Dynamic Risk Measures and Conditional Robust Utility Representation

How can we Understand Risk in a Dynamic Setting?

Samuel Drapeau

IRTG — Disentis Summer School 2008

Juli 22th 2007

Outline

1 Dynamic Risk Measures: Disappointment

2 Preference Orders

3 Conditional Preference Orders

4 Dynamic of Preferences

Outline

1 Dynamic Risk Measures: Disappointment

2 Preference Orders

3 Conditional Preference Orders

4 Dynamic of Preferences

Dynamic Risk Measures: Disappointment

Definition - Static case

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition (Convex Risk Measure — Artzner & al, Föllmer & Schied)

A functional $\rho : \mathbb{L}^\infty :\longrightarrow \mathbb{R}$ is a convex risk measure if it is:

- **Monotone:** For $X, Y \in \mathbb{L}^\infty$, $X \geq Y$ then $\rho(X) \leq \rho(Y)$
- **Translation invariant:** For $X \in \mathbb{L}^\infty$ and $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$
- **Convex:** For $X, Y \in \mathbb{L}^\infty$ and $\lambda \in [0, 1]$:

$$\rho(\lambda X + (1-\lambda) Y) \leq \lambda \rho(X) + (1-\lambda) \rho(Y)$$

- **Normalized:** $\rho(0) = 0$

Dynamic Risk Measures: Disappointment

Definition - Conditional case

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathscr{F}_t$ a sub-σ-algebra of $\mathscr{F}$.

Definition (Conditional Convex Risk Measure)

A functional $\rho_t : \mathbb{L}^\infty \to \mathbb{L}_t^\infty$ is a conditional convex risk measure if it is:

- **Monotone:** For $X, Y \in \mathbb{L}^\infty$, $X \geq Y$ then $\rho_t(X) \leq \rho_t(Y) \quad P$-a.s.
- **Conditionally translation invariant:** For $X \in \mathbb{L}^\infty$ and $m_t \in \mathbb{L}_t^\infty$, $\rho_t(X + m_t) = \rho_t(X) - m_t \quad P$-a.s.
- **Conditionally convexe:** For $X, Y \in \mathbb{L}^\infty$ and $0 \leq \lambda_t \leq 1$ $\mathscr{F}_t$-measurable:

$$\rho_t(\lambda_t X + (1 - \lambda_t) Y) \leq \lambda_t \rho_t(X) + (1 - \lambda_t) \rho_t(Y) \quad P\text{-a.s.}$$

- **Normalized:** $\rho_t(0) = 0 \quad P$-a.s.

Dynamic Risk Measures: Disappointment

Dual representation

An important result concerning convex risk measures is the dual representation (Static case: FÖLLMER and SCHIED. Conditional case: DETLEFSEN and SCANDOLO).

Theorem

If a conditional convex risk measure is continuous from below (i.e. $X_n \searrow X$ implies $\rho_t(X_n) \nearrow \rho_t(X)$) the following representation holds:

$$\rho_t(X) = \operatorname*{ess\,sup}_{\substack{Q \sim P \\ Q = P \text{ over } \mathscr{F}_t}} \left\{ E_Q\left[-X \middle| \mathscr{F}_t \right] - \alpha_t(Q) \right\}$$

where $\alpha_t : \mathcal{M}_1(\Omega, \mathscr{F}, \mathbb{P}) \rightarrow \mathbb{L}^{\infty}_{+}(\Omega, \mathscr{F}_t, P) \cup \infty$ is a penalty function.

Dynamic Risk Measures: Disappointment

Time consistency

Considering a family of conditional risk measures $(\rho_t)_{t\in[0,T]}$ on a filtrated probability space, the property of time consistency is understood as follow:

Definition

The family of conditional convex risk measures, is said to be time consistent if for all $X, Y \in \mathbb{L}^\infty$ and times $0 \leq t \leq s \leq T$, holds:

$$\rho_s(X) \geq \rho_s(Y) \quad P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X) \geq \rho_t(Y) \quad P\text{-a.s.}$$

This definition is equivalent to the following dynamic programing principle:

$$\rho_t(X) = \rho_t(-\rho_s(X))$$

Dynamic Risk Measures: Disappointment

Disappointment

Why are we so disappointed?
The time consistency together with cash invariance impose some very strong conditions in the continuous case such that infinitely many of them lead to some entropic-"like" risk measures, i.e. $\rho_t(X) = 1/\gamma \ln\left(E\left[e^{-\gamma X}\middle|\mathscr{F}_t\right]\right)$.

