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Gap Probabilities for Random Matrix Ensembles

Felix Rubin

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Outline

Introduction

- Main Question: Given a large matrix with random entries, what can be said about the distribution of its eigenvalues?
- In particular: What can be said about the distribution of the largest eigenvalue?
- Started with Physicists in the 50's.
  - Model to understand statistical behavior of slow neutron resonances (Wigner).
- 70's: Applications to number theory (Montgomery).

Gaussian Unitary Ensemble (GUE) I

Definition

A random $N \times N$ Hermitian matrix belongs to the *GUE*, if the diagonal elements x_{jj} and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ $(j < k)$ are independently chosen with normal densities of the form:

$$\frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \sim \mathcal{N}(0, \frac{1}{2}) \text{ (diagonal)},$$

$$\frac{2}{\pi} e^{-2(u_{jk}^2 + v_{jk}^2)} \sim \mathcal{N}(0, \frac{1}{4}) + i\mathcal{N}(0, \frac{1}{4}) \text{ (upper triangular)}$$

Joint p.d.f:

$$p(X) = \prod_{j=1}^{N} \frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \prod_{1 \le j < k \le N} \frac{2}{\pi} e^{-2|x_{jk}|^2} = \frac{1}{Z_N} \exp\{-\mathrm{Tr}(X^2)\}.$$

Gaussian Unitary Ensemble (GUE) II

- Eigenvalue distribution? (Eigenvalues: $(x_1, \ldots, x_N) \subset \mathbb{R}^N$)
- Apply basis transformation and integrate out elements independent of the eigenvalues:

Eigenvalue measure on $\mathbb{R}^N$: If $x_1 < \ldots < x_N$,

$$
\begin{aligned}
u_N(x_1, \ldots, x_N) &= \frac{1}{Z_N} \prod_{1 \le j < k \le N} |x_j - x_k|^2 \exp\left\{ -\sum_{j=1}^{N} x_j^2 \right\} \\
&= \frac{1}{Z_N} \left(\det(p_{j-1}(x_i) e^{(-x_i^2)/2})_{1 \le i,j \le N} \right)^2 \\
&= \det(K_N(x_i, x_j))_{i,j=1}^{N},
\end{aligned}
$$

Gaussian Unitary Ensemble (GUE) III

where

$$K_N(x,y) = \sum_{j=0}^{N-1} p_j^H(x) p_j^H(y) e^{-\frac{x^2+y^2}{2}}$$

$$= \mathrm{const} \cdot e^{-(x^2+y^2)/2} \frac{p_N^H(x)p_{N-1}^H(y) - p_N^H(y)p_{N-1}^H(x)}{x-y}.$$

p_i^H is the i-th normalized Hermite polynomial of degree i.
The eigenvalue distribution can be viewed as a point process on $\mathbb{R}$ via the application $(x_1, \dots, x_N) \mapsto \sum_{i=1}^{N} \delta_{x_i}$. Point processes with a measure of this determinantal form are called *determinantal point processes*.

Gaussian Unitary Ensemble (GUE) IV

Definition

We define the *n-th correlation function* ρ_n by:

$$\rho_n(x_1, \dots, x_n) = \det(K_N(x_i, x_j))_{i,j=1}^{n}, \text{ for } n \leq N.$$

The correlation function can be viewed as a particle density. Namely, if $[x_i, x_i + \Delta x_i]$, $1 \leq i \leq n$, are all disjoint,

$$\rho_n(x_1, \dots, x_n) = \lim_{\Delta x_i \to 0} \frac{P[\text{there is exactly one particle in } [x_i, x_i + \Delta x_i],\ 1 \leq i \leq n]}{\Delta x_1 \dots \Delta x_n}$$

Gap Probabilities and Distribution of Largest Eigenvalue I

Question: $P[\text{ there is no eigenvalue in } (a,b) = 0] = ?,\ a < b \in \mathbb{R}.$

Lemma

Let ϕ be a bounded and measurable function with bounded support B. Then

$$E[\prod_j (1+\phi(x_j))] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \phi(x_j) \rho_n(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

Thus,

$$P[x_{\max} \leq t] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t,\infty)^n} \rho_n(x_1, \ldots, x_n) d^n x.$$

Gap Probabilities and Distribution of Largest Eigenvalue II

The correlation kernel $K_N(x,y)$ can be viewed as the kernel of an integral operator K on $L_2(\mathbb{R})$: If $f \in L_2(\mathbb{R})$,

$$Kf(x) = \int_{\mathbb{R}} K_N(x,y) f(y) dy.$$

One can define the Fredholm determinant of the operator K as:

$$\det(Id - K) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \det(K_N(x_i, x_j))_{i,j=1}^{n} d^n x.$$

If $\rho_n(x_1, \ldots, x_n) = \det(K_N(x_i, x_j))_{1 \le i,j \le n}$, we thus have:

$$P[x_{\max} \le t] = \det(Id - K)|_{L_2(t,\infty)}.$$

Scaling Results and Painlevé I

If one scales around the largest eigenvalue, say $x_{\max}(N)$, of the GUE, one obtains for $N \to \infty$:

$$P\left[x_{\max}(N) \leq \sqrt{2N} + \frac{s}{\sqrt{2}N^{1/6}}\right] \longrightarrow F_{TW}(s) = \det(Id - K_{Airy})|_{L_2(s,\infty)}$$

$$F_{TW}(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)dx\right),$$

q being the solution of a Painlevé-II equation $q'' = sq + 2q^3$ with boundary condition $q(s) \sim Ai(s)$ for $s \to \infty$.

