Semilinear dispersive equations

Daniel Tataru

University of California, Berkeley

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Linear dispersive equations

- The Schrödinger equation:

$$(i\partial_t - \Delta)u = 0, \qquad u(0) = u_0$$

- The wave equation:

$$\Box u = 0, \qquad u(0) = u_0, \quad u_t(0) = u_1$$

- The linearized KdV

$$(\partial_t + \partial_x^3)u = 0, \qquad u(0) = u_0$$

Nonlinear dispersive equations

- The nonlinear Schrödinger equation (NLS):

$$(i\partial_t - \Delta)u = \pm|u|^{p-1}u, \qquad u(0) = u_0$$

- The nonlinear wave equation (NLW):

$$\Box u = \pm|u|^{p-1}u, \qquad u(0) = u_0, \quad u_t(0) = u_1$$

- The generalized KdV (gKdV):

$$(\partial_t + \partial_x^3)u = \partial_x(u^k), \qquad u(0) = u_0$$

Scaling and critical Sobolev spaces

Example:

$$(NLS) \qquad (i\partial_t - \Delta)u = \pm|u|^{p-1}u, \qquad u(0) = u_0 \in \dot{H}^s$$

Scaling law:

$$u(t,x) \to u_\lambda(t,x) = \lambda^\gamma u(\lambda^2 t, \lambda x), \qquad \gamma = \gamma(p)$$

Critical Sobolev space $\dot{H}^{s_c}$:

$$s_c : \qquad \|u\|_{\dot{H}^{s_c}} = \|u_\lambda\|_{\dot{H}^{s_c}}$$

Wave scaling:

$$u(t,x) \to u_\lambda(t,x) = \lambda^\gamma u(\lambda t, \lambda x), \qquad \gamma = \gamma(p)$$

KdV scaling:

$$u(t,x) \to u_\lambda(t,x) = \lambda^\gamma u(\lambda^3 t, \lambda x), \qquad \gamma = \gamma(p)$$

Expected results

a) Ill-posedness:

Theorem

Semilinear dispersive equations are ill-posed in H^s for $s < s_c$.

b) Well-posedness:

Theorem

Semilinear dispersive equations are well-posed in H^s for $s \geq s_c$ if $p \geq p(d)$.

$$p(d) = 1 + \frac{4}{n} \qquad \text{(Schrödinger and KdV)},$$

$$p(d) = 1 + \frac{4}{n-1} \qquad \text{(wave)}$$

Perturbative approach to well-posedness

Equation:

$$(i\partial_t - L)u = N(u), \qquad u(0) = u_0 \in H^s$$

Rephrase as

$$u = SN(u) + e^{itL}u_0$$

$$Sf(t) = \int_0^t e^{i(t-s)L} f(s)ds \qquad \text{(Duhamel term)}$$

X = space of solutions
Y = space of inhomogeneities
Fixed point argument in X provided that:

$$(i) \qquad e^{itL} : H^s \to X$$

$$(i) \qquad N : X \to Y$$

$$(ii) \qquad S : Y \to X$$

From local to global

A. Perturbative analysis for $s > s_c$:

$$T_{max} > \|u_0\|_{H^s}^{\gamma}$$

With conserved H^s "energy" ("energy" subcritical) :

$$T_{max} = \infty$$

B. Perturbative analysis for $s = s_c$:

$$T_{max} = \infty \qquad \text{for} \quad \|u_0\|_{\dot{H}^s} \ll 1$$

$$T_{max} = T_{max}(u_0) \qquad \text{for} \quad \|u_0\|_{\dot{H}^s} \gtrsim 1$$

With conserved H^s "energy" ("energy" critical) : no change.

Dispersive estimates

Equation:

$$(i\partial_t - L(D))u = 0, \qquad u(0) = u_0$$

Homogeneous solution:

$$u(t,x) = \int K(t,x-y)u_0(y)dy, \quad K(t,x) = \int e^{itL(\xi)}e^{ix\xi}\, d\xi$$

Stationary phase method $\Rightarrow$ pointwise bounds on K:

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{n}{2}}\|u(0)\|_{L^1}, \qquad [NLS,\ L(\xi) = \xi^2]$$

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{n-1}{2}}\|u(0)\|_{L^1}, \qquad [NLW,\ L(\xi) = |\xi|,\ |\xi| \approx 1]$$

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}\|u(0)\|_{L^1}, \qquad [KdV,\ L(\xi) = \xi^3,\ |\xi| \approx 1]$$

