The Pole Placement Problem via PI Feedback Controllers.

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Abstract

In this paper we study the pole placement problem using generalized PI controllers of a fixed lag k as compensators. We derive a new strong sufficiency condition which guarantees the arbitrary pole assignability of a given system having m inputs p outputs and McMillan degree n. This sufficiency condition misses the theoretical best possible necessary condition by one degree of freedom. The proof of the main result can be used to derive a numerical procedure. This numerical procedure is however very sensitive in the parameters and further research is required to investigate the conditions under which sensitivity occurs and whether the solution can be made more robust in this regard.

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1 Introduction

Consider an m-input, p-output linear system of McMillan degree n represented through the well known input-output representation

$$y = G(s)u = C(sI_n - A)^{-1}Bu. \tag{1.1}$$

The pole placement problem asks for conditions which guarantee (under certain restrictions) the existence of a feedback compensator

$$u = -K(s)y + v \tag{1.2}$$

such that the poles of the closed loop transfer function

$$G_K(s) := G(s)(I_m + K(s)G(s))^{-1} = (I_p + G(s)K(s))^{-1}G(s) \tag{1.3}$$

have some specific value.

Probably the best studied of all pole placement problems is the so called static pole placement problem where compensators are required to have McMillan degree 0, i.e., one requires that the transfer matrix $K(s)$ is a constant matrix. A simple parameter count then shows that at most mp poles can be placed freely via static compensators, in other words $mp \geq n$ is a necessary condition to guarantee arbitrary pole placement. In particular if the McMillan degree of the plant $G(s)$ is strictly larger than mp then arbitrary pole placement is not possible.

A natural generalization of the static problem is the dynamic pole placement problem which involves (proper) compensators $K(s)$ of McMillan degree q. The closed loop transfer function $G_K(s)$ is in this case a proper transfer function of McMillan degree $n+q$ and the problem therefore requires that $n+q$ poles should be placed at a desired location. Doing again a simple parameter count (Willems & Hesselink 1978) derived the necessary condition

$$q(m+p) + mp \geq n + q \tag{1.4}$$

which reduces to the static condition $mp \geq n$ if $q = 0$.

This paper deals with a pole placement problem where we require that the compensator $K(s)$ represents a PI (Proportional Integral) controller of a fixed lag k. We will treat the most general case of proportional and multiple integral controllers. The main result is formulated in Theorem 2.5 and it provides a new strong sufficiency criterion.

Similar to the the static and the dynamic pole placement problem we are also confronted with a highly nonlinear set of polynomial equations. The fact that the static and the dynamic pole placement problem are actually nonlinear problems was recognized in the seventies by several authors (Brockett & Byrnes 1981, Hermann & Martin 1977, Willems & Hesselink 1978) and this explains in part why only slow progress has been achieved over time in the understanding of these problems. In the sequel we review some of the progress:

A first breakthrough for the static problem was the paper (Brockett & Byrnes 1981). In this paper the authors show that the static pole placement problem is essentially equivalent to a well studied geometric problem in Schubert calculus. Using this connection and using some results from algebraic geometry they were able to show that for a 'generic' set of m-input, p-output linear systems of McMillan degree n arbitrary pole placement is possible if the necessary condition $mp \geq n$ is satisfied. Recall: A nonempty subset $\mathcal{S}$ of some vector space V is called a generic set if the complement of $\mathcal{S}$ in V is contained in the zero locus of a finite set of polynomials. A set of $p \times m$ strictly proper transfer functions of McMillan degree n is called a generic set if the set of all triples of matrices (A, B, C) representing those transfer functions forms a generic set. Note that a generic set $\mathcal{S}$ is automatically dense in V and that the complement of $\mathcal{S}$ in V has measure zero.

The algebro-geometric techniques of (Brockett & Byrnes 1981) were recently extended in (Rosenthal 1994) to the general dynamic problem and it has been shown in (Rosenthal 1994) that the necessary condition (1.4) is again a sufficient condition for the generic system.

Although the results in (Brockett & Byrnes 1981, Rosenthal 1994) were very satisfying in many ways they had the major drawback that compensators defined over the complex numbers $\mathbb{C}$ were required to achieve arbitrary pole placement. Even worse it was pointed out in (Willems & Hesselink 1978) that for the 'real problem', i.e., the problem where compensators are required to be defined over the reals $\mathbb{R}$, the condition $mp \geq n$ may not be sufficient.

For the static problem over $\mathbb{R}$ the major breakthrough was established in (Wang 1992). In this paper Wang showed that $mp > n$ turns out to be a sufficient condition for arbitrary pole placement for a generic set of systems.

The new techniques used in (Wang 1992) were geometric in nature and the arguments in the proof were still quite involved. Seemingly independently it was recognized in (Ariki 1998), (Leventides 1993, Leventides & Karcanias 1995) and (Wang 1996) that the techniques used in (Wang 1992) are mainly based on an elementary linearization technique. This fact did generate a sudden renewed interest in the pole placement problem and several papers were written since that time covering both the static (Leventides & Karcanias 1995, Rosenthal, Schumacher & Willems 1995, Rosenthal & Sottile 1998, Wang 1996) and the dynamic problem (Ariki 1998, Leventides & Karcanias 1996, Rosenthal & Wang 1996).

The paper is structured as follows: In Section 2 we formulate the problem and we provide the main results. The main result (Theorem 2.5) is a very strong sufficiency result for arbitrary pole placement using generalized PI controllers of lag k. The result essentially states that up to $(k+1)mp-1$ closed loop poles can be arbitrarily placed. Since we know from a dimension count that at most $(k+1)mp$ poles can be placed by PI controllers of lag k the difference to the sufficiency criterion given in Theorem 2.5 is at most one degree of freedom. We would like to mention that for $k = 0$ our result specializes to Wang's sufficiency result (Wang 1992) and for $k = 1$ the result is vastly superior to one published in (Novin-Hirbod 1979). (Compare also with the survey article (Munro 1990)).

In Section 3 we homogenize the pole placement map and we provide some general rank conditions which guarantee that the pole placement map is surjective.

In Section 4 we explain the algebraic relation between first order representations and polynomial representations. The section serves as a preliminary for Section 5 where we provide a detailed proof of the main theorem. The proof in Section 5 follows some ideas developed in (Ariki 1998, Leventides & Karcanias 1995, Rosenthal et al. 1995, Wang 1996). The use of Lemma 5.1 is a new idea and it is of independent interest for proving other pole placement results.

Finally in sections 6 and 7 we explain how the proof of the main theorem allows one in principle to compute the closed loop compensator numerically. This numerical scheme is based on a technique called *linearization around a dependent compensator* (see Section 3 for details). Geometrically the compensators computed in this way are near the *base locus* (for this see e.g. (Byrnes 1989)). Since for any desired closed loop characteristic polynomial one finds a solution in any neighborhood of the base locus it follows that the presented numerical procedure can be highly sensitive to the parameter values and even very small numerical inaccuracies can lead to problems. This sensitivity was also observed by one of the referees when he simulated the example of Section 7 and we would like to thank him for doing these simulations.