Dynamic Risk Measures: Disappointment

Disappointment

Why are we so disappointed?
The time consistency together with cash invariance impose some very strong conditions in the continuous case such that infinitely many of them lead to some entropic-"like" risk measures, i.e. $\rho_t(X) = 1/\gamma \ln\left(E\left[e^{-\gamma X}\middle|\mathscr{F}_t\right]\right)$.

Dynamic Risk Measures: Disappointment

Disappointment

Why are we so disappointed?
The time consistency together with cash invariance impose some very strong conditions in the continuous case such that infinitely many of them lead to some entropic-"like" risk measures, i.e. $\rho_t(X) = 1/\gamma \ln\left(E\left[e^{-\gamma X}\middle|\mathscr{F}_t\right]\right)$.

For a subdivision σ_n of the interval $[0, T]$, take as penalty function $\alpha_t(Q) = E\left[\varphi\left(\frac{Z}{Z_t}\right)\middle|\mathscr{F}_t\right]$ for a positive convex function φ twice differentiable in a neighborhood of 1 and with $\inf \varphi(x) = \varphi(1) = 0$. The filtration is generated by a Brownian motion.
If we imposed for the corresponding discrete family of risk measures $\rho_{t_i}^{\sigma_n}$ to be time consistent we have:

Theorem

$$\rho_t^{\sigma_n}(X) \xrightarrow[|\sigma_n|\to 0]{dP\otimes dt} \frac{1}{\gamma} \ln\left(E\left[e^{-\gamma X}\middle|\mathscr{F}_t\right]\right) \tag{2.1}$$

where $\gamma = 2/\varphi''(1)$

Dynamic Risk Measures: Disappointment

Disappointment

Moreover, KUPPER and SCHACHERMAYER proved in the restrictive framework of law invariance a general result:

Theorem

For an infinite family ρ_n of law invariant risk measures on an atom free filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$. If the family is time consistent, there exists then $\gamma \in \mathbb{R}^+ \cup \infty$ such that:

$$\rho_n(X) = \frac{1}{\gamma} \ln \left(E\left[e^{-\gamma X} \middle| \mathscr{F}_n \right] \right)$$

Outline

1 Dynamic Risk Measures: Disappointment

2 Preference Orders

3 Conditional Preference Orders

4 Dynamic of Preferences

Preference Orders

von Neumann J. & Morgenstern O. (1944)[7]

The preference order is defined by a binary relation $\succeq$ on the set of measures with bounded support $\mathcal{M}_b(S, \mathscr{S}) \equiv \mathcal{M}$.

Preference Orders

VON NEUMANN J. & MORGENSTERN O. (1944)[7]

The preference order is defined by a binary relation $\succeq$ on the set of measures with bounded support $\mathcal{M}_b(S, \mathscr{S}) \equiv \mathcal{M}$.

Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Independance:** For any $\mu \succ \nu$ holds:

$$\alpha\mu + (1-\alpha)\lambda \succ \alpha\nu + (1-\alpha)\lambda$$

  for any $\lambda \in \mathcal{M}$ and $\alpha \in]0,1]$.
- **Continuity:** The restriction of $\succeq$ to $\mathscr{M}(B(0,r))$ is continuous w.r.t. the weak topology for any $r > 0$.

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ such that:

$$\mu \succeq \nu \Leftrightarrow U(\mu) \geq U(\nu)$$

where:

$$U(\mu) = \int u(x)\,\mu(dx)$$

Preference Orders

VON NEUMANN J. & MORGENSTERN O. (1944)[7]

The preference order is defined by a binary relation $\succeq$ on the set of measures with bounded support $\mathcal{M}_b(S, \mathscr{S}) \equiv \mathcal{M}$.

Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Independance:** For any $\mu \succ \nu$ holds:

$$\alpha\mu + (1-\alpha)\lambda \succ \alpha\nu + (1-\alpha)\lambda$$

  for any $\lambda \in \mathcal{M}$ and $\alpha \in]0,1]$.
- **Continuity:** The restriction of $\succeq$ to $\mathscr{M}(B(0,r))$ is continuous w.r.t. the weak topology for any $r > 0$.

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ such that:

$$\mu \succeq \nu \Leftrightarrow U(\mu) \geq U(\nu)$$

where:

$$U(\mu) = \int u(x)\,\mu(dx)$$

Preference Orders

SAVAGE L. (1954)

Instead of a preference order on measures he considered it on the set of bounded measurable functions $\mathcal{X}$ defined on a **measurable** space $(\Omega, \mathscr{F})$.

Preference Orders

Savage L. (1954)

Instead of a preference order on measures he considered it on the set of bounded measurable functions $\mathcal{X}$ defined on a **measurable** space $(\Omega, \mathscr{F})$.

Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Independance:** For any $X \succ Y$ holds:

$$\alpha X + (1-\alpha) Z \succ \alpha Y + (1-\alpha) Z$$

for any $Y \in \mathcal{X}$ and $\alpha \in]0,1]$.

- **+ several other technical axioms (archimedian, monotonicity, ...)**

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ and a probability measure $Q \in \mathcal{M}_1(\Omega, \mathscr{F})$ such that:

$$X \succeq Y \Leftrightarrow U(X) \geq U(Y)$$

where:

$$U(X) = E_Q[u(X)]$$

Preference Orders

Savage L. (1954)

Instead of a preference order on measures he considered it on the set of bounded measurable functions $\mathcal{X}$ defined on a **measurable** space $(\Omega, \mathscr{F})$.

Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Independance:** For any $X \succ Y$ holds:

$$\alpha X + (1-\alpha) Z \succ \alpha Y + (1-\alpha) Z$$

  for any $Y \in \mathcal{X}$ and $\alpha \in]0,1]$.
- **+ several other technical axioms (archimedian, monotonicity, ...)**

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ and a probability measure $Q \in \mathcal{M}_1(\Omega, \mathscr{F})$ such that:

$$X \succeq Y \Leftrightarrow U(X) \geq U(Y)$$

where:

$$U(X) = E_Q[u(X)]$$

Preference Orders

Robust version: Gilboa & Schmeidler (89)[3], Maccheroni. . . (04)[5], Föllmer. . . (07)[1][2]¡+

To overcome Elsberg's paradox, the independence axiom will be weakened.

Preference Orders

Robust version: GILBOA & SCHMEIDLER (89)[3], MACCHERONI... (04)[5], FÖLLMER... (07)[1][2]¡+

To overcome Elsberg's paradox, the independence axiom will be weakened. The preference order are now defined on the space $\tilde{\mathcal{X}}$ of uniformly bounded stochastic kernels on the real line $\tilde{X}(\omega, dx)$ in which $\mathcal{X}$ and $\mathcal{M}_b(\mathbb{R})$ are embedded.

Preference Orders

Robust version: GILBOA & SCHMEIDLER (89)[3], MACCHERONI... (04)[5], FÖLLMER... (07)[1][2]¡+

Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Weak Certainty Independance:**

$$\alpha\tilde{X} + (1-\alpha)\mu \succ \alpha\tilde{Y} + (1-\alpha)\mu$$
$$\Downarrow$$
$$\alpha\tilde{X} + (1-\alpha)\nu \succ \alpha\tilde{Y} + (1-\alpha)\nu$$

for any $\nu \in \mathcal{M}_b(\mathbb{R})$.

