^2 + bx + c) = x^2 ax^2 + x^2 bx + x^2 c = \theta^2(a)x^4 + \theta^2(b)x^3 + \theta^2(c)x^2. \]

- We consider the generalized operator evaluation at an element \( b \in \mathbb{F}_{q^m} \) w.r.t. \( a \in \mathbb{F}_{q^m} \):

\[ f(b)_a = \sum_i f_i \mathcal{D}_a^i(b) \]

- The generalized operator evaluation forms an \( \mathbb{F}_q \)-linear map, i.e. we have

\[ f(\beta b + \gamma c)_a = \beta f(b)_a + \gamma f(c)_a, \quad \forall \beta, \gamma \in \mathbb{F}_q, \forall a, b, c \in \mathbb{F}_{q^m} \]

- For a skew polynomial \( f \in \mathbb{F}_{q^m}[x; \theta] \), a vector \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{F}_{q^m}^n \) and \( a \in \mathbb{F}_{q^m} \) we define

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\[ f(\beta b + \gamma c)_a = \beta f(b)_a + \gamma f(c)_a, \quad \forall \beta, \gamma \in \mathbb{F}_q, \forall a, b, c \in \mathbb{F}_{q^m} \]

- For a skew polynomial \( f \in \mathbb{F}_{q^m}[x; \theta] \), a vector \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{F}_{q^m}^n \) and \( a \in \mathbb{F}_{q^m} \) we define

\[ f(b)_a := (f(b_1)_a, \ldots, f(b_n)_a) \]
Skew Polynomials

Example:

\[ x^2 \cdot (ax^2 + bx + c) = x^2ax^2 + x^2bx + x^2c \]
\[ = \theta^2(a)x^4 + \theta^2(b)x^3 + \theta^2(c)x^2. \]

- We consider the **generalized operator evaluation** at an element \( b \in \mathbb{F}_{q^m} \) w.r.t. \( a \in \mathbb{F}_{q^m} \):

\[ f(b)_a = \sum_i f_i D^i_a(b) \]

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Skew Polynomials

Example:

\[ x^2 \cdot (ax^2 + bx + c) = x^2ax^2 + x^2bx + x^2c = \theta^2(a)x^4 + \theta^2(b)x^3 + \theta^2(c)x^2. \]

- We consider the \textit{generalized operator evaluation} at an element \( b \in \mathbb{F}_{q^m} \) w.r.t. \( a \in \mathbb{F}_{q^m} \):
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Linearized Reed–Solomon Codes

Definition

- Define $\beta := (\beta^{(1)} | \cdots | \beta^{(\ell)}) \in \mathbb{F}_{q^m}^n$ with vectors $\beta^{(i)} = (\beta_1^{(i)}, \ldots, \beta_{n_i}^{(i)}) \in \mathbb{F}_{q^m}^{n_i}$ containing $\mathbb{F}_q$-linearly independent elements of $\mathbb{F}_{q^m}$ for all $i = 1, \ldots, \ell$ (i.e. $\text{wt}_{\Sigma R}(\beta) = n$)

- Let $\xi = (\xi_1, \ldots, \xi_\ell) \in \mathbb{F}_{q^m}^\ell$ be a vector containing elements from distinct nontrivial conjugacy classes of $\mathbb{F}_q^m$ and consider a length partition $n = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ s.t. $n = \sum_{i=1}^\ell n_i$

- A linearized Reed–Solomon (LRS) code of length $n$ and dimension $k$ is defined as [MP18, Car19]

  $$\text{LRS}[\theta, \beta, \xi, \ell; n, k] := \left\{ \left( f(\beta^{(1)})_{\xi_1} | \cdots | f(\beta^{(\ell)})_{\xi_\ell} \right) : f \in \mathbb{F}_{q^m}[x; \theta]_{<k} \right\} \subseteq \mathbb{F}_{q^m}^n$$

- Minimum distance: $n - k + 1 \Rightarrow$ LRS codes are maximum sum-rank distance (MSRD) codes

- Efficient unique decoding up to errors of sum-rank weight $t \leq \frac{n-k}{2}$
Linearized Reed–Solomon Codes

