*Applicable Analysis*
Vol. 00, No. 00, January 2008, 1–16

**RESEARCH ARTICLE**

# On some anisotropic, nonlocal, parabolic singular perturbations problems

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(*Received 00 Month 200x; in final form 00 Month 200x*)

This paper is devoted to the study of the anisotropic singular perturbations theory for some quasilinear parabolic problems. Describing the asymptotic behaviour of the solutions of non-local problems yields to show the existence of solution to some integro-differential problems. The closeness between the solutions of these two kinds of problems is established.

**Keywords:** Parabolic problems, anisotropic singular perturbations, asymptotic behaviour.

**AMS Subject Classification**: 35B25, 35B40, 35L15, 35L20, 35L70

## 1. Introduction

Let $\Omega = (0,1)^2$ be the unit square in $\mathbb{R}^2$ and $T > 0$ be a positive constant. If $a$ is a positive, continuous function and $f \in L^2\left(0,T;L^2(\Omega)\right)$, $u^0 \in L^2(\Omega)$ suppose that we are interested in solving the following integro-differential problem: find a function $u_0 = u_0(t,x_1,x_2)$ such that

$$\begin{cases} \partial_t u_0(\cdot,x_1,\cdot) - a\left(\int_\Omega u_0(\cdot,x)\,dx\right)\partial^2_{x_2}u_0(\cdot,x_1,\cdot) = f(\cdot,x_1,\cdot) \\ \qquad\qquad\qquad\qquad \text{in } (0,T)\times(0,1) \text{ a.e } x_1\in(0,1), \\ u_0(\cdot,x_1,\cdot) = 0 \text{ on } (0,T)\times\{0,1\} \text{ a.e } x_1\in(0,1), \\ u_0(0,\cdot) = u^0 \text{ in } \Omega. \end{cases} \tag{1}$$

Since the integral in the first equation of (1) is taken on the whole domain $\Omega$ it is clear that this equation cannot be considered as a parabolic equation parametrized by $x_1$. Moreover, if starting from $w \in L^2\left(0,T;L^2(\Omega)\right)$, one solves for $u = Sw$ the problem parametrized by $x_1$

$$\begin{cases} \partial_t u(\cdot,x_1,\cdot) - a\left(\int_\Omega w(\cdot,x)\,dx\right)\partial^2_{x_2}u(\cdot,x_1,\cdot) = f(\cdot,x_1,\cdot) \\ \qquad\qquad\qquad\qquad \text{in } (0,T)\times(0,1) \text{ a.e } x_1\in(0,1), \\ u(\cdot,x_1,\cdot) = 0 \text{ on } (0,T)\times\{0,1\} \text{ a.e } x_1\in(0,1), \\ u(0,x_1,\cdot) = u^0 \text{ in } (0,1) \text{ a.e } x_1\in(0,1), \end{cases}$$

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ISSN: 0003-6811 print/ISSN 1563-504X online

DOI: 10.1080/0003681YYxxxxxxxx
http://www.informaworld.com

when it comes to finding a fixed point for $S$ some compactness for this mapping is missing. One way to overcome this is to introduce for instance the following anisotropic singular perturbations problem

$$\begin{cases} \partial_t u_\varepsilon - a\left(\int_\Omega u_\varepsilon\left(\cdot, x\right) dx\right)\left(\varepsilon^2 \partial^2_{x_1} u_\varepsilon + \partial^2_{x_2} u_\varepsilon\right) = f \text{ in } (0,T)\times\Omega, \\ u_\varepsilon = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ u_\varepsilon(0) = u^0 \quad \text{in } \Omega, \end{cases} \tag{2}$$

and to see what can be said on the behaviour of $u_\varepsilon$ when $\varepsilon > 0$ goes to 0. This is what we would like to do in this note. For more details about the anisotropic singular perturbations problems from different point of view, the reader is referred to [1, 3–6, 8, 9] (see [10] for the basic theory of singular perturbations).

## 2. A more general setting

Now we suppose that $\Omega$ is a bounded open subset of $\mathbb{R}^n$ and introduce a problem generalizing (2). We split the components of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ into two parts

$$X_1 = (x_1, \ldots, x_p) \quad \text{and} \quad X_2 = (x_{p+1}, \ldots, x_n),$$

so that $x = (X_1, X_2)$ ($p$ and $n$ are positive integers such that $p < n$). Similarly setting

$$\nabla_{X_1} u = (\partial_{x_1} u, \ldots, \partial_{x_p} u)^T, \ \ \nabla_{X_2} u = (\partial_{x_{p+1}} u, \ldots, \partial_{x_n} u)^T,$$

where $T$ denotes the transpose operation and $\partial_{x_i}$ the partial derivative in the direction $x_i$, we have if $\nabla$ denotes the usual gradient operator

$$\nabla u = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}. \tag{3}$$

It will be convenient to simplify the notation to set

$$z = (t, x) \in \mathbb{R}^{n+1}, \ \ Q = (0,T)\times\Omega. \tag{4}$$

Let

$$A = A(z,s) = (a_{ij}(z,s)) \tag{5}$$

be a $n \times n$ matrix defined for $(z,s) \in Q \times \mathbb{R}$ and such that $A$ is of Carathéodory type that is to say

$$z \mapsto a_{ij}(z,s) \text{ is measurable } \forall s \in \mathbb{R}, \quad \forall\, i,j = 1,\ldots,n, \tag{6}$$

$$s \mapsto a_{ij}(z,s) \text{ is continuous a.e. } z \in Q, \quad \forall\, i,j = 1,\ldots,n. \tag{7}$$

Moreover we assume that there exist two positive constants $\lambda, \Lambda$ such that

$$\lambda|\xi|^2 \le A\left(z,s\right)\xi\cdot\xi \le \Lambda|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } z \in Q, \ \forall\, s \in \mathbb{R}. \tag{8}$$

In the inequality above we have denoted the scalar product by a dot and the usual Euclidean norm by $|\cdot|$. Note that (8) implies that the coefficients $a_{ij}$ are uniformly bounded.