Strichartz estimates

Time averaged decay for frequency one data:

$$\|e^{itL}u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_{L^2} \qquad \text{(homogeneous form)}$$

$$\|K(t,\cdot) * f\|_{L_t^p L_x^q} \lesssim \|f\|_{L_t^{p_1'} L_x^{q_1}} \qquad \text{(inhomogeneous form)}$$

Strichartz range for (p,q) and (p_1,q_1):

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} \qquad (NLS)$$

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2} \qquad (NLW)$$

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2} \qquad (KdV)$$

$$p,q \geq 2, \qquad (p,q) \neq (2,\infty)$$

A sample problem

L^2 critical NLS:

$$(i\partial_t + \Delta)u = \pm\, u|u|^{\frac{4}{n}}, \qquad u(0) = u_0 \in L^2$$

Function spaces:

$$X = L_{xt}^{\frac{2(n+2)}{n}}, \qquad Y = L_{xt}^{\frac{2(n+2)}{n+4}}$$

Mapping properties:

$$(i) \qquad e^{it\Delta} : L^2 \to X \qquad \text{(homogeneous Strichartz)}$$

$$(ii) \qquad N : X \to Y \qquad \text{(trivial)}$$

$$(iii) \qquad S : Y \to X \qquad \text{(inhomogeneous Strichartz)}$$

Conclusion: Global well-posedness for small data.

Resonant vs. nonresonant interactions

Bilinear nonlinearities:

$$(i\partial_t + L(D))u = u^2$$

Solutions to linear equations are frequency localized on the characteristic set:

$$\Sigma = \{\tau = L(\xi)\}$$

Worst interactions: both inputs and the output are on Σ:

$$R = \{\xi_1, \xi_2;\ L(\xi_1) + L(\xi_2) = L(\xi_1 + \xi_2)\} \qquad \text{(resonant set)}$$

Favorable case: group velocities are transversal,

$$R_0 = R \cap \{L_\xi(\xi_1), L_\xi(\xi_2), L_\xi(\xi_1 + \xi_2) \text{ not all equal}\}$$

Unfavorable case: equal group velocities.

Bilinear estimates

Let E, F be disjoint sets. Typical estimate:

$$\|e^{itL}\chi_E(D)u_0 \cdot e^{itL}\chi_F(D)v_0\|_{L^2} \lesssim C(E,F)\|u_0\|_{L^2}\|v_0\|_{L^2}$$

- Useful for analysis of bilinear nonresonant estimates.
- Useful for analysis of trilinear transversal estimates
- Requires transversal tangent planes to $\Sigma = \{\tau = L(\xi)\}$ in E, F.
- Does **not** require nonvanishing curvatures.
- Extends easily to $X^{s,b}$ spaces for $b > \frac{1}{2}$.

Bourgain spaces

Characteristic set:

$$\Sigma = \{\tau = L(\xi)\}$$

Bourgain spaces:

$$\|u\|_{X^{s,b}} = \|\hat{u}\langle\xi\rangle^s\langle\tau - L(\xi)\rangle^b\|_{L^2}$$

Mapping properties:

$$e^{itL} : H^s \to X^{s,b}, \qquad b \in \mathbb{R}$$

$$S : X^{s,b-1}_{loc} \to X^{s,b}_{comp}, \qquad b > \frac{1}{2}$$

Strichartz embeddings:

$$X^{s,b} \subset L^p L^q, \qquad b > \frac{1}{2}$$

Bilinear $X^{s,b}$ estimates = convolution estimates in Fourier space

U^p and V^p spaces

1. V^p = functions of bounded p variation

$$\|u\|_{V^p(H)}^p = \sup_{t_k \nearrow} \sum_k \|u(t_{k+1} - u(t_k))\|_H^p$$

2. U^p is an atomic space with atoms

$$a = \sum_k 1_{[t_k, t_{k+1})} u_k, \qquad \sum_k \|u_k\|_H^p \leq 1$$

3. Spaces U_L^p, V_L^p adapted to the L evolution:

$$\|u\|_{U_L^p} = \|e^{-itL} u\|_{U^p L^2}, \qquad \|u\|_{V_L^p} = \|e^{-itL} u\|_{V^p L^2},$$

- U_L^p and V_L^p respect the scaling
- U_L^2 and V_L^2 are “good” replacements of $X^{0,\frac{1}{2}}$.
- Free Strichartz embeddings $U_L^p \subset L_t^p L_x^q$.

Focusing vs. defocusing problems

Defocusing problems.