Definition

- Define $\beta := (\beta^{(1)} | \cdots | \beta^{(\ell)}) \in \mathbb{F}_{q}^{n}$ with vectors $\beta^{(i)} = (\beta_{1}^{(i)}, \ldots, \beta_{n_{i}}^{(i)}) \in \mathbb{F}_{q}^{n_{i}}$ containing $\mathbb{F}_{q}$-linearly independent elements of $\mathbb{F}_{q}$ for all $i = 1, \ldots, \ell$ (i.e. $\text{wt}_{\Sigma R}(\beta) = n$)

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$$

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\]

- Minimum distance: \( n - k + 1 \Rightarrow \) LRS codes are MSRD codes

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Interleaved Linearized Reed–Solomon Codes

Definition

Fix a length partition \( n = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell \) with \( n = \sum_{i=1}^\ell n_i \), an interleaving order \( s \geq 1 \) and \( k = (k_1, \ldots, k_s) \in \mathbb{N}^s \) for \( 1 \leq k_j \leq n \)

Let \( \beta_j = (\beta_j^{(1)} | \cdots | \beta_j^{(\ell)}) \in \mathbb{F}_{q^m}^{n_i} \) with \( \beta_j^{(i)} \in \mathbb{F}_{q^m}^{n_i} \) have \( \text{wt}_{SR}(\beta_j) = n \) for all \( j = 1, \ldots, s \) and define \( \beta = (\beta_1 | \cdots | \beta_s) \).

Let \( \xi_j \in \mathbb{F}_{q^m}^{\ell} \) contain representatives of distinct nontrivial conjugacy classes of \( \mathbb{F}_{q^m} \) and define \( \xi = (\xi_1 | \cdots | \xi_s) \).

An \( s \)-interleaved linearized Reed–Solomon (ILRS) code is defined as

\[
\text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] := \left\{ (c_1, c_2, \ldots, c_s) : c_j \in \text{LRS}[\theta, \beta_j, \xi_j, \ell; n, k_j], \forall j = 1, \ldots, s \right\} \subseteq \mathbb{F}_{q^m}^{sn}.
\]

Minimum distance: \( n - \max_j \{k_j\} + 1 \Rightarrow \text{LRS codes are MSRD codes iff. } k_1 = k_2 = \cdots = k_s \)
Interleaved Linearized Reed–Solomon Codes

Definition

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- Let \( \beta_j = (\beta_j^{(1)} | \cdots | \beta_j^{(\ell)}) \in \mathbb{F}^{n_j}_{q^m} \) with \( \beta_j^{(i)} \in \mathbb{F}^{n_i}_{q^m} \) have \( \text{wt}^{SR}(\beta_j) = n \) for all \( j = 1, \ldots, s \) and define \( \beta = (\beta_1 | \cdots | \beta_s) \).

- Let \( \xi_j \in \mathbb{F}^r_{q^m} \) contain representatives of distinct nontrivial conjugacy classes of \( \mathbb{F}^r_{q^m} \) and define \( \xi = (\xi_1 | \cdots | \xi_s) \).

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\text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] := \left\{ (c_1, c_2, \ldots, c_s) : c_j \in \text{LRS}[\theta, \beta_j, \xi_j, \ell; n, k_j], \quad \forall j = 1, \ldots, s \right\} \subseteq \mathbb{F}^{sn}_{q^m}.
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- Let \( \beta_j = (\beta_j^{(1)} \mid \cdots \mid \beta_j^{(\ell)}) \in \mathbb{F}_{q^m}^{n_j} \) with \( \beta_j^{(i)} \in \mathbb{F}_{q^m}^{n_i} \) have \( \text{wt}_{\Sigma R}(\beta_j) = n \) for all \( j = 1, \ldots, s \) and define \( \beta = (\beta_1 \mid \cdots \mid \beta_s) \).

