CSS-T Codes from Reed-Muller Codes for Quantum Fault-Tolerance

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Coding theory and cryptography

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Joint work with Jessalyn Bolkema.

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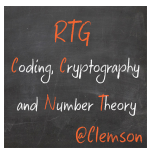

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1. Quantum Computing and Quantum Correction
2. Asymptotic properties of Reed-Muller codes
3. Reed-Muller codes as CSS-T codes

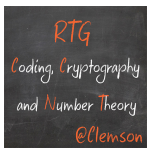
CSS codes

Definition ("CSS codes")

Given C_1 is a binary $[n, k_1, d_1]$ code and C_2 is a binary $[n, k_2, d_2]$ code, then $CSS(C_1, C_2)$ is a $[\![n, k_1 - k_2, \geq \min\{d_1, d_2^{\perp}\}]\!]$ quantum error correcting code.

CSS codes:

- are stabilizer codes generally not optimal;
- correct X and Z errors and their combinations;
- are not fault-tolerance "friendly" because, when considering the universal gates $\{H, S, CX, T\}$, they don't always support the T gate.



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CSS-T codes [Rengaswamy et al., 2020]

Definition

Given C_1 is $[n, k_1, d_1]$ and C_2 is $[n, k_2, d_2]$, the stabilizer code $CSS(C_1, C_2)$ is a CSS-T code if for every $x \in C_2$, $C_1^{\perp}|_x$ contains a self-dual code.

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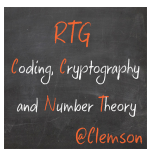
Open Problem: A $[\![n, k_1 - k_2, \min(d_1, d_2^{\perp})]\!]$ family of CSS-T codes such that $\frac{(k_1 - k_2)}{n} = \Omega(1)$ and (ideally) $\frac{\min(d_1, d_2^{\perp})}{n} = \Omega(1)$, where $d_2^{\perp}$ is the minimum distance of $C_2^{\perp}$.



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Reed-Muller Codes

Definition

Let $\mathbb{F}_2[x_1, \ldots, x_m]$, $\mathbb{F}_2^m = \{P_1, \ldots, P_{2^m}\}$. The RM$(r, m)$ is defined as:

$$\mathrm{RM}(r, m) = \{(f(P_1), \ldots, f(P_{2^m})) \mid f \in \mathbb{F}_2[x_1, \ldots, x_m]^r\},$$

where $\mathbb{F}_2[x_1, \ldots, x_m]^r$ is the set of polynomials of total degree at most r.

- A RM(r, m) code is a $[2^m, \sum_{i=0}^{r} \binom{m}{i}, 2^{m-r}]$ linear code.
- For $0 \leq r_2 \leq r_1 \leq m$, we have $\mathrm{RM}(r_2, m) \subseteq \mathrm{RM}(r_1, m)$.
- $\mathrm{RM}(r, m)^{\perp} = \mathrm{RM}(m - r - 1, m)$

RM codes and non-vanishing distance

Proposition

Let $C(s) = RM(r(s), m(s))$ *for* $s \in \mathbb{N}$ *with* $\lim_{s\to\infty} m(s) = \infty$. *The sequence of codes* $(C(s))_{s\in\mathbb{N}}$ *has non-vanishing relative distance if and only if* $\lim_{s\to\infty} r(s) = c$.

Corollary

Let $C_1(s) = RM(r_1(s), m(s))$ *and* $C_2(s) = RM(r_2(s), m(s))$ *for* $s \in \mathbb{N}$ *with* $\lim_{s\to\infty} m(s) = \infty$. *The sequence of CSS codes*

$$(CSS(C_1(s), C_2(s)))_{s\in\mathbb{N}}$$

has always vanishing relative distance.

