Existence of solutions and iterative approximations for nonlinear systems arising in free convection

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Abstract

We study the existence and the regularity of the solutions of some nonlinear partial differential system arising in the study of free convection in a two-dimensional bounded domain, modelizing a porous medium saturated with a fluid. By introducing an iterative method, the closeness of such solutions by solution of linear elliptic problems is given with an exponential rate of convergence.

Keywords. Nonlinear system, elliptic problem, iterative method, Schauder fixed point Theorem, free convection.

1 Introduction

The governing steady problem for the free convection in a two-dimensional bounded domain Ω filled with a fluid saturated porous medium, is given by

$$\partial_x^2 \Psi + \partial_y^2 \Psi = k \partial_y T \tag{1}$$

$$\lambda \left(\partial_x^2 T + \partial_y^2 T \right) = \partial_x T \partial_y \Psi - \partial_y T \partial_x \Psi \tag{2}$$

coupled with some mixed boundary conditions, where (x, y) is the rectangular Cartesian coordinates system. The constants k and λ depend on the density, the viscosity and the thermal expansion coefficient of the fluid, and on the permeability and the thermal diffusivity of the saturated porous medium. This model, written in terms of the stream function Ψ and the temperature T, consists in two strongly coupled partial differential equations (see [1]). More details about the physical background can be found in [5, 6, 7, 8, 11, 13].

The first approach to study this problem allows to introduce similarity variables to reduce the whole system of partial differential equations into one single ordinary differential equation of the third order with appropriate boundary values. This two points boundary value problem can be studied using a shooting method or an auxiliary dynamical system either in the case of prescribed temperature or in the case of prescribed heat flux along a part of the boundary of the domain. For mathematical results about this boundary value problem and the connections with (1)-(2), we refer the reader to [2, 4] and the references therein.

The second natural way, which is the framework of this paper, directly deals with the coupled partial differential equations. Some existence and uniqueness results are given in [1]. The existence result has been proved under very constraining hypothesis and is not satisfactory. In a short note [3], existence of a weak solution has been obtained, under the assumptions used in [1] to get uniqueness. The method consists in defining a suitable contraction (see Remark 4.1 below).

On the one hand, the aim of this paper is to show the existence of weak solutions in a more general case, where we do not consider any smallness hypothesis on the data, as in [1] (and in [3] to a lesser extent). On the other hand, the iterative method that we will introduce allows us to approach such solutions by solutions of linear elliptic problems with an exponential rate of convergence.

The paper is organized as follows: In the second section we give a meaning of the solutions of the problem (1)-(2) in more general case, using a variational formulation. The third section is devoted to the existence result, and in the fourth one, some iterative method is presented to show that we can find a sequence of solutions of linear elliptic problem converging to the solution of this nonlinear partial differential system with an exponential rate of convergence. In the last section we use the regularity results of the linear elliptic problem to study those of our problem.

Let us introduce now the problem we are interesting in. Let Ω be a bounded domain of $\mathbb{R}^2$ with sufficiently smooth boundary Γ, Γ_1 and Γ_2 be two parts of Γ, such that $meas(\Gamma_1) \neq 0$ and

$$\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma, \qquad \Gamma_1 \cap \Gamma_2 = \varnothing. \tag{3}$$

In Ω we consider the boundary value systems defined by

$$\begin{aligned} -\Delta\Psi + K.\nabla H &= F \qquad (4)\\ -\lambda\Delta H + \nabla H.\left(\nabla\Psi\right)^{\perp} + \nabla\Theta.\left(\nabla\Psi\right)^{\perp} &= 0, \qquad (5) \end{aligned}$$

with mixed boundary conditions for Ψ

$$\Psi = 0 \ \text{ on } \Gamma_1 \quad \text{ and } \quad \frac{\partial\Psi}{\partial\nu} = 0 \ \text{ on } \ \Gamma_2, \tag{6}$$

and for H

$$H = 0 \ \text{ on } \Gamma, \tag{7}$$

where $\overrightarrow{\nu}$ is the unit outward normal vector on Γ and $\left(\nabla\Psi\right)^{\perp} = \left(\partial_y\Psi, -\partial_x\Psi\right)$. We suppose that

$$F \in L^2(\Omega) \tag{8}$$

and that the function Θ is the unique solution in $H^2(\Omega)$ of the boundary problem

$$\Delta\Theta = 0 \text{ in } \Omega, \qquad \Theta = h \text{ on } \Gamma, \tag{9}$$

for $h \in H^{\frac{3}{2}}(\Gamma)$. For the coefficients $K = (k_1, k_2)$, it is assumed that

$$K \in L^\infty(\Omega)^2 \qquad \text{and} \qquad \text{div}K \in L^\infty(\Omega). \tag{10}$$

Remark 1.1 *By setting $T = H + \Theta$, $k_1 = 0$ and $F = K.\nabla\Theta$, we recover problem* (1)-(2).

In the following, we will denote by $(\cdot,\cdot)$ the $L^2(\Omega)$-scalar product, and by $\|\cdot\|$ (resp. $|\cdot|_2$, $|\cdot|_\infty$) the norm of $H^1(\Omega)$ (resp. $L^2(\Omega)$, $L^\infty(\Omega)$).

2 Weak formulation

In order to define a variational formulation of the previous problem, let us assume that Ψ and H are classical solutions of (4) and (5) in Ω, such that the boundary conditions (6) and (7) hold. Multiplying (4) and (5) by test functions $u \in H_0^1(\Omega, \Gamma_1)$ and $v \in H_0^1(\Omega)$ respectively, and integrating on Ω, we get

$$\int_\Omega \nabla\Psi.\nabla u\, dx + \int_\Omega u\, K.\nabla H dx = \int_\Omega F u\, dx \tag{11}$$

and

$$\lambda \int_\Omega \nabla H.\nabla v\, dx + \int_\Omega v\, \nabla H. (\nabla\Psi)^\perp dx + \int_\Omega v\, \nabla\Theta. (\nabla\Psi)^\perp dx = 0. \tag{12}$$

(Note that $H_0^1(\Omega, \Gamma_1)$ denotes the subspace of $H^1(\Omega)$ of functions vanishing on Γ_1). If now, we only assume that $\Psi \in H_0^1(\Omega, \Gamma_1)$ and $H \in H_0^1(\Omega)$, the third integral in the latter equality is still well defined (this is due to the fact that $\Theta \in H^2(\Omega)$), whereas, a priori, it is not anymore the case for the second one.

