Localization in Fourier space and Navier-Stokes system

Jean-Yves Chemin
Laboratoire J.-L. Lions, Case 187
Université Pierre et Marie Curie, 75 230 Paris Cedex 05, France
chemin@ann.jussieu.fr

June 27, 2005

Introduction

Presentation of the text

This text consists in notes on a series of lectures given in March 2004 in the De Giorgi center during a semester about "Phase Space Analysis of Partial Differential Equations". The goal of those lectures was to study the incompressible Navier-Stokes system in $\mathbf{R}^d$ with $d = 2$ or $d = 3$ and to show what could be the impact of techniques of localization in frequency space in the study of this system.

The first chapter is devoted to the basic study of this system. Using essentially Sobolev embedding, we prove global wellposedness in $\mathbf{R}^2$ in the energy space and local wellposedness in $L^4_{loc}(\dot{H}^1)$ with initial data in $\dot{H}^{\frac{1}{2}}$ in $\mathbf{R}^3$ and also local wellposedness in $L^3(\mathbf{R}^3)$. Each time, local wellposedness becomes global for small data. The crucial role of the scaling is pointed out. Then we shall study the influence of the special structure of the Navier-Stokes equations: in particular in dimension two, it is globally wellposed and in dimension three, global solutions are globally stable. At the end of this chapter, we develop an elementary L^p approach. We proved in particular global wellposedness results for small initial data in $L^3(\mathbf{R}^3)$.

In the second chapter, we introduced Littlewood-Paley theory, which is the theory that describes the regularity of tempered distributions in terms of decay in Fourier spaces. We expose the basis of Bony's paradifferential calculus, in particular precized product law. We also translate some smallness condition given in the first chapter in terms of Besov spaces.

In the third chapter, we interpret some results of the second chapter in terms of Littlewood-Paley theory. As an illustration, we study the problem of the existence and uniqueness of trajectories for scaling invariant solutions of Navier-Stokes equations. This is an opportunity to revisit Cauchy-Lipschitz theorem.

In the last chapter, we present an anisotropic model of the incompressible Navier-Stokes system coming from the study of geophysical fluids; in this three dimensionnal model, the three dimensionnal laplacian becomes a bidimensionnal laplacian. The purpose of this chapter is to prove wellposedness result for this sytem in scaling invariant spaces. The main point of this chapter is that the structure of the non linear part of the incompressible Navier-Stokes system is used in a crucial way.

Acknowlegments

I want to thank the organizers of the semester and spacially Professor Ferruccio Colombini for their wellcome and for providing exceptionnal conditions for doing mathematics. I want also to thank the listeners of those lectures for the interest they have for the questions adressed in this text.

Contents

Chapter 1

Incompressible Navier-Stokes system with elementary methods

1.1 Introduction

Let us recall what is incompressible Navier-Stokes system.

$$(NS_\nu)\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0, \end{cases}$$

where $v(t,x)$ is a time dependant vector field on $\mathbf{R}^d$, and

$$\operatorname{div} v = \sum_{j=1}^{d}\partial_j v^j\ ,\ \ v\cdot\nabla = \sum_{j=1}^{d} v^j\partial_j \quad \text{and} \quad \Delta = \sum_{j=1}^{d}\partial_j^2.$$

We restrict ourselves to the whole space $\mathbf{R}^d$. In terms of fluid mechanics, it means that we neglect boundary effects. Moreover, we shall only consider the two physical dimensions $d=2$ and $d=3$. For a much more detailed introduction to incompressible Navier-Stokes system, the reader can consult [4], [17] and [54]. For a complete and up to date bibliography, see [4].

In this introduction, we shall point out some very basic facts about this system. The first one is the weak form of the Navier-Stokes system. Using Leibnitz's formula, it is clear that, when the vector field v is smooth enough, we have that

$$v\cdot\nabla v = \operatorname{div}(v\otimes v) \quad \text{where} \quad \operatorname{div}(v\otimes v)^j \stackrel{\text{def}}{=} \sum_{k=1}^{d}\partial_k(v^j v^k) = \operatorname{div}(v^j v).$$

So the Navier-Stokes system may be written as

$$(NS_\nu)\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) - \nu\Delta v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0. \end{cases}$$

The advantage of this formulation is that it makes sense for more singular vector field than the previous one and this will be useful quite frequently.

The second one is the energy estimate. All the following computations are formal ones that will become rigourous in the different chapters of this text. So, taking the scalar product of the system in L^2 with the solution vector field v gives

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + (v\cdot\nabla v|v)_{L^2} - \nu(\Delta v|v)_{L^2} = -(\nabla p|v)_{L^2}.$$

Using formal integrations by part, we may write that

$$\begin{aligned}(v\cdot\nabla v|v)_{L^2} &= \sum_{1\leq j,k\leq d}\int_{\mathbf{R}^d} v^j(\partial_j v^k)v^k dx\\ &= \frac{1}{2}\sum_{1\leq j\leq d}\int_{\mathbf{R}^d} v^j\partial_j(|v|^2)dx\\ &= -\frac{1}{2}\sum_{1\leq j\leq d}\int_{\mathbf{R}^d}(\operatorname{div} v)|v|^2 dx\\ &= 0.\end{aligned}$$

Moreover, we obviously have that $-\nu(\Delta v|v)_{L^2} = \nu\|\nabla u\|_{L^2}^2$. Again by (formal) integration by part, we have that

$$\begin{aligned}-(\nabla p|v)_{L^2} &= -\sum_{j=1}^{d}\int_{\mathbf{R}^2} v^j\partial_j p dx\\ &= \int_{\mathbf{R}^2} p\operatorname{div} v\, dx\\ &= 0.\end{aligned}$$

So, it turns out that, still formally,

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \nu\|\nabla v(t)\|_{L^2}^2 = 0,$$

and by integration that

$$\|v(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla v(t')\|_{L^2}^2 dt' = \|v_0\|_{L^2}^2. \tag{1.1}$$

This basic a priori estimate allowed J. Leray to prove in 1934 the following theorem.

Theorem 1.1.1 *Let u_0 be a divergence free vector field in $L^2(\mathbf{R}^d)$. Then a solution u of (NS_ν) exists in the energy space*

$$L^\infty(\mathbf{R}^+;L^2)\cap L^2(\mathbf{R}^+;\dot{H}^1)$$

such that the energy inequality holds, namely

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2.$$

In this theorem, the concept of solution must be understood, as in all that follows, in the following sense. We shall say that u in a solution of (NS_ν) on $[0,T)\times\mathbf{R}^d$ if u belongs to L^2 locally in $[0,T)\times\mathbf{R}^d$ and if for any function Ψ in $C^1(\mathbf{R}^+;\mathcal{S}(\mathbf{R}^d)^d)$ and divergence free,

$$\begin{aligned}\int_\Omega u(t,x)\cdot\Psi(t,x)\,dx &+ \int_0^t\int_\Omega\Big(\nu\nabla u:\nabla\Psi - u\otimes u:\nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt'\\ &= \int_\Omega u_0(x)\cdot\Psi(0,x)\,dx + \int_0^t\langle f(t'),\Psi(t')\rangle\,dt'.\end{aligned}$$

Moreover, J. Leray proved in 1934 the following theorem.

Theorem 1.1.2 *If $d = 2$, solutions given by the above theorem are unique, continuous with value in $L^2(\mathbf{R}^2)$ and satisfies the energy equality*

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \|u_0\|_{L^2}^2.$$

We shall not prove Theorem 1.1.1. For a proof of it, we refer to the magnificent original paper by J. Leray (see [41]). For a modern proof, see for instance [14] or [17].

As we are working in the whole space $\mathbf{R}^d$, we can compute the pressure. Applying the divergence operator to the system (NS_ν), we get

$$\partial_t \operatorname{div} v + \sum_{1\leq j,k\leq d} \partial_j \partial_k (v^j v^k) - \nu\Delta \operatorname{div} v = -\Delta p.$$

The stationary condition $\operatorname{div} v = 0$ implies that

$$-\Delta p = \sum_{1\leq j,k\leq d} \partial_j \partial_k (v^j v^k).$$

So formally, we have that

$$p = - \sum_{1\leq j,k\leq d} \Delta^{-1} \partial_j \partial_k (v^j v^k) \quad \text{with} \quad \Delta^{-1}\partial_j\partial_k a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(|\xi|^{-2}\xi_j\xi_k \widehat{a}). \tag{1.2}$$

In all this chapter, we shall denote by Q any bilinear map of the form

$$Q^j(u,v) \stackrel{\text{def}}{=} \sum_{k,\ell,m} q_{k,\ell}^{j,m} \partial_m (u^k v^\ell),$$

where $q_{k,\ell}^{j,m}$ are Fourier multipliers of the form

$$q_{k,\ell}^{j,m} a \stackrel{\text{def}}{=} \sum_{n,p} \alpha_{k,\ell}^{j,m,n,p} \mathcal{F}^{-1}\Big(\frac{\xi_n\xi_p}{|\xi|^2}\widehat{a}(\xi)\Big) \quad \text{with} \quad \alpha_{k,\ell}^{j,m,n,p} \in \mathbf{R}.$$

We shall denote by Q_{NS} the particular one related to Navier-Stokes equation, namely

$$Q_{NS}^j(u,v) \stackrel{\text{def}}{=} \operatorname{div}(v^j u) - \sum_{1\leq k,\ell\leq d} \partial_j \Delta^{-1} \partial_k \partial_\ell (u^k v^\ell).$$

Now the incompressible Navier-Stokes system appears as a particular case of

$$(GNS_\nu) \begin{cases} \partial_t v - \nu\Delta v + Q(v,v) &= 0 \\ v_{|t=0} &= v_0. \end{cases}$$

with the quadratic operator Q define above. Let us define $B(u,v)$ (resp. $B_{NS}(u,v)$) by

$$\begin{cases} \partial_t B(u,v) - \nu\Delta B(u,v) = Q(u,v) \text{ (resp. } Q_{NS}(u,v)) \\ B(u,v)_{|t=0} = 0. \end{cases}$$

Now solving (GNS_ν) (resp. (NS_ν)) can be seen as finding a fixed point for the map

$$u \longmapsto e^{\nu t\Delta} u_0 + B(u,u) \text{ (resp. } B_{NS}(u,u)).$$

In all this chapter, we shall solve (GNS_ν) or (NS_ν) using a contraction argument in a well choosen Banach space. It is based on the following classical lemma, that we recall for the reader's convenience.

Lemma 1.1.1 *Let E be a Banach space and $\mathcal{B}$ a bilinear map continuous from $E\times E$ into E and α a positive number such that*

$$\alpha < \frac{1}{4\|\mathcal{B}\|} \quad \textit{with} \quad \|\mathcal{B}\| \stackrel{\text{def}}{=} \sup_{\substack{\|u\|\leq 1\\ \|v\|\leq 1}} \|\mathcal{B}(u,v)\|.$$

Then for any a in the ball $B(0,\alpha)$ of center 0 and radius α of E, a unique x exists in the ball of radius 2α such that

$$x = a + \mathcal{B}(x,x).$$

Moreover, we have $\|x\| \leq 2\|a\|$.

Proof of Lemma 1.1.1 It consists in applying the classical iterative scheme define by

$$x_0 = a \quad \text{and} \quad x_{n+1} = a + \mathcal{B}(x_n, x_n).$$

Let us first prove by induction that $\|x_n\| \leq 2\alpha$. Using this hypothesis on α, we get, by definition of x_{n+1},

$$\|x_{n+1}\| \leq \alpha(1+4\alpha\|\mathcal{B}\|) \leq 2\alpha.$$

Thus the sequence remains in the ball $B(0,2\alpha)$. Then

$$\begin{aligned} x_{n+1} - x_n &= \mathcal{B}(x_n,x_n) - \mathcal{B}(x_{n-1},x_{n-1}) \\ &\leq \mathcal{B}(x_n - x_{n-1}, x_n) + \mathcal{B}(x_{n-1}, x_n - x_{n-1}). \end{aligned}$$

Then we have

$$\|x_{n+1} - x_n\| \leq 4\alpha\|\mathcal{B}\|\,\|x_n - x_{n-1}\|.$$

The hypothesis $4\alpha\|\mathcal{B}\| < 1$, makes that $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in E the limit of which is a fixed point of $x \mapsto a + \mathcal{B}(x,x)$ in the ball $B(0,2\alpha)$. This fixed point is unique because if x and y are two such fixed points, then

$$\|x-y\| \leq \|\mathcal{B}(x-y,y) + \mathcal{B}(x,x-y)\| \leq 4\alpha\|\mathcal{B}\|\,\|x-y\|.$$

The lemma is proved.

If we want to get global solutions of (GNS_ν) or (NS_ν) with such method, we need to use spaces the norm of which is invariant under the transformations that preserve the set of global solutions. This set contains the following transformations, called in the literature "scaling transformations". They are defined by

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$$

The use of the fixed point method explained in Lemma 1.1.1 imposes to consider only scaling invariant spaces. Let us give examples of such spaces:

$$L^\infty(\mathbf{R}^+; L^d(\mathbf{R}^d)),\ L^\infty(\mathbf{R}^+; \dot H^{\frac{d}{2}-1}),\ L^4(\mathbf{R}^+; \dot H^{\frac{d-1}{2}}),\ L^\infty(\mathbf{R}^+; \dot H^{\frac{d}{2}-1}) \cap L^2(\mathbf{R}^+; \dot H^{\frac{d}{2}}).$$

Let us point out that, when $d=2$, the scaling invariant space

$$L^\infty(\mathbf{R}^+; L^2) \cap L^2(\mathbf{R}^+; \dot H^1).$$

is the energy space, the norm of which appears in the formal conservation of energy (1.1). This is the key point in the proof of Theorem 1.1.2.

In the case when $d = 3$, this energy space is below the regularity which provides scaling invariant spaces, namely $H^{\frac{1}{2}}$. We can interpret this saying that in dimension $d = 2$, the global existence of regular solutions of Navier-Stokes system is a critical problem, solved by J. Leray in 1934. In the case when $d = 3$, this can be interpreted as a supercritical problem. This is the core of the difficulty. As we shall see, one of the challenge is to be able to use the special structure of the equation together with scaling invariant spaces.

In the second section of this chapter, we shall use Lemma 1.1.1 to prove local wellposedness of (GNS_ν) for initial data in $\dot{H}^{\frac{d}{2}-1}$ and global wellposedness for small data in $\dot{H}^{\frac{d}{2}-1}$. As everything done in this context of Sobolev spaces rely on the Sobolev embeddings, we give a proof of it.

In the third section, we shall see how to use the special structure of the Navier-Stokes system. First, we shall prove Leray uniqueness theorem in dimension 2. We shall also prove in dimension 3 a result about asymptotics of possible large global solutions. This results will imply in particular that the set of initial data which give rise to global solutions in $L^4_{loc}(\mathbf{R}^+; \dot{H}^1)$ is an open subset of $\dot{H}^{\frac{1}{2}}$.

In the forth section, we prove that (GNS_ν) is locally wellposed in $L^3(\mathbf{R}^3)$ and globally for small data, using essentially Young' inequalities.

Let us point out that up to now, all the theorems for this chapter are proved with elementary method: nothing more than classical Sobolev embeddings and Young's and Hölder's inequalities.

1.2 Wellposedness in Sobolev spaces

The purpose of this section is to investigate the local and global wellposedness of the Navier-Stokes type system, namely the family of systems (GNS_ν). As claimed above, everything in this context relies on Sobolev embeddings. Let us first recall the definition of homogeneous sobolev spaces.

Definition 1.2.1 *Let s be a real number. The homogeneous Sobolev space $\dot{H}^s(\mathbf{R}^d)$ is the set of tempered distributions u the Fourier transform of which $\widehat{u}$ belongs to $L^1_{loc}(\mathbf{R}^d)$ and satisfies*

$$\|u\|^2_{\dot{H}^s} \stackrel{\text{def}}{=} \int_{\mathbf{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

The aim of this paragraph is to study the embeddings of $H^s(\mathbf{R}^d)$ spaces into $L^p(\mathbf{R}^d)$ spaces. We shall prove the following theorem.

Theorem 1.2.1 *If s belongs to $[0, \frac{d}{2}[$, then the space $\dot{H}^s(\mathbf{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbf{R}^d)$.*

Proof of Theorem 1.2.1 First of all, let us show how to find the critical index $p = 2d/(d-2s)$. Let us use a scaling argument. Let v be a function on $\mathbf{R}^d$ and let us denote by v_λ the function $v_\lambda(x) \stackrel{\text{def}}{=} v(\lambda x)$. We have

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}} \|v\|_{L^p}.$$

and

$$\|v_\lambda\|^2_{\dot{H}^s} = \int |\xi|^{2s} |\widehat{v_\lambda}(\xi)|^2 d\xi$$

$$
\begin{aligned}
&= \lambda^{-2d} \int |\xi|^{2s} |\widehat{v}(\lambda^{-1}\xi)|^2 d\xi \\
&= \lambda^{-d+2s} \|v\|^2_{\dot{H}^s}.
\end{aligned}
$$

If an inequality of the type $\|v\|_{L^p} \leq C\|v\|_{\dot{H}^s}$ for any smooth function v is true, it is also true for v_λ for any λ. Then it is obvious that we must have $p = 2d/(d-2s)$.

Let us now prove the theorem. Without any loss of generality, we can assume that $\|f\|_{\dot{H}^s}$ is equal to 1. First we can observe, thanks to Fubini theorem, that for any $p \in [1, +\infty[$ and any measurable function f, we have

$$
\begin{aligned}
\|f\|^p_{L^p} &\stackrel{\text{def}}{=} \int_{\mathbf{R}^d} |f(x)|^p dx \\
&= p \int_{\mathbf{R}^d} \int_0^{|f(x)|} \lambda^{p-1} d\lambda dx \\
&= p \int_0^\infty \lambda^{p-1} m(|f| > \lambda) d\lambda.
\end{aligned}
$$

We shall decompose the function in low and high frequencies. More precisely, we shall write

$$
f = f_{1,A} + f_{2,A} \quad \text{with} \quad f_{1,A} = \mathcal{F}^{-1}\Big(\mathbf{1}_{B(0,A)}\widehat{f}\Big) \quad \text{and} \quad f_{2,A} = \mathcal{F}^{-1}\Big(\mathbf{1}_{B^c(0,A)}\widehat{f}\Big)\cdot \tag{1.3}
$$

As Supp $\widehat{f}_{1,A}$ is compact, the function $f_{1,A}$ is bounded and more precisely we have

$$
\begin{aligned}
\|f_{1,A}\|_{L^\infty} &\leq (2\pi)^{-d} \|\widehat{f_{1,A}}\|_{L^1} \\
&\leq (2\pi)^{-d} \int_{B(0,A)} |\xi|^{-s} |\xi|^s |\widehat{f}(\xi)| d\xi \\
&\leq (2\pi)^{-d} \left(\int_{B(0,A)} |\xi|^{-2s} d\xi\right)^{\frac{1}{2}} \\
&\leq \frac{C}{(d-2s)^{\frac{1}{2}}} A^{\frac{d}{2}-s}.
\end{aligned} \tag{1.4}
$$

The triangle inequality implies that for any positive A, we have

$$
(|f| > \lambda) \subset \Big(|f_{1,A}| > \frac{\lambda}{2}\Big) \cup \Big(|f_{2,A}| > \frac{\lambda}{2}\Big)\cdot
$$

From the above Inequality (1.4), we infer that

$$
A = A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda(d-2s)^{\frac{1}{2}}}{4C}\right)^{\frac{p}{d}} \Longrightarrow m\left(|f_{1,A}| > \frac{\lambda}{2}\right) = 0.
$$

From this we deduce that

$$
\|f\|^p_{L^p} = p \int_0^\infty \lambda^{p-1} m\Big(|f_{2,A_\lambda}| > \frac{\lambda}{2}\Big) d\lambda.
$$

It is well known (it is the so called Bienaimé-Tchebychev inequality) that

$$
\begin{aligned}
m\left(|f_{2,A_\lambda}| > \frac{\lambda}{2}\right) &= \int_{\left(|f_{2,A_\lambda}| > \frac{\lambda}{2}\right)} dx \\
&\leq \int_{\left(|f_{2,A_\lambda}| > \frac{\lambda}{2}\right)} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2} dx \\
&\leq 4 \frac{\|f_{2,A_\lambda}\|^2_{L^2}}{\lambda^2}.
\end{aligned}
$$

So we have that

$$\|f\|_{L^p}^p \leq 4p \int_0^\infty \lambda^{p-3} \|f_{2,A_\lambda}\|_{L^2}^2 d\lambda.$$

But we know that the Fourier transform is (up to a constant) a unitary transform of $L^2(\mathbf{R}^d)$. Thus we have

$$\|f\|_{L^p}^p \leq 4p(2\pi)^{-d} \int_0^\infty \lambda^{p-3} \int_{(|\xi| \geq A_\lambda)} |\widehat{f}(\xi)|^2 d\xi d\lambda. \tag{1.5}$$

Then by definition of A_λ, we have

$$|\xi| \geq A_\lambda \Longleftrightarrow \lambda \leq C_\xi \stackrel{\text{def}}{=} \frac{4C}{(d-2s)^{\frac{1}{2}}} |\xi|^{\frac{d}{p}}.$$

Fubini's theorem implies that

$$\begin{aligned} \|f\|_{L^p}^p &\leq 4p(2\pi)^{-d} \int_{\mathbf{R}^d} \left(\int_0^{C_\xi} \lambda^{p-3} d\lambda \right) |\widehat{f}(\xi)|^2 d\xi \\ &\leq 4\frac{p(2\pi)^d}{p-2} \left(\frac{4C}{(d-2s)^{\frac{1}{2}}} \right)^{p-2} \int_{\mathbf{R}^d} |\xi|^{\frac{d(p-2)}{p}} |\widehat{f}(\xi)|^2 d\xi. \end{aligned}$$

As $2s = \dfrac{d(p-2)}{p}$ the theorem is proved.

Corollary 1.2.1 *If p belongs to $]1,2]$, then*

$$L^p(\mathbf{R}^d) \subset \dot{H}^s(\mathbf{R}^d) \quad \textit{with} \quad s = d\left(\frac{1}{2} - \frac{1}{p}\right) \cdot$$

Proof of Corollary 1.2.1 This corollary is proved by duality. Let us write that

$$\|a\|_{\dot{H}^s} = \sup_{\|\varphi\|_{\dot{H}^{-s}(\mathbf{R}^d)} \leq 1} \langle a, \varphi \rangle.$$

As $s = d\left(\dfrac{1}{2} - \dfrac{1}{p}\right) = d\left(1 - \dfrac{1}{p} - \dfrac{1}{2}\right)$, we have by Theorem 1.2.1,

$$\|\varphi\|_{L^{\overline{p}}} \leq C\|\varphi\|_{\dot{H}^{-s}}$$

where $\overline{p}$ is the conjugate of p defined by $\dfrac{1}{p} + \dfrac{1}{\overline{p}} = 1$ and thus

$$\begin{aligned} \|a\|_{\dot{H}^s} &\leq C \sup_{\|\varphi\|_{L^{\overline{p}}} \leq 1} \langle a, \varphi \rangle \\ &\leq C\|a\|_{L^p}. \end{aligned}$$

The corollary is proved.

The main theorem in the framework of Sobolev spaces is the following.

Theorem 1.2.2 *Let u_0 be in $\dot{H}^{\frac{d}{2}-1}$. A positive time T exists such that the system (GNS_ν) has a unique solution u in $L^4([0,T];\dot{H}^{\frac{d-1}{2}})$which also belongs to*

$$C([0,T];\dot{H}^{\frac{d}{2}-1}) \cap L^2([0,T];\dot{H}^{\frac{d}{2}}).$$

If T_{u_0} denotes the maximal time of existence of such a solution, we have

- *the existence of a constant c such that*

$$\|u_0\|_{\dot H^{\frac d2-1}} \leq c\nu \Longrightarrow T_{u_0} = +\infty.$$

- *If T_{u_0} is finite, then*

$$\int_0^{T_{u_0}} \|u(t)\|^4_{\dot H^{\frac{d-1}{2}}}\, dt = +\infty. \tag{1.6}$$

Moreover, the solutions are stable in the following sense: if u and v are two solutions, then we have

$$\begin{aligned}\|u(t)-v(t)\|^2_{\dot H^{\frac d2-1}} + \nu\int_0^t \|u(t')-v(t')\|^2_{\dot H^{\frac d2}}\, dt' &\leq \|u_0-v_0\|^2_{\dot H^{\frac d2-1}}\\ &\quad\times \exp\biggl(\frac{C}{\nu^3}\int_0^t \Bigl(\|u(t')\|^4_{\dot H^{\frac{d-1}{2}}} + \|v(t')\|^4_{\dot H^{\frac{d-1}{2}}}\Bigr)dt'\biggr).\end{aligned}$$

Proof of Theorem 1.2.2 We shall proof that the map

$$u \longmapsto e^{\nu t\Delta}u_0 + B(u,u)$$

has a unique fixed point in the space $L^4([0,T];\dot H^{\frac{d-1}{2}})$ for an appropriate T. It basically relies on the following two lemmas. The first one is nothing more than a variation about Sobolev embeddings.

Lemma 1.2.1 *A constant C exists such that*

$$\|Q(a,b)\|_{\dot H^{\frac d2-2}} \leq C\|a\|_{\dot H^{\frac{d-1}{2}}}\|b\|_{\dot H^{\frac{d-1}{2}}}.$$

Proof of Lemma 1.2.1 The case when $d=2$ is different from the case when $d=3$. If $d=2$, thanks to Sobolev embedding (see Theorem 1.2.1 page 11), we have

$$\begin{aligned}\|Q(a,b)\|_{\dot H^{-1}} &\leq C\|ab\|_{L^2}\\ &\leq C\|a\|_{L^4}\|b\|_{L^4}\\ &\leq C\|a\|_{\dot H^{\frac12}}\|b\|_{\dot H^{\frac12}}.\end{aligned}$$

If $d=3$, we have by definition of Q,

$$\|Q(a,b)\|_{\dot H^{-\frac12}} \leq C\sup_{k,\ell}(\|a^k\partial b^\ell\|_{\dot H^{-\frac12}} + \|b^\ell\partial a^k\|_{\dot H^{-\frac12}}).$$

Thanks to the dual Sobolev embedding, (see Corollary 1.2.1 page 13), and Sobolev embedding, we have

$$\begin{aligned}\|Q(a,b)\|_{\dot H^{-\frac12}} &\leq C\sup_{k,\ell}(\|a^k\partial b^\ell\|_{L^{\frac32}} + \|b^\ell\partial a^k\|_{L^{\frac32}})\\ &\leq C(\|a\|_{L^6}\|b\|_{\dot H^1} + \|a\|_{\dot H^1}\|b\|_{L^6})\\ &\leq C\|a\|_{\dot H^1}\|b\|_{\dot H^1}.\end{aligned}$$

This proves the lemma.

The second lemma describes an aspect of the smoothing effect of the heat flow.

Lemma 1.2.2 *Let v be the solution in the set of continuous functions in time with values in $\mathcal{S}'$ of*

$$\begin{cases} \partial_t v - \nu\Delta v = f \\ v_{|t=0} = v_0 \end{cases}$$

with f in $L^2([0,T];\dot H^{s-1})$ and v_0 in $\dot H^s$. Then

$$v \in \bigcap_{p=2}^{\infty} L^p([0,T];\dot H^{s+\frac{2}{p}}).$$

Moreover, we have the following estimates

$$\begin{aligned} \|v(t)\|^2_{\dot H^s} + 2\nu\int_0^t \|\nabla v(t')\|^2_{\dot H^s}dt' &= 2\int_0^t \langle f(t'),v(t')\rangle_s dt'\,, &(1.7)\\ \int_{\mathbf{R}^d}|\xi|^{2s}\Big(\sup_{0\le t'\le t}|\widehat v(t',\xi)|\Big)^2 d\xi &\le \|v_0\|^2_{\dot H^s} + \frac{1}{(2\nu)^{\frac12}}\|f\|^2_{L^2([0,T];\dot H^{s-1})} \quad \textit{and} &(1.8)\\ \|v(t)\|_{L^p([0,T];\dot H^{s+\frac{2}{p}})} &\le \frac{1}{(2\nu)^{\frac1p}}\Big(\|v_0\|_{\dot H^s} + \frac{1}{(2\nu)^{\frac12}}\|f\|_{L^2([0,T];\dot H^{s-1})}\Big) &(1.9) \end{aligned}$$

with $\langle a,b\rangle_s \overset{\text{def}}{=} (2\pi)^{-d}\displaystyle\int |\xi|^{2s}\widehat a(\xi)\widehat b(-\xi)d\xi.$

Proof of Lemma 1.2.2 It consists mainely in writing Duhamel's formula in Fourier space, namely

$$\widehat v(t,\xi) = e^{-\nu t|\xi|^2}\widehat v_0(\xi) + \int_0^t e^{-\nu(t-t')|\xi|^2}\widehat f(t',\xi)dt'.$$

The Cauchy-Schwarz inequality implies that

$$\sup_{0\le t'\le t}|\widehat v(t',\xi)| \le |\widehat v_0(\xi)| + \frac{1}{\sqrt{2\nu|\xi|^2}}\|\widehat f(\cdot,\xi)\|_{L^2([0,t])}.$$

Then taking the L^2 norm with respect to $|\xi|^{2s}d\xi$ allows to conclude that

$$\begin{aligned} \Big(\int_{\mathbf{R}^d}\Big(\sup_{0\le t'\le t}|\widehat v(t',\xi)|\Big)^2|\xi|^{2s}d\xi\Big)^{\frac12} &\le \|v_0\|_{\dot H^s} + \frac{1}{(2\nu)^{\frac12}}\Big(\int_{\mathbf{R}^d}\|\widehat f(\cdot,\xi)\|^2_{L^2([0,t])}|\xi|^{2s-2}d\xi\Big)^{\frac12}\\ &\le \|v_0\|_{\dot H^s} + \frac{1}{(2\nu)^{\frac12}}\Big(\int_{[0,t]\times\mathbf{R}^d}|\widehat f(t',\xi)|^2|\xi|^{2s-2}d\xi dt'\Big)^{\frac12}\\ &\le \|v_0\|_{\dot H^s} + \frac{1}{(2\nu)^{\frac12}}\|f\|_{L^2([0,t];\dot H^{s-1})}. \end{aligned}$$

Then we get the result by energy estimate in $\dot H^s$ and interpolation.

