Localization in Nonlinear Lattice Schrödinger Equations with Analytic Quasi-Periodic Potentials

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The equations

We consider a family of quasi-periodic lattice nonlinear Schrödinger equations

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n + |q_n|^2 q_n = 0, \quad n \in \mathbb{Z} \tag{1}$$

where

- $0 < \epsilon \ll 1$,
- V is a non-constant real analytic function on $\mathbb{R}/\mathbb{Z}$,
- α satisfying the Diophantine condition.
- or the potential is independent identically distributed random variables.

Physical Background

- Evolution of Bose-Einstein condensate in a disordered lattice nonlinear optics
- Wave propagation in a nonlinear disordered media

Localization

We will study the localization problem of Lattice nonlinear Schrödinger equations. A solution is said to be localized if, for any $s > 0$,

$$\sup_t \sum_{n\in\mathbb{Z}} |n|^{2s} |q_n(t)|^2 < \infty.$$

In general, to understand the dynamics of an infinite dimensional equation

$$\mathrm{i}\dot{x} = Ax + \mu f(x),$$

it is necessary to understand the operator A and the linear equation.

Schrödinger operators and linear Schrödinger equations

Schrödinger operators:

$$H : l^2 \to l^2, \quad (Hq)_n = \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n.$$

- Central problem: Pure point spectrum and Anderson localization.

Linear equations:

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n = 0, \quad n \in \mathbb{Z}$$

- Central problem: Dynamical localization.
  All solutions with exponentially localized initial value $q(0) = \{q_n(0)\}_{n\in\mathbb{Z}}$ are localized for all t.

Contributors

Gordon, Germinet, Simon, Kotani, Last, Krasovsky, Klein, Klopp, Sorets, Dinaburg, Herman, Johnson, Fröhlich, Spencer, Wittwer, Craig, Goldstein, Puig, Bourgain, Bjerklöv, Gilbert, Pearson, Schlag, Jitomirskaya, Pöschel, Avila, Bourgain, Bellissard, Lima, Scoppola, Eliasson, $\cdots$ (far from complete)

Conclusions

If the potential is sufficiently "random", then almost surely ,

the spectrum of the Schrödinger operators is pure point;

typical solutions of linear equations are quasi-periodic or almost-periodic;

the linear equations have dynamical localization.

Localization problem in lattice NSEs

Recall the lattice nonlinear Schrödinger equations

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V_n q_n + |q_n|^2 q_n = 0, \quad n \in \mathbb{Z}, \tag{2}$$

where V_n is independent identically distributed random variables (i.i.d.), or quasi-periodic $V_n = V(n\alpha + x)$ with Diophantine α.

- In general, disorder of potential helps the localization, while non-linearity will destroy the localization.
- Dynamical localization, i.e., all solutions with localized initial data are localized all the time, is not expectable for nonlinear systems.

- Frölich-Spence-Wayne(1985, J. Stat.Phys), Bellissard-Vittot(1985, CPT-Marseille), Pöschel (1990, Commun. Math. Phys.)

$$\mathrm{i}\dot{q}_n + V_n q_n + \delta|q_{n+1}|^2 q_n + \delta|q_{n-1}|^2 q_n = 0, \quad n \in \mathbb{Z},$$

  where $\{V_n\}$ are i.i.d. . The existence of one (localized) almost-periodic solution for "typical V_n" was proved.
- Yuan(2002, Commun. Math. Phys.), Bambusi-Vella (2002, DCDS). Existence of a family of quasi-periodic solutions was proved.
- Geng-Viveros-Yi (2008, Physica D.) Same result holds for

$$\mathrm{i}\dot{q}_n + V_n q_n + \sum_m e^{-|n-m|}|q_m|^2 q_n = 0,$$

  where $\{V_n\}$ are positive and satisfy the asymptotic condition $V_n \geq \frac{1}{n^2}$.

A conjecture by Frölich-Spencer-Wayne

Consider an ergodic family of lattice Schrödinger equations

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V_n q_n + |q_n|^2 q_n = 0, \quad n \in \mathbb{Z}. \tag{3}$$

where V_n is either i.i.d., or quasi-periodic $V_n = V(n\alpha + x)$.

Conjecture: For almost all equations in the family, "most" solutions with

small initial data of finite support

are localized.

Mathematical results for lattice Schrödinger

- Bourgain-Wang(2008, J. Eur. Math. Soc.)

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V_n q_n + \delta |q_n|^2 q_n = 0,$$

  where $\{V_n\}_{n\in\mathbb{Z}}$, the potential, is a family of i.i.d. random variables. They proved existence of a (localized) quasi-periodic solution for "most" equations.