Let us clarify this point. To this end, for $u, v, w \in H^1(\Omega)$ such that $u\nabla v.(\nabla w)^\perp \in L^1(\Omega)$, let us set

$$a(u, v, w) = \int_\Omega u\nabla v. (\nabla w)^\perp dx = \left(u\nabla v, (\nabla w)^\perp\right) \tag{13}$$

and let us show the following results.

Lemma 2.1 *Let $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ such that one of them vanishes on the boundary of Ω. For $w \in H^1(\Omega)$ we have*

$$a(u, v, w) = -a(v, u, w). \tag{14}$$

In particular, for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and every $w \in H^1(\Omega)$ we have : $a(u, u, w) = 0$.

PROOF. For $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ and $w \in H^1(\Omega)$ the quantities $a(u, v, w)$ and $a(v, u, w)$ are well defined. Since, moreover, $uv \in H_0^1(\Omega)$, we have :

$$\begin{aligned} a(u, v, w) + a(v, u, w) &= \left(u\nabla v + v\nabla u, (\nabla w)^\perp\right) = \left(\nabla(uv), (\nabla w)^\perp\right) \\ &= -\left(\operatorname{div}\left((\nabla w)^\perp\right), uv\right)_{H^{-1}(\Omega), H_0^1(\Omega)} = 0 \end{aligned}$$

because $\operatorname{div}\left((\nabla w)^\perp\right) = 0$. □

Lemma 2.2 *Let $u, v \in H_0^1(\Omega)$. For $w \in H^2(\Omega)$ we have*

$$a(u, v, w) = -a(v, u, w). \tag{15}$$

In particular, for every $u \in H_0^1(\Omega)$ and every $w \in H^2(\Omega)$ we have : $a(u, u, w) = 0$.

Proof. First, because $H^1(\Omega) \hookrightarrow L^4(\Omega)$, the quantities $a(u, v, w)$ and $a(v, u, w)$ are well defined for all $u, v \in H_0^1(\Omega)$ and $w \in H^2(\Omega)$. On the other hand, by Lemma 2.1, for all $\varphi, \psi \in \mathcal{D}(\Omega)$ and all $w \in H^2(\Omega)$, we have : $a(\varphi, \psi, w) = -a(\psi, \varphi, w)$; the conclusion then follows from the density of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. □

Taking into account Lemma 2.1, we can replace the second integral in (12) by

$$-\int_\Omega H\nabla v.\,(\nabla\Psi)^\perp\,dx$$

which is well defined, if $H \in L^\infty(\Omega)$. Having that in mind, we will say that a couple (Ψ, H) such that $\Psi \in H_0^1(\Omega, \Gamma_1)$ and $H \in L^\infty(\Omega) \cap H_0^1(\Omega)$ is a weak solution of the problem (4)-(7) if the integral identities

$$\begin{aligned}(\nabla\Psi, \nabla u) + (K.\nabla H, u) &= (F, u) &(16)\\ \lambda(\nabla H, \nabla v) - a(H, v, \Psi) + a(v, \Theta, \Psi) &= 0 &(17)\end{aligned}$$

hold for any $u \in H_0^1(\Omega, \Gamma_1)$ and for any $v \in H_0^1(\Omega)$.

3 Existence of weak solutions

The proof of the existence theorem is given in two steps. In the first one we use Schauder fixed point theorem to study a slightly modified partial differential system. Next some asymptotic technique is employed to show the existence of the solution of our system.

3.1 Fixed point theorem for a smoothing system

Let ρ be a smooth function with compact support, for instance we assume that $\rho \in \mathcal{D}(\mathbb{R}^2)$, $E : L^2(\Omega) \to L^2(\mathbb{R}^2)$ be a continuous extension operator, such that the restriction $E : H^1(\Omega) \to H^1(\mathbb{R}^2)$ is also continuous (according to the H^1-norm, see for instance Theorem 1 p. 254 in [9] and [12]). Let us consider the following system

$$\left\{\begin{array}{lr}(\nabla\Psi, \nabla u) = -(K.\nabla H, u) + (F, u) & \forall u \in H_0^1(\Omega, \Gamma_1)\\ \lambda(\nabla H, \nabla v) - a(H, v, \rho * E\Psi) = -a(v, \Theta, \rho * E\Psi) & \forall v \in H_0^1(\Omega).\end{array}\right. \tag{18}$$

where $*$ denotes the convolution product. Our goal in this section is to prove, by using a fixed point argument for a suitable map, that (18) has a solution $(\Psi, H) = (\Psi_\rho, H_\rho)$.

The definition of the mapping Φ Let $\Psi \in L^2(\Omega)$ and consider the following linear problem

$$\lambda (\nabla H, \nabla v) - a(H, v, \rho * E\Psi) = -a(v, \Theta, \rho * E\Psi), \qquad \forall v \in H_0^1(\Omega). \tag{19}$$

Let us show that this problem has a unique solution $H \in H_0^1(\Omega)$. The bilinear form $a(\cdot, \cdot, \rho * E\Psi)$ is well defined and is continuous in $H_0^1(\Omega)$, since we have

$$|a(H, v, \rho * E\Psi)| \leq |\nabla(\rho * E\Psi)|_\infty \|H\| \|v\|.$$

The bilinear form $\lambda(\nabla\cdot, \nabla\cdot) - a(\cdot, \cdot, \rho * E\Psi)$ is coercive, since by using Lemma 2.2, we have $a(v, v, \rho * E\Psi) = 0$ for any $v \in H_0^1(\Omega)$ and hence

$$\lambda(\nabla v, \nabla v) - a(v, v, \rho * E\Psi) = \lambda|\nabla v|_2^2, \qquad \forall v \in H_0^1(\Omega).$$

In the right hand side of (19), the linear form $v \mapsto -a(v, \Theta, \rho * E\Psi)$ is continuous, since $\nabla\Theta.\nabla(\rho * E\Psi)^\perp \in L^2(\Omega)$. Applying the Lax-Milgram Lemma, we obtain that there exists one and only one $H \in H_0^1(\Omega)$ satisfying (19). In this way, we have defined a map $\Phi_1 : L^2(\Omega) \to H_0^1(\Omega)$ such that $\Phi_1(\Psi) = H$.

Next, let $\Psi' \in H_0^1(\Omega, \Gamma_1)$ be the solution of the problem

$$(\nabla\Psi', \nabla u) = -(K.\nabla H, u) + (F, u), \qquad \forall u \in H_0^1(\Omega, \Gamma_1). \tag{20}$$

Obviously, according to the Lax-Milgram lemma, the previous problem admits a unique solution $\Psi' \in H_0^1(\Omega, \Gamma_1)$. Thus, we have defined a map $\Phi_2 : H_0^1(\Omega) \to L^2(\Omega)$ such that $\Phi_2(H) = \Psi'$.