As an immediat corollary, we have

Corollary 1.2.2 *A constant C_0 exists such that*

$$\|B(u,v)\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})} \le \frac{C_0}{\nu^{\frac34}}\|u\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})}\|v\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})}.$$

Now, using Lemma 1.1.1, we know that if

$$\|e^{\nu t\Delta}u_0\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})} \leq \frac{\nu^{\frac34}}{2C_0}, \tag{1.10}$$

then we have the existence of a unique solution of (GNS_ν) in the ball of center 0, radius $(2C_0^{-1}\nu^{\frac34})$ in the space $L^4([0,T];\dot H^{\frac{d-1}{2}})$.

Let us investigate when the condition (1.10) is satisfied. As we have

$$\begin{aligned} \forall t\geq 0\,,\ \|e^{\nu t\Delta}u_0\|_{\dot H^{\frac d2-1}} &\leq \|u_0\|_{\dot H^{\frac d2-1}} \quad \text{and}\\ \int_0^\infty \|e^{\nu t\Delta}u_0\|^2_{\dot H^{\frac d2}}\,dt &\leq \frac{1}{2\nu}\|u_0\|^2_{\dot H^{\frac d2-1}}, \end{aligned}$$

we have by interpolation

$$\|e^{\nu t\Delta}u_0\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})} \leq \frac{1}{(2\nu)^{\frac14}}\|u_0\|_{\dot H^{\frac d2-1}}. \tag{1.11}$$

Thus, if $\|u_0\|_{\dot H^{\frac d2-1}} \leq (2C_0)^{-1}\nu$, the smallness condition (1.10) is satisfied and we have a global solution.

Let us now investigate the case of large initial data. We shall decompose it in a small part in $\dot H^{\frac d2-1}$ and in a large part, the Fourier transform of which will be compactly supported. More precisely, if u_0 is in $\dot H^{\frac d2-1}$, a positive real number ρ_{u_0} exists such that

$$\Big(\int_{|\xi|\geq\rho_{u_0}}|\xi|^{d-2}|\widehat u_0(\xi)|^2d\xi\Big)^{\frac12} \leq \frac{\nu}{4C_0}.$$

Thus, we have

$$\|e^{\nu t\Delta}u_0\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})} \leq \frac{\nu^{\frac34}}{4C_0} + \|e^{\nu t\Delta}u_0^\flat\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})}$$

where $u_0^\flat \stackrel{\text{def}}{=} \mathcal F^{-1}(\mathbf 1_{B(0,\rho_{u_0})}(\xi)\widehat u_0(\xi))$. Let us write that

$$\begin{aligned} \|e^{\nu t\Delta}u_0^\flat\|_{L^4([0,T];\dot H^{\frac{d-1}{2}})} &\leq \rho_{u_0}^{\frac12}\|e^{\nu t\Delta}u_0^\flat\|_{L^4([0,T];\dot H^{\frac d2-1})}\\ &\leq (\rho_{u_0}^2T)^{\frac14}\|u_0\|_{\dot H^{\frac d2-1}}. \end{aligned}$$

Thus, if

$$T\leq\Big(\frac{\nu}{4C_0\rho_{u_0}^{\frac12}\|u_0\|_{\dot H^{\frac d2-1}}}\Big)^4, \tag{1.12}$$

we have proved the existence of a unique solution in the ball of center 0 and radius $\nu/2C_0$ of the space $L^4([0,T];\dot H^{\frac{d-1}{2}})$.

In order to prove the whole theorem, let us observe that, if u is a solution of (GNS_ν) in $L^4([0,T];\dot H^{\frac{d-1}{2}})$, Lemma 1.2.1 implies that $Q(u,u)$ belongs to $L^2([0,T];\dot H^{\frac d2-2})$; then Lemma 1.2.2 implies that the solution u belongs to

$$C([0,T];\dot H^{\frac d2-1})\cap L^2([0,T];\dot H^{\frac d2}).$$

In order to prove the stability estimate, let us consider two solutions u and v and let us denote by w their difference. It is the solution of

$$\begin{cases} \partial_t w - \nu\Delta w = Q(w,w) + Q(w, u+v) \\ w_{|t=0} = w_0 = u_0 - v_0. \end{cases}$$

Thus, by energy estimate in $\dot H^{\frac{d}{2}-1}$of Lemma 1.2.2, we have

$$\begin{aligned} \Delta_w(t) &\stackrel{\text{def}}{=} \|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\nu\int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' \\ &\leq \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t \langle Q(w(t'), u(t')+v(t')), w(t')\rangle_{\frac{d}{2}-1} dt'. \end{aligned}$$

The non linear term is treated through the following lemma.

Lemma 1.2.3 *A constant C exists such that*

$$\langle Q(a,b), c\rangle_{\frac{d}{2}-1} \leq C\|a\|_{\dot H^{\frac{d-1}{2}}}\|b\|_{\dot H^{\frac{d-1}{2}}}\|\nabla c\|_{\dot H^{\frac{d}{2}-1}}.$$

Proof of Lemma 1.2.3 By definition of the $\dot H^{\frac{d}{2}-1}$ scalar product, we have, thanks to the Cauchy-Schwarz inequality,

$$\begin{aligned} \langle \alpha, \beta\rangle_{\frac{d}{2}-1} &= \int \widehat\alpha(\xi)\overline{\widehat\beta}(\xi)|\xi|^{d-2} d\xi \\ &= \int |\xi|^{\frac{d}{2}-2}\widehat\alpha(\xi)|\xi|^{\frac{d}{2}}\overline{\widehat\beta}(\xi) d\xi \\ &\leq \|\alpha\|_{\dot H^{\frac{d}{2}-2}}\|\nabla\beta\|_{\dot H^{\frac{d}{2}-1}}. \end{aligned}$$

Then Lemma 1.2.1 implies the result.

Let us go back to the proof of the stability. We deduce from the above lemma that

$$\Delta_w(t) \leq \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C\int_0^t \|w(t')\|_{\dot H^{\frac{d-1}{2}}} N(t')\|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'$$

with $N(t) \stackrel{\text{def}}{=} \|u(t)\|_{\dot H^{\frac{d-1}{2}}} + \|v(t)\|_{\dot H^{\frac{d-1}{2}}}$. By interpolation inequality between $\dot H^{\frac{d}{2}-1}$ and $\dot H^{\frac{d}{2}}$, we infer that

$$\Delta_w(t) \leq \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C\int_0^t \|w(t')\|^{\frac12}_{\dot H^{\frac{d}{2}-1}} N(t')\|\nabla w(t')\|^{\frac32}_{\dot H^{\frac{d}{2}-1}} dt'.$$

Using the convexity inequality $ab \leq \dfrac14 a^4 + \dfrac34 b^{\frac43}$, we deduce that

$$\Delta_w(t) \leq \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} + \frac{C}{\nu^3}\int_0^t \|w(t')\|^2_{\dot H^{\frac{d}{2}-1}} N^4(t') dt' + \nu\int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'.$$

By definition of Δ_w, this can be written

$$\|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + \nu\int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' \leq \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} + \frac{C}{\nu^3}\int_0^t \|w(t')\|^2_{\dot H^{\frac{d}{2}-1}} N^4(t') dt'.$$

Using Gronwall lemma, we infer

$$\|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + \nu\int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' \leq \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} \exp\Big(\frac{C}{\nu^3}\int_0^t N^4(t') dt'\Big).$$

The whole theorem is proved but the blow up criteria. Let us assume that we have a solution of (GNS_ν) on a time interval $[0,T[$ such that

$$\int_0^T \|u(t)\|^4_{\dot H^{\frac{d-1}{2}}}\, dt < \infty.$$

We shall prove that T_{u_0} is greater than T. Thanks to Lemmas 1.2.1 and 1.2.2, we have

$$\int_{\mathbf{R}^d} |\xi|^{d-2}\Big(\sup_{t\in[0,T)} |\widehat{u}(t,\xi)|\Big)^2 d\xi < \infty.$$

Thus a positive number ρ exists such that

$$\forall t \in [0,T[\,,\ \int_{|\xi|\geq\rho} |\xi|^{d-2}|\widehat{u}(t,\xi)|^2 d\xi < \frac{c\nu}{2}\cdot$$

As u belongs to $L^\infty([0,T);\dot H^{\frac{d}{2}-1})$, condition (1.12) implies that the maximal time for a solution of (GNS_ν) with initial data $u(t)$ is bounded from below by a positive real number uniformely on $[0,T[$. Thus $T_{u_0} > T$. The whole Theorem 1.2.2 is now proved.

Now we shall establish a property of small solutions which tells that for the system (GNS_ν), the $\dot H^{\frac{d}{2}-1}$ norm is a Liapounov function near 0.

Proposition 1.2.1 *Let u_0 be in the ball of center* 0 *and radius $c\nu$ of the space $\dot H^{\frac{d}{2}-1}$. Then the function*

$$t \longmapsto \|u(t)\|_{\dot H^{\frac{d}{2}-1}}$$

is a decreasing function.

Proof of Proposition 1.2.1 We use again the fact that

$$\partial_t u - \nu\Delta u = Q(u,u) \quad \text{with} \quad Q(u,u) \in L^2(\mathbf{R}^+;\dot H^{\frac{d}{2}-2}).$$

Thus thanks to Lemma 1.2.2, we infer that

$$\|u(t)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\nu\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' = \|u_0\|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t \langle Q(u(t'),u(t')),u(t')\rangle_{\frac{d}{2}-1} dt'.$$

Using Lemma 1.2.3 and interpolation inequality, we get, for any positive $t_1 \leq t_2$,

$$\begin{aligned}\|u(t_2)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' &\leq \|u(t_1)\|^2_{\dot H^{\frac{d}{2}-1}} + C\int_{t_1}^{t_2} \|u(t')\|^2_{\dot H^{\frac{d-1}{2}}}\|\nabla u(t')\|_{\dot H^{\frac{d}{2}-1}} dt'\\ &\leq \|u(t_1)\|^2_{\dot H^{\frac{d}{2}-1}} + C\int_{t_1}^{t_2} \|u(t')\|_{\dot H^{\frac{d}{2}-1}}\|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'.\end{aligned}$$

As we know, thanks to Theorem 1.2.2, that $u(t)$ remains in the ball of center 0 and of radius $2c\nu$. Thus, if c is small enough, we get that

$$\|u(t_2)\|^2_{\dot H^{\frac{d}{2}-1}} + \nu\int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' \leq \|u(t_1)\|^2_{\dot H^{\frac{d}{2}-1}}.$$

This proves the proposition.

1.3 Consequences of the structure of the Navier-Stokes system

In this section, we shall investigate the particular properties of the Navier-Stokes system, namely results the proof of which uses the energy estimate.

Let us start with the case of dimension two. The energy estimate will allow us to prove that (NS_ν) is globally wellposed for initial data in L^2. The precise theorem is the following which is almost the same statement as Theorem 1.1.2 of the introduction.

Theorem 1.3.1 *Let u_0 be in $L^2(\mathbf{R}^2)$. Then a unique solution exists in $L^4(\mathbf{R}^+;\dot H^{\frac12})$ which belongs to*

$$C_b(\mathbf{R}^+;L^2)\cap L^2(\mathbf{R}^+;\dot H^1)$$

and satisfies the energy equality

$$\|u(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla u(t')\|_{L^2}^2dt'=\|u_0\|_{L^2}^2.$$

Proof of Theorem 1.3.1 It is easy considering what we have done in the preceeding section. Indeed, let us consider the solution u given on the interval $[0,T_{u_0}[$ by Theorem 1.2.2. Thanks to Lemma 1.2.1, we know that $Q(u,u)$ belongs to $L^2_{loc}([0,T_{u_0}[;\dot H^{-1})$. Then Lemma 1.2.2 implies that u is continuous with value in L^2 and that

$$\|u(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla u(t')\|_{L^2}^2dt'=\|u_0\|_{L^2}^2+2\int_0^t\langle Q_{NS}(u(t'),u(t')),u(t')\rangle_{\dot H^{-1}\times\dot H^1}dt'.$$

For any H^1 divergence free vector field v, we have, in fact for $d=2$ or 3,

$$\langle Q_{NS}(v,v),v\rangle=\sum_{k,\ell}\int_{\mathbf{R}^d}v^k\partial_kv^\ell v^\ell dx=-\frac12\int_{\mathbf{R}^d}(\operatorname{div}v)|v|^2dx=0. \tag{1.13}$$

We deduce that, for any $t<T_{u_0}$,

$$\|u(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla u(t')\|_{L^2}^2dt'=\|u_0\|_{L^2}^2.$$

Thanks to the above energy estimate and using interpolation inequality between L^2 and $\dot H^1$, we have, for any $T<T_{u_0}$,

$$\begin{aligned}\int_0^T\|u(t)\|_{\dot H^{\frac12}}^4dt &\leq \|u_0\|_{L^2}^2\int_0^T\|\nabla u(t)\|_{L^2}^2dt\\ &\leq \frac{1}{2\nu}\|u_0\|_{L^2}^4.\end{aligned}$$

Then the blow up condition (1.6) implies the theorem.

Remark The key point here is that the control of the energy estimate implies the control of scaling invariant quantities.

The case of dimension three is much more complicated. The global wellposedness of (NS_ν) for large data in $\dot H^{\frac12}$ remains open. The purpose of this section is first to prove the energy equality for solution of (NS_ν) given by Theorem 1.2.2 and then to state that any global solution is stable.

Proposition 1.3.1 *Let us consider an initial data u_0 in $H^{\frac{1}{2}}$. If u denotes the solution given by Theorem 1.2.2, then u is continuous with value in L^2 and satisfies the energy equality*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \|u_0\|_{L^2}^2.$$

Proof of Proposition 1.3.1 As the solution u belongs to

$$L^\infty_{loc}([0, T_{u_0}[; \dot{H}^{\frac{1}{2}}) \cap L^4_{loc}([0, T_{u_0}[; \dot{H}^1),$$

interpolation between Sobolev spaces implies that u belongs to $L^8_{loc}([0, T_{u_0}[; \dot{H}^{\frac{3}{4}})$ which is obviously a subspace of $L^4_{loc}([0, T_{u_0}[; \dot{H}^{\frac{3}{4}})$. Using Sobolev embedding, we infer that, for any j and k in $\{1, \cdots, d\}$, we have

$$u^j u^k \in L^2_{loc}([0, T_{u_0}[; L^2) \quad \text{and thus} \quad Q(u,u) \in L^2_{loc}([0, T_{u_0}[; \dot{H}^{-1}). \tag{1.14}$$

Lemma 1.2.2 allows to conclude the proof of the proposition.

We shall prove that any global solution, even for large initial data (if it exists) is stable.

Theorem 1.3.2 *Let u be a global solution of (NS_ν) in $L^4_{loc}(\mathbf{R}^+; \dot{H}^1)$. Then we have*

$$\lim_{t\to\infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = 0 \quad \textit{and} \quad \int_0^\infty \|u(t)\|_{\dot{H}^1}^4 dt < \infty.$$

Remark If u_0 belongs also to $L^2(\mathbf{R}^3)$, this theorem is an immediat consequence of Proposition 1.2.1 because thanks to energy estimate, we have

$$\int_{\mathbf{R}^+} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^4 dt \leq \frac{1}{2\nu}\|u_0\|_{L^2}^4.$$

Proof of Theorem 1.3.2 We shall decompose the initial data u_0. A positive real number ρ being given, let us state

$$u_0 = u_{0,h} + u_{0,\ell} \quad \text{with} \quad u_{0,\ell} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B(0,\rho)}(\xi)\widehat{u}_0(\xi)).$$

Let ε be any positive real number. We can choose ρ such that

$$\|u_{0,\ell}\|_{\dot{H}^{\frac{1}{2}}} \leq \min\Big\{c\nu, \ \frac{\varepsilon}{2}\Big\}\cdot$$

Let us denote by u_ℓ the global solution of (NS_ν) given by Theorem 1.2.2 for the initial data $u_{0,\ell}$. Thanks to Proposition 1.2.1, we have

$$\forall t \in \mathbf{R}^+\,, \ \|u_\ell(t)\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{\varepsilon}{2}\cdot \tag{1.15}$$

Let us define $u_h \stackrel{\text{def}}{=} u - u_\ell$. It satisfies

$$\begin{cases} \partial_t u_h - \nu\Delta u_h = Q_{NS}(u, u_h) + Q_{NS}(u_h, u_\ell) \\ \qquad\qquad u_{h|t=0} = u_{0,h}. \end{cases}$$

Obviously, u_h belongs to L^2 (of course with an L^2 norm which does depend on ρ and thus on ε). Moreover, Assertion (1.14) claims that both $Q_{NS}(u, u_h)$ and $Q_{NS}(u_h, u_\ell)$ belongs to $L^2_{loc}(\mathbf{R}^+; \dot{H}^{-1})$. Applying Lemma 1.2.2 and the fundamental relation (1.13) gives

$$\|u_h(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_h(t')\|_{L^2}^2 dt' = \|u_{0,h}\|_{L^2}^2 + 2\int_0^t \langle Q_{NS}(u_h(t'), u_\ell(t')), u_h(t')\rangle dt'.$$

Using Sobolev embedding, we claim that

$$\begin{aligned} |\langle Q_{NS}(u_h(t), u_\ell(t')), u_h(t)\rangle| &\leq C\|u_h(t)u_\ell(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \\ &\leq C\|u_h(t)\|_{L^6}\|u_\ell(t)\|_{L^3}\|\nabla u_h(t)\|_{L^2} \\ &\leq C\|u_\ell(t)\|_{\dot{H}^{\frac{1}{2}}}\|\nabla u_h(t)\|_{L^2}^2. \end{aligned}$$

Then we deduce that

$$\|u_h(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_h(t')\|_{L^2}^2 dt' \leq \|u_{0,h}\|_{L^2}^2 + C\varepsilon \int_0^t \|\nabla u_h(t')\|_{L^2}^2 dt'.$$

Choosing ε small enough ensures that

$$\|u_h(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u_h(t')\|_{L^2}^2 dt' \leq \|u_{0,h}\|_{L^2}^2.$$

This implies that a positive time t_ε exists such that $\|u_h(t)\|_{\dot{H}^{\frac{1}{2}}} < \varepsilon/2$. Thus $\|u(t_\varepsilon)\|_{\dot{H}^{\frac{1}{2}}} \leq \varepsilon$. Theorem 1.2.2 and Proposition 1.2.1 allows to conclude the proof.

Let us state the following corollary of Theorem 1.3.2.

Corollary 1.3.1 *The set of initial data u_0 such that the solution u given by Theorem 1.2.2 is global is an open subset of $\dot{H}^{\frac{1}{2}}$.*

Proof of corollary 1.3.1 Let us consider u_0 in $\dot{H}^{\frac{1}{2}}$ such that the associated solution is global. Let us consider w_0 in $\dot{H}^{\frac{1}{2}}$ and the (a priori) local solution v associated with the initial data $v_0 \stackrel{\text{def}}{=} u_0 + w_0$. The function $w \stackrel{\text{def}}{=} v - u$ is solution of

$$\begin{cases} \partial_t w - \nu\Delta w + Q_{NS}(u, w) + Q_{NS}(w, u) + Q_{NS}(w, w) = 0 \\ w_{|t=0} = w_0. \end{cases}$$

Lemma 1.2.3 together with interpolation inequality gives

$$\begin{aligned} \langle Q_{NS}(u, w) + Q_{NS}(w, u), w\rangle_{\frac{1}{2}} &\leq C\|u\|_{\dot{H}^1}\|w\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}}\|\nabla w\|_{\dot{H}^{\frac{1}{2}}}^{\frac{3}{2}} \quad \text{and} \\ \langle Q_{NS}(w, w), w\rangle_{\frac{1}{2}} &\leq C\|w\|_{\dot{H}^{\frac{1}{2}}}\|\nabla w\|_{\dot{H}^{\frac{1}{2}}}^2. \end{aligned}$$

Let us assume that $\|w_0\|_{\dot{H}^{\frac{1}{2}}} \leq \dfrac{\nu}{8C}$ and define

$$T_{w_0} \stackrel{\text{def}}{=} \sup\Big\{t \,/\, \max_{0\leq t'\leq t} \|w(t')\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{\nu}{4C}\Big\}\cdot$$

Then, using Lemma 1.2.2 and the convexity inequality, we infer that, for any $t < T_{w_0}$,

$$\|w(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu \int_0^t \|\nabla w(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' \leq \|w_0\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{C}{\nu^3}\int_0^t \|u(t')\|_{\dot{H}^1}^4 \|w(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt'.$$

Gronwall's Lemma and Theorem 1.3.2 imply that, for any $t < T_{w_0}$,

$$\|w(t)\|^2_{\dot H^{\frac12}} + \nu \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac12}} dt' \leq \|w_0\|^2_{\dot H^{\frac12}} \exp\Big(\frac{C}{\nu^3}\int_0^\infty \|u(t)\|^4_{\dot H^1} dt\Big).$$

If the smallness condition

$$\|w_0\|^2_{\dot H^{\frac12}} \exp\Big(\frac{C}{\nu^3}\int_0^\infty \|u(t)\|^4_{\dot H^1} dt\Big) \leq \frac{\nu^2}{16C^2}\,,$$

is satisfied, the blow up condition for v is never satisfied. Corollary 1.3.1 is proved.

1.4 An elementary L^p approach

As announced in the introduction of this chapter, the purpose of this section is the proof of a local wellposedness result for initial data in $L^3(\mathbf{R}^3)$. The main result is the following.

Theorem 1.4.1 *Let u_0 be in $L^3(\mathbf{R}^3)$. Then a positive T exists such that a unique solution u exists in the space $C([0,T];L^3)$. Moreover, a constant c exists such that T can be choosen equal at infinity if $\|u_0\|_{L^3}$ is less or equal to $c\nu$.*

The proof of this theorem cannot be done directly by a fixed point argument in the space $L^\infty([0,T];L^3)$ because B_{NS} is not continuous from $L^\infty([0,T];L^3) \times L^\infty([0,T];L^3)$ into $L^\infty([0,T];L^3)$. This has been shown by F. Oru in [46].

As in the preeceding section, we shall use the smoothing effect of the heat equation to define space in which a fixed point method will work. This leads to spaces often called in the literature "Kato spaces".

Definition 1.4.1 *If p is in $[1,\infty]\setminus\{3\}$ and T in $]0,\infty]$, let us define $K_p(T)$ by*

$$K_p(T) \stackrel{\rm def}{=} \Big\{u \in C(]0,T];L^p) \,/\, \|u\|_{K_p(T)} \stackrel{\rm def}{=} \sup_{t\in[0,T]} (\nu t)^{\frac12\left(1-\frac3p\right)}\|u(t)\|_{L^p} < \infty\Big\}.$$

We shall denote by $K_3(T)$ the space of bounded continuous functions from $]0,T]$ with value L^3 equipped with the norm $\|\cdot\|_{L^\infty(]0,T];L^3)}$.

Remarks 1) This space is obviously a Banach space. Moreover, if $T=+\infty$, it is invariant under the scaling of the Navier-Stokes.

2) Let us consider u_0 in L^3. As

$$e^{\nu t\Delta}u_0 = \frac{1}{(4\pi t)^{\frac32}} e^{-\frac{|\cdot|^2}{4\nu t}} \star u_0,$$

we have, thanks to the Young inequality,

$$\|e^{\nu t\Delta}u\|_{L^p} \leq \frac{1}{(4\pi t)^{\frac32}}\|e^{-\frac{|\cdot|^2}{4\nu t}}\|_{L^r}\|u_0\|_{L^3} \quad \text{with} \quad \frac1r = \frac23+\frac1p\cdot$$

This gives

$$\|e^{\nu t\Delta}u\|_{L^p} \leq c(\nu t)^{-\frac12\left(1-\frac3p\right)}\|u_0\|_{L^3} \quad \text{and thus} \quad \|e^{\nu t\Delta}u_0\|_{K_p(\infty)} \leq C\|u_0\|_{L^3}. \tag{1.16}$$

Let us point out that if u_0 belongs to L^3, one can find, for any positive ε, a function ϕ in $\mathcal{S}$ such that $\|u_0 - \phi\|_{L^3} \leq \varepsilon$. This implies in particular that

$$\|e^{\nu t\Delta}(u_0 - \phi)\|_{K_p(\infty)} \leq C\varepsilon.$$

Then observing that $\|e^{\nu t\Delta}\phi\|_{L^p} \leq \|\phi\|_{L^p}$, we get that

$$\|e^{\nu t\Delta}u_0\|_{K_p(T)} \leq C\varepsilon + T^{\frac{1}{2}\left(1-\frac{3}{p}\right)}\|\phi\|_{L^p}. \tag{1.17}$$

3) Let us give an example of a sequence $(\phi_n)_{n\in\mathbf{N}}$ such that the L^3 norm is constant, the $\dot{H}^{\frac{1}{2}}$ tends to infinity and the $K_p(\infty)$ norm of $e^{\nu t\Delta}\phi_n$ tends to 0 when $p > 3$. Let us consider, for some ω in the unit sphere, the sequence

$$\phi_n(x) \stackrel{\text{def}}{=} e^{in(x|\omega)}\phi(x)$$

for some function ϕ in $\mathcal{S}$ the Fourier transform of which is compactly supported. Obviously, we have $\|\phi_n\|_{\dot{H}^{\frac{1}{2}}} \geq cn$. Straightforward computations give

$$e^{\nu t\Delta}\phi_n(x) = e^{in(x|\omega)}\int e^{i(x|\eta)}e^{-\nu t|\eta+n\omega|^2}\widehat{\phi}(\eta)d\eta.$$

Thus, if n is large enough, we have

$$\begin{aligned} t^{\frac{1}{2}}\|e^{\nu t\Delta}\phi_n\|_{L^\infty} &\leq t^{\frac{1}{2}}e^{-\frac{\nu}{2}tn^2}\|\widehat{\phi}\|_{L^1} \\ &\leq \frac{C}{n}\|\widehat{\phi}\|_{L^1}. \end{aligned}$$

As $\|\phi_n\|_{L^3} = \|\phi\|_{L^3}$, we have the announced example.

4) As we shall see in Chapter 2, $\|e^{\nu t\Delta}u_0\|_{K_p(\infty)}$ is equivalent, when $p > 3$ to the homogeneous Besov norm $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ (see Theorem 2.3.1 page 40).

In fact, Theorem 1.4.1 will appear mainely as a corollary of the following theorem.

Theorem 1.4.2 *For any p in $]3,\infty[$, a constant c exists which satisfies the following properties. Let u_0 be an initial data in $\mathcal{S}'$ such that, for some positive T,*

$$\|e^{\nu t\Delta}u_0\|_{K_p(T)} \leq c\nu. \tag{1.18}$$

Then a unique solution u of (GNS_ν) exists in the ball of center 0 and radius $2c\nu$ in the Banach space $K_p(T)$.

Remark Thanks to Inequality (1.17), this theorem implies that, for an initial data in L^3, we have local solution. Thanks to Inequality (1.16), this solution is global if $\|u_0\|_{L^3}$ is small enough.

Proof of Theorem 1.4.2 We shall prove that the classical fixed point procedure can be used in $K_p(T)$ in order to find u such that

$$u = e^{\nu t\Delta}u_0 + B(u,u).$$

This works provided we prove the following lemma.

Lemma 1.4.1 *For any p, q and r such that*

$$0 < \frac{1}{p} + \frac{1}{q} \leq 1 \quad \text{and} \quad \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} < \frac{1}{3} + \frac{1}{r}\cdot$$

Then, for any positive T, the bilinear map B maps $K_p(T) \times K_q(T)$ into $K_r(T)$. Moreover, a constant C (independant of T) exists such that

$$\|B(u,v)\|_{K_r(T)} \leq \frac{C}{\nu} \|u\|_{K_p(T)} \|v\|_{K_q(T)}.$$

Proof of Lemma 1.4.1 The method consists in computing B as a convolution operator. More precisely, we have the following proposition.

Proposition 1.4.1 *We have*

$$B^j(u,v)(t,x) = \sum_{k,\ell} \int_0^t \Gamma^j_{k,\ell}(t-t',\cdot) \star \Big(u^j(t',\cdot) v^\ell(t',\cdot) \Big) dt'$$

where the functions $\Gamma^j_{k,\ell}$ belongs to $C(]0,\infty[; L^s)$ for any s in $[1,\infty[$ and satisfies, for any j, k and ℓ,

$$\|\Gamma^j_{k,\ell}(t,\cdot)\|_{L^s} \leq \frac{C}{(\nu t)^{2-\frac{3}{2s}}}\cdot$$

Proof of Proposition 1.4.1 In Fourier space, we have

$$\mathcal{F}B^j(u,v)(t,\xi) = i \int_0^t e^{-\nu(t-t')|\xi|^2} \sum_{k,\ell} \alpha_{j,k\ell} \xi_j \xi_k \xi_\ell |\xi|^{-2} \mathcal{F}Q(u(t'),v(t'))(\xi) dt'.$$

In order to write this operator as a convolution operator, it is enough to compute the inverse Fourier transform of $\xi_j \xi_k \xi_\ell |\xi|^{-2} e^{-\nu t|\xi|^2}$. Using the fact that

$$e^{-\nu t|\xi|^2} |\xi|^{-2} = \nu \int_t^\infty e^{-\nu t'|\xi|^2} dt',$$

we get that

$$\begin{aligned} \Gamma^j_{k,\ell}(t,x) &= \nu i \int_t^\infty \int_{\mathbf{R}^3} \xi_j \xi_k \xi_\ell e^{i(x|\xi) - \nu t'|\xi|^2} dt' d\xi \\ &= \nu \partial_j \partial_k \partial_\ell \int_t^\infty \int_{\mathbf{R}^3} e^{i(x|\xi) - \nu t'|\xi|^2} dt' d\xi. \end{aligned}$$

Using the formula about the Fourier transform of the Gaussian functions, we get

$$\begin{aligned} \Gamma^j_{k,\ell}(t,x) &= \nu \partial_j \partial_k \partial_\ell \int_t^\infty \frac{1}{(4\pi\nu t')^{\frac{3}{2}}} e^{-\frac{|x|^2}{4\nu t'}} dt' \\ &= \frac{\nu}{\pi^{\frac{3}{2}}} \int_t^\infty \frac{1}{(4\nu t')^3} \Psi^j_{k,\ell}\Big(\frac{x}{\sqrt{4\nu t'}}\Big) dt' \quad \text{with} \\ \Psi^j_{k,\ell}(z) &\stackrel{\text{def}}{=} \partial_j \partial_k \partial_\ell e^{-|z|^2}. \end{aligned}$$

Changing variable $r = (4\nu t')^{-1}|x|^2$ gives

$$|\Gamma^j_{k,\ell}(t,x)| \leq \frac{\nu}{\pi^{\frac{3}{2}}} \frac{1}{|x|^4} \int_0^{\frac{|x|^2}{4\nu t}} r \Psi^j_{k,\ell}\Big(\frac{x}{|x|} r\Big) dr.$$

This implies that

$$|\Gamma^j_{k,\ell}(t,x)| \leq c\min\Bigl\{\frac{1}{(\nu t)^2}, \frac{1}{|x|^4}\Bigr\} \quad \text{and thus} \quad \|\Gamma^j_{k,\ell}(t,\cdot)\|_{L^s} \leq \frac{C}{(\nu t)^{2-\frac{3}{2s}}}\cdot$$

In order to prove the continuity, let us observe that, for $0 \leq c \leq t_1 \leq t_2$, we have

$$|\Gamma^j_{k,\ell}(t_2,x) - \Gamma^j_{k,\ell}(t_1,x)| \leq \frac{C}{|x|^4}\int_{\frac{|x|^2}{4\nu t_2}}^{\frac{|x|^2}{4\nu t_1}} re^{-r}dr.$$

This implies that

$$|\Gamma^j_{k,\ell}(t_2,x) - \Gamma^j_{k,\ell}(t_1,x)| \leq C\min\Bigl\{\frac{t_2^2-t_1^2}{(\nu t_1 t_2)^2}, \frac{1}{|x|^4}\Bigr\}.$$

The proposition is proved.

