On Output Feedback Invariants and Cascade Equivalence of Systems

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Abstract

The paper is concerned with the algebraic classification problem of linear systems up to static output feedback. Using tools from geometric invariant theory it is shown that there exists a quasi projective variety whose points parameterize the output feedback orbits in a unique way. If the McMillan degree $n \geq mp$ then the set of output feedback orbits of nondegenerate transfer functions of size $p \times m$ is a quasi projective variety.

1 Introduction

Consider a time invariant linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

having m inputs, p outputs, and n states. The (full) feedback group is the group generated through the feedback action:

$$u \longmapsto u + Fy \tag{2}$$

and through the change of basis in state space, input space and output space respectively, i.e. through the transformations:

$$x \longmapsto Sx, \qquad S \in Gl_n \tag{3}$$
$$u \longmapsto T_1 u, \qquad T_1 \in Gl_m \tag{4}$$
$$y \longmapsto T_2 y, \qquad T_2 \in Gl_p. \tag{5}$$

The orbits under the full feedback group are referred to as the *output feedback orbits.* In order to fully understand effect of output feedback on the structure of linear systems it is of fundamental interest to (i) classify the feedback orbits, (ii) to determine a complete set invariants for output feedback and (iii) to obtain a detailed description of the adherence order (orbit closure inclusion) of the different orbits.

These obviously important problems have already been studied by many authors (see e.g. [3, 4, 5, 6, 9, 10]) and despite many partial results the problem is still far from being solved.

The transformations induced by the actions (2), (3), (4), (5) describe a group action on the vector space of all matrix triples (A, B, C) which is a vector space of dimension $n(m+n+p)$. There is an extensive mathematical literature on the classification of orbits arising from group actions on vector spaces and more general algebraic varieties and we refer to Section 3 for some more details. If the number of orbits is finite then this

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study generally seeks a discrete set of invariants classifying the finite number of orbits. There are some instances in the control literature where the set of orbits is finite and as example we refer to [2, 10, 11].

In the problem at hand the number of feedback orbits is in general infinite and this makes the problem difficult. In order to classify all orbits it will thus be necessary to derive a 'continuous set of invariants'.

The application of tools from geometric invariant theory (see e.g. [15, 16]) often enables one to derive for a given group action a set of invariants in a systematic way. From a geometric point of view this amounts to describing an algebraic variety whose points parameterize uniquely the closed feedback orbits.

In this paper we construct, using tools from geometric invariant theory, such a quasi-projective algebraic variety, whose points parameterize output feedback orbits in a unique way. Since a quasi-projective variety can be embedded into affine space using e.g. semialgebraic functions our result implies the existence of a complete set of semi-algebraic invariants for output feedback. It also helps to construct such a complete set of invariants; however this problem will not be addressed here.

In order to achieve the result it is crucial to first extend the output feedback action to an action that operates on a compactification of the space of proper transfer functions of McMillan degree n. This process is explained in the next section.

2 Cascade equivalence and the extended feedback group

The notion of *cascade equivalence* was introduced by Byrnes and Helton in [4] and it is closely related to the feedback classification problem. In our context this notion can be equivalently described in the following way:

Consider a time invariant linear **proper** system of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du. \tag{6}$$

In addition to the feedback action (2) and the basis transformations (3), (4) and (5) we will also allow a feedforward transformation

$$y \longmapsto y + Gu. \tag{7}$$

The collection of all those transformations will be called the *extended full feedback group*. The actions in (2), (4), (5) and (7) describe invertible transformations on the space of external variables $\left[u^t \; y^t\right]^t$. In this way one can view these actions as elements of the general linear group $T \in Gl_{m+p}$ and the collection of these transformations is compactly described through:

$$\begin{bmatrix} u \\ y \end{bmatrix} \longmapsto \begin{bmatrix} T_1 & F \\ G & T_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = T \begin{bmatrix} u \\ y \end{bmatrix}, \tag{8}$$

where $T \in Gl_{m+p}$. This shows that (2), (4), (5) and (7) generate the whole general linear group $T \in Gl_{m+p}$.