- Geng-Zhao(2011, preprint)

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + \tan\pi(x + n\alpha) q_n + \epsilon |q_n|^2 q_n = 0,$$

  where α Diophantine. They proved existence of a family of quasi-periodic solutions for a large class of equations.

In the analytic quasi-periodic potential case, it is much more difficult to obtain localization since the potential is less random.

Numerical results

Many numerical results by physicists saying that the delocalization occurs with some rates, i.e., $t^{\frac{1}{3}}$, $t^{\frac{2}{5}}$.

The numerical experiments were carried out for large initial data.

Main Result

Theorem

Consider a family of lattice Schrödinger equations

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n + |q_n|^2 q_n = 0, \quad n \in \mathbb{Z} \tag{4}$$

where α is Diophantine, V is analytic, ϵ is small depending on α and V. Then for almost every x, the following holds:
Arbitrarily take $\{n_1, \cdots, n_b\}$. Let

$$q^0 = (\cdots, 0, q^0_{n_1}, 0, \cdots, 0, q^0_{n_i}, 0, \cdots, 0, q^0_{n_b}, 0, 0, \cdots)$$

with

$$q^0_{n_i} \in (0, \epsilon^{\frac{3}{5}}], i = 1, \cdots, b.$$

Then, with probability $(1 - O(\epsilon^{\frac{1}{3}}))^b$, the solution $q(t, q_0)$ is localized in space and quasi-periodic in time.

Remarks

- The results are true for more general equations, such as

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n + |q_n|^{2p} q_n = 0,$$

and

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n + |q_n|^{2p}(q_{n+1} + q_n + q_{n-1}) = 0,$$

where $n \in \mathbb{Z}, p \in \mathbb{Z}_+$.

- The smallness of ϵ is necessary. Otherwise, there is no localization even for linear equations due to the absolute continuous spectrum. This is different from the random case.

Remarks-continued

- The potential V can be in the Gevrey class

$$\sup_{x\in\mathbb{R}/\mathbb{Z}} |\partial^m V(x)| \leq CL^m m!, \quad m \geq 0,$$

for some C, $L > 0$, and satisfying the transversality condition

$$\max_{0\leq m\leq \tilde{s}} |\partial_\varphi^m (V(x+\varphi) - V(x))| \geq \tilde{\xi} > 0, \quad \forall x, \forall \varphi,$$
$$\max_{0\leq m\leq \tilde{s}} |\partial_x^m (V(x+\varphi) - V(x))| \geq \tilde{\xi} |\varphi|_1, \quad \forall x, \forall \varphi,$$

for some $\tilde{\xi}$, $\tilde{s} > 0$.

The method and remarks

- Method: KAM theory.
- Difficulties: Dense eigenvalues, non-uniformly decaying eigen-states.
- The smallness of the perturbation does not depend on the dimension of torus.
- Advantage: Spatial structure, Short coupling.

The Hamiltonian

The Hamiltonian associated with the lattice equation is

$$H(x) = H_0(x) + G, \tag{5}$$

where

$$H_0(x) := \epsilon \sum_{n\in\mathbb{Z}} \bar{q}_n(q_{n+1} + q_{n-1}) + \sum_{n\in\mathbb{Z}} V_n(x) q_n \bar{q}_n = \epsilon\langle Sq, \bar{q}\rangle + \langle \Lambda(x) q, \bar{q}\rangle,$$

with Λ and S satisfying the shift condition

$$\Lambda_{mn}(x + k\alpha) = \Lambda_{m+k,n+k}(x), \quad S_{mn}(x + k\alpha) = S_{m+k,n+k}(x),$$

and

$$G := \frac{1}{2}\sum_{n\in\mathbb{Z}} |q_n|^4.$$

The Action-Angle variables

Arbitrarily take $\mathcal{J} = \{n_1, \cdots, n_b\} \subset \mathbb{Z}$ and then fixed.

We introduce action-angle variables and parameters to the Hamiltonian function (5),

$$q_i = \sqrt{I_i + \xi_i} e^{\mathrm{i}\theta_i}, \quad \bar{q}_i = \sqrt{I_i + \xi_i} e^{-\mathrm{i}\theta_i}, \quad i \in \mathcal{J},$$

where $\xi = (\xi_{n_1}, \cdots, \xi_{n_b}) \in [\epsilon^{\frac{3}{5}}, 2\epsilon^{\frac{3}{5}}]^b$ is a parameter.