Let us go back to proof of the lemma. Thanks to Young's and Hölder inequality and the condition

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} \leq 1,$$

we have, using Proposition 1.4.1 with s defined by $1 + \dfrac{1}{r} = \dfrac{1}{s} + \dfrac{1}{p} + \dfrac{1}{q}$,

$$\|B(u,v)(t)\|_{L^r} \leq C\int_0^t \frac{1}{\sqrt{\nu(t-t')}^{4-3\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}}\|u(t')\|_{L^p}\|v(t')\|_{L^q}dt'.$$

By definition of the $K_p(T)$ norms, we get that

$$\begin{aligned}
\|B(u,v)(t)\|_{L^r} &\leq \|u\|_{K_p(T)}\|v\|_{K_q(T)}\int_0^t \frac{1}{\sqrt{\nu(t-t')}^{1-3\left(\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}}\frac{1}{\sqrt{\nu t'}^{2-3\left(\frac{1}{p}+\frac{1}{q}\right)}}dt'\\
&\leq \frac{C}{\nu}\frac{1}{(\nu t)^{\frac{1}{2}\left(1-\frac{3}{r}\right)}}\|u\|_{K_p(T)}\|v\|_{K_q(T)}.
\end{aligned}$$

Lemma 1.4.1 is proved.

Now Lemma 1.1.1 implies Theorem 1.4.2.

Proof of Theorem 1.4.1 Thanks to (1.16) and (1.17), we can apply Theorem 1.4.2 with p equal to 6 locally for any initial data and globally for small initial data.

We have existence and uniqueness in the space $K_6(T)$ for small enough T of for $T = \infty$ for small enough initial data. The two points which remain unproven are:

- the solution u is continuous with value in L^3,
- this solution is unique among all the continuous functions with value in L^3.

Those two problems are solved using a method which turns out to be important in the study of Navier-Stokes equations or of (GNS_ν): it consists in considering the new unknown

$$w \stackrel{\text{def}}{=} u - e^{\nu t\Delta}u_0.$$

The idea is that w is better behaved than u. Obviously, we have $w = B(u,u)$. Lemma 1.4.1 applied with $p = q = 6$ and $r = 3$ implies that $w \in C(]0,T]; L^3(\mathbf{R}^3))$. The continuity of w in the origin will follow from the fact that, still using Lemma 1.4.1, we have

$$\|w\|_{L^\infty([0,t];L^3)} \leq \frac{C}{\nu}\|u\|^2_{K_6(t)}.$$

But, Lemma 1.1.1 tells us that

$$\|u\|_{K_6(t)} \leq 2\|e^{\nu t\Delta}u_0\|_{K_6(t)}.$$

Remarks (1.16) and (1.17) then implies

$$\lim_{t\to 0}\|w\|_{L^\infty([0,t];L^3)} = 0.$$

As the heat flow is continuous with values in L^3, we have proved that the solution u is continuous with values in L^3.

Let us prove now that there is at most one solution in the space $C([0,T];L^3)$. Let us observe that, applying Lemma 1.4.1 with $p = q = 3$ and $r = 2$ implies that

$$w = B(u,u) \in K_2(T).$$

In particular, w belongs to $C([0,T];L^2)$. Let us consider two solutions u_j of (GNS_ν) in the space $C([0,T];L^3)$ associated with the same initial data and let us denote by u_{21} the difference $u_2 - u_1$, which coincides with the difference $w_2 - w_1$. Thus it belongs to $C([0,T];L^2)$ and satisfies

$$\begin{cases} \partial_t u_{21} - \nu\Delta u_{21} = f_{21} \\ u_{21|t=0} = 0 \end{cases}$$

with

$$f_{21} = Q(e^{\nu t\Delta}u_0, u_{21}) + Q(u_{21}, e^{\nu t\Delta}u_0) + Q(w_2, u_{21}) + Q(u_{21}, w_1).$$

Thanks to Sobolev embeddings, we have

$$\begin{aligned} \|Q(a,b)\|_{\dot H^{-\frac{3}{2}}} &\leq C \sup_{1\leq k,\ell\leq d}\|a^k b^\ell\|_{\dot H^{-\frac{1}{2}}} \\ &\leq C \sup_{1\leq k,\ell\leq d}\|a^k b^\ell\|_{L^{\frac{3}{2}}} \qquad (1.19) \\ &\leq C\|a\|_{L^3}\|b\|_{L^3}. \qquad (1.20) \end{aligned}$$

Thus, the external force f_{21} belongs in particular to $L^2([0,T];\dot H^{-\frac{3}{2}})$. As u_{21} is the unique solution in the space of continuous functions with value in $\mathcal{S}'$, we infer that u_{21} belongs to

$$L^\infty([0,T];\dot H^{-\frac{1}{2}}) \cap L^2([0,T];\dot H^{\frac{1}{2}})$$

and satisfies, thanks to Lemma 1.2.2,

$$\begin{aligned} \|u_{21}(t)\|^2_{\dot H^{-\frac{1}{2}}} + 2\nu\int_0^t \|u_{21}(t')\|^2_{\dot H^{\frac{1}{2}}}dt' &= 2\int_0^t \langle f_{21}(t'), u_{21}(t')\rangle_{\dot H^{-\frac{1}{2}}}dt' \qquad (1.21) \\ &\leq 2\int_0^t \|f_{21}(t')\|_{\dot H^{-\frac{3}{2}}}\|u_{21}(t')\|_{\dot H^{\frac{1}{2}}}dt'. \qquad (1.22) \end{aligned}$$

As the space is continuous and compactly supported functions in dense in L^3, we decompose u_0 as a sum of a small function in the sense of the L^3 norm and a function of L^6.

$$u_0 = u_0^\sharp + u_0^\flat \quad \text{with} \quad \|u_0^\sharp\|_{L^3} \leq c\nu \quad \text{and} \quad u_0^\flat \in L^6. \tag{1.23}$$

Stating

$$g_{21} \stackrel{\text{def}}{=} f_{21} - Q(e^{\nu t\Delta}u_0^\flat, u_{21}) - Q(u_{21}, e^{\nu t\Delta}u_0^\flat)$$

and applying (1.20) gives, again thanks to Sobolev embeddings,

$$\begin{aligned}
A_{21}(t) &\stackrel{\text{def}}{=} \|g_{21}(t)\|_{\dot{H}^{-\frac{3}{2}}} \\
&\leq C\Big(\|e^{\nu t\Delta}u_0^\sharp\|_{L^3} + \|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)}\Big)\|u_{21}(t)\|_{L^3} \\
&\leq C\Big(\|u_0^\sharp\|_{L^3} + \|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)}\Big)\|u_{21}(t)\|_{\dot{H}^{\frac{1}{2}}}.
\end{aligned}$$

If t is small enough, and c choosen small enough in (1.23), we get

$$A_{21}(t) \leq \frac{\nu}{4}\|u_{21}(t)\|_{\dot{H}^{\frac{1}{2}}}. \tag{1.24}$$

Still using the Sobolev embeddings and the Hölder inequality, we can write that

$$\begin{aligned}
B_{21}(t) &\stackrel{\text{def}}{=} \Big\|Q(e^{\nu t\Delta}u_0^\flat, u_{21}) + Q(u_{21}, e^{\nu t\Delta}u_0^\flat)\Big\|_{\dot{H}^{-\frac{3}{2}}} \\
&\leq C \sup_{1\leq k,\ell\leq d} \|(e^{\nu t\Delta}u_0^{\flat,k})u_{21}^\ell\|_{L^{\frac{3}{2}}} \\
&\leq C\|e^{\nu t\Delta}u_0^\flat\|_{L^6}\|u_{21}\|_{L^2}.
\end{aligned}$$

Using the fact that the heat flow is a contraction of the L^p spaces, and then the interpolation inequality between $\dot{H}^{-\frac{1}{2}}$ and $\dot{H}^{\frac{1}{2}}$, we get

$$B_{21}(t) \leq C\|u_0^\flat\|_{L^6}\|u_{21}(t)\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}\|u_{21}(t)\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Then using (1.22) and (1.24), we infer that

$$\|u_{21}(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \frac{3}{2}\nu\int_0^t \|u_{21}(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' \leq C\|u_0^\flat\|_{L^6}\int_0^t \|u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}\|u_{21}(t')\|_{\dot{H}^{\frac{1}{2}}}^{\frac{3}{2}} dt'.$$

Then, using the classical convexity inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, we get

$$\|u_{21}(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \nu\int_0^t \|u_{21}(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' \leq \frac{C}{\nu^3}\|u_0^\flat\|_{L^6}^4\int_0^t \|u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}^2 dt'.$$

Gronwall lemma implies that $u_{21} \equiv 0$. Theorem 1.4.1 is proved.

1.5 References and Remarks

The mathematical theory of the incompressible Navier-Stokes system has been founded by J. Leray in 1934 in his famous paper [41]. The concept of weak solutions is introduced and the existence of such solutions is proved. The regularity properties of those weak solutions has been studied (see in particular [2]). In this seminal paper [41], J. Leray also proved that if the initial data satisfies a smallness condition of the type

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \leq c\nu^2 \quad \text{or} \quad \|u_0\|_{L^2}^2\|\nabla u_0\|_{L^\infty} \leq c\nu^3,$$

then the solution exists in a space which ensures the uniqueness of such a solution. The smallness condition has been improved by H. Fujita and T. Kato in 1964. In [22], they essentially proved Theorem 1.2.2. The proof presented here relies mainely on Sobolev inequalities. The proof of these classical inequalities given here comes from [8].

The global stability Theorem 1.3.2 has been proved by I. Gallagher, D. Iftimie and F. Planchon in [26] and the idea of Corollary 1.3.1 can be founded in [48]. The existence part of Theorem 1.4.1 is closed to T. Kato's Theorem of 1972 proved in [36]. The uniqueness of solutions continuous solutions with value in L^3 has been proved by G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo in [23]. The proof of Proposition 1.4.1 follows the computations done for instance by F. Vigneron in [57].

Chapter 2

Littlewood-Paley theory

2.1 Localization in frequency space

The very basic idea of this theory consists in a localization procedure in the frequency space. The interest of this method is that the derivatives (or more generally Fourier multipliers) act in a very special way on distributions the Fourier transform of which is supported in a ball or a ring. More precisely, we have the following lemma.

Lemma 2.1.1 *Let $\mathcal{C}$ be a ring, B a ball. A constant C exists so that, for any non negative integer k, any smooth homogeneous function σ of degree m, any couple of real (a, b) so that $b \geq a \geq 1$ and any function u of L^a, we have*

$$\text{Supp } \widehat{u} \subset \lambda B \Rightarrow \sup_{\alpha = k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a};$$

$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^a} \leq \sup_{\alpha = k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \lambda^k \|u\|_{L^a};$$

$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow \|\sigma(D) u\|_{L^b} \leq C_{\sigma,m} \lambda^{m+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}.$$

Proof of Lemma 2.1.1 Using a dilation of size λ, we can assume all along the proof that $\lambda = 1$. Let ϕ be a function of $\mathcal{D}(\mathbf{R}^d)$, the value of which is 1 near B. As $\widehat{u}(\xi) = \phi(\xi)\widehat{u}(\xi)$, we can write, if g denotes the inverse fourier transform of ϕ,

$$\partial^\alpha u = \partial^\alpha g \star u.$$

Applying Young inequalities the result follows through

$$\begin{aligned} \|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \\ &\leq 2\|(1 + |\cdot|^2)^d \partial^\alpha g\|_{L^\infty} \\ &\leq 2\|(\mathrm{Id} - \Delta)^d ((\cdot)^\alpha \phi)\|_{L^1} \\ &\leq C^{k+1}. \end{aligned}$$

To prove the second assertion, let us consider a function $\widetilde{\phi}$ which belongs to $\mathcal{D}(\mathbf{R}^d \setminus \{0\})$ the value of which is identically 1 near the ring $\mathcal{C}$. Using the algebraic identity

$$\begin{aligned} |\xi|^{2k} &= \sum_{1 \leq j_1, \cdots, j_k \leq d} \xi_{j_1}^2 \cdots \xi_{j_k}^2 \\ &= \sum_{|\alpha| = k} (i\xi)^\alpha (-i\xi)^\alpha, \end{aligned} \tag{2.1}$$

and stating $g_\alpha \stackrel{\text{def}}{=} \mathcal{F}^{-1}(i\xi_j)^\alpha|\xi|^{-2k}\widetilde{\phi}(\xi)$, we can write, as $\widehat{u} = \widetilde{\phi}\widehat{u}$ that

$$\widehat{u} = \sum_{|\alpha|=k} (-i\xi)^\alpha \widehat{g}_\alpha \widehat{u},$$

which implies that

$$u = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha u \tag{2.2}$$

and then the result. In order to prove the third assertion, let us observe that the function $\widetilde{\phi}\sigma$ is smooth and compactly supported. Thus stating $g_\sigma \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\widetilde{\phi}\sigma)$, we have that $\sigma(D)u = g_\sigma \star u$ and then

$$\|\sigma(D)u\|_{L^b} \leq C\|u\|_{L^b} \leq C\|u\|_{L^a}.$$

This proves the whole lemma.

The following lemma is in the same spirit. It describes the action of the semi-group of the heat equation on distributions the Fourier transform of which is supported in a ring.

Lemma 2.1.2 *Let $\mathcal{C}$ be a ring. Two positive constants c and C exist such that, for any real a greater than 1, any couple (t, λ) of positive real numbers, we have*

$$\text{Supp } \widehat{u} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}u\|_{L^a} \leq Ce^{-ct\lambda^2}\|u\|_{L^a}.$$

Proof of Lemma 2.1.2 Again, let us consider a function ϕ of $\mathcal{D}(\mathbf{R}^d\setminus\{0\})$, the value of which is identically 1 near the ring $\mathcal{C}$. Let us also assume that $\lambda = 1$. Then, we have

$$\begin{aligned} e^{t\Delta}u &= \phi(D)e^{t\Delta}u \\ &= \mathcal{F}^{-1}\left(\phi(\xi)e^{-t|\xi|^2}\widehat{u}(\xi)\right) \\ &= g(t,\cdot)\star u \quad \text{with} \\ g(t,x) &\stackrel{\text{def}}{=} (2\pi)^{-d}\int e^{i(x|\xi)}\phi(\xi)e^{-t|\xi|^2}d\xi. \end{aligned} \tag{2.3}$$

If we prove that two strictly positive real numbers c and C exist such that, for all strictly positive t, we have

$$\|g(t,\cdot)\|_{L^1} \leq Ce^{-ct}, \tag{2.4}$$

then the lemma is proved. Let us do integrations by part in (2.3) . We get

$$\begin{aligned} g(t,x) &= (1+|x|^2)^{-d}\int(1+|x|^2)^d e^{i(x|\xi)}\phi(\xi)e^{-t|\xi|^2}d\xi \\ &= (1+|x|^2)^{-d}\int\left((\mathrm{Id}-\Delta_\xi)^d e^{i(x|\xi)}\right)\phi(\xi)e^{-t|\xi|^2}d\xi \\ &= (1+|x|^2)^{-d}\int_{\mathbf{R}^d} e^{i(x|\xi)}(\mathrm{Id}-\Delta_\xi)^d\left(\phi(\xi)e^{-t|\xi|^2}\right)d\xi. \end{aligned}$$

Through Leibnitz's formula, we obtain

$$(\mathrm{Id}-\Delta)^d\left(\phi(\xi)e^{-t|\xi|^2}\right) = \sum_{\beta\leq|\alpha|\leq 2d} C^\alpha_\beta\left(\partial^{(\alpha-\beta)}\phi(\xi)\right)\left(\partial^\beta e^{-t|\xi|^2}\right).$$

The Faà-di-Bruno's formula tells us that

$$e^{t|\xi|^2}\partial^\beta(e^{-t|\xi|^2}) = \sum_{\substack{\beta_1+\cdots+\beta_m=\beta \\ |\beta_j|\geq 1}} (-t)^m \prod_{j=1}^{m} \partial^{\beta_j}(|\xi|^2).$$

As the support of ϕ is included in a ring, it turns out that it exists a couple (c, C) of strictly positive real numbers such that, for any ξ in the support of ϕ,

$$\begin{aligned} \left|\left(\partial^{(\alpha-\beta)}\phi(\xi)\right)\left(\partial^\beta e^{-t|\xi|^2}\right)\right| &\leq C(1+t)^{|\beta|}e^{-t|\xi|^2} \\ &\leq C(1+t)^{|\beta|}e^{-ct}. \end{aligned}$$

Thus we have proved that $|g(t,x))| \leq (1+|x|^2)^{-d}e^{-ct}$, which proves Inequality (2.4).

Using Lemmas 2.1.1 and 2.1.2 together with Duhamel's formula, we infer immediately the following corollary.

Corollary 2.1.1 *Let $\mathcal{C}$ be a ring. Two positive constants c and C exist such that, for any real a greater than 1, any positive λ and any f satisfying, for any $t \in [0,T]$, $\mathrm{Supp}\ \widehat{f}(t) \subset \lambda\mathcal{C}$, we have for u the solution of*

$$\partial_t u - \nu\Delta u = f \quad \text{and} \quad u_{|t=0} = 0.$$

and for any $(a,b,p,q) \in [1,\infty]^4$ such that $b \geq a$ and $p \geq q$

$$\|u\|_{L^q([0,T];L^b)} \leq C(\nu\lambda^2)^{-1+\left(\frac{1}{p}-\frac{1}{q}\right)}\lambda^{d\left(\frac{1}{a}-\frac{1}{b}\right)}\|f\|_{L^p([0,T];L^a)}.$$

Now, let us define a dyadic partition of unity. We shall use it all along this text.

Proposition 2.1.1 *Let us define by $\mathcal{C}$ the ring of center 0, of small radius 3/4 and great radius 8/3. It exists two radial functions χ and φ the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$ such that*

$$\forall \xi \in \mathbf{R}^d\,,\ \chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1, \tag{2.5}$$

$$\forall \xi \in \mathbf{R}^d\setminus\{0\}\,,\ \sum_{j\in\mathbf{Z}}\varphi(2^{-j}\xi) = 1, \tag{2.6}$$

$$|j-j'| \geq 2 \Rightarrow \mathrm{Supp}\ \varphi(2^{-j}\cdot)\cap \mathrm{Supp}\ \varphi(2^{-j'}\cdot) = \emptyset, \tag{2.7}$$

$$j \geq 1 \Rightarrow \mathrm{Supp}\ \chi \cap \mathrm{Supp}\ \varphi(2^{-j}\cdot) = \emptyset, \tag{2.8}$$

If $\widetilde{\mathcal{C}} = B(0,2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|j-j'| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset, \tag{2.9}$$

$$\forall \xi \in \mathbf{R}^d\,,\ \frac{1}{3} \leq \chi^2(\xi) + \sum_{j\geq 0}\varphi^2(2^{-j}\xi) \leq 1, \tag{2.10}$$

$$\forall \xi \in \mathbf{R}^d\setminus\{0\}\,,\ \frac{1}{2} \leq \sum_{j\in\mathbf{Z}}\varphi^2(2^{-j}\xi) \leq 1. \tag{2.11}$$

Proof of Proposition 2.1.1 Let us choose α in the interval $]1, 4/3[$ let us denote by $\mathcal{C}'$ the ring of small radius α^{-1} and big radius 2α. Let us choose a smooth function θ, radial with value in $[0, 1]$, supported in $\mathcal{C}$ with value 1 in the neighbourhood of $\mathcal{C}'$. The important point is the following. For any couple of integers (p, q) we have

$$|j - j'| \geq 2 \Rightarrow 2^j\mathcal{C} \cap 2^{j'}\mathcal{C} = \emptyset. \tag{2.12}$$

Let us suppose that $2^{j'}\mathcal{C} \cap 2^j\mathcal{C} \neq \emptyset$ and that $j' \geq j$. It turns out that $2^{j'} \times 3/4 \leq 4 \times 2^{j+1}/3$, which implies that $j' - j \leq 1$. Now let us state

$$S(\xi) = \sum_{j\in\mathbf{Z}} \theta(2^{-j}\xi).$$

Thanks to (2.12), this sum is locally finite on the space $\mathbf{R}^d\backslash\{0\}$. Thus the function S is smooth on this space. As α is greater than 1,

$$\bigcup_{j\in\mathbf{Z}} 2^j\mathcal{C}' = \mathbf{R}^d\backslash\{0\}.$$

As the function θ is non negative and has value 1 near $\mathcal{C}'$, it comes from the above covering property that the above function is positive. Then let us state

$$\varphi = \frac{\theta}{S}\cdot \tag{2.13}$$

Let us check that φ fits. It is obvious that $\varphi \in \mathcal{D}(\mathcal{C})$. The function $1 - \sum_{j\geq 0} \varphi(2^{-j}\xi)$ is smooth thanks to (2.12). As the support of θ is included in $\mathcal{C}$, we have

$$|\xi| \geq \frac{4}{3} \Rightarrow \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1. \tag{2.14}$$

Thus stating

$$\chi(\xi) = 1 - \sum_{j\geq 0} \varphi(2^{-j}\xi), \tag{2.15}$$

we get Identites (2.5)and (2.7). Identity (2.8) is a obvious consequence of (2.12) and of (2.14). Now let us prove (2.9) which will be useful in Section 2.5. It is clear that the ring $\widetilde{\mathcal{C}}$ is the ring of center 0, of small radius 1/12 and of big radius 10/3. Then it turns out that

$$2^p\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} \neq \emptyset \Rightarrow \Big(\frac{3}{4} \times 2^j \leq 2^p \times \frac{10}{3} \quad \text{or} \quad \frac{1}{12} \times 2^p \leq 2^j\frac{8}{3}\Big),$$

and (2.9) is proved. Now let us prove (2.10). As χ and φ have their values in $[0, 1]$, it is clear that

$$\chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1. \tag{2.16}$$

Let us bound from below the sum of squares. The notation $a \equiv b(2)$ means that $a - b$ is even. So we have

$$\begin{gathered} 1 = (\chi(\xi) + \Sigma_0(\xi) + \Sigma_1(\xi))^2 \quad \text{with} \\ \Sigma_0(\xi) = \sum_{j\equiv 0(2), j\geq 0} \varphi(2^{-j}\xi) \quad \text{and} \quad \Sigma_1(\xi) = \sum_{j\equiv 1(2), q\geq 0} \varphi(2^{-j}\xi). \end{gathered}$$

From this it comes that $1 \leq 3(\chi^2(\xi) + \Sigma_0^2(\xi) + \Sigma_1^2(\xi))$. But thanks to (2.7), we get

$$\Sigma_i^2(\xi) = \sum_{j\geq 0, q\equiv i(2)} \varphi^2(2^{-j}\xi)$$

and the proposition is proved.

We shall consider all along this text two fixed functions χ and φ satisfying the assertions (2.5)–(2.10). Now let us fix the notations that will be used in all the following of this text.

Notations

$$
\begin{gathered}
h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi, \\
\Delta_{-1}u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)), \\
\text{if} \quad j \geq 0\,,\ \Delta_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbf{R}^d} h(2^j y)u(x-y)dy, \\
\text{if} \quad j \leq -2\,,\ \Delta_j u = 0, \\
S_j u = \sum_{j' \leq j-1} \Delta_p u = \chi(2^{-j}D)u = 2^{jd}\int_{\mathbf{R}^d} \widetilde{h}(2^j y)u(x-y)dy, \\
\text{if} \quad j \in \mathbf{Z}\,,\ \dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbf{R}^d} h(2^j y)u(x-y)dy, \\
\text{if} \quad j \in \mathbf{Z}\,,\ \dot{S}_j u = \sum_{j' \leq j-1} \dot{\Delta}_p u.
\end{gathered}
$$

Remark Let us point that all the above operators Δ_j and S_j maps L^p into L^p with norms which do not depend on q. This fact will be used all along this book.

Now let us have a look of the case when we may write

$$
\mathrm{Id} = \sum_j \Delta_j \quad \text{or} \quad \mathrm{Id} = \sum_j \dot{\Delta}_j .
$$

This is described by the following proposition, the proof of which is left as an exercise.

Proposition 2.1.2 *Let u be in $\mathcal{S}'(\mathbf{R}^d)$. Then, we have, in the sense of the convergence in the space $\mathcal{S}'(\mathbf{R}^d)$,*

$$
u = \lim_{j\to\infty} S_j u.
$$

The following proposition tells us that the condition of convergence in $\mathcal{S}'$ is somehow weak for series, the Fourier transform of which is supported in dyadic rings.

Proposition 2.1.3 *Let $(u_j)_{j\in\mathbf{N}}$ be a sequence of bounded functions such that the Fourier transform of u_j is supported in $2^j\widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}}$ is a given ring. Let us assume that*

$$
\|u_j\|_{L^\infty} \leq C2^{jN}.
$$

Then the series $(u_j)_{j\in\mathbf{N}}$ is convergent in $\mathcal{S}'$.

Proof of Proposition 2.1.3 Let us use the relation (2.2). After rescaling it can be written

$$
u_j = 2^{-jk}\sum_{|\alpha|=k} 2^{jd}g_\alpha(2^j\cdot)\star\partial^\alpha u_j .
$$

Then for any test function ϕ in $\mathcal{S}$, let us write that

$$
\begin{aligned}
\langle u_j, \phi\rangle &= 2^{-jk}\sum_{|\alpha|=k}\langle u_j, 2^{jd}\check{g}_\alpha(2^j\cdot)\star(-\partial)^\alpha\phi\rangle \qquad (2.17)\\
&\leq C2^{-jk}\sum_{|\alpha|=k} 2^{jN}\|\partial^\alpha\phi\|_{L^1}.
\end{aligned}
$$

Let us choose $k > N$. Then $(\langle u_j, \phi\rangle)_{j\in\mathbf{N}}$ is a convergent series, the sum of which is less than $C\|\phi\|_{M,\mathcal{S}}$ for some integer M. Thus the formula

$$\langle u, \phi\rangle \stackrel{\text{def}}{=} \lim_{j\to\infty} \sum_{j'\le j} \langle \Delta_{j'} u, \phi\rangle$$

defines a tempered distribution.

For the case of the operators $\dot\Delta_j$, the problem is a little bit more delicate. Obviously, it is not true forn $u = \mathbf{1}$ because, for any integer j, we have $\dot\Delta_j \mathbf{1} = 0$. This leads to the following definition.

Definition 2.1.1 *Let us denote by $\mathcal{S}'_h$ the space of tempered distribution such that*

$$\lim_{j\to-\infty} \dot S_j u = 0 \quad in \quad \mathcal{S}'.$$

Examples

- If a tempered distribution u is such that its Fourier transform $\widehat{u}$ is locally integrable near 0, then u belongs to $\mathcal{S}'_h$.
- If u is a tempered distribution such that for some function θ in $\mathcal{D}(\mathbf{R}^d)$ with value 1 near the origin, we have $\theta(D)u$ in L^p for some $p \in [1, +\infty[$, then u belongs to $\mathcal{S}'_h$.
- A non zero constant function u does not belong to $\mathcal{S}'_h$ because $\dot S_j u = u$ for any j in $\mathbf{Z}$.

Remarks

- The space $\mathcal{S}'_h$ is exactly the space of tempered distributions for which we may write

$$u = \sum_j \dot\Delta_j u.$$

- The fact that u belongs to $\mathcal{S}'_h$ or not is an information about low frequencies.
- The space $\mathcal{S}'_h$ is not a closed subspace of $\mathcal{S}'$ for the topology of weak convergence.
- It is an exercice left to the reader to prove that u belongs to $\mathcal{S}'_h$ if and only if, for any θ in $\mathcal{D}(\mathbf{R}^d)$ with value 1 near the origin, we have $\lim_{\lambda\to\infty} \theta(\lambda D)u = 0$ in $\mathcal{S}'$.

2.2 Homogeneous Besov spaces

Definition 2.2.1 *Let u be a tempered distribution, s a real number, and $(p, r) \in [1, +\infty]^2$. The space $\dot B^s_{p,r}$ is the space of distribution in $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot B^s_{p,r}} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbf{Z}} 2^{rqs}\|\dot\Delta_j u\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

There are two important facts to point out. The first one is about the homogeneity. If u is a tempered distribution, then let us consider for any integer N, the tempered distribution u_N defined by $u_N \stackrel{\text{def}}{=} u(2^N\cdot)$. We have the following proposition.

Proposition 2.2.1 *If $\|u\|_{\dot{B}^s_{p,r}}$ is finite, so it is for u_N and we have*

$$\|u_N\|_{\dot{B}^s_{p,r}} = 2^{N(s-\frac{d}{p})}\|u\|_{\dot{B}^s_{p,r}}.$$

Proof of Proposition 2.2.1 We go back to the definition of the operator $\dot{\Delta}_j$. This gives

$$\begin{aligned}\dot{\Delta}_j u_N(x) &= 2^{jd}\int h(2^j(x-y)u_N(y)dy\\ &= 2^{jd}\int h(2^j(x-y))u(2^N y)dy.\end{aligned}$$

By the change of variables $z = 2^N y$, we get that

$$\begin{aligned}\dot{\Delta}_j u_N(x) &= 2^{(j-N)d}\int h(2^{j-N}(2^N x - z))u(z)dz\\ &= (\dot{\Delta}_{j-N}u)(2^N x).\end{aligned}$$

So it turns out that $\|\dot{\Delta}_j u_N\|_{L^p} = 2^{-N\frac{d}{p}}\|\dot{\Delta}_{j-N}u\|_{L^p}$. We deduce from this that

$$\|2^{js}\dot{\Delta}_j u_N\|_{L^p} = 2^{N(s-\frac{d}{p})}2^{(j-N)s}\|\dot{\Delta}_{j-N}u\|_{L^p}.$$

And the proposition follows immediately by summation.