Note that the linear transformation $T \in Gl_{m+p}$ induces the notion of cascade equivalence on the set of proper transfer functions. The following Lemma is easily established:

Lemma 2.1 *There is a bijective correspondence between the set of equivalence classes of the form* (6) *under the extended full feedback group and the set of equivalence classes of the form* (1) *under the full feedback group.*

This Lemma now enables us to concentrate on the linear transformation (8). Instead of working with a state space description we also can work with polynomial matrices. For this let

$$D^{-1}(s)N(s) := G(s) := C(sI - A)^{-1}B + D$$

be a left coprime factorization of the transfer function of system (6). Then the linear transformation (8) is equivalently described through:

$$(D(s)\ N(s)) \longmapsto (D(s)\ N(s))\, T^{-1}, \quad T \in Gl_{m+p}. \tag{9}$$

3 Some basic notions from geometric invariant theory

In this section we explain an important result from geometric invariant theory which we will use later in the paper to derive a set of continuous feedback invariants. The interested reader will find details on geometric invariant theory in the book of Newstead [16].

Let X be a projective variety, i.e. X is the zero locus of a finite set of homogeneous polynomial equations. We will assume that X is embedded into the projective space $\mathbb{P}^N$. Let $G \subset Gl_{N+1}$ be a reductive group (e.g. a group isomorphic to a general linear group) which acts on the projective space $\mathbb{P}^N$ and induces an action on the variety X.

In this situation one has the following general result: (see [16, Theorem 3.14]).

Theorem 3.1 *There exists a Zariski open set X^{ss} of so called* semi-stable *points, a projective variety Y and an algebraic morphism $\phi : X^{ss} \to Y$ having the property that $\phi^{-1}(y)$ contains exactly one closed G–orbit for every $y \in Y$. Moreover there is a Zariski open set $Y^s \subset Y$ such that $\phi^{-1}(y)$ contains one and only one orbit for every $y \in Y^s$.*

The set $X^s := \phi^{-1}(Y^s)$ is the so called set of stable points and both X^s and X^{ss} are Zariski open sets of the variety X. It is possible that X^{ss} is the empty set in which case Theorem 3.1 does not give any insight.

Theorem 3.1 is significant in several ways. First the points of the variety Y may be viewed as a continuous family of invariants capable of distinguishing orbits inside X^s. In addition the variety Y is characterized through some universal properties. Because of this reason one sometimes also speaks about the *categorical quotient* Y.

The fact that this variety Y is projective is surprising. It is our goal in the next section to apply Theorem 3.1 to the feedback orbit classification problem.

4 The projective variety of homogeneous autoregressive systems

In order to apply Theorem 3.1 it will be necessary to compactify the space of all $p \times (m+p)$ autoregressive systems of the form $P(s) = (D(s)\ N(s))$. Such a compactification was provided in [17] and we shortly review the details.

Consider a $p \times (m+p)$ polynomial matrix

$$P(s,t) := \begin{pmatrix} f_{1,1}(s,t) & \dots & f_{1,(m+p)}(s,t) \\ f_{2,1}(s,t) & \dots & f_{2,(m+p)}(s,t) \\ \vdots & & \vdots \\ f_{p,1}(s,t) & \dots & f_{p,(m+p)}(s,t) \end{pmatrix}. \tag{10}$$

We say $P(s,t)$ is homogeneous if each element $f_{i,j}(s,t)$ is a homogeneous polynomial of degree ν_i. We say two homogeneous matrices $P(s,t)$ and $\tilde{P}(s,t)$ are equivalent if they have the same row-degrees and if there is a unimodular matrix $U(s,t)$, whose entries $u_{ij}(s,t)$ are homogeneous polynomials of degree $\nu_i - \nu_j$. Using this equivalence relation we define:

Definition 4.1 An equivalence class of full rank homogeneous polynomial matrices $P(s,t)$ will be called a *homogeneous autoregressive system.* The McMillan degree of a homogeneous autoregressive system is defined as the sum of the row degrees, i.e. through $n := \sum_{i=1}^{p} \nu_i$. The set of all homogeneous autoregressive systems of size $p \times (m+p)$ and McMillan degree n will be denoted by $\mathcal{H}^n_{p,m}$.

