Isotropic Ornstein-Uhlenbeck Flows

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(joint work with Georgi Dimitroff)

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Disentis

Outline

Motivating Example

- Consider the following stochastic differential equation

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = A\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} 17 & 0 \\ 0 & 42 \end{pmatrix} d\begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}$$

  $X_s = x$, $Y_s = y$, A is a real matrix.

- The Solution to this equation is of course:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = e^{(t-s)A}\begin{pmatrix} x \\ y \end{pmatrix} + e^{(t-s)A}\int_0^t e^{-(u-s)A}\begin{pmatrix} 17 \\ 42 \end{pmatrix} dW_u$$

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The flow property

- Consider the solution as a function of the initial value.

$$\Phi_{s,t} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto e^{(t-s)A} \begin{pmatrix} x \\ y \end{pmatrix} + e^{(t-s)A} \int_0^t e^{-(u-s)A} \begin{pmatrix} 17 \\ 42 \end{pmatrix} dW_u$$

- The function $\Phi = \Phi_{s,t}(\cdot, \omega)$ satisfies:
- it is a diffeomorphism for any ω, s, t
- $\Phi_{t,t}(\cdot, \omega)$ is the identity for all ω and t
- $\Phi_{s,t}(\cdot, \omega) = \Phi_{u,t}(\cdot, \omega) \circ \Phi_{s,u}(\cdot, \omega)$
- These properties state that Φ is a stochstic flow of diffeomorphisms.

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Kunita-Type SDEs

- Let is write

$$M\left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) = A \begin{pmatrix} x \\ y \end{pmatrix} t + \begin{pmatrix} 17 & 0 \\ 0 & 42 \end{pmatrix} W_t$$

- Then the SDE becomes

$$d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = M\left(dt, \begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right), X_s = x, Y_s = y$$

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- $M = M(t, \begin{pmatrix} x \\ y \end{pmatrix}))$ satisfies:
- it is a semimartingale for fixed x, y
- it has smooth covariations in x, y for fixed t
- the part of finite variaton is smooth
- This states that is a semimartingale field

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Stochstic Flows And SDEs

$$\phi_{s,t}(x) = x + \int_s^t M(du, \phi_{s,u}(x))$$

- The solution of an SDE driven by a sufficiently smooth semimartingale field generates a stochastic flow.
- For a sufficiently smooth stochastic flow there is a semimartingale field that generates it via an SDE.

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Isotropic Covariance Tensors, Isotropic Brownian Fields

Definition

A function $b : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is an isotropic covariance tensor if:

1. $x \mapsto b(x)$ is smooth enough and the derivatives are bounded.
2. $b(0) = E_d$ (the d-dimensional identity)
3. $x \mapsto b(x)$ is not constant.
4. $b(x) = O^* b(Ox) O$ for any $x \in \mathbb{R}^d$ and $O \in O(d)$

Definition

Let b be as above. $\left\{ M(t,x) : t \geq 0, x \in \mathbb{R}^d \right\}$ is an isotropic Brownian field if:

1. $(t,x) \mapsto M(t,x)$ is a centered Gaussian process.
2. $cov(M(s,x), M(t,y)) = (s \wedge t) b(x-y)$
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Isotropic Brownian Flows (IBFs)

Properties

- translation invariance
- rotation invariance
- one-point motion is a d-dimensional standard Brownian Motion
- SDEs for two-point-distance,...
- Lyapunov-Exponents are known, deterministic and constant
- Lebesgue measure is invariant
- No straightforward entropy definition

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Isotropic Ornstein-Uhlenbeck Flows (IOUFs)

Introduce a drift into the SDE

$$\phi_{s,t}(x) = x + \int_s^t M(du, \phi_{s,u}(x)) - c \int_s^t \phi_{s,u}(x) du \qquad (1)$$

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Spatial Regularity Lemma

Lemma

Let $\phi = \phi_{0,1} : \mathbb{R}^d \to \mathbb{R}^d$ *be as in* (1) *(with* $s = 0$ *and* $t = 1$*). Then we have a.s.*

1. $$\lim_{R\to\infty} \sup_{||x||\geq R} \frac{||\phi(x) - e^{-c}x||}{||x||} = 0 \tag{2}$$

2. $$\lim_{R\to\infty} \sup_{||x||\geq R} \frac{||\phi(x)||}{||x||} = e^{-c} \tag{3}$$

Reformulation To Compact State Space

- It is sufficient to show

$$\lim_{R\to\infty} \sup_{R\leq||x||\leq R+1} \frac{||\phi(x) - e^{-c}x||}{||x||} = 0$$

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The Chaining Lemma

$(\mathbb{X}, d)$ compact metric space, $\phi\colon \mathbb{X} \to \mathbb{R}_+$ be a.s. cont. func.,
$(\delta_i)_{i\geq 0}$ positive real numbers with $\sum_{i=0}^{\infty} \delta_i < \infty$,
$(\chi_i)_{i=0}^{\infty}$ δ_i-dense in $\mathbb{X}$ with $\chi_0 = \{x_0\}$, with $d(x, x_0) \leq \delta_0 \forall x \in \mathbb{X}$.

Lemma (Cranston, Scheutzow and Steinsaltz '00)

For arbitrary positive $\epsilon, z \geq 0$ and an arbitrary sequence of positive reals $(\epsilon_i)_{i\geq 0}$ such that $\epsilon + \sum_{i=0}^{\infty} \epsilon_i = 1$ we have

$$\begin{aligned} &\mathbb{P}\Big(\sup_{x\in\mathbb{X}} \phi(x) > z\Big) \\ \leq\ &\mathbb{P}\big(\phi(x_0) > \epsilon z\big) + \sum_{i=0}^{\infty} |\chi_{i+1}| \sup_{d(x,y)\leq\delta_i} \mathbb{P}\big(|\phi(x) - \phi(y)| > \epsilon_i z\big). \end{aligned}$$

Estimates

- $$\mathbb{P}\left[||\phi(x_0) - e^{-c}x_0|| > \frac{\tilde{\epsilon}R}{2}\right] \leq c_4 e^{-\frac{\tilde{\epsilon}^2}{8d^2}R^2}$$

- $$\begin{aligned} & \mathbb{P}\left[|||\phi(x) - e^{-c}x|| - ||\phi(y) - e^{-c}y||| > 2^{-j-2}\tilde{\epsilon}R\right] \\ \leq\; & \mathbb{P}\left[|||\phi(x) - e^{-c}x|| - ||\phi(y) - e^{-c}y||| > 2^{-j-2}\tilde{\epsilon}R3^j|x-y|\right] \\ \leq\; & \mathbb{P}\left[|||\phi(x) - \phi(y)||| > 2^{-j-3}\tilde{\epsilon}R3^j|x-y|\right] \qquad (4) \\ \leq\; & \mathbb{P}\left[B_1^* \geq \frac{\log(2^{-3-j}\tilde{\epsilon}R3^j) - \lambda}{\sigma}\right] \\ \leq\; & c_5(2^{-j-3}\tilde{\epsilon}R3^j)^{-\frac{\log(2^{-3-j}\tilde{\epsilon}R3^j)-2\lambda}{2\sigma^2}} \end{aligned}$$

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Open Problems - Directions Of Research

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- Meaningful definition of the entropy for IBFs
- Pesin Theory for IBFs

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References

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