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- Minimum distance: $n - \max_j \{k_j\} + 1 \Rightarrow$ LRS codes are MSRD codes iff. $k_1 = k_2 = \cdots = k_s$.
Interleaved Linearized Reed–Solomon Codes

Definition

- The sum-rank weight of a horizontally interleaved vector

\[ x = \left( (x_1^{(1)} \mid \cdots \mid x_1^{(\ell)}), (x_2^{(1)} \mid \cdots \mid x_2^{(\ell)}), \ldots, (x_s^{(1)} \mid \cdots \mid x_s^{(\ell)}) \right) \in \mathbb{F}_{q^m}^{sn} \]

is defined as

\[
\text{wt}_{\Sigma R}(x) := \sum_{i=1}^{\ell} \text{rk}_q(x_1^{(i)}, \ldots, x_s^{(i)})
\]

We call ILRS codes

- **locator-homogeneous**, if the code locators \( \beta \) of the component codes are equal, i.e.

\[
\text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] = \left\{ \left( f_1(\beta)_{\xi_1}, \ldots, f_s(\beta)_{\xi_s} \right) : f_j \in \mathbb{F}_{q^m}[x; \theta]_{<k_j}, \forall j = 1, \ldots, s \right\}
\]

- **evaluation-homogeneous**, if the component codes use the same evaluation parameters \( \xi \)

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  \[ \text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] = \{ (f_1(\beta)\xi_1, \ldots, f_s(\beta)\xi_s) : f_j \in \mathbb{F}_{q^m}[x; \theta]_{<kj}, \forall j = 1, \ldots, s \} \]

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Interleaved Linearized Reed–Solomon Codes

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Evaluation-Homogeneous ILRS Codes

- Consider an evaluation-homogeneous ILRS code \( \text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] \) where the code locators \( \beta_1, \ldots, \beta_s \) of the component codes span the same \( \mathbb{F}_q \)-linear space block-wise, i.e. we have

\[
\langle \beta_1^{(i)} \rangle_q = \langle \beta_2^{(i)} \rangle_q = \cdots = \langle \beta_s^{(i)} \rangle_q, \quad \forall i = 1, \ldots, \ell
\]

- Let \( \beta_* := (\beta_1^{(1)} | \cdots | \beta_\ell^{(\ell)}) \in \mathbb{F}_{q^m}^n \) be such that

\[
\langle \beta_*^{(i)} \rangle_q = \langle \beta_j^{(i)} \rangle_q, \quad \forall i = 1, \ldots, \ell, \forall j = 1, \ldots, s
\]

- Then there exist full-rank matrices \( W_j^{(i)} \in \mathbb{F}_q^{n_j \times n_i} \) such that

\[
\beta_j^{(i)} = \beta_*^{(i)} W_j^{(i)}, \quad \forall i = 1, \ldots, \ell, \forall j = 1, \ldots, s
\]

- Hence, \( \beta_j W_j = \beta_* \) with \( W_j = \text{diag}(W_1^{(1)}, \ldots, W_\ell^{(\ell)}) \) for all \( j = 1, \ldots, s \)
Evaluation-Homogeneous ILRS Codes

Consider an evaluation-homogeneous ILRS code $\text{ILRS}[\theta, \beta, \xi, \ell; s, n, k]$ where the code locators $\beta_1, \ldots, \beta_s$ of the component codes span the same $\mathbb{F}_q$-linear space block-wise, i.e. we have

$$\langle \beta^{(i)}_1 \rangle_q = \langle \beta^{(i)}_2 \rangle_q = \cdots = \langle \beta^{(i)}_s \rangle_q, \quad \forall i = 1, \ldots, \ell$$

Let $\beta_* := (\beta^{(1)}_* | \cdots | \beta^{(\ell)}_*) \in \mathbb{F}_q^n$ be such that

$$\langle \beta^{(i)}_* \rangle_q = \langle \beta^{(i)}_{j} \rangle_q, \quad \forall i = 1, \ldots, \ell, \forall j = 1, \ldots, s$$

Then there exist full-rank matrices $W^{(i)}_j \in \mathbb{F}_q^{n_j \times n_i}$ such that