Theorem 2.2.1 *The space $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a normed space. Moreover, if $s < \dfrac{d}{p}$, then $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a Banach space. For any p, the space $\dot{B}^{\frac{d}{p}}_{p,1}$ is also a Banach space.*

Proof of Theorem 2.2.1 It is obvious that $\|\cdot\|_{\dot{B}^s_{p,r}}$ is a semi-norm. Let us assume that for some u in $\mathcal{S}'_h$, $\|u\|_{\dot{B}^s_{p,r}} = 0$. This implies that the support of $\widehat{u}$ is included in $\{0\}$ and thus that, for any $j \in \mathbf{Z}$, $\dot{S}_j u = u$. As u belongs to $\mathcal{S}'_h$, this implies that $u = 0$.

Let us prove the second part of the theorem. First let us prove that those spaces are continuously embedded in $\mathcal{S}'$. Thanks to Lemma 2.1.1, we have

$$\|\dot{\Delta}_j u\|_{L^\infty} \leq C2^{j\frac{d}{p}}\|\dot{\Delta}_j u\|_{L^p}. \tag{2.18}$$

Thus, if u belongs to $\dot{B}^{\frac{d}{p}}_{p,1}$, the series $(\dot{\Delta}_j u)_{j\in\mathbf{Z}}$ is convergent in L^∞. As u belongs to $\mathcal{S}'_h$, this implies that u belongs to L^∞ and that

$$\|u\|_{L^\infty} \leq C\|u\|_{\dot{B}^{\frac{d}{p}}_{p,1}}. \tag{2.19}$$

In particular, the space $\dot{B}^{\frac{d}{p}}_{p,1}$ is continuously embedded in L^∞ (and thus in $\mathcal{S}'$). In the case when $s < d/p$, let us write that, for negative j and for large enough M,

$$\begin{aligned}|\langle\dot{\Delta}_j u, \phi\rangle| &\leq \|\dot{\Delta}_j u\|_{L^\infty}\|\phi\|_{L^1}\\ &\leq 2^{j\frac{d}{p}}\|\dot{\Delta}_j u\|_{L^p}\|\phi\|_{L^1}\\ &\leq C2^{j\left(\frac{d}{p}-s\right)}\|u\|_{\dot{B}^s_{p,r}}\|\phi\|_{M,\mathcal{S}}.\end{aligned} \tag{2.20}$$

For non negative j, formula (2.2) applied with $u = \dot\Delta_j u$ gives (after a dilation by 2^j)

$$\dot\Delta_j u = 2^{-jk} \sum_{|\alpha|=k} \partial^\alpha (2^{jd} g_\alpha(2^j \cdot) \star \dot\Delta_j u) \quad \text{with} \quad g_\alpha = \mathcal{F}^{-1}(i\xi)^\alpha |\xi|^{-2k} \widetilde{\phi}(\xi).$$

Thus we infer that

$$\begin{aligned}
\langle \dot\Delta_j u, \phi\rangle &= 2^{-jk} \sum_{|\alpha|=k} \langle \partial^\alpha (2^{jd} g_\alpha(2^j \cdot) \star \dot\Delta_j), \phi\rangle \\
&= 2^{-jk} \sum_{|\alpha|=k} \langle \dot\Delta_j, 2^{jd} \check{g}_\alpha(2^j \cdot) \star (-\partial)^\alpha \phi\rangle \\
&\leq \|\dot\Delta_j u\|_{L^\infty} 2^{-jk} \|\phi\|_{M_k, \mathcal{S}}
\end{aligned}$$

for large enough M_k. By definition of $\dot B^s_{p,r}$, this gives $\langle \dot\Delta_j u, \phi\rangle \leq C 2^{j\left(s - \frac{d}{p} - k\right)} \|u\|_{\dot B^s_{p,r}} \|\phi\|_{M_k, \mathcal{S}}$. Choosing k greater than $s - \dfrac{d}{p}$ and then M_k large enough, gives, using the fact that u is in $\mathcal{S}'_h$ and the inequality (2.20), gives

$$\langle u, \phi\rangle \leq C \|u\|_{\dot B^s_{p,r}} \|\phi\|_{M_k, \mathcal{S}}. \tag{2.21}$$

Let (s,p,r) satisfying the hypothesis of the theorem and let us consider a Cauchy sequence $(u_n)_{n\in\mathbf{N}}$ in $\dot B^s_{p,r}$. Using (2.19) or (2.21), this implies that a tempered distribution u exists such that the sequence $(u_n)_{n\in\mathbf{N}}$ converges to u in $\mathcal{S}'$. The main point of the proof consists in proving that this distribution u belongs to $\mathcal{S}'_h$. If $s < d/p$, as, for any n, u_n belongs to $\mathcal{S}'_h$, we have, thanks to (2.21),

$$\forall j \in \mathbf{Z}\,,\ \forall n \in \mathbf{N}\,,\ |\langle \dot S_j u_n, \phi\rangle| \leq C_s 2^{j\left(\frac{d}{p} - s\right)} \sup_n \|u_n\|_{\dot B^s_{p,r}} \|\phi\|_{M, \mathcal{S}}.$$

As the sequence $(u_n)_{n\in\mathbf{N}}$ tends to u in $\mathcal{S}'$, we have

$$\forall j \in \mathbf{Z}\,,\ |\langle \dot S_j u, \phi\rangle| \leq C_s 2^{j\left(\frac{d}{p} - s\right)} \sup_n \|u_n\|_{\dot B^s_{p,r}} \|\phi\|_{M, \mathcal{S}}.$$

Thus u belongs to $\mathcal{S}'_h$. The case when u belongs to $\dot B^{\frac{d}{p}}_{p,1}$ is a little bit different. As $(u_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $\dot B^{\frac{d}{p}}_{p,1}$ and using (2.19), we claim that

$$\forall \varepsilon\,,\ \exists n_0 \,/\, \forall j \in \mathbf{Z}\,,\ \forall n \geq n_0\,,\ \sum_{k\leq j} \|\dot\Delta_k u_n\|_{L^\infty} \leq \frac{\varepsilon}{2} + \sum_{k\leq j} \|\dot\Delta_k u_{n_0}\|_{L^\infty}.$$

Let us choose j_0 small enough such that

$$\forall j \leq j_0\,,\ \sum_{k\leq j} \|\dot\Delta_k u_{n_0}\|_{L^\infty} \leq \frac{\varepsilon}{2}.$$

As u_n belongs to $\mathcal{S}'_h$, we have, for any $j \leq j_0$ and any $n \geq n_0$, $\|\dot S_j u_n\|_{L^\infty} \leq \varepsilon$. We know that the sequence $(u_n)_{n\in\mathbf{N}}$ tends to u in L^∞. This implies that for any $j \leq j_0$, $\|\dot S_j u\|_{L^\infty} \leq \varepsilon$. This proves that u belongs to $\mathcal{S}'_h$. By definition of the norm of $\dot B^s_{p,r}$ the sequence $(\dot\Delta_j u^{(n)})_{n\in\mathbf{N}}$ is a Cauchy one in L^p for any j. Thus an element u_j of L^p exists such that $(\dot\Delta_j u^{(n)})_{n\in\mathbf{N}}$ converges to u_j in L^p. As $(u^{(n)})_{n\in\mathbf{N}}$ converges to u in $\mathcal{S}'$ we have $\dot\Delta_j u = u_j$. Let us define

$$a_j^{(n)} = 2^{js} \|\dot\Delta_j u^{(n)}\|_{L^p} \quad \text{and} \quad a_j = 2^{js} \|\dot\Delta_j u\|_{L^p}.$$

For any j, $\lim\limits_{n\to\infty} a_j^{(n)} = a_j$. As $(a_j^{(n)})_{n\in\mathbf{N}}$ is a bounded sequence of $\ell^r(\mathbf{Z})$, $a \stackrel{\text{def}}{=} (a_j)_{j\in\mathbf{Z}}$ is in $\ell^r(\mathbf{Z})$ and thus $u \in \dot{B}^s_{p,r}$. As $(u^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence in $\dot{B}^s_{p,r}$, we have,

$$\forall \varepsilon > 0\,,\ \exists n_0 \,/\, \forall n \geq n_0\,,\ \forall m\ \|a^{(n+m)} - a^{(n)}\|_{\ell^r(\mathbf{Z})} \leq \varepsilon.$$

As $(a^{(n)})$ tends weakly to a in $\ell^r(\mathbf{Z})$, we get, passing to the limit in m in the above inequality that $\|u^{(n)} - u\|_{\dot{B}^s_{p,r}} = \|a - a^{(n)}\|_{\ell^r(\mathbf{Z})} \leq \varepsilon$. This ends the proof of the theorem.

Let us give the first example for Besov space, the Sobolev spaces $\dot{H}^s$.

Proposition 2.2.2 *The two spaces $\dot{H}^s$ and $\dot{B}^s_{2,2}$ are equal and the two norms satisfies*

$$\frac{1}{C^{|s|+1}}\|u\|_{\dot{B}^s_{2,2}} \leq \|u\|_{\dot{H}^s} \leq C^{|s|+1}\|u\|_{\dot{B}^s_{2,2}}.$$

Proof of Proposition 2.2.2 As the support of the Fourier transform of $\dot{\Delta}_j u$ is included in the ring $2^j\mathcal{C}$, it is clear, as $j \geq 0$, that a constant C exists such that, for any real s and any u such that $\widehat{u}$ belongs to L^2_{loc},

$$\frac{1}{C^{|s|+1}} 2^{2js}\|\dot{\Delta}_j u\|^2_{L^2} \leq \|\dot{\Delta}_j u\|^2_{\dot{H}^s} \leq C^{|s|+1} 2^{2js}\|\dot{\Delta}_j u\|^2_{L^2}. \tag{2.22}$$

Using Identity (2.11), we get

$$\frac{1}{2}\|u\|^2_{\dot{H}^s} \leq \sum_{j\in\mathbf{Z}} \int \varphi^2(2^{-j}\xi)|\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi \leq \|u\|^2_{\dot{H}^s}$$

which proves the proposition.

Let us give an example of a function which belongs to a large class of Besov spaces. Let us give an example of an L^1_{loc} function which belongs to $\dot{B}^s_{p,r}$.

Proposition 2.2.3 *Let σ be in $]0,d[$. Then we have, for any p in $[1,\infty]$,* $\dfrac{1}{|\cdot|^\sigma} \in \dot{B}^{\frac{d}{p}-\sigma}_{p,\infty}$.

Proof of Proposition 2.2.3 It is well known that the Fourier transform of $|\cdot|^{-\sigma}$ is $c_d|\xi|^{\sigma-d}$ and thus belongs to the space L^1 near the origin. Thus u is in $\mathcal{S}'_h$. Now let us compute $\dot{\Delta}_j|\cdot|^{-\sigma}$. By definition of the operator $\dot{\Delta}_j$, we have

$$\begin{aligned}
\dot{\Delta}_j(|\cdot|^{-\sigma})(x) &= 2^{jd}\int_{\mathbf{R}^d} h(2^j(x-y))|y|^{-\sigma}dy \\
&= 2^{j\sigma}h_\sigma(2^j x) \quad \text{with} \\
h_\sigma(y) &\stackrel{\text{def}}{=} \int_{\mathbf{R}^d} h(y-z)|z|^{-\sigma}dz.
\end{aligned}$$

As $\widehat{h}_\sigma(\xi) = \varphi(\xi)\mathcal{F}(|\cdot|^{-\sigma}) = c_d\varphi(\xi)|\xi|^{\sigma-d}$, the function $\widehat{h}_\sigma$ belongs to $\mathcal{D}(\mathbf{R}^d)$. In particular, h_σ is in L^p for any $p \in [1,\infty]$. Then let us write that

$$\|\dot{\Delta}_j(|\cdot|^{-\sigma})\|_{L^p} = 2^{j\left(\sigma-\frac{d}{p}\right)}\|h_\sigma\|_{L^p}.$$

This proves the proposition.

Lemma 2.2.1 *Let $\mathcal{C}'$ be a ring in $\mathbf{R}^d$; let (s,p,r) be as in Theorem 2.2.1. Let $(u_j)_{j\in\mathbf{Z}}$ be a sequence of smooth functions such that*

$$\text{Supp }\ \widehat{u}_j \subset 2^j\mathcal{C}' \quad \textit{and} \quad \Big\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbf{Z}}\Big\|_{\ell^r} < +\infty.$$

Then we have $u = \sum_{j\in\mathbf{Z}} u_j \in \dot{B}^s_{p,r}$ *and* $\|u\|_{\dot{B}^s_{p,r}} \leq C_s\Big\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbf{Z}}\Big\|_{\ell^r}$.

This immediately implies the following corollary.

Corollary 2.2.1 *Let (s,p,r) be as above; then the space $\dot{B}^s_{p,r}$ does not depend on the choice of the functions χ and φ used in the Definition 2.2.1.*

Proof of Lemma 2.2.1 Let us first observe that, using Lemma 2.1.1, we have $(u_j)_{j\leq 0}$ is a convergent series in L^∞. Let us denote by u^- its limit. It is obvious that u^- belongs to $\mathcal{S}'_h$. Using again Lemma 2.1.1, we get that $\|u_j\|_{L^\infty} \leq C2^{j\left(\frac{d}{p}-s\right)}$. Proposition 2.1.3 implies that $(u_j)_{j>0}$ is a convergente series in $\mathcal{S}'$. Let us denote by u^+ its limit. The support of the Fourier transform of u^+ does not contain the origin. Thus u^+ is in $\mathcal{S}'_h$. So does $u \stackrel{\text{def}}{=} u^- + u^+$. Then, let us study $\Delta_{j'}u$. As $\mathcal{C}$ and $\mathcal{C}'$ are two rings, an integer N_0 exists so that $|j'-j| \geq N_0$ then $2^j\mathcal{C} \cap 2^{j'}\mathcal{C}' = \emptyset$. Here $\mathcal{C}$ is the ring defined in the Proposition 2.1.1. Now, it is clear that if $|j'-j| \geq N_0$, then $\Delta_{j'}u_j = 0$. Then we can write that

$$\begin{aligned}
\|\dot{\Delta}_{j'}u\|_{L^p} &= \|\sum_{|j-j'|<N_0} \dot{\Delta}_{j'}u_j\|_{L^p}\\
&\leq C\sum_{|j-j'|<N_0} \|u_j\|_{L^p}.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
2^{j's}\|\dot{\Delta}_{j'}u\|_{L^p} &\leq C\sum_{\substack{j'\geq -1\\ |j'-j|\leq N_0}} 2^{j's}\|u_j\|_{L^p}\\
&\leq C\sum_{\substack{j'\geq -1\\ |j-j'|\leq N_0}} 2^{js}\|u_j\|_{L^p}.
\end{aligned}$$

We deduce from this that

$$2^{j's}\|\dot{\Delta}_{j'}u\|_{L^p} \leq (c_k)_{k\in\mathbf{Z}} \star (d_\ell)_{\ell\in\mathbf{Z}} \quad \text{with} \quad c_k = \mathbf{1}_{[-N_0,N_0]}(k) \quad \text{and} \quad d_\ell = \mathbf{1}_{\mathbf{N}}(\ell)2^{\ell s}\|u_\ell\|_{L^p}.$$

The classical property of convolution between $\ell^1(\mathbf{Z})$ and $\ell^r(\mathbf{Z})$ gives that

$$\|u\|_{\dot{B}^s_{p,r}} \leq C\Big\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbf{Z}}\Big\|_{\ell^r}.$$

This proves the lemma.

The following theorem is the equivalent of Sobolev embedding (see Theorem 1.2.1 page 11).

Theorem 2.2.2 *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any real number s the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

Proof of Theorem 2.2.2 In order to prove this result, we simply apply Lemma 2.1.1 which

$$\|\dot{\Delta}_j u\|_{L^{p_2}} \leq C2^{jd\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\dot{\Delta}_j u\|_{L^{p_1}}.$$

Considering that $\ell^{r_1}(\mathbf{Z}) \subset \ell^{r_2}(\mathbf{Z})$, the theorem is proved.

Now let us study the way Fourier multipliers acts of Besov spaces.

Proposition 2.2.4 *Let σ be a smooth function on $\mathbf{R}^d$ which is homogeneous of degree m. Then for any $(s,p,r) \in \mathbf{R} \times [1,+\infty]^2$ such that $\dot B^{s-m}_{p,r}$ is a Banach space, the operator $\sigma(D)$ maps continuously $\dot B^s_{p,r}$ into $\dot B^{s-m}_{p,r}$.*

Proof of Proposition 2.2.4 Lemma 2.1.1 tells us that $\|\sigma(D)\dot\Delta_j u\|_{L^p} \leq C2^{jm}\|\dot\Delta_j u\|_{L^p}$. Then Lemma 2.2.1 implies the proposition.

Remark Let us point out that this proof is very simple compared with the similar result on L^p spaces when p belongs to $]1,+\infty[$. Moreover, as we shall see in the next section, Fourier multipliers does not map L^∞ into L^∞ in general. From this point of view Besov spaces are much easier to use that classical L^p spaces or Sobolev spaces modeled on L^p.

Theorem 2.2.3 *A constant C exists which satisfies the following properties. If s_1 and s_2 are two real numbers such that $s_1 < s_2$, if $\theta \in]0,1[$, if r is in $[1,\infty]$, then we have, for any $u \in \mathcal{S}'_h$,*

$$
\begin{aligned}
\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,r}} &\leq \|u\|^{\theta}_{\dot B^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot B^{s_2}_{p,r}} \quad \textit{and} \\
\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}} &\leq \frac{C}{s_2-s_1}\Big(\frac{1}{\theta}+\frac{1}{1-\theta}\Big) \|u\|^{\theta}_{\dot B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot B^{s_2}_{p,\infty}}.
\end{aligned}
$$

Proof of Theorem 2.2.3 For the first inequality, let us write that

$$
2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} \leq \Big(2^{js_1}\|\dot\Delta_j u\|_{L^p}\Big)^{\theta}\Big(2^{js_2}\|\dot\Delta_j u\|_{L^p}\Big)^{1-\theta}.
$$

The Hölder inequality implies the first inequality of the theorem. For the second one, (as quite often in this text) we shall estimate in a different way low frequencies and high frequencies. More precisely, let us write

$$
\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}} = \sum_{j\leq N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} + \sum_{j>N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p}.
$$

By definition of the Besov norms, we have

$$
\begin{aligned}
2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} &\leq 2^{j(1-\theta)(s_2-s_1)}\|u\|_{\dot B^{s_1}_{p,\infty}} \quad \text{and} \\
2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} &\leq 2^{-j\theta(s_2-s_1)}\|u\|_{\dot B^{s_2}_{p,\infty}}.
\end{aligned}
$$

Thus we infer that

$$
\begin{aligned}
\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}} &\leq \|u\|_{\dot B^{s_1}_{p,\infty}} \sum_{j\leq N} 2^{j(1-\theta)(s_2-s_1)} + \|u\|_{\dot B^{s_2}_{p,\infty}} \sum_{j>N} 2^{-j\theta(s_2-s_1)} \\
&\leq \|u\|_{\dot B^{s_1}_{p,\infty}} \frac{2^{N(1-\theta)(s_2-s_1)}}{2^{(1-\theta)(s_2-s_1)}-1} + \|u\|_{\dot B^{s_2}_{p,\infty}} \frac{2^{-N\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}}.
\end{aligned}
$$

Choosing N such that

$$
\frac{\|u\|_{\dot B^{s_2}_{p,\infty}}}{\|u\|_{\dot B^{s_1}_{p,\infty}}} \leq 2^{N(s_2-s_1)} < 2\frac{\|u\|_{\dot B^{s_2}_{p,\infty}}}{\|u\|_{\dot B^{s_1}_{p,\infty}}}
$$

implies the theorem.

Proposition 2.2.5 *A constant C exists which satisfies the following properties. Let (s,p,r) be in $(\mathbf{R}^- \setminus\{0\}) \times [1,\infty]^2$ and u a distribution in $\mathcal{S}'_h$. This distribution u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$(2^{js}\|\dot{S}_j u\|_{L^p})_{j\in\mathbf{N}} \in \ell^r.$$

Moreover, we have

$$C^{-|s|+1}\|u\|_{\dot{B}^s_{p,r}} \leq \Big\|(2^{js}\|\dot{S}_j u\|_{L^p})_j\Big\|_{\ell^r} \leq C\Big(1+\frac{1}{|s|}\Big)\|u\|_{\dot{B}^s_{p,r}}.$$

Proof of Proposition 2.2.5 Let us write that

$$\begin{aligned} 2^{js}\|\dot{\Delta}_j u\|_{L^p} &\leq 2^{js}(\|\dot{S}_{j+1}u\|_{L^p} + \|\dot{S}_j u\|_{L^p}) \\ &\leq 2^{-s}2^{(j+1)s}\|\dot{S}_{j+1}u\|_{L^p} + 2^{js}\|\dot{S}_j u\|_{L^p}. \end{aligned}$$

This proves the inequality on the left. For the one on the right, let write that

$$\begin{aligned} 2^{js}\|\dot{S}_j u\|_{L^p} &\leq 2^{js}\sum_{j'\leq j-1}\|\dot{\Delta}_{j'}u\|_{L^p} \\ &\leq \sum_{j'\leq j-1} 2^{(j-j')s}2^{j's}\|\dot{\Delta}_{j'}u\|_{L^p}. \end{aligned}$$

As s is negative, we get the result.

2.3 Characterization of homogeneous Besov spaces

We shall give equivalent definitions of the Besov norm. These definitions does not use the localisation in frequency space. The first one concerns negative indices and uses the heat flow.

Theorem 2.3.1 *Let s be a positive real number and $(p,r) \in [1,\infty]^2$. A constant C exists which satisfies the following property. For u in $\mathcal{S}'_h$, we have*

$$C^{-1}\|u\|_{\dot{B}^{-2s}_{p,r}} \leq \Big\|\|t^s e^{t\Delta}u\|_{L^p}\Big\|_{L^r(\mathbf{R}^+,\frac{dt}{t})} \leq C\|u\|_{\dot{B}^{-2s}_{p,r}}.$$

Proof of Theorem 2.3.1 The proof relies on Lemma 2.1.2. Let us estimate $\|t^s\dot{\Delta}_j e^{t\Delta}u\|_{L^p}$. Using Lemma 2.1.2, we can write

$$\|t^s\dot{\Delta}_j e^{t\Delta}u\|_{L^p} \leq Ct^s 2^{2js}e^{-ct2^{2j}}2^{-2js}\|\dot{\Delta}_j u\|_{L^p}.$$

Using that u belongs to $\mathcal{S}'_h$ and the definition of the homogeneous Besov (semi) norm, we have

$$\begin{aligned} \|t^s e^{t\Delta}u\|_{L^p} &\leq \sum_{j\in\mathbf{Z}}\|t^s\dot{\Delta}_j e^{t\Delta}u\|_{L^p} \\ &\leq C\|u\|_{\dot{B}^{-2s}_{p,r}}\sum_{j\in\mathbf{Z}} t^s 2^{2js}e^{-ct2^{2j}}c_{r,j} \end{aligned}$$

where $(c_{r,j})_{j\in\mathbf{Z}}$ denotes, as in all this proof, a generic element of the unit sphere of $\ell^r(\mathbf{Z})$. If $r=\infty$, the inequality comes immedialtely form the following lemma, the proof of which is an exercice left to the reader.

Lemma 2.3.1 *For any positive s, we have*

$$\sup_{t>0}\sum_{j\in\mathbf{Z}} t^s 2^{2js} e^{-ct2^{2j}} < \infty.$$

If $r<\infty$, using the Hölder inequality with the weight $2^{2js}e^{-ct2^{2j}}$, the above lemma and Fubini's theorem, we obtain

$$\begin{aligned}
\int_0^\infty t^{rs}\|e^{t\Delta}u\|_{L^p}^r\frac{dt}{t} &\le C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\Big(\sum_{j\in\mathbf{Z}} t^s2^{2js}e^{-ct2^{2j}}c_{r,j}\Big)^r\frac{dt}{t}\\
&\le C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\Big(\sum_{j\in\mathbf{Z}} t^s2^{2js}e^{-ct2^{2j}}\Big)^{r-1}\Big(\sum_{j\in\mathbf{Z}} t^s2^{2js}e^{-ct2^{2j}}c^r_{r,j}\Big)\frac{dt}{t}\\
&\le C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\sum_{j\in\mathbf{Z}} t^s2^{2js}e^{-ct2^{2j}}c^r_{r,j}\frac{dt}{t}\\
&\le C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\sum_{j\in\mathbf{Z}}c^r_{r,j}\int_0^\infty t^s2^{2js}e^{-ct2^{2j}}\frac{dt}{t}\\
&\le C_s\|u\|^r_{\dot{B}^{-2s}_{p,r}}\quad\text{with}\quad C_s\stackrel{\text{def}}{=}\int_0^\infty t^{s-1}e^{-t}dt.
\end{aligned}$$

To prove the other inequality, by definition of C_s, we have

$$\dot{\Delta}_j u = C_{s+1}^{-1}\int_0^\infty t^s(-\Delta)^{s+1}e^{t\Delta}\dot{\Delta}_j u dt.$$

As $e^{t\Delta}u = e^{\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}u$, we can write, using Lemmas 2.1.1 and 2.1.2,

$$\begin{aligned}
\|\dot{\Delta}_j u\|_{L^p} &\le C\int_0^\infty t^s2^{2j(s+1)}e^{-ct2^{2j}}\|\dot{\Delta}_j e^{\frac{t}{2}\Delta}u\|_{L^p}dt\\
&\le C\int_0^\infty t^s2^{2j(s+1)}e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}dt.
\end{aligned}$$

If $r=\infty$, we have

$$\begin{aligned}
\|\dot{\Delta}_j u\|_{L^p} &\le C\Big(\sup_{t>0}t^s\|e^{t\Delta}u\|_{L^p}\Big)\int_0^\infty 2^{2j(s+1)}e^{-ct2^{2j}}dt\\
&\le C2^{2js}\Big(\sup_{t>0}t^s\|e^{t\Delta}u\|_{L^p}\Big).
\end{aligned}$$

If $r<\infty$, let us write that

$$\sum_j 2^{-2jsr}\|\dot{\Delta}_j u\|^r_{L^p}\le C\sum_{j\in\mathbf{Z}}2^{2jr}\Big(\int_0^\infty t^s e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}dt\Big)^r.$$

Hölder inequality with the weight $e^{-ct2^{2j}}$ implies that

$$\begin{aligned}
\Big(\int_0^\infty t^s e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}dt\Big)^r &\le \Big(\int_0^\infty e^{-ct2^{2j}}dt\Big)^{r-1}\int_0^\infty t^{rs}e^{-ct2^{2j}}\|e^{t\Delta}u\|^r_{L^p}dt\\
&\le C2^{-2j(r-1)}\int_0^\infty t^{rs}e^{-ct2^{2j}}\|e^{t\Delta}u\|^r_{L^p}dt.
\end{aligned}$$

Thanks to Lemma 2.3.1 and Fubini's theorem, we get

$$
\begin{aligned}
\sum_j 2^{-2jsr}\|\dot\Delta_j u\|_{L^p}^r &\leq C\sum_{j\in\mathbf{Z}} 2^{2j}\int_0^\infty t^{rs}e^{-ct2^{2j}}\|e^{t\Delta}u\|_{L^p}^r dt\\
&\leq C\int_0^\infty\Big(\sum_{j\in\mathbf{Z}} t2^{2j}e^{-ct2^{2j}}\Big)t^{rs}\|e^{t\Delta}u\|_{L^p}^r\frac{dt}{t}\\
&\leq C\int_0^\infty t^{rs}\|e^{t\Delta}u\|_{L^p}^r\frac{dt}{t}\cdot
\end{aligned}
$$

The theorem is proved.

The other characterization deals with indices s in $]0,1[$.