The main result established in [17] is as follows:

Theorem 4.2 ([17]) *$\mathcal{H}^n_{p,m}$ is a smooth projective variety containing the set of proper transfer functions as a Zariski dense subset.*

The embedding of the set of proper transfer functions into the set of homogeneous autoregressive systems is established in the following way: Let $D^{-1}(s)N(s)$ be a left coprime factorization of the transfer function $G(s)$. Consider the autoregressive system $(D(s)\ N(s))$. Without loss of generality we assume that $(D(s)\ N(s))$ is in row proper form having ordered observability indices $\nu_1 \geq \ldots \geq \nu_p$. A row by row homogenization results in a homogeneous autoregressive system, i.e. a point inside the variety $\mathcal{H}^n_{p,m}$.

Note that the group action (8) naturally extends to $\mathcal{H}^n_{p,m}$ and that $\mathcal{H}^n_{p,m}$ has a natural embedding into projective space. In order to apply Theorem 3.1 it is necessary to show that the Zariski open set $X^{ss} \subset \mathcal{H}^n_{p,m}$ of semistable orbits is not empty. For this recall the concept of nondegeneracy as first introduced in [1].

Definition 4.3 A $p \times (m+p)$ homogeneous autoregressive system $P(s,t)$ is called *nondegenerate* if there is no full rank $m \times (m+p)$ matrix K with entries in the complex numbers $\mathbb{C}$ such that

$$\det \begin{pmatrix} P(s,t) \\ K \end{pmatrix} = 0.$$

One verifies that nondegenerate systems cannot exist if the McMillan degree is 'small'. The following result shows when nondegenerate systems are open and dense inside the variety $\mathcal{H}^n_{p,m}$.

Theorem 4.4 ([1]) *If the McMillan degree $n \geq mp$ then the variety $\mathcal{H}^n_{p,m}$ contains a Zariski open set of nondegenerate systems.*

Using [13, Lemma 7.2.2.], we can now show the following:

Theorem 4.5 *The set of semistable orbits $X^{ss} \subset \mathcal{H}^n_{p,m}$ contains the set of nondegenerate systems. In particular if $n \geq mp$ then X^{ss} is nonempty and there exists a projective variety Y and a morphism $\phi : X^{ss} \to Y$ having the property that $\phi^{-1}(y)$ contains exactly one closed Gl_{m+p} orbit for every $y \in Y$.*

Corollary 4.6 *Consider the set of m input, p output systems of McMillan degree n as introduced in (1). If $n \geq mp$ then the set of output feedback orbits of nondegenerate transfer functions is a quasi projective variety.*

5 The feedback orbit classification problem in terms of generalized first order systems

The set of homogeneous autoregressive systems can be described through some generalized first order systems and we refer to [17, 19]. In this section we describe the extended feedback group in terms of those generalized first order system. In this way we make the connection to some work of Hinrichsen and O'Halloran [11].

Following the exposition in [12, 18] consider $(n+p) \times n$ matrices K, L and a $(n+p) \times (m+p)$ matrix M. Those matrices define a generalized state space system through

$$K\dot{x}(t) + Lx(t) + Mw(t) = 0, \qquad x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^{m+p}. \tag{11}$$

The system (11) is called admissible if the homogeneous pencil $[sK + tL]$ has generically full column rank. An admissible system is called controllable if the pencil $[sK + tL\ M]$ has full row rank for all $(s,t) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