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- Let $\beta_* := (\beta_1^{(1)} | \cdots | \beta_\ell^{(\ell)}) \in \mathbb{F}_{q^m}$ be such that
  \[ \langle \beta_*^{(i)} \rangle_q = \langle \beta_j^{(i)} \rangle_q, \quad \forall i = 1, \ldots, \ell, \forall j = 1, \ldots, s \]
- Then there exist full-rank matrices $W_j^{(i)} \in \mathbb{F}_q^{n_i \times n_i}$ such that
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Consider an evaluation-homogeneous ILRS code $\text{ILRS}[\theta, \beta, \xi, \ell, s; n, k]$ where the code locators $\beta_1, \ldots, \beta_s$ of the component codes span the same $\mathbb{F}_q$-linear space block-wise, i.e. we have

$$\langle \beta_1^{(i)} \rangle_q = \langle \beta_2^{(i)} \rangle_q = \cdots = \langle \beta_s^{(i)} \rangle_q, \quad \forall i = 1, \ldots, \ell$$

Let $\beta_* := (\beta_*^{(1)} | \cdots | \beta_*^{(\ell)}) \in \mathbb{F}_{q^m}$ be such that

$$\langle \beta_*^{(i)} \rangle_q = \langle \beta_j^{(i)} \rangle_q, \quad \forall i = 1, \ldots, \ell, \forall j = 1, \ldots, s$$

Then there exist full-rank matrices $W_j^{(i)} \in \mathbb{F}_q^{n_i \times n_i}$ such that

$$\beta_j^{(i)} = \beta_*^{(i)} W_j^{(i)}, \quad \forall i = 1, \ldots, \ell, \forall j = 1, \ldots, s$$

Hence, $\beta_j W_j = \beta_*$ with $W_j = \text{diag}(W_j^{(1)}, \ldots, W_j^{(\ell)})$ for all $j = 1, \ldots, s$
Evaluation-Homogeneous ILRS Codes

Lemma (Transformed Codewords)

For any \( c = (c_1, \ldots, c_s) \in \text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] \) we have that

\[
\tilde{c} = (c_1 W_1, \ldots, c_s W_s) \in \text{ILRS}[\theta, \beta^*, \xi, \ell, s; n, k]
\]

The statement follows from the \( \mathbb{F}_q \)-linearity of the generalized operator evaluation, i.e. for all \( j = 1, \ldots, s \) we have:

\[
c_j W_j = \left( f_j(\beta_j^{(1)})_{\xi_j,1} W_j^{(1)} \mid \cdots \mid f_j(\beta_j^{(\ell)})_{\xi_j,\ell} W_j^{(\ell)} \right) = \left( f_j(\beta_j^{(1)} W_j^{(1)})_{\xi_j,1} \mid \cdots \mid f_j(\beta_j^{(\ell)} W_j^{(\ell)})_{\xi_j,\ell} \right).
\]

\[
= \beta_j^{(1)} \mid \cdots \mid \beta_j^{(\ell)}
\]
Evaluation-Homogeneous ILRS Codes

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For any $c = (c_1, \ldots, c_s) \in \text{ILRS}[\theta, \beta, \xi, \ell, s; n, k]$ we have that

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$$= \beta_j^{(1)} \xi_{j,1} \mid \cdots \mid \beta_j^{(\ell)} \xi_{j,\ell}.$$
Decoding of Evaluation-Homogeneous ILRS Codes

Consider the transmission of \( c \in \text{ILRS}[\theta, \beta, \xi, \ell; s, n, k] \) over a sum-rank channel

\[
y = c + e \quad \text{with} \quad \text{wt}_{\Sigma R}(e) = t.
\]

Syndrome Decoding procedure:
- Compute syndromes using the parity-check matrices \( H_1, \ldots, H_s \) of the component codes
- Solve key equations (multi-sequence skew-feedback shift-register synthesis)
- Recover estimate \( \hat{e} \) of the error vector and return \( \hat{c} = y - \hat{e} \)