Theorem 2.3.2 *Let s be in $]0,1[$ and $(p,r)\in[1,\infty]^2$. A constant C exists such that, for any u in $\mathcal{S}_h'$,*

$$
C^{-1}\|u\|_{\dot B^s_{p,r}} \leq \Big\|\frac{\|\tau_{-z}u-u\|_{L^p}}{|z|^s}\Big\|_{L^r(\mathbf{R}^d;\frac{dz}{|z|^d})} \leq C\|u\|_{\dot B^s_{p,r}}
$$

Proof of Theorem 2.3.2 To prove the right inside inequality, we estimate $\|\tau_{-z}\dot\Delta_j u-\dot\Delta_j u\|_{L^p}$. By definition of $\dot\Delta_j$, we have

$$
\begin{aligned}
(\tau_{-z}\dot\Delta_j u-\dot\Delta_j u)(x) &= 2^{jd}\sum_{\ell=1}^d 2^j z_\ell\Big(\int_0^1 h_{\ell,j}(2^j\cdot,tz)dt\Big)\star u \quad\text{with}\\
h_{\ell,j}(X,Z) &\stackrel{\text{def}}{=} \partial_{x_\ell}h(X+2^jZ).
\end{aligned}
$$

The support of the Fourier transform of $h_{\ell,j}(\cdot,Z)$ is, for any Z, included in the ring $\mathcal{C}$. Thus

$$
(\tau_{-z}\dot\Delta_j u-\dot\Delta_j u)(x) = 2^{jd}\sum_{\ell=1}^d\sum_{|j-j'|\leq 1} 2^j z_\ell\Big(\int_0^1 h_{\ell,j}(2^j\cdot,tz)dt\Big)\star\dot\Delta_{j'}u.
$$

As for any Z, $\|h_{\ell,j}(\cdot,Z)\|_{L^1}=\|\partial_{x_\ell}h\|_{L^1}$, we have

$$
\begin{aligned}
\|\tau_{-z}\dot\Delta_j u-\dot\Delta_j u\|_{L^p} &\leq C2^j|z|\sum_{|j-j'|\leq 1}\|\dot\Delta_{j'}u\|_{L^p}\\
&\leq Cc_{r,j}2^{j(1-s)}|z|\|u\|_{\dot B^s_{p,r}}
\end{aligned}
$$

where $(c_{r,j})_{j\in\mathbf{Z}}$ is, as in the whole proof, any element of the unit sphere of $\ell^r(\mathbf{Z})$. We also have

$$
\begin{aligned}
\|\tau_{-z}\dot\Delta_j u-\dot\Delta_j u\|_{L^p} &\leq 2\|\dot\Delta_j u\|_{L^p}\\
&\leq Cc_{r,j}2^{-js}\|u\|_{\dot B^s_{p,r}}.
\end{aligned}
$$

We infer that, for any integer j',

$$
\|\tau_{-z}u-u\|_{L^p}\leq C\|u\|_{\dot B^s_{p,r}}\Big(|z|\sum_{j\leq j'}c_{r,j}2^{j(1-s)}+\sum_{j>j'}c_{r,j}2^{-js}\Big).
$$

Let us choose $j'=j_z$ such that $\dfrac{1}{|z|}\leq 2^{j_z}<2\dfrac{1}{|z|}$. If $r=\infty$, we get that, for any z in $\mathbf{R}^d$,

$$
\|\tau_{-z}u-u\|_{L^p}\leq C|z|^s\|u\|_{\dot B^s_{p,r}}.
$$

If $r<\infty$, let us write that

$$
\begin{aligned}
\Big\|\frac{\|\tau_{-z}u-u\|_{L^p}}{|z|^s}\Big\|^r_{L^r(\mathbf{R}^d;\frac{dz}{|z|^d})} &\leq C2^r\|u\|^r_{\dot B^s_{p,r}}(I_1+I_2)\quad\text{with}\\
I_1 &\stackrel{\text{def}}{=} \int_{\mathbf{R}^d}\Big(\sum_{j\leq j_z} c_{r,j}2^{j(1-s)}\Big)^r|z|^{-d+r(1-s)}dz\quad\text{and}\\
I_2 &\stackrel{\text{def}}{=} \int_{\mathbf{R}^d}\Big(\sum_{j>j_z} c_{r,j}2^{-js}\Big)^r|z|^{-d-rs}dz.
\end{aligned}
$$

Hölder inequality with the weight $2^{j(1-s)}$ and definition of j_z imply that

$$
\begin{aligned}
\Big(\sum_{j\leq j_z} c_{r,j}2^{j(1-s)}\Big)^r &\leq \Big(\sum_{j\leq j_z}2^{j(1-s)}\Big)^{r-1}\sum_{j\leq j_z}c^r_{r,j}2^{j(1-s)}\\
&\leq C|z|^{-(1-s)(r-1)}\sum_{j\leq j_z}c^r_{r,j}2^{j(1-s)}.
\end{aligned}
$$

By Fubini's theorem, we deduce that

$$
I_1\leq C\sum_j\Big(\int_{B(0,2^{-j+1})}|z|^{-d+1-s}dz\Big)2^{j(1-s)}c^r_{r,j}\leq C.
$$

The estimate on I_2 is strictly analogous. Now let us prove the other inequality. Thanks to the fact that the function h is of meanvalue 0, let us write that

$$
\begin{aligned}
\dot\Delta_j u(x) &= 2^{jd}\int_{\mathbf{R}^d}h(2^jy)\tau_y u(x)dy\\
&= 2^{jd}\int h(2^jy)(\tau_yu(x)-u(x))dy.
\end{aligned}
$$

When $r=\infty$, we have

$$
\begin{aligned}
2^{js}\|\dot\Delta_ju\|_{L^p} &\leq 2^{jd}\int_{\mathbf{R}^d}2^{js}|h(2^jy)|\,\|\tau_yu-u\|_{L^p}dy\\
&\leq 2^{jd}\int_{\mathbf{R}^d}2^{js}|y|^s|h(2^jy)|dy\,\sup_{y\in\mathbf{R}^d}\frac{\|\tau_yu-u\|_{L^p}}{|y|^s}\\
&\leq C\sup_{y\in\mathbf{R}^d}\frac{\|\tau_yu-u\|_{L^p}}{|y|^s}\cdot
\end{aligned}
$$

When $r<\infty$, let us write that

$$
\begin{aligned}
\sum_j 2^{jsr}\|\dot\Delta_ju\|^r_{L^p} &\leq 2^r(\Sigma_1+\Sigma_2)\quad\text{with}\\
\Sigma_1 &\stackrel{\text{def}}{=} \sum_j 2^{jsr}\Big(\int_{2^j|y|\leq 1}2^{jd}|h(2^jy)|\,\|\tau_yu-u\|_{L^p}dy\Big)^r\quad\text{and}\\
\Sigma_2 &\stackrel{\text{def}}{=} \sum_j 2^{jsr}\Big(\int_{2^j|y|\geq 1}2^{jd}|h(2^jy)|\,\|\tau_yu-u\|_{L^p}dy\Big)^r.
\end{aligned}
$$

Hölder inequality implies that

$$
\Big(\int_{2^j|y|\leq 1}2^{jd}|h(2^jy)|\,\|\tau_yu-u\|_{L^p}dy\Big)^r \leq \Big(\int_{2^j|y|\leq 1}2^{jdr'}|h(2^jy)|^{r'}dy\Big)^{r-1}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \times \int_{2^j|y|\leq 1} \|\tau_y u - u\|_{L^p}^r dy\\
&\leq\ C 2^{jd} \int_{2^j|y|\leq 1} \|\tau_y u - u\|_{L^p}^r dy.
\end{aligned}
$$

Using Fubini's theorem, we get that

$$
\begin{aligned}
\Sigma_1 &\leq C\int_{\mathbf{R}^d}\Bigl(\sum_{j/2^j|y|\leq 1} 2^{j(rs+d)}\Bigr)\|\tau_y u - u\|_{L^p}^r dy\\
&\leq C\int_{\mathbf{R}^d} \frac{\|\tau_y u - u\|_{L^p}^r}{|y|^{rs}}\frac{dy}{|y|^d}\cdot
\end{aligned}
$$

In order to estimate Σ_2, let us write, using Hölder inequality with the measure $|y|^{-d}dy$, that

$$
\begin{aligned}
\Bigl(\int_{2^j|y|\geq 1} 2^{jd}|h(2^j y)|\,\|\tau_y u - u\|_{L^p} dy\Bigr)^r &\leq 2^{-jr}\Bigl(\int_{2^j|y|\geq 1} |2^j y|^{d+1}|h(2^j y)|\,\frac{\|\tau_y u - u\|_{L^p}}{|y|}\frac{dy}{|y|^d}\Bigr)^r\\
&\leq 2^{-jr}\int_{2^j|y|\geq 1} \frac{\|\tau_y u - u\|_{L^p}^r}{|y|^r}\frac{dy}{|y|^d}\cdot
\end{aligned}
$$

Then, using Fubini's theorem, we infer that

$$
\begin{aligned}
\Sigma_2 &\leq C\int_{\mathbf{R}^d}\Bigl(\sum_{j/2^j|y|\geq 1} 2^{-jr(1-s)}\Bigr)\frac{\|\tau_y u - u\|_{L^p}^r}{|y|^r}\frac{dy}{|y|^d}\\
&\leq C\int_{\mathbf{R}^d} \frac{\|\tau_y u - u\|_{L^p}^r}{|y|^{rs}}\frac{dy}{|y|^d}\cdot
\end{aligned}
$$

The theorem is proved.

2.4 Precised Sobolev inequalities

The basic lemma is the following one.

Lemma 2.4.1 *Let $1 \leq q < p < \infty$ and α a positive real number. A constant C exists such that*

$$
\|f\|_{L^p} \leq C\|f\|_{\dot B^{-\alpha}_{\infty,\infty}}^{1-\theta}\|f\|_{\dot B^{\beta}_{q,q}}^{\theta} \quad \textit{with} \quad \beta = \alpha\Bigl(\frac{p}{q}-1\Bigr) \quad \textit{and} \quad \theta = \frac{q}{p}\cdot
$$

Proof of Lemma 2.4.1 The proof of this lemma follows exactly the same lines as the proof of the Sobolev inequality (see Theorem 1.2.1 page 11) which appears as a particular case of this above lemma for $q = 2$ and $\alpha = d/2 - \beta$. Let us write that

$$
\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}\mu(|f| > \lambda)d\lambda \quad \text{and} \quad f = \dot S_j f + (\mathrm{Id} - \dot S_j)f.
$$

By definition of the semi-norm $\|\cdot\|_{\dot B^{-\alpha}_{\infty,\infty}}$, we have $\|\dot S_j f\|_{L^\infty} \leq C2^{j\alpha}\|f\|_{\dot B^{-\alpha}_{\infty,\infty}}$. Without any loss of generality, we can assume that $\|f\|_{\dot B^{-\alpha}_{\infty,\infty}} = 1$. As we have

$$
(|f| > \lambda) \subset \Bigl(|\dot S_j f| > \frac{\lambda}{2}\Bigr) \cup \Bigl(|(\mathrm{Id} - \dot S_j)f| > \frac{\lambda}{2}\Bigr),
$$

choosing j_λ in $\mathbf{Z}$ such that

$$
\frac{1}{2}\Bigl(\frac{\lambda}{4C}\Bigr)^{\frac{1}{\alpha}} < 2^{j_\lambda} \leq \Bigl(\frac{\lambda}{4C}\Bigr)^{\frac{1}{\alpha}}, \tag{2.23}
$$

we infer that $(|f| > \lambda) \subset \Big(|(\mathrm{Id} - \dot{S}_{j_\lambda})f| > \dfrac{\lambda}{2}\Big)$. Then, thanks to Bienaymé-Tchebitchev inequality, we have

$$\begin{aligned}
\|f\|_{L^p}^p &\leq p\int_0^\infty \lambda^{p-1}\mu\Big(|(\mathrm{Id} - \dot{S}_{j_\lambda})f| > \frac{\lambda}{2}\Big)d\lambda \\
&\leq p2^q\int_0^\infty \lambda^{p-q-1}\|(\mathrm{Id} - \dot{S}_{j_\lambda})f\|_{L^q}^q d\lambda.
\end{aligned}$$

Let us estimate $\|(\mathrm{Id} - \dot{S}_{j_\lambda})f\|_{L^q}$. By definition of the semi-norm $\|\cdot\|_{\dot{B}^\beta_{q,q}}$, we have

$$\begin{aligned}
\|(\mathrm{Id} - \dot{S}_{j_\lambda})f\|_{L^q} &\leq \sum_{j\geq j_\lambda}\|\dot{\Delta}_j f\|_{L^q} \\
&\leq \sum_{j\geq j_\lambda} 2^{-j\beta}2^{j\beta}\|\dot{\Delta}_j f\|_{L^q} \\
&\leq C\|f\|_{\dot{B}^\beta_{q,q}}\sum_{j\geq j_\lambda}2^{-j\beta}c_j \quad \text{with} \quad \|(c_j)\|_{\ell^q} = 1.
\end{aligned}$$

Thus we get

$$\|f\|_{L^p}^p \leq C\|f\|_{\dot{B}^\beta_{q,q}}^q\int_0^\infty \lambda^{p-q-1}\Big(\sum_{j\geq j_\lambda}2^{-j\beta}c_j\Big)^q d\lambda.$$

Using Hölder inequality in the sum (with the weight $2^{-j\beta}$), we get, by Definition (2.23) of j_λ,

$$\begin{aligned}
\Big(\sum_{j\geq j_\lambda}2^{-j\beta}c_j\Big)^q &\leq \Big(\sum_{j\geq j_\lambda}2^{-j\beta}\Big)^{q-1}\sum_{j\geq j_\lambda}2^{-j\beta}c_j^q \\
&\leq C2^{-j_\lambda\beta(q-1)}\sum_{j\geq j_\lambda}2^{-j\beta}c_j^q \\
&\leq C\lambda^{-(q-1)\frac{\beta}{\alpha}}\sum_{j\geq j_\lambda}2^{-j\beta}c_j^q.
\end{aligned}$$

Then it turns out that

$$\|f\|_{L^p}^p \leq C\|f\|_{\dot{B}^\beta_{q,q}}^q\int_0^\infty\sum_j 2^{-j\beta}\mathbf{1}_{j\geq j_\lambda}c_j^q\lambda^{p-q-(q-1)\frac{\beta}{\alpha}-1}d\lambda.$$

By Definition (2.23) of j_λ, we have using Fubini's theorem

$$\begin{aligned}
\|f\|_{L^p}^p &\leq C\|f\|_{\dot{B}^\beta_{q,q}}^q\sum_j 2^{-j\beta}c_j^q\int_0^{4C2^{j\alpha}}\lambda^{p-q-(q-1)\frac{\beta}{\alpha}-1}d\lambda \\
&\leq C\|f\|_{\dot{B}^\beta_{q,q}}^q\sum_j 2^{-j\beta}c_j^q 2^{j\alpha\left(p-q-(q-1)\frac{\beta}{\alpha}\right)} \quad \text{with} \quad \|(c_j)\|_{\ell^q} = 1.
\end{aligned}$$

As $\beta = \alpha\Big(\frac{p}{q} - 1\Big)$ and $\|(c_j)\|_{\ell^q} = 1$, we get that $\|f\|_{L^p}^p \leq C\|f\|_{\dot{B}^\beta_{q,q}}^q$ and the lemma is proved.

2.5 Paradifferential calculus

Let us study the way the product acts on Besov spaces. In order to do so, we shall use the dyadic decomposition presented in the Section 2.1 to construct a homogenenous version of the paradifferential calculus.

Considering two tempered distributions u and v in $\mathcal{S}'_h$, let us write

$$u = \sum_j \dot\Delta_j u \quad \text{and} \quad v = \sum_j \dot\Delta_j v.$$

Formally, the product writes $uv = \sum_{j',j} \dot\Delta_{j'} u \dot\Delta_j v$. Let us introduce Bony's decomposition.

Definition 2.5.1 *We call homogeneous paraproduct of v by u and denote by $\dot T_u v$ the bilinear operator*

$$\dot T_u v \stackrel{\text{def}}{=} \sum_j \dot S_{j-1} u \dot\Delta_j v.$$

We call homogeneous remainder of u and v and denote by $\dot R(u,v)$ the bilinear operator:

$$\dot R(u,v) = \sum_{|j-j'|\leq 1} \dot\Delta_{j'} u \dot\Delta_j v.$$

Just by looking at the definition, it is clear that (still formally),

$$uv = \dot T_u v + \dot T_v u + \dot R(u,v). \tag{2.24}$$

The way how paraproduct acts on Besov spaces is described by the following theorem.

Theorem 2.5.1 *Let (s,p,r_1) such that $\dot B^s_{p,r_1}$ is a Banach space. Then the paraproduct $\dot T$ maps continuously $L^\infty \times \dot B^s_{p,r_1}$ into $\dot B^s_{p,r}$. Moreover, if t is negative and r_2 such that*

$$\frac{1}{r_1} + \frac{1}{r_2} \stackrel{\text{def}}{=} \frac{1}{r} \leq 1,$$

and if $\dot B^{s+t}_{p,r}$ is a Banach space, then $\dot T$ maps continuously $\dot B^t_{\infty,r_1} \times \dot B^s_{p,r_2}$ into $\dot B^{s+t}_{p,r}$.

Proof of Theorem 2.5.1 From the assertion (2.9), the Fourier transform of $\dot S_{j-1} u \dot\Delta_j v$ and also of $\dot S_{j-1} v \dot\Delta_j u$ is supported in $2^j\widetilde{\mathcal{C}}$. So, the only thing that we have to do is to estimate $\|\dot S_{j-1} u \dot\Delta_j v\|_{L^p}$. Lemma 2.1.1 and Proposition 2.2.5 claim that, for any integer j,

$$\|\dot S_{j-1} u\|_{L^\infty} \leq C\|u\|_{L^\infty} \quad \text{and} \quad \|\dot S_{j-1} u\|_{L^\infty} \leq C_t c_{j,r_1} 2^{-jt} \|u\|_{\dot B^t_{\infty,r_1}} \tag{2.25}$$

where $(c_{j,r})_{j\in\mathbf{Z}}$ denotes an element of the unit sphere of $\ell^r(\mathbf{Z})$. Using Lemma 2.2.1, the estimates about paraproduct are proved.

Now we shall study the behaviour of operators R. Here we have to consider terms of the type $\dot\Delta_j u \dot\Delta_j v$. The Fourier transform of such terms is not supported in rings but in balls of the type $2^j B$. Thus to prove that remainder terms belong to some Besov spaces, we need the following lemma.

Lemma 2.5.1 *Let B be a ball of $\mathbf{R}^d$, s a positive real number and (p,r) in $[1,\infty]^2$ such that $s < d/p$ or $s = d/p$ and $r = 1$. Let $(u_j)_{j\in\mathbf{Z}}$ be a sequence of smooth functions such that*

$$\text{Supp } \widehat{u}_j \subset 2^j B \quad \text{and} \quad \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbf{Z}}\right\|_{\ell^r} < +\infty.$$

Then we have $u = \sum_{j\in\mathbf{Z}} u_j \in \dot B^s_{p,r}$ *and* $\|u\|_{\dot B^s_{p,r}} \leq C_s \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbf{Z}}\right\|_{\ell^r}$.

Proof of Lemma 2.5.1 We have for any j, we have

$$\|u_j\|_{L^p} \leq C2^{-js} \quad \text{and} \quad \|u_j\|_{L^\infty} \leq C2^{j(\frac{d}{p}-s)}.$$

By hypothesis, for any N, the series $(u_j)_{j\leq N}$ is convergent in L^∞ and $(u_j)_{j>N}$ is convergent in L^p. Thus the series $(u_j)_{j\in\mathbf{Z}}$ is convergent in $\mathcal{S}'$ and its limit u belongs to $\mathcal{S}'_h$. Then let us study $\Delta_{j'}u_j$. As $\mathcal{C}$ is a ring (defined in the proposition 2.1.1) and B is a ball, an integer N_1 exists so that, if $j' \geq j + N_1$, then $2^{j'}\mathcal{C} \cap 2^j B = \emptyset$. So it is clear that if $j' \geq j + N_1$, then $\Delta_{j'}u_j = 0$. Now, we write that

$$\begin{aligned} \|\Delta_{j'}u\|_{L^p} &= \Big\| \sum_{j\geq j'-N_1} \Delta_{j'}u_j \Big\|_{L^p} \\ &\leq C \sum_{j\geq j'-N_1} \|u_j\|_{L^p}. \end{aligned}$$

So, we get that

$$\begin{aligned} 2^{j's}\|\Delta_{j'}u\|_{L^p} &\leq C \sum_{j\geq j'-N_1} 2^{j's}\|u_j\|_{L^p} \\ &\leq C \sum_{j\geq j'-N_1} 2^{(j'-j)s}2^{js}\|u_j\|_{L^p}. \end{aligned}$$

So, we deduce from this that

$$2^{j's}\|\Delta_{j'}u\|_{L^p} \leq (c_k) \star (d_\ell) \quad \text{with} \quad c_k = \mathbf{1}_{[-N_1,+\infty[}(k)2^{-ks} \quad \text{and} \quad d_\ell = 2^{\ell s}\|u_\ell\|_{L^p}.$$

So the lemma is proved.

Theorem 2.5.2 *Let (p_k, r_k) (for $k \in \{1,2\}$) such that*

$$\frac{1}{p_1} + \frac{1}{p_2} \stackrel{\text{def}}{=} \frac{1}{p} \leq 1 \quad \textit{and} \quad \frac{1}{r_1} + \frac{1}{r_2} \stackrel{\text{def}}{=} \frac{1}{r} \leq 1.$$

Let $(s_1, s_2) \in \mathbf{R}^2$ such that $s_1 + s_2 \in]0, d/p[$, the operator $\dot{R}$ maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ into $\dot{B}^{s_1+s_2}_{p,r}$. Moreover, if $s_1 + s_2 = 0$ and $r = 1$, the operator $\dot{R}$ maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ into $\dot{B}^{0}_{p,\infty}$. And if $s_1 + s_2 = d/p$ and $r = 1$, the operator $\dot{R}$ maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ into $B^{\frac{d}{p}}_{p,1}$.

Proof of Theorem 2.5.2 By definition of the remainder operator,

$$\dot{R}(u,v) = \sum_{j'} \dot{R}_{j'} \quad \text{with} \quad \dot{R}_{j'} = \sum_{i=-1}^{1} \dot{\Delta}_{j'-i}u\dot{\Delta}_{j'}v.$$

By definition of $\dot{\Delta}_j$, Supp $\mathcal{F}\dot{R}_{j'} \subset 2^{j'}B(0,24)$. So, an integer N_0 exists such that if $j' < j - N_0$, then $\dot{\Delta}_j R_{j'} = 0$. From this, we deduce that

$$\dot{\Delta}_j \dot{R}(u,v) = \sum_{j'\geq j-N_0} \dot{\Delta}_j \dot{R}_{j'}. \tag{2.26}$$

Thus we can write

$$\begin{aligned} \|\dot{\Delta}_j R_{j'}\|_{L^p} &\leq \sum_{i=-1}^{1} \|\dot{\Delta}_{j'-i}u\|_{L^{p_1}}\|\dot{\Delta}_{j'}v\|_{L^{p_2}} \\ &\leq C2^{-j(s_1+s_2)} \sum_{i=-1}^{1} 2^{-(j'-j)(s_1+s_2)}2^{(j'-i)s_1}\|\dot{\Delta}_{j'-i}u\|_{L^{p_1}}2^{j's_2}\|\dot{\Delta}_{j'}v\|_{L^{p_2}}. \end{aligned}$$

Let us define $(r_j)_{j\in\mathbf{Z}}$ by $r_j \stackrel{\text{def}}{=} 2^{-j(s_1+s_2)}\|\dot\Delta_j \dot R(u,v)\|_{L^p}$. Using the assertion (2.26), we have

$$r_j \leq C(b^{(1)}\star b^{(2)})_j \quad \text{with} \quad b_j^{(1)} = 2^{-j(s_1+s_2)}\mathbf{1}_{\mathbf{N}-N_0}(j) \quad \text{and}$$
$$b_j^{(2)} = \sum_{i=-1}^{1} 2^{(j-i)s_1}\|\dot\Delta_{j-i}u\|_{L^{p_1}} 2^{js_2}\|\dot\Delta_j v\|_{L^{p_2}}.$$

If $s_1+s_2\in]0,d/p[$, the sequence $(b_j^{(1)})_{j\in\mathbf{Z}}$ belongs to $\ell^1(\mathbf{Z})$ and the sequence $(b_j^{(2)})_{j\in\mathbf{Z}}$ belongs to $\ell^{12}(\mathbf{Z})$. Thus $(r_j)_{j\in\mathbf{Z}}\in\ell^{r_{12}}(\mathbf{Z})$. Let us assume now that $r_{12}=1$. If $s_1+s_2=0$, $(b_j^{(1)})_{j\in\mathbf{Z}}$ belongs to $\ell^\infty(\mathbf{Z})$ and $(b_j^{(2)})_{j\in\mathbf{Z}}$ to $\ell^1(\mathbf{Z})$. The theorem is proved on that case. If $s_1+s_2=d/p$, then both $(b_j^{(1)})_{j\in\mathbf{Z}}$ and $(b_j^{(2)})_{j\in\mathbf{Z}}$ belong to $\ell^1(\mathbf{Z})$ and the whole theorem is proved.

Now, we are going to infer from this theorem the following two corollaries, the proof of which is nothing but the systematic use of Bony's decomposition and the application of Theorems 2.5.1, 2.5.2 and 2.2.2.

Corollary 2.5.1 *For any positive s and any (p,r) such that $\dot B^s_{p,r}$ is a Banach space, a constant C exists such that*

$$\|uv\|_{\dot B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}} + \|u\|_{\dot B^s_{p,r}}\|v\|_{L^\infty}).$$

In particular, for any $p\in[1,\infty[$, the space $\dot B^{\frac{d}{p}}_{p,1}$ is an algebra.

Corollary 2.5.2 *Let s_k, p_k, r_k (for $k\in\{1,2\}$) and p be such that*

$$s_k < \frac{d}{p_k}\,,\ \ \frac{1}{r_1}+\frac{1}{r_2} \stackrel{\text{def}}{=} \frac{1}{r} \leq 1 \quad \text{and} \quad p\geq \max\{p_1,p_2\}$$

If $s_1+s_2>0$, the product maps $\dot B^{s_1}_{p_1,r_1}\times\dot B^{s_2}_{p_2,r_2}$ into $\dot B^{s_1+s_2-d\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right)}_{p,r}$. If $s_1=-s_2$ and $r=1$, the product maps $\dot B^{s_1}_{p_1,r_1}\times\dot B^{s_2}_{p_2,r_2}$ into $\dot B^{-d\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right)}_{p,\infty}$.

As an application of the above product laws, we shall prove Hardy inequalities.

Theorem 2.5.3 *For any real $s\in\left[0,\frac{d}{2}\right[$, a constant C exists such that for any f in $\dot H^s(\mathbf{R}^d)$,*

$$\int_{\mathbf{R}^d}\frac{|f(x)|^2}{|x|^{2s}}dx \leq C\|f\|^2_{\dot H^s}. \tag{2.27}$$

Proof of Theorem 2.5.3 Let us define

$$I_s(f) \stackrel{\text{def}}{=} \int_{\mathbf{R}^d}\frac{|f(x)|^2}{|x|^{2s}}dx = \langle|\cdot|^{-2s}, f^2\rangle.$$

Using Littewood-Paley decomposition, we can write, as f^2 belongs to $\mathcal{S}'_h$

$$\begin{aligned} I_s(f) &= \sum_{|j-j'|\leq 2}\langle\dot\Delta_j|\cdot|^{-2s},\dot\Delta_{j'}f^2\rangle \\ &= \sum_{|j-j'|\leq 2}\langle 2^{j\left(\frac{d}{2}-2s\right)}\dot\Delta_j|\cdot|^{-2s}, 2^{-j'\left(\frac{d}{2}-2s\right)}\dot\Delta_{j'}f^2\rangle. \end{aligned}$$

Proposition 2.2.3 claims that $|\cdot|^{-2s}$ belongs to $\dot B^{\frac{d}{2}-2s}_{2,\infty}$ and Corollary 2.5.2 claims in particular that $\|f^2\|_{\dot B^{2s-\frac{d}{2}}_{2,1}} \leq C\|f\|^2_{\dot H^s}$. Thus $I_s(f)\leq C\|f\|^2_{\dot H^s}$.

2.6 Around the space $\dot{B}^1_{\infty,\infty}$

Theorem 2.6.1 *The space $\dot{B}^1_{\infty,\infty}$ is not included in the space $C^{0,1}$ of Lipschitz functions.*

Let us exhibit a counterexample in $\mathbf{R}^2$ coming from incompressible bidimensionnal fluid mechanics. If H denotes the Heavyside function, let us state

$$\omega(x) \stackrel{\text{def}}{=} H(x_1)H(1-x_1)H(x_2)H(1-x_2) \quad \text{and} \tag{2.28}$$

$$v(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy \quad \text{where} \quad (z_1,z_2)^\perp \stackrel{\text{def}}{=} (-z_2,z_1). \tag{2.29}$$

In fact v is the divergence free vector field the vorticity of which is ω. The theorem is implies by the following proposition, proved in [7].

Proposition 2.6.1 *The above vector field v belongs to $\dot{B}^1_{\infty,\infty}$ but not to $C^{0,1}$.*

Nevertheless in this case when $k=1$, it is possible to characterize $\dot{B}^1_{\infty,\infty}$. In order to do so, let us introduce the following space, called Zygmund space.

Definition 2.6.1 *For a continuous function u, let us define*

$$\|u\|_{\dot{C}^1_\star} \stackrel{\text{def}}{=} \sup_{\substack{(x,y)\in\mathbf{R}^{2d}\\ y\neq 0}} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|} < +\infty.$$

Proposition 2.6.2 *The seminorms $\|\cdot\|_{\dot{C}^1_\star}$ and $\|\cdot\|_{\dot{B}^1_{\infty,\infty}}$ are equivalent.*

Proof of Proposition 2.6.2 Let us consider a function u in $\dot{B}^1_{\infty,\infty}$ and a point y in $\mathbf{R}^d$. For any integer j,

$$\begin{aligned}
|u(x+y)+u(x-y)-2u(x)| &\leq |\dot{S}_j u(x+y)+\dot{S}_j u(x-y)-2\dot{S}_j u(x)| + 4\sum_{j'\geq j}\|\dot{\Delta}_{j'}u\|_{L^\infty}\\
&\leq |\dot{S}_j u(x+y)+\dot{S}_j u(x-y)-2\dot{S}_j u(x)|\\
&\qquad\qquad + 4\sum_{j'\geq j} 2^{-j'}(2^{j'}\|\dot{\Delta}_{j'}u\|_{L^\infty})\\
&\leq |\dot{S}_j u(x+y)+\dot{S}_j u(x-y)-2\dot{S}_j u(x)| + 2^{3-j}\|u\|_{\dot{B}^1_{\infty,\infty}}.
\end{aligned}$$

Using Taylor inequality at order 2 we get that

$$|\dot{S}_j u(x+y)+\dot{S}_j u(x-y)-2\dot{S}_j u(x)| \leq |y|^2\|D^2\dot{S}_j u\|_{L^\infty}.$$

But, using Lemma 2.1.1, we get that

$$\begin{aligned}
\|D^2\dot{S}_j u\|_{L^\infty} &\leq \sum_{j'\leq j-1}\|D^2\dot{\Delta}_{j'}u\|_{L^\infty}\\
&\leq \sum_{j'\leq j-1} 2^{j'}(2^{j'}\|\dot{\Delta}_{j'}u\|_{L^\infty})\\
&\leq 2^j\|u\|_{\dot{B}^1_{\infty,\infty}}.
\end{aligned}$$

Thus we get that for any positive integer j,

$$|u(x+y)+u(x-y)-2u(x)| \leq \Big(2^j|y|^2+2^{2-j}\Big)\|u\|_{\dot{B}^1_{\infty,\infty}}.$$

Choosing again $j=[-\log_2 y]+1$, we get

$$|u(x+y)+u(x-y)-2u(x)| \leq C\|u\|_{B^1_{\infty,\infty}}|y|.$$

Now let us consider a function u in $C^1_\star$. As the function φ given by proposition 2.1.1 is radial (thus even) we have

$$2^{jd}(h(2^j\cdot)\star u)(x) = 2^{jd}\int h(2^jy)u(x+y)dy.$$

As the integral of h on $\mathbf{R}^d$ is 0 we have

$$2^{jd}(h(2^j\cdot)\star u)(x) = 2^{jd-1}\int h(2^jy)(u(x+y)+u(x-y)-2u(x))dy.$$

As the function $z\mapsto |z|h(z)$ is integrable, we get

$$\|\dot{\Delta}_j u\|_{L^\infty} \leq C2^{-j}\sup_{y\in\mathbf{R}^d}\frac{|u(x+y)+u(x-y)-2u(x)|}{|y|}\cdot$$

The proposition is proved. As we shall see in the next chapter, this type of space can play a role in fluid mechanics.