We split $A$ into four blocks in such way that

$$A = \begin{pmatrix} A_{11} \; A_{12} \\ A_{21} \; A_{22} \end{pmatrix}, \tag{9}$$

where $A_{11}$, $A_{22}$ are respectively $p \times p$ and $(n-p) \times (n-p)$ matrices. We then set for $\varepsilon > 0$

$$A_\varepsilon = A_\varepsilon(z,s) = \begin{pmatrix} \varepsilon^2 A_{11} \; \varepsilon A_{12} \\ \varepsilon A_{21} \;\; A_{22} \end{pmatrix}. \tag{10}$$

If $\xi = \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix} \in \mathbb{R}^n$ where $\bar{\xi}_1 = (\xi_1, \dots, \xi_p)^T$, $\bar{\xi}_2 = (\xi_{p+1}, \dots, \xi_n)^T$ by (8) one has for $\xi_\varepsilon = \begin{pmatrix} \varepsilon\bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix}$

$$\lambda|\xi_\varepsilon|^2 = \lambda\{\varepsilon^2|\bar{\xi}_1|^2 + |\bar{\xi}_2|^2\} \le A\xi_\varepsilon \cdot \xi_\varepsilon = A_\varepsilon\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } z \in Q, \; \forall s \in \mathbb{R}, \tag{11}$$

$$\lambda|\bar{\xi}_2|^2 \le A_{22}\bar{\xi}_2 \cdot \bar{\xi}_2 \quad \forall \bar{\xi}_2 \in \mathbb{R}^{n-p}, \;\text{ a.e. } \; z \in Q, \; \forall\, s \in \mathbb{R}. \tag{12}$$

We would like to consider the family of parabolic, nonlinear, nonlocal problems

$$\begin{cases} \partial_t u_\varepsilon - \nabla \cdot \left(A_\varepsilon\left(\cdot, l\left(u_\varepsilon\right)\right) \nabla u_\varepsilon\right) = f \quad \text{in} \quad Q, \\ u_\varepsilon = 0 \quad \text{on} \quad (0,T) \times \partial\Omega, \\ u_\varepsilon\left(0, \cdot\right) = u^0 \quad \text{in } \Omega, \end{cases} \tag{13}$$

where

$$l\left(u\right) = \int_\Omega h\left(x\right) u\left(t,x\right) dx \tag{14}$$

for some $h \in L^2\left(\Omega\right)$ i.e. $l$ is a continuous linear form on $L^2\left(\Omega\right)$. We are in particular interested to show that $u_\varepsilon$ - perhaps up to subsequence - possesses a limit when $\varepsilon \to 0$. Note that for $A\left(z,s\right) = a\left(s\right) Id$ where $Id$ is the identity matrix in $\mathbb{R}^2$, $h \equiv 1$ one recognizes the problem (2) of the introduction and the limit of $u_\varepsilon$ will provide us with a solution to problem (1).

We suppose here that

$$f \in L^2(Q), \; u^0 \in L^2(\Omega). \tag{15}$$

Then we have

**Theorem 2.1:** *Under the assumptions above for any $\varepsilon > 0$ there exists a weak* *solution $u_\varepsilon$ to problem (13) that is to say a function $u_\varepsilon = u_\varepsilon(t,x)$ such that*

$$\begin{cases} u_\varepsilon \in L^2\left(0,T;H_0^1(\Omega)\right) \cap \mathcal{C}\left([0,T];L^2(\Omega)\right),\ \partial_t u_\varepsilon \in L^2\left(0,T;H^{-1}(\Omega)\right), \\ \langle \partial_t u_\varepsilon, v\rangle + \int_\Omega A_\varepsilon\left(t,x,l\left(u_\varepsilon\right)\right)\nabla u_\varepsilon \cdot \nabla v\, dx = (f,v) \\ \qquad\qquad\qquad\qquad \text{in } \mathcal{D}'(0,T), \quad \forall\, v \in H_0^1(\Omega), \\ u(0,\cdot) = u^0. \end{cases} \tag{16}$$

We refer the reader to [2], [7] for the different spaces introduced above. $\mathcal{D}'$ denotes the space of distributions, $\langle\cdot,\cdot\rangle$ the duality bracket between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $(\cdot,\cdot)$ the usual scalar product in $L^2(\Omega)$.

**Proof:** The proof is a simple application of the Schauder fixed point theorem. Indeed for $w \in L^2(Q)$ one considers $u = Sw$ the solution to

$$\begin{cases} u \in L^2\left(0,T;H_0^1(\Omega)\right) \cap \mathcal{C}\left([0,T];L^2(\Omega)\right),\ \partial_t u \in L^2\left(0,T;H^{-1}(\Omega)\right), \\ \langle \partial_t u, v\rangle + \int_\Omega A_\varepsilon\left(t,x,l(w)\right)\nabla u \cdot \nabla v\, dx = (f,v) \quad \text{in } \mathcal{D}'(0,T), \quad \forall\, v \in H_0^1(\Omega), \\ u(0,\cdot) = u^0. \end{cases} \tag{17}$$

Then it is easy to show (cf. [2]) that $w \to Sw$ has a fixed point $u_\varepsilon$. This completes the proof of the theorem. □

## 3. Passage to the limit

We are now going to analyse the behaviour of $u_\varepsilon$ when $\varepsilon \to 0$. For this purpose we introduce $\Omega_{X_1}$ the section of $\Omega$ above $X_1$ defined as

$$\Omega_{X_1} = \{\, X_2 \mid (X_1, X_2) \in \Omega \,\} \tag{18}$$

and

$$Q_{X_1} = (0,T) \times \Omega_{X_1}. \tag{19}$$

Then - if we let $\varepsilon$ go to 0 very formally - we see that a reasonable candidate for the limit of $u_\varepsilon$ when $\varepsilon \to 0$ is given by

$$u_0 = u_0(t, X_1, X_2) \tag{20}$$

where $u_0$ satisfies for a.e. $X_1 \in \Pi_1(\Omega) = \{X_1 \mid \exists X_2 \text{ such that } (X_1,X_2) \in \Omega\,\}$

$$\begin{cases} \partial_t u_0(\cdot, X_1, \cdot) - \nabla_{X_2} \cdot \left(A_{22}\left(\cdot, X_1, \cdot, l\left(u_0\right)\right)\nabla_{X_2} u_0\right)(\cdot, X_1, \cdot) \\ \qquad\qquad\qquad\qquad = f(\cdot, X_1, \cdot) \quad \text{in } \ Q_{X_1}, \\ u_0(\cdot, X_1, \cdot) = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_{X_1}, \\ u_0(0, X_1, \cdot) = u^0(X_1, \cdot) \quad \text{in } \Omega_{X_1}. \end{cases} \tag{21}$$