- **Uncertainty Aversion:** For $\tilde{X} \sim \tilde{Y}$ and $\alpha \in [0,1]$ holds: $\alpha\tilde{X} + (1-\alpha)\tilde{Y} \succeq \tilde{X}$
- **+ technical axioms (archimedian, monotonicity, continuity from above)**

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ and a penalty function $\alpha : \mathcal{M}_1(\Omega, \mathscr{F}) \mapsto \mathbb{R} \cup \infty$ such that:

$$X \succeq Y \Leftrightarrow U(X) \geq U(Y)$$

where:

$$U(X) = \inf_{Q \in \mathcal{M}_1(\Omega,\mathscr{F})} \{E_Q[u(X)] + \alpha(Q)\}$$

In particular:

$$U(X) = -\rho^{conv}(u(X))$$

Preference Orders

Robust version: Gilboa & Schmeidler (89)[3], Maccheroni... (04)[5], Föllmer... (07)[1][2]¡+

Preference Axioms

- **Weak Preference Order:** $\succeq$ is reflexive, transitive and complete.
- **Weak Certainty Independance:**

$$\begin{array}{ccc} \alpha\tilde{X}+(1-\alpha)\mu & \succ & \alpha\tilde{Y}+(1-\alpha)\mu \\ & \Downarrow & \\ \alpha\tilde{X}+(1-\alpha)\nu & \succ & \alpha\tilde{Y}+(1-\alpha)\nu \end{array}$$

  for any $\nu \in \mathcal{M}_b(\mathbb{R})$.
- **Uncertainty Aversion:** For $\tilde{X} \sim \tilde{Y}$ and $\alpha \in [0,1]$ holds: $\alpha\tilde{X}+(1-\alpha)\tilde{Y} \succeq \tilde{X}$
- **+ technical axioms (archimedian, monotonicity, continuity from above)**

Numerical Representation

There exist a continuous function $u : \mathbb{R} \mapsto \mathbb{R}$ and a penalty function $\alpha : \mathcal{M}_1(\Omega, \mathscr{F}) \mapsto \mathbb{R} \cup \infty$ such that:

$$X \succeq Y \Leftrightarrow U(X) \geq U(Y)$$

where:

$$U(X) = \inf_{Q \in \mathcal{M}_1(\Omega,\mathscr{F})} \{E_Q[u(X)] + \alpha(Q)\}$$

In particular:

$$U(X) = -\rho^{conv}(u(X))$$

Outline

1 Dynamic Risk Measures: Disappointment

2 Preference Orders

3 Conditional Preference Orders

4 Dynamic of Preferences

Conditional Preference Orders

What is a Conditional Preference Order.

The question of a conditional preference order has already emerged in the literature (KREPS & PORTEUS [4], SKIADAS [6], MACHERONI & AL.) but their axiomatic is highly disputable, and is strongly related to their basic setting (Trees).

Conditional Preference Orders

What is a Conditional Preference Order.

The question of a conditional preference order has already emerged in the literature (KREPS & PORTEUS [4], SKIADAS [6], MACHERONI & AL.) but their axiomatic is highly disputable, and is strongly related to their basic setting (Trees).

The key question to address is the completeness, and they are beyond the conditional concept in stochastic many reasons for doubting of this assumption: Indeed, Incompleteness does not reflects an unexceptional trait as pointed out by AUMANN R.J.:

> *Of all the axiom of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life, but unlike them we find it hard to accept even from a normative viewpoint. [...] For example, certain decisions that an individual is asked to make might involve highly hypothetical situations, which he will never face in real life. He might feel that he cannot reach an "honest" decision in such cases. Other decision problems might be extremely complex, too complex for intuitive "insight", and our individual might prefer to make no decision at all in these problems. Is it "rational" to force decision in such cases?*