Finally, we set $\Phi = \Phi_2 \circ \Phi_1$. The mapping Φ is defined from $L^2(\Omega) \to L^2(\Omega)$, and if Ψ is a fixed point of Φ, then $(\Psi, H) := (\Psi, \Phi_1(\Psi))$ is a solution of (18).

A priory estimates In order to study the continuity of Φ, we will need some lemmas.

Lemma 3.1 *Let* $\Psi \in H^2(\Omega)$. *If* $H \in H_0^1(\Omega)$ *satisfies*

$$\lambda(\nabla H, \nabla v) - a(H, v, \Psi) = -a(v, \Theta, \Psi), \qquad \forall v \in H_0^1(\Omega), \tag{21}$$

then

$$\inf_\Gamma h \leq H + \Theta \leq \sup_\Gamma h \tag{22}$$

and

$$\inf_\Gamma h - \sup_\Omega \Theta \leq H \leq \sup_\Gamma h - \inf_\Omega \Theta. \tag{23}$$

In particular, we have $H \in L^\infty(\Omega)$ *and* $|H|_\infty$ *is bounded independently of* Ψ.

PROOF. The ingredients of the proof are in [1, Proposition 3.2] ; for convenience, and because the hypotheses are slightly different, we write it here. Let us set $l = \sup_\Gamma h$ and $H^+ = \sup\{H + \Theta - l\,; 0\}$. Since $H^+ \in H_0^1(\Omega)$, then (21) and the fact that $\Delta\Theta = 0$ imply

$$\begin{aligned}
\lambda(\nabla H^+, \nabla H^+) &= \lambda(\nabla H, \nabla H^+) + \lambda(\nabla\Theta, \nabla H^+) = \lambda(\nabla H, \nabla H^+)\\
&= a(H, H^+, \Psi) - a(H^+, \Theta, \Psi)\\
&= a(H, H^+, \Psi) + a(\Theta, H^+, \Psi)\\
&= a(H + \Theta, H^+, \Psi) = a(H^+, H^+, \Psi) = 0.
\end{aligned}$$

It follows that $|\nabla H^+|_2 = 0$ and hence $H^+ = 0$. This gives the second inequality of (22). To obtain the other one, we set $l' = \inf_\Gamma h$ and $H^- = \inf\{H + \Theta - l'\,; 0\}$ and proceed in the same way. □

Lemma 3.2 *There exists a constant $M > 0$ independent of the choice of $\Psi \in L^2(\Omega)$, such that*

$$\|\Phi(\Psi)\| \le M. \tag{24}$$

Moreover, for every constant $C > 0$, there exists a constant $M' > 0$, such that for all $\Psi \in L^2(\Omega)$ satisfying

$$|\Psi|_2 \le C, \tag{25}$$

we have

$$\|H\| \le M'. \tag{26}$$

PROOF. We take $\Psi' = \Phi(\Psi)$ as a test function in (20), we obtain

$$|\nabla\Psi'|_2^2 = -(K.\nabla H, \Psi') + (F, \Psi'),$$

whence

$$|\nabla\Psi'|_2^2 = (H\mathrm{div}K, \Psi') - (\mathrm{div}\,(H\,K), \Psi') + (F, \Psi').$$

Applying Green formula and taking into account that H vanishes on the boundary Γ, we get

$$|\nabla\Psi'|_2^2 = (H\,\mathrm{div}K, \Psi') + (H\,K, \nabla\Psi') + (F, \Psi').$$

Using the Cauchy-Schwarz inequality and the assumptions (10), it comes

$$\begin{aligned}
|\nabla\Psi'|_2^2 &\le |\mathrm{div}K|_\infty |H|_2 |\Psi'|_2 + |K|_\infty |H|_2 |\nabla\Psi'|_2 + |F|_2 |\Psi'|_2,\\
&\le (|\mathrm{div}K|_\infty |H|_2 + |K|_\infty |H|_2 + |F|_2) \|\Psi'\|. \qquad (27)
\end{aligned}$$

By Poincaré's inequality and (23), we deduce the first estimate in Lemma 3.2. For the last estimate, we replace v by H in (19), it comes

$$\lambda|\nabla H|_2^2 + a(H, H, \rho * E\Psi) = -a(H, \Theta, \rho * E\Psi).$$

But according to Lemma 2.2, we have $a\left(H, H, \rho * E\Psi\right) = 0$, whence if we use the Cauchy-Schwarz inequality and (23), we get

$$\lambda \left|\nabla H\right|_2^2 \leq \left|H\right|_\infty \left|\nabla \Theta\right|_2 \left|\nabla \rho * E\Psi\right|_2 .$$

Thanks to Young's lemma and the continuity of E from $L^2(\Omega)$ into $L^2(\mathbb{R}^2)$, it follows that

$$\begin{aligned} \lambda \left|\nabla H\right|_2^2 &\leq meas\,(\Omega) \left|H\right|_\infty \left|\nabla \Theta\right|_2 \left|\nabla \rho\right|_{\mathbb{R}^2,2} \left|E\Psi\right|_{\mathbb{R}^2,2} \\ &\leq c \left|H\right|_\infty \left|\nabla \Theta\right|_2 \left|\nabla \rho\right|_{\mathbb{R}^2,2} \left|\Psi\right|_2 . \end{aligned} \tag{28}$$

where c is a constant dependent only on E ($\left|\cdot\right|_{\mathbb{R}^2,2}$ denotes the norm of $L^2(\mathbb{R}^2)$). The Poincaré inequality, (23) and (25) give the last estimate. □

The continuity of Φ Let Ψ_n be a converging sequence in $L^2(\Omega)$ i.e.

$$\Psi_n \to \Psi \ \text{ in } \ L^2(\Omega).$$

According to the definition of E, we also have

$$E\Psi_n \to E\Psi \ \text{ in } \ L^2(\mathbb{R}^2). \tag{29}$$

Let us denote by $\Psi_n' = \Phi\left(\Psi_n\right)$ and H_n the solution of (20) when we replace Ψ by Ψ_n i.e. we have

$$\left(\nabla \Psi_n', \nabla u\right) = -\left(K.\nabla H_n, u\right) + \left(F, u\right), \qquad \forall u \in H_0^1\left(\Omega, \Gamma_1\right), \tag{30}$$

and

$$\lambda\left(\nabla H_n, \nabla v\right) - a\left(H_n, v, \rho * E\Psi_n\right) = -a\left(v, \Theta, \rho * E\Psi_n\right), \qquad \forall v \in H_0^1(\Omega). \tag{31}$$