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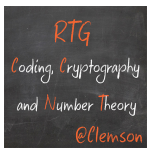
RM codes with non-vanishing rate [Kudekar et al., 2017]

Proposition

Let $C(s) = RM(r(s), m(s))$ *for* $s \in \mathbb{N}$ *such that* $\lim_{s\to\infty} m(s) = \infty$. *Let* $R(C(s))$ *be the rate of code* $C(s)$, *then the asymptotic rate of the sequence* $(C(s))_{s\in\mathbb{N}}$ *is*

$$\lim_{s\to\infty} R(C(s)) = \Phi\left(\lim_{s\to\infty}\left(\frac{2r(s) - m(s)}{\sqrt{m(s)}}\right)\right)$$

where Φ *is the standard normal Gaussian CDF.*



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RM codes with non-vanishing rate - cont'd

Corollary

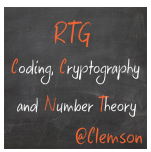
Let $C(m) = RM(\lfloor \frac{m-1}{2} \rfloor + t(m), m)$ *for* $m \in \mathbb{N}$ *with* $t(m)$ *positive. Then the sequence* $(C(m))_{m\in\mathbb{N}}$ *has nonvanishing rate if and only if*

$$\lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} \neq -\infty$$

$$\lim_{m\to\infty} R(C(m)) = \begin{cases} 0 & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} = -\infty \\ \frac{1}{2} & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} = 0 \\ 1 & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} = \infty \end{cases}$$

and

$$\lim_{m\to\infty} R(C(m)) \in \begin{cases} (0, \frac{1}{2}) & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} \in (-\infty, 0) \\ (\frac{1}{2}, 1) & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} \in (0, \infty) \end{cases}$$



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CSS-T codes

Definition (CSS-T)

Let C_1 and C_2 be a $[n, k_1, d_1]$ and a $[n, k_2, d_2]$ binary linear codes respectively. Then $CSS(C_1, C_2)$ is a $[\![n, k_1 - k_2, \geq \min\{d_1, d_2^{\perp}\}]\!]$ quantum stabilizer code. The code $CSS(C_1, C_2)$ is a CSS-T code if for every $x \in C_2$, $C_1^{\perp}|_x$ contains a self-dual code.

Restriction of a code

Definition

We say a codeword, $c \in C$, is supported by some vector $x \in \mathbb{F}_2^n$ if $c \times x = c$, where $\times$ is element-wise multiplication.

Example

$x = (1,1,1,0,0,0)$, $c_1 = (1,1,0,0,0,0)$, $c_2 = (0,0,1,1,0,0)$.
$c_1 \times x = (1,1,0,0,0,0) = c_1$, so c_1 is supported on x.
$c_2 \times x = (0,0,1,0,0,0) \neq c_2$, so c_2 is not supported on x.

Restriction of a code - cont'd

Definition

For some $[n, k]$ binary linear code C and $x \in \mathbb{F}_2^n$, the **restriction of C to (the support of) x**, $C|_x$, is the code created by removing from C all codewords not supported on x followed by removing the entris with coord. where x has zeros.

Example

If $x = (1, 1, 1, 0, 1, 0) \in C_2 \subseteq \mathbb{F}_2^6$ and $C_1^\perp$ has generator matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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$$(0,0,0,0,0,0), (1,1,0,0,0,0), (1,1,1,0,1,0), (0,0,1,0,1,0)$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Notes on duality

- C is self-dual if $C^{\perp} = C \Rightarrow n$ is even and $\dim C = \frac{n}{2}$
- C is self-orthogonal if $C \subseteq C^{\perp}$

Remark

The maximum rate of a CSS-T code defined by RM codes is $\frac{1}{2}$.

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Let $C(m) = RM(\lfloor \frac{m}{2} \rfloor + t(m), m)$ with $m \in \mathbb{N}$

$$\lim_{m\to\infty} R(C(m)) = \begin{cases} 0 & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} = -\infty \\ \frac{1}{2} & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} = 0 \\ 1 & \lim_{m\to\infty} \frac{t(m)}{\sqrt{m}} = \infty \end{cases}$$

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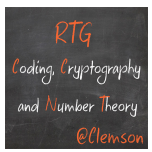

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Notes on duality - again

Proposition

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Notes on shortening and puncturing

Let $C \subseteq \mathbb{F}_2^n$ be a code and $I \subseteq \{1, \dots, n\}$

- $\text{short}(C, I) = C|_x$ where $\text{supp}(x) = \{1, \dots, n\} \setminus I$.
- $\text{punct}(C, I)$: remove from any $x \in C$ the coordinates in I.