Then we have

**Theorem 3.1 :** *Under the assumptions above there exists $u_0$ weak solution to* (21) *in the sense that a.e. $X_1 \in \Pi_1(\Omega)$*

$$\begin{cases} u_0(\cdot, X_1, \cdot) \in L^2\left(0,T; H_0^1(\Omega_{X_1})\right) \cap \mathcal{C}\left([0,T]; L^2(\Omega_{X_1})\right), \ u_0 \in L^2(Q), \\ \partial_t u_0(\cdot, X_1, \cdot) \in L^2\left(0,T; H^{-1}(\Omega_{X_1})\right), \\ u_0(0, X_1, \cdot) = u^0(X_1, \cdot) \quad on\ \Omega_{X_1}, \\ \partial_t (u_0, v)_{\Omega_{X_1}} + \displaystyle\int_{\Omega_{X_1}} A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dX_2 = (f, v)_{\Omega_{X_1}} \\ \qquad\qquad\qquad\qquad in\ \mathcal{D}'(0,T), \ \forall v \in H_0^1(\Omega_{X_1}). \end{cases} \tag{22}$$

$(\cdot,\cdot)_{\Omega_{X_1}}$ *denotes the usual $L^2(\Omega_{X_1})$ scalar product.*

**Remark 1 :** The asymptotic behaviour of the linear parabolic problems can be considered as a particular case. Indeed it is enough to choose $A$ independent of $s$, i.e.

$$A(z,s) = A(z) \quad \forall z \in Q.$$

**Proof:** The solution $u_0$ is going to be obtained as a limit of $u_\varepsilon$ when $\varepsilon \to 0$. That is to say we are going to show the following lemma which at the same time will complete the proof of Theorem 3.1.

**Lemma 3.2:** *Under the assumptions above there exists $u_0 \in L^2(Q)$ and a "subsequence" of $\varepsilon$ still labelled $\varepsilon$ converging toward* 0 *such that*

$$u_\varepsilon \rightharpoonup u_0, \qquad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0, \qquad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \quad in\ L^2(Q),$$
$$\partial_t u_\varepsilon \rightharpoonup \partial_t u_0 \qquad in\ \ L^2\left(0,T; H^{-1}(\Omega)\right),$$

*and $u_0$ is solution to* (22).

(In the above the convergence in $L^2(\Omega)$ means for vectors the convergence of each components).

**Proof:** [Proof of the lemma 3.2]
If we take $v = u_\varepsilon$ in (16) -cf. [2], [7] for the technical details- we get for a.e. $t$

$$\frac{1}{2}\frac{d}{dt}|u_\varepsilon|_{2,\Omega}^2 + \int_\Omega A_\varepsilon(t, \cdot, l(u_\varepsilon)) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx = (f, u_\varepsilon).$$

($|u_\varepsilon|_{2,\Omega}$ denotes the usual $L^2(\Omega)$-norm). Using (11) and the Cauchy-Schwarz inequality we derive

$$\frac{1}{2}\frac{d}{dt}|u_\varepsilon|_{2,\Omega}^2 + \lambda \int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 \, dx \le (f, u_\varepsilon) \le |f(t,\cdot)|_{2,\Omega} |u_\varepsilon|_{2,\Omega}.$$

Since $\Omega$ is bounded -in particular in the directions $X_2$- we have for some constant $C > 0$ independent of $\varepsilon$ a Poincaré inequality of the type

$$|v|_{2,\Omega} \le C \| \nabla_{X_2} v| \,|_{2,\Omega} \quad \forall v \in H_0^1(\Omega). \tag{23}$$

($|\nabla_{X_2} v|$ denotes the Euclidean norm of $\nabla_{X_2} v$). Thus, by the Young inequality, we deduce

$$\frac{1}{2}\frac{d}{dt}\,|u_\varepsilon|_{2,\Omega}^2 + \lambda\int_\Omega \varepsilon^2|\nabla_{X_1}u_\varepsilon|^2 + |\nabla_{X_2}u_\varepsilon|^2dx \le C|f|_{2,\Omega}|\,|\nabla_{X_2}u_\varepsilon|\,|_{2,\Omega}$$
$$\le \frac{C^2}{2\lambda}|f|_{2,\Omega}^2 + \frac{\lambda}{2}|\,|\nabla_{X_2}u_\varepsilon|\,|_{2,\Omega}^2$$

and in particular

$$\frac{1}{2}\frac{d}{dt}\,|u_\varepsilon|_{2,\Omega}^2 + \frac{\lambda}{2}\int_\Omega \varepsilon^2|\nabla_{X_1}u_\varepsilon|^2 + |\nabla_{X_2}u_\varepsilon|^2dx \le \frac{C^2}{2\lambda}|f|_{2,\Omega}^2. \tag{24}$$

Integrating in $t$ between 0 and $T$ we get

$$\frac{1}{2}\,|u_\varepsilon|_{2,\Omega}^2\,(T) + \frac{\lambda}{2}\int_0^T\int_\Omega \varepsilon^2|\nabla_{X_1}u_\varepsilon|^2 + |\nabla_{X_2}u_\varepsilon|^2\,dtdx \le \frac{1}{2}\left|u^0\right|_{2,\Omega}^2 + \frac{C^2}{2\lambda}|f|_{2,Q}^2, \tag{25}$$

and thus

$$u_\varepsilon,\quad |\varepsilon\nabla_{X_1}u_\varepsilon|,\quad |\nabla_{X_2}u_\varepsilon| \quad \text{are bounded in } L^2(Q), \tag{26}$$

(this of course independently of $\varepsilon$). From the equation in (16) we derive for $v \in H_0^1(\Omega)$,

$$|\langle\partial_t u_\varepsilon, v\rangle| \le |(f,v)| + \left|\int_\Omega A_\varepsilon\left(t,\cdot,l\left(u_\varepsilon\right)\right)\nabla u_\varepsilon\cdot\nabla v\,dx\right|$$
$$\le |f\,(t,\cdot)|_{2,\Omega}\,|v|_{2,\Omega} + C\left(|\,|\varepsilon\nabla_{X_1}u_\varepsilon|\,|_{2,\Omega} + |\,|\nabla_{X_2}u_\varepsilon|\,|_{2,\Omega}\right)|\,|\nabla_{X_2}v|\,|_{2,\Omega}$$
$$\le C'\left(|f\,(t,\cdot)|_{2,\Omega} + |\,|\varepsilon\nabla_{X_1}u_\varepsilon|\,|_{2,\Omega} + |\,|\nabla_{X_2}u_\varepsilon|\,|_{2,\Omega}\right)|v|_{H^1(\Omega)},$$

where $C, C'$ are constants independent of $\varepsilon \to 0$, $|v|_{H^1(\Omega)}$ the usual $H^1(\Omega)$ −norm. This implies

$$|\partial_t u_\varepsilon|_{L^2(0,T;H^{-1}(\Omega))} \le C'\left(|f|_{2,Q} + |\,|\varepsilon\nabla_{X_1}u_\varepsilon|\,|_{2,Q} + |\,|\nabla_{X_2}u_\varepsilon|\,|_{2,Q}\right).$$