Now we shall see how Littlewood-Paley theory allows to describe space with various type of modulus of continuity. First let us give the following definition.

Definition 2.6.2 *A function μ from an interval of type $[0,a]$ to $\mathbf{R}_+$ is a modulus of continuity if μ is an increasing continuous function such that $\mu(0)=0$. We say that μ is admissible if and only if the function Γ defined by*

$$\Gamma(y) \stackrel{\text{def}}{=} y\mu\Big(\frac{1}{y}\Big)$$

is non decreasing and satisfies

$$(A) \qquad \int_x^\infty \frac{1}{y^2}\Gamma(y)dy \leq C\frac{\Gamma(x)}{x}\cdot$$

Let us give very basic examples. If $\alpha\in]0,1]$, the functions $\mu(r)=r^\alpha$, $\mu(r)=r(-\log r)^\alpha$ and also $\mu(r)=r(-\log r)(\log(-\log r))^\alpha$ are admissible modulus of continuity.

Definition 2.6.3 *Let μ be a modulus of continuity and (X,d) a metric space. The space C_μ is the space of bounded continuous functions u such that*

$$\|u\|_{C_\mu} \stackrel{\text{def}}{=} \|u\|_{L^\infty(X)} + \sup_{0<d(x,y)\leq a}\frac{\|u(x)-u(y)\|}{\mu(d(x,y))} < \infty.$$

Definition 2.6.4 *Let Γ be an increasing function on $[b,\infty[$. The space B_Γ is the space of bounded continuous functions u on $\mathbf{R}^d$ such that*

$$\|u\|_{B_\Gamma} \stackrel{\text{def}}{=} \|u\|_{L^\infty} + \sup_{j\geq 0}\frac{\|\nabla S_j u\|_{L^\infty}}{\Gamma(2^j)} < \infty.$$

Remarks When $\Gamma(y) = y^{1-\alpha}$, the space B_Γ is equal to $\dot{B}^{\alpha}_{\infty,\infty} \cap L^\infty$.

Proposition 2.6.3 *When μ is an admissible modulus of continuity, the two spaces C_μ and B_Γ are equal (of course on $\mathbf{R}^d$).*

Let us assume that u belongs to B_Γ. As $\nabla\Delta_j = \nabla S_{j+1} - \nabla S_j$ we have

$$\|\nabla\Delta_j u\|_{L^\infty} \leq C\Gamma(2^j)\|u\|_{B_\Gamma}.$$

Using Identity (2.1) page 29 we claim the existence of $(\varphi_j)_{1\leq j\leq d}$ in $\mathcal{D}(\mathbf{R}^d\setminus\{0\})$ such that

$$\varphi(\xi) = \sum_{k=1}^{d} \varphi_k(\xi) i\xi_k \varphi(\xi) \quad \text{and thus} \quad \Delta_j = \sum_{k=1}^{d} 2^{-j}\varphi_k(2^{-j}D)\partial_k\Delta_j.$$

This implies that

$$\|\Delta_j u\|_{L^\infty} \leq C2^{-j}\Gamma(2^j)\|u\|_{B_\Gamma}. \tag{2.30}$$

Now let us write

$$\begin{aligned} |u(x) - u(x')| &\leq \|\nabla S_j u\|_{L^\infty}|x - x'| + 2\sum_{j'\geq j} \|\Delta_{j'}u\|_{L^\infty} \\ &\leq \|\nabla S_j u\|_{L^\infty}|x - x'| + C\|u\|_{B_\Gamma} \sum_{j'\geq j} 2^{-j'}\Gamma(2^{j'}). \end{aligned}$$

Using Condition (A) and the fact then Γ is increasing, we have by definition of $\|\cdot\|_{B_\Gamma}$,

$$\begin{aligned} |u(x) - u(x')| &\leq \|u\|_{B_\Gamma}\Big(\Gamma(2^j)|x - x'| + C\int_{2^j}^{\infty} \frac{1}{y^2}\Gamma(y)dy\Big) \\ &\leq \|u\|_{B_\Gamma}\Big(\Gamma(2^j)|x - x'| + C2^{-j}\Gamma(2^j)\Big). \end{aligned}$$

As usual let us choose $2^{-j} \equiv |x - x'|$. This gives that u is in C_μ. Now let us assume that u belongs to C_μ. By definition of S_j we have

$$\partial_k S_j u(x) = 2^{jd}2^j \int_{\mathbf{R}^d} (\partial_k\widetilde{h})(2^j(x - y))u(y)dy.$$

As $\displaystyle\int_{\mathbf{R}^d} \partial_k\widetilde{h}(y)dy = 0$, we have

$$\begin{aligned} |\partial_k S_j u(x)| &\leq 2^{jd}2^j \int_{\mathbf{R}^d} |\partial_k\widetilde{h}(2^j(x - y))| \times |u(y) - u(x)|dy \\ &\leq \|u\|_\mu 2^{jd}2^j \int_{\mathbf{R}^d} |\partial_k\widetilde{h}(2^j(x - y))|\mu(|y - x|)dy. \end{aligned}$$

Cutting the above integral into two parts we have

$$\begin{aligned} |\partial_k S_j u(x)| \leq \|u\|_\mu 2^{jd}2^j &\int_{|z|\leq 2^{-j}} |\partial_k\widetilde{h}(2^j z)|\mu(|z|)dz \\ &+ 2\|u\|_\mu 2^{jd} \int_{|z|\geq 2^{-j}} |\partial_k\widetilde{h}(2^j z)|\, |2^j z|\Gamma\Big(\frac{1}{z}\Big)dz. \end{aligned}$$

As μ is an increasing function, for any z such that $|z| \leq 2^{-j}$ we have $\mu(|z|) \leq \mu(2^{-j})$. As Γ is also an increasing function, thus for any z such that $|z| \geq 2^{-j}$ we have $\Gamma\Big(\dfrac{1}{|z|}\Big) \leq \Gamma(2^j)$. Thus

$$\begin{aligned} |\partial_k S_j u(x)| \leq \|u\|_\mu 2^{jd}2^j\mu(2^{-j}) &\int_{|z|\leq 2^{-j}} |\partial_k\widetilde{h}(2^j z)|dz \\ &+ 2\|u\|_\mu\Gamma(2^j)2^{jd} \int_{|z|\geq 2^{-j}} |\partial_k\widetilde{h}(2^j z)|\, |2^j z|dz. \end{aligned}$$

We infer that $\|\nabla S_j u\|_{L^\infty} \leq C\|u\|_\mu\Gamma(2^j)$ and the proposition is proved.

2.7 References and Remarks

The Littlewood-Paley theory is a classical theory of harmonic analysis. Its applications to partial differential equations started in 1981 with the fondamental article [1] of J.-M. Bony dedicated to the study of propagation of microlocal singularities in non linear hyperbolic equations and where paradifferential calculus was introduced. The flexibility of this theory makes that it is now a basic tool for the study of evolution partial differential equations. For books presenting this theory in more details, see for instance [7], [49] or [53].

Chapter 3

Besov spaces and Navier-Stokes system

3.1 A wellposedness result in Besov spaces

The purpose of this section is to give an other approach of Theorem 1.4.2 page 23. As said by Theorem 2.3.1 page 40, the smallness condition (1.18) in the case when $T = \infty$ is exactly the smallness condition for the $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ norm. The purpose of this section is to give another approach to Theorem 1.4.2. It relies on Littlewood-Paley theory. This theory allows a description of the smoothing effect of the heat flow which is different from the one used in Chapter 1.

Let us be more specific now. Let us assume that u_0 belongs to $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$. We deduce from Lemma 2.1.2 page 30 that $\|\dot{\Delta}_j e^{\nu t\Delta} u_0\|_{L^p} \leq Ce^{-c\nu t 2^{2j}}\|\dot{\Delta}_j u_0\|_{L^p}$. By time integration, we get

$$\|\dot{\Delta}_j e^{\nu t\Delta} u_0\|_{L^1(\mathbf{R}^+;L^p)} \leq \frac{C}{\nu 2^{2j}} 2^{-js}\|u_0\|_{\dot{B}^s_{p,\infty}}. \tag{3.1}$$

This leads to the following definition.

Definition 3.1.1 *For p in $[1,\infty]$, E_p is the space of functions $u \in L^\infty(\mathbf{R}^+; \dot{B}^{-1+\frac{3}{p}}_{p,\infty})$ such that*

$$\|u\|_{E_p} \stackrel{def}{=} \sup_j 2^{j\left(-1+\frac{3}{p}\right)}\|\dot{\Delta}_j u\|_{L^\infty(\mathbf{R}^+;L^p)} + \sup_j \nu 2^{2j} 2^{j\left(-1+\frac{3}{p}\right)}\|\dot{\Delta}_j u\|_{L^1(\mathbf{R}^+;L^p)} < \infty.$$

Let us remark that the estimate (3.1) implies that

$$\|e^{\nu t\Delta} u_0\|_{E_p} \leq C\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}}.$$

The following theorem is the interpretation in this context of Theorem 1.4.2 in its global version.

Theorem 3.1.1 *Let $p \in [1,+\infty]$. A constant c exists such that, if $\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}} \leq c\nu$, then a unique solution u of (GNS_ν) exists in the ball of center 0 and radius $2c\nu$ of E_p.*

Proof of Theorem 3.1.1 The reader knows that it is enough to prove the following lemma.

Lemma 3.1.1 *For any p in $[1,\infty[$, a constant C exists such that, for any $T \in]0,\infty]$,*

$$\|B(u,v)\|_{E_p} \leq \frac{C}{\nu}\|u\|_{E_p}\|v\|_{E_p}. \tag{3.2}$$

Proof of Lemma 3.1.1 Let us recall that the non linear term $Q(u,v)$ can be written as

$$Q^m(u,v) = \sum_{k,\ell} A^m_{k,\ell}(D)(u^k v^\ell)$$

where the $A^m_{k,\ell}(D)$ are homogeneous Fourier multipliers of degree 1. With the notations of Chapter 2 page 33,

$$u^k v^\ell = \sum_j \dot S_j u^k \dot\Delta_j v^\ell + \sum_j \dot\Delta_j u^k \dot S_{j+1} v^\ell.$$

The fact that the support of the Fourier transform of $\dot S_j u^k \dot\Delta_j v^\ell$ and $\dot\Delta_j u^k \dot S_{j+1} v^\ell$ are included in $2^j \mathcal{B}$ for some ball $\mathcal{B}$ of $\mathbf{R}^3$, an integer N_0 exists such that, if $j' < j - N_0$, then

$$\dot\Delta_j Q(\dot S_{j'} u, \dot\Delta_{j'} v) = \dot\Delta_j Q(\dot\Delta_{j'} u, \dot S_{j'+1} v) = 0.$$

Now, let us decompose B as

$$\begin{aligned} B(u,v) &= B_1(u,v) + B_2(u,v) \quad \text{with} \\ B_1(u,v) &\stackrel{\text{def}}{=} \sum_j B(\dot S_j u, \dot\Delta_j v) \quad \text{and} \\ B_2(u,v) &\stackrel{\text{def}}{=} \sum_j B(\dot\Delta_j u, \dot S_{j+1} v). \end{aligned}$$

By definition of B in Fourier space, an integer N_0 exists such that

$$\dot\Delta_j B_1(u,v) \stackrel{\text{def}}{=} \sum_{j' \geq j - N_0} \dot\Delta_j B(\dot S_{j'} u, \dot\Delta_{j'} v) \quad \text{and} \tag{3.3}$$

$$\dot\Delta_j B_2(u,v) \stackrel{\text{def}}{=} \sum_{j' \geq j - N_0} \dot\Delta_j B(\dot\Delta_{j'} u, \dot S_{j'+1} v). \tag{3.4}$$

We shall treat only B_1 because B_2 is strictly similar. Using Lemma 2.1.1 page 29, we get

$$\|\dot\Delta_j Q(\dot S_{j'} u, \dot\Delta_{j'} v)\|_{L^p} \leq C 2^j \sup_{k,\ell} \|\dot S_{j'} u^k \dot\Delta_{j'} v^\ell\|_{L^p}$$

Using Lemma 2.1.2 page 30, we get

$$\begin{aligned} \|\dot\Delta_j B(\dot S_{j'} u, \dot\Delta_{j'} v)(t)\|_{L^p} &\leq \int_0^t e^{-\nu(t-t')2^{2j}} \|\dot\Delta_j Q(\dot S_{j'} u(t'), \dot\Delta_{j'} v(t'))\|_{L^p} dt' \\ &\leq C 2^j \int_0^t e^{-\nu(t-t')2^{2j}} \sup_{k,\ell} \|\dot S_{j'} u^k(t') \dot\Delta_{j'} v^\ell(t')\|_{L^p} dt' \qquad (3.5) \\ &\leq C 2^j \int_0^t e^{-\nu(t-t')2^{2j}} \|\dot S_{j'} u(t')\|_{L^\infty} \|\dot\Delta_{j'} v(t')\|_{L^p} dt'. \end{aligned}$$

By definitions of the operators $\dot S_j$ and of the E_p norm, we get, thanks to Lemma 2.1.1,

$$\begin{aligned} \|\dot S_{j'} u(t')\|_{L^\infty} &\leq \sum_{j'' < j'} \|\dot\Delta_{j''} u(t')\|_{L^\infty} \\ &\leq \sum_{j'' < j'} 2^{j'' \frac{3}{p}} \|\dot\Delta_{j''} u(t')\|_{L^p} \\ &\leq C 2^{j'} \|u\|_{E_p}. \end{aligned}$$

Thus we deduce that

$$\|\dot\Delta_j B(\dot S_{j'}u,\dot\Delta_{j'}v)(t)\|_{L^p}\leq C\|u\|_{E_p}2^j2^{j'}\int_0^t e^{-\nu(t-t')2^{2j}}\|\dot\Delta_{j'}v(t')\|_{L^p}dt'.$$

Using the Young inequality for the time integral, we obtain by definition of the E_p norm that

$$\begin{aligned}\mathcal{B}_{j,j'}(u,v) &\stackrel{\text{def}}{=} \|\dot\Delta_j B(\dot S_{j'}u,\dot\Delta_{j'}v)\|_{L^\infty(\mathbf{R}^+;L^p)}+\nu2^{2j}\|\dot\Delta_j B(\dot S_{j'}u,\dot\Delta_{j'}v)\|_{L^1(\mathbf{R}^+;L^p)}\\ &\leq C\|u\|_{E_p}2^j2^{j'}\|\dot\Delta_{j'}v\|_{L^1(\mathbf{R}^+;L^p)}\\ &\leq \frac{C}{\nu}\|u\|_{E_p}\|v\|_{E_p}2^j2^{-j'\frac3p}.\end{aligned}$$

Thanks to (3.3) and (3.4), we get

$$\|\dot\Delta_j B_1(u,v)\|_{L^\infty([0,T];L^p)}+\nu2^{2j}\|\dot\Delta_j B_1(u,v)\|_{L^1([0,T];L^p)}\leq\frac{C}{\nu}\|u\|_{E_p}\|v\|_{E_p}\sum_{j'\geq j-N_0}2^{-(j'-j)\frac3p}.$$

The lemma is proved.

3.2 The flow of scaling invariant solutions

The theorem about ordinary differential equations which generalizes the classical Cauchy-Lipschitz theorem is the following. The underlying concept is the Osgood condition.

Definition 3.2.1 *Let μ be a modulus of continuity. We shall say that μ is an Osgood modulus of continuity if and only if*

$$\int_0^a\frac{dr}{\mu(r)}=+\infty.$$

Let us give some examples. The functions

$$\mu(r)=r\,,\ \mu(r)=r(-\log r)^\alpha\quad\text{and}\quad\mu(r)=r(-\log r)(\log(-\log r))^\alpha$$

are Osgood modulus of continuity if $\alpha<1$. The function $\mu(r)=r^\alpha$ with $\alpha<1$ is not. Neither are the functions

$$\mu(r)=r(-\log r)^\alpha\quad\text{and}\quad\mu(r)=r(-\log r)(\log(-\log r))^\alpha$$

if $\alpha\geq1$. The interest of this definition is illustrated by the following theorem.

Theorem 3.2.1 *Let E be a Banach space, Ω an open subset of E, I a open interval of $\mathbf{R}$ and (t_0,x_0) an element of $I\times\Omega$. Let us consider a function $F\in L^1_{loc}(I;\mathcal{C}_\mu(\Omega;E))$. Let us assume that μ is an Osgood modulus of continuity. Then an open interval J exists such that $t_0\in J\subset I$ and such that the equation*

$$(EDO)\quad x(t)=x_0+\int_{t_0}^t F(t',x(t'))dt'$$

has a unique continuous solution defined on J.

Proof of Theorem 3.2.1 Let us begin by proving the uniqueness of the trajectories. Let $x_1(t)$ and $x_2(t)$ two solutions of (EDO) defined in a neighbourhood $\widetilde{J}$ of t_0 with the same initial data x_0. Let us define $\delta(t) = ||x_1(t) - x_2(t)||$. It is obvious that

$$0 \leq \delta(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\delta(t'))dt' \quad \text{with} \quad \gamma \in L^1_{loc}(I) \quad \text{and} \quad \gamma \geq 0. \tag{3.6}$$

Now the key point is the following classical Osgood Lemma which can be understood as a generalization of classical Gronwall Lemma.

Lemma 3.2.1 *Let ρ be a measurable function with value in $[0, a]$, γ a non negative locally integrable function and μ a continuous and non decreasing function. Let us assume that, for a non negative real number c, the function ρ satisfies*

$$\rho(t) \leq c + \int_{t_0}^{t} \gamma(t')\mu(\rho(t'))dt'. \tag{3.7}$$

If c is positive, then we have

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \leq \int_{t_0}^{t} \gamma(t')dt' \quad \textit{with} \quad \mathcal{M}(x) = \int_{x}^{1} \frac{dr}{\mu(r)}. \tag{3.8}$$

If $c = 0$ and if μ is Osgood then the function ρ is identically 0.

Let us admit this lemme for a while. We immediatly get that $\delta \equiv 0$ in (3.6). Now let us prove the existence by considering the classical Picard scheme

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F(t', x_k(t'))dt'.$$

We skip the fact that for J small enough, the sequence $(x_k)_{k\in\mathbf{N}}$ is well defined and bounded in the space $C_b(J, \Omega)$. Let us state $\rho_{k,n}(t) = \sup\limits_{t' \leq t} \|x_{k+n}(t') - x_k(t')\|$. We have that

$$0 \leq \rho_{k+1,n}(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\rho_{k,n}(t'))dt'.$$

Let us state $\rho_k(t) \stackrel{\text{def}}{=} \sup\limits_{n} \rho_{k,n}(t)$. As μ is a non decreasing function we deduce that

$$0 \leq \rho_{k+1}(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\rho_k(t'))dt'.$$

Fatou' Lemma implies now that

$$\widetilde{\rho}(t) \stackrel{\text{def}}{=} \limsup_{k\to\infty} \rho_k(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\widetilde{\rho}(t'))dt'.$$

Lemme 3.2.1 implies that $\widetilde{\rho}(t) \equiv 0$ near t_0; this concluded the proof of Theorem 3.2.1.

Proof of Lemma 3.2.1 Let us state

$$R_c(t) \stackrel{\text{def}}{=} c + \int_{t_0}^{t} \gamma(t')\mu(\rho(t'))dt'.$$

The function R_c is a continuous non decreasing function. So we have

$$\begin{aligned}\frac{dR_c}{dt} &= \gamma(t)\mu(\rho(t))\\ &\leq \gamma(t)\mu(R_c(t)).\end{aligned} \tag{3.9}$$

Let us assume that c is positive. The function R_c is also positive. So we infer from (3.9) that

$$-\frac{d}{dt}\mathcal{M}(R_c(t)) = \frac{dR_c}{dt}\frac{1}{\mu(R_c(t))} \leq \gamma(t).$$

Thus we get (3.8) by integration. Let us assume now that $c = 0$ and that ρ is not identically 0 near t_0. As the function μ is non decreasing, we can consider the function $\widetilde{\rho}(t) \stackrel{\text{def}}{=} \sup_{s\in[t_0,t]} \rho(t')$ instead of ρ. A real number t_1 greater than t_0 exists such that $\rho(t_1)$ is positive. As the function ρ satisfies (3.7) with $c = 0$, it also satisfies this inequality for any positive c' less than $\rho(t_1)$. Then it comes from (3.8) that

$$\forall c' \in]0, \rho(t_1)]\,,\ \mathcal{M}(c') \leq \int_{t_0}^{t_1} \gamma(t')dt' + \mathcal{M}(\rho(t_1)),$$

which implies that $\displaystyle\int_0^1 \frac{dr}{\mu(r)} < +\infty$. Thus the lemma is proved.

Theorem 3.2.1 implies that a flow can be defined. The regularity of the flow can be computed in a general formula.

The following proposition establishes that we have to generalize Osgood Theorem if we want to prove the existence of a flow for solution of (NS_ν) given by Theorem 3.1.1.

Proposition 3.2.1 *Let u_0 be a distribution homogeneous of degree -1 and smooth outside the origin. Let μ any admissible modulus of continuity such that $e^{t\Delta}u_0 \in L^1([0,T]; C_\mu)$ for some positive T. Then μ does not satisfies the Osgood condition.*

Proof of Proposition 3.2.1 The fact that u_0 is homogenenous of degree -1 implies that $\nabla S_j u_0 = 2^{2j} S_0 u_0(2^j\cdot)$ and thus that

$$\|e^{t\Delta}\nabla S_j u_0\|_{L^\infty} = 2^{2j}\|e^{t2^{2j}\Delta}\nabla S_0 u_0\|_{L^\infty}.$$

On the space of functions the Fourier transform of which is compactly supported in a fixed compact, the operator $e^{-c\Delta}$ is bounded on all the L^p spaces. As the function Γ is non decreasing, we have, if j_t denote the greatest integer j such that j tel que $2^{-2j} \geq t$,

$$\sup_j \frac{\|e^{t\Delta}\nabla S_j u_0\|_{L^\infty}}{\Gamma(2^j)} \geq 2^{2j_t}\frac{\|e^{t2^{2j_t}\Delta}\nabla S_0 u_0\|_{L^\infty}}{\Gamma(2^{j_t})} \geq \frac{C}{t}\frac{1}{\Gamma\left(\frac{1}{\sqrt{t}}\right)}\cdot$$

Thus, if $e^{t\Delta}u_0$ is in $L^1([0,T]; C_\mu)$, we have by definition of Γ,

$$\int_0^{\sqrt{T}} \frac{dr}{\mu(r)} = 2\int_0^T \frac{dt}{t\,\Gamma\left(\frac{1}{\sqrt{t}}\right)} \leq c\int_0^T \|e^{t\Delta}u_0\|_{C_\mu}dt.$$

The proposition is proved.

Theorem 3.2.2 *A constant C exists such that, for any v in the space $L^1([0,T];B^{-r}_\infty)$, for some positive r and such that a positive integer j_0 exists such that*

$$N_{j_0}(T,v) \stackrel{\text{def}}{=} \sup_{j\geq j_0} 2^j \|\Delta_j v\|_{L^1([0,T];L^\infty)} < \frac{1}{C},$$

then a unique a continuous map ψ of $[0,T]\times \mathbf{R}^d$ in $\mathbf{R}^d$ exists such that

$$\psi(t,x) = x + \int_0^t v(t',\psi(t',x))dt' \quad \textit{et} \quad \psi(t,\cdot)-\mathrm{Id} \in C^{1-CN_{j_0}(t,v)} \qquad \forall t\leq T.$$

Proof of Theorem 3.2.2 We shall only proof the following lemma.

Lemma 3.2.2 *Under the hypothesis of the above theorem, if γ_j are two continuous functions such that*

$$\gamma_j(t) = x_j + \int_0^t v(t',\gamma_j(t'))dt',$$

we have, if $|x_1-x_2|\leq 2^{-j_0}$,

$$\forall t_0\leq T\,,\ |\gamma_1(t_0)-\gamma_2(t_0)| \leq C|x_1-x_2|^{1-CN_{j_0}(t_0,v)} \exp\Big(2^{j_0(r+1)}\int_0^{t_0} \|v(t,\cdot)\|_{B^{-r}_{\infty,\infty}} dt\Big).$$

Proof of Lemma 3.2.2 Let us decompose v in a low and a high frequency part. This leads to

$$\begin{aligned}
|\gamma_1(t)-\gamma_2(t)| &\leq |x_1-x_2| + \int_0^t |S_j v(t',\gamma_1(t')) - S_j v(t',\gamma_2(t'))|dt' \\
&\qquad\qquad + 2\int_0^t \sum_{j'\geq j} \|\Delta_{j'} v(t')\|_{L^\infty} dt' \\
&\leq |x_1-x_2| + \int_0^t \|\nabla S_j v(t',\cdot)\|_{L^\infty} |\gamma_1(t')-\gamma_2(t')|dt' \\
&\qquad\qquad + 2^{1-j}\sum_{j'\geq j} 2^{j-j'} 2^{j'} \int_0^t \|\Delta_{j'} v(t')\|_{L^\infty} dt'.
\end{aligned}$$

Let us state, for $0\leq t\leq t_0\leq T$, $\rho(t) \stackrel{\text{def}}{=} \sup_{t'\leq t} |\gamma_1(t')-\gamma_2(t')|$ and

$$D_j(t) \stackrel{\text{def}}{=} |x_1-x_2| + 2^{2-j} N_{j_0}(t_0,v) + \int_0^t \|\nabla S_j v(t',\cdot)\|_{L^\infty} |\gamma_1(t')-\gamma_2(t')|dt'.$$

By definition of $N_{j_0}(t,v)$, for any $\forall j\geq j_0$, $\rho(t)\leq D_j(t)$. Then, we have,

$$\forall t\leq t_0\,,\ D_j(t) \leq |x_1-x_2| + 2^{2-j} N_{j_0}(t_0,v) + \int_0^t \|\nabla S_j v(t',\cdot)\|_{L^\infty} D_j(t')dt'.$$

The Gronwall lemma implies that, for any $t\leq t_0$,

$$D_j(t) \leq \Big(|x_1-x_2| + 2^{2-j}N_{j_0}(t_0,v)\Big)\exp\Big(\int_0^t \|\nabla S_j v(t',\cdot)\|_{L^\infty} dt'\Big).$$

Using Lemma 2.1.1 page 29, we have, for any $t \leq t_0$,

$$\int_0^t \|\nabla S_j v(t',\cdot)\|_{L^\infty} dt' \leq \int_0^t \sum_{j'<j_0} 2^{j'} \|\Delta_{j'} v(t',\cdot)\|_{L^\infty} dt' + \sum_{j'=j_0}^{j} \int_0^t 2^{j'} \|\Delta_{j'} v(t',\cdot)\|_{L^\infty} dt'$$
$$\leq 2^{j_0(r+1)} \int_0^t \|v(t',\cdot)\|_{B^{-r}_{\infty,\infty}} dt' + jN_{j_0}(t,v). \quad (3.10)$$

Thus for any integer $j \geq j_0$ and any $t \leq t_0$, we have

$$D_j(t) \leq \Big((|x_1 - x_2| + 2^{2-j} N_{j_0}(t_0, v)\Big) \exp\Big(2^{j_0(r+1)} \int_0^t \|v(t',\cdot)\|_{B^{-r}_{\infty,\infty}} dt' + jN_{j_0}(t,v)\Big).$$

Let us choose $2^j \equiv |x_1 - x_2|^{-1}$; we infer that

$$\rho(t_0) \leq C|x_1 - x_2|^{1-CN_{j_0}(t_0,v)} \exp\Big(2^{j_0(r+1)} \int_0^{t_0} \|v(t',\cdot)\|_{B^{-r}_{\infty,\infty}} dt'\Big)$$

and the lemma is proved.

3.3 References ans Remarks

Theorem 3.1.1 has been proved by M. Cannone, Y. Meyer and F. Planchon in [3] by a different method. A local version and various extensions of Theorem 3.1.1 can be founded in [9]. The rest of this short chapter comes essentially from [11]. For an extensive study if the use of Littlewood-Paley theory in the context of Navier-Stokes equations, we refer to the books [4] by M. Cannone and [39] by P.-G. Lemarié-Rieusset.

Chapter 4

Anisotropic viscosity

The purpose of this section is to study a version of the incompressible Navier-Stokes system in $\mathbf{R}^3$ where the usual Laplacian is substituted by the Laplacian in the horizontal variables,

$$(ANS_\nu)\left\{\begin{array}{rcl}\partial_t u + u\cdot\nabla u - \nu\Delta_h u & = & -\nabla p\\ \operatorname{div} u & = & 0\\ u_{|t=0} & = & u_0,\end{array}\right.$$

where $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$. We refer to [14] for the motivations. As we shall see, it appears to be partly parabolic (in the horizontal variables) and partly hyperbolic (in the vertical variable). The purpose is to prove theorems analogous to the case of classical Navier-Stokes system.

4.1 Wellposedness with one vertical derivative in L^2

To make the basic ideas clearer, we shall first prove a weaker theorem, but the proof of which is simpler. Let us introduce anisotropic Sobolev spaces which are natural here because the horizontal variable $x_h = (x_1, x_2)$ does not play the same role as the vertical one x_3.

Definition 4.1.1 *Let s and s' be two real number. The space $H^{s,s'}$ is the space of tempered distributions u such that $\widehat{u}$ belongs to $L^1_{loc}(\mathbf{R}^3)$ and*

$$\|u\|^2_{H^{s,s'}} \stackrel{\text{def}}{=} \int_{\mathbf{R}^3} (1+|\xi_h|^2)^s (1+\xi_3^2)^{s'} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

It is obvious that $H^{s,s'}$ is a Banach space. Our theorem is the following.

Theorem 4.1.1 *Let u_0 in an initial data in $H^{0,1}(\mathbf{R}^3)$. Then a positive time T exists such that a unique solution u of (ANS_ν) exists in the space*

$$L^\infty([0,T]; H^{0,1}) \cap L^2([0,T]; H^{1,1}).$$

Moreover, the solution u is continuous with value in L^2 and satisfies the energy equality

$$\|u(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla_h u(t')\|^2_{L^2} dt' = \|u_0\|^2_{L^2}.$$

Moreover, if we have

$$\|u_0\|^{\frac12}_{L^2} \|\partial_3 u_0\|^{\frac12}_{L^2} \leq c\nu \tag{4.1}$$

for some small enough constant c, then the solution is global.