- nonlinearity adds to the linear dispersive effects
- positive definite energy
- no nontrivial steady states
- scattering at infinity is expected
- Example: $(i\partial_t + \Delta)u = u|u|^{p-1}$

Focusing problems.

- nonlinearity counters the linear dispersive effects
- energy not positive definite
- nontrivial steady states (solitons) exist
- scattering at infinity cannot hold in general
- Example: $(i\partial_t + \Delta)u = -u|u|^{p-1}$

"Solitons"

Obstructions to global well-posedness:

1. Steady states
   - Example: (NLW)

$$\Box u = |u|^{p-1}u$$

$$\Delta Q = |Q|^{p-1}Q$$

2. Solitons
   - Example: (NLS)

$$(i\partial_t - \Delta)u = -|u|^{p-1}u \qquad u = e^{i\alpha t}Q(x)$$

$$\Delta Q = |Q|^{p-1}Q + \alpha Q, \qquad \alpha > 0$$

3. Self-similar solutions
   - Example: (NLS)

$$(i\partial_t - \Delta)u = -|u|^{p-1}u \qquad u = t^{\alpha}Q(x/\sqrt{t})$$

$$\Delta Q + \frac{i}{2}x\partial_x Q + i\alpha Q = |Q|^{p-1}Q, \qquad \alpha > 0$$

The threshold problem

Equation:

$$(i\partial_t + L(D))u = N(u), \qquad u(0) = u_0 \in \dot{H}^{s_c}$$

Perturbative theory ⇔ small data global result and scattering
"Solitons" ⇔ no large data result

Theorem (threshold conjecture)

Global well-posedness holds for all data with size below a treshold, which is given by the size of the smallest "soliton" (obstruction)

Theorem (defocusing conjecture)

In the absence of any obstructions global well-posedness holds for all data in the critical Sobolev space.

Soliton resolution

Equation:

$$(i\partial_t + L(D))u = N(u), \qquad u(0) = u_0 \in \dot{H}^{s_c}$$

Perturbative theory $\Leftrightarrow$ small data global result and scattering
Large data solutions may contain "solitons" and/or blow-up in finite time.

Conjecture (soliton resolution)

a) Any global in time solution which stays bounded in H^{s_c} resolves as $t \to \infty$ into a sum of solitons plus a dispersive part.

b) Any solution blowing up in finite time which stays bounded in H^{s_c} resolves as $t \to T_{max}$ into a sum of solitons plus a dispersive part.

Harmonic maps $\phi : \mathbb{R}^n \to (M, g)$.

Lagrangian:

$$L(\phi) = \int_{\mathbb{R}^n} |\nabla\phi|_m^2 \, dx$$

Euler-Lagrange equation:

$$\Delta\phi^i = -\Gamma^i_{jk}(\phi)\nabla\phi^j\nabla\phi^k$$

Covariant form:

$$D_k\partial_k\phi = 0,$$

Extrinsic form $(M, g) \subset (\mathbb{R}^N, e)$:

$$\Delta\phi^i = -S^i_{jk}(\phi)\nabla\phi^j\nabla\phi^k$$

Wave maps $\phi : \mathbb{M}^{n+1} \to (M, g)$.

Lagrangian:

$$L(\phi) = \int_{\mathbb{M}^{n+1}} \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_g \, dxdt = \int_{\mathbb{M}^{n+1}} |\partial_t \phi|_g^2 - |\nabla_x \phi|_g^2 \, dxdt$$

Critical points = Wave maps:

$$D^\alpha \partial_\alpha \phi = 0, \qquad \phi(0) = \phi_0, \;\; \partial_t \phi(0) = \phi_1$$

In local coordinates

$$\partial^\alpha \partial_\alpha \phi^i = -\Gamma^i_{jk}(\phi) \partial^\alpha \phi^j \partial_\alpha \phi^k$$

Extrinsic form

$$\partial^\alpha \partial_\alpha \phi^i = -S^i_{jk}(\phi) \partial^\alpha \phi^j \partial_\alpha \phi^k$$

Conserved energy:

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_x \phi|_g^2 + |\partial_t \phi|_g^2 \, dx$$

Harmonic heat flow $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to (M, g)$

Lagrangian:

$$L(\phi) = \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx$$

Gradient flow of L:

$$\partial_t \phi - \Delta \phi = \Gamma^i_{jk}(\phi) \nabla \phi^j \nabla \phi^k, \qquad \phi(0) = \phi_0$$