Matrix Ensembles with Generalized Cauchy Weights I

(Joint work with Joseph Najnudel and Ashkan Nikeghbali)

- Consider the Unitary group $U(N)$ with the Haar measure μ_N. The eigenvalue distribution function here is:

$$\mathrm{const} \cdot \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} d\theta_j,$$

where $e^{i\theta_j}$, $j = 1, \ldots, N$, are the eigenvalues of $U \in U(N)$ with $\theta_j \in [-\pi, \pi]$.

Matrix Ensembles with Generalized Cauchy Weights II

- Generalize this eigenvalue distribution: Introduce a complex parameter s, $\Re s \geq -\frac{1}{2}$, and write:

$$\mathrm{const} \cdot \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} w_U(\theta_j) d\theta_j,$$

  where $w_U(\theta_j) = (1 + e^{i\theta_j})^{\overline{s}}(1 + e^{-i\theta_j})^s$.

- $U(N)$ is linked to $H(N)$ (Hermitian matrices) via the Cayley transform

$$X \in H(N) \mapsto \frac{i - X}{i + X} \in U(N).$$

Matrix Ensembles with Generalized Cauchy Weights III

- The Cayley transform sends the generalized Haar measure to the following Cauchy type measure on $H(N)$:

$$\text{const} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w_H(x_j) dx_j,$$

where $w_H(x_j) = (1 + ix_j)^{-s-N}(1 - ix_j)^{-\bar{s}-N}$.

- The correlation kernel for this eigenvalue process is:

$$K_N(x,y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x-y},$$

with $\phi(x) = \sqrt{Cw_H(x)}p_N(x)$, and $\psi(x) = \sqrt{Cw_H(x)}p_{N-1}(x)$. (Borodin, Olshanski, 2001).

Results: An ODE related to the Painlevé-VI equation I

Consider

$$\frac{d}{dt}\log\det(Id - K_N)|_{L_2(t,\infty)} = \frac{d}{dt}\log P[\text{no eigenvalue inside } (t,\infty)].$$

It is known that this is equal to $R(t,t)$, where $R(x,y) \doteq K_N(1-K_N)^{-1}$ is the resolvent kernel of K_N.
Using a general method given by Tracy, Widom (1994), we prove a differential equation for the above quantity. All one needs to find are the following recurrence equations for ϕ and ψ:

$$\begin{aligned} m(x)\phi'(x) =& A(x)\phi(x) + B(x)\psi(x) \\ m(x)\psi'(x) =& -C(x)\phi(x) - A(x)\psi(x), \end{aligned}$$

where A, B and m are polynomials in x.

Results: An ODE related to the Painlevé-VI equation II

We find:

Theorem

Let $\sigma(t) = (1+t^2)R(t,t) = (1+t^2)\frac{d}{dt}\log\det(Id - K_N)|_{L_2(t,\infty)}$. *Then,*

$$(1+t^2)^2(\sigma'')^2 + 4(1+t^2)(\sigma')^3 - 8t(\sigma')^2\sigma$$
$$+ 4\sigma^2(\sigma' - (\Re s)^2) + 8\Re s(\Re s\, t - \alpha_0)\sigma\sigma'$$
$$+ 4\left[2\alpha_0\Re s\, t - \alpha_0^2 - (\Re s)^2 t^2 + \frac{|s|^2}{(\Re s)^2}N(2\Re s + N)\right](\sigma')^2 = 0,$$

where $\alpha_0 = \Im s(1 + \frac{N}{\Re s})$.
(A similar result for $s \in \mathbb{R}$ *has been established by Witte, Forrester (2000))*

Results: An ODE related to the Painlevé-VI equation III

The solution of this equation can be expressed in terms of the solution of the Painlevé-VI equation via a Bäcklund transformation and the change of variable

$$x = \frac{t+i}{2i}, \qquad \eta(x) = \frac{\sigma(t) - (\Re s)^2 t - \alpha_0 \Re s}{2i}.$$

Further Steps: Scaling results for $N \to \infty$ I

Theorem

If $t = N/\tau$ and $\sigma(N/\tau) = -\theta_N(\tau)(\tau/N + N/\tau)$ in the ODE, we get an ODE for $\theta_N(\tau)$ of the form:

$$\sum_{k\geq 0} f_k(\tau, \theta_N(\tau), \theta_N'(\tau), \theta_N''(\tau))N^{-k} = 0,$$

where the sum is finite and f is rational in all variables. Moreover, f_0 corresponds to the Painlevé-V equation. Thus, θ_N satisfies a differential equation which tends to the Painlevé-V equation if $N \to \infty$.

Further Steps: Scaling results for $N \to \infty$ II

Further steps:

- Does the solution θ_N converge to the solution of the Painlevé-V equation? Ie. does $\det(Id - K_N)|_{L_2(N\tau^{-1},\infty)}$ converge to $\det(Id - K)|_{L_2(\tau^{-1},\infty)}$, where $K(x,y) = \lim_{N\to\infty} K_N(x,y)$?
- How do the ODE and its solution behave, if one also scales the parameter s?