It follows from (26) that

$$\partial_t u_\varepsilon \quad \text{is bounded in} \quad L^2\left(0,T;H^{-1}(\Omega)\right). \tag{27}$$

Up to a subsequence we deduce from (26), (27) that there exist $u_0, u_1, u_2, \in L^2(Q)$ and $u_3 \in L^2\left(0,T;H^{-1}(\Omega)\right)$ - i.e. with components in $L^2(Q)$ for the vectors - such that

$$u_\varepsilon \rightharpoonup u_0,\qquad \varepsilon\nabla_{X_1}u_\varepsilon \rightharpoonup u_1,\qquad \nabla_{X_2}u_\varepsilon \rightharpoonup u_2,\qquad \text{in } L^2(Q),$$
$$\partial_t u_\varepsilon \rightharpoonup u_3 \qquad \text{in} \quad L^2\left(0,T;H^{-1}(\Omega)\right).$$

By the continuity of the derivation in $\mathcal{D}'(Q)$ we derive that

$$u_\varepsilon \rightharpoonup u_0,\qquad \varepsilon\nabla_{X_1}u_\varepsilon \rightharpoonup 0,\qquad \nabla_{X_2}u_\varepsilon \rightharpoonup \nabla_{X_2}u_0 \quad \text{in } L^2(Q), \tag{28}$$
$$\partial_t u_\varepsilon \rightharpoonup \partial_t u_0 \qquad \text{in} \quad L^2\left(0,T;H^{-1}(\Omega)\right). \tag{29}$$

A delicate issue is now to show that

$$u_0(t, X_1, \cdot) \in H_0^1(\Omega_{X_1}) \quad \text{a.e.} \quad (t, X_1) \in (0,T) \times \Pi_1(\Omega).$$

For that we denote by $\mathcal{B}$ an open ball of $\mathbb{R}^{n-p}$ such that

$$\Omega_{X_1} \subset \mathcal{B} \qquad \forall X_1 \in \Pi_1(\Omega). \tag{30}$$

We suppose that $u_\varepsilon(t, \cdot)$ is extended by 0 outside $\Omega$. Then from (26) we derive that

$$\int_0^T \int_{\Pi_1(\Omega)\times\mathcal{B}} |\nabla_{X_2} u_\varepsilon|^2 \, dtdx \le C, \tag{31}$$

where $C$ is a constant independent of $\varepsilon$. This can also be written as

$$|u_\varepsilon|_{L^2((0,T)\times\Pi_1(\Omega);H_0^1(\mathcal{B}))} \le C.$$

Thus there exists a function $\bar{u}_0 \in L^2\left((0,T)\times\Pi_1(\Omega); H_0^1(\mathcal{B})\right)$, a subsequence of the subsequence above and still labelled $\varepsilon$ such that

$$u_\varepsilon \rightharpoonup \bar{u}_0 \quad \text{in } L^2\left((0,T)\times\Pi_1(\Omega); H_0^1(\mathcal{B})\right)$$

and in particular

$$u_\varepsilon \to \bar{u}_0 \quad \text{in } \mathcal{D}'\left((0,T)\times\Pi_1(\Omega)\times\mathcal{B}\right).$$

If $u_0$ is also extended by 0 outside $\Omega$ one has of course also

$$u_\varepsilon \to u_0 \quad \text{in } \mathcal{D}'\left((0,T)\times\Pi_1(\Omega)\times\mathcal{B}\right)$$

and thus

$$u_0 = \bar{u}_0 \in L^2\left((0,T)\times\Pi_1(\Omega); H_0^1(\mathcal{B})\right).$$

It follows that

$$u_0(t, X_1, \cdot) \in H_0^1(\mathcal{B}) \ \text{ a.e. } \ (t, X_1) \in (0,T)\times\Pi_1(\Omega),$$

i.e.

$$u_0(t, X_1, \cdot) \in H_0^1(\Omega_{X_1}) \ \text{ a.e. } \ (t, X_1) \in (0,T)\times\Pi_1(\Omega). \tag{32}$$

We next show that

$$l(u_\varepsilon) \to l(u_0).$$

This will follow from the lemma:

**Lemma 3.3:** *For any $v \in H_0^1(\Omega)$, the functions*

$$t \mapsto \int_\Omega u_\varepsilon v dx, \quad t \mapsto \int_\Omega u_0 v dx \tag{33}$$

*belong to* $H^1(0,T)$ *and for the subsequence above we have*

$$
\begin{aligned}
&(i) \quad \int_\Omega u_\varepsilon v dx \to \int_\Omega u_0 v dx \qquad in\ L^2(0,T),\ \mathcal{C}(0,T),\\
&(ii) \quad \int_\Omega u_\varepsilon v dx \rightharpoonup \int_\Omega u_0 v dx \qquad in\ H^1(0,T),\\
&(iii) \quad l(u_\varepsilon) \to l(u_0) \quad in\ L^2(0,T).
\end{aligned}
$$

*($\mathcal{C}(0,T)$ denotes the space of continuous functions on $[0,T]$ for the uniform norm).*

**Proof:** [Proof of the lemma 3.3]
From (16) we derive for $v \in H_0^1(\Omega)$

$$\frac{d}{dt}(u_\varepsilon(t,\cdot),v) = (f(t,\cdot),v) - \int_\Omega A_\varepsilon(t,x,l(u_\varepsilon))\nabla u_\varepsilon(t,x)\cdot\nabla v(x)\,dx.$$