The normal form

Then the Hamiltonian (5) becomes

$$H(x,\xi) = \tilde{\mathcal{N}}(\theta, I, q, \bar{q}; x, \xi) + \tilde{P}(\theta, I, q, \bar{q}; \xi),$$

where

$$\begin{aligned} \tilde{\mathcal{N}}(\theta, I, q, \bar{q}; x, \xi) &= \sum_{n\in\mathcal{J}} (V_n(x) + \xi_n) I_n + \sum_{n\in\mathbb{Z}_1} V_n(x) q_n \bar{q}_n \\ &\quad + \epsilon \sum_{n\in\mathbb{Z}} (q_n \bar{q}_{n+1} + \bar{q}_n q_{n+1}) \end{aligned}$$

up to an irrelevant constant $\sum_{n\in\mathcal{J}} (V_n(x)\xi_n + \frac{1}{2}\xi_n^2)$.

The perturbation

The perturbation $\tilde{P} = \tilde{P}(\theta, I, q, \bar{q}; \xi)$ is

$$\begin{aligned}
& \frac{1}{2}\sum_{n\in\mathbb{Z}_1}|q_n|^4 + \frac{1}{2}\sum_{n\in\mathcal{J}} I_n^2 + \epsilon \sum_{\substack{n\in\mathcal{J}\\ n+1\notin\mathcal{J}}} \sqrt{I_n+\xi_n}(e^{-\mathrm{i}\theta_n} q_{n+1} + e^{\mathrm{i}\theta_n}\bar{q}_{n+1}) \\
& + \epsilon \sum_{\substack{n\notin\mathcal{J}\\ n+1\in\mathcal{J}}} \sqrt{I_{n+1}+\xi_{n+1}}(e^{-\mathrm{i}\theta_{n+1}} q_n + e^{\mathrm{i}\theta_{n+1}}\bar{q}_n) \\
& + \epsilon \sum_{\substack{n\in\mathcal{J}\\ n+1\in\mathcal{J}}} \sqrt{I_n+\xi_n}\sqrt{I_{n+1}+\xi_{n+1}}(e^{-\mathrm{i}(\theta_n-\theta_{n+1})} + e^{\mathrm{i}(\theta_n-\theta_{n+1})}) \\
= \; & \frac{1}{2}\sum_{n\in\mathbb{Z}_1}|q_n|^4 + \frac{1}{2}\sum_{n\in\mathcal{J}} I_n^2 + \text{finite many } \theta \text{ dependent terms.}
\end{aligned}$$

A technique problem

A technique problem: after introducing the action-angle variables, the shift condition of H in (5) has been destroyed. To solve this problem, we add b variables $q'_{n_1}, \cdots, q'_{n_b}$ and the corresponding conjugate variables $\bar{q}'_{n_1}, \cdots, \bar{q}'_{n_b}$ into this system. Omitting the prime of the newly-added variables for convenience, we re-write the system as

$$\begin{aligned} H &= \tilde{\mathcal{N}} + \tilde{P} \\ &= \sum_{n\in\mathcal{J}} (V_n(x) + \xi_n) I_n + \left[\sum_{n\in\mathbb{Z}} V_n(x)|q_n|^2 + \epsilon \sum_{n\in\mathbb{Z}} (\bar{q}_n q_{n+1} + q_n \bar{q}_{n+1}) \right] \\ &\quad + \langle A(x,\xi) q, \bar{q} \rangle + P, \end{aligned}$$

The normal form-continued

where

$$\begin{aligned}\langle A(x,\xi)q,\bar{q}\rangle &= -\sum_{n\in\mathcal{J}} V_n(x)|q_n|^2 \\ &\quad -\epsilon \sum_{n \text{ or } n+1\in\mathcal{J}} (\bar{q}_n q_{n+1} + q_n \bar{q}_{n+1}).\end{aligned}$$

Note that $\langle A(x,\xi)q,\bar{q}\rangle$ contains only finite many quadratic terms.

The Hamiltonians

To this stage, we arrive at the perturbed Hamiltonians of the form

$$H(x,\xi) = \mathcal{N}(x,\xi) + P(x,\xi), \tag{6}$$

with

$$\mathcal{N}(x,\xi) = \langle \omega(x,\xi), I \rangle + \langle T(x)q, \bar{q} \rangle + \langle A(x,\xi)q, \bar{q} \rangle,$$

where $T(x)$ is the Schrödinger operators, $A(x,\xi)$ is a finite range operator satisfying

$$A_{mn} \equiv 0, \quad \text{if} \;\; |m| \;\; \text{or} \;\; |n| > \hat{N}.$$

For each x, $H(x,\xi)$ is C^1_W smoothly parametrized by $\xi \in \mathcal{O} = \mathcal{O}(x)$.

$$P = \frac{1}{2}\sum_{n\in\mathbb{Z}_1} |q_n|^4 + \frac{1}{2}\sum_{n\in\mathcal{J}} I_n^2 + \text{finite many } \theta \text{ dependent terms.}$$

KAM theorem

Our goal is to prove that, for a.e. $x \in \mathbb{R}/\mathbb{Z}$, the Hamiltonians $H = \mathcal{N} + P$ still admits invariant tori for most of the parameter $\xi \in \mathcal{O}$ corresponding to space localized and time quasi-periodic solutions, provided that $\|X_P\|$ is sufficiently small.

