Random Integer Partitions and the Bose Gas

Mathias Rafler[1]
supervised by Sylvie Rœlly[1] and Hans Zessin[2]

[1]Universität Potsdam

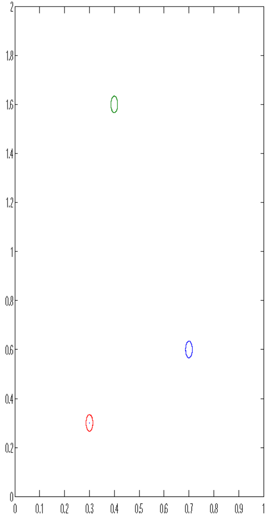
[2]Universität Bielefeld

Disentis Summer School 2008

Outline

Classical Mechanics

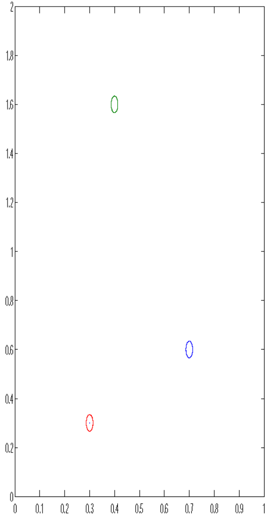
Free Particles in $\mathbb{R}^d$



- given region G, place N point in G without interaction
- given bigger region G', repeat with N' points
- is there a limit for $G \to \mathbb{R}^d$, $\frac{N}{|G|} \to u > 0$?

Classical Mechanics

Free Particles in $\mathbb{R}^d$

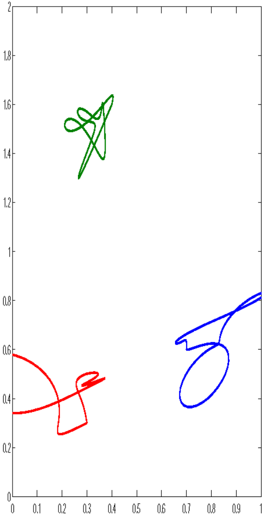


Answer: Yes, Poisson process on $\mathbb{R}^d$! (Nguyen, Zessin, 1976)

- #particles in $G \sim \mathbf{P}_{u|G|}$
- #particles in disjoint regions is independent

Quantum Mechanics

Differences and the Bose Gas

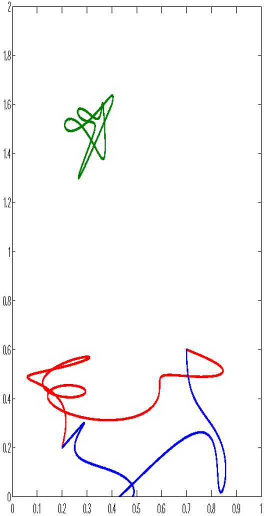


Basic Objects: Loops

- $x : [0, \beta] \rightarrow \mathbb{R}^d$, $x(0) = x(\beta)$
- Brownian bridge of length β

Quantum Mechanics

Differences and the Bose Gas



Indistinguishable particles

- loops may concatenate

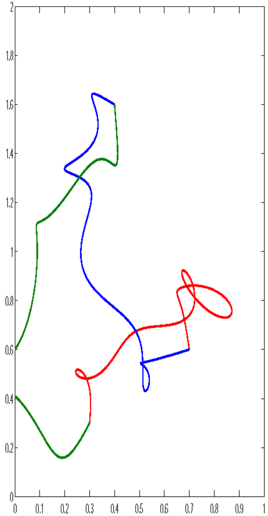
Composite loops

- $x : [0, j\beta] \to \mathbb{R}^d$, $x(0) = x(j\beta)$
- Brownian bridge of length $j\beta$

(Ginibre)

Quantum Mechanics

Differences and the Bose Gas



Indistinguishable particles

- loops may concatenate

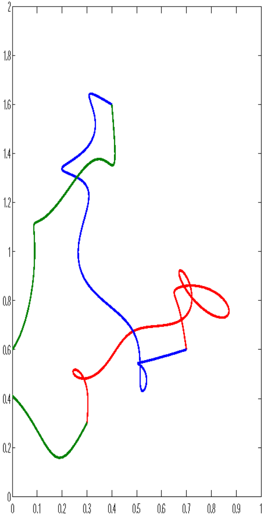
Composite loops

- $x : [0, j\beta] \to \mathbb{R}^d$, $x(0) = x(j\beta)$
- Brownian bridge of length $j\beta$

(Ginibre)

Quantum Mechanics

Differences and the Bose Gas



Ginibre Gas

- $\#j$-loops in $G \sim \mathbf{P}_{cj^{-(1+d/2)}|G|}$
- $\#$loops of different lengths and in disjoint regions is independent

Consequence

- expected $\#$loops $c|G|\sum_{j\geq 1} j^{-(1+d/2)}$
- expected $\#$particles $c|G|\sum_{j\geq 1} j^{-d/2}$

Limit as $\frac{N}{|G|} \to u > 0$?