Covariant form:

$$\partial_t \phi - D_k \partial_k \phi = 0$$

Energy relation:

$$\frac{d}{dt} L(\phi) = - \int_{\mathbb{R}^n} |D_k \partial_k \phi|^2 \, dx$$

Schrödinger maps $\phi : \mathbb{R} \times \mathbb{R}^n \to (M, g, J, \omega)$

Hamiltonian (conserved energy):

$$H(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_x \phi|_g^2 \, dx$$

Symplectic form

$$\omega(v, w) = \int \langle v, Jw \rangle_g dx, \qquad v, w \in T_\phi M$$

Covariant formulation (for Kahler targets):

$$\partial_t \phi = J D_k \partial_k \phi, \qquad \phi(0) = \phi_0$$

Scale invariance

$$
\begin{array}{rll}
\text{(HM)} & \phi(x) \to \phi(\lambda x), & \dot{H}^{\frac{n}{2}} \\
\text{(WM)} & \phi(t,x) \to \phi(\lambda t, \lambda x), & \dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1} \\
\text{(HH)} & \phi(t,x) \to \phi(\lambda^2 t, \lambda x), & \dot{H}^{\frac{n}{2}} \\
\text{(SM)} & \phi(t,x) \to \phi(\lambda^2 t, \lambda x), & \dot{H}^{\frac{n}{2}}
\end{array}
$$

Classification:
$n = 1$: energy subcritical
$n = 2$: energy critical
$n \geq 3$: energy supercritical

Local vs global targets
$\dot{H}^{\frac{n}{2}} \not\subset L^\infty$: local coordinates not so useful
$\dot{H}^{\frac{n}{2}} \subset VMO$: homotopy of maps well-defined

Special target: the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$

$$\text{(HM)} \qquad \Delta\phi = -\phi|\nabla\phi|^2$$

$$\text{(WM)} \qquad \partial^\alpha\partial_\alpha\phi = -\phi(\partial^\alpha\phi \cdot \partial_\alpha\phi)$$

$$\text{(HH)} \qquad \partial_t\phi - \Delta\phi = \phi|\nabla\phi|^2$$

$$\text{(SM)} \qquad \partial_t\phi = \phi \times \Delta\phi$$

Energy space $\dot{H}^1$ splits into connected components = homotopy classes

Harmonic maps $\phi : \mathbb{R}^2 \to \mathbb{S}^2$

Theorem (Helein)

Global regularity of finite energy solutions. (True for any complete target)

Special solutions: k-equivariant harmonic maps,

$$Q_k(r, \theta) = (2 \tan^{-1}(r^k), k\theta), \qquad k \geq 1$$

- Minimizers of the Lagrangian in homotopy classes (indexed by k)
- Unique up to symmetries
- $k = 1$: stereographic projection.

Special target: the hyperbolic space $\mathbb{H}^2 \subset \mathbb{M}^{2+1}$

$$\text{(HM)} \qquad \Delta\phi = -\phi|\nabla\phi|_m^2$$

$$\text{(WM)} \qquad \partial^\alpha\partial_\alpha\phi = -\phi(\partial^\alpha\phi \cdot \partial_\alpha\phi)_m$$

$$\text{(HH)} \qquad \partial_t\phi - \Delta\phi = \phi|\nabla\phi|_m^2$$

$$\text{(SM)} \qquad \partial_t\phi = \phi \times_m \Delta\phi$$

- No nontrivial homotopy classes
- No finite energy harmonic maps

Wave-maps in $2+1$ dimensions

Known results:

1. Small data global well-posedness
   - Tao '01 $\mathbb{S}^2$, Krieger '03 $\mathbb{H}^2$, T. '04 (M, g)
   - complex function spaces, Strichartz, bilinear, null-frames
   - renormalization argument
2. Threshold conjecture
   - Sterbenz-T '08; also Tao '08 $\mathbb{H}^n$, Krieger-Schlag '08 H^2
3. Equivariant blow-up with soliton profile for $\mathbb{S}^2$
   - Krieger-Schlag-T. '05, Rodnianski-Sterbenz '05

Open problems

1. Long time behavior of near soliton solutions
2. The soliton conjecture

Schrödinger maps in $2+1$ dimensions

Known results:

1. Small data global well-posedness
   - Bejenaru-Ionescu-Kenig-T. '08
   - Strichartz, bilinear, lateral frames
   - frame method, gauge selection issues
2. Threshold conjecture for equivariant maps into $\mathbb{H}^2$, $\mathbb{S}^2$.
   - Bejenaru-Ionescu-Kenig-T. '11-12
3. k- equivariant soliton stability in $\dot{H}^1$, $k \geq 3$
   - Gustafson-Nakanishi-Tsai '07-09
4. 1,2- equivariant soliton instability in H^1, stability in stronger topology
   - Bejenaru-T. '10
5. 1-equivariant blow-up with soliton profile for $\mathbb{S}^2$
   - Raphael-Kenig-Merle '11, Perelman '12