In an upcoming book on open problems in systems theory the second author together with Willems (see (Rosenthal & Willems 1998)) did pose it as an important open problem to design new numerical methods which will result in the computation of feedback compensators which are less sensitive to parameter uncertainties.

2 Main Results

On the side of static feedback controllers probably the simplest and most widely implemented controllers are PI (Proportional Integral) controllers. By definition a PI controller is a transfer function

$$K(s) := F_0 + \frac{1}{s}F_1$$

where F_0, F_1 are $m \times p$ constant matrices. It is well known that these controllers are used when the plant is exposed to stepwise disturbances or command signals, in order to ensure asymptotic regulation. Thus by stabilizing the system using first order PI controllers we also ensure asymptotic rejection of step disturbances.

In this paper we will study the pole placement problem using generalized PI controllers of the form

$$K(s) := F_0 + \frac{1}{s}F_1 + \cdots + \frac{1}{s^k}F_k \tag{2.1}$$

and we say $K(s)$ is a *PI controller of lag k*. These higher order PI controller also reject ramps ($\frac{1}{s^2}$) or higher order ($\frac{1}{s^k}$) disturbances. In the time domain such a PI controller is given through the integral representation

$$u(t) = F_0 y(t) + F_1 \int_{t_0}^{t} y(r_1)\ dr_1 + \cdots + F_k \int_{t_0}^{t}\int_{t_0}^{r_1} \cdots \int_{t_0}^{r_{k-1}} y(r_k)\ dr_k \cdots dr_2\ dr_1. \tag{2.2}$$

The lag k represents then in an obvious way the number of integrators needed to compute the feedback law $u(t)$.

Lemma 2.1 *Let $K(s)$ be a PI controller of lag k as introduced in* (2.1). *If F_k has full rank (i.e.,* $\operatorname{rank} F_k = \min(m,p)$*) then $K(s)$ is a transfer function of McMillan degree*

$$\deg K(s) = k \min(m,p).$$

Proof: First assume that $m \leq p$ and let

$$F(s) := s^k F_0 + s^{k-1}F_1 + \cdots + sF_{k-1} + F_k.$$

One readily verifies that $K(s) = (s^k I_m)^{-1} F(s)$ is a left coprime factorization of $K(s)$ and that the McMillan degree is therefore equal to the highest degree of the $m \times m$ minors of the polynomial matrix $\left[s^k I_m \; F(s) \right]$. This degree is equal to km.

Similarly if $m > p$ one verifies that $K(s) = F(s)(s^k I_p)^{-1}$ represents a right coprime factorization and the McMillan degree is therefore in this situation equal to kp. $\square$

The problem we are studying in this paper is now as follows: Let $G(s)$ be a plant of McMillan degree n. Find conditions which guarantee that $G(s)$ is *arbitrarily pole assignable using PI controllers of lag* k, i.e., find conditions which guarantee that for every polynomial $f(s) \in \mathbb{R}[s]$ of degree $n + k\min(m,p)$ there exists a PI controller $K(s)$ of lag k such that the closed loop transfer function $G_K(s) = G(s)(I_m + K(s)G(s))^{-1}$ has characteristic polynomial $f(s)$.

The problem under consideration turns out to be highly nonlinear. Indeed if $k = 0$ the problem reduces to the static output pole placement problem and this problem has been considered quite intractable until recently. For $k = 1$ the problem was probably first studied in (Novin-Hirbod 1979). We would also like to remark that the presented control scheme is different from the Non-Minimal State Space (NMSS) and Proportional Integral Plus (PIP) approach of Young et al. (for this see the recent articles (Chotai, Young, McKenna & Tych 1998, Young, Chotai, McKenna & Tych 1998) and the references therein). By extending the state space to include additional state variables derived from the measured output and input variables Young et al. are lead to a linear design problem which is simpler to handle than the (nonlinear) problem studied in this paper. In particular the problem studied in this paper cannot be transformed into a state variable feedback problem which in turn would be a linear problem. In Remark 3.2 we will however explain that the problem can be reformulated in terms of a specific (ungeneric!) static output pole placement problem. (See Remark 3.3 for more details).

A straight forward parameter count gives the following necessary condition:

Lemma 2.2 *If $G(s)$ has McMillan degree n then it is arbitrarily pole assignable using PI controllers of lag k only if*

$$(k+1)mp \geq k\min(m,p) + n. \tag{2.3}$$

As mentioned earlier a solution for this problem requires in general the solution of a large set of higher degree polynomial equations. There is however one exception: In the MISO as well as in the SIMO situation, the problem under consideration is a linear one. In these situations one has:

Lemma 2.3 *If $\min(m,p) = 1$, i.e. in the MISO and in the SIMO situation, condition (2.3) is not only necessary but also sufficient for arbitrary pole assignment of a generic system using PI controllers of lag k.*

The proof of this result will be given in Remark 3.2.

Remark 2.4 If $m = p = 1$, i.e. in the SISO situation Lemma 2.3 states that arbitrary pole placement with PI controllers of lag k is not possible unless the McMillan degree of the plant is $n \leq 1$. This is due to the fact that in the SISO situation the dimension of the set of PI controllers of lag k is $k + 1$ whereas $n + k$ different closed loop poles have to be assigned.

The main result of this paper is now as follows:

Theorem 2.5 *If*

$$(k+1)mp > k\min(m,p) + n \tag{2.4}$$

then a generic set of $p \times m$ transfer functions $G(s)$ of McMillan degree n is arbitrarily pole assignable using PI controllers of lag k.

The proof of this theorem will occupy a large part of this paper and it will be completed in Section 5. Note that if $k = 0$ the theorem reduces to Wang's sufficiency result (Wang 1992). We would also like to mention that in many cases the sufficiency condition (2.4) is also necessary. One such situation occurs when $k = 0$, $m = p = 2$ and we refer to (Willems & Hesselink 1978) for details. Another situation was given in (Rosenthal & Sottile 1998) and it occurs if $k = 0$, $\min(m,p) = 2$ and $\max(m,p) = 4$. We conclude the section with an illustrative example:

Example 2.6 Assume $m = p = 10$ i.e. the plant $G(s)$ has 10 inputs and 10 outputs and assume that $k = 1$, i.e. the controller is a simple PI controller of the form $K(s) := K_0 + 1/sF_1$. The best sufficiency result as given in (Novin-Hirbod 1979) guarantees that a system of up to McMillan degree 19 can be arbitrarily pole assigned. According to Theorem 2.5 a system of up to McMillan degree 189 can be arbitrarily pole assigned and Lemma 2.2 states that the McMillan degree should not be more than 190 in order to guarantee arbitrary pole assignability.