Conditional Preference Orders

Axiomatic

Axiomatic

- **Partial Weak Order:** $\succeq^{\mathscr{G}}$ is P-a.s. reflexive and transitive.
- **$\mathscr{G}$-consistency:** For all $\tilde{X}, \tilde{Y}$ and family $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{G}$ holds:
  - Intersection consistency: $\exists n \in \mathbb{N}\,,\ \tilde{X} \succ^{\mathscr{G}}_{A_n} \tilde{Y} \quad \Longrightarrow \quad \tilde{X} \succ^{\mathscr{G}}_{\{\bigcap_{n\in\mathbb{N}} A_n\}} \tilde{Y}$
  - Union consistency: $\forall n \in \mathbb{N}\,,\ \tilde{X} \succ^{\mathscr{G}}_{A_n} \tilde{Y} \quad \Longrightarrow \quad \tilde{X} \succ^{\mathscr{G}}_{\{\bigcup_{n\in\mathbb{N}} A_n\}} \tilde{Y}$
  - Least comparison: There exists $A \in \mathscr{G}$ with $P[A] > 0$ such that: $\tilde{X} \succeq^{\mathscr{G}}_{A} \tilde{Y}$ or $\tilde{X} \preceq^{\mathscr{G}}_{A} \tilde{Y}$
- **$\mathscr{G}$-Uncertainty Aversion:** For $\tilde{X} \sim^{\mathscr{G}} \tilde{Y}$ holds $\alpha\tilde{X} + (1-\alpha)\tilde{Y} \succeq^{\mathscr{G}} \tilde{X}$ for all $\mathscr{G}$-measurable function α with $0 \leq \alpha \leq 1$
- **Monotonicity:** If $\tilde{Y}(\omega) \succeq \tilde{X}(\omega)$ P-a.s., then $\tilde{Y} \succeq^{\mathscr{G}} \tilde{X}$. Moreover, for reals x, y, $x < y$ iff $\delta_x \prec^{\mathscr{G}} \delta_y$
- **Weak Certainty Independence**: For $\tilde{X}, \tilde{Y} \in \tilde{X}$, $\tilde{Z}_i \equiv \mu_i \in \mathcal{M}_b(\mathbb{R}, \mathscr{G})$ for $i = 1, 2$ and a $\mathscr{G}$-measurable function α such that $0 < \alpha \leq 1$ we have:

$$\alpha\tilde{X} + (1-\alpha)\tilde{Z}_1 \succ^{\mathscr{G}} \alpha\tilde{Y} + (1-\alpha)\tilde{Z}_1 \quad \Longrightarrow \quad \alpha\tilde{X} + (1-\alpha)\tilde{Z}_2 \succ^{\mathscr{G}} \alpha\tilde{Y} + (1-\alpha)\tilde{Z}_2$$

- **Continuity**: If $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}$ are such that $\tilde{Z} \succ^{\mathscr{G}} \tilde{Y} \succ^{\mathscr{G}} \tilde{X}$, there exists then $\mathscr{G}$-measurable functions α, β with $0 < \alpha, \beta < 1$ such that:

$$\alpha\tilde{Z} + (1-\alpha)\tilde{X} \quad \succ^{\mathscr{G}} \quad \tilde{Y} \quad \succ^{\mathscr{G}} \quad \beta\tilde{Z} + (1-\beta)\tilde{X}$$

Moreover for all $c > 0$, the restriction of $\succeq^{\mathscr{G}}$ to $\mathcal{M}_1([-c, c], \mathscr{G})$ is continuous with respect

Conditional Preference Orders

Conditional von Neumann & Morgenstern

Even if we loose completeness, we can manage to deal with in a good way:

Lemma

Suppose given a weak partial preference order satisfying the first and second axiom aforementioned, then for each $\tilde{X}, \tilde{Y} \in \tilde{\mathscr{X}}$ there exists a partition $A, B, C \in \mathscr{G}$ of Ω such that:

$$\begin{cases} \tilde{X} \succ_A^{\mathscr{G}} \tilde{Y} \\ \tilde{X} \prec_B^{\mathscr{G}} \tilde{Y} \\ \tilde{X} \sim_C^{\mathscr{G}} \tilde{Y} \end{cases}$$

Conditional Preference Orders

Conditional VON NEUMANN & MORGENSTERN

Considering the restriction of $\succeq^{\mathscr{G}}$ on $\mathcal{M}_b(\mathbb{R}, \mathscr{G})$ we get a conditional version of the theorem of VON NEUMANN J. & MORGENSTERN O.:

Theorem

If $\succ^{\mathscr{G}}$ verify the first, second, fifth and sixth axiom aforementioned, there exists then a conditional VON NEUMANN and MORGENSTERN representation of $\succeq^{\mathscr{G}}$:

$$\forall \mu \in \mathcal{M}_b(\mathbb{R}, \mathscr{G}) \text{ , for } P\text{-almost all } \omega \in \Omega \text{ , } U(\mu, \omega) = \int u(x, \omega)\, \mu(dx, \omega) \tag{4.1}$$

where $U(\mu, \cdot)$ is a $\mathscr{G}$-measurable random variable, for all $\omega \in \Omega$, $u(\cdot, \omega)$ is continuous and for all $x \in \mathbb{R}$ $u(x, \cdot)$ is $\mathscr{G}$-measurable.