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$$\text{punct}(C_1, [n] \setminus x) \subseteq \text{short}(C_1^\perp, [n] \setminus x)$$

CSS-T with RM codes

Theorem

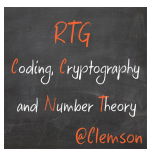
Define $C_1 = RM(\lfloor \frac{m-1}{2} \rfloor - t, m)$ *and* $C_2 = RM(r_2, m)$. *If*

- $r_2 \leq 2t + 1$ *for* m *even, or*
- $r_2 \leq 2t$ *for* m *odd*

then $CSS(C_1, C_2)$ *is a CSS-T code.*

$x = \mathrm{ev}_{\mathbb{F}_2^m}(p) \in C_1$ and $y = \mathrm{ev}_{\mathbb{F}_2^m}(q) \in C_2$

$$\begin{aligned}\mathrm{punct}(y, \mathbb{F}_2^m \setminus \mathrm{supp}(x)) &= \mathrm{punct}(\mathrm{ev}_{\mathbb{F}_2^m}(q), \mathbb{F}_2^m \setminus \mathrm{supp}(x)) \\ &= \mathrm{punct}(\mathrm{ev}_{\mathbb{F}_2^m}((1-p)q) + \mathrm{ev}_{\mathbb{F}_2^m}(pq), \mathbb{F}_2^m \setminus \mathrm{supp}(x)) \\ &= \mathrm{punct}(\mathrm{ev}_{\mathbb{F}_2^m}(pq), \mathbb{F}_2^m \setminus \mathrm{supp}(x)) \\ &= \mathrm{short}(\mathrm{ev}_{\mathbb{F}_2^m}(pq), \mathbb{F}_2^m \setminus \mathrm{supp}(x))\end{aligned}$$



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Conclusion

- CSS-T codes can be used for quantum fault-tolerance.
- CSS-T from Reed-Muller codes first family of CSS-T codes with non-vanishing asymptotic rate up to $\frac{1}{2}$.
- Missing CSS-T families with higher asymptotic rates and non-vanishing relative distance.

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Thank you.

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TBD.

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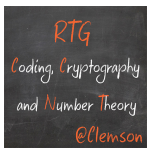

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Single Qubit

The **state** of a qubit can be described as a normalized vector in $\mathbb{C}^2$.

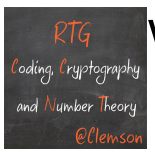
In quantum computing, we define the **standard** or **computational basis** of $\mathbb{C}^2$ to be

$$\left\{ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

General form for the state of a qubit (superposition):

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

with $|\alpha|^2 + |\beta|^2 = 1$.



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n Qubits

- n qubits use $\mathbb{C}^{2^n}$, since $\mathbb{C}^{2^n} = \bigotimes_{i=1}^{n} \mathbb{C}^2$.

Example

Consider a state consisting of 2 qubits:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

The probability mass function of this state is given by

$$p_{|\psi\rangle}(x) = \begin{cases} |\alpha|^2 & \text{if } x = |00\rangle \\ |\beta|^2 & \text{if } x = |01\rangle \\ |\gamma|^2 & \text{if } x = |10\rangle \\ |\delta|^2 & \text{if } x = |11\rangle \end{cases}.$$

Quantum Errors and Quantum Error Correction

Definition ("Errors")

$\boxed{X}$	$\boxed{Z}$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

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Operations

Definition

We say a linear operator U is **unitary** if $UU^\dagger = U^\dagger U = I$.

- All operations in quantum computing are unitary.
- The group of unitary operations on 1 qubit under composition is denoted $\mathbb{U}$.
- The group of unitary operations on n qubits, $\mathbb{U}^n$, is the n-fold tensor product of $\mathbb{U}$

Universality

Example

Some popular Clifford gates are

H	S	X	T
$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}$

The set of gates $\{H, S, CX, T\}$ is universal.