3 Linearization around a Dependent Compensator

In this section we will introduce the *pole placement map* associated to the pole placement problem using PI controllers of lag k. We will also derive some specific sufficiency conditions for a plant $G(s)$ which guarantee the arbitrary pole assignability using PI controllers of lag k.

As a preliminary we compute the characteristic polynomial of the closed loop transfer function $G_K(s)$ and we will show that there is no restriction to assume that $m \leq p$.

For this assume first that $m \leq p$ and assume that the compensator $K(s) = (s^k I_m)^{-1} F(s)$ has McMillan degree km, i.e., $\operatorname{rank} F_k = \min(m,p)$. Let $G(s) = N_r(s) D_r^{-1}(s)$ be a right coprime factorization of the plant. Then the closed loop transfer function is

$$G_K(s) = N_r(s)\big(s^k I_m D_r(s) + F(s) N_r(s)\big)^{-1}(s^k I_m).$$

If no pole zero cancellation did occur the closed loop characteristic polynomial is equal to

$$\det[s^k I_m \;\; F(s)] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}. \tag{3.1}$$

Next assume that $m > p$ and assume that $K(s) = F(s)(s^k I_p)^{-1}$ is a right coprime factorization of the compensator. Let $G(s) = D_l^{-1}(s) N_l(s)$ be a left coprime factorization of the plant. Using this notation the closed loop transfer function can be written as

$$G_K(s) = (s^k I_p)\big(D_l(s) s^k I_p + N_l(s) F(s)\big)^{-1} N_l(s)$$

and the closed loop characteristic polynomial can be written as

$$\det[s^k I_p \;\; F^t(s)] \begin{bmatrix} D_l^t(s) \\ N_l^t(s) \end{bmatrix}. \tag{3.2}$$

Since (3.2) is formally identical with (3.1) we will assume in all further considerations that

$$m \leq p. \tag{3.3}$$

In order to simplify notation we will write $D(s)$ for $D_r(s)$ and $N(s)$ for $N_r(s)$.

Next we will introduce the pole placement map. As it was recognized already in (Brockett & Byrnes 1981) 'singular' compensators are very crucial in the study of any pole placement problem. Because of this reason we will consider compensators having the form

$$Eu = -(F_0 + \frac{1}{s}F_1 + \cdots + \frac{1}{s^k}F_k)y + v, \tag{3.4}$$

where E is a not necessarily invertible $m \times m$ matrix. Note that if E is invertible then (3.4) defines a regular compensator of the form $u = -K(s)y + v$. We will identify the set of all generalized compensators of the form (3.4) with a matrix

$$M := [E, F_0, \ldots, F_k]$$

and we will identify this matrix with a point in the vector space $\mathbb{R}^{(k+1)mp+m^2}$.

Definition 3.1 Let $\mathcal{P}^{n+km}$ denote the set of all polynomials of degree $\leq n+km$. Then the pole placement map

$$\chi : \mathbb{R}^{(k+1)mp+m^2} \longmapsto \mathcal{P}^{n+km} \tag{3.5}$$

is defined through

$$\chi(M) = \det[s^k E \;\; F(s)] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \det[E, F_0, \ldots, F_k] \begin{bmatrix} s^k D(s) \\ s^k N(s) \\ \vdots \\ N(s) \end{bmatrix}. \tag{3.6}$$

Remark 3.2 It is possible to view Equation (3.6) as the (behavioral) interconnection of the static compensator $M := [E, F_0, \ldots, F_k]$ with the system described by

$$\begin{bmatrix} s^k D(s) \\ s^k N(s) \\ \vdots \\ N(s) \end{bmatrix}. \tag{3.7}$$

In this way we can view the problem under consideration as a static output pole placement problem.

If $m = 1$ the matrix appearing in (3.7) represents a column vector with $(1 + (k+1)p)$ entries. If in addition the polynomial $s^k D(s)$ has degree $n + k = (k+1)p$ and if in addition the $(1 + (k+1)p)$ polynomials in (3.7) are linearly independent then it follows that the resultant map appearing in (3.6) is surjective. One verifies that the linear independence of these polynomials is generically satisfied. This provides the proof of Lemma 2.3 if $m = 1$. The proof for $p = 1$ goes similar.

Remark 3.3 The above arguments could led one to believe that Theorem 2.5 can simply be deduced from the results on static output pole placement as derived in (Wang 1992). This is not the case and the reason is as follows: The set of plants having the form as described in (3.7) represents a proper algebraic set (a 'non-generic' set) inside the class of all systems with m inputs, $(k+1)p$ outputs and McMillan degree $k\min(m,p)+n$. Applying Wang's result (Wang 1992) would hence require to verify that among the systems of the form (3.7) there is a plant which is part of the generic set of arbitrarily pole assignable systems as explicitly described in (Rosenthal et al. 1995). Since the polynomial equations described in (Rosenthal et al. 1995) are quite complicated this turns out to be a highly nontrivial task and we therefore decided in this paper to provide a direct proof.

If we identify a polynomial

$$f(s) = f_0 + f_1 s + \cdots + f_N s^N \in \mathbb{R}[s]$$

with the point $f = (f_0, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$ it is possible to identify the set of polynomials $\mathcal{P}^{n+km}$ with the vector space $\mathbb{R}^{n+km+1}$. In this way χ describes a polynomial map between two real vector spaces.

In the following some of the important properties of the map χ are summarized:

1) χ is homogeneous of degree m; i.e. $\chi(\lambda M) = \lambda^m \chi(M)$. Therefore if a point $f \neq 0$ belongs to the range of χ, then the whole line through f and the origin is contained in the range.

2) Although the map depends on particular factorizations of the transfer functions of system and controller, the zeros of $f(s) = \chi([E, F_0, \ldots, F_k])$ (or the line through f and the origin) are uniquely determined by the system and the controller.

3) The closed loop poles can be arbitrarily placed by controllers (3.4) if and only if χ is onto.

Following the language in (Rosenthal et al. 1995, Rosenthal & Wang 1996) we will say M is a *dependent compensator* if $\chi(M) = 0$ and we will say M is a *full compensator* if the Jacobian map $d\chi_M$ is surjective. The following result was also derived in (Ariki 1998, Leventides & Karcanias 1995, Rosenthal et al. 1995, Rosenthal & Wang 1996). For the convenience of the reader we repeat the short simple proof.

Lemma 3.4 *The pole placement map χ introduced in* (3.5) *is surjective as soon as there is a dependent and full compensator M.*

Proof: By the Inverse Function Theorem, χ maps a neighborhood of the dependent compensator M onto a neighborhood of 0 in $\mathbb{R}^{n+km+1}$. Since χ is homogeneous χ is necessarily surjective. □

We would like to remark that there exist always dependent compensators, indeed a matrix M with $\operatorname{rank} M < m$ is dependent. In general such rank deficient dependent compensators are however not full. Indeed if $\operatorname{rank} M = m-1$ then one can prove that the rank of the Jacobian $d\chi_M$ is at most $(k+1)(m+p)-m+1$ and if $\operatorname{rank} M < m-1$ then necessarily $d\chi_M$ is the zero map. (Compare with the description of $d\chi_M$ in the proof of the next theorem).