Conditional Preference Orders

Conditional Robust Representation

Theorem

If the preference order $\succeq^{\mathscr{G}}$ fulfills all the axioms aforementioned, there exists then a conditional numerical representation $\tilde{U}$ which restriction on $\mathcal{M}_b(\mathcal{R},\mathscr{G})$ is a conditional von Morgenstern *and* Neumann *representation.*
If moreover the range of u is P-a.s. equal to $\mathbb{R}$ and the induced preference order $\succeq^{\mathscr{G}}$ on $\mathcal{X}$, viewed as a subset of $\tilde{\mathcal{X}}$ satisfies the following additional continuity property:

$$X \succ^{\mathscr{G}} Y \text{ and } X_n \nearrow X \quad P\text{-a.s.} \qquad \Longrightarrow \qquad X_n \succ^{\mathscr{G}} Y \text{ for all large } n \tag{4.2}$$

There exists then a penalty function $\alpha^{\mathscr{G}}_{min} : \mathcal{M}_1(\Omega,\mathscr{F}) \to \mathbb{L}^{\infty}(\Omega,\mathscr{G},P) \cup \{+\infty\}$ such that we get for the induced preference relation a generalised robust Savage representation on $\mathcal{X}$:

$$U(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}_1(\Omega,\mathscr{F},\equiv P \text{ on } \mathscr{F}_t)} \left\{ E_Q\left[u(X)\,\middle|\,\mathscr{G}\right] + \alpha^{\mathscr{G}}_{min}(Q)\right\} \tag{4.3}$$

Outline

1 Dynamic Risk Measures: Disappointment

2 Preference Orders

3 Conditional Preference Orders

4 Dynamic of Preferences

Conditional Preference Orders

Conditional Robust Representation

We consider here some processes $(X_t)_{t=0,1\dots T}$.

- **Temporal Consistency:** If $X \succeq^{t+1} Y$ and $X = Y$ up to time t, then $X \succeq^t Y$.
  This should delivers the time consistency of the risk measure ρ_t and a recursive definition of the utility function.
- **Information Preference:** For an increasing function $f : \mathbb{N} \mapsto \mathbb{N}$ with $f(s) = s$ for $s \leq t$ and $f(s) \geq s$ for $s > t$, than for any adapted process Y equal to X up to time t and with $\mathcal{L}aw\left(Y \middle| \mathscr{F}_t\right) \sim \mathcal{L}aw\left(X_{f(\cdot)} \middle| \mathscr{F}_t\right)$ we should have $X \succeq^t Y$.

Bibliography

Hans Föllmer and Alexander Schied.
Stochastic Finance. An Introduction in Discrete Time.
de Gruyter Studies in Mathematik. Walter de Gruyter, Berlin, New York, 2 edition, 2004.

Hans Föllmer, Alexander Schied, and Stefan Weber.
Robust Preferences and Robust Portfolio Choice.
Preprint, 2007.

I. Gilboa and D. Schmeidler.
Maximin Expected Utility with a Non-Unique Prior.
Journal of Mathematical Economics, 18:141–153, 1989.

David M Kreps and Evan L Porteus.
Temporal Resolution of Uncertainty and Dynamic Choice Theory.
Econometrica, 46(1):185–200, January 1978.
available at http://ideas.repec.org/a/ecm/emetrp/v46y1978i1p185-200.html.

Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini.
Ambiguity Aversion, Robustness, and the Variational Representation of Preferences.
Econometrica, 74(6):1447–1498, November 2006.

Costis Skiadas.
Conditioning and aggregation of preferences.
Econometrica, 65(2):347–368, March 1997.

John von Neumann and Oskar Morgenstern.
Theory of Games and Economics Behavior.
Princeton University Press, 2nd edition, 1947.