Universal Decoding procedure:
- Compute the transformed received word

\[
\tilde{y} = (y_1 W_1 \ldots y_s W_s) = (c_1 W_1 \ldots c_s W_s) + (e_1 W_1 \ldots e_s W_s)
\]
- Apply a decoder \( \tilde{D} \) for ILRS[\( \theta, \beta, \xi, \ell, s; n, k \)] to \( \tilde{y} \):
  - Interpolation-based: recovers message polynomials \( f_1, \ldots, f_s \) directly (no transformation required)
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**Universal Decoding procedure:**
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$$\tilde{y} = \begin{pmatrix} y_1 W_1 & \cdots & y_s W_s \end{pmatrix} = \begin{pmatrix} c_1 W_1 & \cdots & c_s W_s \end{pmatrix} + \begin{pmatrix} e_1 W_1 & \cdots & e_s W_s \end{pmatrix}$$

- Apply a decoder $\tilde{D}$ for ILRS[$\theta, \beta_*, \xi, \ell, s; n, k$] to $\tilde{y}$:
  - **Interpolation-based:** recovers message polynomials $f_1, \ldots, f_s$ directly (no transformation required)
- Decoding scheme has the same characteristic as decoding $e$ directly
Locator-Homogeneous ILRS Codes

- Consider a locator-homogeneous ILRS code ILRS[\(\theta, \beta, \xi, \ell, s; n, k\)] with code locators 
  \(\beta = (\beta_* | \cdots | \beta_*)\) where \(\beta_* = (\beta_*^{(1)} | \beta_*^{(2)} | \cdots | \beta_*^{(s)}) \in \mathbb{F}_q^n\) with \(\text{wt}_{\Sigma R}(\beta_*) = n\)

- Let \(G_j \in \mathbb{F}_{q^m}^{k_j \times n}\) be a generator matrix of the \(j\)-th component code of the form 
  \[G_j = \lambda_{k_j}(\beta_*)^\gamma = \left( V_{k_j}(\beta_*^{(1)})_{\xi_{j,1}} | V_{k_j}(\beta_*^{(2)})_{\xi_{j,2}} | \cdots | V_{k_j}(\beta_*^{(\ell)})_{\xi_{j,\ell}} \right)\]

**Lemma**

For a vector \(\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{F}_q^r\), an element \(\xi \in \mathbb{F}_q^m\), a nonzero \(c \in \mathbb{F}_q^m\) and \(k \in \mathbb{N}^*\) we have that 
\[V_k(\beta)_{\xi^c} = N_k(c) \cdot V_k(\beta)_{\xi}\]
with \(N_k(c) := \text{diag}(N_0^\theta(\theta(c)c^{-1}), \ldots, N_{k-1}^\theta(\theta(c)c^{-1})) \in \mathbb{F}_{q^m}^{k \times k}\).
Locator-Homogeneous ILRS Codes

Consider a locator-homogeneous ILRS code $\text{ILRS}[\theta, \beta, \xi, \ell, s; n, k]$ with code locators $\beta = (\beta_\ast | \cdots | \beta_\ast)$ where $\beta_\ast = (\beta_\ast^{(1)} | \beta_\ast^{(2)} | \cdots | \beta_\ast^{(s)}) \in \mathbb{F}_q^m$ with $\text{wt}_\Sigma(\beta_\ast) = n$

Let $G_j \in \mathbb{F}_q^{k_j \times n}$ be a generator matrix of the $j$-th component code of the form

$$G_j = \lambda_{k_j}(\beta_\ast^\gamma) = \left( V_{k_j}(\beta_\ast^{(1)})_{\xi_j, 1} | V_{k_j}(\beta_\ast^{(2)})_{\xi_j, 2} | \cdots | V_{k_j}(\beta_\ast^{(\ell)})_{\xi_j, \ell} \right)$$

Lemma

For a vector $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{F}_q^m$, an element $\xi \in \mathbb{F}_q^m$, a nonzero $c \in \mathbb{F}_q^*$ and $k \in \mathbb{N}^*$ we have that

$$V_k(\beta)_{\xi^c} = N_k(c) \cdot V_k(\beta)_{\xi}$$

with $N_k(c) := \text{diag}(N_0^\theta(c\xi^{-1}), \ldots, N_{k-1}^\theta(c\xi^{-1})) \in \mathbb{F}_q^{k \times k}$. 