Quantum Mechanics

Differences and the Bose Gas

Theorem (Ginibre)

Partition function of a system of free particles obeying Bose Statistics at inverse temperature β and fugacity z

$$Z_G = \exp\left[\sum_{j\geq 1} \frac{z^j}{j} \int \alpha_G(\omega)\mathbb{P}_{xx}^{j\beta}(\mathrm{d}\omega)\mathrm{d}x\right]$$

[Ginibre J: Some Applications of functional Integration in Statistical Mechanics. Statist. Mech. and Quantum Field Theory, Les Houches Summer School Theoret. Phys, (Gordon and Breach), 1971, 327-427]

Partition Problem

Extract from Bose Gas

Suppose for $\alpha > 1$ fixed, $j = 1, 2, \ldots,$

$$X_j \sim \mathbf{P}_{tj^{-(1+\alpha)}}$$

independent. Put

$$X = (X_1, X_2, \ldots)$$
$$N(X) = \sum_{j \geq 1} jX_j.$$

Consider

$$\mathbb{P}_{t,N_t} = Law(X|N(X) = N_t).$$

Partition problem: How does $\frac{X}{N_t}$ behave as $t \to \infty$, $\frac{N_t}{t} \to u > 0$?

Partition Problem

Extract from Bose Gas

By construction,

$$\frac{1}{N_t}N(X) = \frac{1}{N_t}\sum_{j\geq 1} jX_j = 1 \quad \mathbb{P}_{t,N_t}\text{-a.s.}$$

Let

$$Y := \lim_{t\to\infty} \frac{X}{N_t},$$

by Fatou's lemma

$$N(Y) = \sum_{j\geq 1} jY_j \leq 1.$$

Is equality preserved in the Limit?

Partition Problem

Extract from Bose Gas

By construction,

$$\frac{1}{N_t}N(X) = \frac{1}{N_t}\sum_{j\geq 1} jX_j = 1 \quad \mathbb{P}_{t,N_t}\text{-a.s.}$$

Let

$$Y := \lim_{t\to\infty} \frac{X}{N_t},$$

by Fatou's lemma

$$N(Y) = \sum_{j\geq 1} jY_j \leq 1.$$

Sometimes.

Partition Problem

Extract from Bose Gas

By construction,

$$\frac{1}{N_t}N(X) = \frac{1}{N_t}\sum_{j\geq 1} jX_j = 1 \quad \mathbb{P}_{t,N_t}\text{-a.s.}$$

Let

$$Y := \lim_{t\to\infty}\frac{X}{N_t},$$

by Fatou's lemma

$$N(Y) = \sum_{j\geq 1} jY_j \leq 1.$$

But not always!

Partition Problem

Extract from Bose Gas

Theorem

Subject to given conditions

$$ujY_j = \begin{cases} \frac{z^j}{j^\alpha} & \text{if } u \leq u^* \\ \frac{1}{j^\alpha} & \text{if } u > u^* \end{cases}$$

where $z = z(u)$ *is the solution of*

$$u \wedge u^* = \sum_{j \geq 1} \frac{z^j}{j^\alpha}.$$

Partition Problem

Extract from Bose Gas

Theorem

Subject to given conditions

$$uN(Y) = \begin{cases} u & \text{if } u \le u^* \\ u^* & \text{if } u > u^* \end{cases}$$

where $z = z(u)$ is the solution of

$$u \wedge u^* = \sum_{j \ge 1} \frac{z^j}{j^\alpha}.$$

Partition Problem

Extract from Bose Gas

Theorem

Subject to given conditions

$$N(Y) = \begin{cases} 1 & \text{if } u \leq u^* \\ \frac{u^*}{u} < 1 & \text{if } u > u^* \end{cases}$$

where $z = z(u)$ is the solution of

$$u \wedge u^* = \sum_{j \geq 1} \frac{z^j}{j^\alpha}.$$

Partition Problem

Extract from Bose Gas

Theorem

Subject to given conditions

$$N(Y) = \begin{cases} 1 & \text{if } u \leq u^* \\ \frac{u^*}{u} < 1 & \text{if } u > u^* \end{cases}$$

Elements of proof

- show LDP for $\frac{X}{N_t}$ to get convergence and to obtain variational problem;
- problems: poor continuity properties, small sets, minimisation problem with constraints

Quantum Mechanics

Interpretation

Competition: density vs. Brownian bridges

- *low density.* interparticle distance too large to build large composite loops
- *high density.* interparticle distance small; even possibility to build infinitely long loops (Bose-Einstein Condensation)

- Ginibre J: Some Applications of functional Integration in Statistical Mechanics. Statist. Mech. and Quantum Field Theory, Les Houches Summer School Theoret. Phys, (Gordon and Breach), 1971, 327-427
- Nguyen X X, Zessin H: Martin-Dynkin boundary of mixed Poisson processes, Z. Wahrscheinlichkeitstheorie verw. Gebiete 37 (1976/77), no. 3, 191–200
- Kingman, J F C: The representation of partition structures, J. London Math. Soc. (2) 18 (1978), no. 2, 374–380
- Föllmer H: Phase transition and Martin Boundary, Seminaire de probabilites (Strasbourg) 9 (1975), 305–17
- R: Martin-Dynkin boundaries of the Bose Gas, Preprint