Thus we have (see (8)) for some constant $C$ independent of $\varepsilon$

$$\left|\frac{d}{dt}(u_\varepsilon,v)\right| \le |f|_{2,\Omega}|v|_{2,\Omega} + C(\varepsilon|\,|\nabla_{X_1}u_\varepsilon|\,|_{2,\Omega} + |\,|\nabla_{X_2}u_\varepsilon|\,|_{2,\Omega})\,|\,|\nabla v|\,|_{2,\Omega}, \qquad (34)$$

whence

$$\frac{d}{dt}(u_\varepsilon,v) \in L^2(0,T).$$

Similarly one has

$$|(u_\varepsilon,v)| \le |u_\varepsilon|_{2,\Omega}|v|_{2,\Omega}$$

and by (26), (34) we conclude that

$$(u_\varepsilon,v) \in H^1(0,T)$$

and for some constant $D$ independent of $\varepsilon$ it holds

$$|(u_\varepsilon,v)|_{H^1(0,T)} \le D.$$

Thus - up to a subsequence of the above subsequence - we have

$$(u_\varepsilon,v) \to L_v \ \text{ in } L^2(0,T),\ \mathcal{C}(0,T), \qquad (35)$$

$$(u_\varepsilon,v) \rightharpoonup L_v \ \text{ in } H^1(0,T). \qquad (36)$$

Let us choose $\varphi \in \mathcal{D}(0,T)$ then one has

$$\int_0^T (u_\varepsilon,v)\varphi dt = \int_0^T\int_\Omega u_\varepsilon v\varphi dt dx \to \int_0^T\int_\Omega u_0 v\varphi dt dx = \int_0^T (u_0,v)\varphi dt,$$

by (28). On the other hand by (35) we have

$$\int_0^T (u_\varepsilon,v)\varphi dt \to \int_0^T L_v\varphi dt.$$

It follows that

$$L_v = \int_\Omega u_0 v dx$$

and by uniqueness of the limit, the convergences in (35), (36) are not up to a subsequence. This completes the proof of $(i), (ii)$ and $(iii)$ for $h \in H_0^1(\Omega)$ (recall (14)). To obtain $(iii)$ for $h \in L^2(\Omega)$ it is enough to rely in the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ taking into account (26). □

**End of the proof of Lemma 3.2 and Theorem 3.1**
For $\varphi \in \mathcal{D}(0,T)$, $v \in H_0^1(\Omega)$ we have by (16)

$$\begin{aligned}
-\int_0^T (u_\varepsilon, v)\varphi'\,dt &+ \int_0^T\int_\Omega \varepsilon A_{11}(z, l(u_\varepsilon))\,\varepsilon\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_1}v\,\varphi\,dt\,dx \\
&+ \int_0^T\int_\Omega A_{12}(z, l(u_\varepsilon))\,\varepsilon\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_1}v\,\varphi\,dt\,dx \\
&+ \int_0^T\int_\Omega A_{21}(z, l(u_\varepsilon))\,\varepsilon\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_2}v\,\varphi\,dt\,dx \\
&+ \int_0^T\int_\Omega A_{22}(z, l(u_\varepsilon))\,\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_2}v\,\varphi\,dt\,dx = \int_0^T\int_\Omega fv\varphi\,dt\,dx.
\end{aligned} \tag{37}$$

Noting $(iii)$ of Lemma 3.3 we have -up to a subsequence-

$$A(z, l(u_\varepsilon)) \to A(z, l(u_0)) \quad \text{a.e. } z \in Q, \tag{38}$$

i.e. we have this convergence for all coefficients of $A$. It follows that for $v \in H_0^1(\Omega)$

$$a_{ij}(\cdot, l(u_\varepsilon))\,\partial_{x_k}v \to a_{ij}(\cdot, l(u_0))\,\partial_{x_k}v \quad \text{in } L^2(Q), \qquad \forall\, i,j,k = 1,\dots,n.$$

Then by (28), (29) one can easily pass to the limit in (37) to get

$$-\int_0^T (u_0, v)\varphi'\,dt + \int_0^T\int_\Omega A_{22}(z, l(u_0))\,\nabla_{X_2}u_0\cdot\nabla_{X_2}v\,\varphi\,dt\,dx = \int_0^T\int_\Omega fv\varphi\,dt\,dx. \tag{39}$$

Using an argument as in [3] or [5] one can easily conclude that

$$\begin{aligned}
\int_0^T (u_0, v)_{\Omega_{X_1}}\varphi'\,dt &+ \int_0^T\int_{\Omega_{X_1}} A_{22}(t, X_1, X_2, l(u_0))\,\nabla_{X_2}u_0\cdot\nabla_{X_2}v\,\varphi\,dt\,dx \\
&= \int_0^T\int_{\Omega_{X_1}} f(t, X_1, X_2)\,v\varphi\,dt\,dx \\
\forall\varphi \in \mathcal{D}(0,T), &\quad \forall\, v \in H_0^1(\Omega_{X_1}), \quad \text{a.e.} \quad X_1 \in \Pi_1(\Omega),
\end{aligned} \tag{40}$$

which is the last equation in (22). (We denoted by $(\cdot,\cdot)_{\Omega_{X_1}}$ the scalar product in $L^2(\Omega_{X_1})$). Clearly (40) implies that for a.e. $X_1 \in \Pi_1(\Omega)$

$$\partial_t u_0 = \nabla_{X_2}\left(A_{22}(\cdot, l(u_0))\,\nabla_{X_2}u_0\right) + f \quad \text{in } \mathcal{D}'((0,T)\times\Omega_{X_1}) \tag{41}$$

and since $u_0 \in L^2\left((0,T)\,;H_0^1\left(\Omega_{X_1}\right)\right)$ we derive

$$\partial_t u_0 \in L^2\left(0,T;H^{-1}\left(\Omega_{X_1}\right)\right) \text{ for a.e. } X_1 \in \Pi_1(\Omega).$$

Since for a.e. $X_1 \in \Pi_1(\Omega)$

$$u_0 \in L^2\left(0,T;L^2\left(\Omega_{X_1}\right)\right) \tag{42}$$

by the usual embedding theorem the two first relations of (22) follow.

For $v \in H_0^1\left(\Omega\right)$ we have when $\varepsilon \to 0$

$$\left(u^0,v\right) = \left(u_\varepsilon\left(0,\cdot\right),v\right) = \int_\Omega u_\varepsilon\left(0,\cdot\right) v dx \to \int_\Omega u_0\left(0,\cdot\right) v dx$$

by $(i)$ of Lemma 3.3. Thus

$$\left(u^0,v\right) = \left(u_0\left(0,\cdot\right),v\right) \quad \forall v \in H_0^1\left(\Omega\right). \tag{43}$$

By density of $H_0^1\left(\Omega\right)$ in $L^2\left(\Omega\right)$, (43) holds for every $v \in L^2\left(\Omega\right)$ and we get

$$u_0\left(0,\cdot\right) = u^0.$$

This completes the proof of Lemma 3.2. □

Of course the proof of Theorem 3.1 is also completed.