From (24), (26) and the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, there exist $\Psi' \in H_0^1\left(\Omega, \Gamma_1\right)$, $H \in H_0^1(\Omega)$ and subsequences $\Psi_{n'}'$, $H_{n'}$ such that :

$$\Psi_{n'}' \rightharpoonup \Psi' \quad \text{in } \ H_0^1\left(\Omega, \Gamma_1\right), \tag{32}$$

$$H_{n'} \rightharpoonup H \quad \text{in } \ H_0^1(\Omega), \tag{33}$$

$$H_{n'} \to H \quad \text{in } \ L^2(\Omega), \tag{34}$$

$$H_{n'} \to H \quad \text{a.e. in } \ \Omega. \tag{35}$$

(The last convergence is ensured by Lebesgue Theorem). Thanks to (32) and (33), we can take the limit in (30) as $n' \to \infty$. Then, we obtain

$$\left(\nabla \Psi', \nabla u\right) + \left(K.\nabla H, u\right) = \left(F, u\right), \qquad \forall u \in H_0^1\left(\Omega, \Gamma_1\right).$$

For (31), on the one hand, we use (33) to get

$$\lambda(\nabla H_{n'}, \nabla v) \to \lambda(\nabla H, \nabla v), \qquad \forall v \in H_0^1(\Omega).$$

On the other hand, as a consequence of (29), using Lebesgue theorem, we get

$$\nabla\left(\rho * E\Psi_{n'}\right)=\nabla\rho * E\Psi_{n'} \rightarrow \nabla\rho * E\Psi=\nabla\left(\rho * E\Psi\right) \text{ in } L^{2}\left(\mathbb{R}^{2}\right). \tag{36}$$

This implies

$$a\left(v,\Theta,\rho * E\Psi_{n'}\right) \rightarrow a\left(v,\Theta,\rho * E\Psi\right), \qquad \forall v \in H_{0}^{1}(\Omega).$$

Then for the second term of the left hand side of (19), we write

$$a\left(H_{n'},v,\rho * E\Psi_{n'}\right)=a\left(H_{n'}-H,v,\rho * E\Psi_{n'}\right)+a\left(H,v,\rho * E\Psi_{n'}\right).$$

From (23) and (35) it follows that $H \in L^{\infty}(\Omega)$, thus $a\left(H,v,\Psi\right)$ is well defined, and using (24), (29) and (34), we deduce

$$\begin{aligned} a\left(H_{n'}-H,v,\rho * E\Psi_{n'}\right) &\rightarrow 0 \\ a\left(H,v,\rho * E\Psi_{n'}\right) &\rightarrow a\left(H,v,\rho * E\Psi\right) \end{aligned}$$

and hence,

$$\lambda\left(\nabla H,\nabla v\right)-a\left(H,v,\rho * E\Psi\right)=-a\left(v,\Theta,\rho * E\Psi\right), \qquad \forall v \in H_{0}^{1}(\Omega).$$

Finally, we deduce that $\Phi(\Psi)=\Psi'$. Thus by the uniqueness of the solution of problems (19) and (20) we can easily show that the whole sequence $\Psi_n'=\Phi(\Psi_n)$ converge to $\Psi'=\Phi(\Psi)$ strongly in $L^2\left(\Omega\right)$ which ensure the continuity of Φ. □

We have the following result.

Proposition 1 *Under the assumptions* (8)-(10)*, the smoothing system* (18) *has at least a solution.*

Proof. Let us set $B=\{\Psi \in H_0^1(\Omega,\Gamma_1)\,;\|\Psi\| \leq M\}$, where M is the constant obtained in Lemma 3.2, and let $\bar{B}$ denote the closure of B in $L^2(\Omega)$.

The function Φ maps $\bar{B}$ into itself, which is compact in $L^2(\Omega)$. Applying the Schauder fixed point theorem, we get that there is $\Psi \in \bar{B}$ such that $\Phi(\Psi)=\Psi$. If we set $H=\Phi_1(\Psi)$, then (Ψ,H) is a solution of (18). □

3.2 Asymptotic method

In the following we take $\rho=\rho_n$ where $\left(\rho_n\right)_n$ is a smoothing sequence. Let (Ψ_n,H_n) denote a weak solution of (18) i.e.

$$\left\{\begin{array}{c} \left(\nabla\Psi_n,\nabla u\right)=-\left(K.\nabla H_n,u\right)+\left(F,u\right), \\ \lambda\left(\nabla H_n,\nabla v\right)-a\left(H_n,v,\rho_n * E\Psi_n\right)=-a\left(v,\Theta,\rho_n * E\Psi_n\right). \end{array}\right. \tag{37}$$

Testing the first identity by Ψ_n, the second one by H_n and arguing as in the proof of Lemma 3.2 to get the boundedness of the sequence $\left(\Psi_n\right)_n$ in $H^1\left(\Omega\right)$ and

$$\begin{aligned} \lambda\left|\nabla H_n\right|_2^2 &\leq meas\left(\Omega\right)\left|H_n\right|_{\infty}\left|\nabla\Theta\right|_2\left|\rho_n\right|_{\mathbb{R}^2,2}\left|\nabla\left(E\Psi_n\right)\right|_{\mathbb{R}^2,2} \\ &\leq c\left|H_n\right|_{\infty}\left|\nabla\Theta\right|_2\left\|\Psi_n\right\|, \end{aligned}$$

(since $E : H^1(\Omega) \to H^1(\mathbb{R}^2)$ is continuous) which shows the boundedness of the sequence $(\Psi_n)_n$ in $H^1(\Omega)$ by using Poincaré's inequality. From this and the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, there exist $\Psi \in H_0^1(\Omega, \Gamma_1)$, $H \in H_0^1(\Omega)$ and subsequences $\Psi_{n'}$, $H_{n'}$ such that :

$$\Psi_{n'} \rightharpoonup \Psi \quad \text{in} \;\; H_0^1(\Omega, \Gamma_1), \tag{38}$$

$$H_{n'} \rightharpoonup H \quad \text{in} \;\; H_0^1(\Omega), \tag{39}$$

$$\Psi_{n'} \to \Psi \quad \text{in} \;\; L^2(\Omega), \tag{40}$$

$$H_{n'} \to H \quad \text{in} \;\; L^2(\Omega). \tag{41}$$

Passing to limit in (37) and using the same argument as above to show that

$$\forall u \in H_0^1(\Omega, \Gamma_1), \qquad (\nabla\Psi, \nabla u) + (K.\nabla H, u) = (F, u).$$