DLR
Locator-Homogeneous ILRS Codes

- Consider a locator-homogeneous ILRS code $\text{ILRS}[\theta, \beta, \xi, \ell, s; n, k]$ with code locators $\beta = (\beta_1 | \cdots | \beta_s)$ where $\beta_* = (\beta_*^{(1)} | \beta_*^{(2)} | \cdots | \beta_*^{(s)}) \in F_{qm}^n$ with $\text{wt}_R(\beta_*) = n$.

- Let $G_j \in F_{q^m}^{k_j \times n}$ be a generator matrix of the $j$-th component code of the form

$$G_j = \lambda_{k_j}(\beta_*^{(j)}) \gamma = \begin{pmatrix} V_{k_j}(\beta_*^{(1)})\xi_{j,1} & V_{k_j}(\beta_*^{(2)})\xi_{j,2} & \cdots & V_{k_j}(\beta_*^{(\ell)})\xi_{j,\ell} \end{pmatrix}$$

Lemma

For a vector $\beta = (\beta_1, \ldots, \beta_r) \in F_{q^m}^r$, an element $\xi \in F_{q^m}$, a nonzero $c \in F_{q^m}^*$ and $k \in \mathbb{N}^*$ we have that

$$V_k(\beta)\xi^c = N_k(c) \cdot V_k(\beta)\xi$$

with $N_k(c) := \text{diag}(N_0^\theta(\theta(c)c^{-1}), \ldots, N_{k-1}^\theta(\theta(c)c^{-1})) \in F_{q^m}^{k \times k}$.
Locator-Homogeneous ILRS Codes

- Let $c_j = (c_{j,1}, \ldots, c_{j,\ell}) \in \mathbb{F}_{q^m}^{\ell}$ contain only nonzero entries and define the conjugate vectors $\xi_j^{c_j} = (\xi_{j,1}^{c_j}, \ldots, \xi_{j,\ell}^{c_j}) \in \mathbb{F}_{q^m}^{\ell}$ for all $j = 1, \ldots, s$ and $\xi^c = (\xi_1^c | \cdots | \xi_s^c) \in \mathbb{F}_{q^m}^{s\ell}$.

- Then there exists a generator matrix for the $j$-th component code of ILRS $[\theta, \beta, \xi^c, \ell, s; n, k]$ of the form

$$
\begin{pmatrix}
N_{kj}(c_{j,1}) \cdot V_{kj}(\beta^{(1)}_{*})_{\xi_{j,1}} & N_{kj}(c_{j,2}) \cdot V_{kj}(\beta^{(2)}_{*})_{\xi_{j,2}} & \cdots & N_{kj}(c_{j,\ell}) \cdot V_{kj}(\beta^{(\ell)}_{*})_{\xi_{j,\ell}}
\end{pmatrix}
$$

(2)

- If $c_{j,1} = c_{j,2} = \cdots = c_{j,\ell}$ or $c_j \in \mathbb{F}_q^{\ell}$, then we can write (2) as

$$
N_{kj}(c_{j,1}) \cdot \left( V_{kj}(\beta^{(1)}_{*})_{\xi_{j,1}} | V_{kj}(\beta^{(2)}_{*})_{\xi_{j,2}} | \cdots | V_{kj}(\beta^{(\ell)}_{*})_{\xi_{j,\ell}} \right) = N_{kj}(c_{j,1}) \cdot G_j.
$$

- Since $N_{kj}(c_{j,1})$ has full rank, $N_{kj}(c_{j,1}) \cdot G_j$ and $G_j$ have the same row space for all $j = 1, \ldots, s$. Hence,

$$
\text{ILRS}[\theta, \beta, \xi^c, \ell, s; n, k] = \text{ILRS}[\theta, \beta, \xi, \ell, s; n, k]
$$
Locator-Homogeneous ILRS Codes

- Let $c_j = (c_{j,1}, \ldots, c_{j,\ell}) \in \mathbb{F}_q^{\ell m}$ contain only nonzero entries and define the conjugate vectors $\xi_{c_j}^j = (\xi_{c_j,1}^j, \ldots, \xi_{c_j,\ell}^j) \in \mathbb{F}_q^{\ell m}$ for all $j = 1, \ldots, s$ and $\xi^c = (\xi_1^c | \cdots | \xi_s^c) \in \mathbb{F}_q^{s \ell m}$.