There is a natural equivalence relation among generalized first order systems: If $U \in Gl_{n+p}$ and $S \in Gl_n$ then

$$(K, L, M) \sim (UKS^{-1}, ULS^{-1}, UM). \tag{12}$$

If the high order coefficient matrix

$$[K\, M]$$

has the property that the first $(n+p) \times (n+p)$ minor is invertible then the system (11) is equivalent to a system having

$$K = \begin{bmatrix} -I \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} A \\ C \end{bmatrix}, \quad M = \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}, \tag{13}$$

i.e. the system is equivalent to a usual state space system of the form

$$\dot{x} = Ax + Bu, \ y = Cx + Du.$$

The connection to the set of homogeneous autoregressive systems is established through:

Theorem 5.1 ([8, 18]) *The categorical quotient of the set of controllable state space systems as introduced in* (11) *under the group action* $Gl_{n+p} \times Gl_n$ *is isomorphic to the smooth projective variety* $\mathcal{H}^n_{m,p}$ *of all* $m\times(m+p)$ *homogeneous autoregressive systems of McMillan degree* n.

Next we define:

Definition 5.2 Two generalized first order systems are equivalent under the extended full feedback group if there are invertible matrices $S \in Gl_n$, $T \in Gl_{m+p}$ and $U \in Gl_{n+p}$ such that

$$(K, L, M) \quad \sim \quad (UKS^{-1}, ULS^{-1}, UMT^{-1}). \quad (14)$$

Note that the linear transformation T introduced in (8) corresponds to a change of basis in the set of external variables $w = \begin{bmatrix} u \\ y \end{bmatrix}$ and it is therefore equal to the transformation T appearing in (14). Finally the group action described in (14) corresponds exactly to the transformations (i), (ii), (iii) considered by Hinrichsen and O'Halloran in [11, p. 2730].

As a consequence of Theorem 5.1 we have:

Theorem 5.3 *The categorical quotient induced by the group action* (14) *is equal to the projective variety* $\mathcal{H}^n_{m,p}/Gl_{m+p}$. *Moreover* $\mathcal{H}^n_{m,p}/Gl_{m+p}$ *represents a continuous parameterization of the feedback orbits under the full feedback group.*

6 A concrete description of the moduli space in the MISO situation

In this section we explain our result in the multi input, single output (i.e. $p = 1$) situation. The variety $\mathcal{H}^n_{1,m}$ consists in this case of all $1 \times (m+1)$ polynomial vectors

$$P(s) = (f_1(s), \ldots, f_{m+1}(s))$$

whose polynomial entries have degree at most n. In this way we can identify the space $\mathcal{H}^n_{1,m}$ with the projective space $\mathbb{P}^N$, where $N = mn + m + n$.

By definition a systems is nondegenerate if the $m+1$ polynomial vectors $\{f_1(s), \ldots, f_{m+1}(s)\} \subset \mathbb{R}[s]$ are linearly independent over $\mathbb{R}$. Clearly this can only happen if the McMillan degree $n \geq m$.

Identify the set of polynomial vectors of degree at most n with the vector space $\mathbb{R}^{n+1}$. A system $P(s) = (f_1(s), \ldots, f_{m+1}(s))$ defines then a linear subspace

$$\operatorname{span}_{\mathbb{R}}\{f_1(s), \ldots, f_{m+1}(s)\} \subset \mathbb{R}^{n+1}.$$

This subspace has dimension $m + 1$ if and only if $P(s)$ describes a nondegenerate system. The semistable points under the extended full feedback group Gl_{m+1} describe therefore a well defined $m+1$ dimensional subspace of $\mathbb{R}^{n+1}$. The categorical quotient $\mathcal{H}^n_{1,m}/Gl_{m+1}$ is in this case exactly the Grassmann variety $\mathrm{Grass}(m + 1, \mathbb{R}^{n+1})$ of $m + 1$ dimensional subspaces in $\mathbb{R}^{n+1}$.

7 Conclusion

In this paper we did show that the output feedback problem is closely related to the study of the moduli space $\mathcal{H}^n_{p,m}/Gl_{m+p}$, where $\mathcal{H}^n_{p,m}$ denotes the set of $p\times(m+p)$ homogeneous autoregressive systems of McMillan degree n. In upcoming work [7] we will describe this moduli space in detail.

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