Open problems

1. Threshold conjecture, nonequivariant case
2. The soliton conjecture

The frame method

Goal: Interpret (SM) as a semilinear Schrödinger system.
For each $(x, t) \in \mathbb{R}^{2+1}$ choose an orthonormal frame (v, w) in $T_{\phi(x,t)}\mathbb{S}^2$. Set

$$\psi_j = \partial_j \phi \cdot v + i \partial_j \phi \cdot w, \qquad j = 1, \cdots, n, n+1$$

Its evolution is described by the real connection coefficients

$$A_j = \partial_j v \cdot w$$

which define the connection $\mathbf{D}_m = \partial_m + i A_m$. Then ψ_m satisfy

$$\mathbf{D}_l \psi_m = \mathbf{D}_m \psi_l.$$

The curvature of the connection is given by

$$\mathbf{D}_l \mathbf{D}_m - \mathbf{D}_m \mathbf{D}_l = i(\partial_l A_m - \partial_m A_l) = i \Im(\psi_l \overline{\psi_m}).$$

Then replace the evolution of ϕ with the evolution of its derivatives ψ_m.

Schrödinger map equation:

$$\psi_{n+1} = i\mathbf{D}_l \psi_l.$$

Differentiating

$$\mathbf{D}_{n+1}\psi_m = i\mathbf{D}_l\mathbf{D}_l\psi_m + \Im(\psi_l\overline{\psi_m})\psi_l$$

and expanding

$$(i\partial_t + \Delta_x)\psi_m = -2iA_l\partial_l\psi_m + \Big(A_{n+1} + (A_l^2 - i\partial_l A_l)\Big)\psi_m - i\psi_l \Im(\overline{\psi}_l \psi_m).$$

This is coupled with the curl system for the A_j's

$$\partial_l A_m - \partial_m A_l = \Im(\psi_l\overline{\psi_m}).$$

and is invariant with respect to the gauge transformation

$$\psi_m \to e^{i\theta}\psi_m, \qquad A_m \to A_m + \partial_m\theta.$$

0. Comparison equation: cubic NLS,

$$(i\partial_t - \Delta)u = \pm\psi|\psi|^2$$

1. Extrinsic gauge. (v, w) are a fixed frame on the sphere

$$(i\partial_t - \Delta)\psi = \psi \cdot \nabla\psi \pm \psi|\psi|^2$$

- No such global frame exists
- Red term is nonperturbative

2. Coulomb gauge. Add elliptic constraint $\partial_j A_j = 0$:

$$(i\partial_t - \Delta)\psi = |\nabla|^{-1}|\psi|^2 \cdot \nabla\psi \pm \psi|\psi|^2$$

- Defined globally for small data
- Blue term is perturbative in high dimension $n \geq 4$.

3. The Caloric gauge. Define frame via a (HH) flow,

$$(i\partial_t - \Delta)\psi = B(\psi, \bar{\psi}) \cdot \nabla\psi \pm \psi|\psi|^2 \qquad B(\xi, \eta) \sim \frac{\xi + \eta}{\xi^2 + \eta^2}$$

The Caloric gauge.

- At each time t solve (HH) with $\phi(t)$ as the initial data,

$$\partial_s \phi - \Delta_x \phi = \phi |\partial_x \phi|^2 \qquad \phi(0,t,x) = \phi(t,x).$$

  As $s \to \infty$ we have $\phi \to P$.

- Choose (v_∞, w_∞) at $s = \infty$ as an arbitrary orthonormal base in $T_P \mathbb{S}^2$, and pull back along the heat flow using parallel transport,

$$w \cdot \partial_s v = 0 \Leftrightarrow A_0 = 0$$

- Derive a heat equation for ψ_m,

$$(\partial_s - \Delta_x)\psi_m = 2iA_l \partial_l \psi_m - (A_l^2 - i\partial_l A_l)\psi_m + i\Im(\psi_m \overline{\psi_l})\psi_l.$$

- Recover the coefficients A_m at $s = 0$ we integrate in

$$\partial_s A_m = \Im(\psi_0 \overline{\psi_m}) = \Im\Big((\partial_l \psi_l + iA_l \psi_l)\overline{\psi_m}\Big)$$