□

## 4. Additional results

We now give additional properties of the solutions to (22). We have indeed

**Theorem 4.1:** *Let $u_0$ be solution to* (22)*, then*

$$u_0 \in V = \left\{v \in L^2\left(Q\right) \mid \left|\nabla_{X_2} v\right| \in L^2\left(Q\right), \partial_t v \in L^2\left(0,T;H^{-1}\left(\Omega\right)\right)\right\}, \tag{44}$$

$$u_0 \in L^\infty\left(0,T;L^2\left(\Omega\right)\right). \tag{45}$$

**Proof:** From (22) we have for a.e. $X_1 \in \Pi_1\left(\Omega\right), \, v \in H_0^1\left(\Omega_{\Omega_{X_1}}\right)$

$$\langle \partial_t u_0, v\rangle_{\Omega_{X_1}} + \int_{\Omega_{X_1}} A_{22}\left(z,l\left(u_0\right)\right)\nabla_{X_2} u_0 \cdot \nabla_{X_2} v\, dX_2 = \int_{\Omega_{X_1}} f v dX_2.$$

($\langle\cdot,\cdot\rangle_{\Omega_{X_1}}$ denotes the duality bracket between $H^{-1}(\Omega_{X_1})$ and $H_0^1(\Omega_{X_1})$). Choosing $v = u_0$ we deduce

$$\frac{1}{2}\frac{d}{dt}\left|u_0\right|^2_{2,\Omega_{X_1}} + \int_{\Omega_{X_1}} A_{22}\left(z,l\left(u_0\right)\right)\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dX_2 = \int_{\Omega_{X_1}} f u_0\, dX_2.$$

Integrating on $(0,t_0)\times\Pi_1(\Omega)$ we get

$$\frac{1}{2}\left|u_0\right|_{2,\Omega}^2(t_0)+\int_0^{t_0}\int_\Omega A_{22}\left(z,l\left(u_0\right)\right)\nabla_{X_2}u_0\cdot\nabla_{X_2}u_0dx$$
$$=\frac{1}{2}\left|u^0\right|_{2,\Omega}^2+\int_0^{t_0}\int_\Omega fu_0\,dtdx. \tag{46}$$

Choosing $t_0=T$ and using (12) we obtain

$$\lambda|\,|\nabla_{X_2}u_0|\,|_{2,Q}^2\le\frac{1}{2}\left|u^0\right|_{2,\Omega}^2+|f|_{2,Q}\left|u_0\right|_{2,Q}.$$

Applying the Poincaré inequality on each section $\Omega_{X_1}$ we obtain for a constant $C$

$$|u_0|_{2,Q}^2\le C|\,|\nabla_{X_2}u_0|\,|_{2,Q}^2$$

and thus by Young's inequality

$$\lambda|\,|\nabla_{X_2}u_0|\,|_{2,Q}^2\le\frac{1}{2}\left|u^0\right|_{2,\Omega}^2+\frac{C^2}{2\lambda}|f|_{2,Q}^2+\frac{\lambda}{2}|\,|\nabla_{X_2}u_0|\,|_{2,Q}^2.$$

From this it follows that

$$|\,|\nabla_{X_2}u_0|\,|_{2,Q}^2\le\frac{1}{\lambda}\left|u^0\right|_{2,\Omega}^2+\frac{C^2}{\lambda^2}|f|_{2,Q}^2.$$

On the other hand integrating the equation in (22) on $\Pi_1(\Omega)$ we have for $v\in H_0^1(\Omega)$

$$\langle\partial_t u_0,v\rangle=\frac{d}{dt}\left(u_0,v\right)=\int_{\Pi_1(\Omega)}\partial_t\left(u_0,v\right)_{\mathcal{B}}dX_1=\int_{\Pi_1(\Omega)}\partial_t\left(u_0,v\right)_{\Omega_{X_1}}dX_1$$
$$=\int_\Omega fv\,dx-\int_\Omega A_{22}\left(\cdot,l\left(u_0\right)\right)\nabla_{X_2}u_0\cdot\nabla_{X_2}v\,dx.$$

where $\mathcal{B}$ is defined by (30) and $u_0$, $v$ are supposed to be extended by 0 outside $\Omega$. Then it follows that

$$|\langle\partial_t u_0,v\rangle|\le|(f,v)|+\left|\int_\Omega A_{22}\left(\cdot,l\left(u_0\right)\right)\nabla u_0\cdot\nabla v\,dx\right|$$
$$\le C\left(\left|f\left(t,X_1,\cdot\right)\right|_{2,\Omega}+|\,|\nabla_{X_2}u_0|\,|_{2,\Omega}\right)|v|_{H^1(\Omega)},$$

where $C$ is a positive constant. Then after integrating on $(0,T)$ we get

$$|\partial_t u_0|_{L^2(0,T;H^{-1}(\Omega))}\le C\left(|f|_{2,Q}+|\,|\nabla_{X_2}u_0|\,|_{2,Q}\right).$$

Thus (44) follows and (45) is then a consequence of (46). □

In the case where the solution of (22) is unique we can yet prove

**Theorem 4.2:** *Suppose that* (22) *has a unique solution then we have*

$$u_\varepsilon\longrightarrow u_0\quad in\ V \tag{47}$$

*i.e.*

$$u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \quad \text{in } L^2(Q), \quad \partial_t u_\varepsilon \longrightarrow \partial_t u_0 \quad \text{in } L^2\left(0,T;H^{-1}(\Omega)\right)$$

*and*

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(Q). \tag{48}$$

*(The convergences hold for the whole sequence).*

**Proof:** We introduce

$$I_\varepsilon = \int_Q A_\varepsilon\left(z, l\left(u_\varepsilon\right)\right) \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}\left(u_\varepsilon - u_0\right) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}\left(u_\varepsilon - u_0\right) \end{pmatrix} dtdx.$$

In what follows we just denote by $A_\varepsilon$ the matrix $A_\varepsilon\left(z, l\left(u_\varepsilon\right)\right)$ and drop the measures of integration to get

$$\begin{aligned}
I_\varepsilon &= \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} - \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla u_\varepsilon \\
&+ \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \\
&= \int_Q f u_\varepsilon - \int_0^T \langle \partial_t u_\varepsilon, u_\varepsilon \rangle - \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} - \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla u_\varepsilon \\
&+ \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \\
&:= \int_Q f u_\varepsilon - \int_0^T \langle \partial_t u_\varepsilon, u_\varepsilon \rangle dt + J_\varepsilon.
\end{aligned} \tag{49}$$