The main sufficiency result of this section is now as follows:

Theorem 3.5 *Let $M = [E, F_0, \ldots, F_k]$ and let $F(s) := s^k F_0 + s^{k-1} F_1 + \cdots + sF_{k-1} + F_k$. If M is a dependent compensator of the system $G(s)$ and if the polynomial entries of the matrix*

$$\begin{bmatrix} s^k D(s) \\ s^k N(s) \\ \vdots \\ N(s) \end{bmatrix} \operatorname{adj}\left(s^k ED(s) + F(s)N(s)\right) \tag{3.8}$$

span the real vector space $\mathcal{P}^{n+km}$ of all polynomials of degree $\leq n + km$ then there exists for every polynomial $f(s)$ of degree $n+km$ a regular PI controller $K(s) = (s^k I_m)^{-1} \tilde{F}(s)$ of lag k such that the the closed loop characteristic polynomial is

$$\det[s^k I_m \;\; \tilde{F}(s)] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = f(s).$$

Proof: By (Rosenthal & Wang 1996, Theorem 3.10) the Jacobian $d\chi_M : \mathbb{R}^{(k+1)mp+m^2} \rightarrow \mathbb{R}^{(n+km+1)}$ is given by

$$X \longmapsto \operatorname{tr}\left(X \begin{bmatrix} s^k D(s) \\ s^k N(s) \\ \vdots \\ N(s) \end{bmatrix} \operatorname{adj}\left(s^k ED(s) + F(s)N(s)\right) \right) \tag{3.9}$$

where X is an arbitrary $m \times ((k+1)p + m)$ matrix i.e., X describes an arbitrary element of the tangent space $\mathbb{R}^{(k+1)mp+m^2}$. By (Rosenthal & Wang 1996, Corollary 3.11) the Jacobian $d\chi_M$ is surjective and M is therefore a full and dependent compensator. By the last Lemma the map χ is surjective. It therefore exists a possibly singular compensator $\left[\tilde{E} \;\; \tilde{F}(s)\right]$ such that

$$\det[s^k \tilde{E} \;\; \tilde{F}(s)] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = f(s).$$

Since $G(s)$ is strictly proper and since $f(s)$ has degree $n + km$ we conclude that E must be invertible, i.e., the compensator is a regular compensator. □

Since M is a dependent compensator $\det(s^k ED(s) + F(s)N(s)) = 0$ and therefore

$$\operatorname{rank} \operatorname{adj}\left(s^k ED(s) + F(s)N(s)\right) \leq 1. \tag{3.10}$$

It therefore exist a polynomial vector $\alpha(s)$ of size $m \times 1$ and a polynomial vector $\beta(s)$ of size $1 \times m$ such that $\operatorname{adj}(s^k ED(s) + F(s)N(s)) = \alpha(s)\beta(s)$. Because of this the Jacobian $d\chi_M$ can be simplified considerably and computations become easier.

4 The associated Smooth Behavior

So far we did work in the frequency domain using polynomial representations. Since our main theorem says something about a generic set (A, B, C) of matrix triples it will be necessary to make the connection to first order systems. The reader can find more backgroung on this section in (Rosenthal & Schumacher 1997).

Associated to a matrix triple is a smooth behavior $\mathcal{B}(A,B,C)$ defined through:

$$\left\{ \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \in C^\infty(\mathbb{R};\mathbb{R}^{m+p}) \mid \exists x(t) \in C^\infty(\mathbb{R};\mathbb{R}^n) : \dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) \right\}.$$

Similarly if $D(s), N(s)$ are two matrices of size $m \times m$ and $p \times m$ respectively then there is an associated smooth behavior $\mathcal{B}(D,N)$ defined through:

$$\left\{ \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \in C^\infty(\mathbb{R};\mathbb{R}^{m+p}) \mid \exists \ell(t) \in C^\infty(\mathbb{R};\mathbb{R}^m) : \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{pmatrix} \ell(t) \right\}.$$

The following theorem provides a condition when those two behaviors are the same.

Theorem 4.1 *Assume (A,B) is a controllable matrix pair. If there exists a $n \times m$ matrix $X(s)$ such that*

$$\ker \begin{bmatrix} sI_n - A & -B & 0 \\ C & 0 & -I_p \end{bmatrix} = \operatorname{im} \begin{bmatrix} X(s) \\ D(s) \\ N(s) \end{bmatrix} \quad \forall s \in \mathbb{C} \tag{4.1}$$

then the smooth behaviors $\mathcal{B}(A,B,C)$ and $\mathcal{B}(D,N)$ are the same.

The theorem can be deduced from (Kuijper & Schumacher 1990, Lemma 4.1). We give here a direct proof:

Proof: Since (A,B) is a controllable pair there are polynomial matrices $P_1(s), P_2(s), P_3(s)$ such that

$$U := \begin{bmatrix} sI_n - A & -B & 0 \\ C & 0 & -I_p \\ P_1(s) & P_2(s) & P_3(s) \end{bmatrix}$$

is a unimodular matrix. From (4.1) it also follows that

$$V(s) := P_1(s)X(s) + P_2(s)D(s) + P_3(s)N(s)$$

is a unimodular matrix. It therefore follows that the behaviors

$$\begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} X(\frac{d}{dt}) \\ D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{bmatrix} \ell(t) \tag{4.2}$$

and

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \\ P_1(\frac{d}{dt})x(t) + P_2(\frac{d}{dt})u(t) + P_3(\frac{d}{dt})y(t) = V(\frac{d}{dt})\ell(t) \end{array} \right\} \tag{4.3}$$

are the same. □

Remark 4.2 If (4.1) is satisfied then in analogy to (3.6) we can express the pole placement map (up to a constant factor) through

$$\chi(M) = \det[s^k E \;\; F(s)] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \text{const. } \det \begin{bmatrix} sI_n - A & -B & 0 \\ C & 0 & -I_p \\ 0 & s^k E & F(s) \end{bmatrix}$$

and in analogy to (3.8) one can derive a rank condition in terms of the matrices (A, B, C) which guarantees the arbitrary pole assignability for a dependent compensator $[s^k E \;\; F(s)]$.

We conclude the section with a technical lemma which we will need in the proof of the main theorem (Theorem 2.5).