Proof of Theorem 4.1.1 The lack of smoothing effect in the vertical variable x_3 prevents both from solving the system by a fixed point method like in Section 1.2 and from using compactness methods based on the L^2 energy estimate. The structure of the proof is the following:

- first, we shall define a family of approximated problems with global smooth solutions,
- then we shall solve globally those approximated problems and prove uniform bounds on this family,
- then, we shall prove that the sequence defined by this procedure is a Cauchy sequence in the energy space
$$L^\infty([0,T];L^2)\cap L^2([0,T];H^{1,0}).$$

Step 1: The family of approximated solutions We use the Friedrichs method: let us define the sequence of operators $(P_n)_{n\in\mathbf{N}}$ by

$$P_n a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B(0,n)}\widehat{a})$$

and let us solve

$$(ANS_{\nu,n})\begin{cases} \partial_t u_n - \nu\Delta_h u_n + P_n(u_n\cdot\nabla u_n) - P_n \displaystyle\sum_{1\leq j,k\leq 3}\nabla\Delta^{-1}\partial_j\partial_k(u_n^j u_n^k) = 0\\ \operatorname{div} u_n = 0\\ u_{n|t=0} = P_n u_0\end{cases}$$

where $\Delta^{-1}\partial_j\partial_k$ is defined precisely in (1.2) page 9. In fact, the system $(ANS_{\nu,n})$ turns out to be an ordinary differential equation on the space

$$L^2_n \stackrel{\text{def}}{=} \Big\{v\in L^2(\mathbf{R}^3)\,/\,\operatorname{div} v = 0 \quad\text{and}\quad \operatorname{Supp}\,\widehat{v}\subset B(0,n)\Big\}$$

with the L^2 norm because we have, for any u and v in L^2_n,

$$\begin{aligned} Q_n(u,v) &\stackrel{\text{def}}{=} \Big\|P_n(u\cdot\nabla v) - P_n\nabla\sum_{1\leq j,k\leq 3}\Delta^{-1}\partial_j\partial_k(u^j v^k)\Big\|_{L^2}\\ &\leq Cn^{\frac{3}{2}+1}\|u\|_{L^2}\|v\|_{L^2}.\end{aligned}$$

Thus, for any n, a maximal solution u_n exists in $C^\infty([0,T_n[;L^2_n)$ with of course $T_n>0$.

Step 2: A priori bounds The first one is easily obtained: it is simply the energy estimate (1.1) page 8 formally done in the introduction and which is now rigorous because u_n is smooth, namely

$$\|u_n(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla_h u_n(t')\|_{L^2}^2 dt' = \|P_n u_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2. \tag{4.2}$$

Let us recall some classical blow up result s for ordinary differential equations. Let us state now a necessary condition for blow up.

Proposition 4.1.1 *Let F be a function from $\mathbf{R}\times E$ into E which satisfy the hypothesis of Theorem 3.2.1 at any point (t_0,x_0) of E. Let us also assume that a locally bounded function M from $\mathbf{R}^+$ into $\mathbf{R}^+$ and a locally integrable function β from $\mathbf{R}^+$ into $\mathbf{R}^+$ exist such that*

$$\|F(t,u)\| \leq \beta(t)M(\|u\|).$$

Then if $]T_\star,T^\star[$ is the maximal intervalle of existence of an integral curve and if $T^\star$ is finite, we have

$$\limsup_{t\to T^\star}\|u(t)\| = \infty.$$

Proof of Proposition 4.1.1 Let us first observe that, if you consider a positive time T such that $\|u(t)\|$ is bounded on $[T_0,T[$, then we can extend the solution on $[T_0,T_1]$ with $T_1>T$. As the function u is bounded on $[T_0,T[$, we deduce from the hypothesis on F that,

$$\forall t\in[T_0,T[\,,\ \|F(t,u(t))\| \leq C\beta(t),$$

the function β being integrable on $[T_0,T]$. Thus for any positive ε, a positive η exists such that, for any t and t' such that $T-t<\eta$ and $T-t'<\eta$,

$$\|u(t)-u(t')\| < \varepsilon.$$

The space E being complete, $u_\star$ exists in E such that $\lim\limits_{t\to T} u(t) = u_\star$. Applying Theorem 3.2.1, we construct a solution of (EDO) on $[T_0,T_1]$.

Corollary 4.1.1 *Under the hypotheses of Proposition 4.1.1, if we have in addition*

$$\|F(t,u)\| \leq M\|u\|^2,$$

then if the maximal time interval of existence is $]T_\star,T^\star[$ and $T^\star$ is finite, then

$$\int_0^{T^\star}\|x(t)\|dt = \infty.$$

Proof of Corollary 4.1.1 The solution satisfies

$$\|x(t)\| \leq \|x_0\| + M\int_0^t\|x(t')\|^2dt'.$$

Gronwall's lemma implies that

$$\|x(t)\| \leq \|x_0\|\exp\Big(M\int_0^t\|x(t')\|dt'\Big).$$

Thanks to Corollary 4.1.1, this implies that, for any n, the solution u_n of $(ANS_{\nu,n})$ is global which means that, for any n, u_n belongs to $C^\infty(\mathbf{R}_+;L^2_n)$.

The second a priori bound is more difficult to obtain. Let us differentiate $(ANS_{\nu,n})$ with respect to ∂_3. This gives, dropping the index n in order to make the notations lighter,

$$\|\partial_3u(t)\|_{L^2}^2 + 2\nu\int_0^t\|\nabla_h\partial_3u(t')\|_{L^2}^2dt' = \|\partial_3P_nu_0\|_{L^2}^2 - 2\sum_{1\leq k,\ell\leq 3} I_{k,\ell}(t) \tag{4.3}$$

with

$$I_{k,\ell}(t) \stackrel{\text{def}}{=} \int_{\mathbf{R}^3}\partial_3u^k(t)\partial_ku^\ell(t)\partial_3u^\ell(t)dx.$$

Let us start with the term $I_{k,\ell}$ with $k\neq 3$, namely the terms which contain only two vertical derivatives, which are the one which are not compensated by any smoothing effect. The following proposition will be useful.

Proposition 4.1.2 *A constant C exists such that*

$$\Big(\int_{\mathbf{R}^3} a(x)b(x)c(x)dx\Big)^2 \leq C\min\Big\{\|a\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|\nabla_h b\|_{L^2}, \|\nabla_h a\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|b\|_{L^2}\Big\}$$
$$\times \|a\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|b\|_{L^2}\|\nabla_h c\|_{L^2}\|c\|_{L^2}.$$

Proof of Proposition 4.1.2 Let us write that

$$\begin{aligned} J(a,b,c) &\stackrel{\text{def}}{=} \int_{\mathbf{R}^3} a(x)b(x)c(x)dx \\ &= \int_{\mathbf{R}} dx_3 \int_{\mathbf{R}^2} a(x_h,x_3)b(x_h,x_3)c(x_h,x_3)dx_h. \end{aligned}$$

The Hölder inequality implies that

$$\begin{aligned} J(a,b,c) &\leq \int_{\mathbf{R}} \|a(\cdot,x_3)\|_{L^2_h}\|b(\cdot,x_3)\|_{L^4_h}\|c(\cdot,x_3)\|_{L^4_h}dx_3 \\ &\leq \|a\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|b\|_{L^2(\mathbf{R}_{x_3};L^4_h)}\|c\|_{L^2(\mathbf{R}_{x_3};L^4_h)}. \end{aligned}$$

Then using the Sobolev inequality, the interpolation inequality and the Cauchy-Schwarz inequality, we get that

$$\begin{aligned} \|b\|^2_{L^2(\mathbf{R}_{x_3};L^4_h)} &\leq \int_{\mathbf{R}} \|\nabla_h b(\cdot,x_3)\|_{L^2_h}\|b(\cdot,x_3)\|_{L^2_h}dx_3 \\ &\leq \|\nabla_h b\|_{L^2}\|b\|_{L^2}. \end{aligned}$$

The proof of the other inequality is similar.

We shall also use a corollary of this proposition.

Corollary 4.1.2 *A constant C exists such that*

$$\Big(\int_{\mathbf{R}^3} a(x)b(x)c(x)dx\Big)^2 \leq C\|\partial_3 a\|_{L^2}\|a\|_{L^2}\|\nabla_h b\|_{L^2}\|b\|_{L^2}\|\nabla_h c\|_{L^2}\|c\|_{L^2}.$$

Proof of Corollary 4.1.2 Let us observe that

$$\begin{aligned} \|a(\cdot,x_3)\|^2_{L^2_h} &= \int_{-\infty}^{x_3} \frac{d}{dy_3}\Big(\int_{\mathbf{R}^2}|a(x_h,y_3)|^2dx_h\Big)dy_3 \\ &= 2\int_{-\infty}^{x_3}\int_{\mathbf{R}^2} a(x_h,y_3)\partial_{y_3}a(x_h,y_3)dx_h dy_3. \end{aligned}$$

Cauchy-Schwarz inequality implies that

$$\forall x_3 \in \mathbf{R}\,,\ \|a(\cdot,x_3)\|^2_{L^2_h} \leq 2\|\partial_3 a\|_{L^2}\|a\|_{L^2}.$$

The corollary is proved.

Let us go back to the proof of Theorem 4.1.1. Applying the above corollary in the case when $a = \partial_k u^\ell$, $b = \partial_3 u^k$ and $\partial_3 u^\ell$ gives

$$I_{k,\ell}(t) \leq C\|\nabla_h\partial_3 u(t)\|^{\frac{3}{2}}_{L^2}\|\partial_3 u(t)\|_{L^2}\|\nabla_h u(t')\|^{\frac{1}{2}}_{L^2}.$$

The estimate of the terms $I_{3,\ell}$ demands the use of the special structure of the system, namely the fact that the non linear term is $u\cdot\nabla u$ and that the vector fields involved are divergence free. The divergence free condition implies that

$$\begin{aligned} I_{3,\ell}(t) &= \int_{\mathbf{R}^3} \partial_3 u^3(t,x)\partial_3 u^\ell(t,x)\partial_3 u^\ell(t,x)dx \\ &= -\int_{\mathbf{R}^3} \operatorname{div}_h u^h(t,x)\partial_3 u^\ell(t,x)\partial_3 u^\ell(t,x)dx. \end{aligned}$$

This term is strictly analogous to the preceeding ones. Thus, we have that, for any k and ℓ,

$$I_{k,\ell}(t) \leq C\|\nabla_h\partial_3 u(t)\|_{L^2}^{\frac32}\|\partial_3 u(t)\|_{L^2}\|\nabla_h u(t)\|_{L^2}^{\frac12}.$$

Plugging this in the energy estimate (4.3) gives

$$\begin{aligned} \|\partial_3 u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla_h\partial_3 u(t')\|_{L^2}^2 dt' &\leq \|\partial_3 u_0\|_{L^2}^2 \\ &\quad + C\int_0^t \|\nabla_h\partial_3 u(t')\|_{L^2}^{\frac32}\|\partial_3 u(t')\|_{L^2}\|\nabla_h u(t')\|_{L^2}^{\frac12}dt'. \end{aligned}$$

Using the convexity inequality $ab \leq \frac14 a^4 + \frac34 b^{\frac43}$, we have

$$\|\partial_3 u(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h\partial_3 u(t')\|_{L^2}^2 dt' \leq \|\partial_3 u_0\|_{L^2}^2 + \frac{C}{\nu^3}\int_0^t \|\partial_3 u(t')\|_{L^2}^4\|\nabla_h u(t')\|_{L^2}^2 dt'. \quad (4.4)$$

Let us reintroduce for the moment the index n and define

$$T_n \overset{\text{def}}{=} \sup\Big\{t>0\,,\ \|\partial_3 u_n\|_{L^\infty([0,t];L^2)}^2 + \nu\|\nabla_h\partial_3 u_n\|_{L^2([0,t];L^2)}^2 \leq 2\|\partial_3 u_0\|_{L^2}^2\Big\}.$$

As the function u_n is continuous with value in H^s for any s and as $\|\partial_3 P_n u_0\|_{L^2} \leq \|\partial_3 u_0\|_{L^2}$ in L^2, the time T_n is positive and for any $t<T_n$, we have

$$\|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h\partial_3 u(t')\|_{L^2}^2 dt' \leq \|\partial_3 u_0\|_{L^2}^2\Big(1+\frac{C}{\nu^3}\|\partial_3 u_0\|_{L^2}^2\int_0^t \|\nabla_h u_n(t')\|_{L^2}^2 dt'\Big). \quad (4.5)$$

Thanks to the energy estimate (4.2), we have, for any $t<T_n$,

$$\|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h\partial_3 u_n(t')\|_{L^2}^2 dt' \leq \|\partial_3 u_0\|_{L^2}^2\Big(1+\frac{C}{\nu^4}\|\partial_3 u_0\|_{L^2}^2\|u_0\|_{L^2}^2\Big).$$

Thus under the smallness condition (4.1), we have that $T_n=+\infty$ and thus,

$$\forall t\geq 0\,,\ \forall n\in\mathbf{N}\,,\ \|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h\partial_3 u_n(t')\|_{L^2}^2 dt' \leq 2\|\partial_3 u_0\|_{L^2}^2.$$

Now, let us investigate the case when the initial data does not satisfy the smallness condition. We shall write u_n as a perturbation of the free solution $u_{N_0,F} \overset{\text{def}}{=} e^{\nu t\Delta}P_nP_{N_0}u_0$ by stating

$$w_n \overset{\text{def}}{=} u_n - u_{N_0,F},$$

the integer N_0 being chosen later on. Inequality (4.5) becomes

$$\begin{aligned} &\|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h\partial_3 u_n(t')\|_{L^2}^2 dt' \\ &\qquad \leq \|\partial_3 u_0\|_{L^2}^2\bigg(1+\frac{C}{\nu^3}\|\partial_3 u_0\|_{L^2}^2\Big(\int_0^t \|\nabla_h u_{N_0,F}(t')\|_{L^2}^2 dt' + \int_0^t \|\nabla_h w_n(t')\|_{L^2}^2 dt'\Big)\bigg). \end{aligned}$$

By definition of $u_{N_0,F}$, we get

$$
\begin{aligned}
&\|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h \partial_3 u_n(t')\|_{L^2}^2 dt'\\
&\qquad\qquad \leq \|\partial_3 u_0\|_{L^2}^2\Big(1+\frac{C}{\nu^3}\|\partial_3 u_0\|_{L^2}^2\Big(tN_0^2\|u_0\|_{L^2}^2 + \int_0^t \|\nabla_h w_n(t')\|_{L^2}^2 dt'\Big)\Big).
\end{aligned}
$$

Let us estimate $\displaystyle\int_0^t \|\nabla_h w_n(t')\|_{L^2}^2 dt'$. By definition of w, we have

$$
\begin{gathered}
\partial_t w_n - \nu\Delta_h w_n + P_n(u_n\cdot\nabla w_n) + P_n(u_n\cdot\nabla u_{n,F}) = -\nabla p_n\\
\operatorname{div} w_n = 0\\
w_{n|t=0} = (\mathrm{Id} - P_{N_0})u_0.
\end{gathered}
$$

Using the divergence free condition, we get by energy estimate that

$$
\nu\int_0^t \|\nabla_h w(t')\|_{L^2}^2 dt' \leq \|(\mathrm{Id}-P_{N_0})u_0\|_{L^2}^2 - 2\int_0^t \langle u_n(t')\cdot\nabla u_{N_0,F}, w_n(t')\rangle dt'.
$$

Let us notice that, using Lemma 2.1.1,

$$
\begin{aligned}
|\langle u_n(t')\cdot\nabla u_{N_0,F}, w_n(t')\rangle| &\leq \|\nabla u_{N_0,F}(t')\|_{L^\infty(\mathbf{R}^3)}\|u_n(t)\|_{L^2}\|w_n(t)\|_{L^2}\\
&\leq C\|u_0\|_{L^2}^2\|\nabla u_{N_0,F}(t')\|_{L^\infty(\mathbf{R}^3)}\\
&\leq CN_0^{\frac{5}{2}}\|u_0\|_{L^2}^3.
\end{aligned}
$$

Thus, for any n,

$$
\nu\int_0^t \|\nabla_h w_n(t')\|_{L^2}^2 dt' \leq \|(\mathrm{Id}-P_{N_0})u_0\|_{L^2}^2 + tN_0^{\frac{5}{2}}\|u_0\|_{L^2}^3.
$$

We infer that

$$
\begin{aligned}
&\|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h \partial_3 u_n(t')\|_{L^2}^2 dt'\\
&\quad \leq \|\partial_3 u_0\|_{L^2}^2\Big(1+\frac{C}{\nu^3}\|\partial_3 u_0\|_{L^2}^2\Big(tN_0^2\|u_0\|_{L^2}^2 + \frac{1}{\nu}\|(\mathrm{Id}-P_{N_0})u_0\|_{L^2}^2 + \frac{1}{\nu}tN_0^{\frac{5}{2}}\|u_0\|_{L^2}^3\Big)\Big).
\end{aligned}
$$

Choosing N_0 and then $\underline{T}$ such that the above quantity is small enough ensures that, for any $t\leq\underline{T}$, we have, for any n

$$
\|\partial_3 u_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla_h \partial_3 u_n(t')\|_{L^2}^2 dt' \leq 2\|\partial_3 u_0\|_{L^2}^2. \tag{4.6}
$$

Classical compactness arguments allows to exibit a solution u of (ANS_ν) which belongs to $L^\infty([0,T];H^{0,1})\cap L^2([0,T];H^{1,1})$. To prove uniqueness, let us prove the following lemma.

Lemma 4.1.1 *Let u_j, $j\in\{1,2\}$ be two solutions of (ANS_ν) in the space*

$$
L^\infty([0,T];H^{0,1})\cap L^2([0,T];H^{1,1}).
$$

Then we have

$$
\|u_2(t)-u_1(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla_h(u_2-u_1)(t')\|_{L^2}^2 dt' \leq \|(u_2-u_1)(0)\|_{L^2}^2 \exp M_{u_j}(t) \quad \textit{with}
$$

$$
M_{u(t)} \stackrel{\mathrm{def}}{=} \frac{C}{\nu^3}\int_0^t \|\partial_3\nabla_h u(t')\|_{L^2}\|\nabla u(t')\|_{L^2} dt'.
$$

Remark As u_j belongs to $L^\infty([0,T];H^{0,1})\cap L^2([0,T];H^{1,1})$, we have

$$M_{u_j}(t)\leq \frac{C}{\nu^3}\|\partial_3\nabla_h u_j\|_{L^2_t(L^2)}\Big(\frac{1}{\sqrt{2\nu}}\|u_j(0)\|_{L^2}+t^{\frac12}\|\partial_3 u_j\|_{L^\infty_t(L^2)}\Big).$$

Proof of Lemma 4.1.1 Stating $u_{21}\stackrel{\rm def}{=}u_2-u_1$, we get, by a L^2 energy estimate

$$\|u_{21}(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla_h u_{21}(t')\|_{L^2}^2dt'=-I^h(t)-I^v(t)$$

with

$$\begin{aligned}I^h(t) &\stackrel{\rm def}{=} \sum_{\substack{1\leq k\leq 2\\ 1\leq \ell\leq 3}}\int_0^t\int_{\mathbf{R}^3}u_{21}^k(t')\partial_k u_1^\ell(t')u_{21}^\ell(t')dt'dx \quad\text{and}\\ I^v(t) &\stackrel{\rm def}{=} \sum_{1\leq \ell\leq 3}\int_0^t\int_{\mathbf{R}^3}u_{21}^3(t')\partial_3 u_1^\ell(t')u_{21}^\ell(t')dt'dx.\end{aligned}$$

Corollary 4.1.2 applied with $a=\partial_k u_1^\ell$, $b=u_{21}^k$ and $c=u_{21}^\ell$, implies that

$$\begin{aligned}I^h(t) &\leq \int_0^t\|\partial_3\nabla_h u_j(t')\|_{L^2}^{\frac12}\|\nabla_h u_j(t')\|_{L^2}^{\frac12}\|\nabla_h u_{21}(t')\|_{L^2}\|u_{21}(t')\|_{L^2}dt'\\ &\leq \frac{\nu}{2}\int_0^t|\nabla_h u_{21}(t')\|_{L^2}^2dt'+\frac{C}{\nu}\int_0^t\|\partial_3\nabla_h u_j(t')\|_{L^2}\|\nabla_h u_j(t')\|_{L^2}\|u_{21}(t')\|_{L^2}^2dt'.\end{aligned}$$

Proposition 4.1.2 applied with $a=u_{21}^3$, $b=\partial_3 u_1^\ell$ and $c=u_{21}^\ell$ gives

$$I^v(t)\leq\int_0^t\|u_{21}^3(t')\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|\partial_3\nabla_h u_j(t')\|_{L^2}^{\frac12}\|\partial_3 u_j(t')\|_{L^2}^{\frac12}\|\nabla_h u_{21}(t')\|_{L^2}^{\frac12}\|u_{21}(t')\|_{L^2}^{\frac12}dt'.$$

The following property is important.

Lemma 4.1.2 *Let v be a divergence free vector field. Then we have*

$$\|v^3\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\leq\sqrt2\|\nabla_h v\|_{L^2}^{\frac12}\|v\|_{L^2}^{\frac12}.$$

Proof of Lemma 4.1.2 Let us write that

$$\begin{aligned}\|v^3(\cdot,x_3)\|_{L^2_h}^2 &= 2\int_{-\infty}^{x_3}\Big(\int_{\mathbf{R}^2}\partial_{y_3}v^3(x_h,y_3)v^3(x_h,y_3)dx_h\Big)dx_3\\ &= -2\int_{-\infty}^{x_3}\Big(\int_{\mathbf{R}^2}\mathrm{div}_h\, v^h(x_h,y_3)v^3(x_h,y_3)dx_h\Big)dx_3.\end{aligned}$$

The Cauchy Schwarz inequality allows to conclude this proof.

Let us go back to the proof of Lemma 4.1.1. Now, we have

$$\begin{aligned}I^v(t) &\leq \int_0^t\|\nabla_h u_{21}(t')\|_{L^2}\|u_{21}(t')\|_{L^2}\|\partial_3\nabla_h u_j(t')\|_{L^2}^{\frac12}\|\partial_3 u_j(t')\|_{L^2}^{\frac12}dt'\\ &\leq \frac{\nu}{2}\int_0^t\|\nabla_h u_{21}(t')\|_{L^2}^2dt+\frac{C}{\nu}\int_0^t\|\partial_3\nabla_h u_j(t')\|_{L^2}\|\partial_3 u_j(t')\|_{L^2}\|u_{21}(t')\|_{L^2}^2dt'.\end{aligned}$$

The application of the Gronwall lemma concludes the proof.

4.2 Anisotropic viscosity and scaling invariant spaces

This study requires a careful use of Littlewood-Paley theory in the vertical variable. Let us consider the partition of unity on $\mathbf{R}$ given by Proposition 2.1.1 page 31. We take the same notation in order to avoid heaviness. We have the following equivalent of Lemma 2.1.1 page 29.

Lemma 4.2.1 *Let $\|\cdot\|_E$ be a semi-norm and B a ball and $\mathcal{C}$ a ring of $\mathbf{R}$. A constant C exists which satisfies the following properties.*

For any positive λ, any $1 \leq p \leq q \leq \infty$, we have, for any function u, the Fourier transform in the horizontal variable of which is supported in λB, we have

$$\|\partial_3^k a\|_{L^q(\mathbf{R}_{x_3};E)} \leq C^{k+1}\lambda^{k+\frac{1}{p}-\frac{1}{q}}\|a\|_{L^p(\mathbf{R}_{x_3};E)}.$$

For any positive λ, for any function u, the Fourier transform in the horizontal variable of which is supported in $\lambda\mathcal{C}$, we have

$$\|a\|_{L^p(\mathbf{R}_{x_3};E)} \leq C^{k+1}\lambda^{-k}\|\partial_3^k a\|_{L^p(\mathbf{R}_{x_3};E)}.$$

The proof is the same has the one of Lemma 2.1.1 page 29 and thus omitted.

Lemma 4.2.2 *Let $\mathcal{C}$ be a ring of $\mathbf{R}$. A constant C exists such that for any divergence free vector field the Fourier transform of which is supported in $\lambda\mathcal{C}$, we have*

$$\|v^3\|_{L^\infty(\mathbf{R}_{x_3};E)} \leq C\lambda^{-\frac{1}{2}}\|\operatorname{div}_h v^h\|_{L^2(\mathbf{R}_{x_3};E)}.$$

Proof of Lemma 4.2.2 Left as an exercice.

Let us define the space we are going to work with.

Definition 4.2.1 *Let us define the following seminorms.*

$$\begin{aligned}
\|a\|_{\dot{\mathcal{B}}^{0,s}} &\stackrel{\text{def}}{=} \sum_{j\in\mathbf{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^2},\\
\|a\|_{\widetilde{L}^p_T(\dot{\mathcal{B}}^{0,s})} &\stackrel{\text{def}}{=} \sum_{j\in\mathbf{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^p_T(L^2)} \quad \textit{and}\\
\|a\|_{T,s} &\stackrel{\text{def}}{=} \|a\|_{\widetilde{L}^\infty_T(\dot{\mathcal{B}}^{0,s})} + \sqrt{2\nu}\|\nabla_h a\|_{\widetilde{L}^2_T(\dot{\mathcal{B}}^{0,s})}.
\end{aligned}$$

Let us define the following spaces inhomogenous spaces.

$$\begin{aligned}
\mathcal{B}^{0,s} &\stackrel{\text{def}}{=} \{u \in L^2 \,/\, \|a\|_{\dot{\mathcal{B}}^{0,s}} < \infty\} \quad \textit{and}\\
\mathcal{B}_{T,s} &\stackrel{\text{def}}{=} \{u \in L^\infty_T(L^2) \,/\, \|a\|_{T,s} < \infty\}.
\end{aligned}$$

Now we can state the main theorem of this chapter.

Theorem 4.2.1 *Let u_0 be in $\mathcal{B}^{0,\frac{1}{2}}$. Then a positive T exists such that a unique solution exists in $\mathcal{B}_T^{0,\frac{1}{2}}$. This solution is continuous with value in $\mathcal{B}^{0,\frac{1}{2}}$. Moreover, if $\|u_0\|_{\dot{\mathcal{B}}^{0,\frac{1}{2}}} \leq c\nu$ with small enough c, then T can be choosen equal to $+\infty$.*

We shall not prove this theorem totally here. We refer to the work [47] of M. Paicu for a complete proof of this theorem. We shall only prove a part of it, namely the following theorem.

Theorem 4.2.2 *A constant c exists such that, for u_0 in $\dot{\mathcal{B}}^{0,1}$, if $\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}} \leq c\nu$, then the solution given by Theorem 4.1.1 is global.*

Proof of Theorem 4.2.2 Let us admit for the time being the following lemma.

Lemma 4.2.3 *For any positive s, a constant C_0 exists such that*

$$\Big|\int_0^T (\dot\Delta_j(u\nabla u)|\dot\Delta_j u)dt\Big| \leq \frac{C_0}{\nu}\|u\|_{T,\frac12}\|u\|_{T,s}^2 c_j^2 2^{-2js}$$

where, as in all that follows in the chapter, $(c_j)_{j\in\mathbf{Z}}$ denotes a generic series of non negative terms the sum of which is 1.

Then, let us consider the sequence $(u_n)_{n\in\mathbf{N}}$ of solutions of $(ANS_{\nu,n})$ and let us define

$$T_n \stackrel{\text{def}}{=} \sup\Big\{t\geq 0 \,/\, \|u_n\|_{t,\frac12} \leq 4\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}}\Big\}.$$

As u_n is a smooth function, it is easy to see that if $\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}}$ is less than $c\nu$, then, for any n, T_n is positive. Applying Lemma 4.2.3 with $s=1/2$ or $s=1$, we get, by (localized in frequency) energy estimate, that, for any $T\leq T_n$,

$$2^{2js}\|\dot\Delta_j u_n\|_{L^\infty_T(L^2)}^2 + 2\nu 2^{2js}\|\dot\Delta_j\nabla_h u_n\|_{L^2_T(L^2)}^2 \leq 2^{2js}\|\dot\Delta_j u_0\|_{L^2}^2 + 4\frac{C_0}{\nu}\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}}\|u_n\|_{T,s}^2 c_j^2.$$

This inequality can be written

$$\Big(2^{js}\|\dot\Delta_j u_n\|_{L^\infty_T(L^2)} + \sqrt{2\nu}2^{js}\|\dot\Delta_j\nabla_h u_n\|_{L^2_T(L^2)}\Big)^2 \leq 2\Big(\|u_0\|_{\dot{\mathcal{B}}^{0,s}} + 4\Big(\frac{C_0\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}}}{\nu}\Big)^{\frac12}\|u_n\|_{T,s}\Big)^2 c_j^2.$$

By definition of $\|u\|_{T,s}$ semi norms, we deduce that for any $T<T_n$,

$$\|u_n\|_{T,s} \leq \sqrt2\|u_0\|_{\dot{\mathcal{B}}^{0,s}} + 2\sqrt2\Big(\frac{C_0\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}}}{\nu}\Big)^{\frac12}\|u_n\|_{T,s}.$$

Thus choosing $\|u_0\|_{\dot{\mathcal{B}}^{0,\frac12}}$ small enough will give, for $s=1/2$ or $s=1$,

$$\forall t<T_n\,,\ \|u_n\|_{T,s} \leq 2\|u_0\|_{\dot{\mathcal{B}}^{0,s}}.$$

Applying this with $s=1/2$ gives $T_n=+\infty$ and then, for $s=1/2$ or $s=1$,

$$\forall T\geq 0\,,\ \|u_n\|_{T,s} \leq 2\|u_0\|_{\dot{\mathcal{B}}^{0,s}}.$$

As obviously

$$\|\partial_3 u_n\|_{L^\infty_T(L^2)}^2 + 2\nu\|\partial_3\nabla_h u_n\|_{L^2_T(L^2)}^2 \leq \|u_n\|_{T,1},$$

Theorem 4.2.2 is proved, provided of course that we prove Lemma 4.2.3.