(We used (16) with $v = u_\varepsilon$). Remark now that

$$\begin{aligned}
\int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} &= \int_Q A_{21}\left(z, l\left(u_\varepsilon\right)\right) \varepsilon \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} u_0 \\
&+ \int_Q A_{22}\left(z, l\left(u_\varepsilon\right)\right) \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0.
\end{aligned}$$

Passing to the limit as we did in (37) we obtain that

$$\lim_{\varepsilon \to 0} \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} = \int_Q A_{22}\left(z, l\left(u_0\right)\right) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0.$$

Note that since the possible limit is unique this is not up to a subsequence. We obtain similarly

$$\lim_{\varepsilon \to 0} \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla u_\varepsilon = \int_Q A_{22}\left(z, l\left(u_0\right)\right) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0,$$

$$\lim_{\varepsilon \to 0} \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} = \int_Q A_{22}\left(z, l\left(u_0\right)\right) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0$$

and thus combining these three passages to the limit

$$\lim_{\varepsilon\to 0} J_\varepsilon = -\int_Q A_{22}\left(z, l\left(u_0\right)\right)\nabla_{X_2}u_0\cdot\nabla_{X_2}u_0. \tag{50}$$

We now consider the other terms in $I_\varepsilon$ -see (49)- in order to pass to the limit. Let $\theta_k \in \mathcal{D}\left(\Omega\right)$ be a sequence of smooth functions independent of $t$ that we will choose later on. We have

$$\begin{aligned}
&\int_0^T \langle\partial_t u_\varepsilon, u_\varepsilon\rangle dt \\
&= \int_0^T \langle\partial_t\left(u_\varepsilon-\theta_k\right), u_\varepsilon-\theta_k\rangle dt + \int_0^T \langle\partial_t\theta_k, u_\varepsilon-\theta_k\rangle dt + \int_0^T\langle\partial_t u_\varepsilon,\theta_k\rangle dt \\
&= \frac{1}{2}\left|u_\varepsilon-\theta_k\right|^2_{2,\Omega}(T) - \frac{1}{2}\left|u^0-\theta_k\right|^2_{2,\Omega} + \int_0^T\langle\partial_t u_\varepsilon,\theta_k\rangle dt.
\end{aligned}$$

Going back to (49) we have

$$R_\varepsilon := I_\varepsilon + \frac{1}{2}\left|u_\varepsilon-\theta_k\right|^2_{2,\Omega}(T) = \frac{1}{2}\left|u^0-\theta_k\right|^2_{2,\Omega} + \int_Q fu_\varepsilon - \int_0^T\langle\partial_t u_\varepsilon,\theta_k\rangle dt + J_\varepsilon. \tag{51}$$

Using (22), (28), (29), (50) we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} R_\varepsilon \\
&= \frac{1}{2}\left|u^0-\theta_k\right|^2_{2,\Omega} + \int_Q fu_0 - \int_Q A_{22}\left(z,l\left(u_0\right)\right)\nabla_{X_2}u_0\cdot\nabla_{X_2}u_0 - \int_0^T\langle\partial_t u_0,\theta_k\rangle dt \\
&= \frac{1}{2}\left|u^0-\theta_k\right|^2_{2,\Omega} + \int_0^T\int_{\Pi_1(\Omega)}\langle\partial_t u_0, u_0\rangle_{\Omega_{X_1}}\left(t, X_1\right)dt - \int_0^T\langle\partial_t u_0,\theta_k\rangle dt.
\end{aligned}$$

Note that since we have

$$\int_0^T\langle\partial_t u_0,\theta_k\rangle dt = \int_0^T\int_{\Pi_1(\Omega)}\langle\partial_t u_0,\theta_k\rangle_{\Omega_{X_1}}\left(t,X_1\right)dtdX_1$$

the limit above becomes

$$\begin{aligned}
\lim_{\varepsilon\to 0} R_\varepsilon &= \frac{1}{2}\left|u^0-\theta_k\right|^2_{2,\Omega} + \int_{\Pi_1(\Omega)}\int_0^T\langle\partial_t\left(u_0-\theta_k\right),\left(u_0-\theta_k\right)\rangle_{\Omega_{X_1}}\left(t,X_1\right)dtdX_1 \\
&= \frac{1}{2}\left|u^0-\theta_k\right|^2_{2,\Omega} + \frac{1}{2}\int_{\Pi_1(\Omega)}\left|u_0-\theta_k\right|^2_{2,\Omega_{X_1}}\left(T,X_1\right) - \left|u^0-\theta_k\right|^2_{2,\Omega_{X_1}}\left(X_1\right)dtdX_1 \\
&= \frac{1}{2}\left|u_0-\theta_k\right|^2_{2,\Omega}(T).
\end{aligned} \tag{52}$$

Thus from (51) we deduce

$$\limsup_{\varepsilon\to 0} I_\varepsilon \leq \lim_{\varepsilon\to 0} R_\varepsilon = \frac{1}{2}\left|u_0-\theta_k\right|^2_{2,\Omega}(T).$$

Choosing $\theta_k$ such that

$$\theta_k \to u_0\left(T\right) \quad \text{in } L^2\left(\Omega\right), \tag{53}$$

we deduce that

$$0 \leq \limsup_{\varepsilon \to 0} I_\varepsilon \leq 0$$

and thus

$$I_\varepsilon \to 0 \quad \text{when } \varepsilon \to 0.$$

Since, by the coerciveness assumption, we have

$$\lambda \int_0^T \int_\Omega |\varepsilon \nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \leq I_\varepsilon,$$

it follows that

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0, \qquad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \quad \text{in } L^2(Q),$$

and thus (by Poincaré's inequality)

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } L^2(Q).$$

It remains to show that

$$\partial_t u_\varepsilon \longrightarrow \partial_t u_0 \quad \text{in } L^2\left(0, T; H^{-1}\left(\Omega\right)\right).$$