For the second identity of (37), we have

$$\rho_{n'} * E\Psi_{n'} \;\; \to \;\; E\Psi \quad \text{in} \;\; L^2\left(\mathbb{R}^2\right), \tag{42}$$

$$\rho_{n'} * E\Psi_{n'} \;\; \rightharpoonup \;\; E\Psi \quad \text{in} \;\; H^1\left(\mathbb{R}^2\right). \tag{43}$$

Indeed, according to the continuity of E and (34), it follows that

$$E\Psi_{n'} \to E\Psi \quad \text{in} \;\; L^2\left(\mathbb{R}^2\right). \tag{44}$$

In the identity

$$\rho_{n'} * E\Psi_{n'} = \rho_{n'} * (E\Psi_{n'} - E\Psi) + \rho_{n'} * E\Psi,$$

the last term converges to $E\Psi$ strongly in $H^1(\mathbb{R}^2)$ since $\rho_{n'}$ is a smoothing sequence i.e.

$$\rho_{n'} * E\Psi \to E\Psi \;\; \text{in} \;\; H^1\left(\mathbb{R}^2\right). \tag{45}$$

By (44) and Young's lemma we have

$$\begin{aligned} |\rho_{n'} * (E\Psi_{n'} - E\Psi)(x)|_{\mathbb{R}^2,2} &\leq |\rho_{n'}|_{\mathbb{R}^2,1} |E\Psi_{n'} - E\Psi|_{\mathbb{R}^2,2} \\ &= |E\Psi_{n'} - E\Psi|_{\mathbb{R}^2,2} \to 0. \end{aligned}$$

i.e.

$$\rho_{n'} * (E\Psi_{n'} - E\Psi) \to 0 \;\; \text{in} \;\; L^2\left(\mathbb{R}^2\right).$$

($|\cdot|_{\mathbb{R}^2,1}$ denotes the norm of $L^1(\mathbb{R}^2)$). Combining this with (45) to obtain (42). To show (43) it suffices to note that $\rho_{n'} * E\Psi_{n'}$ is bounded in $H^1(\mathbb{R}^2)$ and one use the uniqueness of the limit in $L^2(\mathbb{R}^2)$. Next, noting that the scalar product in $L^2(\mathbb{R}^2)$ of strongly converging sequence and weakly converging sequence converges to the scalar product of their limits and arguing as above to show that

$$\lambda(\nabla H, \nabla v) - a(H, v, \Psi) = -a(v, \Theta, \Psi), \qquad \forall v \in H_0^1(\Omega).$$

Finally, the couple (Ψ, H) is a weak solution of the problem (4)-(7). To summarize, we can state the principal theorem of the existence of solutions.

Theorem 1 *Under the assumptions* (8)-(10)*, there exists a weak solution of the problem* (4)-(7)*.*

4 Iterative approximation method

In this section, we introduce an iterative method to approximate the solution of the system (16), (17), based on the idea of solving a sequence of the corresponding linear elliptic problems, so that the solutions of these linear problems converge toward the solution of our system at an exponential rate. To this end, let us assume that

$$C\,|K|_\infty\,|h|_{\infty,\Gamma} < \lambda, \tag{46}$$

where C is Poincaré's constant of Ω. Under this hypothesis, we know from [1] that the problem (4)-(7) has at most one weak solution, and hence by Theorem 1 exactly one weak solution. Next, we start by describing the iterative method.

4.1 The construction of the sequences

Let (α_n) be a sequence of positive numbers, converging to 0.

First step Let $\Psi_1 = \overline{\Psi}_1 \in \mathcal{D}\left(\overline{\Omega}, \Gamma_1\right)$ the space of functions of class C^∞ on $\overline{\Omega}$ vanishing on Γ_1 and let us consider the following linear problem

$$\lambda\left(\nabla H_1, \nabla v\right) - a\left(H_1, v, \overline{\Psi}_1\right) = -a\left(v, \Theta, \overline{\Psi}_1\right), \qquad \forall v \in H_0^1(\Omega). \tag{47}$$

Since the bilinear form $\lambda\left(\nabla\cdot, \nabla\cdot\right) - a\left(\cdot, \cdot, \overline{\Psi}_1\right)$ is well defined, continuous and coercive in $H_0^1(\Omega)$ and in the right hand side of (47), the linear form $v \mapsto -a(v, \Theta, \overline{\Psi}_1)$ is continuous, there exists one and only one $H_1 \in H_0^1(\Omega)$ satisfying (47).

Second step If we replace in (16) H by H_1, then let $\Psi_2 \in H_0^1\left(\Omega, \Gamma_1\right)$ be the solution of the problem

$$\left(\nabla \Psi_2, \nabla u\right) = -\left(K.\nabla H_1, u\right) + \left(F, u\right), \qquad \forall u \in H_0^1\left(\Omega, \Gamma_1\right).$$

Obviously, according to the Lax-Milgram lemma, the previous problem admits a unique solution $\Psi_2 \in H_0^1\left(\Omega, \Gamma_1\right)$.

By the density of $\mathcal{D}\left(\overline{\Omega}, \Gamma_1\right)$ in $H_0^1\left(\Omega, \Gamma_1\right)$, there exists $\overline{\Psi}_2 \in \mathcal{D}\left(\overline{\Omega}, \Gamma_1\right)$ such that

$$\left\|\Psi_2 - \overline{\Psi}_2\right\| \leq \alpha_2.$$

Third step The same way as previously allows us to construct by induction, three sequences of functions $\Psi_n \in H_0^1\left(\Omega, \Gamma_1\right)$, $\overline{\Psi}_n \in \mathcal{D}\left(\overline{\Omega}, \Gamma_1\right)$ and $H_n \in H_0^1(\Omega)$ such that

$$\left(\nabla \Psi_n, \nabla u\right) = -\left(K.\nabla H_{n-1}, u\right) + \left(F, u\right), \qquad \forall u \in H_0^1\left(\Omega, \Gamma_1\right), \tag{48}$$

$$\lambda\left(\nabla H_n, \nabla v\right) - a\left(H_n, v, \overline{\Psi}_n\right) = -a\left(v, \Theta, \overline{\Psi}_n\right), \qquad \forall v \in H_0^1\left(\Omega\right). \tag{49}$$

and

$$\left\|\Psi_n - \overline{\Psi}_n\right\| \leq \alpha_n. \tag{50}$$

Then, we have

Lemma 4.1 *There exists a constant $M > 0$ independent of n, such that*

$$\|\Psi_n\|, \left\|\overline{\Psi}_n\right\|, \|H_n\| \leq M \tag{51}$$

for all $n \geq 1$.