Lemma 4.3 *Let $d(s) := \sum_{i=1}^{\gamma} d_i s^i \in \mathbb{R}^m[s]$ and $n(s) := \sum_{i=1}^{\gamma-1} n_i s^i \in \mathbb{R}^p[s]$ be two polynomial vectors. Then the following conditions are equivalent:*

1. *There exists a polynomial vector $x(s) \in \mathbb{R}^n[s]$ such that*

$$\begin{bmatrix} sI_n - A & -B & 0 \\ C & 0 & -I_p \end{bmatrix} \begin{bmatrix} x(s) \\ d(s) \\ n(s) \end{bmatrix} = 0.$$

2.

$$\begin{aligned} 0 &= Bd_0 + ABd_1 + \cdots + A^{\gamma} B d_{\gamma} \\ n_i &= CBd_{i+1} + CABd_{i+2} + \cdots + CA^{\gamma-i-1} B d_{\gamma}; \quad i = 0, \ldots, \gamma - 1. \end{aligned}$$

Proof: Both equations are equivalent with the linear system

$$x_{t-1} = Ax_t + Bd_t, \;\; n_t = Cx_t, \quad 0 \le t \le \gamma, \;\; x_{\gamma} = 0, \;\; x_{-1} = 0. \tag{4.4}$$

□

5 Proof of the Main Theorem

Before we will go to the technical part of the proof we do explain the basic proof strategy. It is our goal to show that under the condition

$$m \le p \quad \text{and} \quad (k+1)mp > km + n \tag{5.1}$$

there is a generic set of matrices (A, B, C) of size $n \times n$, $n \times m$ and $p \times n$ which can be arbitrarily pole assigned using PI controllers of lag k.

For this we will exhibit a polynomial map

$$\varphi : \;\; (A, B, C) \longmapsto M \tag{5.2}$$

which assigns to every system a dependent compensator M. The rank condition for 'fullness' in Theorem 3.5 (compare also with Remark 4.2) reduces then to an algebraic condition of the systems parameters (A, B, C). Again because of Theorem 3.5 we have demonstrated that a generic set of systems is arbitrarily pole assignable if we can find one particular system $(\bar{A}, \bar{B}, \bar{C})$ having the property that $\bar{M} := \varphi(\bar{A}, \bar{B}, \bar{C})$ is a full and dependent compensator.

This program was carried through for the static pole assignment problem (the case $k = 0$) in (Rosenthal et al. 1995) using an explicit polynomial map φ. In our more general situation we will only show the existence of this polynomial map.

First we need a technical result. Let $a < b$ be two positive integers and let T be a $a \times b$ matrix of rank a. Then the unique projection operator P having the property that $P^2 = P$, $\mathrm{im}\,(P) = \ker\,(T)$ and $\ker\,(P) = \text{row. span}\,(T)$ is given through

$$P = I_b - T^t(TT^t)^{-1}T.$$

If we only require that $\mathrm{im}\,(P) = \ker\,(T)$ and $\ker\,(P) = \text{row. span}\,(T)$ then

$$\tilde{P} = \det(TT^t)I_b - T^t\mathrm{adj}\,(TT^t)T$$

works equally well. Moreover even if T has not full rank it is still true that $\mathrm{im}\,(\tilde{P}) \subset \ker\,(T)$ as one immediately verifies.

Next assume that T depends in a polynomial way on κ different parameters $z_1, \ldots, z_\kappa$, i.e., $T = T(z) := T(z_1, \ldots, z_\kappa)$ and assume that for a particular value $\bar{z} \in \mathbb{R}^\kappa$ the matrix $T(\bar{z})$ has (full) rank a. Let $Mat_{b\times c}(\mathbb{R})$ denote the set of all real $b \times c$ matrices and assume that $c \leq b - a$. Finally let $Y \in Mat_{b\times c}$ be a particular matrix having the property that $T(\bar{z})Y = 0$. Under those conditions we have:

Lemma 5.1 *There exists a polynomial map*

$$\varphi\ : \mathbb{R}^\kappa \longrightarrow Mat_{b\times c}(\mathbb{R})$$

having the properties that

1. $T(z)\varphi(z) = 0$.
2. $\varphi(\bar{z}) = Y$.

Proof: From the previous remarks and from the assumption on the rank of $T(\bar{z})$ it follows the existence of a constant matrix X of size $b \times c$ such that

$$Y = \left(\det(T(\bar{z})T^t(\bar{z}))I_b - T^t(\bar{z})\mathrm{adj}\,(T(\bar{z})T^t(\bar{z}))T(\bar{z})\right)X.$$

$\varphi(z) := (\det(TT^t)I_b - T^t\mathrm{adj}\,(TT^t)T)\,X$ has then all desired properties. $\square$

We are now prepared to complete the proof of Theorem 2.5. First we will describe a particular system $(\bar{A}, \bar{B}, \bar{C})$ and a particular dependent compensator $\bar{M}$ whose Jacobian $d\chi_{\bar{M}}$ is surjective. For this let $r = n - m[n/m]$ and define

$$\nu_1 := [n/m] + 1, \ldots, \nu_r := [n/m] + 1, \nu_{r+1} := [n/m], \ldots, \nu_m := [n/m].$$

Let $\mu_1 = 0, \quad \mu_2 = k, \quad \mu_i = \min(\mu_{i-1} + k + 1, \nu_m - 1)$, for $i = 3, \dots, p$. Then define:

$$D_1(s) := \begin{bmatrix} s^{\nu_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s^{\nu_{m-1}} \\ 0 & \cdots & 0 \end{bmatrix}, \quad \bar{d}(s) := \begin{bmatrix} 0 \\ s^{\mu_m - k} \\ \vdots \\ s^{\mu_3 - k} \\ s^{\nu_m} \end{bmatrix},$$

$$N_1(s) := \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \\ 0 & \cdots & 0 \end{bmatrix}, \qquad \bar{n}(s) := \begin{bmatrix} s^{\mu_p} \\ \vdots \\ s^{\mu_2} \\ s^{\mu_1} \end{bmatrix}. \tag{5.3}$$

One readily verifies that $D(s) := \left[D_1(s)\ \bar{d}(s)\right]$ and $N(s) := [N_1(s)\ \bar{n}(s)]$ describe a right coprime factorization $N(s)D^{-1}(s)$ of a $p \times m$ transfer function $G(s)$ of McMillan degree $n = \sum_{i=0}^{m} \nu_i$. As our particular system we will choose a minimal realization $(\bar{A}, \bar{B}, \bar{C})$ of the transfer function $G(s)$.