- Then there exists a generator matrix for the $j$-th component code of ILRS $[\theta, \beta, \xi^c, \ell, s; n, k]$ of the form

$$
\begin{pmatrix}
N_{kj}(c_{j,1}) \cdot V_{kj}(\beta^{(1)}_*)^j_{\xi_{j,1}} & N_{kj}(c_{j,2}) \cdot V_{kj}(\beta^{(2)}_*^j)^{j,j}_{\xi_{j,2}} & \cdots & N_{kj}(c_{j,\ell}) \cdot V_{kj}(\beta^{(\ell)}_*^j)^{j,j}_{\xi_{j,\ell}}
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Locator-Homogeneous ILRS Codes

- Let $c_j = (c_{j,1}, \ldots, c_{j,\ell}) \in \mathbb{F}_q^\ell$ contain only nonzero entries and define the conjugate vectors $\xi^j = (\xi^j_{c_1}, \ldots, \xi^j_{c_{\ell}}) \in \mathbb{F}_q^\ell$ for all $j = 1, \ldots, s$ and $\xi^c = (\xi^c_1 | \cdots | \xi^c_s) \in \mathbb{F}_q^s$.

- Then there exists a generator matrix for the $j$-th component code of ILRS $[\theta, \beta, \xi^c, \ell, s; n, k]$ of the form

$$
\begin{pmatrix}
N_{k_j}(c_{j,1}) \cdot V_{k_j}(\beta_{1j}^{(1)})\xi_{j,1} & N_{k_j}(c_{j,2}) \cdot V_{k_j}(\beta_{1j}^{(2)})\xi_{j,2} & \cdots & N_{k_j}(c_{j,\ell}) \cdot V_{k_j}(\beta_{1j}^{(\ell)})\xi_{j,\ell}
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\end{pmatrix} = N_{k_j}(c_{j,1}) \cdot G_j.
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Hence,

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- Let $\mathbf{c}_j = (c_{j,1}, \ldots, c_{j,\ell}) \in \mathbb{F}_q^{\ell m}$ contain only nonzero entries and define the conjugate vectors $\xi_{c}^j = (\xi_{c}^{j,1}, \ldots, \xi_{c}^{j,\ell}) \in \mathbb{F}_q^{\ell m}$ for all $j = 1, \ldots, s$ and $\xi^c = (\xi_1^c | \cdots | \xi_s^c) \in \mathbb{F}_q^{s \ell m}$.

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Consider the transmission of \( c \in \text{ILRS}[\theta, \beta, \xi^c, \ell, s; n, k] \) over a sum-rank channel

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\]

where either

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- \( c_j \in \mathbb{F}_q^\ell \) for all \( j = 1, \ldots, s \).

Universal Decoding procedure:

- Since \( \text{ILRS}[\theta, \beta, \xi^c, \ell, s; n, k] = \text{ILRS}[\theta, \beta, \xi, \ell, s; n, k] \) we can apply a decoder \( D \) to \( y \):
  - Interpolation-based: recovers message polynomials \( f_1, \ldots, f_s \) directly (transformation required to account for different encoding)
  - Syndrome-based: Returns \( c = y - e \) (no transformation required)

Decoding scheme has the same performance since the error \( e \) is not changed.
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Summary & Outlook

✓ We considered decoding of *interleaved linearized Reed–Solomon (ILRS)* codes with different component codes in the sum-rank metric
✓ We proposed a *decoding framework* for a class of *evaluation-homogeneous ILRS codes* where the code locators for each block span the same $\mathbb{F}_q$-linear space
✓ We derived a *decoding framework* for a class of *locator-homogeneous ILRS codes*
✓ Any decoder for ILRS codes can be used in the proposed universal decoding framework

Further work:

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