Proof. We take Ψ_n as a test function in (48), we obtain

$$|\nabla\Psi_n|_2^2 = (H_{n-1}\mathrm{div}K, \Psi_n) - (\mathrm{div}\,(H_{n-1}K)\,, \Psi_n) + (F, \Psi_n)\,.$$

Since H_{n-1} vanishes on the boundary Γ, we deduce

$$|\nabla\Psi_n|_2^2 = (H_{n-1}\mathrm{div}K, \Psi_n) + (H_{n-1}K, \nabla\Psi_n) + (F, \Psi_n)\,.$$

Using Poincaré's and Cauchy-Schwarz inequalities (10) and (23), it comes

$$\begin{aligned} |\nabla\Psi_n|_2^2 &\leq |\mathrm{div}K|_\infty |H_{n-1}|_2 |\Psi_n|_2 + |K|_\infty |H_{n-1}|_2 |\nabla\Psi_n|_2 + |F|_2 |\Psi_n|_2\,, \\ &\leq (|\mathrm{div}K|_\infty |H_{n-1}|_2 + |K|_\infty |H_{n-1}|_2 + |F|_2) \|\Psi_n\|\,. \end{aligned} \tag{52}$$

This shows the first estimate in (51). The second estimate is obtained by (50). For the last estimate, we replace v by H_n in (49), it comes

$$\lambda |\nabla H_n|_2^2 + a\left(H_n, H_n, \overline{\Psi}_n\right) = -a\left(H_n, \Theta, \overline{\Psi}_n\right).$$

But according to Lemma 2.2, we have $a\left(H_n, H_n, \overline{\Psi}_n\right) = 0$, whence if we use the Cauchy-Schwarz inequality and (23), we get

$$\lambda |\nabla H_n|_2^2 \leq |H_n|_\infty |\nabla\Theta|_2 \left|\nabla\overline{\Psi}_n\right|_2. \tag{53}$$

The Poincaré inequality, (23) and the second estimate in (51) give the last estimate. □

4.2 The Convergence result

The following theorem shows that the sequences (Ψ_n) and (H_n) converge to Ψ and H in $H^1(\Omega)$ respectively, and give the rate of convergence.

Theorem 2 *Under the assumptions* (8)-(10)*, and if the condition* (46) *holds, then there exists a positive constant A independent of n, such that*

$$|\nabla(\Psi_n - \Psi)|_2 + |\nabla(H_n - H)|_2 \leq A\beta^n \tag{54}$$

where $\beta = \dfrac{C\,|K|_\infty |h|_{\infty,\Gamma}}{\lambda} < 1$.

Proof. Let m and n be positive integers. We derive from (48) and (49) that

$$(\nabla(\Psi_{n+m}-\Psi_n),\nabla u) = -(K.\nabla(H_{n+m-1}-H_{n-1}),u), \qquad \forall u \in H_0^1(\Omega,\Gamma_1)$$

and

$$\lambda(\nabla(H_{n+m}-H_n),\nabla v) - a(H_{n+m},v,\overline{\Psi}_{n+m}) \\ + a(H_n,v,\overline{\Psi}_n) = -a(v,\Theta,\overline{\Psi}_{n+m}-\overline{\Psi}_n), \qquad \forall v \in H_0^1(\Omega).$$

Let us choose $u=\Psi_{n+m}-\Psi_n$ and $v=H_{n+m}-H_n$. Thanks to Lemma 2.1 or 2.2, we can write

$$\begin{aligned} a(H_{n+m},H_{n+m}-H_n,\overline{\Psi}_{n+m}) &= a(H_{n+m}-H_n,H_{n+m}-H_n,\overline{\Psi}_{n+m}) \\ &\quad + a(H_n,H_{n+m}-H_n,\overline{\Psi}_{n+m}) \\ &= a(H_n,H_{n+m}-H_m,\overline{\Psi}_{n+m}), \end{aligned}$$

and hence

$$\lambda\,|\nabla(H_{n+m}-H_n)|_2^2 - a(H_n+\Theta,H_{n+m}-H_n,\overline{\Psi}_{n+m}-\overline{\Psi}_n) = 0. \tag{55}$$

We also have

$$|\nabla(\Psi_{n+m}-\Psi_n)|_2^2 = -(K.\nabla(H_{n+m-1}-H_{n-1}),\Psi_{n+m}-\Psi_n). \tag{56}$$

In (56), by using the Cauchy-Schwarz inequality, the Poincaré inequality and (10), we obtain

$$\begin{aligned} |\nabla(\Psi_{n+m}-\Psi_n)|_2^2 &\leq |K|_\infty\,|\nabla(H_{n+m-1}-H_{n-1})|_2\,|\Psi_{n+m}-\Psi_n|_2 \\ &\leq C\,|K|_\infty\,|\nabla(H_{n+m-1}-H_{n-1})|_2\,|\nabla(\Psi_{n+m}-\Psi_n)|_2 \end{aligned}$$

whence

$$|\nabla(\Psi_{n+m}-\Psi_n)|_2 \leq C\,|K|_\infty\,|\nabla(H_{n+m-1}-H_{n-1})|_2\,. \tag{57}$$

For (55), the Cauchy-Schwarz inequality gives

$$\begin{aligned} \lambda\,|\nabla(H_{n+m}-H_n)|_2^2 &= a\left(H_n+\Theta,H_{n+m}-H_n,\overline{\Psi}_{n+m}-\overline{\Psi}_n\right) \\ &\leq |H_n+\Theta|_\infty\,|\nabla(H_{n+m}-H_n)|_2\left|\nabla\left(\overline{\Psi}_{n+m}-\overline{\Psi}_n\right)\right|_2. \end{aligned}$$

Thanks to (22), we obtain

$$\lambda\,|\nabla(H_{n+m}-H_n)|_2 \leq |h|_{\infty,\Gamma}\left|\nabla\left(\overline{\Psi}_{n+m}-\overline{\Psi}_n\right)\right|_2. \tag{58}$$

If we use (50), then (58) becomes

$$\begin{aligned}\lambda|\nabla\left(H_{n+m}-H_n\right)|_2 \leq |h|_{\infty,\Gamma}\Big\{&|\nabla(\overline{\Psi}_{n+m}-\Psi_{n+m})|_2+|\nabla(\Psi_{n+m}-\Psi_n)|_2\\ &+|\nabla(\Psi_n-\overline{\Psi}_m)|_2\Big\}\\ \leq |h|_{\infty,\Gamma}&\left\{\alpha_{n+m}+\alpha_n+|\nabla(\Psi_{n+m}-\Psi_n)|_2\right\}.\end{aligned}$$