Next we describe a dependent and full compensator of $G(s)$. For this recall that $m \leq p$ and assume that $k \geq 1$. With this we define:

$$E = \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}, F_0 = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}, F_k = \begin{bmatrix} 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & -1 & \ddots & & \vdots \\ \vdots & & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & -1 & 0 \end{bmatrix}, \tag{5.4}$$

$$F_1 = \cdots = F_{k-1} = 0. \tag{5.5}$$

Finally we define

$$\bar{M} := [E, F_0, \dots, F_k].$$

We claim that $\bar{M}$ is a full and dependent compensator of $(\bar{A}, \bar{B}, \bar{C})$. For this we compute

$$s^k E D(s) + F(s) N(s) = \begin{bmatrix} s^{\nu_1 + k} & \cdots & \cdots & 0 & 0 \\ 1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & s^{\nu_{m-1}+k} & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}. \tag{5.6}$$

Since $\det\left[s^k E D(s) + F(s) N(s)\right] = 0$ it follows that the compensator $\bar{M}$ is a dependent compensator. In order to show that $\bar{M}$ is a full compensator we will apply Theorem 3.5 and compute:

$$\operatorname{adj}\left[s^k ED(s)+F(s)N(s)\right]=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}[\beta_1(s),\ldots,\beta_m(s)]=:\alpha(s)\beta(s), \tag{5.7}$$

where the entries of the polynomial vector $\beta(s)$ are given by:

$$\beta_1(s)=1,\quad \beta_i(s)=s^{(i-1)k+\nu_1+\cdots+\nu_{i-1}},\ i=2,\ldots,m.$$

Since all of the monoms $1,s,\ldots,s^{\nu_m+k}$ are among the entries of $s^kd(s),s^kn(s),\ldots,n(s)$ and since $\nu_i\le\nu_m+1$ for all i, the space spanned by the entries of

$$s^kd(s)\beta(s),s^kn(s)\beta(s),\ldots,n(s)\beta(s)$$

is equal to

$$\operatorname{span}\{s^i\beta_j|i=0,1,\ldots\nu_m+k,\ j=1,\ldots,m\}=\mathcal{P}^{n+km}.$$

By Theorem 3.5 $\bar{M}$ is a full and dependent compensator.

Next we describe a polynomial map φ which assigns to a plant (A,B,C) a dependent compensator M and which has the property that $\varphi(\bar{A},\bar{B},\bar{C})=\bar{M}$. We will do this in 2 steps, each time we will make use of Lemma 5.1.

Let $d(s):=\sum_{i=1}^{\nu_m}d_is^i\in\mathbb{R}^m[s]$ and consider the linear transformation defined as:

$$T_1=\left(\begin{array}{cccc|cccccc}
0&\cdots&\cdots&0&B&AB&&\cdots&A^{\gamma-1}B&A^\gamma B\\ \hline
&&&&0&CB&CAB&&\cdots&CA^{\gamma-1}B\\
&&&&&0&CB&&\cdots&CA^{\gamma-2}B\\
&&-I&&&&0&&\ddots&\vdots\\
&&&&&&&&\ddots&CB\\
&&&&&&&&&0
\end{array}\right)$$

By combining Lemma 4.3 and Lemma 5.1 we readily show the existence of a polynomial map

$$\varphi_1\ :\ (A,B,C)\longmapsto(d(s),n(s))$$

which has the property that $\varphi_1(\bar{A},\bar{B},\bar{C})=(\bar{d}(s),\bar{n}(s))$ and which has the property that the entries of $d(s)$ have degree $\le\nu_m$, the entries of $n(s)$ have degree $\le\nu_m-1$ and that there exists a polynomial vector $x(s)$ such that

$$\begin{bmatrix}sI_n-A&-B&0\\ C&0&-I_p\end{bmatrix}\begin{bmatrix}x(s)\\ d(s)\\ n(s)\end{bmatrix}=0$$

whenever $\varphi_1(A,B,C)=(d(s),n(s))$.

Let $<,>$ be the standard inner product of $\mathbb{R}^m$ respectively $\mathbb{R}^p$ and let $e \in \mathbb{R}^m$ and $f(s) = \sum_{i=1}^{k} f_i s^i \in \mathbb{R}^p[s]$. Then define:

$$\begin{array}{rccl} T_2: & \mathbb{R}^{m+(k+1)p} & \longrightarrow & \mathcal{P}^{\nu_m+k} \\ & (e, f(s)) & \longmapsto & < s^k e, d(s) > + < f(s), n(s) > . \end{array}$$

Note that T_2 has full rank $\nu_m + k + 1 = \dim \mathcal{P}^{\nu_m+k}$ if $(d(s), n(s)) = (\bar{d}(s), \bar{n}(s))$. Also by assumption (5.1) we know that $(k+1)p > k + \nu_m$. It therefore follows that $\dim \ker T_2 \geq m$ for all choices of $(d(s), n(s))$. Because of Lemma 5.1 there exists a polynomial map

$$\begin{array}{rccl} \varphi_2: & \mathbb{R}^{\nu_m(m+p)+m} & \longrightarrow & Mat_{m\times((k+1)p+m)} \\ & (d(s), n(s)) & \longmapsto & M \end{array}$$

which has the property that $\varphi(\bar{d}(s), \bar{n}(s)) = \bar{M}$. The polynomial map

$$\varphi := \varphi_2 \circ \varphi_1$$

has all desired properties and this completes the proof of Theorem 2.5.

6 A Numerical Procedure

Theorem 2.5 gives not only an existence result but also comes with a concrete numerical procedure to construct a particular controller which places a given closed loop characteristic polynomial.

The numerical procedure is based on the following: If the Jacobian around a dependent compensator is surjective we can find for any closed loop characteristic polynomial and for any small neighborhood around this dependent compensator a non-dependent compensator which assigns the desired closed loop characteristic polynomial. This non-dependent compensator can be iteratively found using Newton's Method starting with some initial condition M.

We would like to stress that the numerical scheme turns out to be highly unstable. This is due to the fact that geometrically we do a linearization inside the 'base locus' (Compare also with (Rosenthal & Willems 1998)). The computed solutions will hence be in a neighborhood of the base locus as well and the numerical paradigm should probably require that a solution is as far away from the base locus as possible. We consider it as a major challenge for the numerical analysis community to come up with numerical stable algorithms for computing compensators in the area of pole placement. This challenge is formulated in (Rosenthal & Willems 1998).

In order to apply Newton's method for finding a desired controller one has to choose a 'local parameterization'. This can be established by choosing $n + km + 1$ directions

$$\{M_i | i = 0, \dots, n + km\}$$

such that the Jacobian matrix

$$J(z) = \left[\frac{\partial \chi_j(z)}{\partial z_i}\right]$$

is nonsingular at $z = 0$, where $z = (z_0, \dots z_{n+km})$ and $\chi(z) = (\chi_0(z), \dots, \chi_{n+km}(z))$ with the $\{\chi_i(z)\}$ being defined through

$$\chi(M(z)) = \chi_0(z) + \chi_1(z)s + \cdots + \chi_{n+km}(z)s^{n+km}, \quad M(z) = M + \sum_{i=0}^{n+km} M_i z_i.$$