Testing the equation in (22) with $v \in H_0^1\left(\Omega\right)$ and integrating on $\Pi_1\left(\Omega\right)$, we get

$$\int_{\Pi_1(\Omega)} \partial_t \left(u_0, v\right)_{\Omega_{X_1}} dX_1 + \int_\Omega A_{22}\left(z, l\left(u_0\right)\right) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dx = \int_\Omega f v \, dx,$$

whence

$$\frac{d}{dt}\left(u_0, v\right)_\Omega + \int_\Omega A_{22}\left(z, l\left(u_0\right)\right) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dx = \int_\Omega f v \, dx.$$

Then subtracting the above equality from (16) leads to

$$\begin{aligned}
\frac{d}{dt}\left(u_\varepsilon - u_0, v\right) &+ \int_\Omega \varepsilon^2 A_{11}\left(z, l\left(u_\varepsilon\right)\right) \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v \, dx \\
&+ \int_\Omega \varepsilon A_{12}\left(z, l\left(u_\varepsilon\right)\right) \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v \, dx \\
&+ \int_\Omega \varepsilon A_{21}\left(z, l\left(u_\varepsilon\right)\right) \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v \, dx \\
&+ \int_\Omega \left(A_{22}\left(z, l\left(u_\varepsilon\right)\right) \nabla_{X_2} u_\varepsilon - A_{22}\left(z, l\left(u_0\right)\right) \nabla_{X_2} u_0\right) \cdot \nabla_{X_2} v \, dx \\
&= 0.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and by (44), it follows that

$$\begin{aligned}\langle \partial_t (u_\varepsilon - u_0), v\rangle &\le \varepsilon^2 C |\,|\nabla_{X_1} u_\varepsilon|\,|_{2,\Omega}\,|\,|\nabla_{X_1} v|\,|_{2,\Omega} \\ &+ \varepsilon C |\,|\nabla_{X_2} u_\varepsilon|\,|_{2,\Omega}\,|\,|\nabla_{X_1} v|\,|_{2,\Omega} + \varepsilon C |\,|\nabla_{X_1} u_\varepsilon|\,|_{2,\Omega}|\,|\nabla_{X_2} v|\,|_{2,\Omega} \\ &+ |\,|A_{22}(\cdot, l(u_\varepsilon))\nabla_{X_2} u_\varepsilon - A_{22}(\cdot, l(u_0))\nabla_{X_2} u_0|\,|_{2,\Omega}|\,|\nabla_{X_1} v|\,|_{2,\Omega} \\ &\le \left(C\left(\varepsilon^2 |\,|\nabla_{X_1} u_\varepsilon|\,|_{2,\Omega} + \varepsilon |\,|\nabla_{X_2} u_\varepsilon|\,|_{2,\Omega} + \varepsilon |\,|\nabla_{X_1} u_\varepsilon|\,|_{2,\Omega}\right)\right. \\ &+ \left.|\,|A_{22}(\cdot, l(u_\varepsilon))\nabla_{X_2} u_\varepsilon - A_{22}(\cdot, l(u_0))\nabla_{X_2} u_0|\,|_{2,\Omega}\right)|v|_{H^1(\Omega)},\end{aligned}$$

for any $v \in H_0^1(\Omega)$. This implies

$$\begin{aligned}|\partial_t (u_\varepsilon - u_0)|_{H^{-1}(\Omega)} \le & C\left(\varepsilon^2 |\,|\nabla_{X_1} u_\varepsilon|\,|_{2,\Omega} + \varepsilon |\,|\nabla_{X_2} u_\varepsilon|\,|_{2,\Omega} + \varepsilon |\,|\nabla_{X_1} u_\varepsilon|\,|_{2,\Omega}\right) \\ & + |\,|A_{22}(\cdot, l(u_\varepsilon))\nabla_{X_2} u_\varepsilon - A_{22}(\cdot, l(u_0))\nabla_{X_2} u_0|\,|_{2,\Omega}.\end{aligned}$$

Integrating over $(0, T)$, letting $\varepsilon \to 0$ and taking into account $(iii)$ of Lemma 3.3, we deduce

$$|\partial_t (u_\varepsilon - u_0)|_{L^2(0,T;H^{-1}(\Omega))} \to 0.$$

This completes the proof of the theorem. □

**Remark 1 :** We also have

$$u_\varepsilon(t, \cdot) \longrightarrow u_0(t, \cdot) \quad \text{in } L^2(\Omega) \quad \forall t \in (0, T). \tag{54}$$

Indeed, if we replace $T$ by $t_0 \in (0, T)$ in the proof of Theorem 4.2 we get from (51), (52),

$$\limsup_{\varepsilon \to 0} |\theta_k - u_\varepsilon(t_0, \cdot)|_{2,\Omega} \le |\theta_k - u_0(t_0, \cdot)|_{2,\Omega},$$

thus

$$\begin{aligned}\limsup_{\varepsilon \to 0} |u_\varepsilon(t_0, \cdot) - u_0(t_0, \cdot)|_{2,\Omega} &\le |\theta_k - u_0(t_0, \cdot)|_{2,\Omega} + \limsup_{\varepsilon \to 0} |\theta_k - u_\varepsilon(t_0, \cdot)|_{2,\Omega} \\ &\le 2\,|\theta_k - u_0(t_0, \cdot)|_{2,\Omega}.\end{aligned} \tag{55}$$

Then passing to the limit in the inequality above when $k \to \infty$ and taking into account (53) for $T = t_0$, we obtain

$$\limsup_{\varepsilon \to 0} |u_\varepsilon(t_0, \cdot) - u_0(t_0, \cdot)|_{2,\Omega} \le 0,$$

whence

$$\lim_{\varepsilon \to 0} |u_\varepsilon(t_0, \cdot) - u_0(t_0, \cdot)|_{2,\Omega} = 0.$$

It follows that

$$u_\varepsilon(t, \cdot) \longrightarrow u_0(t, \cdot) \quad \text{in } L^2(\Omega) \;\; \forall t \in (0, T).$$

**Remark 2 :** In the case when $u_0$ is not unique one can show that the set of solutions to (22) lies in a neighborhood of

$$\{u_\varepsilon \mid \varepsilon < \varepsilon_0\}$$

for the topology of $V$, that is to say from any subsequence of $u_\varepsilon$ there exists another subsequence such that (47), (48) hold. In addition if there exists a subsequence of $u_\varepsilon$, still labeled by $u_\varepsilon$, and a function $u_0$ such that

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } \mathcal{D}'(Q),$$

then $u_0$ is a solution of (22) and (47), (48) hold.

***Acknowledgements***

The authors have been supported by the Swiss National Science Foundation under the contracts #20 − 113287/1 and #20 − 117614/1. They are very grateful to this institution.

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