This and (57) imply

$$\begin{aligned}|\nabla(H_{n+m}-H_n)|_2 &\leq \frac{1}{\lambda}|h|_{\infty,\Gamma}\left\{\alpha_{n+m}+\alpha_n+C\,|K|_\infty\,|\nabla(H_{n+m-1}-H_{n-1})|_2\right\}\\ &=\left(\frac{1}{\lambda}|h|_{\infty,\Gamma}\right)(\alpha_{n+m}+\alpha_n)+\beta\,|\nabla(H_{n+m-1}-H_{n-1})|_2.\end{aligned}$$

By induction it follows

$$|\nabla(H_{n+m}-H_n)|_2 \leq \beta^n\left\{|\nabla(H_m-H_0)|_2+\left(\frac{1}{\lambda}|h|_{\infty,\Gamma}\right)\sum_{j=1}^{n}\frac{\alpha_{j+m}+\alpha_j}{\beta^j}\right\}$$

If the sequence (α_n) is chosen such that $\alpha_n=\left(\frac{\beta}{2}\right)^n$ then we obtain

$$|\nabla(H_{n+m}-H_n)|_2 \leq \left(2M+\frac{2}{\lambda}|h|_{\infty,\Gamma}\right)\beta^n \tag{59}$$

where M is the constant from (51). Now, (57) gives

$$|\nabla(\Psi_{n+m}-\Psi_n)|_2 \leq C|K|_\infty\left(2M+\frac{2}{\lambda}|h|_{\infty,\Gamma}\right)\beta^{n-1} \tag{60}$$

and it follows that (H_n) and (Ψ_n) are Cauchy's sequences in $H^1(\Omega)$; therefore they converge and their limits necessarily are H and Ψ. Finally taking the limit as $m\to+\infty$ in the inequalities (59) and (60), we obtain (54). □

Remark 4.1 *As we said in the introduction, under the hypothesis* (46)*, it is possible to prove by a fix point argument that problem* (4)-(7) *has exactly one weak solution. This is done in* [3]*. Let us give a brief idea of the proof. Let* $\mathbf{W}=H_0^1(\Omega,\Gamma_1)\times H_0^1(\Omega)$*. On* $\mathbf{W}$ *we define the norm* $\|\ \|_{\mathbf{W}}$ *by*

$$\|(\Psi,H)\|_{\mathbf{W}}=\kappa|\nabla\Psi|_2+|\nabla H|_2$$

where $\kappa>0$ *is a suitable constant.*

Let $\mathbf{D} = \mathcal{D}(\Omega, \Gamma_1) \times \mathcal{D}(\Omega)$ *and let* $\mathbf{F} : \mathbf{D} \to \mathbf{W}$ *be the application defined in the following way. If* $(\Psi, H) \in \mathbf{D}$*, then* $\mathbf{F}(\Psi, H) = (\tilde{\Psi}, \tilde{H})$ *where* $\tilde{\Psi}$ *and* $\tilde{H}$ *are the solutions of the linear problems*

$$(\nabla\tilde{\Psi}, \nabla u) = -(K.\nabla H, u) + (F, u), \qquad \forall u \in H_0^1(\Omega, \Gamma_1), \tag{61}$$

$$\lambda(\nabla\tilde{H}, \nabla v) - a(\tilde{H}, v, \Psi) = -a(v, \Theta, \Psi), \qquad \forall v \in H_0^1(\Omega). \tag{62}$$

If we choose $\kappa = \sqrt{\frac{|h|_{\infty,\Gamma}}{\lambda C|K|_\infty}}$*, then* $\mathbf{F} : \mathbf{D} \to \mathbf{W}$ *is Lipschitz continuous, with the Lipschitz constant*

$$\beta = \sqrt{\frac{C|K|_\infty |h|_{\infty,\Gamma}}{\lambda}}.$$

Since $\mathbf{D}$ *is dense in the Banach space* $\mathbf{W}$*, the map* $\mathbf{F}$ *can be extended to* $\overline{\mathbf{F}} : \mathbf{W} \to \mathbf{W}$ *which is still Lipschitz continuous, with the same Lipschitz constant* β*.*

From (46)*, we have* $\beta < 1$*, and hence* $\overline{\mathbf{F}}$ *is a* contraction *and has a unique fixed point, which is the unique weak solution of the problem* (4)-(7)*.*

5 Regularity of the solutions

We shall study in this section, under appropriate smooth condition on the boundary Γ, the existence of higher order weak derivatives of solutions (Ψ, H) of the elliptic problem. To simplify, we consider the Dirichlet problem, i.e.

$$\Gamma = \Gamma_1 \tag{63}$$

We start by the regularity of the couple (Ψ, H) in the framework of Theorem 2 i.e. if (46) is held.

Theorem 3 *In addition to the hypotheses of Theorem* 2*, if we assume that* (63) *holds and* Γ *is of class* $\mathcal{C}^2$*, then*

(i) *the elements of the approximative sequence* (Ψ_n, H_n) *satisfy*

$$\Psi_n, H_n \in H^2(\Omega),$$

(ii) *the first component of the weak solution* $(\Psi, H) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *to problem* (4)-(7) *is smooth, i.e.*

$$\Psi \in H^2(\Omega).$$

(iii) Ψ_n *converges towards* Ψ *at an exponential rate in* $H^2(\Omega)$*, i.e.*

$$\|\Psi_n - \Psi\|_{H^2(\Omega)} \leq C\beta^n$$

where C *is a positive constant independent of* n *and* β *is defined in Theorem* 2*.*

PROOF. Since (48) is equivalent to

$$-\Delta\Psi_n = F - K.\nabla H_{n-1}, \tag{64}$$

and $F - K.\nabla H_{n-1} \in L^2(\Omega)$, we deduce from the regularity theorem of elliptic problem that $\Psi_n \in H^2(\Omega)$ (see for instance [10]). For the regularity of H_n, we can write (49) as

$$-\lambda\Delta H_n + \nabla H_n.\left(\nabla\overline{\Psi}_n\right)^{\perp} = -\nabla\Theta.\left(\nabla\overline{\Psi}_n\right)^{\perp}, \tag{65}$$

with $\nabla\overline{\Psi}_n \in [\mathcal{C}^{\infty}(\Omega)]^n$ and $-\nabla\Theta.\left(\nabla\overline{\Psi}_n\right)^{\perp} \in L^2(\Omega)$. Again, applying the regularity theorem of elliptic problem, we deduce that $H_n \in H^2(\Omega)$.