For any closed loop characteristic polynomial

$$f(s) = f_0 + f_1 s + \cdots + f_{n+km}s^{n+km},$$

let

$$f = (f_0, \dots, f_{n+km}),$$

and choose a δ with sufficiently small absolute value $|\delta|$. Starting with the initial value $z^0 = 0$ Newton's method computes iteratively

$$z^{i+1} = z^i - (\chi(z^i) - \delta f)J^{-1}(z^i).$$

As long as δ is chosen sufficiently close to zero the procedure converges and in the limit one has:

$$\lim_{i \mapsto \infty} \chi(z^i) = \delta f.$$

For a strictly proper system, $\det D(s)$ is the only $m \times m$ minor in

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

with degree n. Let $\det D(s) = b_0 + \cdots + b_n s^n$ (which is the open loop characteristic polynomial). Then for any monic $f(s)$ the solution of

$$\det(s^k ED(s) + s^k F_0 N(s) + \cdots + F_k N(s)) = \delta f(s)$$

must have $\det E = \delta / b_n$, which means that some of the coefficients of the controller

$$u(s) = E^{-1}(F_0 + \cdots + \frac{1}{s^k}F_k)y(s)$$

may be very large if the absolute value $|\delta|$ is too small. To find the solution of

$$\chi(z) = \delta f$$

for not too small value $|\delta|$, one can choose a sequence $\delta_1, \delta_2, \dots$ and use Newton's method to solve

$$\chi(z) = \delta_i f$$

with initial point z^0 the solution of

$$\chi(z) = \delta_{i-1} f.$$

At each step, as long as the Jacobian has full rank and $|\delta_i - \delta_{i-1}|$ is not too large, Newton's method will always converge to the solution.

It is important to recall that the above method works if the dependent compensator is full. Although the existence of such compensator is generically guaranteed due to Theorem 2.5, there is no systematic way for its selection. However according to (Leventides & Karcanias 1996) there is a way to avoid some cases of dependent compensators that are not full. If we take $m(s) = C(s).g(s)$ to be a d degree polynomial vector in the column span of $C(s) = \left[s^k D(s)^t, ..., N(s)^t\right]^t$ and construct a dependent compensator M such that $Mm(s) = 0$, then for M to be full, the inequality:

$$d \geq k + \left[\frac{n}{m}\right]$$

must hold. Therefore the above inequality restricts the set of possible $m(s)$ for the construction of full dependent compensators and in many cases may significantly reduce the search.

7 An illustrative Example

In this section we illustrate the developed techniques as well as the numerical procedure on a concrete example. For this consider the 3-input, 3-output system of McMillan degree 14:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -5 & 0 & -2 & 2 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & -1 & 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & -7 & 12 & 1 & 5 & -5 & 0 & -5 & -10 & -5 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 5 & 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 & 2 & -1 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x(t).$$

Let

$$X(s) := \begin{bmatrix} 1 & 0 & 0 \\ s & 0 & 0 \\ s^2 & 0 & 0 \\ s^3 & 0 & 0 \\ s^4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 0 \\ 0 & s^2 & 0 \\ 0 & s^3 & 0 \\ 0 & s^4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & s \\ 0 & 0 & s^2 \\ 0 & 0 & s^3 \end{bmatrix}, \quad \begin{array}{l} D(s) := \begin{bmatrix} s^5-2s^4+1 & 2s^5+1 & 0 \\ 0 & s^5+s^3-s^2-2 & s^2+2s+1 \\ 2s^5+3s^4-2s+1 & -s^5-s & s^4 \end{bmatrix} \\ \\ N(s) := \begin{bmatrix} -1 & s^4+s^3+s+1 & s^3+2s^2+s \\ s^4-s^2+s & -s^4-2s-1 & s \\ 2s^3-s+1 & -s^3-2 & 1 \end{bmatrix}. \end{array}$$

One verifies that relation (4.1) is satisfied and the system is therefore equivalently described by

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{pmatrix} \ell(t)$$

The smallest k which satisfies

$$(k+1)mp - k\min(m,p) > n$$

is $k=1$. The main theorem states that for a generic set of systems having 3 inputs, 3 outputs and McMillan degree 14 it is possible to do arbitrary pole assignment using PI controllers of order 1 only. Based on this we attempt to pole assign the system with PI controllers of order 1. The pole placement map $\chi(M) = \chi[E, F_0, F_1]$ is given by

$$\begin{aligned} \chi([E,F_0,F_1]) \;=\; \det\Bigg(& E \begin{bmatrix} s^6-2s^5+s & 2s^6+s & 0 \\ 0 & s^6+s^4-s^3-2s & s^3+2s^2+s \\ 2s^6+3s^5-2s^2+s & -s^6-s^2 & s^5 \end{bmatrix} \\ & +F_0 \begin{bmatrix} -s & s^5+s^4+s^2+s & s^4+2s^3+s^2 \\ 2s^5-s^3+s^2 & -s^5-2s^2-s & s^2 \\ 2s^4-s^2+s & -s^4-2s & s \end{bmatrix} \\ & +F_1 \begin{bmatrix} -1 & s^4+s^3+s+1 & s^3+2s^2+s \\ 2s^4-s^2+s & -s^4-2s-1 & s \\ 2s^3-s+1 & -s^3-2 & 1 \end{bmatrix} \Bigg). \end{aligned}$$

To find an M such that $\chi(M)=0$, we solve the equation

$$E \begin{bmatrix} 0 \\ s^3+2s^2+s \\ s^5 \end{bmatrix} + F_0 \begin{bmatrix} s^4+2s^3+s^2 \\ s^2 \\ s \end{bmatrix} + F_1 \begin{bmatrix} s^3+2s^2+s \\ s \\ 1 \end{bmatrix} = 0.$$

The unique (up to row equivalent) full rank solution is given by

$$M = [E \mid F_0 \mid F_1] = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right].$$

Since M is a dependent compensator one necessarily has that E is singular. In order to make sure that the pole placement map is surjective in a neighborhood of M we will have to verify that M is a full compensator (compare with Section 3). For this we compute:

$$\begin{aligned} &\begin{bmatrix} sD(s) \\ sN(s) \\ N(s) \end{bmatrix} \operatorname{adj}\left[sED(s) + sF_0N(s) + F_1N(s)\right] \\ = &\begin{bmatrix} 0 \\ s^3+2s^2+s \\ s^5 \\ s^4+2s^3+s^2 \\ s^2 \\ s \\ s^3+2s^2+s \\ s \\ 1 \end{bmatrix} [1,\ -s^6+2s^5-s,\ (s^6-2s^5+s)(s^6-2s^3-3s-1)]. \end{aligned}$$

Since the entries of this last matrix span the space $\mathcal{P}^{17}$ it follows by Theorem 3.5 that M is a full and dependent compensator. In particular there exists in any neighborhood of M and for any closed loop characteristic polynomial $f(s)$ of degree 17 a PI controller of order 1 such that the closed loop system has characteristic polynomial $f(s)$.