Proof of Lemma 4.2.3 It relies on anisotropic paradifferential calculus. Moreover, we have to distinguish betweeen horizontal and vertical derivates. This leads to introduce the following notations.

$$
\begin{aligned}
I^{1,h}_{j,j'}(T) &\stackrel{\text{def}}{=} \sum_{k=1}^{2}\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot S_{j'-1}u^k\partial_k\dot\Delta_{j'}u)\dot\Delta_j u\,dxdt\\
I^{2,h}_{j,j'}(T) &\stackrel{\text{def}}{=} \sum_{k=1}^{2}\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot\Delta_{j'}u^k\partial_k\dot S_{j'+2}u)\dot\Delta_j\, udxdt\\
I^{1,v}_{j,j'}(T) &\stackrel{\text{def}}{=} \int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot S_{j'-1}u^3\partial_3\dot\Delta_{j'}u)\dot\Delta_j u\,dxdt\\
I^{2,v}_{j,j'}(T) &\stackrel{\text{def}}{=} \int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot\Delta_{j'}u^3\partial_3\dot S_{j'+2}u)\dot\Delta_j u\,dxdt.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
I_j(T) &\stackrel{\text{def}}{=} \int_0^T(\dot\Delta_j(u\cdot\nabla u)|\dot\Delta_j u)dt\\
&= \sum_{j'}\Big(I^{1,h}_{j,j'}(T)+I^{2,h}_{j,j'}(T)+I^{1,v}_{j,j'}(T)+I^{2,v}_{j,j'}(T)\Big).
\end{aligned}
$$

As the terms $I^{1,h}_{j,j'}(T)$ and $I^{2,h}_{j,j'}(T)$ are analogous, we shall only prove estimates on $I^{2,h}_{j,j'}(T)$.

Proposition 4.2.1 *Let $(s_1,s_2)\in\mathbf{R}^2$ such that $s_1>0$. A constant C exists such that, if*

$$
I^{2,h}_j(a,b,c)(T)\stackrel{\text{def}}{=}\sum_{j'}\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot\Delta_{j'}a\partial_k\dot S_{j'+2}b)\dot\Delta_j c\,dtdx,
$$

then, we have

$$
I^{2,h}_j(a,b,c)(T)\le\frac{C}{\nu}c_j^2 2^{-j(s_1+s_2)}\|a\|_{T,s_1}\|b\|_{T,\frac12}\|c\|_{T,s_2}.
$$

Proof of Proposition 4.2.1 Thanks to Proposition 4.1.2, we have

$$
\begin{aligned}
I^{2,h}_{j,j'}(a,b,c)(T) &\stackrel{\text{def}}{=} \left|\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot\Delta_{j'}a\partial_k\dot S_{j'+2}b)\dot\Delta_j c\,dtdx\right|\\
&\le \int_0^T\|\nabla_h\dot S_{j'+2}b(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|\nabla_h\dot\Delta_{j'}a(t)\|^{\frac12}_{L^2}\|\dot\Delta_{j'}a(t)\|^{\frac12}_{L^2}\\
&\qquad\qquad\times\|\nabla_h\dot\Delta_j c(t)\|^{\frac12}_{L^2}\|\dot\Delta_j c(t)\|^{\frac12}_{L^2}.
\end{aligned}
$$

By definition of $\|\cdot\|_{T,s}$, we have

$$
\begin{aligned}
I^{2,h}_{j,j'}(a,b,c)(T)\le c_{j'}^{\frac12}c_j^{\frac12}2^{-\frac{j}{2}s_1-\frac{j'}{2}s_2}\|a\|^{\frac12}_{T,s_1}\|c\|^{\frac12}_{T,s_2}\int_0^T\|\nabla_h\dot S_{j'+2}b(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\\
\times\|\nabla_h\dot\Delta_{j'}a(t)\|^{\frac12}_{L^2}\|\nabla_h\dot\Delta_j c(t)\|^{\frac12}_{L^2}dt.
\end{aligned}
$$

We shall often use the following lemma which is a straightforward consequence of Lemma 4.2.1 and of the definition of the semi norm $\|\cdot\|_{T,s}$.

Lemma 4.2.4 *A constant C exists such that, for any j*

$$\|\dot S_j a\|_{L^\infty_T(L^\infty(\mathbf{R}_{x_3};L^2_h))} + \sqrt{\nu}\|\dot S_j \nabla_h a\|_{L^2_T(L^\infty(\mathbf{R}_{x_3};L^2_h))} \leq C\|a\|_{T,\frac12}.$$

For any σ less than $1/2$, a constant C exists such that

$$\|\dot S_j a\|_{L^\infty_T(L^\infty(\mathbf{R}_{x_3};L^2_h))} + \sqrt{\nu}\|\dot S_j \nabla_h a\|_{L^2_T(L^\infty(\mathbf{R}_{x_3};L^2_h))} \leq C\|a\|_{T,\sigma}c_j 2^{j(\frac12-\sigma)}$$

where, as in this whole chapter, $(c_j)_{j\in\mathbf{Z}}$ denotes a non negative series the sum of which is 1.

Let us go back to the proof of Proposition 4.2.1. Using the above lemma, the Hölder inequality, Lemma 4.2.4 and the definition of $\|\cdot\|_{T,s}$, we get

$$\begin{aligned} I^{2,h}_{j,j'}(a,b,c)(T) &\leq c_{j'}^{\frac12}c_j^{\frac12}2^{-\frac{j'}{2}s_1-\frac{j}{2}s_2}\|a\|_{T,s_1}^{\frac12}\|c\|_{T,s_2}^{\frac12}\|\nabla_h\dot S_{j'+2}b\|_{L^2_T(L^\infty(\mathbf{R}_{x_3};L^2_h))}\\ &\qquad\qquad \times\|\nabla_h\dot\Delta_{j'}a(t)\|^{\frac12}_{L^2_T(L^2)}\|\nabla_h\dot\Delta_j c(t)\|^{\frac12}_{L^2_T(L^2)}\\ &\leq \frac{C}{\nu}c_{j'}c_j 2^{-j's_1-js_2}\|a\|_{T,s_1}\|b\|_{T,\frac12}\|c\|_{T,s_2}. \end{aligned}$$

Now let us observe that, as the support of the vertical Fourier transform of $\dot\Delta_{j'}a\partial_k\dot S_{j'+2}b$ is supported in a ball of type $2^{j'}B$, then

$$\begin{aligned} 2^{j(s_1+s_2)}I^{2,h}_j(a,b,c)(T) &\leq \sum_{j'>j-N_0}2^{j(s_1+s_2)}I^{2,h}_{j,j'}(a,b,c)(T)\\ &\leq \frac{C}{\nu}c_j\|a\|_{T,s_1}\|b\|_{T,\frac12}\|c\|_{T,s_2}\sum_{j'>j-N_0}c_{j'}2^{-(j'-j)s_1}. \end{aligned}$$

Proposition 4.2.1 is proved.

Along the same lines, we have the following proposition.

Proposition 4.2.2 *Let $(s_1,s_2)\in\mathbf{R}^2$ such that $s_2>0$. A constant C exists such that, if*

$$I^{1,h}_j(a,b,c)(T)\stackrel{\mathrm{def}}{=}\sum_{j'}\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot S_{j'-1}a\partial_k\dot\Delta_{j'}b)\dot\Delta_j c\,dtdx,$$

then, we have

$$I^{1,h}_j(a,b,c)(T)\leq\frac{C}{\nu}c_j^2 2^{-j(s_1+s_2)}\|a\|_{T,\frac12}\|b\|_{T,s_1}\|c\|_{T,s_2}.$$

Now let us estimate the terms that involves vertical derivatives. We shall use the structure of the non linear term. The main point is that whenever ∂_3 shows up, so does u^3.

Proposition 4.2.3 *Let $(s_1,s_2)\in\mathbf{R}^2$ $s_1>0$ is positive. A constant C exists such that, if w is a divergence free vector field, then*

$$I^{2,v}_j(w,a,b)(T)\stackrel{\mathrm{def}}{=}\sum_{j'}\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot\Delta_{j'}w^3\partial_3\dot S_{j'+2}a)\dot\Delta_j b\,dtdx,$$

then, we have

$$I^{2,v}_j(w,a,b)(T)\leq\frac{C}{\nu}c_j^2 2^{-j(s_1+s_2)}\|w\|_{T,s_1}\|a\|_{T,\frac12}\|b\|_{T,s_2}.$$

Proof of Proposition 4.2.3 Applying Proposition 4.1.2 and then Lemma 4.2.2, we get

$$\begin{aligned}
I^{2,v}_{j,j'}(w,a,b)(T) &\stackrel{\text{def}}{=} \Big|\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot\Delta_{j'}w^3\partial_3\dot S_{j'+2}a)\dot\Delta_j b\,dtdx\Big|\\
&\leq \int_0^T\|\dot\Delta_{j'}w^3\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|\partial_3\nabla_h\dot S_{j'+2}a(t)\|_{L^2}^{\frac12}\|\partial_3\dot S_{j'+2}a(t)\|_{L^2}^{\frac12}\\
&\qquad\qquad\times\|\nabla_h\dot\Delta_j b(t)\|_{L^2}^{\frac12}\|\dot\Delta_j b(t)\|_{L^2}^{\frac12}dt\\
&\leq 2^{-\frac{j'}{2}}\int_0^T\|\dot\Delta_{j'}\nabla_h w\|_{L^2}\|\partial_3\nabla_h\dot S_{j'+2}a(t)\|_{L^2}^{\frac12}\|\partial_3\dot S_{j'+2}a(t)\|_{L^2}^{\frac12}\\
&\qquad\qquad\times\|\nabla_h\dot\Delta_j b(t)\|_{L^2}^{\frac12}\|\dot\Delta_j b(t)\|_{L^2}^{\frac12}dt.
\end{aligned}$$

Using the Hölder inequality and then Lemma 4.2.4 and the definition of $\|\cdot\|_{T,s}$, we get

$$\begin{aligned}
I^{2,v}_{j,j'}(w,a,b)(T) &\leq c_j^{\frac12}2^{-\frac{j}{2}s_2}\|a\|_{T,\frac12}^{\frac12}\|b\|_{T,s_2}^{\frac12}\\
&\qquad\times\int_0^T\|\dot\Delta_{j'}\nabla_h w\|_{L^2}\|\partial_3\nabla_h\dot S_{j'+2}a(t)\|_{L^2}^{\frac12}\|\nabla_h\dot\Delta_j b(t)\|_{L^2}^{\frac12}dt\\
&\leq \frac{C}{\nu}c_jc_{j'}2^{-j's_1-js_2}\|w\|_{T,s_1}\|a\|_{T,\frac12}\|b\|_{T,s_2}.
\end{aligned}$$

Then we infer

$$I^{2,v}_j(w,a,b)(T)\leq\frac{C}{\nu}c_j2^{-j(s_1+s_2)}\|w\|_{T,s_1}\|a\|_{T,\frac12}\|b\|_{T,s_2}\sum_{j'\geq j-N_0}c_{j'}2^{-(j'-j)s_1}.$$

The proposition is proved.

Let us estimate the term $I^{1,v}$. This term looks the worse because the vertical derivative acts on the term where the vertical frequencies are high. Again, the particular structure of the non linear term is demanded.

Proposition 4.2.4 *Let s be a real number. Then a positive constant C exists such that, if w is a divergence free vector field, then, if*

$$I^{1,v}_j(w,a)(T)\stackrel{\text{def}}{=}\sum_{j'}\int_0^T\int_{\mathbf{R}^3}\dot\Delta_j(\dot S_{j'-1}w^3\partial_3\dot\Delta_{j'}a)\dot\Delta_j a\,dtdx,$$

we have

$$I^{1,v}_j(w,a)(T)\leq\frac{C}{\nu}c_j^2 2^{-2js}\|w\|_{T,\frac12}\|a\|^2_{T,s}.$$

Proof of Proposition 4.2.4 The following lemma will transform vertical derivative acting on high vertical frequencies term into a sum of terms either of type $I^{2,v}_j(T)$ or of terms which contains horizontal derivatives.

Lemma 4.2.5 *We have*

$$\begin{aligned}
I^{1,v}_j(w,a)(T) &= \sum_{m=1}^3\int_0^T\int_{\mathbf{R}^3}K^m_j(t)dtdx\quad\textit{with}\\
K^1_j(t) &\stackrel{\text{def}}{=} \sum_{j'}(\dot S_{j'-1}-\dot S_{j-1})w^3(t)\partial_3\Delta_{j'}\Delta_j a(t)\dot\Delta_j a(t)\,,\\
K^2_j(t) &\stackrel{\text{def}}{=} \sum_{j'}[\dot\Delta_j,\dot S_{j'-1}w^3(t)]\partial_3\dot\Delta_{j'}a(t)\dot\Delta_j a(t)\quad\textit{and}\\
K^3_j(t) &\stackrel{\text{def}}{=} -\frac12\dot S_{j-1}\,\mathrm{div}_h\,w^h(t)(\dot\Delta_j a(t))^2.
\end{aligned}$$

Proof of Lemma 4.2.5 Let us begin by writing the following computations.

$$\begin{aligned}
\mathcal{B}_j^{1,v}(w,a)(t) &\stackrel{\text{def}}{=} \sum_{j'} \dot\Delta_j\Big(\dot S_{j'-1}w^3(t)\partial_3\dot\Delta_{j'}a(t)\Big)\\
&= \sum_{j'} \dot S_{j'-1}w^3(t)\partial_3\dot\Delta_j\dot\Delta_{j'}a(t) + [\dot\Delta_j, \dot S_{j'-1}w^3(t)]\partial_3\dot\Delta_{j'}a(t)\\
&= \sum_{j'} (\dot S_{j'-1}-\dot S_{j-1})w^3(t)\partial_3\dot\Delta_j\dot\Delta_{j'}a(t) + \sum_{j'} \dot S_{j-1}w^3(t)\partial_3\dot\Delta_j\dot\Delta_{j'}a(t)\\
&\qquad\qquad + \sum_{j'} [\dot\Delta_j, \dot S_{j'-1}w^3(t)]\partial_3\dot\Delta_{j'}a(t)\\
&= K_j^1(t) + K_j^2(t) + \dot S_{j-1}w^3(t)\partial_3\dot\Delta_j a(t).
\end{aligned}$$

Thanks to the fact that $\partial_3 w^3 = -\operatorname{div}_h w^h$, an integration by part gives

$$\int_{\mathbf{R}^3} \dot S_{j-1}w^3(t)\partial_3\dot\Delta_j a(t)\dot\Delta_j a(t)dx = \frac{1}{2}\int_{\mathbf{R}^3} \dot S_{j-1}\operatorname{div}_h w^h(t)(\dot\Delta_j a(t))^2dx.$$

This proves the lemma.

Let us go back to the proof of Proposition 4.2.4. Using Proposition 4.1.2 and then Lemma 4.2.2, we get

$$\begin{aligned}
K_j^1(t) &\leq C2^j \sum_{\substack{j''\in(j'-1,j-1)\\ |j'-j|\leq N}} \|\dot\Delta_{j''}w^3(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|\nabla_h\dot\Delta_j a(t)\|_{L^2}\|\dot\Delta_j a(t)\|_{L^2}\\
&\leq \sum_{|j''-j|\leq N} 2^{\frac{j''}{2}}\|\dot\Delta_{j''}\nabla_h w(t)\|_{L^2}\|\nabla_h\dot\Delta_j a(t)\|_{L^2}\|\dot\Delta_j a(t)\|_{L^2}.
\end{aligned}$$

Thus, by definition of $\|\cdot\|_{T,s}$, we have

$$\int_0^T K_j^1(t)dt \leq \frac{C}{\nu}c_j^2 2^{-2js}\|w\|_{T,\frac12}\|a\|_{T,s}^2. \tag{4.7}$$

Using Proposition 4.1.2 we get

$$K_j^3(t) \leq C\|\dot S_j \operatorname{div}_h w^h(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\|\nabla_h\dot\Delta_j a(t)\|_{L^2}\|\dot\Delta_j a(t)\|_{L^2}.$$

Then applying Lemma 4.2.4, we get, by definition of $\|\cdot\|_{T,s}$,

$$\int_0^T K_j^3(t)dt \leq \frac{C}{\nu}c_j^2 2^{-2js}\|w\|_{T,\frac12}\|a\|_{T,s}^2. \tag{4.8}$$

In order to estimate $K_j^2(T)$, we need a control on the commutator. As $\dot\Delta_j$ is a convolution operator, the point is to describe the commutation between a convolution and a multiplication in an anisotropic way.

Lemma 4.2.6 *Let E, F and G three Banach spaces continuously included in $\mathcal{S}'(\mathbf{R}^2)$ such that*

$$\|\alpha\beta\|_G \leq \|\alpha\|_E\|\beta\|_F.$$

A constant C exists such that, for any $p\in[1,+\infty]$, any lipschitz function α from $\mathbf{R}_{x_3}$ into E any function β in $L^p(\mathbf{R}_{x_3};F)$, we have

$$\|[\dot\Delta_j,\alpha]\beta\|_{L^p(\mathbf{R}_{x_3},G)} \leq C2^{-j}\|\partial_3\alpha\|_{L^\infty(\mathbf{R}_{x_3};E)}\|\beta\|_{L^p(\mathbf{R}_{x_3};F)}.$$

Proof of Lemma 4.2.6 By definition of $\dot\Delta_j$, we have

$$\begin{aligned}\Big([\dot\Delta_j,\alpha]\beta\Big)(x_h,x_3) &= \dot\Delta_j(\alpha\beta)(x_h,x_3)-\alpha(x_h,x_3)\dot\Delta_j\beta(x_h,x_3)\\ &= 2^j\int_{\mathbf{R}^d} h(2^j(x_3-y_3))(\alpha(x_h,y_3)-\alpha(x_h,x_3))\beta(x_h,y_3)dy_3.\end{aligned}$$

As the function a is supposed to be lipschitzian with respect to the vertical variable x_3, we have

$$\|\alpha(\cdot,y_3)-\alpha(\cdot,x_3)\|_E\leq\|\partial_3\alpha\|_{L^\infty(\mathbf{R}_{x_3};E)}|y_3-x_3|.$$

It turns out that

$$\begin{aligned}\|([\dot\Delta_j,\alpha]\beta)(\cdot,x_3)\|_G &\leq C2^j\int_{\mathbf{R}^d}|h(2^j(x_3-y_3))|\,\|\alpha(\cdot,y_3)-\alpha(\cdot,x_3)\|_E\|\beta(\cdot,y_3)\|_F dy_3\\ &\leq C2^{-j}\|\partial_3\alpha\|_{L^\infty(\mathbf{R}_{x_3};E)}2^j\int_{\mathbf{R}^d}|h(2^j(x_3-y_3))|\,2^j|y_3-x_3|\|\beta(\cdot,y_3)\|_F dy_3.\end{aligned}$$

Then Young inequality implies that

$$\|[\dot\Delta_j,\alpha]\beta\|_{L^p(\mathbf{R}_{x_3};G)}\leq C2^{-j}\|\partial_3\alpha\|_{L^\infty(\mathbf{R}_{x_3};E)}\|h(\cdot)|\cdot|\,\|_{L^1(\mathbf{R})}\|b\|_{L^p(\mathbf{R}_{x_3};F)}.$$

This concludes the proof of the lemma.

Remark This lemma can be interpreted as a gain of one derivative by commutation between the operator $\dot\Delta_j$ and the multiplication by a lipschitzian function.

Let us go back to the proof of Proposition 4.2.4. We have, using the Hölder inequalities and Lemma 4.2.6 with $E=L^4_h$, $F=L^2_h$ and $G=L^{\frac43}_h$,

$$\begin{aligned}K^2_{j,j'}(t) &\stackrel{\mathrm{def}}{=} \Big|\int_{\mathbf{R}^3}\Big([\dot\Delta_j,\dot S_{j'-1}w^3(t)]\partial_3\dot\Delta_{j'}a(t)\Big)(x_h,x_3)\dot\Delta_j a(t)(x_h,x_3)dx_hdx_3\Big|\\ &\leq \int_{\mathbf{R}}\Big\|[\dot\Delta_j,\dot S_{j'-1}w^3(t)]\partial_3\dot\Delta_{j'}a(t)(\cdot,x_3)\Big\|_{L^{\frac43}_h}\|\dot\Delta_j a(t,\cdot,x_3)\|_{L^4_h}dx_3\\ &\leq C2^{-j}\|\dot S_{j'-1}\partial_3 w^3(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\int_{\mathbf{R}}\|\partial_3\dot\Delta_{j'}a(t,\cdot,x_3)\|_{L^4_h}\|\dot\Delta_j a(t,\cdot,x_3)\|_{L^4_h}dx_3\\ &\leq C2^{-j}\|\dot S_{j'-1}\operatorname{div}_h w^h(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\int_{\mathbf{R}}\|\partial_3\dot\Delta_{j'}a(t,\cdot,x_3)\|_{L^4_h}\|\dot\Delta_j a(t,\cdot,x_3)\|_{L^4_h}dx_3.\end{aligned}$$

Sobolev embeddings, interpolation inequality together with Lemma 4.2.1 and Lemma 4.2.2 gives, by definition of $\|\cdot\|_{T,s}$ and thanks to Lemma 4.2.4,

$$\begin{aligned}\sum_{|j'-j|\leq5}\int_0^T K^2_{j,j'}(t)dt &= C\sum_{|j'-j|\leq5}2^{j'-j}\int_0^T\|\dot S_{j'-1}\nabla_h w(t)\|_{L^\infty(\mathbf{R}_{x_3};L^2_h)}\int_{\mathbf{R}}\|\nabla_h\dot\Delta_{j'}a(t,\cdot,x_3)\|^{\frac12}_{L^2_h}\\ &\qquad\times\|\dot\Delta_{j'}a(t,\cdot,x_3)\|^{\frac12}_{L^2_h}\|\dot\Delta_j\nabla_h a(t,\cdot,x_3)\|^{\frac12}_{L^2}\|\dot\Delta_j a(t,\cdot,x_3)\|^{\frac12}_{L^2}dx_3dt\\ &\leq \frac{C}{\nu}c_j^2 2^{-2js}\|w\|_{T,\frac12}\|a\|^2_{T,s}.\end{aligned}$$

This concludes the proof of Proposition 4.2.4.

Propositions 4.2.1–4.2.4 implies Lemma 4.2.3. The proof of Theorem 4.2.2 is now complete.

4.3 References and Remarks

The use of anisotropic Sobolev spaces is not recent in partial differential equations if we have in mind boundary value problem (see for instance the book [33]). An anisotropic paradifferential calculus has been built by M. Sablé-Tougeron in [50]. Anisotropic Sobolev spaces have been introduced in the context of incompressible Navier-Stokes system by D. Iftimie in [34]. The study of anisotropic incompressible Navier-Stokes system has been initiated in [13] and in [35]. The sharp scaling invariant result (Theorem 4.2.1) has been proved by M. Paicu in [47].

Bibliography

[1] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l'École Normale Supérieure*, **14**, 1981, pages 209–246.

[2] L. Caffarelli, R. Kohn et L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Communications on Pure and Applied Mathematics*, **35**, 1982, pages 771–831.

[3] M. Cannone, Y. Meyer et F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire "Équations aux Dérivées Partielles de l'École polytechnique, Exposé VIII, 1993–1994.

[4] M. Cannone, *Ondelettes, paraproduits et Navier-Stokes*, Diderot Éditeur, Arts et Sciences, 1995.

[5] J.-Y. Chemin, Remarques sur l'existence pour le système de Navier-Stokes incompressible, *SIAM Journal of Mathematical Analysis*, **23**, 1992, pages 20–28.

[6] J.-Y. Chemin et N. Lerner, Flot de champs de vecteurs non-lipschitziens et équations de Navier-Stokes, *Journal of Differential Equations*, **121**, 1995, pages 314–328.

[7] J.-Y. Chemin, *Fluides parfaits incompressibles*, Astérisque, **230**, 1995.

[8] J.-Y. Chemin and C.-J. Xu, Inclusions de Sobolev en calcul de Weyl-Hörmander et systèmes sous-elliptiques, *Annales de l'École Normale Supérieure*, **30**, 1997, pages 719–751.

[9] J.-Y. Chemin, A remark on the inviscid limit for two-dimensionnal incompressible fluid, *Communication in Partial Differential Equations*, **21**, 1996, pages 1771–1779.

[10] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *Journal d'Analyse Mathématique*, **77**, 1999, pages 27–50.

[11] J.-Y. Chemin, Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray, *Prépublication du centre de Mathématiques de l'École polytechnique, à paraître dans Séminaire et Congrès, 2004, page 99–123.*

[12] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Anisotropy and dispersion in rotating fluids, in Non linear Partial Differential Equations and their Applications, *Studies in Mathematics and its applications*, North Holland, 2002, pages 171–192.

[13] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *Modélisation Mathématique et Analyse Numérique*,**34**, 2000, pages 315–335.

[14] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Basics of Mathematical Geophysics, preprint of École polytechnique

[15] J.-Y. Chemin and N. Masmoudi, About the life span of regular solutions of equations related to viscoelastic fluids, *SIAM Journal of Mathematical Analysis*, **33**, 2001, pages 84–112.

[16] F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, *Duke Mathematical Journal*, **77**, 1995, pages 657–698.

[17] P. Constantin et C. Foias, *Navier-Stokes equations*, Chigago University Press, 1988.

[18] B. Desjardins, A few remarks on ordinary differential equations, *Communications in Partial Differential Equations*, **21**, 1996, pages 1667–1703.

[19] B. Desjardins, Linear transport equations with values in Sobolev spaces and application to the Navier-Stokes equations, *Differential and Integral Equations*, **10**, 1997, pages 577–586.

[20] R. Di Perna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Inventiones Mathematicae*, **98**, 1989, pages 511–549.

[21] T.M. Fleet, *Differential analysis*, Cambridge University Press, 1980.

[22] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archiv for Rational Mechanic Analysis*, **16**, 1964, pages 269–315.

[23] G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo, Unicité des solutions mild des équations de Navier-Stokes dans $L^3(\mathbf{R}^3)$ et d'autres espaces limites, *Revista Matematica Iberoamericana*, **16**, 2000, pages 605–667.

[24] I. Gallagher, Un résultat de stabilité pour les solutions faibles des équations des fluides tournants, *Notes aux Comptes-Rendus de l'Académie des Sciences de Paris*, **324**, Série 1, 1996, pages 183–186.

[25] I. Gallagher, The tridimensional Navier-Stokes equations with almost bidimensional data: stability, uniqueness and life span, *International Mathematical Research Notices*, **18**, 1997, pages 919–935.

[26] I. Gallagher, D. Iftimie, and F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations, *Annales de l' Institut Fourier*, **53**, 2003, pages 1387–1424.

[27] P. Gérard, Y, Meyer et F. Oru, Inégalités de Sobolev précisées, *Séminaire EDP de l'École Polytechnique*, 4, 1996-1997.

[28] Z. Grujic et I. Kukavica, Space analycity for the Navier-Stokes and related equations with initial data in L^p, *Journal of Functional Analysis*, **152**, 1998, pages 247–466.

[29] Y. Giga, Solutions for semi linear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *Journal of Differential Equations*, **61**, 1986, pages 186–212

[30] Y. Giga and R. Kohn, About self similar *Communications in Pure and Applied Mathematics*, **38**, 19985, pages 297–319.

[31] Y. Giga et T. Miyakawa, Solutions in L^r of the Navier-Stokes initial value problem, *Archiv for Rational Mechanics and Analysis*, **89**, 1985, pages 267–281.

[32] Y. Giga et T. Miyakawa, Navier-Stokes flows in $\mathbf{R}^3$ and Morrey spaces, *Communications in Partial differential Equations*, **14**, 1989, pages 577–618.

[33] L. Hörmander, *Linear Partial Differential Operators*, Grundlehren der mathematischen Wissenschaften, **116**, Springer-Verlag, 1963.

[34] D. Iftimie, The resolution of the Navier–Stokes equations in anisotropic spaces, *Revista Matematica Ibero–Americana*, **15**, 1999, pages 1-36.

[35] D. Iftimie, A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, *SIAM Journal of Mathematical Analysis*, **33**, 2002, page 1483–1493.

[36] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$, *Journal of functionnal Analysis*, **9**, 1972, pages 296–305.

[37] T. Kato et G. Ponce, Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbf{R}^2)$, *Revista Matemática Iberoamericana*, **2**, 1986, pages 73–88.

[38] H. Kozono et M. Yamazaki, Semilinear heat equations and the Navier-Stokes equations with distributions in new function spaces as initial data, *Communications in Partial Differential Equations*, **19**, 1994, pages 959–1014.

[39] P.-G. Lemari-Rieusset, *Recent developments in the Navier-Stokes problem*, Research Notes in Mathematics, **431**, 2002.

[40] J. Leray, Étude de diverses équations intégrales non linéaires et quelques problèmes que pose l'hydrodynamique, *Journal de Mathématiques Pures et Appliquées*, **9**(12),1933, pages 1–82.

[41] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63**, 1933, pages 193–248.

[42] P.-L. Lions, Équations différentielles ordinaires et équations de transport avec des coefficients irréguliers, *Séminaire Équations aux Dérivées Partielles de l'École Polytechnique*, 1988-1989.

[43] Malek, J. Necas, M. Rokyta et M. Ruzicka, *Weak and measure-valued solutions to evolutionnary partial differential equations*, Applied Mathematics and Mathematical Computation, **13**, Chapman Hall, 1996.

[44] J. Neças, , M. Ružička and V. Šverak, On Leray's self-similar solutions of the Navier-Stokes equations *Acta Mathematica*, **176**, 1996, pages 283–294.

[45] Y. Meyer, *Ondelettes et opérateurs tome 2*, Hermann, 1991.

[46] F. Oru, PhD dissertation.

[47] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Prépublication du CMLS, École polytechnique*, **13**, 2002.

[48] G. Ponce, R. Racke, T.C. Sideris, and E.S. Titi, Global stability of large solutions to the 3D Navier-Stokes equations, *Communications in Mathematical Physics*, **159**, 1994, pages 329–341.

[49] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, 3. Walter de Gruyter, Berlin, 1996.

[50] M. Sablé-Tougeron, Régularité microlocale pour des problèmes aux limites non linéaires, *Annales de l'Institut Fourier*, **36**, 1986, pages 39–82.

[51] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[52] E.M. Stein, *Harnomic Analysis*, Princeton University Press, 1993.

[53] M. Taylor, *Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials*, Mathematical Surveys and Monographs, 81, American Mathematical Society, Providence, 2000.

[54] R. Temam, *Navier-Stokes equations, Theory and Numerical analysis*, North Holland, Amsterdam, **1984**.

[55] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland (1978).

[56] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

[57] F. Vigneron, Spatial decay of the velocity field of an incompressible viscous field in $\mathbf{R}^d$. *Preprint 2004–07 of CMLS, École polytechnique,*

[58] W. von Wahl, *The equations of Navier-Stokes and abstract parabolic equations*, Aspect der Mathematik, Vieweg & Sohn, Wiesbaden, 1985.

[59] M. Wiegner, Decay results for the weak solutions of the Navier-Stokes equations on $\mathbf{R}^n$, *Journal of the London Mathematical Society*, **35**, 1987, pages 303–313.