For the second and the third points, let m and n be positive integers. We derive from (48) that

$$\begin{cases} -\Delta\left(\Psi_{n+m} - \Psi_n\right) = -K.\nabla\left(H_{n+m-1} - H_{n-1}\right) \quad \text{in } \Omega \\ \Psi_{n+m} - \Psi_n = 0 \quad \text{on } \Gamma. \end{cases}$$

This means that $(\Psi_{n+m} - \Psi_n)$ is a solution of an homogenous Dirichlet problem of an elliptic equation. Since $-K.\nabla\left(H_{n+m-1} - H_{n-1}\right) \in L^2(\Omega)$, we deduce from the regularity theorem of elliptic problem that

$$\begin{aligned} \|\Psi_{n+m} - \Psi_n\|_{H^2(\Omega)} &\leq C\left(|\Psi_{n+m} - \Psi_n|_2 + |K.\nabla\left(H_{n+m-1} - H_{n-1}\right)|_2\right) \\ &\leq C\left(|\Psi_{n+m} - \Psi_n|_2 + |K|_{\infty}|\nabla\left(H_{n+m-1} - H_{n-1}\right)|_2\right). \end{aligned}$$

The proof is complete according to (54). □

The weakness of regularity of $\nabla\Psi_n$ influenced the regularity of H and prevented the use of the regularity theorem of elliptic problem. If there exists a subsequence of H_n, which is bounded in the norm of $H^2(\Omega)$, the problem (4)-(7) admits a solution $(\Psi, H) \in H^2(\Omega) \times H^2(\Omega)$.

We shall now answer the following question in the theorem below: what about the regularity of all the solutions of (4)-(7) in the general case?

Theorem 4 *In addition to the assumptions* (8)-(10)*, let us assume that* (63) *holds and that* Γ *is of class* $\mathcal{C}^2$*. If* $(\Psi, H) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *is a weak solution of the problem* (4)-(7)*, then* $\Psi \in H^2(\Omega)$*. Moreover if*

$$c_0^2 C_{\lambda}\|\Psi\|_{H^2(\Omega)} < 1, \tag{66}$$

where the constants c_0 *and* C_{λ} *are some constants given below, then we have* $(\Psi, H) \in H^2(\Omega) \times H^2(\Omega)$.

PROOF. Let $(H, \Psi) \in \left[H_0^1(\Omega)\right]^2$ be a weak solution to problem (4)-(7). According to the regularity theorem of elliptic problem and since we have

$$-\Delta\Psi = F - K.\nabla H, \tag{67}$$

and $F - K.\nabla H \in L^2(\Omega)$, we deduce that $\Psi \in H^2(\Omega)$. For the regularity of H, we cannot directly apply the regularity theorem of elliptic problem. However, if we choose a sequence $\Psi'_n \in \mathcal{D}\left(\overline{\Omega}\right)$ satisfying

$$\|\Psi'_n - \Psi\|_{H^2(\Omega)} \leq \alpha_n, \tag{68}$$

then, the elliptic boundary value problem

$$\begin{aligned} -\lambda \Delta \Xi &= -\nabla \Xi . \left(\nabla \Psi'_n\right)^{\perp} - \nabla \Theta . \left(\nabla \Psi'_n\right)^{\perp} \\ \Xi &= 0 \quad \text{on } \Gamma, \end{aligned}$$

admits a unique solution $H'_n \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$\begin{aligned} \|H'_n\|_{H^2(\Omega)} &\leq C_\lambda \left(|H'_n|_2 + \left|\nabla \Theta . \left(\nabla \Psi'_n\right)^{\perp}\right|_2 + \left|\nabla H'_n . \left(\nabla \Psi'_n\right)^{\perp}\right|_2 \right) \\ &\leq C_\lambda \left(|H'_n|_2 + \left|\nabla \overline{\Psi}_n\right|_{L^4(\Omega)} |\nabla \Theta|_{L^4(\Omega)} + |\nabla \Psi'_n|_{L^4(\Omega)} |\nabla H_n|_{L^4(\Omega)} \right), \end{aligned}$$

where C_λ is the best constant of the operator $-\lambda\Delta$ given in Theorem 8.12 in [10] dependent only on Γ and λ. It follows, from the imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ (c_0 is the norm of this injection), that

$$\begin{aligned} \|H'_n\|_{H^2(\Omega)} &\leq C_\lambda \left(|H'_n|_2 + c_0^2 \|\nabla \Psi'_n\| \|\nabla \Theta\| + c_0^2 \|\nabla \Psi'_n\| \|\nabla H'_n\| \right) \\ &\leq C_\lambda \left(|H'_n|_2 + c_0^2 \|\Psi'_n\|_{H^2(\Omega)} \|\Theta\|_{H^2(\Omega)} + c_0^2 \|\Psi'_n\|_{H^2(\Omega)} \|H'_n\|_{H^2(\Omega)} \right). \end{aligned}$$

Using (68) to get

$$\|H'_n\|_{H^2(\Omega)} \leq C_\lambda \left(|H'_n|_2 + c_0^2 \|\Psi'_n\|_{H^2(\Omega)} \|\Theta\|_{H^2(\Omega)} + c_0^2 \left(\|\Psi\|_{H^2(\Omega)} + \alpha_n \right) \|H'_n\|_{H^2(\Omega)} \right).$$

Since $c_0^2 C_\lambda \|\Psi\|_{H^2(\Omega)} < 1$, all the terms of α_n, except maybe a finite number of them, satisfying for some μ: $c_0^2 C_\lambda \|\Psi\|_{H^2(\Omega)} + c_0^2 C_\lambda \alpha_n < \mu < 1$, then it follows that

$$(1-\mu) \|H'_n\|_{H^2(\Omega)} \leq C_\lambda \left(|H'_n|_2 + c_0^2 \left(\|\Psi\|_{H^2(\Omega)} + \alpha_n \right) \|\Theta\|_{H^2(\Omega)} \right).$$

We have therefore, from the boundedness of $|H'_n|_2$ (we can verify that H_n satisfy (23)), the boundedness of $\|H'_n\|_{H^2(\Omega)}$. Then there exists a subsequence $H'_{n'}$ converging weakly in $H^2(\Omega)$ strongly in $H^1(\Omega)$ to $H' \in H^2(\Omega)$. We can verify that $H' \in H^1_0(\Omega)$ is a solution of (5). Since the elliptic problem (5) coupled with the boundary condition (7) has a unique solution, we deduce that $H' = H$ and

$$\begin{aligned} H'_n &\rightharpoonup H \quad \text{in} \quad H^2(\Omega), \\ H'_n &\to H \quad \text{in} \quad H^1(\Omega). \end{aligned}$$

This completes the proof. □

Remark 5.1 *If we use estimates like* (23), (52) *and* (53), *we can replace* (66) *by another hypothesis only depending on the data.*

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