We have computed the PI controllers placing the closed loop poles to the following 17 complex conjugate points in the left half complex plane:

$$\left\{-1,\ \cos\left(\frac{18-k}{18}\pi\right) \pm i \sin\left(\frac{18-k}{18}\pi\right),\quad k = 1, \ldots, 8\right\}. \tag{7.1}$$

Let $f \in \mathbb{R}^{18}$ be the vector whose entries coincide with the coefficients of the unique monic polynomial whose root consists of the numbers in (7.1) i.e. the entries of f are the coefficients of the polynomial

$$(s+1)\prod_{k=1}^{8}((s+\cos(k\pi/18))^2 + \sin^2(k\pi/18)).$$

Consider the parameter vector

$$z = (z_1, \ldots, z_{18})$$

and define the local paramtererization

$$M(z) := [E(z) \mid F_0(z) \mid F_1(z)] := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & z_1 & z_2 & z_3 & 0 & z_4 & z_5 & z_6 \\ 0 & 1 & z_7 & z_8 & z_9 & 0 & z_{10}-1 & z_{11} & z_{12} \\ 0 & 0 & z_{13} & z_{14} & z_{15} & 1 & z_{16} & z_{17}-1 & z_{18} \end{array}\right].$$

Then the Jacobian matrix

$$J(z) = \left[\frac{\partial \chi_j(z)}{\partial z_i}\right]$$

is nonsingular at $z = 0$, where $\{\chi_i(z)\}$ are defined by

$$\chi(M(z)) = \chi_0(z) + \chi_1(z)s + \cdots + \chi_{n+km}(z)s^{n+km}.$$

We have chosen 7 δ's range from $\delta_1 = 0.0000000001$ to $\delta_7 = 0.001$, and used Newton's iteration method

$$z^{i+1} = z^i - (\chi(z^i) - \delta f)J^{-1}(z^i)$$

to compute the solution of

$$\chi(z) = \delta f.$$

The initial point z^0 was 0 for δ_1, and was the solution of

$$\chi(z) = \delta_{j-1} f$$

for δ_j, $j = 2, \ldots, 7$. The iteration was stopped when

$$\| (1/\delta)\chi(z^i) - f \|_2 \leq 10^{-10}.$$

Let $\bar{z}$ be the value computed by the iteration scheme. The PI controller is then given by

$$K(s) = E(\bar{z})^{-1}F_0(\bar{z}) + \frac{1}{s}E(\bar{z})^{-1}F_1(\bar{z}). \tag{7.2}$$

For this purpose we first compute

$$E(z)^{-1}F_0(z) = \tfrac{1}{z_{13}}\begin{bmatrix} z_2z_{13} - z_1z_{14} & z_3z_{13} - z_1z_{15} & -z_1 \\ z_8z_{13} - z_7z_{14} & z_9z_{13} - z_7z_{15} & -z_7 \\ z_{14} & z_{15} & 1 \end{bmatrix},$$

$$E(z)^{-1}F_1(z) = \tfrac{1}{z_{13}}\begin{bmatrix} z_4z_{13} - z_1z_{16} & z_5z_{13} - z_1z_{17} + z_1 & z_6z_{13} - z_1z_{18} \\ z_{10}z_{13} - z_{13} - z_7z_{16} & z_{11}z_{13} - z_7z_{17} + z_7 & z_{12}z_{13} - z_7z_{18} \\ z_{16} & z_{17} - 1 & z_{18} \end{bmatrix}.$$

Once $\bar{z}$ is computed it is possible to compute the control law $u(t)$ from the identity

$$u(t) = E(\bar{z})^{-1}F_0(\bar{z})y(t) + E(\bar{z})^{-1}F_1(\bar{z})\int_0^t y(r)\,dr.$$

Following is the computed controller which we obtained with the help of the computer package MatLab using seven iterations and using $\delta_7 = 0.001$:

$$E(\bar{z})^{-1}F_0(\bar{z}) = \begin{bmatrix} 12.81125702878 & -43.06500585945 & -350.13207303523 \\ 7.23355325637 & -24.52237268211 & -197.64167583523 \\ -34.85086536755 & 127.42909838229 & 1000.00000000000 \end{bmatrix},$$

$$E(\bar{z})^{-1}F_1(\bar{z}) = \begin{bmatrix} 1.46475447582 & 350.84099505228 & 57.64299234455 \\ -0.02796359744 & 198.72746054141 & 33.04861987006 \\ -5.73872704249 & -997.40580776538 & -163.13959523207 \end{bmatrix}.$$

Using the 'fsolve' routine of the computer package Maple we did compute the achieved closed loop poles. The following table contains the obtained results and a comparison with the desired pole location.

Desired Pole Location		Achieved Value
$\cos(\pi)$	-1	-1.0014042
$\cos\left(\frac{17}{18}\pi\right) \pm i\sin\left(\frac{17}{18}\pi\right)$	$-.9848078 \pm .1736483i$	$-.9838290 \pm .1731961i$
$\cos\left(\frac{16}{18}\pi\right) \pm i\sin\left(\frac{16}{18}\pi\right)$	$-.9396926 \pm .3420202i$	$-.9398808 \pm .3424523i$
$\cos\left(\frac{15}{18}\pi\right) \pm i\sin\left(\frac{15}{18}\pi\right)$	$-.8660255 \pm .5i$	$-.8662690 \pm .4998789i$
$\cos\left(\frac{14}{18}\pi\right) \pm i\sin\left(\frac{14}{18}\pi\right)$	$-.7660444 \pm .6427876i$	$-.7658117 \pm .6427269i$
$\cos\left(\frac{13}{18}\pi\right) \pm i\sin\left(\frac{13}{18}\pi\right)$	$-.6427876 \pm .7660446i$	$-.6428850 \pm .7661021i$
$\cos\left(\frac{12}{18}\pi\right) \pm i\sin\left(\frac{12}{18}\pi\right)$	$-.5 \pm .8660255i$	$-.4999778 \pm .8660087i$
$\cos\left(\frac{11}{18}\pi\right) \pm i\sin\left(\frac{11}{18}\pi\right)$	$-.3420201 \pm .9396928i$	$-.3420227 \pm .9396941i$
$\cos\left(\frac{10}{18}\pi\right) \pm i\sin\left(\frac{10}{18}\pi\right)$	$-.1736482 \pm .9848077i$	$-.1736481 \pm .9848079i$

It should be pointed out that because we are solving a system of 18 polynomial equations of degree 3, the solution is highly sensitive in the parameters. One needs to keep at least 4 digits after the decimal point to have a stable closed loop system and this explains the many digits which we did provide. Our computational experiments also revealed that the sensitivity turns out to be higher if we try to assign a closed loop characteristic polynomial with multiple poles, because in this case the positions of the poles are very sensitive to the coefficients of the polynomial.

8 Conclusion

In this paper we did study the output pole assignment problem using PI feedback compensators of a fixed lag k. The main result is a strong sufficiency result which guarantees arbitrary pole assignability for a generic set of systems. The main proof technique is based on the method of 'linearization around a dependent compensator'. This method was recently developed by the authors and it has been successfully applied to other pole placement problems in the past. In a later part of the paper we did explain that the proof comes with a natural numerical algorithm. This algorithm however results in solutions which are highly sensitive in the parameters and the design of stable numerical algorithms for computing feedback compensators (which theoretically do exist) remains a challenge for the future.

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