KAM theorem

Theorem

For the above Hamiltonians, if $\varepsilon = \varepsilon(\alpha, V)$ is small, there is a full-measure subset $\tilde{\mathcal{X}}$ of $\mathbb{R}/\mathbb{Z}$ such that for every $x \in \tilde{\mathcal{X}}$, the following holds. There exists a positive measure Cantor set $\mathcal{O}_\epsilon = \mathcal{O}_\epsilon(x) \subset \mathcal{O}$ with

$$meas(\mathcal{O}_\epsilon) = (1 - O(\epsilon^{\frac{1}{3}}))^b meas(\mathcal{O}),$$

such that for $\xi \in \mathcal{O}_\epsilon$, the Hamiltonian H has a b-dimensional inviant torus correponding to the b-frequency quasi-periodic solution $q(t, \xi)$ of the lattice Schrödinger equations. Moreover, for any $s > 0$,

$$\sup_t \sum_{n \in \mathbb{Z}} |n|^{2s} |q_n(t) - q(0)|^2 < \infty.$$

Two ingredients in KAM

- The cohomological equation.
- Measure estimates

The homological equation

In each step of KAM iteration, the key is to solve homological equtions of the form

$$(\langle k, \omega \rangle I + T + A)F_1 = f_1$$

$$\langle k, \omega \rangle F_2 + [T + A, F_2] = f_2$$

where T is the Schrödinger operator and A is of finite range.

Small divisor conditions

The small divisor condition will be

$$|\langle k, \omega \rangle I + \Omega_i| > \frac{\gamma}{|k|^{\tau(k)}}$$

and

$$|\langle k, \omega \rangle I + \Omega_i - \Omega_j| > \frac{\gamma}{|k|^{\tau(k)}},$$

where $\tau(k)$ is depends on the support of k, and $\{\Omega_n\}$ are eigenvalues of $T + A$.

KAM step: Property of T(by Eliasson)

Lemma

Fix any $x \in \mathbb{R}/\mathbb{Z}$. There exists a sequences of orthogonal matrices U_ν on $\mathbb{R}/\mathbb{Z}$, $\nu \in \mathbb{N}$, with $|(U_\nu - I_{\mathbb{Z}})_{mn}| \le \varepsilon_0^{\frac{1}{2}} e^{-\frac{1}{2}\sigma_\nu |m-n|}$, such that

$$U_\nu^*(D_0 + Z_0)U_\nu = D_\nu + Z_\nu,$$

where $Z_\nu = (Z_{mn}^\nu)_{m,n\in\mathbb{Z}}$ is a symmetric matrix satisfying

$$|Z_{mn}^\nu| \le \varepsilon_\nu e^{-\sigma_\nu |m-n|},$$

and D_ν is a real symmetric matrix which can be block-diagonalized via an orthogonal matrix Q_ν with $Q_{mn}^\nu = 0$ if $|m-n| > N_\nu$. More precisely,

$$\tilde{D}_\nu = Q_\nu^* D_\nu Q_\nu = \prod_j \tilde{D}_{\Lambda_j^\nu}^\nu \ \ with \ \ \sharp\Lambda_j^\nu \le M_\nu, \ \ \mathrm{diam}\Lambda_j^\nu \le M_\nu N_\nu.$$

Further problems

Prove similar results for

- One dimension lattice Schrödinger with random potential

$$\mathrm{i}\dot{q}_n + q_{n+1} + q_{n-1} + V_n q_n + |q_n|^2 q_n = 0, \quad n \in \mathbb{Z}^1, \tag{7}$$

  i.e., V_n is independent identically distributed random variables.
- higher dimensional lattice Schrödinger with random potentials

$$\mathrm{i}\dot{q}_n + \epsilon \sum_{|j-n|=1} q_j + V_n q_n + |q_n|^2 q_n = 0, \quad n, j \in \mathbb{Z}^d \tag{8}$$

- Diffusion orbits and diffusion rates.

Thank You